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QUANTUM ELECTRODYNAMICS AT INFINITE MOMENTUM: SCATTERING FROM AN EXTERNAL FIELD†

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ABSTRACT

Using a formulation of quantum electrodynamics in the infinite momentum frame, we develop a theory to describe the scattering of energetic electrons or photons off an external field. A physical picture emerges which proves to be a realization of Feynman's "parton" ideas. In this picture the incoming electron is composed of bare constituents (the quanta of the Schroedinger fields) which, at high laboratory energies, interact slowly with one another. Each bare constituent is scattered from the external field in a simple way, then the constituents again interact among themselves to form the final state. This formalism is applied to elastic electron and photon scattering, bremsstrahlung and pair production, and deep inelastic electroproduction of lepton pairs, and the results of Cheng and Wu and others are recovered in a simple way. In these applications, perturbation theory is used to construct the wave functions of the constituents in the initial and final states.

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I. INTRODUCTION

Recently considerable progress has been made in evaluating amplitudes for high energy electromagnetic processes. Various authors[1] have found, using conventional calculational techniques and considerable labor, that these amplitudes have several unifying features. First, when two electromagnetic particles having large relative momenta exchange a fixed amount of momentum, the interaction can be viewed as occurring between the bare quanta which compose the incoming and outgoing scattering states. And furthermore, the interaction between these constituents is simply a relativistic generalization of the eikonal amplitude familiar from nonrelativistic scattering processes.[2] Thus, a physical picture for these scattering processes emerges which is similar to Feynman's "parton" ideas.[3] We wish to show in this paper that these interesting features can be easily understood and derived from a recent formulation of quantum electrodynamics in the infinite momentum frame developed by two of the authors.[4]

The motivation for developing a formal theory of quantum electrodynamics in the infinite momentum frame, hereafter referred to as I., was the hope that this exact theory would lead to an approximate ultrarelativistic theory which could provide a simple description of extremely high energy phenomena, just as nonrelativistic field theories provide understanding of low energy phenomena. For example, the nonrelativistic limit of quantum electrodynamics possesses tremendous computational simplifications and intuitive insights into low energy electromagnetic processes. It was shown in I. that quantum electrodynamics in the infinite momentum frame, although formally equivalent to quantum electrodynamics developed in an ordinary reference frame, possesses several simplifying features itself. These include the formal absence of vacuum pair creation, computational simplicities, and a nonrelativistic analogy which should become a basis for intuition into high energy phenomena. However, just as the nonrelativistic limit of quantum electrodynamics has certain deficiencies, its ultrarelativistic limit will inherit several limitations already contained in I. For example, the renormalization procedure becomes more difficult, old-fashioned perturbation theory must be used, and manifest covariance is lost. Nonetheless, we will see in this article that for a limited range of applications, specifically the calculation of high energy amplitudes, the formulation of quantum electrodynamics in the infinite momentum frame possesses distinct advantages over the conventional theory.

The plan of this paper will be to review the formalism of quantum electrodynamics in the infinite momentum frame developed in I., and present a heuristic derivation of the salient features of that paper in a "nonrelativistic" fashion. We next introduce an external field into the theory and derive a closed form for the scattering operator, formally valid as the energies of incident and produced particles tend to infinity. We then apply this formalism to several electrodynamic processes and obtain the results of Cheng and Wu and others.

II. REVIEW OF THE INFINITE-MOMENTUM FORMALISM

The trajectories of particles in nonrelativistic processes cluster about a single direction in space-time, which is generally taken to be the time axis. The trajectories in extreme-relativistic processes likewise cluster about a direction in space-time, which can be conventionally taken to be a null vector in the t-z plane. It is sensible to describe nonrelativistic processes in the coordinate system $(t, x, y, \mathfrak{z})$. It is likewise sensible to describe extreme-relativistic processes using the coordinates $\tau = 2^{-1/2}(t+z)$, $x, y, \mathfrak{z} = 2^{-1/2}(t-z)$, since in this coordinate system the particle trajectories cluster about the new "time" axis.

In I., quantum electrodynamics was reformulated in this new coordinate system

$$x^{\mu} = (\tau, x^1, x^2, \mathfrak{z}) = C^{\mu}_{\nu} \hat{x}^{\nu} = g^{\mu\nu} x_{\nu} \quad , \tag{II.1}$$

with x^{ν} the usual space-time coordinates and

$$C^{\mu}_{\nu} = \begin{pmatrix} 2^{-1/2} & 0 & 0 & 2^{-1/2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2^{-1/2} & 0 & 0 & -2^{-1/2} \end{pmatrix} , \qquad g^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{II.2.}$$

The corresponding momenta are $H = p_0 = 2^{-1/2}(E - p_z)$, $\eta = p_3 = 2^{-1/2}(E + p_z)$, and $\underset{\sim}{p} = (p_x, p_y)$. Since H generates τ-translations, it plays the role of a Hamiltonian.

In brief, the procedure used in I. is:

1. Change variables in the Lagrangian. The equations of motion, being form-invariant, remain unchanged.

2. Choose the gauge $A^o = A_3 = 0$.

3. Identify the independent field components and quantize them with the known equal-τ commutation relations satisfied by the corresponding free field components. Only the two transverse components of the electromagnetic potential are independent variables; the component A_3 is zero by the gauge choice, and the component A_0 is eliminated in a way similar to that of conventional Coulomb-gauge electrodynamics. In a similar way, we find that only two of the four components of the Dirac field ψ are independent. Once the equal-τ commutation relations among the independent field components have been specified, all of the equal-τ commutators in the theory can be calculated using Maxwell's equations and the Dirac equation.

4. Construct the Hamiltonian.

5. For the perturbation expansion of the S-matrix, use "old-fashioned" Heitler perturbation theory. This procedure is seen to give a perturbative solution to the field theory identical to the more familiar Feynman expansion.

The infinite momentum analysis of I. led naturally to the use of four-component spinors and polarization vectors which, when boosted to (almost) the speed of light in the z-direction, became eigenstates of helicity as measured in the lab. (Thus, if we choose to describe processes involving particles with almost infinite momentum in the +z direction, this notion of infinite momentum helicity coincides with the familiar description of helicity.) It turns out that the matrix elements of the Hamiltonian of I. are remarkably simple if one chooses the incoming and outgoing particles to be in infinite momentum helicity states.

Instead of simply evaluating the relevant matrix elements in the context of I., we find it instructive and intriguing to rederive these results in a simple heuristic fashion which takes full advantage of the nonrelativistic structure present in the infinite momentum frame. (The connection between the formalism of I. and the formalism to be presented here is given in the Appendix.)

We begin with the mass shell condition for a free electron, $p_\mu p^\mu = m^2$, or $2\eta H - \underset{\sim}{p}^2 = m^2$. If we make the usual identification $p_\mu \rightarrow i\partial_\mu$ we arrive at the equation of motion for the free electron field (the Klein-Gordon equation):

$$i\,\partial_0 \Psi(x) = \frac{1}{2\eta}\,(\underset{\sim}{p}^2 + m^2)\,\Psi(x)\ , \tag{II.3}$$

where $1/\eta$ is the integral operator

$$\left[\frac{1}{\eta}\Psi\right](x) = \frac{1}{2i}\int d\xi\ \epsilon(\mathfrak{z}-\xi)\ \Psi(\tau, \underset{\sim}{x}, \xi). \tag{II.4}$$

As we will see, it suffices to let $\Psi(x)$ have only two components. The two components are postulated to satisfy the equal-τ anticommutation relations

$$\left\{\Psi_\alpha(x),\ \Psi_\beta(x')\right\}_{\tau=\tau'} = \delta_{\alpha\beta}\,\delta(\mathfrak{z}-\mathfrak{z}')\ \delta^2(\underset{\sim}{x}-\underset{\sim}{x}')\ . \tag{II.5}$$

Free photons are described by the two transverse components $\underset{\sim}{A}(x)$ of the electromagnetic potential. As in paper I, we use the infinite momentum gauge, $A^0 = A_3 = 0$. The equal-τ commutation relations satisfied by $\underset{\sim}{A}(x)$

are

$$\left[A^j(x),\ A^k(x')\right]_{\tau=\tau'} = \delta_{jk}\ i\Delta(x-x')_{\tau=\tau'}$$

$$= \delta_{jk}\ \frac{1}{4i}\ \epsilon(\mathscr{y}-\mathscr{y}')\ \delta^2(\underset{\sim}{x}-\underset{\sim}{x}') \ . \tag{II.6}$$

The free photon Hamiltonian is

$$H_\gamma = \frac{1}{2} \sum_{k=1}^{2} \int d\underset{\sim}{x}\ dz\ A^k(x)\ \underset{\sim}{p}^2\ A^k(x) \ . \tag{II.7}$$

Using the commutation relations (II.6), this Hamiltonian leads to the expected equation of motion,

$$\left[A^k(x),\ H_\gamma\right] = i\partial_0\ A^k(x) = \frac{1}{2\eta}\ \underset{\sim}{p}^2\ A^k(x) \ . \tag{II.8}$$

The natural two component spinors w(s) and polarization vectors $\underset{\sim}{\epsilon}(\lambda)$ in this description are

$$w(+1/2) = \begin{pmatrix}1\\0\end{pmatrix} \qquad w(-1/2) = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\underset{\sim}{\epsilon}(+1) = 2^{-1/2}(1, i) \qquad \underset{\sim}{\epsilon}(-1) = 2^{-1/2}(1,-i), \tag{II.9}$$

where the arguments s, λ refer to the infinite momentum helicity discussed earlier. Using these wave functions, the Fourier expansion of the fields Ψ, A take the form[5]

$$\Psi(x) = (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta}\ \sum_s \left\{\sqrt{2\eta}\ w(s)\ e^{-ip\cdot x}\ b(p,s) + \sqrt{2\eta}\ w(-s)\ e^{+ip\cdot x} d^\dagger(p,s)\right\} , \tag{II.10}$$

$$\underset{\sim}{A}(x) = (2\pi)^{-3}\int d\underset{\sim}{p}\int_0^\infty \frac{d\eta}{2\eta}\ \sum_\lambda \left\{\underset{\sim}{e}(\lambda)\ e^{-ip\cdot x}\ a(p,\lambda) + \underset{\sim}{e}(\lambda)^*\ e^{+ip\cdot x}\ a^\dagger(p,\lambda)\right\} \ . \tag{II.11}$$

The operator $b^\dagger(p,s)$, $d^\dagger(p,s)$, and $a^\dagger(p,s)$ are creation operators for electrons, positrons, and photons respectively. They satisfy the commutation relations

$$\left\{b(p,s),\ b^\dagger(p',s')\right\} = \delta_{ss'}(2\pi)^3\ 2\eta\ \delta(\eta-\eta')\ \delta^2(\underset{\sim}{p}-\underset{\sim}{p}')$$

$$\left\{d(p,s),\ d^\dagger(p',s')\right\} = \delta_{ss'}(2\pi)^3\ 2\eta\ \delta(\eta-\eta')\ \delta^2(\underset{\sim}{p}-\underset{\sim}{p}')$$

$$\left[a(p,\lambda),\ a^\dagger(p',\lambda')\right] = \delta_{\lambda\lambda'}(2\pi)^3\ 2\eta\ \delta(\eta-\eta')\ \delta^2(\underset{\sim}{p}-\underset{\sim}{p}') \ . \tag{II.12}$$

The electrodynamic interaction can be introduced into this formalism by writing the free electron wave equation in the form[6]

$$i\partial_0 \Psi = (m - i\underset{\sim}{\sigma}\cdot \underset{\sim}{p}) \frac{1}{2\eta} (m + i\underset{\sim}{\sigma}\cdot \underset{\sim}{p})\Psi \ , \tag{II.13}$$

then making the gauge-invariant substitution $p_\mu \longrightarrow p_\mu - e A_\mu$. Then, using the gauge choice $A_3 = 0$, the wave equation with interactions becomes

$$i\partial_0 \Psi = eA_0\Psi + (m - i\underset{\sim}{\sigma}\cdot [\underset{\sim}{p} - e\underset{\sim}{A}]) \frac{1}{2\eta} (m + i\underset{\sim}{\sigma}\cdot [\underset{\sim}{p} - e\underset{\sim}{A}])\Psi \tag{II.14}$$

The dependent variable A_0 is eliminated with the help of Maxwell's equations, $\partial^\mu F_{\nu\mu} = \partial^\mu \partial_\mu A_\nu - \partial_\nu \partial^\mu A_\mu = J_\nu$. Choosing $\nu = 3$ and recalling that $A_3 = 0$, we find that $-\partial_3(\partial_3 A_0 - \underset{\sim}{\nabla}\cdot \underset{\sim}{A}) = J_3$. From I., we find that $J_3 = J^0 = e\Psi^\dagger\Psi$. Therefore

$$A_0 = \frac{1}{\eta^2} e\Psi^\dagger\Psi + \frac{1}{\eta} \underset{\sim}{p}\cdot\underset{\sim}{A} \ , \tag{II.15}$$

where $1/\eta^2$ is the integral operator

$$\left[\frac{1}{\eta^2}\Psi\right](x) = -\frac{1}{2}\int d\xi \, |\not{\xi} - \xi| \quad \Psi(\tau, \underset{\sim}{x}, \xi) \, . \tag{II.16}$$

Now the equation of motion for Ψ reads

$$i\partial_0 \Psi = \Psi \frac{e^2}{\eta^2} \Psi^\dagger\Psi + \Psi \frac{e}{\eta} \underset{\sim}{p}\cdot\underset{\sim}{A} + (m - i\underset{\sim}{\sigma}\cdot [\underset{\sim}{p} - e\underset{\sim}{A}]) \frac{1}{2\eta} (m + i\underset{\sim}{\sigma}\cdot [\underset{\sim}{p} - e\underset{\sim}{A}])\Psi \tag{II.17}$$

Finally, from Eqs. (II.5),(II.17), and the Heisenberg relation $[iH, \Psi] = \partial_0\Psi$, we can conjecture that the Hamiltonian for the theory is

$$\begin{aligned} H = \int d\underset{\sim}{x}\, d\not{\xi} \Bigg\{ & \frac{e^2}{2} \Psi^\dagger\Psi \frac{1}{\eta^2} \Psi^\dagger\Psi + e\,\Psi^\dagger\Psi \frac{1}{\eta} \underset{\sim}{p}\cdot\underset{\sim}{A} \\ & + \Psi^\dagger(m - i\underset{\sim}{\sigma}\cdot[\underset{\sim}{p} - e\underset{\sim}{A}]) \frac{1}{2\eta} (m + i\underset{\sim}{\sigma}\cdot[\underset{\sim}{p} - e\underset{\sim}{A}])\Psi \\ & + \frac{1}{2}\sum_{k=1}^{2} A^k \underset{\sim}{p}^2 A^k \Bigg\} \end{aligned} \tag{II.18}$$

$$= h_0 + h_I \tag{II.19}$$

with $h_0 = H_{e=0}$.

As we have mentioned, the matrix elements of H are very simple when taken between the "infinite momentum helicity" states created by the operators

$b^\dagger(p, s)$, $d^\dagger(p, s)$, $a^\dagger(p,\lambda)$. The matrix elements are easily calculated using the expansions (II.10) and (II.11) of the fields:

1) Single photon emission (Fig. 1a):

$$\langle e^-(p', s')\ \gamma(q, \lambda)|H|\ e^-(p, s)\rangle$$
$$= (2\pi)^3 \delta(\eta_{out} - \eta_{in})\, \delta^2(\mathbf{p}_{out} - \mathbf{p}_{in})\, [2\eta']^{1/2} [2\eta]^{-1/2}\ e\, w^\dagger(s')\, \mathbf{j}(\mathbf{p}', \mathbf{p}) \cdot \boldsymbol{\epsilon}^*(\lambda)\, w(s) \ , \tag{II.20}$$

where

$$w^\dagger(s')\,\mathbf{j}(p', p) \cdot \boldsymbol{\epsilon}^*(\lambda)\ w(s)$$
$$= w^\dagger(s')\Big\{\eta_q^{-1}\, \mathbf{q} \cdot \boldsymbol{\epsilon}^*(\lambda) - \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^*(\lambda)\, [2\eta]^{-1}\, \boldsymbol{\sigma} \cdot \mathbf{p}$$
$$- \boldsymbol{\sigma} \cdot \mathbf{p}'\, [2\eta']^{-1}\, \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^*(\lambda) - \frac{1}{2}\, im\ \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^*(\lambda)\Big[\eta'^{-1} - \eta^{-1}\Big]\Big\}\, w(s) \ . \tag{II.21}$$

In Table I, we list all of the possible matrix elements $w^\dagger \mathbf{j} \cdot \boldsymbol{\epsilon}^* w$.[7]

The matrix elements for other processes involving two fermions and one photon can be obtained by the usual substitution rules. For instance, the matrix element for $\gamma \to e^- e^+$ is

$$\langle e^-(p', s')\ e^+(p, s)|H|\ \gamma(q, \lambda)\rangle$$
$$= (2\pi)^3\ \delta(\eta_{out} - \eta_{in})\ \delta^2(\mathbf{p}_{out} - \mathbf{p}_{in})\ [2\eta]^{1/2}\ [2\eta']^{1/2}$$
$$\times\ \ e\, w^\dagger(s')\ \mathbf{j}(\mathbf{p}', -\mathbf{p}) \cdot \boldsymbol{\epsilon}^*(-\lambda)\ w(-s) \tag{II.22}$$

2) Instantaneous electron exchange (Fig. 1b):

$$\langle e^-(p_4, s_4)\ \gamma(p_3, \lambda_3)\, |H|\, e^-(p_1, s_1)\ \gamma(p_2, \lambda_2)\rangle$$
$$= (2\pi)^3\ \delta(\eta_{out} - \eta_{in})\, \delta^2(\mathbf{p}_{out} - \mathbf{p}_{in})\, [2\eta_4]^{1/2}\ [2\eta_1]^{1/2}$$
$$\times\, e^2\ w^\dagger(s_4)\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}(\lambda_2)\ [2\eta_0]^{-1}\ \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^*(\lambda_3)\ w(s_1) \ . \tag{II.23}$$

The spinor product is very simple:

$$w^\dagger(s_4)\, \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}(\lambda_2)\, [2\eta_0]^{-1}\, \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon}^*(\lambda_3)\ w(s_1)$$
$$= \begin{cases} 1/\eta_0 & \text{if all the particles are right handed or if all the particles are left handed} \\ 0 & \text{otherwise} \end{cases} \tag{II.24}$$

3) Instantaneous scalar photon exchange (Fig. 1c):

$$\langle e^-(p_3, s_3)\, e^-(p_4, s_4)|H|\, e^-(p_1, s_1)\, e^-(p_2, s_2)\rangle$$
$$= (2\pi)^3\, \delta(\eta_{out} - \eta_{in})\, \delta^2(\underset{\sim}{p}_{out} - \underset{\sim}{p}_{in})\, [2\eta_1\, 2\eta_2\, 2\eta_3\, 2\eta_4]^{1/2}\, e^2 (\eta_0)^{-2} \delta_{s_1 s_3} \delta_{s_2 s_4}$$
$$+ \text{contribution from crossed diagram .} \qquad \text{(II.25)}$$

The veteran field theorist, armed with this information, will be able to construct the rules for old-fashioned perturbation diagrams by whatever formal methods suit his taste:

1) A factor $(H_f - H + i\epsilon)^{-1}$ for each intermediate state;

2) An overall factor $-2\pi i\, \delta(H_f - H_i)$;

3) For each internal line, a sum over spins and an integration $(2\pi)^{-3} \int d\underset{\sim}{p} \int_0^\infty d\eta/(2\eta)$;

4) For each vertex,

   a) a factor $(2\pi)^3\, \delta(\eta_{out} - \eta_{in})\, \delta(\underset{\sim}{p}_{out} - \underset{\sim}{p}_{in})$,

   b) a factor $[2\eta]^{1/2}$ for each fermion line entering or leaving the vertex (the factors $[2\eta]^{1/2}$ associated with each internal fermion line have the effect of removing the factor $1/(2\eta)$ from the phase space integral),

   c) a simple matrix element (e.g., $e w^\dagger \underset{\sim}{j} \cdot \epsilon^* w$) .

5) These rules give the S-matrix element[8] $\langle f|S|i\rangle$. One obtains the differential cross section from the S-matrix in the conventional fashion.

The heuristic approach presented here shows that with some imagination and a little guesswork (along with considerable hindsight), one can obtain these simple results in a simple way.

TABLE I

MATRIX ELEMENTS FOR PHOTON EMISSION

$$p_{\pm} = 2^{-1/2}(p^2 \pm ip^2)\ ,\ q = p - p'$$

s	s'	λ	$w^{\dagger}(s')\,\mathbf{j}(p',p)\cdot\epsilon^*(\lambda)\,w(s)$
1/2	1/2	1	$(q_-/\eta_q) - (p'_-/\eta')$
1/2	1/2	-1	$(q_+/\eta_q) - (p_+/\eta)$
1/2	-1/2	1	$-2^{-1/2}\,\mathrm{im}\,\eta_q/(\eta\eta')$
1/2	-1/2	-1	0
-1/2	1/2	1	0
-1/2	1/2	-1	$-2^{-1/2}\,\mathrm{im}\,\eta_q/(\eta\eta')$
-1/2	-1/2	1	$(q_-/\eta_q) - (p_-/\eta)$
-1/2	-1/2	-1	$(q_+/\eta_q) - (p'_+/\eta')$

III. HIGH ENERGY SCATTERING FROM AN EXTERNAL POTENTIAL

The reformulation of quantum electrodynamics described in I. and above was motivated by a desire to develop limiting theories to describe high energy scattering. We will develop here such a theory to describe the scattering of high energy electrons and photons in a prescribed external electromagnetic potential $a_\mu(x)$. We have derived the results of this section using the complete canonical formalism of I. with the external potential included in the Lagrangian. However, the same results can be obtained by extending the heuristic discussion of Section II. Since the heuristic method is somewhat simpler, we present it here.

Begin by introducing the potential a_μ into the electron wave equation (II.14) according to the gauge-invariant substitution $p_\mu \longrightarrow p_\mu - ea_\mu$. Then the equation of motion reads

$$(i\partial_0 - eA_0 - ea_0)\Psi = \left[m - i\underset{\sim}{\gamma}\cdot(\underset{\sim}{p} - e\underset{\sim}{A} - e\underset{\sim}{a})\right] \frac{1}{2(\eta - ea_3)} \left[m + i\underset{\sim}{\gamma}\cdot(\underset{\sim}{p} - e\underset{\sim}{A} - e\underset{\sim}{a})\right]\Psi . \qquad \text{(III. 1)}$$

Here $\left[\eta - ea_3\right]^{-1}$ is the integral operator

$$\left[\frac{1}{\eta - ea_3}\Psi\right](x) = \int d\xi \, \frac{1}{2i} \epsilon(\mathscr{z} - \xi) \exp\left(-i\int_\xi^{\mathscr{z}} d\xi' \, a_3(\tau, x, \xi')\right) \Psi(\tau, x, \xi) . \qquad \text{(III. 2)}$$

Now, just as in Section III, we can eliminate the dependent variable A_0 using Maxwell's equations and find the Hamiltonian H which gives $i[H, \Psi] = \partial_0 \Psi$. The result is

$$H(\tau) = \int d\underset{\sim}{x}\, d\mathscr{z} \Big\{ ea_0 \Psi^\dagger\Psi + \frac{e^2}{2}\Psi^\dagger\Psi \frac{1}{\eta^2} \Psi^\dagger\Psi + e\Psi^\dagger\Psi \frac{1}{\eta}\underset{\sim}{p}\cdot\underset{\sim}{A}$$

$$+ \Psi^\dagger[m - i\underset{\sim}{\gamma}\cdot(\underset{\sim}{p} - e\underset{\sim}{A} - e\underset{\sim}{a})] \frac{1}{2(\eta - ea_3)} [m + i\underset{\sim}{\gamma}\cdot(\underset{\sim}{p} - e\underset{\sim}{A} - e\underset{\sim}{a})]\Psi$$

$$+ \frac{1}{2}\sum_{k=1}^{2} A^k \underset{\sim}{p}^2 A^k \Big\} . \qquad \text{(III. 3)}$$

It will be convenient to imagine writing H in the form $H(\tau) = H_0(\tau) + V(\tau)$, where $H_0(\tau)$ is given by (III.3) with $a_\mu = 0$ and

$$V(\tau) = H(\tau) - H_0(\tau) \tag{III.4}$$

Thus H_0 is the full Hamiltonian for quantum electrodynamics with no external potential, and V gives the additional effect of the potential.

Now let us look at the scattering matrix in the interaction picture with V as the interaction Hamiltonian. Define the interaction picture fields by

$$\Psi_I(\tau, \underset{\sim}{x}, y) = \exp(+i H_0(0)\tau) \;\; \Psi(0, x_T, y) \; \exp(-i H_0(0)\tau)$$

$$\underset{\sim}{A}_I(\tau, x, y) = \exp(+i H_0(0)\tau) \;\; \underset{\sim}{A}(0, x_T, y) \; \exp(-i H_0(0)\tau) \,, \tag{III.5}$$

and let $V_I(\tau)$ be given by (III.3) and (III.4) with $\Psi_I(x)$ and $\underset{\sim}{A}_I(x)$ substituted for $\Psi(x)$ and $A(x)$. Then it is a familiar exercise to show that the scattering matrix can be written in the form

$$S_{fi} = \langle f | T \exp\left(-i \int d\tau\, V_I(\tau)\right) | i \rangle \,, \tag{III.6}$$

where T indicates τ-ordering and $|f\rangle$ and $|i\rangle$ are appropriate eigenstates of $H_0(0)$ (which may be evaluated in perturbation theory).

We are interested in the high energy limit of S_{fi} as $\eta_i, \eta_f \longrightarrow \infty$. To study this limit we let $|i_0\rangle$ and $|f_0\rangle$ be fixed states and calculate S_{fi} between the high energy states $|i\rangle = e^{-i\omega K_3}|i_0\rangle$ and $|f\rangle = e^{-i\omega K_3}|f_0\rangle$, where K_3 is the generator of Lorentz boosts in the z direction. Thus we want to calculate

$$S_{fi} = \langle f_0 | e^{i\omega K_3} \, T\left\{ \exp\left(-i \int d\tau\, V_I(\tau)\right)\right\} e^{-i\omega K_3} | i_0 \rangle$$

$$= \langle f_0 | T \left\{ \exp\left(-i \int d\tau\, e^{i\omega K_3}\, V_I(\tau)\, e^{-i\omega K_3}\right)\right\} | i_0 \rangle \tag{III.7}$$

in the limit $\omega \longrightarrow \infty$.

We recall from I. that the boost operator K_3 is given by

$$K_3 = \int d\underset{\sim}{x}\, d y \left[\Psi^\dagger \, \frac{i}{2} \overleftrightarrow{\partial}_3 \Psi + (\partial_3 \underset{\sim}{A}) \cdot (\partial_3 \underset{\sim}{A}) \right]_{\tau = 0} \,, \tag{III.8}$$

and that the fields transform very simply under boosts:

$$e^{i\omega K_3}\,\Psi_I(\tau,\underset{\sim}{x},\mathscr{y})\,e^{-i\omega K_3} = e^{\omega/2}\,\Psi_I(e^{-\omega}\tau,\underset{\sim}{x},e^{\omega}\mathscr{y})$$
$$e^{i\omega K_3}\,\underset{\sim}{A}_I(\tau,\underset{\sim}{x},\mathscr{y})\,e^{-i\omega K_3} = \underset{\sim}{A}_I(e^{-\omega}\tau,\underset{\sim}{x},e^{\omega}\mathscr{y})\ . \qquad \text{(III.9)}$$

It is thus easy to calculate the effect of the boost operator on $V_I(\tau)$. The term $ea_0\Psi^\dagger\Psi$ remains finite in the limit $\omega\to\infty$ and the rest of the terms are of order $e^{-\omega}$; we indeed find that

$$e^{i\omega K_3}V_I(\tau)\,e^{-i\omega K_3} = \int d\underset{\sim}{x}d\mathscr{y}\,e^{\omega}ea_0(\tau,\underset{\sim}{x},\mathscr{y})\,\Psi_I^\dagger(e^{-\omega}\tau,\underset{\sim}{x},e^{\omega}\mathscr{y})\,\Psi_I(e^{-\omega}\tau,\underset{\sim}{x},e^{\omega}\mathscr{y}) + O(e^{-\omega})$$
$$= \int d\underset{\sim}{x}d\mathscr{y}\,ea_0(\tau,\underset{\sim}{x},e^{-\omega}\mathscr{y})\,\Psi_I^\dagger(e^{-\omega}\tau,\underset{\sim}{x},\mathscr{y})\,\Psi_I(e^{-\omega}\tau,\underset{\sim}{x},\mathscr{y}) + O(e^{-\omega})$$
$$\text{(III.10)}$$

Upon going to the limit the operators are all evaluated at $\tau = 0$, so the τ-ordering can be ignored. (This may be checked by examining the power series expansion.) Thus we obtain as $\tau \to$

$$S_{fi} = \langle f_0|\mathbb{F}|i_0\rangle + O(e^{-\omega})$$
$$= \langle f|\mathbb{F}|i\rangle + O(e^{-\omega})\ , \qquad \text{(III.11)}$$

where

$$\mathbb{F} = \exp\left(-i\int d\tau\,d\underset{\sim}{x}d\mathscr{y}\,ea_0(\tau,\underset{\sim}{x},0)\,\Psi_I^\dagger(0,\underset{\sim}{x},\mathscr{y})\,\Psi_I(0,\underset{\sim}{x},\mathscr{y})\right)$$
$$= \exp\left\{-i\int d\underset{\sim}{x}\;\chi(\underset{\sim}{x})\;\rho(\underset{\sim}{x})\right\} \qquad \text{(III.12)}$$

and

$$\chi(\underset{\sim}{x}) = e\int d\tau\;a_0(\tau,\underset{\sim}{x},0) \qquad \text{(III.13)}$$
$$\rho(\underset{\sim}{x}) = \int d\mathscr{y}\,\Psi^\dagger(0,\underset{\sim}{x},\mathscr{y})\,\Psi(0,\underset{\sim}{x},\mathscr{y})\ . \qquad \text{(III.14)}$$

This (formally) closed expression for the limiting form of the scattering operator is in fact the eikonal approximation, and also establishes a connection with parton ideas. The initial state $|i\rangle$ is an eigenstate of H_0, the Hamiltonian for quantum electrodynamics with no external field. Thus it is a "dressed" electron, photon, or whatever. Imagine expanding $|i\rangle$ in terms of the "bare" quanta associated with the fields $\psi(0,\underset{\sim}{x},\mathscr{y})$, $\underset{\sim}{A}(0,\underset{\sim}{x},\mathscr{y})$ at time $\tau = 0$:

$$|i\rangle = +\int d\underset{\sim}{p} \int_0^\infty \frac{d\eta}{2\eta} \sum_\lambda g(\underset{\sim}{p},\eta,\lambda)\, a^\dagger(\underset{\sim}{p},\eta,\lambda)|0\rangle + \ldots$$

$$+\int d\underset{\sim}{p}_1 \int_0^\infty \frac{d\eta_1}{2\eta_1} \int d\underset{\sim}{p}_2 \int_0^\infty \frac{d\eta_2}{2\eta_2} \sum_{s_1 s_2} h(\underset{\sim}{p}_1,\eta_1,s_1;\underset{\sim}{p}_2,\eta_2,s_2)$$

$$\times\, b^\dagger(\underset{\sim}{p}_1,\eta_1,s_1)\, d^\dagger(\underset{\sim}{p}_2,\eta_2,s_2)\,|0\rangle + \ldots \,. \qquad \text{(III.15)}$$

Here, for example, $h(\underset{\sim}{p}_1,\eta_1,s_1;\underset{\sim}{p}_2,\eta_2,s_2)$ is the amplitude for the state $|i\rangle$ to contain a bare electron with momentum $\underset{\sim}{p}_1,\eta_1$ and spin s_1, and a bare positron with momentum $\underset{\sim}{p}_2,\eta_2$ and spin s_2.

We also imagine the final scattering state $|f\rangle$ to be expanded in terms of bare quanta ("partons") in the same way. If we know all the amplitudes g, h, etc., we can then evaluate S_{fi} by moving $\mathbb{F}$ to the right past all of the parton creation operators until $\mathbb{F}$ acts on the vacuum state $|0\rangle$. That is, we write

$$\mathbb{F}\, b^\dagger \circ\circ\circ\, a^\dagger|0\rangle = \mathbb{F} b^\dagger \mathbb{F}^{-1} \circ\circ\circ\, \mathbb{F} a^\dagger \mathbb{F}^{-1}\, \mathbb{F}|0\rangle \quad . \qquad \text{(III.16)}$$

We note that $\mathbb{F}$ is invariant under $\mathscr{y}$-translations, and thus commutes with the momentum operator η. Since $|0\rangle$ is the only state with $\eta = 0$, we conclude that $\mathbb{F}|0\rangle = |0\rangle$. (This result can be formally assured by considering the operators in $\rho(x)$ to be normal-ordered.) The effect of $\mathbb{F}$ on the creation operators $b^\dagger$, $d^\dagger$, $a^\dagger$ is easily calculated using the equal-τ commutation relations (II.5). We find first that

$$\mathbb{F}\, \psi^\dagger(0,\underset{\sim}{x},\mathscr{y})\, \mathbb{F}^{-1} = e^{-i\chi(\underset{\sim}{x})}\, \psi^\dagger(0,\underset{\sim}{x},\mathscr{y}) \quad . \qquad \text{(III.17)}$$

Upon Fourier-transforming this relation we obtain the convolution integral

$$\mathbb{F} b^\dagger(\underset{\sim}{p},\eta;s)\, \mathbb{F}^{-1} = \int \frac{d\underset{\sim}{p}'}{(2\pi)^2}\, b^\dagger(\underset{\sim}{p}',\eta;s)\, F(\underset{\sim}{p}'-\underset{\sim}{p}) \quad , \qquad \text{(III.18)}$$

where

$$F(\underset{\sim}{q}) = \int d\underset{\sim}{x}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}\, e^{-i\chi(\underset{\sim}{x})} \quad . \tag{III.19}$$

Thus when a high energy bare electron passes through the potential at position $\underset{\sim}{x}$, the only effect of the potential is to multiply the electron wave function by an eikonal phase factor $\exp(-i\chi(\underset{\sim}{x}))$. (Note that the phase $\chi(\underset{\sim}{x})$ is simply the integral of the potential along the trajectory of the electron.) The momentum component η of the bare electron and its infinite momentum helicity s are conserved in the process, and no pairs are created.

The effect of F on the positron creation operators is equally simple. In passing through the potential each bare positron receives the opposite phase:

$$\mathbb{F}\, d^\dagger(\underset{\sim}{p},\eta;s)\, \mathbb{F}^{-1} = \int \frac{d\underset{\sim}{p}'}{(2\pi)^2}\, d^\dagger(\underset{\sim}{p}',\eta;s)\, F_c(\underset{\sim}{p}' - \underset{\sim}{p}) \ , \tag{III.20}$$

where

$$F_c(\underset{\sim}{q}) = \int d\underset{\sim}{x}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}}\, e^{+i\chi(\underset{\sim}{x})} \quad . \tag{III.21}$$

Finally, we find that the bare photons are unaffected by the potential:

$$\mathbb{F}\, a^\dagger(\underset{\sim}{p},\eta;\lambda)\, \mathbb{F}^{-1} = a^\dagger(\underset{\sim}{p},\eta;\lambda) \ . \tag{III.22}$$

After we have moved F to the right past all of the parton creation operators, we are left with an expansion of the state $\mathbb{F}|i\rangle$ in terms of parton states (similar to the expansion (III.15) of $|i\rangle$). Assuming that the expansion of the final state $|f\rangle$ is also known, it is then a simple matter to compute the overlap S_{fi} of $|f\rangle$ with $\mathbb{F}|i\rangle$.

Of course we do not in fact know the amplitudes involved in the expansions of the states $|i\rangle$ and $|f\rangle$ in terms of bare particle states. In the examples treated in the next section we are forced to use approximate amplitudes calculated from perturbation theory. What we wish to emphasize here is the physical picture that emerges from the present discussion:

1) The scattering of high energy physical particles from the external potential is not simple. For example, it is not described by a single eikonal phase.
2) The physical particles can be viewed as being composed of certain constituent particles (called partons in the language of Feynman). In the present case the partons are the "bare" quanta created by the fields ψ and $\underset{\sim}{A}$ at $\tau = 0$.
3) The scattering of high energy partons from the potential is simple.
4) The interaction of the partons among themselves is complicated, but at high energies these interactions are slowed down by relativistic time dilation. Therefore no parton-parton interactions take place during the finite time interval during which the partons interact with the external field.

Thus the scattering of high energy particles from the external field occurs in three steps. First the partons in the initial state interact among themselves during the infinite time interval $-\infty < \tau < 0$. Then each individual parton scatters in a simple way from the external potential. Finally, the partons again interact among themselves during the infinite time interval $0 < \tau < \infty$.

IV. EXAMPLES

In this section we calculate the high energy limits of the cross sections for several interesting scattering processes. As we have seen, the contribution to the high energy limit of the S-matrix from the scattering of the individual partons off the external field can be calculated exactly. However, the interactions among the partons in the initial and final states do not simplify in the high energy limit. Thus we include these interactions only to a finite order in perturbation theory.

Nevertheless, the required calculations in perturbation theory are quite easy because of the simple form of the matrix elements of the Hamiltonian in the infinite momentum frame.

We begin with a short discussion of the methods involved in the calculations, then proceed to the calculation of cross sections for electron scattering with second order vertex corrections, bremsstrahlung, pair production, Delbruck scattering, and electroproduction of μ-pairs in an external field.

A. Calculational Methods

In all of our applications we must compute the amplitudes involved in the expansions (III.15) of the initial and final states in terms of bare particle states. To do this, we recall the definition of the unitary evolution operator $U(\tau', \tau) = \exp(ih_0\tau') \exp(-i[h_0 + h_I][\tau' - \tau]) \exp(-ih_0\tau)$, where h_0 is the free particle Hamiltonian and $h_0 + h_I$ is the full Hamiltonain for quantum electrodynamics with no external potential. The final physical scattering state $|f(b)\rangle$ consisting of outgoing particles with momenta and helicities labeled by 'b' is related to the corresponding bare particle state $|b\rangle$ by $\langle f(b)| = \langle b|U(\infty, 0)$. Similarly, the physical initial state $|i(a)\rangle$ is related to the corresponding bare particle state $|a\rangle$ by $|i(a)\rangle = U(0, -\infty)|a\rangle$. Thus the high energy limit of the scattering matrix, Eq. (III.11), can be written as

$$\langle b|S|a\rangle = \langle f(b)|\mathbb{F}|i(a)\rangle = \langle b|U(\infty, 0)\, \mathbb{F}\, U(0, -\infty)|a\rangle . \qquad \text{(IV.1)}$$

We need the expansion (III.15) of $|f(b)\rangle$ in terms of bare particle states $|n\rangle$: $\langle f(b)| = \sum_n \langle b|U(\infty, 0)|n\rangle \langle n|$. The amplitudes $\langle b|U(\infty, 0)|n\rangle$ can be calculated to a finite order in perturbation theory using the familiar perturbation expansion of $U(\infty, 0)$:

$$\langle f(b)| = \langle b| + \sum_n \langle b|h_I|n\rangle \frac{1}{H_f - H_n + i\epsilon} \langle n|$$

$$+ \sum_{m,n} \langle b|h_I|m\rangle \frac{1}{H_f - H_m + i\epsilon} \langle m|h_I|n\rangle \frac{1}{H_f - H_n + i\epsilon} \langle n| + \ldots, \tag{IV.2}$$

where H_f is the energy of the final state and $h_0|m\rangle = H_m|m\rangle$.

Similarly, the initial state can be written as

$$|i(a)\rangle = \sum_n |n\rangle \langle n|U(0, -\infty)|a\rangle = |a\rangle + \sum_n |n\rangle \frac{1}{H_i - H_n + i\epsilon} \langle n|h_I|a\rangle + \ldots$$

However, since the initial state in our examples is always a one particle state, it is convenient to factor the wave function renormalization constant $\sqrt{Z_a}$ out of this expansion[9]:

$$|i(a)\rangle = \sqrt{Z_a} \Big\{ |a\rangle + \sum_n{}' |n\rangle \frac{1}{H_i - H_n} \langle n|h_I|a\rangle$$

$$+ \sum_n{}' \sum_m{}' |n\rangle \frac{1}{H_i - H_n} \langle n|h_I|m\rangle \frac{1}{H_i - H_m} \langle m|h_I|a\rangle + \ldots \tag{IV.3}$$

If $|a\rangle$ is, say, a one electron state then the sums Σ' exclude one electron states; the $i\epsilon$ terms in the energy denominators are then irrelevant. Since $U(0, -\infty)$ is unitary, the renormalization constant $\sqrt{Z_a}$ can be determined from the requirement

$$\langle i(a)|i(a')\rangle = \langle a|a'\rangle \quad . \tag{IV.4}$$

Let us return now to the formula (IV.1) for $\langle b|S|a\rangle$. It will prove convenient to explicitly separate the uninteresting "no scattering" term $\langle b|a\rangle$ from $\langle b|S|a\rangle$ before doing any calculations. This can be accomplished by noting that $\langle b|U(\infty, 0)\, \mathbb{1}\, U(0, -\infty)|a\rangle = \langle b|U(\infty, -\infty)|a\rangle$ is the S-matrix for quantum electrodynamics with no external potential, which is simply $\langle b|U(\infty, -\infty)|a\rangle = \langle b|a\rangle$ if $|a\rangle$ is a (stable) one particle state. Thus

$$\langle b|S|a\rangle = \langle b|a\rangle + \langle b|U(\infty, 0)\,[\mathbb{F} - \mathbb{1}]\,U(0, -\infty)|a\rangle \quad . \tag{IV.5}$$

It is, of course, only the second term in (IV.5) which is related to cross sections. With the normalization conventions used in this paper, the exact relationship is[10]

$$d\sigma = \frac{1}{2\eta_a} \frac{d\underset{\sim}{p}_1\, d\eta_1}{(2\pi)^3\, 2\eta_1} \cdots \frac{d\underset{\sim}{p}_N\, d\eta_N}{(2\pi)^3\, 2\eta_N} (2\pi)\, \delta\left(\eta_a - \sum_{j=1}^{N} \eta_j\right) \left|\langle b|\mathscr{T}|a\rangle\right|^2 , \quad \text{(IV.6)}$$

where the transition amplitude $\langle b|\mathscr{T}|a\rangle$ is defined by

$$\langle b|U(\infty, 0)\, [\mathbb{F} - \mathbb{1}]\; U(0, -\infty)|a\rangle = (2\pi)\, \delta(\eta_a - \eta_b) \langle b|\mathscr{T}|a\rangle \; . \quad \text{(IV.7)}$$

B. Electron Scattering

We wish to calculate the amplitude

$$S_{fi} - \delta_{fi} = \langle e^-(p', s')|U(\infty, 0)\, [\mathbb{F} - \mathbb{1}]\; U(0, -\infty)|e^-(p, s)\rangle \quad \text{(IV.8)}$$

for high energy electron scattering off an external field. We will calculate the amplitude to second order in the structure of the physical electron. Using the expansion (IV.3) for $\langle e^-|U(\ \ , 0)$ and $U(0, -\infty)\, e^-\rangle$ and keeping terms to order e^2 we find with the help of (III.18) that

$$S_{fi} - \delta_{fi} = (2\pi)\, \delta(\eta - \eta')\, 2\eta\, Z_2 \left[F(\underset{\sim}{p}' - \underset{\sim}{p}) - (2\pi)^2\, \delta^2(\underset{\sim}{p}' - \underset{\sim}{p})\right]$$

$$\times \Bigg\{ \delta_{ss'} + (2\pi)^{-3} \int d\underset{\sim}{p}_2 \int_0^{\eta} \frac{d\eta_2}{2\eta_2} \sum_{s_1, \lambda_2} e^2$$

$$\times \frac{w^\dagger(s')\, \underset{\sim}{j}(p', p' - p_2) \cdot \underset{\sim}{\epsilon}(\lambda_2) w(s_1) \quad w^\dagger(s_1)\, \underset{\sim}{j}(p - p_2, p) \cdot \underset{\sim}{\epsilon}^*(\lambda_2)\, w(s)}{[H(p') - H(p' - p_2) - \omega(p_2)]\; [H(p) - H(p - p_2) - \omega(p_2)]} \Bigg\} . \quad \text{(IV.9)}$$

Here $H(p) = (\underset{\sim}{p}^2 + m^2)/2\eta$ is the free electron Hamiltonian, $\omega(p) = \underset{\sim}{p}^2/2\eta$ is the free photon Hamiltonian, and Z_2 is the electron wave function renormalization constant (to be calculated to order e^2). The two terms in Eq. (IV.9) are represented by τ-ordered diagrams in Fig. 2a and 2b. The figures also clarify the kinematic notation chosen here. The black dots in the diagrams refer to the eikonal factor $[F(\underset{\sim}{p}' - \underset{\sim}{p}) - (2\pi)^2\, \delta^2(\underset{\sim}{p}' - \underset{\sim}{p})]$.

In order to discuss the general form of the scattering amplitude, let us write (IV.9) in the abbreviated form

$$S_{fi} - \delta_{fi} = (2\pi)\,\delta(\eta-\eta')\,2\eta\left[F(\underset{\sim}{q}) - (2\pi)^2\,\delta^2(\underset{\sim}{q})\right] \quad w^\dagger(s')\,M(\underset{\sim}{p}',\eta;\,\underset{\sim}{p},\eta)\,w(s) \qquad \text{(IV.10)}$$

where $q^\mu = p'^\mu - p^\mu$. One important result which we notice immediately is that the second order vertex correction does not destroy the proportionality between the scattering amplitude and the eikonal factor that one finds if the electron structure is neglected altogether.[1] However, it should be pointed out that if the scattering amplitude were calculated to fourth order in the structure of the electron, a diagram like Fig. 3 would appear and this proportionality would be lost.[11]

The effects of the electron structure are contained in the factor $w^\dagger Mw$. It will come as no surprise that the four matrix elements of M are simply related to two invariant form factors $F_{1,2}(q^2)$. It is instructive to derive this relation using the invariance principles which appear naturally in the infinite momentum frame. Using Eq. (IV.10) and the table of matrix elements, Table I, then we can easily verify that $w^\dagger Mw$ is invariant under the following symmetry operations:

1) Lorentz z-boosts: momenta transform according to $(\eta, \underset{\sim}{p}) \longrightarrow (e^\omega \eta, \underset{\sim}{p})$, helicities remain unchanged.

2) "Galilean boosts": momenta transform according to $(\eta, \underset{\sim}{p}) \rightarrow (\eta, \underset{\sim}{p} + \eta u)$, helicities remain unchanged.

3) Rotations in the (x^1, x^2)-plane.

4) "Parity": momenta transform according to $(\eta, p^1, p^2) \longrightarrow (\eta, p^1, -p^2)$, helicities are reversed.

For $q \neq 0$, the four matrices $\mathbb{1}$, $\underset{\sim}{q}\cdot\underset{\sim}{\sigma}$, $\underset{\sim}{q}\times\underset{\sim}{\sigma} = q^1\sigma^2 - q^2\sigma^1$, σ_z are linearly independent. Thus M can be written in the form

$$M(p', p) = a\,\mathbb{1} + b\,\underset{\sim}{q}\cdot\underset{\sim}{\sigma} + c\,\underset{\sim}{q}\times\underset{\sim}{\sigma} + d\sigma_z\ . \qquad \text{(IV.11)}$$

The coefficients a, b, c, d will then be functions of p' and p, or, equivalently, of $\eta(=\eta')$, $\underset{\sim}{p}' + \underset{\sim}{p}$, $\Theta_q = \tan^{-1}(q^2/q^1)$, and $\underset{\sim}{q}^2$. But the invariance of $w^\dagger M w$ under Lorentz z-boosts implies that the coefficients are independent of η; invariance under "Galilean boosts" implies that they are independent of $\underset{\sim}{p}' + \underset{\sim}{p}$; and rotational invariance implies that they are independent of Θ_q. Thus each coefficient is a function of $\underset{\sim}{q}^2$ only. Finally, invariance of $w^\dagger M w$ under the "parity" operation implies that $c(\underset{\sim}{q}^2) = -c(\underset{\sim}{q}^2)$ and $d(\underset{\sim}{q}^2) = -d(\underset{\sim}{q}^2)$; hence c = d = 0. The remaining form factors a and b are functions of $\underset{\sim}{q}^2$; but since $\eta_q = 0$,

$$q^2 \equiv q^\mu q_\mu = 2\eta_q H_q - \underset{\sim}{q}^2 = -\underset{\sim}{q}^2 \tag{IV.12}$$

Therefore the expansion of M takes the form

$$M(p', p) = a(q^2)\, \mathbb{1} + b(q^2)\, \underset{\sim}{q} \cdot \underset{\sim}{\sigma} \tag{IV.13}$$

This analysis can be compared to the general analysis of electron scattering from a weak external field which concludes that the S-matrix, calculated to first order in the external potential and all orders in the structure of the electron, takes the form

$$S_{fi} - \delta_{fi} = -i \int d^4x\, ea_\mu(x)\, e^{iq\cdot x}$$

$$\times\, \overline{U}(p', s') \left\{ \gamma^\mu F_1(q^2) + \frac{i}{2m} \sigma^{\mu\nu} q_\nu F_2(q^2) \right\} U(p, s) . \tag{IV.14}$$

In the high energy limit, Eq. (IV.14) becomes

$$S_{fi} - \delta_{fi} = -2\pi i\, \delta(\eta' - \eta) \left[\int d\underset{\sim}{x}\, e^{-i\underset{\sim}{q}\cdot\underset{\sim}{x}} \int d\tau\, ea_0(\tau, \underset{\sim}{x}, 0) \right]$$

$$\overline{U}(p', s') \left\{ \gamma^o F_1(q^2) + \frac{i}{2m} \sigma^{o\nu} q_\nu F_2(q^2) \right\} U(p, s).$$

When this result is converted to the notation used in this paper, it reads

$$S_{fi} - \delta_{fi} = (2\pi)\, \delta(\eta' - \eta)\, 2\eta \left[-i \int d\underset{\sim}{x}\, \chi(\underset{\sim}{x}) \exp(-i\underset{\sim}{q}\cdot\underset{\sim}{x}) \right]$$

$$\times\, w^\dagger(s') \left\{ F_1(q^2)\, \mathbb{1} + \frac{i}{2m} F_2(q^2)\, \underset{\sim}{q}\cdot\underset{\sim}{\sigma} \right\} w(s). \tag{IV.15}$$

Comparison of this result with (IV.10) and (IV.13) shows that the form factor $a(q^2)$ can be identified with $F_1(q^2)$ and $b(q^2)$ can be identified with $i/(2m)]F_2(q^2)$. Thus our result is

$$S_{fi} - \delta_{fi} = (2\pi)\, \delta(\eta' - \eta)\, 2\eta \left[F(\underset{\sim}{q}) - (2\pi)^2\, \delta^2(\underset{\sim}{q})\right]$$

$$\times\, w^\dagger\,(s') \left\{F_1(q^2)\ \mathbb{1} + \frac{i}{2m}\ F_2(q^2)\ \underset{\sim}{q}\cdot\underset{\sim}{\sigma}\right\} w(s). \qquad \text{(IV.16)}$$

Apparently the amplitude for scattering with no change in helicity is proportional to $F_1(q^2)$, whereas the helicity flip amplitudes are proportional to $F_2(q^2)$. For instance

$$S_{fi}\left(s'=\frac{1}{2},\ s=\frac{1}{2}\right) = \delta_{fi} + (2\pi)\,\delta(\eta'-\eta)\, 2\eta\left[F(\underset{\sim}{q}) - (2\pi)^2\ \delta^2(\underset{\sim}{q})\right] F_1(q^2) \qquad \text{IV.17)}$$

$$S_{fi}\left(s'=-\frac{1}{2}, s=\frac{1}{2}\right) = (2\pi)\,\delta(\eta'-\eta)\ 2\eta\ F(\underset{\sim}{q}) - (2\pi)^2\ \delta^2(\underset{\sim}{q})\ \frac{iq_+}{\sqrt{2}m}\ F_2(q^2), \qquad \text{(IV.18)}$$

where $q_\pm = 2^{-1/2}(q^1 \pm iq^2)$.

We are now in a position to return to Eq. (IV.9) in order to calculate the electron form factors. We begine with the helicity flip amplitude and the form factor F_2. It is convenient to choose a coordinate system (by transforming the coordinates with a "Galilean boost" if necessary) so that

$$p^\mu = (\eta, -\underset{\sim}{p}', H), \qquad p'^\mu = (\eta, \underset{\sim}{p}', H), \qquad q^\mu = (0, 2\underset{\sim}{p}', 0)\ .$$

Then the energy denominators in (IV.9) become

$$H(p') - H(p' - p_2) - \omega(p_2) = -\frac{1}{2\eta}\left[\frac{(\underset{\sim}{p}_2 - \beta\underset{\sim}{p}')^2 + \beta^2 m^2}{\beta(1-\beta)}\right]$$

$$H(p) - H(p - p_2) - \omega(p_2) = -\frac{1}{2\eta}\left[\frac{(\underset{\sim}{p}_2 + \beta\underset{\sim}{p}')^2 + \beta^2 m^2}{\beta(1-\beta)}\right], \qquad \text{(IV.19)}$$

where

$$\beta = \eta_2/\eta\ .$$

The numerator factor in the helicity flip amplitude is trivially calculated with the aid of Table I:

$$\sum_{s_1,\lambda_2} w^{\dagger}(-1/2)\, \underset{\sim}{j}\cdot\underset{\sim}{\epsilon}\, w\, w^{\dagger}\underset{\sim}{j}\cdot\underset{\sim}{\epsilon}^{*}\, w(+1/2)$$

$$= \frac{im\ \eta_2}{\sqrt{2}\,\eta(\eta-\eta_2)}\left(\frac{p_{2+}}{\eta_2}-\frac{p'_+}{\eta}\right)+\left(\frac{p_{2+}}{\eta_2}-\frac{p'_+}{\eta}\right)\frac{-im\,\eta_2}{\sqrt{2}\,\eta(\eta-\eta_2)}$$

$$= \frac{1}{\eta^o}\sqrt{2}\,im\,\frac{\beta}{1-\beta}\ p'_+ \ . \qquad \text{(IV.20)}$$

If we insert these results (IV.19) and (IV.20) back into (IV.9) and use (IV.18) to identify $F_2(q^2)$ we find[12]

$$F_2(\underset{\sim}{q}^2) = \frac{4\alpha m^2}{(2\pi)^2}\int_0^1 d\beta\, \beta^2(1-\beta)\int d\underset{\sim}{p}_2 \times \left[\left(\underset{\sim}{p}_2^2+\beta^2\left[\frac{1}{4}\underset{\sim}{q}^2+m^2\right]\right)^2-\beta^2(\underset{\sim}{p}_2\cdot\underset{\sim}{q})^2\right]^{-1}. \qquad \text{(IV.21)}$$

The integrals are elementary and we find without difficulty

$$F_2(\underset{\sim}{q}^2) = \frac{\alpha}{2\pi}\left[\frac{2m^2}{|\underset{\sim}{q}|\,(\underset{\sim}{q}^2+4m^2)^{1/2}}\ \log\left(\frac{(\underset{\sim}{q}^2+4m^2)^{1/2}+|\underset{\sim}{q}|}{(\underset{\sim}{q}^2+4m^2)^{1/2}-|\underset{\sim}{q}|}\right)\right]. \qquad \text{(IV.22)}$$

We recognize this equation as a familiar expression for the second order contribution ot $F_2(q^2)$.[13] Letting $\underset{\sim}{q}^2$ 0, we obtain

$$F_2(0) = \frac{\alpha}{2\pi} \qquad \text{(IV.23)}$$

which is the famous anomalous magnetic moment of the electron.

Before turning to consider the form factor $F_1(q^2)$, we shall point out the calculational advantages that the formulation of infinite momentum perturbation theory used here has over others that have appeared in the literature.[14] First, no high energy approximation has to be used to extract the important pieces of the energy denominators and vertices. This occurs because of the simple scaling behavior our kinematic variables have under boosts in the z-direction. Secondly, the electrodynamic vertices between infinite momentum helicity states are so simple that traces can be altogether avoided.

We now turn our attention to the helicity nonflip amplitude and the form factor F_1. Using Table I, we calculate the numerator factor in the amplitude (IV.9):

$$\sum_{s_1, \lambda_2} w^{\dagger}(+1/2)\, \mathbf{j} \cdot \boldsymbol{\epsilon}\, w\, w^{\dagger} \mathbf{j} \cdot \boldsymbol{\epsilon}^* \, w(+1/2)$$

$$= \left(\frac{p_{2+}}{\eta_2} - \frac{p'_+ - p_{2+}}{\eta - \eta_2}\right)\left(\frac{p_{2-}}{\eta_2} + \frac{p'_- + p_{2-}}{\eta - \eta_2}\right)$$

$$+ \left(\frac{p_{2-}}{\eta_2} - \frac{p'_-}{\eta}\right)\left(\frac{p_{2+}}{\eta_2} + \frac{p'_+}{\eta}\right) + \left(\frac{m}{\sqrt{2}}\, \frac{\eta_2}{\eta(\eta - \eta_2)}\right)^2$$

$$= \left[2\eta^2 \beta^2 (1-\beta)^2\right]^{-1} \Big\{ \left(\mathbf{p}_2^2 - \beta^2 \mathbf{p}'^2\right)(1 + (1-\beta)^2) + m^2 \beta^4$$

$$- 2i(\mathbf{p}' \times \mathbf{p}_2)\ \beta^2 (\beta - 2) \Big\}, \qquad \text{(IV.24)}$$

where we have used the fact that $2k_+p_- = \mathbf{k} \cdot \mathbf{p} - i\mathbf{k} \times \mathbf{p}$.

If we substitute the expressions (IV.24) and (IV.19) for the numerator and energy denominators in (IV.9) and use (IV.17) to identify $F_1(q^2)$, we find

$$F_1(\mathbf{q}^2) = Z_2(1 + I(\mathbf{q}^2)), \qquad \text{(IV.25)}$$

where

$$I(\mathbf{q}^2) = \frac{2\alpha}{(2\pi)^2} \int_0^1 d\beta \int d\mathbf{p}_2\, \beta^{-1} \left[\left(\mathbf{p}_2^2 - \tfrac{1}{4}\beta^2 \mathbf{q}^2\right)(1 + (1-\beta)^2) + m^2 \beta^4\right]$$

$$\times \left[\left(\mathbf{p}_2^2 + \beta^2\left[m^2 + \tfrac{1}{4}\mathbf{q}^2\right]\right)^2 - \beta^2 (\mathbf{p}_2 \cdot \mathbf{q})^2\right]^{-1} \qquad \text{(IV.26)}$$

In (IV.26) we have used the fact that the term in the numerator proportional to $\mathbf{p}_2 \times \mathbf{q}$ will not contribute to the integral.

The integral defining $I(q^2)$ diverges as $\beta \to 0$ and as $\mathbf{p}_2^2 \to \infty$. However these divergences are cancelled by corresponding divergences in Z_2, just as in conventional treatments of the second order vertex. If we calculate Z_2 to order

α using

$$Z_2 \left\langle e^-\left(p', \tfrac{1}{2}\right) \middle| U(\infty, 0)\, U(0, -\infty) \middle| e^-\left(p, -\tfrac{1}{2}\right)\right\rangle = (2\pi)^3\, 2\eta\, \delta(\eta' - \eta)\, \delta^2(\underset{\sim}{p}' - \underset{\sim}{p}), \tag{IV.27}$$

we find easily that

$$Z_2 = (1 + I(0))^{-1} \; . \tag{IV.28}$$

Thus $F_1(\underset{\sim}{q}^2)$, calculated to order α, is

$$\begin{aligned} F_1(\underset{\sim}{q}^2) &= (1 + I(0))^{-1}\,(1 + I(\underset{\sim}{q}^2)) \\ &= 1 + (I(\underset{\sim}{q}^2) - I(0)) \end{aligned} \tag{IV.29}$$

The integral defining $I(\underset{\sim}{q}^2)_{\text{renormalized}} = I(\underset{\sim}{q}^2) - I(0)$ is now better defined: the β-integral converges for fixed $\underset{\sim}{p}_2$ and the $\underset{\sim}{p}_2$-integral converges for fixed β. However, the integral still has the familiar infrared divergence coming from the region near $\beta = 0$, $\underset{\sim}{p}_2 = 0$. In an explicit evaluation of $F_1(q^2)$, this infrared divergence could be eliminated by inserting a small photon mass in the energy denominators.

Before proceeding to the next example, we should point out that the use of the eikonal approximation in (IV.8) is self-consistent, even though Fig. 2b includes a loop. This is true because the loop integrals are well behaved in the region $\beta \approx 1$, where the electron in the intermediate state is no longer a "right mover." If the integrals had diverged at the endpoint $\beta = 1$, the claim that Eq. (IV.8) closely approximates the effect of external field on the physical particle would have been unjustified.

C. Bremsstrahlung

In this section we shall calculate the helicity amplitudes for the experimentally interesting process of bremsstrahlung off an external field. The matrix element of interest is then,

$$S_{fi} = \langle\, e(p', s')\gamma(k,\lambda) \mid U(\infty, 0)(\mathbb{F}-1)\, U(0, -\infty) \mid e(p, s) \,\rangle \;. \qquad \text{(IV.30)}$$

If we insert our expression for the physical states from Section IV.A accurate to terms of order e, we readily find

$$\begin{aligned} S_{fi} = {} & (2\pi)\,\delta(\eta - \eta' - \eta_k)\, 2\left[\eta\eta'\right]^{\frac{1}{2}} \left[F(\underset{\sim}{p}' + \underset{\sim}{k} - \underset{\sim}{p}) - (2\pi)^2 \delta^2(\underset{\sim}{p}' + \underset{\sim}{k} - \underset{\sim}{p})\right] \\ & \times e\left[\frac{w^{\dagger}(s')\underset{\sim}{j}(p', p' + k)\cdot \underset{\sim}{\epsilon}^{*}(\lambda) w(s)}{H(p') + \omega(k) - H(p' + k)} + \frac{w^{\dagger}(s')\underset{\sim}{j}(p - k, p)\cdot \underset{\sim}{\epsilon}^{*}(\lambda)\, w(s)}{H(p) - \omega(k) - H(p - k)}\right] . \end{aligned} \qquad \text{(IV.31)}$$

The terms in this expression can be visualized with the aid of Fig. 4a and b respectively.

In order to discuss bremsstrahlung conveniently we choose a coordinate system with its z-axis along the direction of the outgoing photon. The energy denominators in Eq. (III.32) become,

$$H(p') + \omega(k) - H(p' + k) = \frac{\eta_k}{2\eta\eta'}\,(\underset{\sim}{p}'^2 + m^2)$$

$$H(p) - \omega(k) - H(p - k) = -\frac{\eta_k}{2\eta\eta'}\,(\underset{\sim}{p}^2 + m^2) \;.$$

Finally, if we choose definite helicities for the incoming and outgoing particles we obtain, with the aid of Table 1, the infinite momentum helicity amplitudes for bremsstrahlung,

$$S_{fi} = (2\pi)\delta(\eta - \eta' - \eta_k)\, 2[\eta\eta']^{\frac{1}{2}} \left[F(\underset{\sim}{p}' - \underset{\sim}{p}) - (2\pi)^2\delta^2(\underset{\sim}{p}' - \underset{\sim}{p})\right] e\, M(s \to s', \lambda)$$

$$M(\tfrac{1}{2} \to \tfrac{1}{2}, 1) = \frac{2\eta}{\eta_k}\left\{-\frac{p'_-}{\underset{\sim}{p}'^2 + m^2} + \frac{p_-}{\underset{\sim}{p}^2 + m^2}\right\}$$

$$M(\tfrac{1}{2} \to \tfrac{1}{2}, -1) = \frac{2\eta'}{\eta_k}\left\{-\frac{p'_+}{\underset{\sim}{p}'^2 + m^2} + \frac{p_+}{\underset{\sim}{p}^2 + m^2}\right\} \qquad \text{(IV.32)}$$

$$M(\tfrac{1}{2} \to -\tfrac{1}{2}, 1) = \sqrt{2}\; i m \left\{-\frac{1}{\underset{\sim}{p}'^2 + m^2} + \frac{1}{\underset{\sim}{p}^2 + m^2}\right\}$$

$$M(\tfrac{1}{2} \to -\tfrac{1}{2}, -1) = 0 \; .$$

These results should prove useful in detailed calculations with specified external fields. For cases in which the external field can be treated perturbatively, one can easily show that Eqs. (4.32) lead to the high energy limit of the Bethe-Heitler formula.

D. Pair Production

We wish to calculate the scattering amplitude

$$S_{fi} = \langle\, e^-(p_1, s_1)\, e^+(p_2, s_2) \mid U(\infty, 0)\, [\mathbb{F} - \mathbb{1}]\;\; U(0, -\infty) \mid \gamma(k, \lambda)\,\rangle, \qquad \text{(IV.33)}$$

Proceeding along familiar lines, we insert perturbation expansions of the physical states accurate to first order in e and find

$$S_{fi} = (2\pi)\,\delta(\eta_k - \eta_1 - \eta_2)\; 2\left[\eta_1\eta_2\right]^{\frac{1}{2}}\; e\; \frac{d\underset{\sim}{p}}{(2\pi)^2}\; \frac{w^{\dagger}(s_1)\,\underset{\sim}{j}\,(p, p-k)\cdot \underset{\sim}{\epsilon}(\lambda)\, w(-s_2)}{\omega(k) - H(p) - H(k-p)}$$

$$\times \left[F(\underset{\sim}{p}_1 - \underset{\sim}{p})\, F_c(\underset{\sim}{p}_2 + \underset{\sim}{p} - \underset{\sim}{k}) - (2\pi)^4 \delta^2(\underset{\sim}{p}_1 - \underset{\sim}{p})\,\delta^2(\underset{\sim}{p}_2 + \underset{\sim}{p} - \underset{\sim}{k})\right], \tag{IV.34}$$

which can be visualized with the aid of Fig. 5.

If we now choose the z-axis along the direction of the photon and calculate helicity amplitudes, we find

$$S_{fi} = (2\pi)\delta(\eta_k - \eta_1 - \eta_2)\; 2\left[\eta_1\eta_2\right]^{\frac{1}{2}}\; e\,(2\pi)^{-2} \int d\underset{\sim}{p}\; M(\lambda \rightarrow s_1, s_2)$$

$$\times \left[F(\underset{\sim}{p}_1 - \underset{\sim}{p})\, F_c(\underset{\sim}{p}_2 + \underset{\sim}{p}) - (2\pi)^4 \delta^2(\underset{\sim}{p}_1 - \underset{\sim}{p})\,\delta^2(\underset{\sim}{p}_2 + \underset{\sim}{p})\right], \tag{IV.35}$$

where

$$M(1 \rightarrow \tfrac{1}{2}, -\tfrac{1}{2}) = \left(\frac{-2\eta_1}{\eta_k}\right) \frac{p_+}{\underset{\sim}{p}^2 + m^2}$$

$$M(1 \rightarrow -\tfrac{1}{2}, \tfrac{1}{2}) = \left(\frac{2\eta_2}{\eta_k}\right) \frac{p_+}{\underset{\sim}{p}^2 + m^2}$$

$$M(1 \rightarrow \tfrac{1}{2}, \tfrac{1}{2}) = \left(\sqrt{2}\, i m\right) \frac{1}{\underset{\sim}{p}^2 + m^2}$$

$$M(1 \rightarrow -\tfrac{1}{2}, -\tfrac{1}{2}) = 0 \quad .$$

It is interesting to convert the momentum integration in (IV.35) to an integration in coordinate space in order to appreciate the two-dimensional Galilean invariance group which manifests itself in the infinite momentum frame. To begin, we drop the special requirement that the transverse momentum $\underset{\sim}{k}$ of the photon be zero and return to the energy denominator in (IV.34):

$$\omega(k, \eta_k) - H(\underset{\sim}{p}, \eta_1) - H(\underset{\sim}{k} - \underset{\sim}{p}, \eta_2) = (2\eta_k)^{-1}\left[\underset{\sim}{p} + (\underset{\sim}{k} - \underset{\sim}{p})\right]^2 - (2\eta_1)^{-1}\left[\underset{\sim}{p}^2 + m^2\right] - (2\eta_2)^{-1}\left[(\underset{\sim}{k} - \underset{\sim}{p})^2 + m^2\right] .$$

This is a rather messy function of the momentum $\underset{\sim}{p}$ of the electron and the momentum $(\underset{\sim}{k} - \underset{\sim}{p})$ of the positron in the intermediate state. As is usual with two body problems in "non-relativistic" quantum mechanics, it pays to change variables to the total momentum, $\underset{\sim}{k}$, of the two particles and their relative momentum. Since η plays the role of particle mass in the nonrelativistic analogy, the relative momentum is

$$\underset{\sim}{q} = \bar{\eta}\left\{\frac{\underset{\sim}{p}}{\eta_1} - \frac{\underset{\sim}{k} - \underset{\sim}{p}}{\eta_2}\right\} , \tag{IV.36}$$

where

$$\bar{\eta} = \eta_1\eta_2 / (\eta_1 + \eta_2)$$

is the "reduced mass" of the pair. When written as a function of $\underset{\sim}{k}$ and $\underset{\sim}{q}$, the energy denominator is independent of $\underset{\sim}{k}$:

$$\omega(k, \eta_k) - H(\underset{\sim}{p}, \eta_1) - H(\underset{\sim}{k} - \underset{\sim}{p}, \eta_2) = -(2\bar{\eta})^{-1}\left[\underset{\sim}{q}^2 + m^2\right] \tag{IV.37}$$

(In non-relativistic terms, this is minus the "internal energy" of the pair.) Similarly, the vertex matrix element $w^\dagger \underset{\sim}{j}\,\epsilon \cdot w$ in (IV.34) is a function of the relative momentum q only. After a little algebra we obtain the explicit form,

$$\frac{e\, w^{\dagger}(s_1)\, \underset{\sim}{j}\,(\underset{\sim}{p}, \eta_1; p-k, -\eta_2)\cdot \underset{\sim}{\epsilon}(\lambda)\, w(-s_2)}{\omega(k, \eta_k) - H(p, \eta_1) - H(\underset{\sim}{k}-\underset{\sim}{p}, \eta_2)} \equiv w^{\dagger}(s_1)\, \underset{\sim}{\widetilde{G}}(\underset{\sim}{q}; \eta_1, \eta_2)\, w(-s_2)\cdot \underset{\sim}{\epsilon}(\lambda) \ ,$$

$$\underset{\sim}{\widetilde{G}}(\underset{\sim}{q}; \eta_1, \eta_2) = e\left\{\left(\frac{\eta_2 - \eta_1}{\eta_k}\right) q - i(\underset{\sim}{q} \times \hat{z})\, \sigma_z + im\underset{\sim}{\sigma}\right\} (\underset{\sim}{q}^2 + m^2)^{-1} \ , \tag{IV.38}$$

where $\underset{\sim}{q} \times \hat{z} = (q^2, -q^1)$.

Using these results, we can write (IV.34) as a coordinate-space integral. Let $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$ be the coordinates of the electron and positron respectively in the Fourier expansions (III.19) and (III.20) of the eikonal factors, and define

$$\begin{aligned} \underset{\sim}{R} &= \eta_k^{-1}(\eta_1 \underset{\sim}{x}_1 + \eta_2 \underset{\sim}{x}_2) = \text{coordinate of the center of "mass" of the pair} \\ \underset{\sim}{r} &= \underset{\sim}{x}_1 - \underset{\sim}{x}_2 = \text{relative coordinate.} \end{aligned} \tag{IV.39}$$

Then we find

$$\begin{aligned} S_{fi} &= (2\pi)\,\delta(\eta_k - \eta_1 - \eta_2)\, 2\left[\eta_1 \eta_2\right]^{\frac{1}{2}} \int d\underset{\sim}{x}_1 d\underset{\sim}{x}_2\, e^{-i\underset{\sim}{p}_1\cdot \underset{\sim}{x}_1}\, e^{-i\underset{\sim}{p}_2\cdot \underset{\sim}{x}_2} \left[e^{-i\chi(\underset{\sim}{x}_1)}\, e^{+i\chi(\underset{\sim}{x}_2)} - 1\right] \\ &\times w^{\dagger}(s_1)\, \underset{\sim}{G}(\underset{\sim}{r}; \eta_1, \eta_2)\, w(-s_2) \cdot \underset{\sim}{\epsilon}(\lambda)\, e^{i\underset{\sim}{k}\cdot \underset{\sim}{R}} \ , \end{aligned} \tag{IV.40}$$

where

$$\underset{\sim}{G}(\underset{\sim}{r}; \eta_1, \eta_2) = (2\pi)^{-2} \int d\underset{\sim}{q}\, e^{i\underset{\sim}{q}\cdot \underset{\sim}{r}}\, \underset{\sim}{\widetilde{G}}(\underset{\sim}{q}; \eta_1, \eta_2) \ .$$

It is interesting to interpret the various factors in (IV.40). First, $\underset{\sim}{\epsilon}(\lambda) \exp(i\underset{\sim}{k}\cdot\underset{\sim}{R})$ is the wave function of the initial bare photon. Multiplying this by $\underset{\sim}{G}(\underset{\sim}{r})$ tells us the composition of the physical photon in terms of its constituents, which, to first order, are an electron and a positron.[15] Hence we might refer to $\underset{\sim}{G}(\underset{\sim}{r}) \cdot \underset{\sim}{\epsilon}(\lambda) \exp(i\underset{\sim}{k}\cdot\underset{\sim}{R})$ as the first order approximation to the wave function of the physical photon. The "internal" wave function $\underset{\sim}{G}(\underset{\sim}{r})$ satisfies a two-dimensional Schroedinger equation with a point source,

$$\left(-\frac{1}{2\eta}\nabla^2 + \frac{m^2}{2\eta}\right)\underset{\sim}{G}(\underset{\sim}{r}) = \frac{e}{2\eta}\left\{-i\left(\frac{\eta_2-\eta_1}{\eta_k}\right)\underset{\sim}{\nabla} - (\underset{\sim}{\nabla}\times\hat{z})\,\sigma_z + i m \underset{\sim}{\sigma}\right\}\delta^2(\underset{\sim}{r}) .$$

The solution of this equation which vanishes as $|\,r\,| \to \infty$ is simply related to the modified Bessel function K_0 :

$$\underset{\sim}{G}(\underset{\sim}{r}) = \frac{e}{2\pi}\left\{-i\left(\frac{\eta_2-\eta_1}{\eta_k}\right)\underset{\sim}{\nabla} - (\underset{\sim}{\nabla}\times\hat{z})\,\sigma_z + i m\underset{\sim}{\sigma}\right\} K_0(m\,|\,\underset{\sim}{r}\,|) .$$

The next factor in Eq. (IV.40), the eikonal phase factor, tells us how the constituents of the physical photon interact with the external field. Finally, the factors $w^\dagger(s_1)\exp(-i\underset{\sim}{p}_1\cdot\underset{\sim}{x}_1)$ and $w(-s_2)\exp(-i\underset{\sim}{p}_2\cdot\underset{\sim}{x}_2)$ are the wave functions of the final electron and positron (calculated to zeroth order). Evaluation of the S-matrix is completed by integrating over the coordinates $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$ of the electron and positron and multiplying by (2π) times an η-conserving delta function and by a fermion normalization factor $(2\eta_1)^{\frac{1}{2}}\,(2\eta_2)^{\frac{1}{2}}$.

E. Delbruck Scattering

Let us turn our attention now to the problem of photon scattering off an external field. We shall see that our scattering theory gives a clear and concise derivation of the amplitude for this process.

The matrix element we wish to calculate is

$$S_{fi} - \delta_{fi} = \langle\,\gamma(p',\lambda')\,|\,U(\infty,0)\,\,[\mathbb{F}-1]\,U(0,-\infty)\,|\,\gamma(p,\lambda)\,\rangle . \qquad \text{(IV.41)}$$

If we insert the expansion of the physical photon state into (IV.41) and calculate to order e^2, we find

$$S_{fi} - \delta_{fi} = e^2 (2\pi)^{-4} \, \delta(\eta' - \eta) \int_0^{\eta} d\eta_1 \int d\underset{\sim}{p}_1 \, d\underset{\sim}{p}_1' \sum_{s_1, s_2}$$

$$\times \left[F(\underset{\sim}{p}_1' - \underset{\sim}{p}_1) \, F_c(\underset{\sim}{p}_2' - \underset{\sim}{p}_2) - (2\pi)^4 \delta^2(\underset{\sim}{p}_1 - \underset{\sim}{p}_1') \delta^2(\underset{\sim}{p}_2' - \underset{\sim}{p}_2) \right]$$

$$\times w^{\dagger}(s_1) \, \underset{\sim}{j} \, (p_1, -p_2) \cdot \underset{\sim}{\epsilon}(\lambda) \, w(-s_2) \, w^{\dagger}(-s_2) \, j\,(-p_2', p_1') \cdot \epsilon^{*}(\lambda') w(s_1) \tag{IV.42}$$

$$\times \left[\omega(p) - H(p_1) - H(p_2) \right]^{-1} \left[\omega(p') - H(p_1') - H(p_2') \right]^{-1}$$

where

$$p_1 = (\underset{\sim}{p}_1, \eta_1) \qquad p_1' = (\underset{\sim}{p}_1', \eta_1)$$

$$p_2 = (\underset{\sim}{p} - \underset{\sim}{p}_1, \eta - \eta_1) \qquad p_2' = (\underset{\sim}{p}' - \underset{\sim}{p}_1', \eta - \eta_1)$$

This formula is visualized, and its kinematics are defined, in the τ-ordered diagram Fig. 6.

We are now faced with two related problems. First, the integrand in (IV.42) is a very messy function of the independent momenta $\underset{\sim}{p}_1$ and $\underset{\sim}{p}_1'$. Second, the momentum integration is divergent: if the integrals are cut off in an arbitrary non-covariant fashion, the result will depend on the cutoff parameter. The remedy is simple. Since S_{fi} is invariant under the Galilean symmetry group discussed in I and in Section IV.B, it will be to our advantage to use integration variables which are invariant under this group.

We choose to make use of four Galilean-invariant momenta $\underset{\sim}{r}$, $\underset{\sim}{q}$, $\underset{\sim}{\ell}$ and $\underset{\sim}{Q}$. The momenta $\underset{\sim}{r}$ and $\underset{\sim}{q}$ are defined so that the momentum transfer from the external potential to the electron in the intermediate state is $\underset{\sim}{r} + \underset{\sim}{q}$ and the momentum transfer to the positron is $\underset{\sim}{r} - \underset{\sim}{q}$:

$$\begin{aligned} \underset{\sim}{p}_1' - \underset{\sim}{p}_1 &= \underset{\sim}{r} + \underset{\sim}{q} \\ \underset{\sim}{p}_2' - \underset{\sim}{p}_2 &= \underset{\sim}{r} - \underset{\sim}{q} \ . \end{aligned} \tag{IV.43}$$

The momenta $\underset{\sim}{\ell}$ and $\underset{\sim}{Q}$ are defined so that the "relative momentum" of the electron-positron pair is $\underset{\sim}{\ell} - \underset{\sim}{Q}$ before the interaction with the external field, and $\underset{\sim}{\ell} + \underset{\sim}{Q}$ after the interaction:

$$\begin{aligned} \underset{\sim}{\ell} - \underset{\sim}{Q} &= \bar{\eta} \left[\frac{\underset{\sim}{p}_1}{\eta_1} - \frac{\underset{\sim}{p}_2}{\eta_2} \right] \\ \underset{\sim}{\ell} + \underset{\sim}{Q} &= \bar{\eta} \left[\frac{\underset{\sim}{p}_1'}{\eta_1} - \frac{\underset{\sim}{p}_2'}{\eta_2} \right] , \end{aligned} \tag{IV.44}$$

where $\bar{\eta} = \eta_1 \eta_2 / \eta$ is the "reduced mass" of the pair. We will use $\underset{\sim}{q}$ and $\underset{\sim}{\ell}$ as integration variables instead of $\underset{\sim}{p}_1$ and $\underset{\sim}{p}_1'$. The momentum $\underset{\sim}{r}$ is, of course, fixed by the external momenta: $2\underset{\sim}{r} = \underset{\sim}{p}' - \underset{\sim}{p}$. We find with a little algebra that $\underset{\sim}{Q}$ is given in terms of $\underset{\sim}{r}$ and $\underset{\sim}{q}$ by

$$\underset{\sim}{Q} = \tfrac{1}{2} (\underset{\sim}{r} + \underset{\sim}{q}) - \alpha \underset{\sim}{r} , \tag{IV.45}$$

where we have defined

$$\alpha = \eta_1 / \eta \ . \tag{IV.46}$$

When this change of variables has been made, the scattering matrix takes the form

$$S_{fi} - \delta_{fi} = e^2 (2\pi)^{-4} \eta\, \delta(\eta' - \eta) \int d\underset{\sim}{q} \left[F(\underset{\sim}{r} + \underset{\sim}{q})\, F_c(\underset{\sim}{r} - \underset{\sim}{q}) - (2\pi)^4 \delta(\underset{\sim}{r} + \underset{\sim}{q}) \delta(\underset{\sim}{r} - \underset{\sim}{q}) \right] M_\Lambda(\underset{\sim}{q}, \underset{\sim}{r}; \lambda, \lambda') \tag{IV.47}$$

where

$$M_\Lambda(\underset{\sim}{q}, \underset{\sim}{r}; \lambda, \lambda') = \int_0^1 d\alpha \int^\Lambda d\underset{\sim}{\ell} \sum_{s_1 s_2}$$

$$w^\dagger(s_1)\, \underset{\sim}{j}\,(p_1, -p_2) \cdot \underset{\sim}{\epsilon}(\lambda)\, w(-s_2)\; w^\dagger(-s_2)\, \underset{\sim}{j}\,(-p_2', p_1') \cdot \underset{\sim}{\epsilon}^*(\lambda') w(s_1) \tag{IV.48}$$

$$\left[\omega(p) - H(p_1) - H(p_2)\right]^{-1} \left[\omega(p') - H(p_1') - H(p_2')\right] \; .$$

Eq. (IV.47) has the attractive property that the integrand of the $\underset{\sim}{q}$-integration decomposes into two factors: one describing the interaction with the external field, and a second, called the photon impact factor by Cheng and Wu[16], describing the composition of the physical photon as a bare pair.

A technical complication arises because the impact factor M depends on a cutoff Λ in the $\underset{\sim}{\ell}$- integration. However, we will see that the cutoff does not affect the scattering amplitude, and therefore has no physical significance.

It is quite easy to write down the explicit form of M_Λ using the variables $\underset{\sim}{\ell}$ and $\underset{\sim}{Q} = \frac{1}{2}(\underset{\sim}{r} + \underset{\sim}{q}) - \alpha \underset{\sim}{r}$. The energy denominators are

$$\omega(p) - H(p_1) - H(p_2) = -(2\bar{\eta})^{-1} \left[(\underset{\sim}{\ell} - \underset{\sim}{Q})^2 + m^2\right] = -\left[2\eta\, \alpha(1-\alpha)\right]^{-1} \left[(\underset{\sim}{\ell} - \underset{\sim}{Q})^2 + m^2\right]$$

$$\omega(p') - H(p_1') - H(p_2') = -(2\bar{\eta})^{-1} \left[(\underset{\sim}{\ell} + \underset{\sim}{Q})^2 + m^2\right] = -\left[2\eta\, \alpha(1-\alpha)\right]^{-1} \left[(\underset{\sim}{\ell} + \underset{\sim}{Q})^2 + m^2\right] .$$

By making use of the Galilean invariance of the numerator factors $w^{\dagger}\underset{\sim}{j}\,w$, we can write them in terms of $\underset{\sim}{\ell}$ and $\underset{\sim}{Q}$ immediately:

$$w^{\dagger}(s_1)\underset{\sim}{j}(p_1,-p_2)\,w(-s_2) = w^{\dagger}(s_1)\underset{\sim}{j}(\underset{\sim}{\ell}-\underset{\sim}{Q},\eta_1;\underset{\sim}{\ell}-\underset{\sim}{Q},-\eta_2)\,w(-s_2)$$

$$w^{\dagger}(-s_2)\underset{\sim}{j}(-p_2',p_1')\,w(s_1) = w^{\dagger}(-s_2)\underset{\sim}{j}(\underset{\sim}{\ell}+\underset{\sim}{Q},-\eta_2;\underset{\sim}{\ell}+\underset{\sim}{Q},\eta_1)\,w(s_1)\,.$$

Thus M_Λ takes the form

$$M_\Lambda(\underset{\sim}{q},\underset{\sim}{r};\lambda,\lambda') = \int_0^1 d\alpha \int^{\Lambda} d\underset{\sim}{\ell}\left[2\eta\,\alpha(1-\alpha)\right]^2 n(\underset{\sim}{\ell},\underset{\sim}{Q};\lambda,\lambda')\left[(\underset{\sim}{\ell}-\underset{\sim}{Q})^2+m^2\right]^{-1}\left[(\underset{\sim}{\ell}+\underset{\sim}{Q})^2+m^2\right]^{-1}, \tag{IV.49}$$

where

$$\begin{aligned} n(\underset{\sim}{\ell},\underset{\sim}{Q};\lambda,\lambda') = \sum_{s_1 s_2} & w^{\dagger}(s_1)\underset{\sim}{j}(\underset{\sim}{\ell}-\underset{\sim}{Q},\eta_1;\underset{\sim}{\ell}-\underset{\sim}{Q},-\eta_2)\cdot\underset{\sim}{\epsilon}(\lambda)w(-s_2) \\ & \times w^{\dagger}(-s_2)\underset{\sim}{j}(\underset{\sim}{\ell}+\underset{\sim}{Q},-\eta_2;\underset{\sim}{\ell}+\underset{\sim}{Q},\eta_1)\cdot\underset{\sim}{\epsilon}^*(\lambda')w(s_1)\quad . \end{aligned} \tag{IV.50}$$

Let us consider the helicity flip case first. Reading from Table I, we find

$$n(\underset{\sim}{\ell},\underset{\sim}{Q};1,-1) = -2(\eta_1\eta_2)^{-1}(\ell_+-Q_+)(\ell_++Q_+) = -2\eta^{-2}\left[\alpha(1-\alpha)\right]^{-1}\left[\ell_+\ell_+-Q_+Q_+\right]\,. \tag{IV.51}$$

Thus

$$M_\Lambda(\underset{\sim}{q},\underset{\sim}{r},+1,-1) = -8\int_0^1 d\alpha\,\alpha(1-\alpha)\int^{\Lambda} d\underset{\sim}{\ell}\left[\ell_+\ell_+-Q_+Q_+\right]\left[(\underset{\sim}{\ell}-\underset{\sim}{Q})^2+m^2\right]^{-1}\left[(\underset{\sim}{\ell}+\underset{\sim}{Q})^2+m^2\right]^{-1}. \tag{IV.52}$$

The helicity non-flip amplitude is also quite simple. Reading from Table I, we find

$$n(\underset{\sim}{\ell}, \underset{\sim}{Q}; +1, +1) = \left[\eta_1^{-2} + \eta_2^{-2}\right](\ell_+ - Q_+)(\ell_- + Q_-) + \tfrac{1}{2} m^2 \bar{\eta}^{-2}$$

$$= \tfrac{1}{2}\left[\eta\,\alpha(1-\alpha)\right]^{-2} \left\{ \left[\alpha^2 + (1-\alpha)^2\right] (\underset{\sim}{\ell}^2 - \underset{\sim}{Q}^2 - 2i\underset{\sim}{\ell} \times \underset{\sim}{Q}) + m^2 \right\} . \qquad \text{(IV.53)}$$

The term proportional to $\underset{\sim}{\ell} \times \underset{\sim}{Q}$ can be dropped since it will not contribute to M_Λ. Thus we obtain

$$M\,(\underset{\sim}{q}, \underset{\sim}{r}; +1, +1) = 2\int_0^1 d\alpha \int^\Lambda d\underset{\sim}{\ell} \left[(\alpha^2 + [1-\alpha]^2)(\underset{\sim}{\ell}^2 - \underset{\sim}{Q}^2) + m^2\right]\left[(\underset{\sim}{\ell} - \underset{\sim}{Q})^2 + m^2\right]^{-1}\left[(\underset{\sim}{\ell} + \underset{\sim}{Q})^2 + m^2\right]^{-1} . \qquad \text{(IV.54)}$$

As mentioned earlier, the impact factors M_Λ given in (IV.52) and (IV.54) depend on the cutoff parameter Λ used to avoid the logarithmic divergence in the $\underset{\sim}{\ell}$- integration. However, we can verify that the cutoff does not affect the scattering amplitude in the limit $\Lambda \rightarrow \infty$ by writing M_Λ in the form

$$M_\Lambda(\underset{\sim}{q}, \underset{\sim}{r}; \lambda, \lambda') = \widetilde{M}_\Lambda(\underset{\sim}{q}, \underset{\sim}{r}; \lambda, \lambda') + M_\Lambda(\underset{\sim}{r}, \underset{\sim}{r}; \lambda, \lambda') . \qquad \text{(IV.55)}$$

The term $\widetilde{M}_\Lambda$ defined by (IV.55) is evidently finite in the limit $\Lambda \rightarrow \infty$. If we use the simple observation that

$$\int d\underset{\sim}{q} \left[F(\underset{\sim}{r} + \underset{\sim}{q})\, F_c(\underset{\sim}{r} - \underset{\sim}{q}) - (2\pi)^4 \delta^2(\underset{\sim}{r} + \underset{\sim}{q})\, \delta^2(\underset{\sim}{r} - \underset{\sim}{q})\right] = 0 ,$$

we see that the cutoff dependent part of $M_\Lambda(\underset{\sim}{q}, r; \lambda, \lambda')$, namely $M_\Lambda(\underset{\sim}{r}, \underset{\sim}{r}; \lambda, \lambda')$, does not contribute to the scattering amplitude (IV.47) and therefore has no special significance.

In addition, we may note that because of its definition $\widetilde{M}_\Lambda(\underset{\sim}{q},\underset{\sim}{r};\lambda,\lambda')$ is zero at $\underset{\sim}{q}=\underset{\sim}{r}$. It is also zero at $\underset{\sim}{q}=-\underset{\sim}{r}$. (Indeed, it is an even function of $\underset{\sim}{q}$, as can be verified by making the change of variables $\alpha \longrightarrow (1-\alpha)$ in (IV.52) and (IV.54).) Thus the scattering amplitude (IV.47) remains finite even if the eikonal factors are singular at $\underset{\sim}{q} = \pm r$, as they are in the case that $A_\mu(x)$ is a static Coulomb potential. The renormalized impact factors $\widetilde{M}_\infty(\underset{\sim}{q},\underset{\sim}{r};\lambda,\lambda')$ are identical (aside from a factor $-e^4(2\pi)^{-3}$) to the impact factors for the photon found by other techniques by Cheng and Wu.[16]

G. Electroproduction of μ Pairs; Scaling

We wish to discuss here a "model" calculation which, hopefully, has important features in common with electron-nucleon inelastic scattering. We imagine the process pictured in Figs. 7a and 7b: a virtual photon, produced by the scattered electron, creates a pair of muons which diffract through an external field (e.g. a nucleus). In the spirit of inelastic electron-nucleon scattering we put eikonal phases only on the members of the pair and treat all particles as distinguishable.

One purpose of the model is to investigate the scaling property recently discovered in electron-nucleon scattering.[17] To do this, we assume that only the final electron is observed and construct the cross section $d\sigma/dQ^2 d\nu$, where Q^2 is the four-momentum transfer from the electron line and ν is the energy transfer. We then ask whether the diffractive mechanism envisioned here leads to scale invariant expressions for the form factors σ_T and σ_S in the limit $Q^2 \longrightarrow \infty$.[18]

To begin, we construct the scattering amplitude corresponding to Figs. 7a and 7b:

$$S_{fi} = e^2 (2\pi)\, \delta(\eta - \eta' - \eta_1 - \eta_2) \left[2\eta 2\eta' 2\eta_1 2\eta_2\right]^{\frac{1}{2}} \int \frac{d\underset{\sim}{p}_1'}{(2\pi)^2}$$

$$\times \left\{ \frac{\Sigma_\lambda w^\dagger(s')\, \underset{\sim}{j}\,(p', p) \cdot \epsilon^*(\lambda)\, w(s) \quad w^\dagger(s_1)\, \underset{\sim}{j}\,(p_1', -p_2') \cdot \underset{\sim}{\epsilon}\,(\lambda)\, w(-s_2)}{(2\eta_q)\,(H(p) - H(p') - \omega(q))} + (\eta_q)^{-2} \delta_{s', s}\, \delta_{s_1, -s_2} \right\}$$

$$\times \left[H(p) - H(p') - H_\mu(p_1') - H_\mu(p_2')\right]^{-1} \left[F(\underset{\sim}{p}_1 - \underset{\sim}{p}_1')\, F_c(\underset{\sim}{p}_2 - \underset{\sim}{p}_2') - (2\pi)^4 \delta^2(\underset{\sim}{p}_1 - \underset{\sim}{p}_1')\, \delta^2(\underset{\sim}{p}_2 - \underset{\sim}{p}_2')\right], \tag{IV.56}$$

where

$$\underset{\sim}{q} = \underset{\sim}{p} - \underset{\sim}{p}', \qquad \eta_q = \eta - \eta'$$

$$\underset{\sim}{p}_2' = -\underset{\sim}{p}_1' + \underset{\sim}{q}, \qquad \eta_1' = \eta_1,\ \eta_2' = \eta_2 .$$

The first term in braces in (IV.56) corresponds to exchange of transverse photons (Fig. 7a); the second term corresponds to the exchange of a "scalar photon" (Fig. 7b). The function $H_\mu(p)$ refers to the free muon Hamiltonian $(\underset{\sim}{p}^2 + \mu^2)/2\eta$, where μ is the muon mass.

Before proceeding further, it is convenient (as usual) to change variables in the momentum integration from $\underset{\sim}{p}_1'$ to $\underset{\sim}{k}$, where $\underset{\sim}{k}$ is the "relative momentum" of the virtual μ-pair:

$$\underset{\sim}{k} = \frac{\eta_1 \eta_2}{\eta_q} \left(\frac{\underset{\sim}{p}_1'}{\eta_1} - \frac{\underset{\sim}{p}_2'}{\eta_2}\right) = \underset{\sim}{p}_1' - \alpha \underset{\sim}{q}, \tag{IV.57}$$

where

$$\alpha = \eta_1 / \eta_q . \tag{IV.58}$$

It is also convenient to let $-Q^2$ stand for the square of the 4-momentum transferred from the electron line:

$$-Q^2 = (p - p')^{\mu} (p - p')_{\mu} \quad . \tag{IV.59}$$

In terms of these variables, the energy denominators in (IV.56) have the simple forms

$$\begin{aligned} H(p) - H(p') - \omega(q) &= -Q^2/(2\eta_q) \quad , \\ H(p) - H(p') - H_{\mu}(p_1') - H_{\mu}(p_2') &= -\frac{Q^2}{2\eta_q} - \frac{\eta_q}{2\eta_1\eta_2}(\underset{\sim}{k}^2 + \mu^2) \quad . \end{aligned} \tag{IV.60}$$

The numerator functions $w^{\dagger}\underset{\sim}{j}\cdot\underset{\sim}{\epsilon}^* w \;\; w^{\dagger}\underset{\sim}{j}\cdot\underset{\sim}{\epsilon} w$ can be read from Table I, and are also simple functions of $\underset{\sim}{k}$.

We are now prepared to write out S_{fi} in a form suitable for calculating the cross section. Let us choose the z-axis in the direction of the beam, so $\underset{\sim}{p} = 0$, and consider S_{fi} for the choice of spins $s = s' = s_1 = \frac{1}{2}$, $s_2 = -\frac{1}{2}$. Then when we substitute the expressions from Table I and Eq. (IV.60) into (IV.56) we obtain

$$S_{fi} = (2\pi)\,\delta(\eta - \eta' - \eta_1 - \eta_2)\left[2\eta\, 2\eta' 2\eta_1 2\eta_2\right]^{\frac{1}{2}} \left(\frac{-2e^2}{Q^2\eta_q}\right) \widetilde{M}(\underset{\sim}{p}_1, \underset{\sim}{p}_2) \quad , \tag{IV.61}$$

where

$$\widetilde{M}(\underset{\sim}{p}_1, \underset{\sim}{p}_2) = (2\pi)^{-2}\int d\underset{\sim}{k}\, \widetilde{f}(\underset{\sim}{k})$$

$$\left[F(\underset{\sim}{p}_1 - \alpha\underset{\sim}{q} - \underset{\sim}{k})\, F_c(\underset{\sim}{p}_2 - (1-\alpha)\underset{\sim}{q} + \underset{\sim}{k}) - (2\pi)^4 \delta^2(\underset{\sim}{p}_1 - \alpha\underset{\sim}{q} - \underset{\sim}{k})\,\delta^2(\underset{\sim}{p}_2 - (1-\alpha)\underset{\sim}{q} + \underset{\sim}{k})\right] \quad , \tag{IV.62}$$

and

$$\begin{aligned}\widetilde{f}(\underset{\sim}{k}) &= \widetilde{f}_R(\underset{\sim}{k}) + \widetilde{f}_L(\underset{\sim}{k}) + \widetilde{f}_S(\underset{\sim}{k}) \\ &= \left\{\left(\frac{\eta}{\eta'} p'_-\right) \alpha k_+ - p'_+ \left[1-\alpha\right]k_- + \alpha(1-\alpha)\, Q^2\right\}\left[\underset{\sim}{k}^2 + \alpha(1-\alpha)Q^2 + \mu^2\right]^{-1} .\end{aligned} \qquad \text{(IV.63)}$$

The three terms in $f(\underset{\sim}{k})$ arise from exchange of a right handed photon, a left handed photon, and a "scalar photon" respectively.

The physics of the amplitude $\widetilde{M}(p_1, p_2)$ is more apparent if we write it as a Fourier transform by inserting the expansions of the eikonal factors into (IV.62). The resulting structure of $\widetilde{M}(\underset{\sim}{p}_1, \underset{\sim}{p}_2)$, and its physical interpretation will be familiar from the discussion of pair production by real photons in Section IV.D. We find

$$\begin{aligned}\widetilde{M}(\underset{\sim}{p}_1, \underset{\sim}{p}_2) &= \int d\underset{\sim}{x}_1 d\underset{\sim}{x}_2\, e^{-i\underset{\sim}{p}_1\cdot\underset{\sim}{x}_1}\, e^{-i\underset{\sim}{p}_2\cdot\underset{\sim}{x}_2}\, M(\underset{\sim}{x}_1, \underset{\sim}{x}_2) \\ &= \int d\underset{\sim}{x}_1 d\underset{\sim}{x}_2\, e^{-i\underset{\sim}{p}_1\cdot\underset{\sim}{x}_1}\, e^{-i\underset{\sim}{p}_2\cdot\underset{\sim}{x}_2}\left[\exp\left(-i\chi(\underset{\sim}{x}_1)\right)\exp\left(+i\lambda(\underset{\sim}{x}_2)\right)-1\right] f(\underset{\sim}{x}_1-\underset{\sim}{x}_2)\, e^{i\underset{\sim}{q}\cdot\underset{\sim}{R}}\end{aligned} \qquad \text{(IV.64)}$$

where $\underset{\sim}{R} = \eta_q^{-1}(\eta_1\underset{\sim}{x}_1 + \eta_2\underset{\sim}{x}_2)$ and $f(\underset{\sim}{r})$ is the Fourier transform of $\widetilde{f}(\underset{\sim}{k})$. Explicit evaluation gives the wave function of the virtual muon pair, $f(\underset{\sim}{r})$, in terms of modified Bessel functions K_0 and K_1:

$$f(\underset{\sim}{r}) = (2\pi)^{-2}\int d\underset{\sim}{k}\, e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\, \widetilde{f}_R(\underset{\sim}{k}) + \widetilde{f}_L(\underset{\sim}{k}) + \widetilde{f}_S(\underset{\sim}{k}) = f_R(\underset{\sim}{r}) + f_L(\underset{\sim}{r}) + f_S(\underset{\sim}{r}) \; , \qquad \text{(IV.65)}$$

$$f_R(\underset{\sim}{r}) = \frac{i}{2\pi}\left(\frac{\eta}{\eta'}p'_-\right)\alpha\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}}\frac{r_+}{r}K_1\left(\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}}r\right)$$

$$f_L(\underset{\sim}{r}) = -\frac{i}{2\pi}p'_+(1-\alpha)\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}}\frac{r_-}{r}K_1\left(\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}}r\right) \qquad \text{(IV.66)}$$

$$f_S(\underset{\sim}{r}) = \frac{1}{2\pi}\alpha(1-\alpha)Q^2 K_0\left(\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}}r\right).$$

We will see in the sequel that, for our purposes, this expression for $f(\underset{\sim}{r})$ is not as formidable as it seems.

With a useable expression for S_{fi} now at hand, we are ready to construct the cross section $d\sigma$ integrated over the unobserved momenta of the muon pair. Using (IV.61) in Eq. (IV.6) we obtain

$$d\sigma = d\underset{\sim}{p}'\, d\eta'\left(\frac{4e^4}{(2\pi)^4Q^4\eta_q}\right)\int_0^1 d\alpha\,(2\pi)^{-4}\int d\underset{\sim}{p}_1\, d\underset{\sim}{p}_2\,|\,\widetilde{M}(p_1,p_2)\,|^2 . \qquad \text{(IV.67)}$$

Since $M(\underset{\sim}{x}_1,\underset{\sim}{x}_2)$ is simpler than $\widetilde{M}(\underset{\sim}{p}_1,\underset{\sim}{p}_2)$, we write the $\underset{\sim}{p}_1$, $\underset{\sim}{p}_2$-integral as

$$\begin{aligned}(2\pi)^{-4}\int d\underset{\sim}{p}_1\, d\underset{\sim}{p}_2\,|\,\widetilde{M}(\underset{\sim}{p}_1,\underset{\sim}{p}_2)\,|^2 &= \int d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2\,|\,M(\underset{\sim}{x}_1,\underset{\sim}{x}_2)\,|^2\\ &= \int d\underset{\sim}{x}_1\, d\underset{\sim}{x}_2\,|\,f(\underset{\sim}{x}_1-\underset{\sim}{x}_2)\,|^2\left[2-2\cos\left(\chi(\underset{\sim}{x}_1)-\chi(\underset{\sim}{x}_2)\right)\right] \qquad \text{(IV.68)}\\ &= \int d\underset{\sim}{r}\,|\,f(\underset{\sim}{r})\,|^2\int d\underset{\sim}{b}\left[2-2\cos\left(\chi(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{r})-\chi(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{r})\right)\right]\end{aligned}$$

Assuming that the potential has cylindrical symmetry about the z-axis, we can replace $|\,f(\underset{\sim}{r})\,|^2$ by $|\,f_R(\underset{\sim}{r})\,|^2+|\,f_L(\underset{\sim}{r})\,|^2+|\,f_S(\underset{\sim}{r})\,|^2$ in (IV.68), since the various cross terms will vanish when the integration over the angle of $\underset{\sim}{r}$ is performed. Thus the cross section separates into a part due to the exchange of a "transverse photon", $d\sigma_T = d\sigma_R + d\sigma_L$, and a part due to the exchange of a "scalar photon", $d\sigma_S$. If we substitute the expressions for $|f_R|^2$, $|f_L|^2$, and $|f_S|^2$ obtained from (IV.66) into (IV.68) and (IV.67) and interchange the roles of α and $(1-\alpha)$ in $d\sigma_L$, we obtain

$$
\begin{aligned}
d\sigma &= d\sigma_T + d\sigma_S \\
&= d\underset{\sim}{p}'\, d\eta' \left(4e^4(2\pi)^{-6} Q^{-4}\eta_q^{-1}\right) \int_0^1 d\alpha \int d\underset{\sim}{r} \\
&\quad \left\{\tfrac{1}{4}\left[\left(\frac{\eta}{\eta'}\right)^2 + 1\right] \underset{\sim}{p}'^2\, \alpha^2 \left[\alpha(1-\alpha)Q^2 + \mu^2\right]\left[K_1\left(\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}} r\right)\right]^2 \right. \\
&\quad \left. + \alpha^2(1-\alpha)^2 Q^4 \left[K_0\left(\left[\alpha(1-\alpha)Q^2+\mu^2\right]^{\frac{1}{2}} r\right)\right]^2 \right\} \\
&\quad \int d\underset{\sim}{b}\left[2 - 2\cos\left(\chi(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{r}) - \chi(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{r})\right)\right] .
\end{aligned}
\tag{IV.69}
$$

This expression gives the cross section in the high energy limit discussed in Section III, i.e. in the limit $\eta, \eta' \longrightarrow \infty$ with η/η' and Q^2 fixed. It remains now to evaluate $d\sigma$ in the limit $Q^2 \longrightarrow \infty$. To take this limit we have only to note that the modified Bessel functions appearing in (IV.69) are large only for small values of their arguments, so that the main contribution to the $\underset{\sim}{r}$-integral comes from the region $\underset{\sim}{r}^2 < \left[\alpha(1-\alpha)Q^2 + \mu^2\right]^{-1}$.

Physically, this means that for large Q^2 the transverse separation r between the muons as they pass through the external potential is small. If the separation were zero the two muons would receive exactly opposite eikonal phases; thus for small r the net phase received by the muon pair is proportional not to χ but to $\nabla\chi$.

Mathematically, this means that the $Q^2 \longrightarrow \infty$ limit of $d\sigma$ can be obtained by substituting for the $\underset{\sim}{b}$-integral in (IV.69) its limiting form as $r \longrightarrow 0$.[19] This limiting form is easily evaluated:

$$\int d\underset{\sim}{b}\left[2-2\cos\left(\chi(\underset{\sim}{b}+\tfrac{1}{2}\underset{\sim}{r})-\chi(\underset{\sim}{b}-\tfrac{1}{2}\underset{\sim}{r})\right)\right]\sim\int d\underset{\sim}{b}\left[2-2\cos\left(\underset{\sim}{r}\cdot\underset{\sim}{\nabla}\chi(\underset{\sim}{b})\right)\right]$$

$$\sim\int d\underset{\sim}{b}\left[\underset{\sim}{r}\cdot\underset{\sim}{\nabla}\chi(\underset{\sim}{b})\right]^2 \qquad (IV.70)$$

$$=\tfrac{1}{2}r^2\int d\underset{\sim}{b}\left[\underset{\sim}{\nabla}\chi(\underset{\sim}{b})\right]^2 .$$

(In the last step we have used the assumed cylindrical symmetry of $\chi(\underset{\sim}{b})$.)

Once the limiting form (IV.70) of the $\underset{\sim}{b}$-integral has been substituted into (IV.69), the $\underset{\sim}{r}$-integral can be evaluated using the formula[20]

$$\int_0^\infty dx\left[K_J(x)\right]^2 x^{s-1} = 2^{s-3}\left[\Gamma(\tfrac{1}{2}s)\right]^2\frac{(\tfrac{1}{2}s+J)\Gamma(\tfrac{1}{2}s-J)}{\Gamma(s)} .$$

This leads to

$$d\sigma = d\underset{\sim}{p}'d\eta'\left(\tfrac{2}{3}e^4(2\pi)^{-5}Q^{-4}\eta_q^{-1}\right)\int d\underset{\sim}{b}\left[\underset{\sim}{\nabla}\chi(\underset{\sim}{b})\right]^2$$

$$\times\left\{\tfrac{1}{2}\left[\left(\frac{\eta}{\eta'}\right)^2+1\right]\underset{\sim}{p}'^2\int_0^1 d\alpha\,\frac{\alpha^2}{\alpha(1-\alpha)Q^2+\mu^2}+\int_0^1 d\alpha\left[\frac{\alpha(1-\alpha)Q^2}{\alpha(1-\alpha)Q^2+\mu^2}\right]^2\right\} . \qquad (IV.71)$$

Evaluating the α-integrals in the limit $Q^2\to\infty$, we find

$$d\sigma = d\sigma_T + d\sigma_S$$

$$\sim d\underset{\sim}{p}'\,d\eta'\,\frac{2e^4}{3(2\pi)^5Q^4\eta_q}\int d\underset{\sim}{b}\left[\underset{\sim}{\nabla}\chi(\underset{\sim}{b})\right]^2 \qquad (IV.72)$$

$$\times\left\{\tfrac{1}{2}\left[\left(\frac{\eta}{\eta'}\right)^2+1\right]\left(\frac{\underset{\sim}{p}'^2}{Q^2}\right)\left[\log\frac{Q^2}{\mu^2}+0(1)\right]+1\right\} .$$

We recall that this is the cross section for the choice of spins $s = s' = \frac{1}{2}$, $s_1 = \frac{1}{2}$, $s_2 = -\frac{1}{2}$. It is not difficult to see that the choice $s = s' = \frac{1}{2}$, $s_1 = -\frac{1}{2}$, $s_2 = +\frac{1}{2}$ leads to the same cross section. Each of the other six possible choices for the spins of the final particles gives a cross section $d\sigma_S = 0$ and a cross section $d\sigma_T$ which is small compared to the cross section in (IV.72) as $Q^2 \to \infty$.[21] Thus the limiting cross section for $s = \frac{1}{2}$ (or $s = -\frac{1}{2}$), summed over final spins, is two times the cross section in (IV.72).

In order to make contact with standard notation and identify the form factors $\sigma_T(Q^2, \nu)$, $\sigma_S(Q^2, \nu)$, let us define

$$\begin{aligned} E &= \text{lab energy of the incident electron} = 2^{-\frac{1}{2}}\left[\eta + H(\underset{\sim}{p})\right] \\ E' &= \text{lab energy of the scattered electron} = 2^{-\frac{1}{2}}\left[\eta' + H(\underset{\sim}{p}')\right] \\ \nu &= E - E' \quad . \end{aligned} \tag{IV.73}$$

Apparently in the high energy limit

$$\eta = 2^{\frac{1}{2}} E, \quad \eta' = 2^{\frac{1}{2}} E', \quad \eta_q = 2^{\frac{1}{2}} \nu \quad . \tag{IV.74}$$

We recall also the definition of Q^2:

$$Q^2 = -(p-p')^\mu (p-p')_\mu = -2\eta_q\Big(H(p) - H(p')\Big) + \underset{\sim}{p}'^2 = \frac{\eta}{\eta'}\, \underset{\sim}{p}'^2 + \frac{\eta_q^2}{\eta\eta'}\, m^2 \, . \tag{IV.75}$$

Thus in the high energy limit, and neglecting m^2 compared to Q^2, we can replace $\underset{\sim}{p}'^2$ by

$$\underset{\sim}{p}'^2 = \frac{E'}{E}\, Q^2 \, . \tag{IV.76}$$

When we make these replacements we find for the cross section summed over final spins,

$$\frac{d\sigma}{dQ^2 d\nu} = \frac{d\sigma_T}{dQ^2 d\nu} + \frac{d\sigma_S}{dQ^2 d\nu}$$
$$\sim \frac{2\alpha^2}{3\pi^2} \frac{1}{\nu Q^4} \frac{E'}{E} \left\{ \frac{E^2 + E'^2}{2EE'} \log\left(\frac{Q^2}{\mu^2}\right) + 1 \right\} \int d\underset{\sim}{b} \left[\underset{\sim}{\nabla} \chi(\underset{\sim}{b})\right]^2 \quad . \tag{IV.77}$$

Using (IV.77) we can extract the form factors σ_S and σ_T[22]:

$$\sigma_S(Q^2, \nu/Q^2) \sim \frac{2\alpha}{3\pi} \frac{1}{Q^2} \int d\underset{\sim}{b} \left[\underset{\sim}{\nabla} \chi(\underset{\sim}{b})\right]^2$$
$$\sigma_T(Q^2, \nu/Q^2) \sim \frac{2\alpha}{3\pi} \frac{1}{Q^2} \log\left(\frac{Q^2}{\mu^2}\right) \int d\underset{\sim}{b} \left[\underset{\sim}{\nabla} \chi(\underset{\sim}{b})\right]^2 \tag{IV.78}$$

It is interesting to compare the behavior of σ_S and σ_T in the present model with the famous scaling behavior of the same form factors for deep inelastic electron-nucleon scattering.[23] In this model, $\sigma_S(Q^2, \nu/Q^2)$ is scale invariant: for large ν and Q^2, $Q^2\sigma_S$ is a function of (ν/Q^2) only. However, the factor $\log(Q^2/\mu^2)$ spoils the scaling behavior of σ_T.[24]

In the somewhat hypothetical limit of an external field which varies in space slowly compared the lepton Compton-wavelength $(1/\chi|\nabla\chi| \ll \mu^{-1})$, the formula (IV.71) for σ_S/σ_T is valid for all Q^2. The direct evaluation is shown in Fig. 8; we see that σ_S/σ_T is never larger than 0.26.

It is not clear what direct connection these calculations have with respect to hadron electroproduction. While there appears to be a diffractive mechanism[25] operating in both cases, the details (e.g., the scaling behavior of σ_T) are different. However it may be that some features of the process, such as the importance of small transverse distances $(\Delta x)^2 \lesssim Q^{-2}$ at large Q^2 are common to both.

V. FUTURE PROBLEMS AND POSSIBLE LIMITATIONS

Throughout this paper we have found support for a simple physical picture for high energy scattering processes. However, this picture is couched in perturbation theory, and one may wonder whether it is generally valid. For example, to what extent does this picture apply to strong coupling field theories? Or, more modestly, will this picture survive higher order calculations in quantum electrodynamics?

Studies of diagrams such as shown in Fig. 9 indicate that the complete situation is not as simple as we suggest in this paper.[26] Using these or other methods it is not difficult to find that this diagram diverges logarithmically as $\eta \rightarrow \infty$, where η refers to the incoming electron. The logarithm comes from a loop integral and receives a large contribution from that region of phase space in which the internal partons are (almost) "wee". This example raises two problems. First, if we apply perturbation theory to very high orders, we must be equipped to deal with such logarithms, which in sufficiently high order violate s-channel unitarity. And secondly, since the internal photons in this example are (almost) "wee," one can question the applicability of the eikonal approximation to this diagram. The true situation may be somewhat like using purely nonrelativistic methods to calculate the Lamb shift: they work up to a certain point, and contribute a great deal of insight into the physics. However beyond that point they fail utterly. In the present case there is very likely a similar boundary, associated with wee partons, beyond which the simple methods of this paper fail. It remains for the future to see how much of the physics lies on the simple side of the boundary and how sharply the properties of the boundary region can be delineated.

APPENDIX

The two component formalism described in Section II suggests, in the interest of overall simplicity and uniformity, a change in notation, mainly in normalization factors, from that used in Paper I. This appendix is devoted to clarifying the connection between the old and new formalism.

We begin by discussing the electromagnetic potential. The operator $\mathbf{A}(x)$, as discussed in (II.6) and below, may be directly identified with $\mathbf{A}_T(x)$, of I.

$$\text{new } \mathbf{A}(x) = \mathbf{A}_T(x) \text{ old .} \tag{A.1}$$

However, the plane wave expansion (II.11) of $\mathbf{A}(x)$ differs from Eq. (IV.37) of I by a factor $\left[2(2\pi)^3\right]^{\frac{1}{2}}$; the comparison yields

$$\text{new } a(p, \ ) = \left[2(2\pi)^3\right]^{\frac{1}{2}} a(p, .) \text{ old .} \tag{A.2}$$

The connection between the new 2-component electron field $\psi(x)$ and the old 4-component $\Psi(x)$ is more disagreeable. Not only is there a change in normalization but also there is a unitary rotation. The essential connection is between

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{A.3}$$

and the independent dynamical variables of I.,

$$\Psi_+ = \begin{pmatrix} \Psi_1 \\ 0 \\ 0 \\ \Psi_4 \end{pmatrix} . \tag{A.4}$$

By comparing the anticommutation relations (IV.36) of I. with (II.5) of this paper, we see that the normalizations of the field operators differ by a relative factor $2^{\frac{1}{4}}$. If we choose phases such that

$$\text{new } \psi_1(x) = 2^{\frac{1}{4}} \Psi_1(x) \text{ old }, \tag{A.5}$$

then we find it is best to make the identification

$$\text{new } \psi_2(x) = i2^{\frac{1}{4}} \Psi_4(x) \text{ old }, \tag{A.6}$$

We verify the connection by comparing the equations of motion for Ψ_+ and ψ. Elimination of Ψ_- from Eq. (IV.18) of I. produces

$$(i\partial_0 - eA_0)\Psi_+ = \left[-(\underset{\sim}{p} - e\underset{\sim}{A})\cdot\underset{\sim}{\gamma} + m\right]\frac{1}{2\eta}\left[+(\underset{\sim}{p} - e\underset{\sim}{A})\cdot\underset{\sim}{\gamma} + m\right]\Psi_+ \tag{A.7}$$

Using the γ-matrices (IV.9) of I., we see that, as 2×2 matrices acting on the first and fourth components of Ψ_+, the matrices γ^1 and γ^2 are

$$\gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} .$$

If we combine (A.5) into the two component spinor relation

$$\begin{aligned} \psi(x) &= 2^{\frac{1}{4}} U \,\Psi(x) \\ U &= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \end{aligned} \tag{A.8}$$

and insert this relation into (A.7) we obtain

$$(i\partial_0 - eA_0)\psi = \left[-(\underset{\sim}{p} - e\underset{\sim}{A})\cdot U\underset{\sim}{\gamma}U^{-1} + m \right] \frac{1}{2\eta} \left[(\underset{\sim}{p} - e\underset{\sim}{A})\cdot U\underset{\sim}{\gamma}U^{-1} + m \right] \psi \, . \tag{A.9}$$

But

$$U\gamma^j U^{-1} = i\sigma^j \, , \tag{A.10}$$

so that (A.9) is identical to the equation of motion (II.14) for ψ.

The unitary matrix U introduces relative phases in the comparison of the elements of the plane wave expansions of ψ and Ψ_+. By definition, the new spinors w(s) appearing in the expansion (II.10) of ψ are equal to the old two component spinors w(s) appearing in the expansion (IV.32) of Ψ'_+ in I. Thus the creation and distruction operators in (II.10) must absorb, in addition to a normalization, the phase introduced by the presence of U. The comparison between (II.10) and (IV.32) of I, using Eq. (A.8), yields

$$\begin{aligned}
\text{new } b(p, +\tfrac{1}{2}) &= \left[2(2\pi)^3\right]^{\frac{1}{2}} b(p, +\tfrac{1}{2}) \text{ old} \\
\text{new } b(p, -\tfrac{1}{2}) &= i\left[2(2\pi)^3\right]^{\frac{1}{2}} b(p, -\tfrac{1}{2}) \text{ old} \\
\text{new } d^\dagger(p, +\tfrac{1}{2}) &= i\left[2(2\pi)^3\right]^{\frac{1}{2}} d^\dagger(\tilde{p}, +\tfrac{1}{2}) \text{ old} \\
\text{new } d^\dagger(p, -\tfrac{1}{2}) &= \left[2(2\pi)^3\right]^{\frac{1}{2}} d^\dagger(p, -\tfrac{1}{2}) \text{ old.}
\end{aligned} \tag{A.11}$$

This completes the correspondence-relations between the old and new notations. It is now straightforward to check that the new formalism is consistent with the old, including rules for diagrams.

We must apologize for changing notation in midstream. However, many disagreeable factors of $\sqrt{2}$, $(2\pi)^{3/2}$, etc. have thereby been purged, and a consistent mnemonic now exists for the factors "2" occuring in the rules for perturbation diagrams at the end of Section II: for every factor τ a factor 2, for every factor η a factor 2.

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2. R. J. Glauber, <u>Lectures in Theoretical Physics</u>, edited by W. E. Britten <u>et al.</u>, (Wiley-Interscience Inc., New York, 1959), Vol. 1.
3. R. P. Feynman, invited paper at the Third Topical Conf. on High Energy Collisions of Hadrons, Stony Brook, New York, Sept. 1969; Phys. Rev. Letters <u>23</u>, 1415 (1969).
4. J. B. Kogut and D. E. Soper, Phys. Rev. <u>D1</u>, 2901 (1970).
5. Note that $d\underset{\sim}{p}\ d\ /2\eta$ is the Lorentz-invariant surface element $dp_x dp_y dp_z/2E$ on the mass shell. The η-integration runs from 0 to ∞, thus covering the forward mass shell.
6. Recall that the nonrelativistic equation of motion is written $i\partial_0\psi = \frac{1}{2m}\underset{\sim}{\sigma}\cdot\underset{\sim}{p}\,\underset{\sim}{\sigma}\cdot\underset{\sim}{p}\,\psi$ before introducing the minimal substitution in order to obtain the correct $\underset{\sim}{\sigma}\cdot\underset{\sim}{B}$ term.
7. Readers familiar with the discussion in I. of the Galilean subgroup of the Lorentz group will note that such combinations in Table I as $(\underset{\sim}{q}/\eta_q) - (\underset{\sim}{p}/\eta)$ transform under this subgroup like (momentum/mass) - (momentum/mass) and are therefore invariant under "Galilean boosts." This invariance can often be used to practical advantage in calculations.
8. With the present normalization conventions, $\langle f|S\ i\rangle = (2\pi)^4\delta^4(p_f - p_i)M$, where M is the invariant amplitude calculated with the conventions of Bjorken and

Drell using Dirac spinors normalized to $\bar{u}u = 2m$. See J. D. Bjorken and S. D. Drell, Relativistic Quantum Fields, (McGraw Hill, New York, 1965); Appendix B.

9. We also use this formula for a one particle final state.

10. This relationship can be obtained by using a wave packet for the initial state (cf., M. L. Goldberger and K. M. Watson, Collision Theory (John Wiley, New York, 1964); Sec. 3.3)). In the high energy limit in which $\eta \sim \sqrt{2}$ E this reduces to the more familiar result with η replaced everywhere by E in (IV.6) and (IV.7).

11. Cf., H. Cheng and T. T. Wu, Phys. Rev. 184, 1868 (1969).

12. To calculate F_2 to order e^2, we can use the value $Z_2 = 1$, which is correct to order e^0.

13. S. J. Chang and S. K. Ma have used different infinite momentum techniques to obtain this result, Phys. Rev. 180, 1506 (1969).

14. S. D. Drell, D. J. Levy, T. M. Yan, Phys. Rev. Letters 22, 744 (1969); H. Cheng and T. T. Wu, Phys. Rev. D1, 1069 (1970); S. J. Chang and S. K. Ma, op. cit.

15. The amplitude $\sqrt{Z_3}$ for a physical photon to be a bare photon is 1 to lowest order, but does not, of course, contribute to pair production.

16. H. Cheng and T. T. Wu, Phys. Rev. 182, 1852 (1969).

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18. Note that the limit $\nu \to \infty$ is already implicit in our formalism.

19. More precisely: let $d\sigma'$ be the limiting form of $d\sigma$ so obtained. Then it is not difficult to prove that $d\sigma_S = d\sigma'_S[1 + 0(1/Q^2)]$, $d\sigma_T = d\sigma'_T[1 + 0(1/\log Q^2)]$ as $Q^2 \to \infty$, assuming that the potential is sufficiently well behaved.

20. A. Erdelyi, ed., Tables of Integral Transforms, (McGraw Hill, New York, 1954); Vol. 1, pg. 334.

21. If the helicity is flipped on the electron line $d\sigma_T$ is suppressed by a factor (m^2/Q^2) as $Q^2 \to \infty$. If the helicity is flipped on the muon line, $d\sigma_T$ is suppressed by a factor $[1/\log(Q^2/\mu^2)]$.

22. J. D. Bjorken, Phys. Rev. D1, 1376 (1970).

23. J. D. Bjorken, Phys. Rev. 179, 1547 (1969).

24. Strictly speaking, scale invariance for σ_T means that $Q^2 \sigma_T$ approaches a finite limit as $Q^2 \to \infty$ with (ν/Q^2) held constant. However, we have evaluated σ_T in this model in the limit $(\nu/Q^2) \to \infty$ with Q^2 held constant, and then we have let $Q^2 \to \infty$. It is not impossible for σ_T to exhibit scale invariance in the limit $Q^2 \to \infty$, $(\nu/Q^2) = \text{const.}$, but not in the reversed limit used here.

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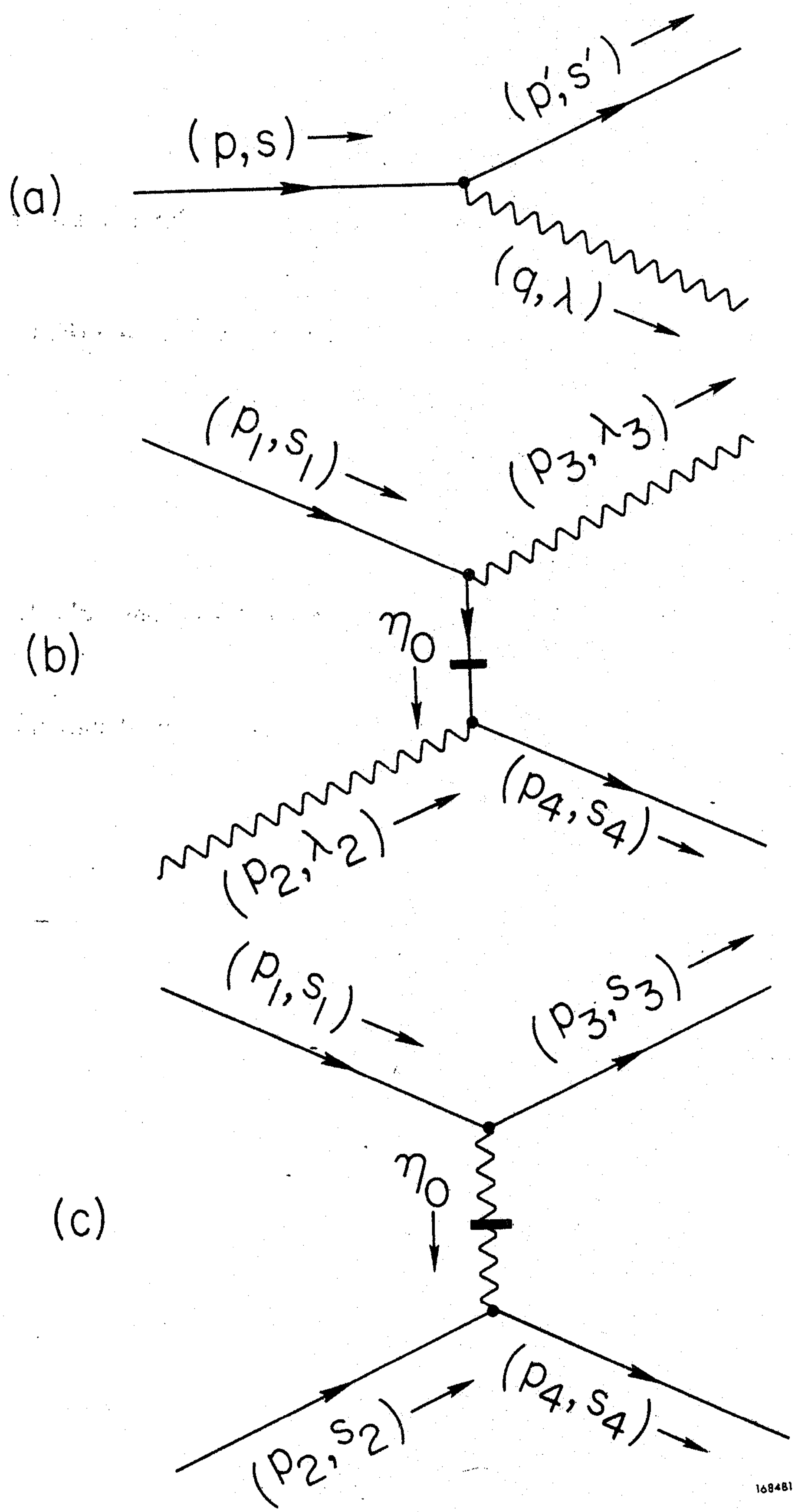
FIGURE CAPTIONS

1. Vertices in the infinite momentum frame.
2. Electron scattering off an external field. (a) Zeroth order in electron structure, (b) second order in electron structure.
3. Higher order contribution to electron scattering off an external field.
4. Bremsstrahlung off an external field.
5. Pair production on an external field.
6. Delbruck scattering.
7. Muon pair production off an external field.
8. σ_S/σ_T for high energy electroproduction of lepton pairs from a slowly varying external field.
9. Electron scattering diagram contributing $\eta \log \eta$ term to the S-matrix.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(p,s)
(p′,s′)
(q,λ)
(b)
(p1,s1)
(p3,λ3)
η0
(p2,λ2)
(p4,s4)
(c)
(p1,s1)
(p3,s3)
η0
(p2,s2)
(p4,s4)
168481

Fig. 1

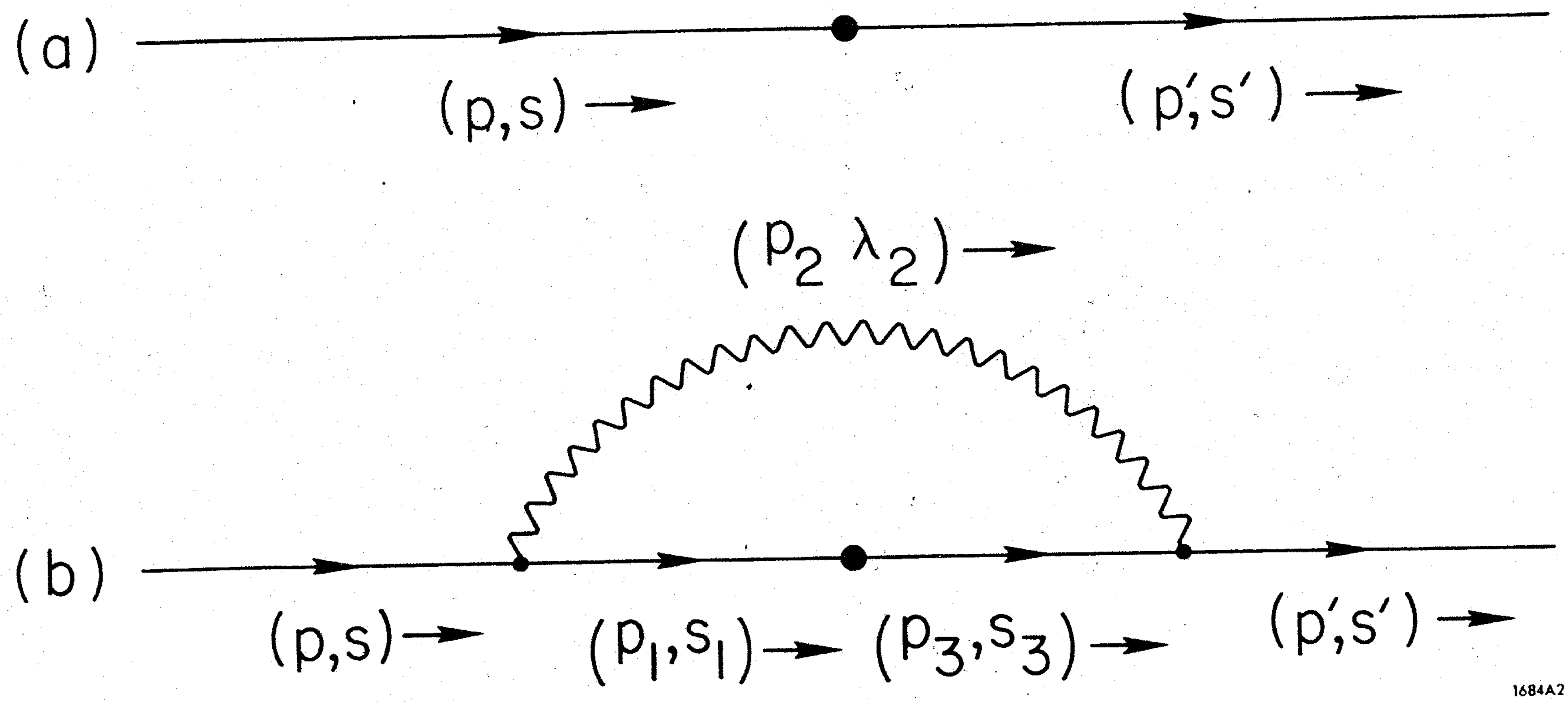
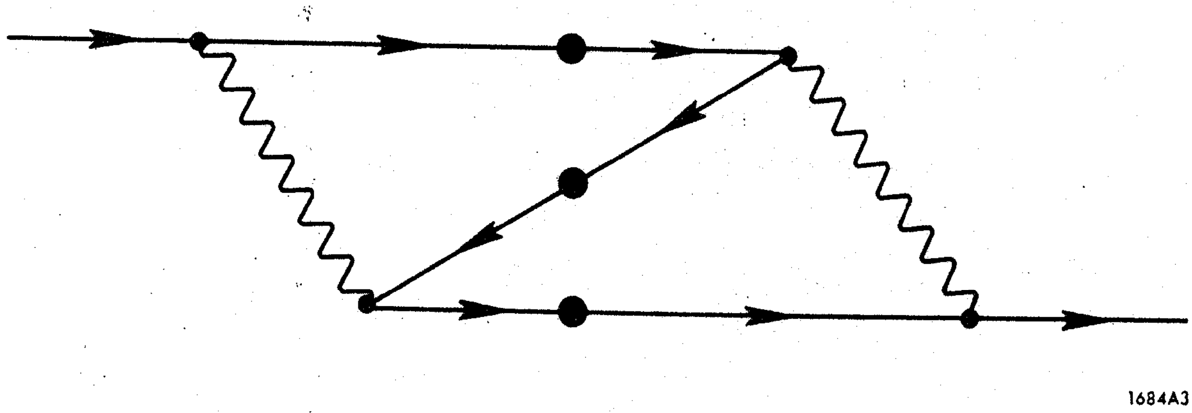


Fig. 2



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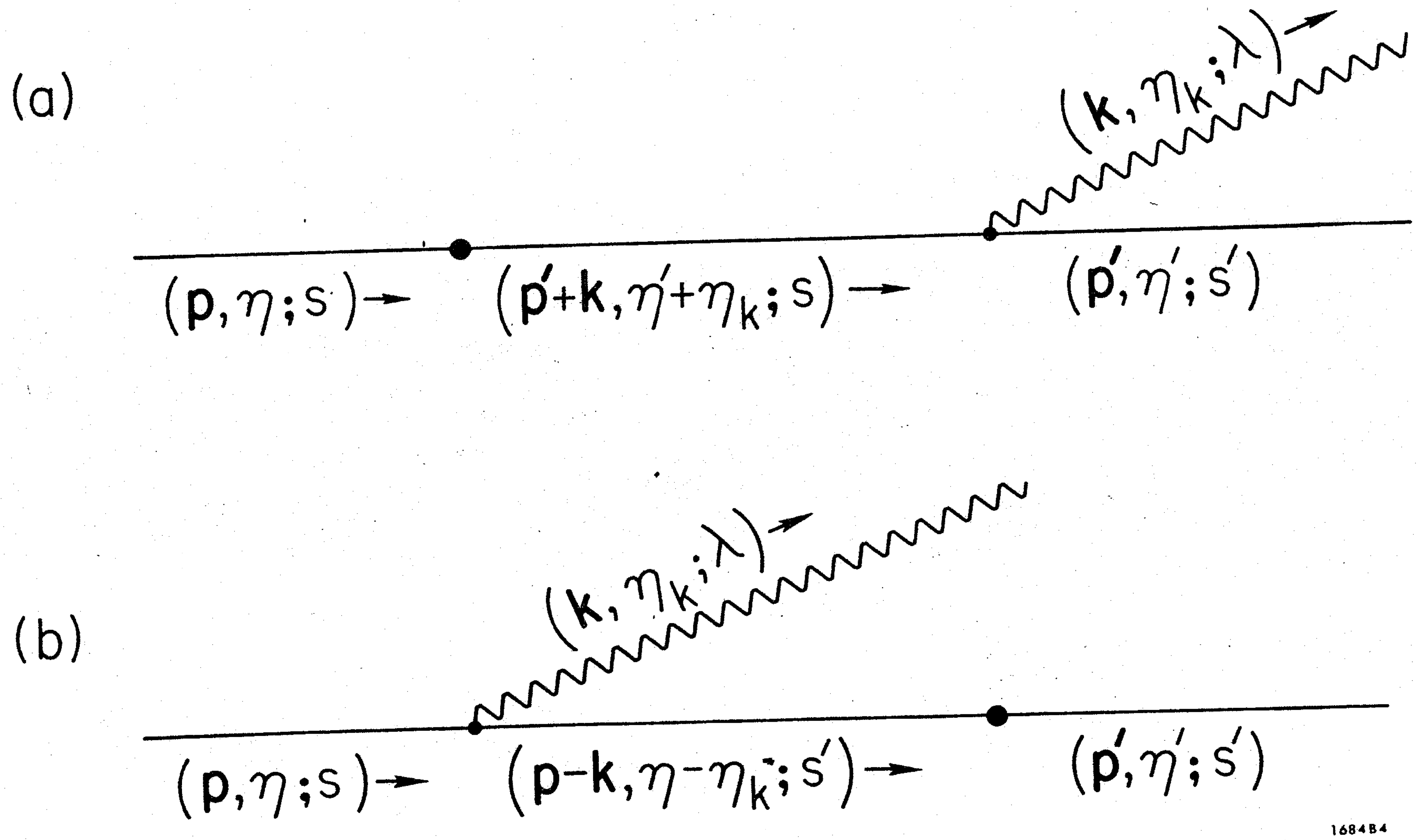

(a)
$(\mathbf{k}, \eta_k; \lambda)$
$(\mathbf{p}, \eta; s) \rightarrow$
$(\mathbf{p}' + \mathbf{k}, \eta' + \eta_k; s) \rightarrow$
$(\mathbf{p}', \eta'; s')$
(b)
$(\mathbf{k}, \eta_k; \lambda)$
$(\mathbf{p}, \eta; s) \rightarrow$
$(\mathbf{p} - \mathbf{k}, \eta - \eta_k; s') \rightarrow$
$(\mathbf{p}', \eta'; s')$
1684B4

Fig. 4

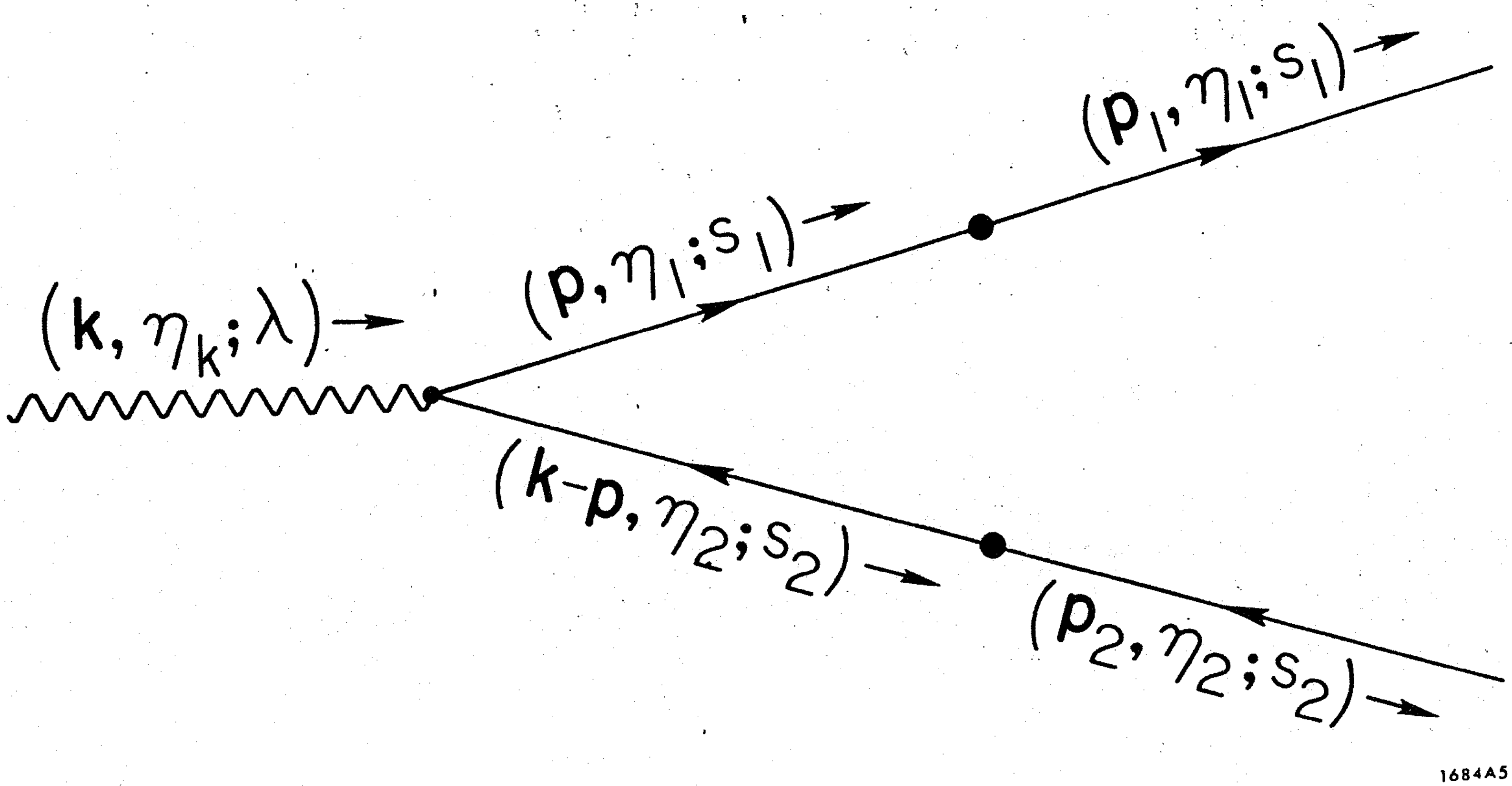

Fig. 5

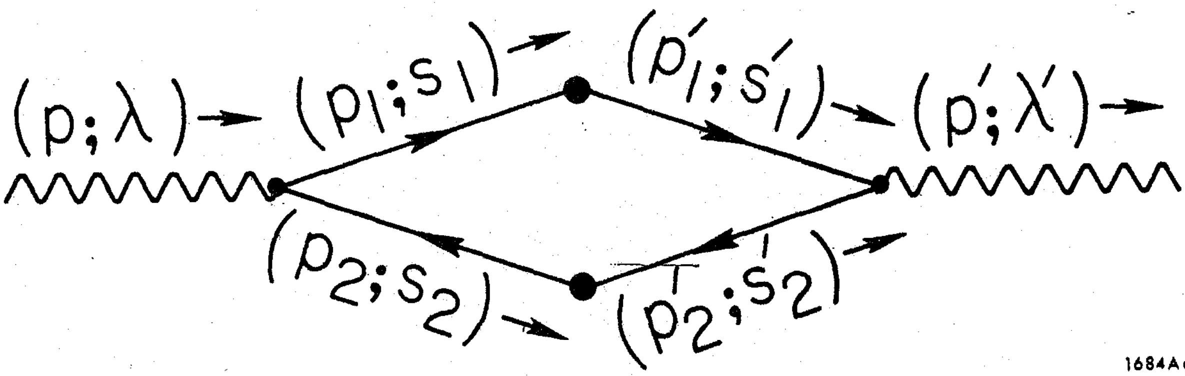

Fig. 6

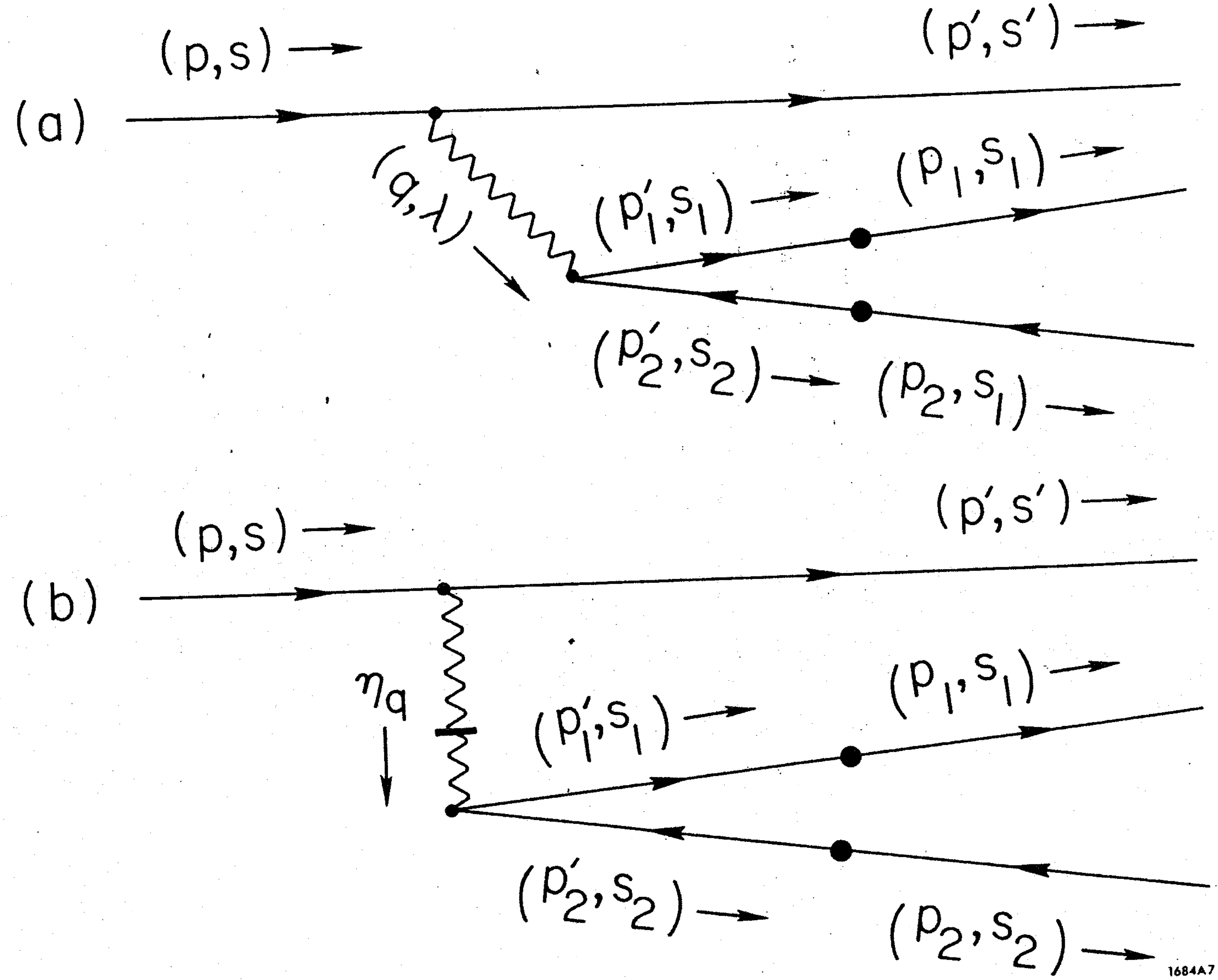
(a)
(p,s)
(p′,s′)
(q,λ)
(p′₁,s₁)
(p₁,s₁)
(p′₂,s₂)
(p₂,s₁)
(b)
(p,s)
(p′,s′)
η_q
(p′₁,s₁)
(p₁,s₁)
(p′₂,s₂)
(p₂,s₂)
1684A7

Fig. 7

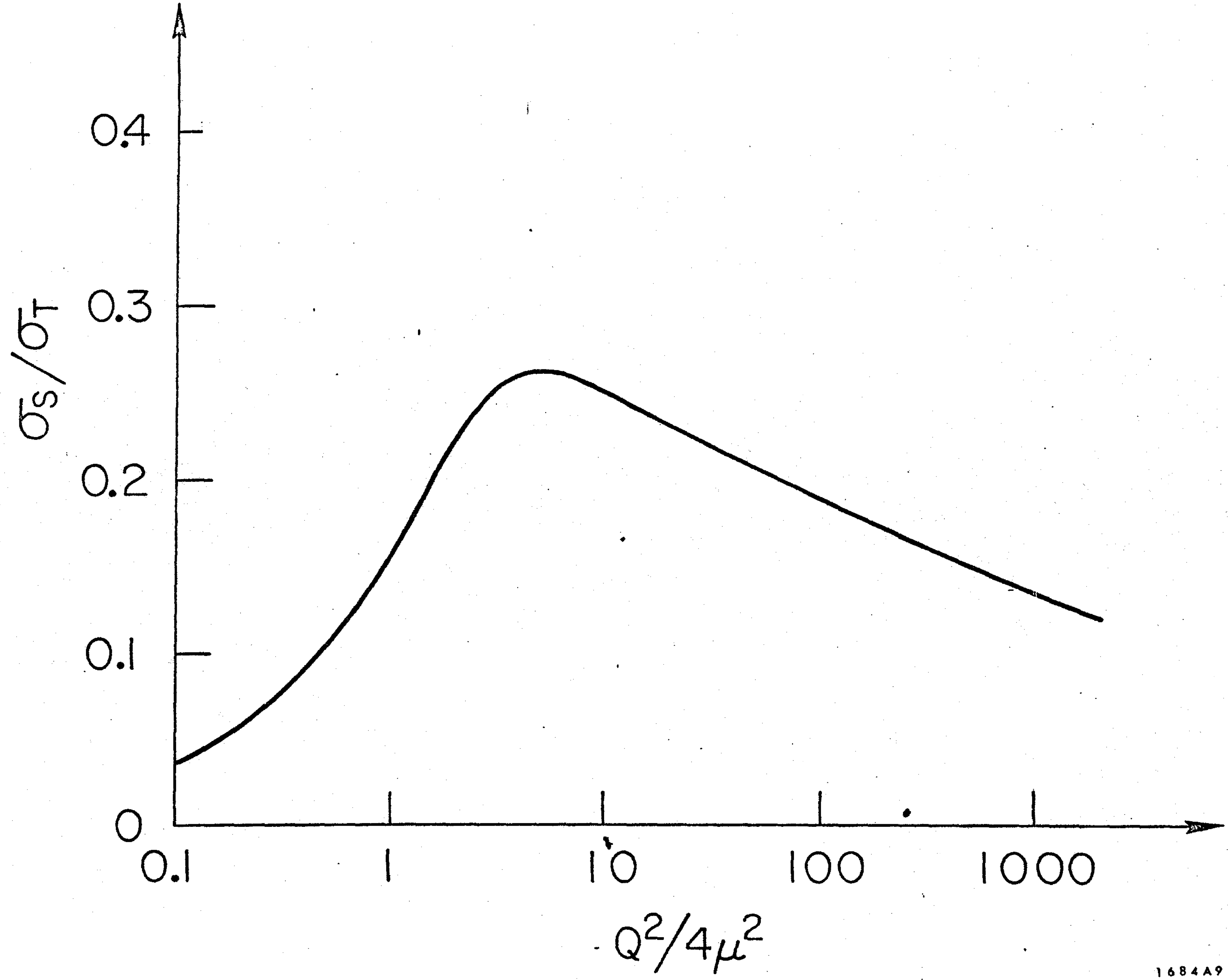

Fig. 8

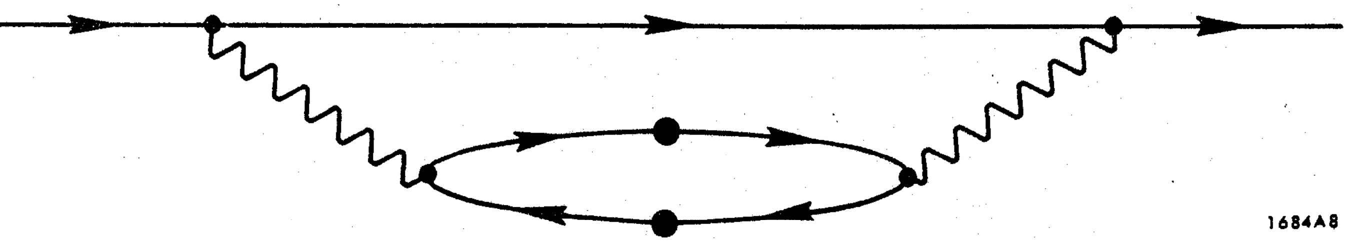

Fig. 9