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Young diagrams in an $N \times M$ box and the KP hierarchy

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Abstract

In this paper, we firstly find that the multiplications $H_M(u_1)H_M(u_2)\cdots H_M(u_N)$ are tau functions of the KP hierarchy for any positive integer numbers $M,\ N$, where $H_M(z)$ is the generating function of Young diagrams in an $1 \times M$ box. Then as applications, we find that the general states and the scalar products of general states in the phase model and the XX0 model are all tau functions of the KP hierarchy. We also discuss the wave functions in these models.

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1. Introduction

Symmetric functions are an attractive research object, which were used to determine irreducible characters of highest weight representations of the classical groups [1–3]. Recently Symmetric functions appear in mathematical physics, especially in integrable models. In [4], the group in the Kyoto school uses Schur functions in a remarkable way to understand the KP and KdV hierarchies. In [5], Tsuda defined the UC hierarchy which is a generalization of the KP hierarchy and obtained that tau functions of the UC hierarchy can be realized in terms of the universal characters. In [6,7], Tsilevich and Sułkowski, respectively, give the realization of the phase model in the algebra of Schur functions and build the relations between the q-boson model and Hall–Littlewood functions. In [8], the authors give that the wave functions of the XXZ Heisenberg chain in two specific limits ($\triangle \to 0$ and $\triangle \to -\infty$) are expressed by means of Schur functions.

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Since these quantum integrable models and classical integrable systems can both be realized in the algebras of symmetric functions, they should have some relations. In [9], the authors find that the partition function of the six vertex model on a finite-size square lattice with domain wall boundary conditions is a tau function of the KP hierarchy. In [10,11], the authors give that the scalar product of a general state and a Bethe eigenstate in a finite-length XXZ spin-$\frac{1}{2}$ chain is a tau function of the KP hierarchy. In paper [11], the author also obtain that the scalar product of Bethe states in the phase model and the 4-vertex model becomes a tau function of the 2-KP hierarchy. In [12], the author derive that the scalar product of two general states in the phase model is a tau function of the 2-toda hierarchy. In this paper, we find that the general states and the scalar products of two general states in the phase model and the XX0 model are all tau functions of the KP hierarchy.

The first result of this paper is that the multiplications $H_M(u_1)H_M(u_2)\cdots H_M(u_N)$ are tau functions of the KP hierarchy for any positive integer numbers M, N, where $H_M(z)$ is the generating function of Young diagrams in an $1 \times M$ box. We know that if $M \to \infty$ the generating function $H(z) = H_\infty(z)$ is the vertex operator

$$H(z) = e^{\xi(\mathbf{x},z)}, \quad H^{\perp}(z) = e^{\xi(\tilde{\partial}_{\mathbf{x}},z^{-1})}$$

where $\xi(\mathbf{x}, z) = \sum_{n=1}^{\infty} x_n z^n$ and

$$\tilde{\partial}_{\mathbf{x}} = (\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \frac{1}{3}\partial_{x_3}, \cdots), \quad \partial_{x_i} = \frac{\partial}{\partial x_i}.$$

Here we do not distinguish between Schur function $S_\lambda(\mathbf{x})$ and Young diagram λ. Explicitly $H_M(z)$ is the truncated expansion of $H(z)$ (including all the terms z^j satisfying $0 \le j \le M$), i.e., $H_M(z) = 1 + zh_1 + \cdots + z^M h_M$, where $h_j(\mathbf{x})$ can be written as

$$h_j(\mathbf{x}) = \sum_{k_1+2k_2+\cdots jk_j=j} \frac{x_1^{k_1} x_2^{k_2} \cdots x_j^{k_j}}{k_1! k_2! \cdots k_j!}.$$

We find that the multiplication $H_M(u_1)H_M(u_2)\cdots H_M(u_N)$ equals

$$e^{X_1(u)} \cdots e^{X_N(u)} |0\rangle$$

with

$$X_k(u_1, \cdots, u_N) = (-1)^{k-1} \sum_{j=1}^{M} S_{(j,1^{k-1})}(u_1, \cdots, u_N) \psi_{-\frac{2j-1}{2}} \psi^*_{-\frac{2k-1}{2}}, \quad k = 1, 2, \cdots, N$$

where ψ_j, ψ_j^* are Fermions. This result tells us that the multiplications

$$H_M(u_1)H_M(u_2)\cdots H_M(u_N)$$

are tau functions of the KP hierarchy.

The second result of this paper is that the general states and the scalar products of two general states in the phase model and the XX0 model are tau functions of the KP hierarchy. The general state of the phase model is [13]

$$|\Psi(u_1, \cdots, u_N)\rangle = \prod_{i=1}^{N} B(u_j)|0\rangle.$$

In papers [6,7], the authors give that the operator $B(u)$ acts on Schur functions as the operator of multiplication by $u^M H_M(u^2)$, then the state $|\Psi(u_1,\cdots,u_N)\rangle$ with variables $\mathbf{x}$ and the scalar product of two general states are tau functions of the KP hierarchy.

As to the XX0 model, from papers [8,14], we know that the general state $|\Psi_N(u_1,\cdots,u_N)\rangle$ equals

$$L_N H_{M+1-N}(\tilde{u}_1)\cdots H_{M+1-N}(\tilde{u}_N)|0\rangle$$

where $\tilde{u} = i\coth u$ and

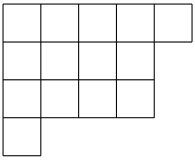
$$L_N = \prod_{j=1}^{N} i[u_j]^M (1 + S_{(1)}(\tilde{u}_1,\cdots,\tilde{u}_{j-1})\tilde{u}_j + \cdots + S_{(1^{j-1})}(\tilde{u}_1,\cdots,\tilde{u}_{j-1})\tilde{u}_j^{j-1}).$$

Then we have that the general state $|\Psi(u_1,\cdots,u_N)\rangle$ of Hamiltonian in the XX0 model with variables $\mathbf{x}$ and the scalar product of two general states are tau functions of the KP hierarchy.

The paper is organized as follows. In section 2, we recall Schur function, then we recall the definition of the KP hierarchy and its tau functions. In section 3, we prove that the multiplications $H_M(u_1)H_M(u_2)\cdots H_M(u_N)$ are tau function of the KP hierarchy. In section 4, we recall the phase model and give that the general states and the scalar products of two general states in the phase model are all tau functions of the KP hierarchy. In section 5, we recall the XX0 model and give that the general states and the scalar products of two general states in the XX0 model are all tau functions of the KP hierarchy.

2. Schur functions and the KP hierarchy

A partition λ is a sequence $(\lambda_1,\lambda_2,\cdots,\lambda_l)$ of non-negative integers in decreasing order [3]: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$. Replacing numbers by squares, we obtain the Young diagram of partition λ. For example, the Young diagram of partition $\lambda = (5,4,4,1)$ is



(1)

The row index of Young diagram is labeled from top to bottom and the column index is labeled from left to right. The conjugate λ' of Young diagram λ is the transpose of λ. For example, $\lambda' = (4,3,3,3,1)$ if $\lambda = (5,4,4,1)$. Let the number of squares on the main diagonal be r, then Young diagram λ is denoted by

$$(m_1,m_2,\cdots,m_r|n_1,n_2,\cdots,n_r) \tag{2}$$

where $m_i = \lambda_i - i,\ n_i = \lambda'_i - i$ for $i = 1,2,\cdots,r$. For example, $\lambda = (5,4,4,1)$ can also be denoted by $(4,2,1|3,1,0)$.

Corresponding to every Young diagram, there are many symmetric functions, we only consider Schur functions. In the following, we review the definition of Schur function $S_\lambda(\mathbf{x})$. Let $\mathbf{x} = (x_1,x_2,\cdots)$. The operators $h_n(\mathbf{x})$ are determined by the generating function:

$$H(z) = \sum_{n=0}^{\infty} h_n(\mathbf{x})z^n = e^{\xi(\mathbf{x},z)}, \quad \xi(\mathbf{x},z) = \sum_{n=1}^{\infty} x_n z^n \tag{3}$$

and set $h_n(\mathbf{x}) = 0$ for $n < 0$. Note that $h_n(\mathbf{x})$ is the complete homogeneous symmetric function by the Miwa transform, i.e., replacing ix_i with the power sum p_i.

For Young diagram $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, the Schur function $S_\lambda = S_\lambda(\mathbf{x})$ is a polynomial of variables $\mathbf{x}$ in $\mathbb{C}[\mathbf{x}]$ defined by the Jacobi–Trudi formula [15]:

$$S_\lambda(\mathbf{x}) = \det\left(h_{\lambda_i - i + j}(\mathbf{x})\right)_{1 \le i, j \le l}. \tag{4}$$

Write λ in the form of $(m_1, m_2, \cdots, m_r | n_1, n_2, \cdots, n_r)$, we have [4]

$$S_\lambda(\mathbf{x}) = \det\left(S_{(m_i|n_j)}(\mathbf{x})\right)_{1 \le i, j \le r}. \tag{5}$$

For Young diagrams $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_l)$, let $\lambda_i \ge \mu_i$ for all $1 \le i \le l$, the skew Schur function

$$S_{\lambda/\mu}(\mathbf{x}) = \sum_\nu C^\lambda_{\mu\nu} S_\nu(\mathbf{x}) = \det(h_{\lambda_i - \mu_j - i + j})_{1 \le i, j \le l} \tag{6}$$

where $C^\lambda_{\mu\nu}$ is the coefficient of $S_\lambda(\mathbf{x})$ in $S_\mu(\mathbf{x})S_\nu(\mathbf{x})$ and given by Littlewood–Richardson rule which one can find in books [2,3].

Introduce Fermions $\psi^*_j,\ \psi_j,\ j \in \mathbb{Z} + \frac{1}{2}$, which satisfy the anti-commutation relations

$$\psi_j\psi_k + \psi_k\psi_j = 0, \quad \psi^*_j\psi^*_k + \psi^*_k\psi^*_j = 0, \quad \psi^*_j\psi_k + \psi_k\psi^*_j = \delta_{j+k,0}. \tag{7}$$

The Fock representation space of Fermions is the space of Maya diagrams. A Maya diagram is made up of black and white stones lined up along the real line with the convention that all the stones are black far away to the right, whereas all the stones are white far away to the left. For example, the following is a Maya diagram

$$\begin{array}{cccccccccc} \cdots & \circ & \circ & \bullet & \bullet & \circ & \bullet & \circ & \bullet & \bullet & \cdots \\ & -\frac{7}{2} & -\frac{5}{2} & -\frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} & \end{array} \tag{8}$$

By writing half integers $k_1, k_2, \cdots$ for the positions of the black stones, a Maya diagram is described as an increasing sequence of half integers

$$\mathbf{k} = \{k_n\}_{n \ge 1} \quad \text{with} \quad k_1 < k_2 < k_3 < \cdots.$$

For example, the Maya diagram in (8) is denoted by

$$-\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{9}{2}, \cdots$$

Define the charge p of a Maya diagram as the number of white stones on the right half line minus the number of black stones on the left half line. For example, the charge of Maya diagram in (8) is zero.

We set $\mathcal{F}$ to be the vector space based by the set of Maya diagrams, and call it the Fermionic Fock space. The basis vector is written as $|\mathbf{k}\rangle$. Specially,

$$|0\rangle = |\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\rangle.$$

The left action of the Fermions on $\mathcal{F}$ is defined as follows:

$$\psi_j|\mathbf{k}\rangle=\begin{cases}(-1)^{n-1}|\cdots,k_{n-1},k_{n+1},\cdots\rangle & \text{if } k_n=-j;\\ 0 & \text{otherwise;}\end{cases} \tag{9}$$

$$\psi_j^*|\mathbf{k}\rangle=\begin{cases}(-1)^{n}|\cdots,k_{n},j,k_{n+1},\cdots\rangle & \text{if } k_n<j<k_{n+1};\\ 0 & \text{otherwise.}\end{cases} \tag{10}$$

Any Maya diagram corresponds to the vector (up to sign)

$$\psi_{j_1}\cdots\psi_{j_r}\psi_{k_1}^*\cdots\psi_{k_s}^*|vac\rangle \quad \text{for} \quad j_1<\cdots j_r<0 \quad \text{and} \quad k_1<\cdots k_s<0.$$

Write λ in the form of

$$\lambda=(m_1,m_2,\cdots,m_r|n_1,n_2,\cdots,n_r)$$

Under Boson–Fermion correspondence, the polynomial $S_\lambda(\mathbf{x})$ goes over into the basis vector

$$\psi_{-m_1-\frac{1}{2}}\cdots\psi_{-m_r-\frac{1}{2}}\psi^*_{-n_1-\frac{1}{2}}\cdots\psi^*_{-n_r-\frac{1}{2}}|0\rangle \tag{11}$$

in Fock space $\mathcal{F}$ of charge 0 multiplied by $(-1)^{\sum_{j=1}^r n_j+r(r-1)/2}$.

Therefore, we have three vector spaces which are isomorphic to each other [4], the polynomial ring $\mathbb{C}[\mathbf{x}]=\mathbb{C}[x_1,x_2,\cdots]$ of infinitely many variables $\mathbf{x}=(x_1,x_2,\cdots)$, the charge zero part of the Fermionic Fock space $\mathcal{F}$, and the vector space based by the set of Young diagrams. In the following, we do not distinguish them, that is, we write

$$\lambda=S_\lambda(\mathbf{x})=(-1)^{\sum_{j=1}^r n_j+r(r-1)/2}\psi_{-m_1-\frac{1}{2}}\cdots\psi_{-m_r-\frac{1}{2}}\psi^*_{-n_1-\frac{1}{2}}\cdots\psi^*_{-n_r-\frac{1}{2}}|0\rangle.$$

In [4,16], the KP hierarchy is defined by

$$\sum_{j\in\mathbb{Z}+\frac{1}{2}}\psi_j^*\tau\otimes\psi_{-j}\tau=0 \tag{12}$$

where τ is a vector in $\mathcal{F}$.

Introduce the operators

$$X_A=\sum_{j,k}a_{jk}:\psi_{-j}\psi_k^*:$$

the matrix $A=(a_{ij})$ satisfy the following condition: there exists $N>0$ such that $a_{ij}=0$ for all i,j with $|i-j|>N$. The Lie algebra $gl(\infty)$ is defined to be the vector space

$$gl(\infty)=\{X_A\}\oplus\mathbb{C} \tag{13}$$

which can be found in [4] and the group G corresponding to the Lie algebra $gl(\infty)$ is defined to be

$$G=\{e^{X_1}\cdots e^{X_n}|X_i\in gl(\infty)\}. \tag{14}$$

It is proved that $g|0\rangle$ is a solution (tau function) of the KP hierarchy for any $g\in G$.

Another description of the KP hierarchy is the following which one can also find in book [4]. Introduce a pseudodifferential operator

$$L=\partial+\sum_{j=1}^{\infty}f_j\partial^{-j} \tag{15}$$

where $\partial = \partial_{x_1}$ and ∂^{-1} satisfies $\partial^{-1}\partial = \partial\partial^{-1} = 1$. In section 4, we will discuss the wave functions w of the KP hierarchy in the phase model, the wave functions w are the eigenstates of L:

$$Lw = kw \tag{16}$$

where k is a eigenvalue. The wave functions w also satisfy

$$\frac{\partial w}{\partial x_j} = B_j w \quad \text{where} \quad B_j = (L^j)_+ \tag{17}$$

where $(L^j)_+$ is obtained from L^j by removing all terms ∂^{-j} for all $j > 0$. The compatibility condition between (16) and (17) gives

$$\frac{\partial L}{\partial x_j} = [B_j, L] \tag{18}$$

this is an infinite set of nonlinear evolution equations in infinitely many functions $f_1, f_2, \cdots$ of the infinitely many variables $(x_1, x_2, \cdots)$. Equation (18) is called the KP hierarchy. The functions $f_1, f_2, \cdots$ are obtained from tau function $\tau = \tau(x_1, x_2, \cdots)$ which equals $g|0\rangle$ for some $g \in G$ under Boson–Fermion correspondence.

The wave functions w can be written as

$$w = \frac{\tau\left(x_1 - \frac{1}{k}, x_2 - \frac{1}{2k^2}, x_3 - \frac{1}{3k^3}, \cdots\right)}{\tau(x_1, x_2, x_3, \cdots)} e^{\xi(\mathbf{x},k)} = (1 + \sum_{j=1}^{\infty} w_j \partial^{-j}) e^{\xi(\mathbf{x},k)}, \tag{19}$$

$$w^* = \frac{\tau\left(x_1 + \frac{1}{k}, x_2 + \frac{1}{2k^2}, x_3 + \frac{1}{3k^3}, \cdots\right)}{\tau(x_1, x_2, x_3, \cdots)} e^{-\xi(\mathbf{x},k)} = (1 + \sum_{j=1}^{\infty} w_j^* \partial^{-j}) e^{-\xi(\mathbf{x},k)} \tag{20}$$

which satisfy the bilinear identity

$$\oint \frac{dk}{2\pi i} w w^* = 0. \tag{21}$$

From (21), the equation (17) can be obtained.

Introduce the generating function

$$H^{\perp}(z) = \sum_{n=0}^{\infty} h_n^{\perp}(\mathbf{x}) z^n = e^{\xi(\tilde{\partial}_{\mathbf{x}}, z)}, \quad \tilde{\partial}_{\mathbf{x}} = (\partial_{x_1}, \frac{1}{2}\partial_{x_2}, \cdots, \frac{1}{n}\partial_{x_n}, \cdots) \tag{22}$$

and set $h_n^{\perp}(\mathbf{x}) = 0$ for $n < 0$, where the operator $h_n^{\perp} = h_n^{\perp}(\mathbf{x})$ is the adjoint of multiplication by $h_n = h_n(\mathbf{x})$ in the sense of

$$(h_n S_\lambda, S_\mu) = (S_\lambda, h_n^{\perp} S_\mu). \tag{23}$$

Note that $h_n^{\perp}(\mathbf{x}) = h_n(\tilde{\partial}_{\mathbf{x}})$.

Introduce the generating functions

$$E(z) = \sum_{n=0}^{\infty} (-1)^n e_n(\mathbf{x}) z^n = e^{-\xi(\mathbf{x},z)}, \quad E^{\perp}(z) = \sum_{n=0}^{\infty} (-1)^n e_n^{\perp}(\mathbf{x}) z^n = e^{-\xi(\tilde{\partial}_{\mathbf{x}}, z)} \tag{24}$$

and set $e_n(\mathbf{x}) = e_n^{\perp}(\mathbf{x}) = 0$ for $n < 0$, where $e_n(\mathbf{x}) = S_{(1^n)}(\mathbf{x})$ and the operator $e_n^{\perp}(\mathbf{x})$ is the adjoint of multiplication by $e_n(\mathbf{x})$. Then we get the following property of the KP wave functions which we will discuss in the next section.

Proposition 2.1. *The functions $w_j,\ w_j^*$ in equations (19) and (20) satisfy*

$$e_n^{\perp}\tau = (-1)^n \tau w_n, \tag{25}$$

$$h_n^{\perp}\tau = (-1)^n \tau w_n^*. \tag{26}$$

Proof. These results hold since

$$\tau\left(x_1 - \frac{1}{k}, x_2 - \frac{1}{2k^2}, x_3 - \frac{1}{3k^3}, \cdots\right) = E^{\perp}(k^{-1})\tau(x_1, x_2, x_3, \cdots),$$

$$\tau\left(x_1 + \frac{1}{k}, x_2 + \frac{1}{2k^2}, x_3 + \frac{1}{3k^3}, \cdots\right) = H^{\perp}(k^{-1})\tau(x_1, x_2, x_3, \cdots). \quad \square$$

3. Young diagrams in an N x M box

In the following, we consider the Young diagram in an $N \times M$ box, which means that this Young diagram has at most N rows and at most M columns.

From [3], we know that the multiplication of Schur functions $S_\lambda(\mathbf{x})$ and $S_\mu(\mathbf{x})$ for any Young diagrams $\lambda,\ \mu$ satisfies the Littlewood–Richardson rule. For example,

$$S_{(3)}(\mathbf{x})S_{(1)}(\mathbf{x}) = S_{(4)}(\mathbf{x}) + S_{(3,1)}(\mathbf{x}),$$

$$S_{(2)}(\mathbf{x})S_{(2)}(\mathbf{x}) = S_{(4)}(\mathbf{x}) + S_{(3,1)}(\mathbf{x}) + S_{(2,2)}(\mathbf{x}).$$

In this paper, we restrict all the Young diagrams in an $N \times M$ box. For example, if $M = 3,\ N = 2$,

$$S_{(3)}(\mathbf{x})S_{(1)}(\mathbf{x}) = S_{(3,1)}(\mathbf{x}),$$

$$S_{(2)}(\mathbf{x})S_{(2)}(\mathbf{x}) = S_{(3,1)}(\mathbf{x}) + S_{(2,2)}(\mathbf{x}),$$

where the term $S_{(4)}(\mathbf{x})$ is deleted since the Young diagram (4) is not in the 2 × 3 box.

Let

$$H_M(z) = \sum_{n=0}^{M} h_n(\mathbf{x})z^n \tag{27}$$

where $h_n(\mathbf{x})$ is introduced in (3). This kind of truncated generating function is firstly given by Tsilevich in paper [6]. The commutation relation of $H_M^{\perp}(z)$ and $H_M(w)$

$$H_M^{\perp}(z)H_M(w) = \frac{1}{1 - zw}\left(H_M(w)H_M^{\perp}(z) - (zw)^{M+1}(H_M(w^{-1})H_M^{\perp}(z^{-1})\right) \tag{28}$$

is obtained from the Yang–Baxter relation of the phase model, where vertex operator

$$H_M^{\perp}(z) = \sum_{n=0}^{M} h_n^{\perp}(\mathbf{x})z^n. \tag{29}$$

In the $M \to \infty$ limit and let $|zw| < 1$, the equation (28) turns into the well-known relation [2]

$$H^{\perp}(z)H(w) = \frac{1}{1 - zw}H(w)H^{\perp}(z) \tag{30}$$

where

$$H^{\perp}(z) = \lim_{M\to\infty} H_M^{\perp}(z), \qquad H(z) = \lim_{M\to\infty} H_M(z)$$

and $H(z)$ is the same as in (3) and $H^{\perp}(z)$ is the same as in (22).

Then Tsilevich derived the following result in paper [6]

$$H_M(u_1)H_M(u_2)\cdots H_M(u_N)=\sum_{\lambda\in N\times M} S_\lambda(u_1,\cdots,u_N)S_\lambda(\mathbf{x}), \tag{31}$$

here $S_\lambda(\mathbf{x})$ is the Schur function defined in (4), but $S_\lambda(u_1,\cdots,u_N)$ is the Schur function defined in (4) replacing x_i by

$$\frac{1}{i}(u_1^i+u_2^i+\cdots+u_N^i),$$

that is, the Schur function $S_\lambda(u_1,\cdots,u_N)$ is the same with that defined in [3]. We take an example.

Example 3.1. Let $M=3,\ N=2,$

$$\begin{aligned}
&H_M(u_1)H_M(u_2)\\
&\quad=(1+h_1u_1+h_2u_1^2+h_3u_1^3)(1+h_1u_2+h_2u_2^2+h_3u_2^3)\\
&\quad=1+h_1u_2+h_2u_2^2+h_3u_2^3+h_1u_1+h_1h_1u_1u_2+h_1h_2u_1u_2^2+h_1h_3u_1u_2^3+h_2u_1^2\\
&\qquad+h_2h_1u_1^2u_2+h_2h_2u_1^2u_2^2+h_2h_3u_1^2u_2^3+h_3u_1^3+h_3h_1u_1^3u_2+h_3h_2u_1^3u_2^2+h_3h_3u_1^3u_2^3\\
&\quad=1+S_{(1)}(u_1+u_2)+S_{(2)}(u_1^2+u_1u_2+u_2^2)+S_{(1^2)}u_1u_2\\
&\qquad+S_{(3)}(u_1^3+u_1^2u_2+u_1u_2^2+u_2^3)+S_{(2,1)}(u_1^2u_2+u_1u_2^2)\\
&\qquad+S_{(3,1)}(u_1u_2^3+u_1^3u_2+u_1^2u_2^2)+S_{(2,2)}u_1^2u_2^2\\
&\qquad+S_{(3,2)}(u_1^2u_2^3+u_1^3u_2^2)+S_{(3,3)}u_1^3u_2^3\\
&\quad=\sum_{\lambda\in 2\times 3} S_\lambda(u_1,u_2)S_\lambda(\mathbf{x}).
\end{aligned}$$

From the result in equation (31), and by Boson–Fermion correspondence, we get the following result:

Proposition 3.2. *For any positive integers $M,\ N$, the following relations hold*

$$\begin{aligned}
&H_M(u_1)\cdots H_M(u_N)\cdot 1\\
&\quad=(1+X_1(u_1,\cdots,u_N))\cdots(1+X_N(u_1,\cdots,u_N))|0\rangle &(32)\\
&\quad=e^{X_1(u_1,\cdots,u_N)}\cdots e^{X_N(u_1,\cdots,u_N)}|0\rangle &(33)
\end{aligned}$$

where

$$X_k(u_1,\cdots,u_N)=(-1)^{k-1}\sum_{j=1}^{M} S_{(j,1^{k-1})}(u_1,\cdots,u_N)\psi_{-\frac{2j-1}{2}}\psi^*_{-\frac{2k-1}{2}},\qquad k=1,2,\cdots,N.$$

Proof. Since

$$H_M(u_1)H_M(u_2)\cdots H_M(u_N)=\sum_{\lambda\in N\times M} S_\lambda(u_1,\cdots,u_N)S_\lambda(\mathbf{x})$$

The Young diagram $\lambda\in N\times M$ can be written as the form

$$(m_1,m_2,\cdots,m_r|n_1,n_2,\cdots,n_r)$$

for $0 \le r \le \min\{M, N\}$. We know that

$$S_\lambda(\mathbf{x}) = (-1)^{\sum_{j=1}^{r} n_j + r(r-1)/2}\psi_{-m_1-\frac{1}{2}}\cdots\psi_{-m_r-\frac{1}{2}}\psi^*_{-n_1-\frac{1}{2}}\cdots\psi^*_{-n_r-\frac{1}{2}}|0\rangle, \tag{34}$$

then

$$(-1)^{\sum_{i=1}^{r}(k_i-1)+r(r-1)/2}\psi_{-\frac{2j_1-1}{2}}\cdots\psi_{-\frac{2j_r-1}{2}}\psi^*_{-\frac{2k_1-1}{2}}\cdots\psi^*_{-\frac{2k_r-1}{2}}|0\rangle \tag{35}$$

equals to Schur function $S_\lambda(\mathbf{x})$ with

$$\lambda = (j_r - 1, j_{r-1} - 1, \cdots, j_1 - 1|k_r - 1, k_{r-1} - 1, \cdots, k_1 - 1)$$

for integers

$$0 < j_1 < j_2 < \cdots < j_r, \quad 0 < k_1 < k_2 < \cdots < k_r.$$

Note that in equation (35), we rearranged the order of Fermions which is different from the order of Fermions in equation (34).

Since Fermions ψ_j, ψ^*_j satisfy

$$\psi_j^2 = 0, \quad \psi_j^{*2} = 0,$$

and $S_{(j,1^{k-1})}(\mathbf{x}) = S_{(j-1|k-1)}(\mathbf{x})$, then we have

$$\begin{aligned}
&(1 + X_1(u_1, \cdots, u_N))\cdots(1 + X_N(u_1, \cdots, u_N)) \\
&= \sum_{\substack{0\le r\le\min\{M,N\}\\ 1\le j_1,\cdots,j_r\le M\\ 1\le k_1<\cdots<k_r\le N}} \prod_{i=1}^{r}(-1)^{k_i-1}S_{(j_i-1|k_i-1)}\psi_{-\frac{2j_i-1}{2}}\psi^*_{-\frac{2k_i-1}{2}} &(36)\\
&= \sum_{\substack{0\le r\le\min\{M,N\}\\ 1\le j_1,\cdots,j_r\le M\\ 1\le k_1<\cdots<k_r\le N}} \begin{matrix}(-1)^{\sum_{i=1}^{r}(k_i-1)}S_{(j_1-1|k_1-1)}\cdots S_{(j_r-1|k_r-1)}\\ \cdot\psi_{-\frac{2j_1-1}{2}}\psi^*_{-\frac{2k_1-1}{2}}\cdots\psi_{-\frac{2j_r-1}{2}}\psi^*_{-\frac{2k_r-1}{2}}\end{matrix} &(37)\\
&= \sum_{\substack{0\le r\le\min\{M,N\}\\ 1\le j_1,\cdots,j_r\le M\\ 1\le k_1<\cdots<k_r\le N}} \begin{matrix}(-1)^{\sum_{i=1}^{r}(k_i-1)+r(r-1)/2}S_{(j_1-1|k_1-1)}\cdots S_{(j_r-1|k_r-1)}\\ \cdot\psi_{-\frac{2j_1-1}{2}}\cdots\psi_{-\frac{2j_r-1}{2}}\psi^*_{-\frac{2k_1-1}{2}}\cdots\psi^*_{-\frac{2k_r-1}{2}}\end{matrix} &(38)\\
&= \sum_{\substack{0\le r\le\min\{M,N\}\\ 1\le j_1<\cdots<j_r\le M\\ 1\le k_1<\cdots<k_r\le N}} \begin{matrix}\sum_\sigma sign(\sigma)S_{(j_{\sigma(1)}-1|k_1-1)}\cdots S_{(j_{\sigma(r)}-1|k_r-1)}\\ (-1)^{\sum_{i=1}^{r}(k_i-1)+r(r-1)/2}\psi_{-\frac{2j_1-1}{2}}\cdots\psi_{-\frac{2j_r-1}{2}}\psi^*_{-\frac{2k_1-1}{2}}\cdots\psi^*_{-\frac{2k_r-1}{2}}\end{matrix}
\end{aligned}$$

Note that $\{j_1, j_2, \cdots, j_r\}$ in equations (36), (37), (38) are different from that in the last equation above. In the last equation, we combine the same terms together and arrange Fermions $\psi_{-\frac{2j-1}{2}}$ in the order of j increasing. The following term in the last equation above

$$(-1)^{\sum_{i=1}^{r}(k_i-1)+r(r-1)/2}\psi_{-\frac{2j_1-1}{2}}\cdots\psi_{-\frac{2j_r-1}{2}}\psi^*_{-\frac{2k_1-1}{2}}\cdots\psi^*_{-\frac{2k_r-1}{2}}|0\rangle$$

corresponds to $S_\lambda(\mathbf{x})$ with

$$\lambda = (j_r - 1, j_{r-1} - 1, \cdots, j_1 - 1|k_r - 1, k_{r-1} - 1, \cdots, k_1 - 1),$$

and

$$\sum_{\sigma} sign(\sigma) S_{(j_{\sigma(1)}-1|k_1-1)} \cdots S_{(j_{\sigma(r)}-1|k_r-1)}$$

equals

$$\det(S_{(j_s-1|k_t-1)}(\mathbf{x}))_{1\leq s,t\leq r} = S_\lambda(\mathbf{x}).$$

Since

$$j_r \leq M, \quad k_r \leq N$$

we have $\lambda \in N \times M$. Then we get the result. □

From this result, we obtain that

Proposition 3.3. *For any integers* $M, \ N$, *the polynomial*

$$H_M(u_1)H_M(u_2)\cdots H_M(u_N) = \sum_{\lambda\in N\times M} S_\lambda(u_1,\cdots,u_N)S_\lambda(\mathbf{x}) \tag{39}$$

with variables $\mathbf{x}$ *are tau functions of the KP hierarchy.*

Note that the method we used here to obtain tau functions of the KP hierarchy and the method Foda et al. used in paper [9] are equivalent, which are two ways to describe tau functions of the KP hierarchy, i.e., tau function equals

$$\tau = \sum_{\lambda} c_\lambda S_\lambda(\mathbf{x}), \text{ where } c_\lambda \text{ satisfy Plüker relations,} \tag{40}$$

$$= g|0\rangle, \text{where } g \in G \text{ which is introduced in equation (14).} \tag{41}$$

One can find these in book [4]. Foda et al. [9] use the method in equation (40) above to get tau functions, we use the method in equation (41) above to get tau functions. These two methods are equivalent in essence. But there are two differences between the results in [9] and our results: firstly $\lambda \in N \times (N-1)$ box in paper [9], and $\lambda \in N \times M$ box in our paper for any positive integers M, N; Secondly the coefficients of polynomials corresponding Young diagram λ in these two papers are different.

The tau function in equation (39) depends only on finite variables $\{x_1, x_2, \cdots, x_{M+N-1}\}$, this is because the polynomial $h_n(\mathbf{x})$ depends only on variables $\{x_1, x_2, \cdots, x_n\}$ and the Schur function $S_\lambda(\mathbf{x})$ is obtained from $h_n(\mathbf{x})$ by definition (4).

Here, we give the following property of KP wave functions which I will be calling on later when I deal with KP wave functions in section 4.

Proposition 3.4. *Let tau function*

$$\tau = \sum_{\lambda\in N\times M} S_\lambda(u_1,\cdots,u_N)S_\lambda(\mathbf{x}). \tag{42}$$

For $1 \leq m \leq M, \ 1 \leq n \leq N$, *we have*

$$e_n^\perp \tau = (-1)^n \tau w_n = \sum_{\lambda\in N\times M} S_\lambda(u_1,\cdots,u_N)S_{\lambda/(1^n)}(\mathbf{x}), \tag{43}$$

$$h_m^\perp \tau = (-1)^m \tau w_m^* = \sum_{\lambda\in N\times M} S_\lambda(u_1,\cdots,u_N)S_{\lambda/(m)}(\mathbf{x}). \tag{44}$$

4. The phase model

We begin this section with a bosonic system based on the following algebra [6,7]

$$[N,\phi]=-\phi, \quad [N,\phi^\dagger]=\phi^\dagger, \quad [\phi,\phi^\dagger]=\pi, \tag{45}$$

where $\pi=|0\rangle\langle 0|$ is the vacuum projection. The operator ϕ is one-sided isometry

$$\phi\phi^\dagger=1, \qquad \phi^\dagger\phi=1-\pi.$$

This algebra can be represented in the Fock space $\mathcal{F}$ consisting of n-particle states $|n\rangle$, the operators ϕ, $\phi^\dagger$ and N acting as the phase operators and the number of particles operator, respectively,

$$\phi^\dagger|n\rangle=|n+1\rangle, \quad \phi|n\rangle=|n-1\rangle, \quad \phi|0\rangle=0, \quad N|n\rangle=n|n\rangle,$$

where $|0\rangle$ is the vacuum state, the special case $n=0$ of the n particle state.

Let the tensor product

$$\mathcal{F}=\bigotimes_{i=0}^{M}\mathcal{F}_i, \tag{46}$$

be $M+1$ copies of the Fock space. Denote by $\phi_i, \phi_i^\dagger, N_i$ the operators that act as $\phi, \phi^\dagger, N$ in (45), respectively, in the ith space and identically in the other spaces.

The phase model is a model of a periodic chain with the Hamiltonian

$$H=-\frac{1}{2}\sum_{i=0}^{M}\left(\phi_i^\dagger\phi_{i+1}+\phi_i\phi_{i+1}^\dagger-2N_i\right). \tag{47}$$

Define the operator of the total number of particles by

$$\hat{N}=\sum_{i=0}^{M}N_i.$$

Then the N-particle vectors in this space are of the form

$$\bigotimes_{i=0}^{M}|n_i\rangle_i, \qquad \text{where } |n_i\rangle_i=(\phi_i^\dagger)^{n_i}|0\rangle_i, \quad N=\sum_{i=0}^{M}n_i, \tag{48}$$

the numbers n_i are called the occupation numbers of the state (48).

From [17], we know that the phase model is integrable. Introduce the L-matrix

$$L_i(u)=\begin{pmatrix} u^{-1} & \phi_i^\dagger \\ \phi_i & u \end{pmatrix}, \qquad i=0,\cdots,M,$$

where u is a scalar parameter, here we treat u as uI with I being the identity operator in $\mathcal{F}$. For every $i=0,\cdots,M$, the L-matrix satisfies the bilinear equation

$$R(u,v)\big(L_i(u)\otimes L_i(v)\big)=\big(L_i(v)\otimes L_i(u)\big)R(u,v), \tag{49}$$

where R-matrix $R(u,v)$ is a 4×4 matrix given by

$$R(u,v)=\begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix}, \tag{50}$$

with

$$f(v,u)=\frac{u^2}{u^2-v^2}, \quad g(v,u)=\frac{uv}{u^2-v^2}.$$

Define the monodromy matrix by

$$T(u)=L_M(u)L_{M-1}(u)\cdots L_0(u),$$

which gives the solution of the phase model. It also satisfies the bilinear equation

$$R(u,v)\big(T(u)\otimes T(v)\big)=\big(T(v)\otimes T(u)\big)R(u,v). \tag{51}$$

Let

$$T(u)=\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}, \tag{52}$$

we have

$$\hat{N}B(u)=B(u)(\hat{N}+1), \qquad \hat{N}C(u)=C(u)(\hat{N}-1). \tag{53}$$

Therefore, we call $B(u)$ the creation operator and $C(u)$ the annihilation operator. The operators $A(u)$ and $D(u)$ do not change the total number of particles.

Denote by $|0\rangle_j$ the vacuum vector in $\mathcal{F}_j$ and by $|0\rangle=\otimes_{i=0}^{M}|0\rangle_i$. The general states are

$$|\Psi(u_1,\cdots,u_N)\rangle=\prod_{i=1}^{N}B(u_j)|0\rangle. \tag{54}$$

The general states (54) turn into the eigen-states (which are called Bethe States) of the Hamiltonian (47) if the parameters $u_1,\cdots,u_N$ satisfy Bethe equations. One can find the details about Bethe equation in the book [17], we omit these since we only consider the general states in the following.

There is the following isometry between the states (48) and the Schur functions

$$\bigotimes_{i=0}^{M}|n_i\rangle_i \mapsto S_\lambda(\mathbf{x}), \quad \lambda=1^{n_1}2^{n_2}\cdots. \tag{55}$$

In papers [6,7], the authors obtained that the operator $B(u)$ acts on Schur functions as the operator of multiplication by $u^{-M}H_M(u^2)$, where $H_M(z)=\sum_{k=0}^{M}z^k h_k$, which is defined in (27), and the operator $C(u)$ acts on Schur functions as the operator $u^M H_M^{\perp}(u^{-2})$, where $H_M^{\perp}(z)=\sum_{k=0}^{M}z^k h_k^{\perp}$ is defined in (29). They also derived that the general states have the following expansions

$$\begin{aligned} |\Psi(u_1,\cdots,u_N)\rangle &= (u_1\cdots u_N)^{-M}\sum_{\lambda\in N\times M}S_\lambda(u_1^2,\cdots,u_N^2)\bigotimes_{i=0}^{M}|n_i\rangle_i, \\ &= (u_1\cdots u_N)^{-M}H_1(u_1^2)H_M(u_2^2)\cdots H_M(u_N^2), \quad \lambda=1^{n_1}2^{n_2}\cdots \end{aligned} \tag{56}$$

and

$$\begin{aligned}\langle\Psi(v_1,\cdots,v_N)| &= \langle 0|C(v_N)\cdots C(v_2)C(v_1)\\ &= (v_1\cdots v_N)^M \sum_{\lambda\in N\times M} S_\lambda(v_1^{-2},\cdots,v_N^{-2})\bigotimes_{i=0}^{M}\langle n_i|_i, \quad \lambda=1^{n_1}2^{n_2}\cdots \end{aligned}\tag{57}$$

The results above are obtained in paper [6,7]. From these results and Proposition 3.2, we obtain the following result:

Proposition 4.1. *For any positive integers M, N, the general states $|\Psi(u_1,\cdots,u_N)\rangle$ of Hamiltonian in the phase model with variables* $\mathbf{x}$ *are tau functions of the KP hierarchy.*

The equation (56) can also be written as

$$|\Psi(u_1,\cdots,u_N)\rangle = (u_1\cdots u_N)^{-M}\sum_{\lambda\in N\times M} S_\lambda(u_1^2,\cdots,u_N^2)S_\lambda(\mathbf{x}).\tag{58}$$

Note that this tau function depend only on finite variables $\{x_1,x_2,\cdots,x_{M+N-1}\}$.

From equation (58), we have

Lemma 4.2. *For $m>M$,*

$$h_m^{\perp}|\Psi(u_1,\cdots,u_N)\rangle = 0.\tag{59}$$

For $0<m\le M$,

$$h_m^{\perp}|\Psi(u_1,\cdots,u_N)\rangle = (u_1\cdots u_N)^{-M}\sum_{\lambda\in N\times M} S_\lambda(u_1^2,\cdots,u_N^2)S_{\lambda/(m)}(\mathbf{x}).\tag{60}$$

Here the operator $h_m^{\perp}=h_m^{\perp}(\mathbf{x})$ acts on $S_\lambda(\mathbf{x})$ or $\bigotimes_{i=0}^{M}|n_i\rangle_i$.

Since $C(v)=v^M\sum_{k=0}^{M}v^{-2k}h_k^{\perp}$, we have

Proposition 4.3. *The following relations hold*

$$\begin{aligned}&C(v)|\Psi(u_1,\cdots,u_N)\rangle\\ &\quad= v^M H_M^{\perp}(v^{-2})|\Psi(u_1,\cdots,u_N)\rangle\\ &\quad= \left(\frac{v}{u_1\cdots u_N}\right)^M \sum_{\lambda\in N\times M} S_\lambda(u_1^2,\cdots,u_N^2)S_\lambda(x_1+\frac{1}{v^2},\cdots,x_n+\frac{1}{nv^{2n}},\cdots).\end{aligned}\tag{61}$$

Proof. Since

$$\begin{aligned}&H_M^{\perp}(k^{-1})|\Psi(u_1,\cdots,u_N)\rangle\\ &\quad= H^{\perp}(k^{-1})|\Psi(u_1,\cdots,u_N)\rangle &(62)\\ &\quad= (u_1\cdots u_N)^{-M}e^{\xi(\partial_{\mathbf{x}},k^{-1})}\sum_{\lambda\in N\times M} S_\lambda(u_1^2,\cdots,u_N^2)S_\lambda(\mathbf{x}) &(63)\\ &\quad= (u_1\cdots u_N)^{-M}\sum_{\lambda\in N\times M} S_\lambda(u_1^2,\cdots,u_N^2)S_\lambda(x_1+\frac{1}{k},\cdots,x_n+\frac{1}{nk^n},\cdots). &(64)\end{aligned}$$

Then we get the result. □

Denote $\tau_\Psi = |\Psi(u_1, \cdots, u_N)\rangle$ which is a tau function of the KP hierarchy. From equation (44), we obtain

$$h_k^\perp |\Psi(u_1, \cdots, u_N)\rangle = h_k^\perp \tau_\Psi = (-1)^k \tau_\Psi w_k^*.$$

Using the result in equation (59), we get

Proposition 4.4. *For tau function $\tau_\Psi = |\Psi(u_1, \cdots, u_N)\rangle$ of the KP hierarchy, the term w_m^* in the expansion of wave function w^* defined in (20) equals zero for all $m > M$.*

For the similar reason, we get

Proposition 4.5. *For tau function $\tau_\Psi = |\Psi(u_1, \cdots, u_N)\rangle$ of the KP hierarchy, the term w_n in the expansion of wave function w defined in (19) equals zero for all $n > N$.*

From Proposition 4.4, we obtain

Proposition 4.6. *The following relation holds*

$$C(v)|\Psi(u_1, \cdots, u_N)\rangle = v^M |\Psi(u_1, \cdots, u_N)\rangle (1 + \sum_{k=1}^{\infty} (-1)^k \frac{w_k^*}{v^{2k}}) \tag{65}$$

$$= v^M |\Psi(u_1, \cdots, u_N)\rangle (1 + \sum_{k=1}^{M} (-1)^k \frac{w_k^*}{v^{2k}}). \tag{66}$$

In the following, we will discuss another kind of tau functions of the KP hierarchy which are obtained in the phase model. From equations (56) and (57), the scalar product

$$\langle \Psi(v_1, \cdots, v_N)|\Psi(u_1, \cdots, u_N)\rangle$$
$$= \left(\frac{v_1 \cdots v_N}{u_1 \cdots u_N} \right)^M \sum_{\lambda \in N \times M} S_\lambda(v_1^{-2}, \cdots, v_N^{-2}) S_\lambda(u_1^2, \cdots, u_N^2) \tag{67}$$

From Proposition 3.2, we get that

Proposition 4.7. *Let*

$$x_n = \frac{1}{n} \sum_{k=1}^{M} u_k^{2n}, \quad or \quad x_n = \frac{1}{n} \sum_{k=1}^{M} v_k^{-2n}, \tag{68}$$

the scalar products $\langle \Psi(v_1, \cdots, v_N)|\Psi(u_1, \cdots, u_N)\rangle$, which equal

$$\sum_{\lambda \in N \times M} S_\lambda(v_1^{-2}, \cdots, v_N^{-2}) S_\lambda(\mathbf{x}), \quad or \quad \sum_{\lambda \in N \times M} S_\lambda(u_1^2, \cdots, u_N^2) S_\lambda(\mathbf{x}), \tag{69}$$

of the phase model for any positive integers N and M are also tau functions of the KP hierarchy with variables $\mathbf{x}$, up to an overall factor

$$\left(\frac{v_1 \cdots v_N}{u_1 \cdots u_N} \right)^M.$$

Note that in paper [11], the author obtained that the scalar product of Bethe states in the phase model and the 4-vertex model becomes, up to a simple factor, a tau function of the 2-KP hierarchy; In paper [12], the author derived that the scalar product of two general states in the phase model is a tau function of the 2-toda hierarchy up to a simple factor; Here we get that the scalar product of two general states in the phase model is a tau function of the KP hierarchy.

We denote the sums in equation (69) by $\tau_{\Psi\Psi}$ and only discuss

$$\tau_{\Psi\Psi} = \sum_{\lambda\in N\times M} S_\lambda(v_1^{-2},\cdots,v_N^{-2})S_\lambda(\mathbf{x}). \tag{70}$$

Note that $\tau_{\Psi\Psi}$ is a restricted tau function of the KP hierarchy in the sense that $\mathbf{x}$ in $\tau_{\Psi\Psi}$ are not free variables but depend on $u_1,\cdots,u_N$.

From (44), we have

$$h_k^\perp \tau_{\Psi\Psi} = (-1)^k \tau_{\Psi\Psi} w_k^* = \sum_{\lambda\in N\times M} S_\lambda(v_1^{-2},\cdots,v_N^{-2})S_{\lambda/(k)}(\mathbf{x}) \tag{71}$$

where $\lambda/(k)$ is a skew Young diagram. The following correlation function has the same form with that of the sum in the right hand side of the equation above.

Proposition 4.8. *The correlation function*

$$\langle 0|C(v_N)\cdots C(v_1)B(u_1)\cdots B(u_{N-1})\phi_k^\dagger|0\rangle = \sum_\lambda S_\lambda(v_1^{-2},\cdots,v_N^{-2})S_{\lambda/(k)}(\mathbf{x}) \tag{72}$$

up to an overall factor, where the sum is taken over all $\lambda\in N\times M$, $(k)\subset\lambda$ and $\lambda/(k)\in(N-1)\times M$.

Note that the variables $\mathbf{x}$ here depend on $u_1,\cdots,u_{N-1}$.

Proof. Since $\phi_k^\dagger|0\rangle = |(k)\rangle$, we have

$$B(u_1)\cdots B(u_{N-1})\phi_k^\dagger|0\rangle = (u_1\cdots u_{N-1})^{-M}\sum_{\mu\in(N-1)\times M} S_\mu(u_1^2,\cdots,u_{N-1}^2)\sum_{\nu\in N\times M} C_{\mu(k)}^\nu|\nu\rangle$$

where $C_{\mu(k)}^\nu$ is the coefficient of Schur function $S_\nu(\mathbf{x})$ in the multiplication $S_\mu(\mathbf{x})S_{(k)}(\mathbf{x})$ and given by Littlewood–Richardson rule which one can find in book [2,3]. Then

$$\begin{aligned}
&\langle 0|C(v_N)\cdots C(v_1)B(u_1)\cdots B(u_{N-1})\phi_k^\dagger|0\rangle\\
&=\left(\frac{v_1\cdots v_N}{u_1\cdots u_{N-1}}\right)^M \sum_{\lambda,\mu}\sum_\nu C_{\mu(k)}^\nu S_\lambda(v_1^{-2},\cdots,v_N^{-2})S_\mu(u_1^2,\cdots,u_{N-1}^2)\langle\lambda|\nu\rangle\\
&=\left(\frac{v_1\cdots v_N}{u_1\cdots u_{N-1}}\right)^M \sum_\lambda S_\lambda(v_1^{-2},\cdots,v_N^{-2})S_{\lambda/(k)}(u_1^2,\cdots,u_{N-1}^2),
\end{aligned}$$

where $\lambda\in N\times M$, $\mu\in(N-1)\times M$, $\nu\in N\times M$. We get the result. □

Since $B(u) = u^{-M}H_M(u^2)$, acting on the vacuum, we obtain

$$B(u_N)|0\rangle = u_N^{-M}\sum_{k=0}^M u_N^{2k}|(k)\rangle = u_N^{-M}\sum_{k=0}^M u_N^{2k}\phi_k^\dagger|0\rangle. \tag{73}$$

Then

$$
\begin{aligned}
|\Psi(u_1,\cdots,u_N)\rangle &= \sum_{k=0}^{M} u_N^{2k-M} B(u_1)\cdots B(u_{N-1})|(k)\rangle \\
&= \sum_{k=0}^{M} u_N^{2k-M} B(u_1)\cdots B(u_{N-1})\phi_k^{\dagger}|0\rangle,
\end{aligned}
$$

therefore the scalar product satisfy the following relation

$$
\langle\Psi(v_1,\cdots,v_N)|\Psi(u_1,\cdots,u_N)\rangle = \sum_{k=0}^{M} u_N^{2k-M} \langle\Psi(v_1,\cdots,v_N)|\Psi(u_1,\cdots,u_{N-1})\phi_k^{\dagger}|0\rangle. \tag{74}
$$

From (43), we have

$$
e_k^{\perp}\tau_{\Psi\Psi} = (-1)^k \tau_{\Psi\Psi} w_k = \sum_{\lambda\in N\times M} S_\lambda(v_1^{-2},\cdots,v_N^{-2}) S_{\lambda/(1^k)}(\mathbf{x}) \tag{75}
$$

where $\lambda/(1^k)$ is a skew Young diagram. The following correlation function has the same form with that of the sum in the right hand side of the equation above.

Proposition 4.9. *The correlation function*

$$
\langle 0|C(v_N)\cdots C(v_1)B(u_1)\cdots B(u_{N-k})(\phi_1^{\dagger})^k|0\rangle = \sum_{\lambda} S_\lambda(v_1^{-2},\cdots,v_N^{-2}) S_{\lambda/(1^k)}(\mathbf{x}) \tag{76}
$$

up to an overall factor, where the sum is taken over all $\lambda \in N\times M$, $(1^k)\subset\lambda$ *and* $\lambda/(1^k)\in (N-k)\times M$.

Note that the variables $\mathbf{x}$ here depend on $u_1,\cdots,u_{N-k}$.

Proof. Since $(\phi_1^{\dagger})^k|0\rangle = |(1^k)\rangle$, we have

$$
\begin{aligned}
&B(u_1)\cdots B(u_{N-k})(\phi_1^{\dagger})^k|0\rangle \\
&= (u_1\cdots u_{N-k})^{-M} \sum_{\mu\in(N-k)\times M} S_\mu(u_1^2,\cdots,u_{N-k}^2) \sum_{\nu\in N\times M} C_{\mu(1^k)}^{\nu}|\nu\rangle
\end{aligned}
$$

where $C_{\mu(1^k)}^{\nu}$ is the coefficient of Schur function $S_\nu(\mathbf{x})$ in the multiplication $S_\mu(\mathbf{x})S_{(1^k)}(\mathbf{x})$. Then

$$
\begin{aligned}
&\langle 0|C(v_N)\cdots C(v_1)B(u_1)\cdots B(u_{N-k})(\phi_1^{\dagger})^k|0\rangle \\
&= \left(\frac{v_1\cdots v_N}{u_1\cdots u_{N-k}}\right)^M \sum_{\lambda,\mu}\sum_{\nu} C_{\mu(1^k)}^{\nu} S_\lambda(v_1^{-2},\cdots,v_N^{-2}) S_\mu(u_1^2,\cdots,u_{N-k}^2)\langle\lambda|\nu\rangle \\
&= \left(\frac{v_1\cdots v_N}{u_1\cdots u_{N-k}}\right)^M \sum_{\lambda} S_\lambda(v_1^{-2},\cdots,v_N^{-2}) S_{\lambda/(1^k)}(u_1^2,\cdots,u_{N-k}^2),
\end{aligned}
$$

where $\lambda\in N\times M,\ \mu\in(N-k)\times M,\ \nu\in N\times M$. We get the result. $\square$

We can similarly discuss the tau function

$$\tau'_{\Psi\Psi} = \sum_{\lambda \in N \times M} S_\lambda(u_1^2, \cdots, u_N^2) S_\lambda(\mathbf{x}) \tag{77}$$

in equation (69). The wave functions $w_n^*,\ w_n$ for this tau function have the same form with the correlation functions

$$\langle 0|\phi_k C(v_{N-1}) \cdots C(v_1) B(u_1) \cdots B(u_N)|0\rangle, \tag{78}$$

and

$$\langle 0|(\phi_1)^k C(v_{N-k}) \cdots C(v_1) B(u_1) \cdots B(u_N)|0\rangle. \tag{79}$$

Other correlation relations can be calculated in this way. There are similar calculations in paper [12], our calculations are different from the results in [12] since our calculations have close relations with the tau functions and wave functions of the KP hierarchy and the results in [12] are related to the 2-toda hierarchy.

5. The XX0 model

Let

$$\sigma_i^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_i^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_i^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{80}$$

be the Pauli matrices acting on V_i isomorphic to $\mathbb{C}^2$, they are understood to act as identity elsewhere. The Hamiltonian of a periodic length-$M+1$ XX0 spin-$\frac{1}{2}$ chain is

$$H = \sum_{i=0}^{M} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y\right). \tag{81}$$

The eigenstates and eigenvalues of H can be obtained from the algebraic Bethe Ansatz.

Introduce the R-matrix

$$R_{ab}(u,v) = \begin{pmatrix} i\cosh(u-v) & 0 & 0 & 0 \\ 0 & \sinh(u-v) & i & 0 \\ 0 & i & \sinh(u-v) & 0 \\ 0 & 0 & 0 & i\cosh(u-v) \end{pmatrix}_{ab}, \tag{82}$$

where ab means $R_{ab}(u,v)$ acts on $V_a \otimes V_b$.

The L-matrix is

$$L_{ab}(u,v) = R_{ab}(u,v) = \begin{pmatrix} \sinh(u - v + i\frac{\pi}{2}\tilde{\sigma}_b) & i\sigma_b^- \\ i\sigma_b^+ & \sinh(u - v + i\frac{\pi}{2}\tilde{\sigma}'_b) \end{pmatrix}_a, \tag{83}$$

where

$$\sigma^\pm = (\sigma^x \pm i\sigma^y)/2, \ \tilde{\sigma} = (1+\sigma^z)/2, \ \tilde{\sigma}' = (1-\sigma^z)/2.$$

Then the local intertwining relation holds

$$R_{ab}(u,v) L_{ac}(u,w) L_{bc}(v,w) = L_{bc}(v,w) L_{ac}(u,w) R_{ab}(u,v). \tag{84}$$

Define the monodromy matrix by

$$\begin{aligned} T_a(u,\{w\}_M) &= L_{a0}(u,w_1)L_{a1}(u,w_2)\cdots L_{aM}(u,w_M) \\ &= \begin{pmatrix} A(u,\{w\}_M) & B(u,\{w\}_M) \\ C(u,\{w\}_M) & D(u,\{w\}_M) \end{pmatrix}_a . \end{aligned}$$

It is easy to see that the operators $A(u,\{w\}_M), B(u,\{w\}_M), C(u,\{w\}_M), D(u,\{w\}_M)$ acts in $V_0 \otimes V_1 \otimes \cdots \otimes V_M$. One can check that the following relation also holds

$$R_{ab}(u,v)T_a(u,\{w\}_M)T_b(v,\{w\}_M) = T_b(v,\{w\}_M)T_a(u,\{w\}_M)R_{ab}(u,v). \tag{85}$$

Let

$$|0\rangle = \otimes^M \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

the general state

$$|\Psi_N(u_1,\cdots,u_N)\rangle = B(u_1,\{w\}_M)\cdots B(u_N,\{w\}_M)|0\rangle \tag{86}$$

is the eigenstate (which is called Bethe eigenstate) of XX0 Hamiltonian H if the following equations, which are called the Bethe equations, hold

$$(-1)^{N-1}\frac{a(u_i,\{w\}_M)}{d(u_i,\{w\}_M)} = 1 \tag{87}$$

for $1 \le i \le N$, where $a(u,\{w\}_M)$ and $d(u,\{w\}_M)$ are defined in

$$A(u,\{w\}_M)|0\rangle = a(u,\{w\}_M)|0\rangle, \quad D(u,\{w\}_M)|0\rangle = d(u,\{w\}_M)|0\rangle, \tag{88}$$

clearly,

$$a(u,\{w\}_M) = \prod_{i=1}^{M} i\cosh(u-w_i), \quad d(u,\{w\}_M) = \prod_{i=1}^{M}\sinh(u-w_i).$$

In the following, we let $w_i = 0$ for $0 \le i \le M$, and we only consider the general state

$$|\Psi_N(u_1,\cdots,u_N)\rangle = B(u_1)\cdots B(u_N)|0\rangle \tag{89}$$

for any parameters $u_1,\cdots,u_N$. Denote

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |\Uparrow_n^m\rangle = \otimes_{j=n}^m |\uparrow\rangle_i, \quad |\Downarrow_n^m\rangle = \otimes_{j=n}^m |\downarrow\rangle_i$$

for any integers n, m satisfying $0 \le n \le m \le M$. The basis in linear space $(\mathbb{C}^2)^{M+1}$ can be written as

$$\prod_{k=0}^{M}(\sigma_k^-)^{e_k}|\Uparrow_0^M\rangle \tag{90}$$

where e_k equals 0 or 1. Denote $N = e_0 + e_1 + \cdots + e_M$ which is called the particle number, then $0 \le N \le M+1$.

Define a map from states (90) to Young diagrams (Schur functions):

$$\jmath : \prod_{k=0}^{M}(\sigma_k^-)^{e_k}|\Uparrow_0^M\rangle \mapsto \lambda, \quad \lambda = \mu - \delta \tag{91}$$

where $\mu = (0^{e_0}1^{e_1}\cdots M^{e_M})$ and $\delta = (N-1, N-2, \cdots, 1, 0)$ with $N = e_0 + e_1 + \cdots + e_M$. This map is not unique. For example, the images of the states $\sigma_1^- | \Uparrow_0^M\rangle$ and $\sigma_0^- \sigma_2^- | \Uparrow_0^M\rangle$ are both $\lambda = (1)$. However, if the particle number N is fixed, the state corresponding λ under the map $\jmath$ is unique.

For parameter u, we denote $[u] = \sinh u$, $[u]_+ = \cosh u$, $\tilde{u} = i \coth u$ and denote $\{u\}_N = (u_1, \cdots, u_N)$, $\{\tilde{u}\}_N = (\tilde{u}_1, \cdots, \tilde{u}_N)$. From the equation (28) in [8] and solution in [14], we have that the state $|\Psi_N(u_1, \cdots, u_N)\rangle$ equals

$$
\begin{aligned}
&B(u_1)\cdots B(u_N)| \Uparrow_0^M\rangle \\
&\quad = L_N H_{M+1-N}(\tilde{u}_1)\cdots H_{M+1-N}(\tilde{u}_N)|0\rangle && (92)\\
&\quad = L_N \sum_{\lambda\in N\times(M+1-N)} S_\lambda(\{\tilde{u}\}_N) S_\lambda(\mathbf{x}) \\
&\quad = L_N \sum_{\lambda\in N\times(M+1-N)} S_\lambda(\{\tilde{u}\}_N) \prod_{k=0}^{M} (\sigma_k^-)^{e_k} | \Uparrow_0^M\rangle && (93)
\end{aligned}
$$

with $\sum_{k=0}^{M} e_k = N$, where

$$L_N = \prod_{j=1}^{N} i[u_j]^M (1 + S_{(1)}(\{\tilde{u}\}_{j-1})\tilde{u}_j + \cdots + S_{(1^{j-1})}(\{\tilde{u}\}_{j-1})\tilde{u}_j^{j-1}). \tag{94}$$

The state $\langle\Psi_N(v_1, \cdots, v_N)|$ equals

$$
\begin{aligned}
&\langle\Uparrow_0^M |C(v_1)\cdots C(v_N) \\
&\quad = L'_N \langle 0| H^{\perp}_{M+1-N}(\tilde{v}_1^{-1})\cdots H^{\perp}_{M+1-N}(\tilde{v}_N^{-1}) && (95)\\
&\quad = L'_N \sum_{\lambda\in N\times(M+1-N)} S_\lambda(\{v^{-1}\}_N) \langle\Uparrow_0^M | \prod_{k=0}^{M} (\sigma_k^+)^{e_k} && (96)
\end{aligned}
$$

with $\sum_{k=0}^{M} e_k = N$, where the coefficient L'_N equals

$$L'_N = \prod_{j=1}^{N} i(i[v_j]_+)^M (1 + S_{(1)}(\{v^{-1}\}_{j-1})\tilde{v}_j^{-1} + \cdots + S_{(1^{j-1})}(\{v^{-1}\}_{j-1})\tilde{v}_j^{-j+1}). \tag{97}$$

One can find the results above in papers [8,14]. From these results, we can discuss the relations between the XX0 model and the KP hierarchy as the last section, the results include:

Proposition 5.1. *For any positive integers M, N, the general states $|\Psi(u_1, \cdots, u_N)\rangle$ of Hamiltonian in the XX0 model with variables* $\mathbf{x}$ *are tau functions of the KP hierarchy.*

Note that the result in Proposition 5.1 has been obtained in our paper [14], the new results that discuss the relations between the XX0 model and the KP hierarchy in this paper is the following.

Denote the tau function $|\Psi(u_1, \cdots, u_N)\rangle$ in Proposition 5.1 by τ_Ψ^X, which depend only on finite variables $\{x_1, x_2, \cdots, x_M\}$ for any positive integer N. We can calculate $h_n^\perp \tau_\Psi^X$ to obtain the wave functions. In the last section, we have obtained that we can get wave functions of the KP hierarchy by calculating

$$C(v)|\Psi(u_1, \cdots, u_N)\rangle = C(v)B(u_1)\cdots B(u_N)|0\rangle$$

in the phase model. But this is not true for the XX0 model. For example,

Example 5.2. Let $M = 3,\ N = 2,$

$$\begin{aligned}
&B(u_1)B(u_2)|0\rangle \\
&\quad = L_2(u_1,u_2)\sum_{\lambda\in 2\times 2} S_\lambda(u_1,u_2)S_\lambda(\mathbf{x}) \\
&\quad = L_2(u_1,u_2)(|\downarrow\downarrow\uparrow\uparrow\rangle + S_{(1)}(u_1,u_2)|\downarrow\uparrow\downarrow\uparrow\rangle + S_{(2)}(u_1,u_2)|\downarrow\uparrow\uparrow\downarrow\rangle \\
&\qquad + S_{(1^2)}(u_1,u_2)|\uparrow\downarrow\downarrow\uparrow\rangle + S_{(2,1)}(u_1,u_2)|\uparrow\downarrow\uparrow\downarrow\rangle + S_{(2,2)}(u_1,u_2)|\uparrow\uparrow\downarrow\downarrow\rangle).
\end{aligned}$$

By calculation,

$$\begin{aligned}
&C(v)|\downarrow\downarrow\uparrow\uparrow\rangle = i(i[v]_+)^2[v]\,(|\uparrow\downarrow\uparrow\uparrow\rangle + \tilde{v}|\downarrow\uparrow\uparrow\uparrow\rangle)\,, \\
&C(v)|\uparrow\downarrow\downarrow\uparrow\rangle = i(i[v]_+)[v]^2\,(|\uparrow\uparrow\downarrow\uparrow\rangle + \tilde{v}|\uparrow\downarrow\uparrow\uparrow\rangle)\,.
\end{aligned}$$

Then we find that $C(v)S_\lambda(\mathbf{x})$ is not equal to the sum of $h_n^\perp S_\lambda(\mathbf{x})$.

However, we get the following result: Define

$$\imath : \prod_{k=0}^{M}(\sigma_k^-)^{e_k}|\Uparrow_0^M\rangle \mapsto \mu, \tag{98}$$

where $\mu = (0^{e_0}1^{e_1}\cdots M^{e_M})$, here μ is a strict Young diagram in the sense of $\mu_1 > \mu_2 > \cdots$. If the particle number N is fixed, the correspondence between $\prod_{k=0}^{M}(\sigma_k^-)^{e_k}|\Uparrow_0^M\rangle$ and λ under the map $\jmath$ and the correspondence between $\prod_{k=0}^{M}(\sigma_k^-)^{e_k}|\Uparrow_0^M\rangle$ and μ under the map $\imath$ are unique. Write

$$\begin{aligned}
B(u_1)\cdots B(u_N)|\Uparrow_0^M\rangle &= L_N(\{u\}_N)\sum_{\lambda\in N\times(M+1-N)} S_\lambda(\{\tilde{u}\}_N)\prod_{k=0}^{M}(\sigma_k^-)^{e_k}|\Uparrow_0^M\rangle \\
&= L_N(\{u\}_N)\sum_{\lambda\in N\times(M+1-N)} S_\lambda(\{\tilde{u}\}_N)S_\mu(\mathbf{x}),
\end{aligned}$$

here, the row number $l(\mu)$ of Young diagram μ can only be $N-1$ or N. We find that

Proposition 5.3. *In the calculation of* $C(v)|\Psi_N(u_1,\cdots,u_N)\rangle$, *if* $l(\mu) = N-1$, *we have*

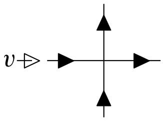
$$C(v)S_\mu(\mathbf{x}) = \sum_{n=0}^{M} f_\nu S_\nu(\mathbf{x}) \tag{99}$$

where the Young diagrams ν *are in* $\mu/(n)$ *and are strict Young diagrams, if* $l(\mu) = N$, *we have*

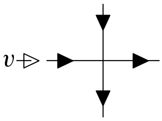
$$C(v)S_\mu(\mathbf{x}) = \sum_{n=1}^{M} g_\nu S_\nu(\mathbf{x}) \tag{100}$$

where the Young diagrams ν *are in* $\mu/(n)$ *and are strict Young diagrams, the row number of* ν *is* $N-1$.

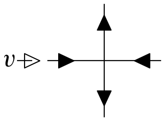
Proof. These results can be obtained directly from the correspondence between the six-vertex model and the XX0 model. The entries of R-matrix in (82) are associated with vertices as shown in the following



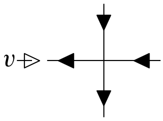
$a_+(v)=i\cosh(v)$



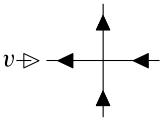
$b_+(v)=\sinh(v)$



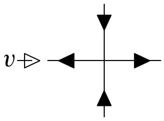
$c_+(v)=i$



$a_-(v)=i\cosh(v)$



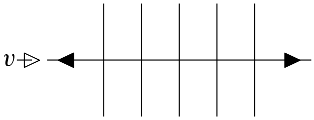
$b_-(v)=\sinh(v)$



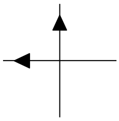
$c_-(v)=i$

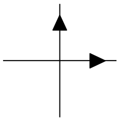
(101)

and all types of vertex not shown are weighted to zero. Then $C(v)$ is associated with

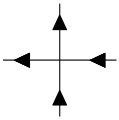


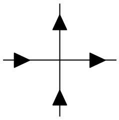
The vertices





equal





respectively. Then we get the conclusion. □

The coefficients $f_\nu,\ g_\nu$ in equations (99) and (100) can be calculated by

$$\langle\nu|C(v)|\mu\rangle.$$

Example 5.4. In $C(v)|\Psi_N(u_1,\cdots,u_N)\rangle$, let $N=3$, $|\downarrow\downarrow\uparrow\downarrow\uparrow\rangle\mapsto\mu=(3,1)$, we have

$$C(v)(3,1)=f_{(3,1)}(3,1)+f_{(3)}(3)+f_{(2,1)}(2,1)+f_{(2)}(2)+f_{(1)}(1)$$

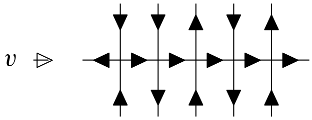
which corresponds to

$$
\begin{aligned}
&C(v)|\downarrow\downarrow\uparrow\downarrow\uparrow\rangle \\
&\quad = f_{(3,1)}|\uparrow\downarrow\uparrow\downarrow\uparrow\rangle + f_{(3)}|\downarrow\uparrow\uparrow\downarrow\uparrow\rangle + f_{(2,1)}|\uparrow\downarrow\downarrow\uparrow\uparrow\rangle + f_{(2)}|\downarrow\uparrow\downarrow\uparrow\uparrow\rangle + f_{(1)}|\downarrow\downarrow\uparrow\uparrow\uparrow\rangle,
\end{aligned}
$$

the coefficient

$$f_{(3,1)} = \langle\uparrow\downarrow\uparrow\downarrow\uparrow|C(v)|\downarrow\downarrow\uparrow\downarrow\uparrow\rangle$$

which equals the weight of



then $f_{(3,1)} = i(i[v]_+)^2[v]^2$. Other coefficients can be calculated by this method: $f_{(2,1)} = i^3(i[v]_+)[v]$, $f_{(3)} = i(i[v]_+)^3[v]$, $f_{(2)} = i^3(i[v]_+)^2$, $f_{(1)} = i(i[v]_+)^3[v]$.

Let $N = 2$, $|\uparrow\downarrow\uparrow\downarrow\uparrow\rangle \mapsto \mu = (3, 1)$, we have

$$C(v)(3,1) = g_{(3)}(3) + g_{(2)}(2) + g_{(1)}(1)$$

which corresponds to

$$C(v)|\downarrow\downarrow\uparrow\downarrow\uparrow\rangle = g_{(3)}|\uparrow\uparrow\uparrow\downarrow\uparrow\rangle + g_{(2)}|\uparrow\uparrow\downarrow\uparrow\uparrow\rangle + g_{(1)}|\uparrow\downarrow\uparrow\uparrow\uparrow\rangle,$$

the coefficient $g_{(3)} = i(i[v]_+)^2([v])^2$, $g_{(2)} = i^3(i[v]_+)([v])$, $g_{(1)} = i(i[v]_+)^2([v])^2$.

From the example above, we can see that the calculations in (99) and (100) are similar to that in the algebra of Schur's Q-functions, but they are different. In order to see the difference clearly, we briefly review the calculation in the algebra of Schur's Q-functions, which can be found in book [3]. Let Q_λ denote the Schur's Q-function corresponding to strict Young diagram λ, which is the special case $t = -1$ of Hall–Littlewood symmetric function $Q_\lambda(\mathbf{x}, t)$, and $q_r = Q_{(r)}$, we know that $q_r^\perp Q_\lambda = Q_{\lambda/(r)}$ is a skew Schur's Q-function, then we have

$$
\begin{aligned}
&(a_0 + a_1 q_1^\perp + a_2 q_2^\perp + a_3 q_3^\perp)Q_{(3,1)} \\
&\quad = a_0 Q_{(3,1)} + 2a_1(Q_{(3)} + Q_{(2,1)}) + 4a_2 Q_{(2)} + 2a_3 Q_{(1)}.
\end{aligned}
\tag{102}
$$

In general,

$$q_n^\perp Q_\lambda = \sum_\mu 2^{a(\mu-\lambda)} Q_\mu \tag{103}$$

where $a(\mu - \lambda)$ is the number of integers $i \geq 1$ such that $\mu - \lambda$ has a square in the ith column but not in the $(i+1)$st column, and the sum is over strict Young diagram μ such that $\lambda \supset \mu$ and $\mu - \lambda$ is a horizontal strip. Because of the difference, we still use Schur function $S_\mu(\mathbf{x})$ in (99) and (100).

Another kind of tau functions of the KP hierarchy in the XX0 model is derived from

$$
\begin{aligned}
&\langle\Psi_N(v_1, \cdots, v_N)|\Psi_N(u_1, \cdots, u_N)\rangle \\
&\quad = L_N L_N' \sum_{\lambda \in N\times(M+1-N)} S_\lambda(\tilde{v}_1^{-1}, \cdots, \tilde{v}_N^{-1}) S_\lambda(\tilde{u}_1, \cdots, \tilde{u}_N),
\end{aligned}
$$

we have

Proposition 5.5. *Let*

$$x_n = \frac{1}{n}\sum_{k=1}^{N} \tilde{u}_k^n, \quad or \quad x_n = \frac{1}{n}\sum_{k=1}^{M} \tilde{v}_k^{-n}, \tag{104}$$

the scalar products $\langle \Psi(v_1, \cdots, v_N) | \Psi(u_1, \cdots, u_N) \rangle$, *which equal*

$$\sum_{\lambda \in N \times (M+1-N)} S_\lambda(\tilde{v}_1^{-1}, \cdots, \tilde{v}_N^{-1}) S_\lambda(\mathbf{x}), \quad or \quad \sum_{\lambda \in N \times (M+1-N)} S_\lambda(\tilde{u}_1, \cdots, \tilde{u}_N) S_\lambda(\mathbf{x}), \tag{105}$$

in the XX0 model for any positive integers N *and* M *are also tau functions of the KP hierarchy with variables* $\mathbf{x}$, *up to an overall factor* $L_N L'_N$.

For these tau functions, we can still calculate the wave functions, but which can not be obtained from the calculations of

$$\langle 0 | C(v_N) \cdots C(v_1) B(u_1) \cdots B(u_{N-1}) \sigma_k^- | 0 \rangle.$$

Note that from the results in [10] and [11], we know that the scalar product of a general state and a Bethe eigenstate in a finite-length XXZ spin-$\frac{1}{2}$ chain is a tau function of the KP hierarchy. But here, our result is that the scalar product of two general states in the XX0 model is a tau function of the KP hierarchy.

6. Concluding remarks

In this paper we present new results on the relation between the quantum integrable models and classical integrable systems. The main result of this paper is that

$$H_M(u_1) H_M(u_2) \cdots H_M(u_N) = \sum_{\lambda \in N \times M} S_\lambda(u_1, \cdots, u_N) S_\lambda(\mathbf{x})$$

is a tau function of the KP hierarchy. Since the general states and the scalar products of two general states in the phase model and the XX0 model can be written into this form, they are tau function of the KP hierarchy. We also discuss the wave function of the KP hierarchy in the phase model and the XX0 model.

Clearly, there are fermion realizations of the phase model and the XX0 model, even the XXZ model, and the scalar product of a general state and a Bethe state could possibly be Boson–Fermion correspondence. In the following, we will discuss these clearly and we want to find more general relations between the quantum integrable model and classical integrable systems.

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