



Standard Model in Weyl conformal geometry

To the memory of my physics teacher

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Abstract We study the Standard Model (SM) in Weyl conformal geometry. This embedding is truly minimal *with no new fields* beyond the SM spectrum and Weyl geometry. The action inherits a gauged scale symmetry $D(1)$ (known as Weyl gauge symmetry) from the underlying geometry. The associated Weyl quadratic gravity undergoes spontaneous breaking of $D(1)$ by a geometric Stueckelberg mechanism in which the Weyl gauge field (ω_μ) acquires mass by “absorbing” the spin-zero mode of the \tilde{R}^2 term in the action. This mode also generates the Planck scale and the cosmological constant. The Einstein-Proca action emerges in the broken phase. In the presence of the SM, this mechanism receives corrections (from the Higgs) and it can induce electroweak (EW) symmetry breaking. The EW scale is proportional to the vev of the Stueckelberg field. The Higgs field (σ) has direct couplings to the Weyl gauge field ($\sigma^2 \omega_\mu \omega^\mu$). The SM fermions only acquire such couplings for non-vanishing kinetic mixing of the gauge fields of $D(1) \times U(1)_Y$. If this mixing is present, part of the mass of Z boson is not due to the usual Higgs mechanism, but to its mixing with massive ω_μ . Precision measurements of Z mass then set lower bounds on the mass of ω_μ which can be light (few TeV). In the early Universe the Higgs field can have a geometric origin, by Weyl vector fusion, and the Higgs potential can drive inflation. The dependence of the tensor-to-scalar ratio r on the spectral index n_s is similar to that in Starobinsky inflation but mildly shifted to lower r by the Higgs non-minimal coupling to Weyl geometry.

1 Motivation

The Standard Model (SM) with the Higgs mass parameter set to zero has a scale symmetry. This may indicate that this symmetry plays a role in model building for physics beyond the SM [1]. Scale symmetry is natural in physics at higher

scales or in the early Universe when all states are essentially massless. In such scenario, the scales of the theory such as the Planck scale and the electroweak (EW) scale will be generated by the vacuum expectations values (vev's) of some scalar fields. In this work we consider the SM with a *gauged* scale symmetry (also called Weyl gauge symmetry) [2–4] which we prefer to the more popular *global* scale symmetry, since the latter is broken by black-hole physics [5]. A natural framework for this symmetry is the Weyl conformal geometry [2–4] where this symmetry is built in. We thus consider the SM embedded in the Weyl conformal geometry and study the implications.

The Weyl geometry is defined by classes of equivalence $(g_{\alpha\beta}, \omega_\mu)$ of the metric $(g_{\alpha\beta})$ and the Weyl gauge field (ω_μ) , related by the Weyl gauge transformation, see (a) below. If matter is present, (a) must be extended by transformation (b) of the scalars (ϕ) and fermions (ψ)

$$(a) \hat{g}_{\mu\nu} = \Sigma^d g_{\mu\nu}, \quad \hat{\omega}_\mu = \omega_\mu - \frac{1}{\alpha} \partial_\mu \ln \Sigma, \\ \sqrt{\hat{g}} = \Sigma^{2d} \sqrt{g}, \\ (b) \hat{\phi} = \Sigma^{-d/2} \phi, \quad \hat{\psi} = \Sigma^{-3d/4} \psi, \quad (d = 1). \quad (1)$$

Here d is the Weyl charge of $g_{\mu\nu}$, α is the Weyl gauge coupling,¹ $g = |\det g_{\mu\nu}|$ and $\Sigma > 0$. This is a non-compact gauged dilatation symmetry, denoted $D(1)$. Since it is Abelian, the normalization of the charge d is not fixed.² In this paper we take $d = 1$. The case of arbitrary d is recovered from our results by simply replacing $\alpha \rightarrow d \alpha$. A discussion on symmetry (1) and a brief introduction to Weyl geometry are found in Appendix A.

To study the SM in Weyl geometry, all one needs to know for the purpose of this work is the expression of the connection ($\tilde{\Gamma}$) of this geometry, which differs from the Levi-

¹ Our convention is $g_{\mu\nu} = (+, -, -, -)$ while the curvature tensors are defined as in [6].

² For example $d = 1$ is a convention used in e.g. [7] while $d = 2$ was considered in [8,9].

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Civita connection (Γ) of (pseudo-)Riemannian case used in Einstein gravity. The Weyl connection is a solution to $\tilde{\nabla}_\lambda g_{\mu\nu} = -\alpha \omega_\lambda g_{\mu\nu}$ where $\tilde{\nabla}_\mu$ is defined by $\tilde{\Gamma}_{\mu\nu}^\lambda$. This solution is (see Appendix)

$$\tilde{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda + (1/2) \alpha [\delta_\mu^\lambda \omega_\nu + \delta_\nu^\lambda \omega_\mu - g_{\mu\nu} \omega^\lambda]. \quad (2)$$

$\tilde{\Gamma}$ is invariant under (1), as it should be, since the parallel transport of a vector must be gauge independent. Taking the trace in (2), with a notation $\tilde{\Gamma}_\mu = \tilde{\Gamma}_{\mu\nu}^\nu$ and $\Gamma_\mu = \Gamma_{\mu\nu}^\nu$, then

$$\omega_\mu \propto \tilde{\Gamma}_\mu - \Gamma_\mu. \quad (3)$$

The Weyl field is thus a measure of the (trace of the) deviation from a Levi–Civita connection.

The general quadratic gravity action defined by Weyl geometry [2–4], invariant under (1), is written in terms of scalar and tensor curvatures of this geometry. Using $\tilde{\Gamma}$ of (2) and standard formulae one can express these curvatures in terms of their Riemannian counterparts and re-write the action in a more familiar Riemannian notation (as we shall do). In the limit $\omega_\mu = 0$ i.e. if: (i) ω_μ is ‘pure gauge’ or if (ii) ω_μ becomes massive and decouples, then $\tilde{\Gamma} = \Gamma$ and then *Weyl geometry becomes Riemannian!* This is an interesting transition, relevant later. In (i) invariance under (1) reduces to local scale invariance (no ω_μ).

The role of Weyl gauge symmetry in model building beyond SM was studied before [7–27]. We go beyond these models which were limited to actions *linear* in the scalar curvature \tilde{R} of Weyl geometry and also introduced *additional states* (scalar fields beyond the Higgs field) to maintain symmetry (1) and to generate the mass scales (Planck, etc) of the theory.

Our approach here to model building is truly *minimal*, in the sense that *no new fields* are added to the SM spectrum – we simply embed the SM in Weyl geometry! Note that the Weyl gauge field present here is part of the underlying geometry and of Weyl gravity.³ The gravity part of the action is fixed by the Weyl geometry [2–4], is actually *quadratic* and is automatically invariant under (1) (a) (since $\tilde{\Gamma}$ is invariant). This minimal approach builds on our recent results in [28, 29] (also [30–32]) that showed that the original Weyl quadratic gravity action *in the absence of matter* is broken spontaneously to the Einstein–Proca action. Therefore, *this breaking is geometric in nature* (no scalar field is added to this purpose).

With this result, embedding the SM in Weyl geometry is very natural: one sets the Higgs mass parameter to zero and ‘upgrades’ the SM covariant derivatives, to respect symmetry (1) inherited from Weyl geometry. Thus, both the Lagrangian and its underlying geometry ($\tilde{\Gamma}$) have the same Weyl gauge

³ The literature often calls Weyl gravity the square of the Weyl tensor in Riemannian geometry. We actually consider the original Weyl quadratic gravity in Weyl geometry which has additional terms (Sect. 2.1).

symmetry. This is a *unique* feature, not present in models with local scale symmetry based on Riemannian geometry (i.e. with no ω_μ). It adds mathematical consistency to the model and motivated this study. Hereafter we refer to this model as SMW.

There is additional motivation to study the SMW and the Weyl geometry:

- (a) Einstein gravity emerges naturally. After a Stueckelberg mechanism, the Weyl gauge field ω_μ acquires a mass $m_\omega \sim \alpha M_p$ (M_p : Planck scale) by ‘eating’ the spin zero-mode (ϕ_0) of geometric origin propagated by the $(1/\xi^2)\tilde{R}^2$ term in the action of coupling ξ . The gauge fixing of symmetry (1) is dynamical, as shown by the equations of motion. After ω_μ decouples, the Einstein action is naturally obtained as a broken phase of Weyl gravity. M_p and the cosmological constant (Λ) are both generated by $\langle\phi_0\rangle$ and are related: $\Lambda/M_p^2 = (3/2)\xi^2$.
- (b) The theory has a symmetry $D(1) \times U(1)_Y \times SU(2)_L \times SU(3)$. A gauge kinetic mixing of ω_μ with the hypercharge field B_μ of $U(1)_Y$ is not forbidden by this symmetry.
- (c) The Higgs has couplings to ω_μ , of type $\sigma^2 \omega_\mu \omega^\mu$. The SM gauge bosons and fermions do not couple to ω_μ [8, 9]. Only if a gauge kinetic mixing exists, can fermions couple to ω_μ .
- (d) The SM Higgs potential is recovered for small Higgs field values (relative to Planck scale). The EW symmetry breaking is then induced by gravitational effects, with the Higgs mass and electroweak scale obtained for perturbative couplings of the Weyl quadratic gravity.
- (e) If a gauge kinetic mixing is present, part of the Z boson mass is not due to the Higgs mechanism, but to the geometric Stueckelberg mechanism (giving mass to ω_μ). Experimental data on m_Z provide constraints on the Weyl gauge coupling α and on the mass of ω .
- (f) The Higgs potential at large field values drives inflation. Interestingly, the origin of the Higgs field in the early Universe is geometrical, from the Weyl boson fusion, see c). The prediction for the tensor-to-scalar ratio (r) (for given spectral index n_s) is bounded from above by that in the Starobinsky model with similar dependence $r(n_s)$, due to the \tilde{R}^2 term.
- (g) The SMW can provide a successful alternative to the Λ CDM, as discussed in [33].

These interesting properties of the SMW are studied in Sect. 2. The relation to other scale-invariant models follows (Sect. 3). The Conclusions are in Sect. 4. The Appendix has an introduction to Weyl conformal geometry and additional calculations for Sect. 2.

2 SM in Weyl conformal geometry

2.1 Einstein action from spontaneous breaking of Weyl quadratic gravity

Consider first the original Weyl gravity action [2–4] and here we follow [28,29]. The action is

$$\mathcal{L}_0 = \sqrt{g} \left[\frac{1}{4!} \frac{1}{\xi^2} \tilde{R}^2 - \frac{1}{4} F_{\mu\nu}^2 - \frac{1}{\eta^2} \tilde{C}_{\mu\nu\rho\sigma}^2 \right], \quad (4)$$

with couplings $\xi, \eta \leq 1$. Here $F_{\mu\nu} = \tilde{\nabla}_\mu \omega_\nu - \tilde{\nabla}_\nu \omega_\mu$ is the field strength of ω_μ , with $\tilde{\nabla}_\mu \omega_\nu = \partial_\mu \omega_\nu - \tilde{\Gamma}_{\mu\nu}^\rho \omega_\rho$. Since $\tilde{\Gamma}_{\mu\nu}^\alpha = \tilde{\Gamma}_{\nu\mu}^\alpha$ is symmetric, $F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$. $\tilde{C}_{\mu\nu\rho\sigma}$ and \tilde{R} are the Weyl tensor and scalar curvature in Weyl geometry, derived from Eq. (2). Their relations to Riemannian Weyl-tensor $C_{\mu\nu\rho\sigma}$ and scalar curvature R are shown in Eqs. (A-11), (A-14):

$$\begin{aligned} \tilde{C}_{\mu\nu\rho\sigma}^2 &= C_{\mu\nu\rho\sigma}^2 + \frac{3}{2} \alpha^2 F_{\mu\nu}^2, \\ \tilde{R} &= R - 3\alpha \nabla_\mu \omega^\mu - \frac{3}{2} \alpha^2 \omega_\mu \omega^\mu. \end{aligned} \quad (5)$$

The rhs of these equations is in a Riemannian notation, so $\nabla_\mu \omega^\lambda = \partial_\mu \omega^\lambda + \Gamma_{\mu\rho}^\lambda \omega^\rho$.

Each term in \mathcal{L}_0 is invariant under $D(1)$ of (1). Indeed, \tilde{R} transforms as $\tilde{R} \rightarrow (1/\Sigma) \tilde{R}$ (see Appendix), so $\sqrt{g} \tilde{R}^2$ is invariant. Also $\sqrt{g} C_{\mu\nu\rho\sigma}^2$ and $F_{\mu\nu}^2 \sqrt{g}$ are invariant; similar for $\sqrt{g} \tilde{C}_{\mu\nu\rho\sigma}^2$. The term $\tilde{C}_{\mu\nu\rho\sigma}^2$ ensures that \mathcal{L}_0 is a general Weyl action and is largely spectator under the transformations below, so its impact could be analysed separately. But it is needed at a quantum level, so we included it here anyway (it brings a massive spin-2 ghost [34]).

In \mathcal{L}_0 we replace $\tilde{R}^2 \rightarrow -2\phi_0^2 \tilde{R} - \phi_0^4$ with ϕ_0 a scalar field. Doing so gives a classically equivalent \mathcal{L}_0 , since by using the solution $\phi_0^2 = -\tilde{R}$ of the equation of motion of ϕ_0 in the modified \mathcal{L}_0 , one recovers action (4). With Eq. (5), \mathcal{L}_0 becomes in a Riemannian notation

$$\begin{aligned} \mathcal{L}_0 = \sqrt{g} \left\{ \frac{-1}{12\xi^2} \phi_0^2 \left[R - 3\alpha \nabla_\mu \omega^\mu - \frac{3}{2} \alpha^2 \omega_\mu \omega^\mu \right] \right. \\ \left. - \frac{\phi_0^4}{4! \xi^2} - \frac{1}{4} \left[1 + 6 \frac{\alpha^2}{\eta^2} \right] F_{\mu\nu}^2 - \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 \right\} \quad (6) \end{aligned}$$

or, making the symmetry manifest

$$\begin{aligned} \mathcal{L}_0 = \sqrt{g} \left\{ \frac{-1}{2\xi^2} \left[\frac{1}{6} \phi_0^2 R + (\partial_\mu \phi_0)^2 - \frac{\alpha}{2} \nabla_\mu (\omega^\mu \phi_0^2) \right] \right. \\ \left. - \frac{\phi_0^4}{4! \xi^2} + \frac{\alpha^2}{8\xi^2} \phi_0^2 \left[\omega_\mu - \frac{1}{\alpha} \partial_\mu \ln \phi_0^2 \right]^2 \right. \\ \left. - \frac{1}{4\gamma^2} F_{\mu\nu}^2 - \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 \right\}, \\ \text{with } 1/\gamma^2 \equiv 1 + 6\alpha^2/\eta^2 \geq 1. \end{aligned} \quad (7)$$

Every term of coefficient $\propto 1/\xi^2$ and the entire \mathcal{L}_0 are invariant under (1); we must then “fix the gauge” of this symmetry. This follows dynamically from the equations of motion of ϕ_0 , ω_μ (see later), while at the level of the Lagrangian this is done by applying to \mathcal{L}_0 a specific form of transformation (1) that is scale-dependent $\Sigma = \phi_0^2/(\phi_0^2)$ which is fixing ϕ_0 to its vev (we shall see shortly how ϕ_0 acquires a vev); naively, one simply sets $\phi_0 \rightarrow \langle \phi_0 \rangle$ in (7). In terms of the transformed fields (with a “hat”), \mathcal{L}_0 becomes

$$\begin{aligned} \mathcal{L}_0 = \sqrt{\hat{g}} \left[-\frac{1}{2} M_p^2 \hat{R} + \frac{3}{4} M_p^2 \alpha^2 \gamma^2 \hat{\omega}_\mu \hat{\omega}^\mu - \frac{1}{4} \langle \phi_0^2 \rangle M_p^2 \right. \\ \left. - \frac{1}{4} \hat{F}_{\mu\nu}^2 - \frac{1}{\eta^2} \hat{C}_{\mu\nu\rho\sigma}^2 \right], M_p^2 \equiv \frac{\langle \phi_0^2 \rangle}{6\xi^2}. \end{aligned} \quad (8)$$

In (8) a total divergence in the action, $\delta S = \alpha/(4\xi^2) \langle \phi_0^2 \rangle \int d^4x \sqrt{\hat{g}} \nabla_\mu \hat{\omega}^\mu$ was ignored – it may be replaced by a *local* condition $\nabla_\mu \hat{\omega}^\mu = 0$. This constraint will be obtained shortly from the current conservation of the symmetric phase, Eqs. (6), (7).

In (8) we identify M_p with the Planck scale. Equation (8) is the Einstein gauge (frame) and also the unitary gauge of action (7). By Stueckelberg mechanism [35–37], ω_μ has become a massive Proca field, after “eating” the derivative $\partial_\mu \ln \phi_0$ of the Stueckelberg field ($\ln \phi_0$) in (7) [28,29]. The number of degrees of freedom is conserved: in addition to the graviton, the massless ϕ_0 and massless ω_μ were replaced by massive ω_μ with a mass $m_\omega^2 = (3/2)\alpha^2 \gamma^2 M_p^2$. We expect m_ω to be near Planck scale M_p , but the Weyl gravity coupling may be naturally $\alpha \ll 1$, so $m_\omega \ll M_p$.

The Einstein–Proca action in (8) is a broken phase of \mathcal{L}_0 of (7). After ω_μ decouples from (8), below m_ω the Einstein–Hilbert action is obtained as a ‘low-energy’ effective theory of Weyl gravity [28,29]. Hence, Einstein gravity appears to be the “Einstein gauge”-fixed version of the Weyl action. However, *the breaking is more profound and is not the result of a mere ‘gauge choice’*: it is accompanied by a Stueckelberg mechanism and by a transition from Weyl to Riemannian geometry: indeed, when massive ω_μ decouples then $\tilde{\Gamma}$ of (2) is replaced by Γ .

In the other case, when ω_μ is very light ($\alpha \ll 1$), it does not decouple in the flat space-time limit and may exist at low energies provided that non-metricity bounds (usually TeV-like [38–40]) do not forbid this. It could also act as a dark matter candidate, see e.g. [41].

Note that the Stueckelberg term in (7)

$$\begin{aligned} (\alpha^2/4) \phi_0^2 [\omega_\mu - (1/\alpha) \partial_\mu \ln \phi_0^2]^2 &= (\tilde{D}_\mu \phi_0)^2, \\ \tilde{D}_\mu \phi_0 &\equiv [\partial_\mu - \alpha/2 \omega_\mu] \phi_0, \end{aligned} \quad (9)$$

is simply a Weyl-covariant kinetic term of the Stueckelberg field that became the mass term of ω_μ in (8). That is, a Weyl gauge-invariant kinetic term of a (Weyl-charged)

scalar in Weyl geometry is a mass term for ω_μ in the (pseudo)Riemannian geometry underlying (8). This gives an interesting geometric interpretation to the origin of mass, as a transition from Weyl to Riemannian geometry, without any scalar field present in the final spectrum. The field ϕ_0 also generated the Planck mass and was “extracted” from the \tilde{R}^2 term i.e. is of geometric origin (like ω_μ), giving an elegant breaking mechanism.

This leaves the question of how ϕ_0 acquired a vev and how the “gauge fixing” (for ω_μ) emerges. This is seen from the equations of motion shown in Appendix B. First, from (6) one immediately writes the equation of motion of $g^{\mu\nu}$ and takes its trace. From this “traced” equation, one subtracts the equation of motion of ϕ_0 itself, also obtained from (6). As shown in Appendix B, one immediately finds

$$\square K = 0, \quad \text{where } \square = \nabla_\mu \nabla^\mu, \quad K = \phi_0^2, \quad (10)$$

so there is a conserved (onshell) current $\nabla^\mu K_\mu = 0$, with $K_\mu = \partial_\mu \phi_0^2$ in our Weyl gauge invariant model, similar to the global scale invariant case [42–46]. For a Friedmann–Robertson–Walker (FRW) Universe, with $g_{\mu\nu} = (1, -a(t)^2, -a(t)^2, -a(t)^2)$, Eq. (10) gives $\ddot{K} + 3H\dot{K} = 0$ which has a solution $\phi_0^2(t) = c_1 \int_{t_0}^t d\tau/a(\tau)^3 + c_2$, with constants $c_{1,2}$ and $H = \dot{a}/a$. At large t , $\phi_0(t)$ evolves to a constant and this is how ϕ_0 acquires a vev $\langle \phi_0 \rangle$, similar to [44–46].

Further, from (6) one writes the equation of motion of ω_μ and applying ∇_ρ to it, one finds a conserved current [28, 29] (see also Appendix B)

$$J_\rho = \xi \alpha \phi_0 (\partial_\rho - \alpha/2 \omega_\rho) \phi_0, \quad \nabla_\mu J^\mu = 0. \quad (11)$$

Using (10), the last equation simply gives $\nabla_\mu (\omega^\mu \phi_0^2) = 0$. With ϕ_0 replaced by its vev $\langle \phi_0 \rangle$, then $\nabla_\mu \omega^\mu = 0$; this the “gauge fixing” condition, specific to a massive Proca field, that emerged from the conserved current of the Weyl gauge symmetry (it was also inferred earlier from Eq. (8)). This concludes our discussion on the “gauge fixing”.

Finally, one may ask what Weyl geometry tells us about the cosmological constant (Λ). From Lagrangians (7) and (8) we find

$$\Lambda = \frac{1}{4} \langle \phi_0^2 \rangle, \quad \frac{\Lambda}{M_p^2} = \frac{3}{2} \xi^2. \quad (12)$$

Both the cosmological constant and the Planck scale are generated by ϕ_0 and are thus related; hence, in models based on Weyl geometry $\Lambda > 0$. In the formal limit $\langle \phi_0 \rangle \rightarrow 0$ then $\Lambda, M_p \rightarrow 0$ and the Weyl gauge symmetry is restored.⁴

In conclusion, Weyl action (4), (7) is more fundamental than Einstein–Proca action (8) which is its “low-energy”, broken phase. When the massive Weyl gauge boson decouples,

⁴ This limit is formal, since the linearisation of (4) with $\phi_0 = -\tilde{R}$ implicitly assumes that ϕ_0 is non-zero.

the geometry becomes Riemannian and the Einstein gravity is recovered. In some sense this picture is entirely geometrical, since we did not include matter. Thus, ultimately the underlying geometry of our Universe may actually be Weyl conformal geometry. Its Weyl gauge symmetry could then explain a small (non-vanishing, positive) cosmological constant.

2.2 Weyl quadratic gravity and “photon”: photon mixing

Consider now \mathcal{L}_0 in the presence of the SM hypercharge gauge group $U(1)_Y$. A kinetic mixing of ω_μ (Weyl “photon”) with the B_μ gauge field of $U(1)_Y$ is allowed by the direct product symmetry $U(1)_Y \times D(1)$. Such mixing was mentioned in the literature [21] but not investigated. Consider then⁵

$$\mathcal{L}_1 = \sqrt{g} \left\{ \frac{1}{4! \xi^2} \tilde{R}^2 - \frac{1}{4} \left[F_{\mu\nu}^2 + 2 \sin \chi F_{\mu\nu} F_y^{\mu\nu} + F_y^2 \right] - \frac{1}{\eta^2} \tilde{C}_{\mu\nu\rho\sigma}^2 \right\}. \quad (13)$$

where F_y is the field strength of B_μ . The source of B_μ is the SM fermionic Lagrangian (not shown in Eq. (13)) which is invariant under (1) and is independent of the Weyl gauge field [8, 9] (see next section).

We repeat the steps in Sect. 2.1 and after transformation (1) under which B_μ is invariant, $\hat{B}_\mu = B_\mu$, we find \mathcal{L}_1 in terms of the new fields (with a hat):

$$\mathcal{L}_1 = \sqrt{g} \left\{ -\frac{1}{2} M_p^2 \hat{R} + \frac{3}{4} M_p^2 \alpha^2 \hat{\omega}_\mu \hat{\omega}^\mu - \frac{3\xi^2}{2} M_p^4 - \frac{1}{4} \left[\frac{1}{\gamma^2} \hat{F}_{\mu\nu}^2 + 2 \sin \chi \hat{F}_{\mu\nu} \hat{F}_y^{\mu\nu} + \hat{F}_y^2 \right] - \frac{1}{\eta^2} \hat{C}_{\mu\nu\rho\sigma}^2 \right\}. \quad (14)$$

The kinetic mixing is removed by a transformation [47] to new (‘primed’) fields

$$\hat{\omega}_\mu = \gamma \omega'_\mu \sec \tilde{\chi}, \quad \hat{B}_\mu = B'_\mu - \omega'_\mu \tan \tilde{\chi}, \quad \text{with} \\ \sin \tilde{\chi} \equiv \gamma \sin \chi, \quad (15)$$

⁵ Note that ω_μ is C-even [8, 9] and the photon is C-odd; the mixing violates C and CP. Global or discrete symmetries (C, CP, Z_2 etc) can be used to forbid the kinetic mixing; such symmetries can however be broken by black-hole physics/gravity [5]. Also, the CPT invariance theorem applies only if the theory is local, unitary and in flat space-time, so it cannot be used here: the Weyl-geometry actions are neither unitary (\tilde{C}^2 term in Weyl action has a ghost) nor in flat space-time. The consequences of $\chi \neq 0$ are further studied in Sect. 2.7.

where, for a simpler notation, we introduced $\sin \tilde{\chi}$ (note that $\gamma \leq 1$).⁶ The result is

$$\mathcal{L}_1 = \sqrt{\hat{g}} \left\{ -\frac{1}{2} M_p^2 \hat{R} + \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 + \frac{3}{4} M_p^2 \alpha^2 \gamma^2 \sec \tilde{\chi}^2 \omega'_\mu \omega'^\mu - \frac{1}{4} (F'_{\mu\nu}^2 + F'_{\nu\mu}^2) - \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 \right\}, \quad (16)$$

with $F'_{\mu\nu} = \nabla_\mu B'_\nu - \nabla_\nu B'_\mu = \partial_\mu B'_\nu - \partial_\nu B'_\mu$ and $F'_{\mu\nu} = \partial_\mu \omega'_\nu - \partial_\nu \omega'_\mu$.

As in the previous section, we obtain again the Einstein–Proca action but with diagonal gauge kinetic terms for both gauge fields. However, the final, canonical hypercharge gauge field B'_μ has acquired a dependence on the Weyl gauge field, see (15), due to the initial kinetic mixing. In the full model, upon the electroweak symmetry breaking the photon field (A_μ) is a mixing of the hypercharge (B'_μ) with $SU(2)_L$ neutral gauge field (A_μ^3)

$$A_\mu = B'_\mu \cos \theta_w + A_\mu^3 \sin \theta_w = [\hat{B}_\mu + \hat{\omega}_\mu \sin \chi] \cos \theta_w + \sin \theta_w A_\mu^3. \quad (17)$$

where θ_w is the Weinberg angle and in the second step we used Eq. (15).

Due to the gauge kinetic mixing the photon field includes a small component of the initial Weyl gauge field, suppressed by $\sin \chi$ and by the mass ($\sim M_p$) of ω_μ , but still present;⁷ however, it exists only in the presence of matter e.g. fermionic fields that act as the source of B_μ . Such mixing in models with Abelian gauge fields beyond the hypercharge exists in string models, with similar massive *and* anomaly-free gauge fields (as ω_μ , see later) and similar mass mechanism [48, 49]. However, here ω_μ is a gauge field of a space-time (dilatation) symmetry. The mixing is not forbidden by the Coleman–Mandula theorem – the overall symmetry is always a direct product $U(1)_Y \times D(1)$ and both symmetries are subsequently broken spontaneously.⁸

2.3 Fermions

Consider now the SM fermions (ψ) in Weyl geometry and examine their action. To begin with, to avoid a complicated

⁶ In the limit $\gamma = 1$ there is no $\tilde{C}_{\mu\nu\rho\sigma}^2$ term in the initial action (formally $\eta \rightarrow \infty$).

⁷ In some sense this says that Weyl's unfortunate attempt to identify ω_μ to the photon was not entirely wrong, if the aforementioned mixing is present.

⁸ The theorem implies that $D(1)$ cannot be part of an internal non-Abelian symmetry so d cannot be fixed.

notation we do not display the SM gauge group dependence:

$$\mathcal{L}_f = \frac{1}{2} \sqrt{g} \bar{\psi} i \gamma^a e_a^\mu \tilde{\nabla}_\mu \psi + h.c.,$$

$$\tilde{\nabla}_\mu \psi = \left(\partial_\mu - \frac{3}{4} \alpha \hat{\omega}_\mu + \frac{1}{2} \tilde{s}_\mu^{ab} \sigma_{ab} \right) \psi. \quad (18)$$

Here \tilde{s}_μ^{ab} is the Weyl geometry spin connection. In (18), the Weyl charge of the fermions is $(-3/4)$ according to our convention in (1) ($d = 1$). The relation of the Weyl spin connection to the spin connection s_μ^{ab} of (pseudo-)Riemannian geometry is (see Appendix A)

$$\tilde{s}_\mu^{ab} = s_\mu^{ab} + \frac{1}{2} \alpha (e_\mu^a e_\nu^b - e_\mu^b e_\nu^a) \hat{\omega}_\nu,$$

$$s_\mu^{ab} = (-1) e^{\lambda b} (\partial_\mu e_\lambda^a - e_\nu^a \Gamma_{\mu\lambda}^\nu), \quad (19)$$

where $\sigma_{ab} = \frac{1}{4} [\gamma_a, \gamma_b]$ while $\Gamma_{\mu\lambda}^\nu$ is the Levi–Civita connection, $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$ and $e_\mu^\mu e_\nu^\nu = \delta_\nu^\mu$. It can be checked that, similar to the Weyl connection ($\tilde{\Gamma}$), the Weyl spin connection \tilde{s}_μ^{ab} is invariant under (1). This is seen by using that s_μ^{ab} transforms under (1) as

$$\hat{s}_\mu^{ab} = s_\mu^{ab} + (e_\mu^a e_\nu^b - e_\mu^b e_\nu^a) \partial_\nu \ln \Sigma^{1/2}. \quad (20)$$

With \tilde{s}_μ^{ab} invariant, one checks that \mathcal{L}_f is Weyl gauge invariant. In fact one can easily show that $(-3/4)\alpha \hat{\omega}_\mu$ in $\gamma^\mu \tilde{\nabla}_\mu \psi$ is cancelled by the $\hat{\omega}_\mu$ -presence in the Weyl spin connection. This cancellation also happens between fermions and anti-fermions [8, 9] (eqs. 36, 37).⁹ This is so because both fermions and anti-fermions have the same *real* Weyl charge (no i factor in $\tilde{\nabla}_\mu \psi$). As a result, we have

$$\mathcal{L}_f = \frac{1}{2} \sqrt{g} \bar{\psi} i \gamma^a e_a^\mu \nabla_\mu \psi + h.c.,$$

$$\nabla_\mu \psi = \left(\partial_\mu + \frac{1}{2} s_\mu^{ab} \sigma_{ab} \right) \psi. \quad (21)$$

Thus the SM fermions do not couple [8, 9] to the Weyl field ω_μ and there is no gauge anomaly.

We can now restore the SM gauge group dependence and the Lagrangian becomes

$$\mathcal{L}_f = \frac{1}{2} \sqrt{g} \bar{\psi} i \gamma^a e_a^\mu \left[\partial_\mu - ig \vec{T} \vec{A}_\mu - i Y g' \hat{B}_\mu + \frac{1}{2} s_\mu^{ab} \sigma_{ab} \right] \psi + h.c., \quad (22)$$

with the usual quantum numbers of the fermions under the SM group (not shown), $\vec{T} = \vec{\sigma}/2$, and with g and g' the gauge couplings of $SU(2)_L$ and $U(1)_Y$. But this is not the final result.

⁹ But only for the Weyl charge in (18) can we write a Weyl invariant \mathcal{L}_f without a scalar compensator in [8, 9].

Since the fermions are $U(1)_Y$ charged and the initial field \hat{B}_μ in (22) is shifted by the gauge kinetic mixing, as seen in Eq. (15), then ω'_μ is still present in \mathcal{L}_f :

$$\mathcal{L}_f = \frac{1}{2} \sqrt{g} \bar{\psi} i \gamma^a e_a^\mu \left[\partial_\mu - ig \vec{T} \vec{A}_\mu - i Y g' \left(B'_\mu - \omega'_\mu \tan \tilde{\chi} \right) + \frac{1}{2} s_\mu^{ab} \sigma_{ab} \right] \psi + h.c. \quad (23)$$

We found a new coupling of the SM fermions to ω'_μ , of strength $Y g' \tan \tilde{\chi}$. This coupling comes with the usual fermions hypercharge assignment (which is anomaly-free). After the electroweak symmetry breaking B'_μ is replaced in terms of the mass eigenstates A_μ , Z_μ , Z_μ^ω and ω'_μ is a combination of Z_μ , Z_μ^ω (see later, Eq. (47)). If $\chi \sim \tilde{\chi} = 0$, the fermions Lagrangian is identical to that in the (pseudo)Riemannian case, with no Weyl gauge symmetry.

Regarding the Yukawa interactions notice that the SM Lagrangian is invariant under (1)

$$\mathcal{L}_Y = \sqrt{g} \sum_{\psi=l,q} \left[\bar{\psi}_L Y_\psi H \psi_R + \bar{\psi}_L Y'_\psi \tilde{H} \psi'_R \right] + h.c. \quad (24)$$

where H is the Higgs $SU(2)_L$ doublet and $\tilde{H} = i\sigma_2 H^\dagger$, the sum is over leptons and quarks; Y , Y' are the SM Yukawa matrices. \mathcal{L}_Y is invariant under (1): indeed, since the Weyl charge is real, the sum of charges of the fields in each Yukawa term is vanishing: two fermions (charge $2 \times (-3)/4$), the Higgs (charge $-1/2$) and \sqrt{g} (charge 2). Hence the Yukawa interactions have the same form as in SM in the (pseudo)Riemannian space-time.

2.4 Gauge bosons

Regarding the SM gauge bosons, their SM action is invariant under transformation (1) [8,9]. A way to understand this is that a gauge boson of the SM enters under the corresponding covariant derivative acting on a field charged under it and should transform (have same weight) as ∂_μ acting on that field; since coordinates are kept fixed under (1), the gauge fields do not transform either. Their kinetic terms are then similar to those of the SM in flat space-time, since the Weyl connection is symmetric. Explicitly, this is seen from the equation below, where the sum is over the SM gauge group factors: $SU(3) \times SU(2)_L \times U(1)_Y$

$$\mathcal{L}_g = - \sum_{\text{groups}} \frac{\sqrt{g}}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma}, \quad (25)$$

$F_{\mu\nu}$ involves the difference $\tilde{\nabla}_\mu A_\nu - \tilde{\nabla}_\nu A_\mu$, where A is a generic notation for a SM gauge boson and since $\tilde{\nabla}_\mu A_\nu = \partial_\mu A_\nu - \tilde{\Gamma}_{\mu\nu}^\rho A_\rho$, then for a symmetric $\tilde{\Gamma}_{\mu\nu}^\rho = \tilde{\Gamma}_{\nu\mu}^\rho$ one sees that $\tilde{\Gamma}$ and its ω_μ -dependence cancel out in the field strength

$F_{\mu\nu}$. Hence, \mathcal{L}_g does not depend on ω_μ and has the same form in Weyl and in (pseudo)Riemannian geometries.

2.5 Higgs sector

• **The action:** Let us now consider the SM Higgs doublet (H) in Weyl conformal geometry:

$$\mathcal{L}_H = \sqrt{g} \left\{ \frac{\tilde{R}^2}{4! \xi^2} - \frac{\tilde{C}_{\mu\nu\rho\sigma}^2}{\eta^2} - \frac{\xi_h}{6} |H|^2 \tilde{R} + |\tilde{D}_\mu H|^2 - \lambda |H|^4 - \frac{1}{4} \left(F_{\mu\nu}^2 + 2 \sin \chi F_{\mu\nu} F_y^{\mu\nu} + F_{y\mu\nu}^2 \right) \right\}. \quad (26)$$

The $SU(2)_L \times U(1)_Y \times D(1)$ derivative acting on H is

$$\tilde{D}_\mu H = [\partial_\mu - i \mathcal{A}_\mu - (1/2) \alpha \omega_\mu] H, \quad (27)$$

where $\mathcal{A}_\mu = (g/2) \vec{\sigma} \cdot \vec{A}_\mu + (g'/2) B_\mu$; \vec{A}_μ is the $SU(2)_L$ gauge boson, B_μ is the $U(1)_Y$ boson. The case of no gauge kinetic mixing in (26) ($\chi = 0$) is obvious; we keep $\chi \neq 0$ for generality.

We consider the electroweak unitary gauge where $H = (1/\sqrt{2}) h \zeta$, with $\zeta^T \equiv (0, 1)$. Then

$$|\tilde{D}_\mu H|^2 = |(\partial_\mu - \alpha/2 \omega_\mu) H|^2 + H^\dagger \mathcal{A}_\mu \mathcal{A}^\mu H, \quad (28)$$

with

$$H^\dagger \mathcal{A}_\mu \mathcal{A}^\mu H = (h^2/8) \mathcal{Z}, \quad \mathcal{Z} \equiv [g^2 (A_\mu^{12} + A_\mu^{22}) + (g A_\mu^3 - g' B_\mu)^2]. \quad (29)$$

As done earlier, in \mathcal{L}_H replace $\tilde{R}^2 \rightarrow -2\phi_0^2 \tilde{R} - \phi_0^4$ to find a classically equivalent action; using the equation of motion of ϕ_0 and its solution $\phi_0^2 = -\tilde{R}$ back in the action, one recovers (26). After this replacement, the non-minimal coupling term in (26) is modified

$$-\frac{1}{6} \xi_h |H|^2 \tilde{R} \rightarrow \frac{-1}{12} \left(\frac{1}{\xi^2} \phi_0^2 + \xi_h h^2 \right) \tilde{R}. \quad (30)$$

It is interesting to notice that the initial term in the action, $(1/\xi^2) \tilde{R}^2$, (where $\xi < 1$) in (26) was replaced by a term above with a large non-minimal coupling $1/\xi^2 > 1$ (plus an additional ϕ_0^4). For details, the Lagrangian \mathcal{L}_H after step (30) is presented in the Appendix, see Eq. (C-1).

Next, to fix the gauge, apply transformation (1) to \mathcal{L}_H with a special *scale-dependent* Σ which fixes the fields combination $(\phi_0^2/\xi^2 + \xi_h h^2)$ to a constant:

$$\hat{g}_{\mu\nu} = \Sigma g_{\mu\nu}, \quad \hat{\phi}_0^2 = \frac{\phi_0^2}{\Sigma}, \quad \hat{\omega}_\mu = \omega_\mu - \frac{1}{\alpha} \partial_\mu \ln \Sigma, \quad \hat{B}_\mu = B_\mu, \\ \hat{\mathcal{A}}_\mu = \mathcal{A}_\mu, \quad \Sigma = \frac{\phi_0^2/\xi^2 + \xi_h h^2}{\langle \phi_0^2/\xi^2 + \xi_h h^2 \rangle}. \quad (31)$$

In terms of the transformed fields and metric (with a ‘hat’), \mathcal{L}_H becomes

$$\mathcal{L}_H = \sqrt{\hat{g}} \left\{ -\frac{1}{2} M_p^2 \left[\hat{R} - 3\alpha \nabla_\mu \hat{\omega}^\mu - \frac{3}{2} \alpha^2 \hat{\omega}_\mu \hat{\omega}^\mu \right] - \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 \right. \\ \left. + \frac{1}{2} |(\partial_\mu - \alpha/2 \hat{\omega}_\mu) \hat{h}|^2 + \frac{1}{8} \hat{h}^2 \hat{\mathcal{Z}} \right. \\ \left. - \hat{V} - \frac{1}{4} \left(\frac{1}{\gamma^2} \hat{F}_{\mu\nu}^2 + 2 \sin \chi \hat{F}_{\mu\nu} \hat{F}_y^{\mu\nu} + \hat{F}_y^2 \right) \right\}, \quad (32)$$

where we used (5), the notation $\hat{\mathcal{Z}} = \mathcal{Z}(B_\mu \rightarrow \hat{B}_\mu, \vec{A}_\mu \rightarrow \hat{\vec{A}}_\mu)$, with $\gamma \leq 1$ defined in (7) and

$$M_p^2 \equiv \frac{1}{6} \left\{ \frac{1}{\xi^2} \langle \phi_0^2 \rangle + \xi_h \langle h^2 \rangle \right\}, \quad (33)$$

and finally

$$\hat{V} = \frac{1}{4!} \left[6\lambda \hat{h}^4 + \xi^2 (6M_p^2 - \xi_h \hat{h}^2)^2 \right]. \quad (34)$$

We found again a massive ω_μ in (32) by Stueckelberg mechanism after ‘eating’ the radial direction field $(1/\xi^2) \phi_0^2 + \xi_h h^2$, with constraint $\nabla_\mu \omega^\mu = 0$. We identify M_p with the Planck scale; M_p and thus also m_ω receive contributions from both the Higgs and ϕ_0 (due to \hat{R}^2).

The term proportional to ξ^2 in \hat{V} is ultimately due to the $(1/\xi^2) \hat{R}^2$ term in the action and is ultimately responsible for the EW symmetry breaking and for inflation, see later.

Equation (32) contains a mixing term $\hat{\omega}^\mu \partial_\mu \hat{h}$ from the Weyl-covariant derivative of \hat{h} . We choose the unitary gauge for the D(1) symmetry i.e. eliminate this term by replacing

$$\hat{h} = M_p \sqrt{6} \sinh \frac{\sigma}{M_p \sqrt{6}}, \\ \hat{\omega}_\mu = \hat{\omega}'_\mu + \frac{1}{\alpha} \partial_\mu \ln \cosh^2 \frac{\sigma}{M_p \sqrt{6}}. \quad (35)$$

Then \mathcal{L}_H becomes

$$\mathcal{L}_H = \sqrt{\hat{g}} \left\{ -\frac{1}{2} M_p^2 \hat{R} - \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 \right. \\ \left. + \frac{3}{4} M_p^2 \alpha^2 \hat{\omega}'_\mu \hat{\omega}'^\mu \cosh^2 \frac{\sigma}{M_p \sqrt{6}} \right. \\ \left. + \frac{1}{2} (\partial_\mu \sigma)^2 - \hat{V} + \frac{3}{4} M_p^2 \hat{\mathcal{Z}} \sinh^2 \frac{\sigma}{M_p \sqrt{6}} \right. \\ \left. - \frac{1}{4} \left(\frac{1}{\gamma^2} \hat{F}_{\mu\nu}^2 + 2 \sin \chi \hat{F}'_{\mu\nu} \hat{F}_y^{\mu\nu} + \hat{F}_y^2 \right) \right\}. \quad (36)$$

with the potential \hat{V} expressed now in terms of the field σ , using (34), (35).

The term $(3/4) M_p^2 \alpha^2 \hat{\omega}'_\mu \hat{\omega}'^\mu \cosh^2 \sigma / (M_p \sqrt{6})$ contains a leading coupling $(1/8) \alpha^2 \hat{\omega}'_\mu \hat{\omega}'^\mu \sigma^2$ (expand for $\sigma \leq M_p$), with additional corrections suppressed by M_p^2 . If there is no

kinetic mixing, $\chi = 0$, this is the only coupling of ω_μ to the Higgs and the SM states!

• **Kinetic mixing:** Finally, remove the gauge kinetic mixing in \mathcal{L}_H by replacing $\hat{\omega}'_\mu, \hat{B}_\mu$ by

$$\hat{\omega}'_\mu = \gamma \omega'_\mu \sec \tilde{\chi}, \quad \hat{B}_\mu = B'_\mu - \omega'_\mu \tan \tilde{\chi}, \\ (\sin \tilde{\chi} \equiv \gamma \sin \chi); \quad (37)$$

and \mathcal{L}_H becomes:

$$\mathcal{L}_H = \sqrt{\hat{g}} \left\{ -\frac{1}{2} M_p^2 \hat{R} - \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 \right. \\ \left. + \frac{3}{4} M_p^2 \alpha^2 \gamma^2 (\sec^2 \tilde{\chi}) \omega'_\mu \omega'^\mu + \frac{1}{2} (\partial_\mu \sigma)^2 - \hat{V} \right. \\ \left. + \frac{3}{4} M_p^2 \left[\mathcal{Z}' + \alpha^2 \gamma^2 (\sec^2 \tilde{\chi}) \omega'_\mu \omega'^\mu \right] \sinh^2 \frac{\sigma}{M_p \sqrt{6}} \right. \\ \left. - \frac{1}{4} (F'_{\mu\nu}^2 + F'_y \mu\nu^2) \right\}. \quad (38)$$

where $F' (F'_y)$ is the field strength of $\omega' (B')$ and

$$\mathcal{Z}' = \left[g' (B'_\mu - \omega'_\mu \tan \tilde{\chi}) - g \hat{A}_\mu^3 \right]^2 + g^2 (\hat{A}_\mu^{12} + \hat{A}_\mu^{22}) \quad (39)$$

Note the presence in \mathcal{L}_H of a coupling $\Delta \mathcal{L}_H = (1/8) \sigma^2 \omega'_\mu \omega'^\mu (g'^2 \tan^2 \tilde{\chi} + \alpha^2 \gamma^2 \sec^2 \tilde{\chi})$; this is due to 1) the gauge kinetic mixing χ and 2) to the Higgs coupling to ω_μ , Eq. (27). This coupling is non-zero even if there is no gauge kinetic mixing ($\chi = 0$) when it becomes $\Delta \mathcal{L}_H = (\alpha^2 \gamma^2 / 8) \sigma^2 \omega'_\mu \omega'^\mu$. This is relevant for Higgs physics and can constrain α .

• **Higgs potential:** One may write \mathcal{L}_H in a more compact form

$$\mathcal{L}_H = \sqrt{\hat{g}} \left\{ -\frac{1}{2} M_p^2 \hat{R} - \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 - \frac{1}{4} (F'_{\mu\nu}^2 + F'_y \mu\nu^2) \right. \\ \left. + \mathcal{L}_h + m_W^2(\sigma) W_\mu^- W^{+\mu} + \frac{1}{2} X^T \mathcal{M}^2(\sigma) X \right\} \quad (40)$$

with the σ -dependent mass $m_W(\sigma)$ of $SU(2)_L$ bosons $W_\mu^\pm = 1/\sqrt{2} (A_\mu^1 \mp i A_\mu^2)$ given by

$$m_W^2(\sigma) = \frac{3g^2}{2} M_p^2 \sinh^2 \frac{\sigma}{M_p \sqrt{6}} = \frac{g^2}{4} \sigma^2 + \mathcal{O}(\sigma^4 / M_p^2). \quad (41)$$

The σ -dependent matrix $\mathcal{M}(\sigma)$ written in Eq. (40) in the basis $X \equiv (B'_\mu, A_\mu^3, \omega'_\mu)$ is presented in the Appendix, Eq. (C-3). Finally we have

$$\mathcal{L}_h = \frac{1}{2} (\partial_\mu \sigma)^2 - \hat{V}(\sigma) \quad (42)$$

and

$$\hat{V}(\sigma) = \frac{3}{2} M_p^4 \left\{ 6\lambda \sinh^4 \frac{\sigma}{M_p \sqrt{6}} + \xi^2 \left(1 - \xi_h \sinh^2 \frac{\sigma}{M_p \sqrt{6}} \right)^2 \right\} \quad (43)$$

$$= \frac{1}{4} \left(\lambda - \frac{1}{9} \xi_h \xi^2 + \frac{1}{6} \xi_h^2 \xi^2 \right) \sigma^4 - \frac{1}{2} \xi_h \xi^2 M_p^2 \sigma^2 + \frac{3}{2} \xi^2 M_p^4 + \mathcal{O}(\sigma^6/M_p^2). \quad (44)$$

This is the Higgs potential in our SMW model in the unitary gauge for the EW and $D(1)$ symmetries. The second line is valid for small field values $\sigma \ll M_p$ when we recover a Higgs potential similar to that in the SM; the quadratic term has a negative coefficient (with $\xi_h > 0$, as needed for inflation, see later). This follows when the Higgs field contributes positively to the Planck scale, Eq. (33) and “to compensate” for its contribution to M_p , a negative sign emerges in (34) and in $\hat{V}(\sigma)$. The EW symmetry is thus broken at tree level.

2.6 EW scale and Higgs mass

The small field regime $\sigma \ll M_p$ in (44) gives realistic predictions in the limit $\xi_h \xi^2 \ll 1$; indeed, in this case the quartic Higgs coupling becomes λ and the EW scale $\langle \sigma \rangle$ and the Higgs mass are

$$\langle \sigma \rangle^2 = \frac{1}{\lambda} \xi_h \xi^2 M_p^2, \quad m_\sigma^2 = 2 \xi_h \xi^2 M_p^2. \quad (45)$$

To comply with the values of the Higgs mass and EW vev we must set $\xi \sqrt{\xi_h} \sim 3.5 \times 10^{-17}$. This means one or both perturbative couplings ξ_h and ξ take small values, while $\lambda \sim 0.12$ as in the SM and the regime $\sigma \ll M_p$ is respected. Recall that ξ is the coupling of the term $(1/\xi^2)\tilde{R}^2$, hence we see the relevance of this term for the hierarchy of scales.

The SMW model with the Higgs action as in Eqs. (26), (40) has similarities to Agravity [50, 51] which is a global scale invariant model. Unlike in Agravity, we only have the Higgs scalar, while the role of the second scalar field (s) in [50], that generated the Planck scale and Higgs mass in Agravity is played in our model by the “geometric” Stueckelberg field (ϕ_0). This field was not added “ad-hoc” and cannot couple to the Higgs field, being extracted from the \tilde{R}^2 term itself (see Eq. (26)). Hence, there is no classical coupling between the Higgs field and the field generating M_p in SMW, while in [50] a coupling $\lambda_H s h^2 s^2$ is present.

However, the SMW contains the field ω_μ (part of Weyl geometry), not present in [50]. Our preference here for a local, gauged scale symmetry, that brought in the Weyl gauge field, is motivated by three aspects: firstly, we already have a “geometric” mass generation mechanism which does not need adding ad-hoc an extra scalar; secondly, global sym-

metries do not survive black-hole physics [5] and finally, the Weyl gauge symmetry of the action is also a symmetry of the underlying geometry (connection $\tilde{\Gamma}$), as it should be the case.

Although the quantum corrections to m_σ deserve a separate study, note that large corrections to m_σ could arise from quantum corrections due to ω_μ , via coupling $\omega_\mu w^\mu \sigma^2$. But ω_μ may in principle be light ($m_\omega \sim \alpha M_p$) possibly near the TeV scale [39, 40], rather than near M_p , if $\alpha \ll 1$; this is possible if the Weyl gauge symmetry breaking scale is low. The mass of ω_μ is then the only physical scale for the low-energy observer above which the full gauged scale invariant action is restored together with its ultraviolet (UV) protection role for m_σ . Hence, if the mass of ω_μ is not far above TeV-scale, its loop corrections to m_σ can be under control. In this way the Weyl gauge symmetry may protect the Higgs sector.

From (45), using the Planck scale expression Eq. (33) then

$$\langle \sigma \rangle^2 \approx \frac{\xi_h}{6\lambda} \langle \phi_0^2 \rangle. \quad (46)$$

With $\xi \sqrt{\xi_h} \sim 3 \times 10^{-17}$ fixed earlier, one still has a freedom of either a hierarchy or comparable values of these two vev’s, depending on the exact values of $\xi_h < 1$. Equation (46) relates the EW scale physics to the underlying Weyl geometry represented by the \tilde{R}^2 term in the action (from which ϕ_0 is “extracted”).

2.7 Constraints from Z mass

Let us now compute the eigenvalues of the Higgs-dependent matrix $\mathcal{M}^2(\sigma)$, Eqs. (40), (C-3), and examine the constraints from the mass of Z on the model parameters α and χ . Since Z_μ and ω_μ mix, part of Z boson mass is not due the Higgs mechanism, but to this mixing and ultimately, to the Stueckelberg mechanism giving mass to ω_μ . After the electroweak symmetry breaking, in the mass eigenstates basis of $\mathcal{M}^2(\langle \sigma \rangle)$, one has the photon field (A_μ) (it is massless, since $\det \mathcal{M}^2 = 0$), the neutral gauge boson (Z) and the Weyl field (Z^ω).

$\mathcal{M}^2(\sigma)$ is brought to diagonal form by two rotations (C-4), (C-5) giving

$$\begin{pmatrix} B'_\mu \\ A_\mu^3 \\ \omega'_\mu \end{pmatrix} = \begin{pmatrix} \cos \theta_w & -\sin \theta_w \cos \zeta & -\sin \theta_w \sin \zeta \\ \sin \theta_w & \cos \theta_w \cos \zeta & \cos \theta_w \sin \zeta \\ 0 & -\sin \zeta & \cos \zeta \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_\mu \\ Z_\mu^\omega \end{pmatrix} \quad (47)$$

Denote by U the matrix relating the gauge eigenstates (B'_μ , A_μ^3 , ω'_μ) to the mass eigenstates (A_μ , Z_μ , Z_μ^ω); then $\mathcal{M}^2(\sigma)$ is diagonalised into $\mathcal{M}_d^2 = U^T \mathcal{M}^2 U$ for a suitable ζ

$$\tan 2\zeta = \frac{-2g'(g^2 + g'^2)^{1/2}}{g^2(1 - 2\delta^2)\csc 2\tilde{\chi} + (g^2 + 2g'^2)\cot 2\tilde{\chi}} \quad \text{with}$$

$$\delta^2 = \frac{\alpha^2\gamma^2}{g^2} \coth^2 \frac{\langle\sigma\rangle}{M_p\sqrt{6}}. \quad (48)$$

The masses of Z boson (m_Z) and Weyl gauge field (m_ω) are then found¹⁰

$$m_{Z,\omega}^2 = \frac{3}{4}M_p^2 \sinh^2 \frac{\langle\sigma\rangle^2}{M_p\sqrt{6}} \left\{ g^2 + \frac{1}{2}\sec^2 \tilde{\chi} \left[2g'^2 + 2\alpha^2\gamma^2 \coth^2 \frac{\langle\sigma\rangle}{M_p} \pm \sqrt{\mathcal{P}} \right] \right\}, \quad (50)$$

where $\mathcal{P} = 4g'^2(g^2 + g'^2)\sin^2 2\tilde{\chi}$

$$+ \left[g^2(1 - 2\delta^2) + (g^2 + 2g'^2)\cos 2\tilde{\chi} \right]^2. \quad (51)$$

Since $\langle\sigma\rangle \ll M_p$ (see conditions after Eq. (45))

$$m_Z^2 = \frac{1}{4}(g^2 + g'^2)\langle\sigma\rangle^2 \left\{ 1 + \frac{\langle\sigma\rangle^2}{18M_p^2} \left[1 - \frac{3g'^2}{\alpha^2} \sin^2 \chi \right] + \mathcal{O}(\langle\sigma\rangle^4/M_p^4) \right\}. \quad (52)$$

The factor in front is the mass of Z boson (hereafter m_{Z^0}) in the SM; m_Z has a negligible correction from Einstein gravity ($\propto \langle\sigma^2\rangle/M^2$). But there is also a correction ($\propto \sin^2 \chi/\alpha^2$) from the Weyl field i.e. due to deviations from Einstein gravity induced by Weyl geometry. This can be significant and it reduces m_Z by a relative amount:

$$\varepsilon \equiv \frac{\Delta m_Z}{m_{Z^0}}$$

$$= -\frac{g'^2\langle\sigma\rangle^2}{12M_p^2} \frac{\sin^2 \chi}{\alpha^2} + \mathcal{O}\left(\frac{\langle\sigma\rangle^4}{M_p^4}\right)$$

$$= -\frac{1}{8}\left(\frac{\langle\sigma\rangle}{m_\omega}\right)^2 (g'\tan\tilde{\chi})^2 + \mathcal{O}\left(\frac{\langle\sigma\rangle^4}{m_w^4}\right). \quad (53)$$

In the second step we replaced the mass of ω and the definition of $\tilde{\chi}$ in Eq. (37).

The effect in (53) is significant if $\sin \chi/\alpha \gg 1$. From the mass of Z boson and with Δm_Z at 1σ deviation, one has

¹⁰ If there is no mixing, $\chi = 0$, then in Eq. (48), also (47), $\zeta = 0$, and with M_p of (33) and h of (35) then

$$m_\omega^2 = \frac{3\alpha^2}{2}\gamma^2 M_p^2 \left[1 + \sinh^2 \frac{\langle\sigma\rangle}{M_p\sqrt{6}} \right] \quad (49)$$

$$= \frac{\alpha^2}{4}\gamma^2 \left[(1 + \xi_h)\langle h \rangle^2 + \frac{\langle\phi_0\rangle^2}{\xi^2} \right],$$

$$m_Z^2 = \frac{3}{2}(g^2 + g'^2)M_p^2 \sinh^2 \frac{\langle\sigma\rangle}{M_p\sqrt{6}}.$$

$|\varepsilon| \leq 2.3 \times 10^{-5}$, then Eq. (53) gives a lower bound on the Weyl gauge coupling α , for a given non-zero gauge kinetic mixing:

$$\alpha \geq 2.17 \times 10^{-15} \sin \chi. \quad (54)$$

Note that for an arbitrary charge d of the metric, the results depending on α are modified by replacing $\alpha \rightarrow d \times \alpha$. In terms of the mass of ω_μ one finds

$$\frac{m_\omega}{\text{TeV}} \geq 6.35 \times \tan \tilde{\chi}. \quad (55)$$

This gives a lower bound on the mass of the Weyl field in terms of the mixing angle χ and γ . A larger m_ω allows a larger amount of mixing. For a mixing angle of e.g. $\tilde{\chi} = \pi/4$ then $m_\omega \geq 6.35$ TeV. Note that if there is no term $(1/\eta)\tilde{C}_{\mu\nu\rho\sigma}^2$ in the original gravity action, then $\gamma = 1$ and then $\chi = \tilde{\chi}$. Alternatively, using the current lower bound on the non-metricity scale (represented by m_ω) which is of the order of the TeV scale [39, 40], then

$$\tan \tilde{\chi} \leq 0.16 \quad (56)$$

This is consistent with the non-metricity constraint.

These bounds are significant and affect other phenomenological studies. To give an example, consider the impact of ω_μ on the $g - 2$ muon magnetic moment, due to the new coupling of ω_μ in \mathcal{L}_f , Eq. (23). Using [52, 53] an estimate of the correction of ω_μ to Δa_μ is

$$\Delta a_\mu \sim \frac{1}{12\pi^2} \frac{m_\mu^2}{m_\omega^2} (g'\tan\tilde{\chi})^2 = 2.56 \times 10^{-13}, \quad (57)$$

where we used constraints (53), (55). These do not allow Δa_μ to account for the SM discrepancy with the experiment [54]; however, this discrepancy may be only apparent, according to lattice-based results [55]. One can also use these constraints when studying the role of ω_μ for phenomenology in other examples, such as the dark matter problem [41], in which case it may even provide a solution (of geometric origin!) to this problem; other implications can be for example in the birefringence of the vacuum induced by ω_μ . This can impact on the propagation of the observed polarization of the gamma-ray bursts [56]¹¹ or of the CMB [57].

2.8 Inflation

The SMW model can have successful inflation. The Higgs potential in (43) can drive inflation as discussed in [30, 31, 58]. But who “ordered” the Higgs in the early Universe? the Higgs could initially be produced by the Weyl gauge boson fusion, by the coupling $\omega_\mu\omega^\mu HH^\dagger$ dictated by the symmetry, Eq. (26). This means, rather interestingly, that the Higgs can

¹¹ I thank Tiberiu Harko (Babeş-Bolyai University) for bringing this paper to my attention.

be regarded as having a *geometric origin* like ω_μ which is part of the Weyl connection.¹²

As seen from (38) this coupling becomes $\omega_\mu \omega^\mu f(\sigma)$ with σ the neutral Higgs. But in a Friedmann–Robertson–Walker universe considered below, $g_{\mu\nu} = (1, -a(t)^2, -a(t)^2, -a(t)^2)$, the vector field background compatible with the metric is $\omega_\mu(t) = 0$ [31]. The fluctuations of σ and of (longitudinal component of) ω_μ do not mix since $\omega_\mu(t) \delta \omega^\mu \delta \sigma$ is then vanishing. As a result, the single-field inflation formalism in the Einstein gravity applies, with σ as the inflaton. Since M_p is simply the scale of Weyl gauge symmetry breaking, $\sigma > M_p$ is natural.

The predictions of the Higgs inflation are then [30,31]

$$r = 3(1 - n_s)^2 - \frac{16}{3} \xi_h^2 + \mathcal{O}(\xi_h^3). \quad (58)$$

Here r is the tensor-to-scalar ratio and n_s is the scalar spectral index. Up to small corrections from ξ_h that can be neglected for $\xi_h < 10^{-3}$, the above dependence $r = r(n_s)$ is similar to that in the Starobinsky model [59,60] of inflation where $r = 3(1 - n_s)^2$. For mildly larger $\xi_h \sim 10^{-3} - 10^{-2}$, Eq. (58) departs from the Starobinsky model prediction and r is mildly reduced relative to its value in the Starobinsky case, for given n_s . These results require a hierarchy $\lambda \ll \xi_h^2 \xi^2$ which may be respected by a sufficiently small λ and¹³ $\xi_h \sim 10^{-3} - 10^{-2}$.

A relatively very small λ means that it is actually the squared term in (43) that is multiplied by ξ^2 (see also (34)) that is mostly responsible for inflation, and that is ultimately due to the initial term ϕ_0^4 “extracted” from the initial quadratic curvature $(1/\xi^2) \tilde{R}^2$ term in (26); this then explains the close similarities to the Starobinsky R^2 -inflation. Thus, we actually have a Starobinsky-Higgs inflation. The initial Higgs field h (which has $\xi_h \neq 0$) still plays a role as it brings a minimum in¹⁴ $\hat{V}(\sigma)$ of (43). In conclusion, a negligible λ is required for successful inflation (as the numerical values of r below also show it). This is consistent with SM prediction for λ at the high scales, while a value of λ at the EW scale as in the SM can then be induced by the SM quantum corrections.

The numerical results give that for $N = 60$ efolds and with $n_s = 0.9670 \pm 0.0037$ at 68% CL (TT, TE, EE+low E + lensing + BK14 + BAO) [62] then [30,31,58]¹⁵

$$0.00257 \leq r \leq 0.00303, \quad (59)$$

¹² In a sense this is also true for fermions, by subsequent Higgs decay (24), or for B_μ by Higgs- $\omega_\mu \rightarrow B_\mu$ -Higgs.

¹³ From the normalization of the CMB anisotropy one also finds that $\xi^2 < 1.45 \times 10^{-9}$ [30].

¹⁴ The Higgs and Starobinsky/ R^2 inflation usually mix, especially at the quantum level [61].

¹⁵ Our results quoted above from [30,31] were obtained from potential $\hat{V}(\sigma)$ of Eq. (43) and they agree both analytically and numerically to those in [58] obtained by a different method using a two-field analysis.

while for n_s at 95% CL:

$$0.00227 \leq r \leq 0.00303. \quad (60)$$

The case of Starobinsky model for $N = 60$ corresponds to the upper limit of r above and is reached for the smallest ξ_h , when this limit is saturated, according to relation (58).

The small value of r found above may be reached by the next generation of CMB experiments CMB-S4 [63,64], LiteBIRD [65,66], PICO [67], PIXIE [68] that have sensitivity to r values as low as 0.0005. Such sensitivity will be able to test this inflation model and to distinguish it from other models. For example, similarly small but distinct values of r are found in other models with Weyl gauge symmetry [31,32] based on the Palatini approach to gravity action (4) used in this paper; however these models do not respect relation (58) and the slope of the curve $r(n_s)$ is different, due to their different vectorial non-metricity. The above experiments also have the sensitivity to distinguish inflation in this model from the Starobinsky model for $\xi_h \sim 10^{-2}$ when the curve $r(n_s)$ is shifted by ξ_h below that of the Starobinsky model, towards smaller r (for fixed n_s).

3 SMW and its properties

In this section we discuss some features of our model and the differences from other SM-like models with local scale invariance. The main aspect of our model is that scale symmetry is *gauged*, Eq. (1). The Weyl gauge symmetry is not only a symmetry of the action but also of the underlying Weyl geometry; indeed, the Weyl connection is invariant under (1). This adds consistency to SMW and distinguishes it from models with an action that is Weyl or conformal invariant (with no ω_μ) and built in a (pseudo-)Riemannian space – their connection and thus their underlying geometry do not share this symmetry of the action.

An important feature of the SMW is the spontaneous breaking of Weyl gauge symmetry even in the *absence of matter*, as seen in Sect. 2.1. Hence, this breaking is ultimately of geometric origin. This is different from previous models with this symmetry [7–27] where some scalar fields were introduced “ad-hoc” to induce spontaneous breaking of their symmetry and to generate M_p and Einstein action from a $\phi^2 R$ term. In the SMW the necessary scalar field (ϕ_0) is “extracted” from the (geometric) R^2 -term, plays the role of the Stueckelberg field and is eaten by ω_μ which becomes massive. This was possible since the model was quadratic in curvature – this is another difference from models [7–27] which were linear-only in R . Therefore, the Einstein–Proca action and the Planck scale emerge in the broken phase of the SMW.

The breaking of the Weyl gauge symmetry is accompanied by a change of the underlying geometry. When massive ω_μ

decouples at some (high) scale, the Weyl connection becomes Levi–Civita, so Weyl geometry becomes Riemannian and the theory is then metric.¹⁶ Thus, the breaking of the symmetry in Sect. 2.1 (see [28, 29]) is not just a result of a “gauge fixing” to the Einstein frame, as it happens in Weyl or conformal theories with no ω_μ ; it is accompanied by the Stueckelberg mechanism and by a change of the underlying geometry.¹⁷

The SMW avoids some situations present in interesting models with local scale invariance (without ω_μ), like a negative kinetic term of the scalar field [73] (also [74–76]), or an imaginary vev [77–79] of the scalar that generates¹⁸ M_p . Such situations may not be a cause of concern, see however the discussion in [16, 18]. Gauging the scale symmetry avoids such situations – in SMW this scalar field plays the role of a would-be Goldstone of the Weyl gauge symmetry (eaten by ω_μ). See also Eq. (7) where the (negative) kinetic term in the first square bracket is cancelled by that in the second square bracket corresponding to a Stueckelberg mechanism.¹⁹

In local scale invariant models (without ω_μ) the associated current can be trivial, leading to so-called “fake conformal symmetry” [81, 82]; in the SMW the current is non-trivial even in the absence of matter [28, 29] due to dynamical ω_μ . If ω_μ were not dynamical ($F_{\mu\nu} = 0$) it could be integrated out algebraically to leave a local scale-invariant action [28, 29]; in this case Weyl geometry would be integrable and metric, see e.g. [17, 18]. But since ω_μ is dynamical, the theory is also non-metric. This non-metricity would indeed be a physical problem if ω_μ were massless (assuming this, non-metricity of a theory was used as an argument against such theory by Einstein²⁰ [2]). However, non-metricity became here an advantage, since Weyl geometry with dynamical ω_μ enabled the Stueckelberg breaking mechanism, ω_μ acquired

¹⁶ A similar Weyl gauge symmetry breaking and change of geometry exists in a Palatini version [31, 32].

¹⁷ An aspect of models with Weyl gauge symmetry relates to their geodesic completeness, see [16, 18]. In conformal/Weyl invariant models (without ω_μ) this aspect seems possible in the (metric) Riemannian spacetime where geodesic completeness or incompleteness is related to a gauge choice (and singularities due to an unphysical conformal frame) [69–71]. In models in Weyl geometry, the geodesics are determined by the affine structure. Differential geometry demands the existence of the Weyl gauge field [72] for the construction of the affine connection, because this ensures that geodesics are invariant (as necessary on physical grounds, the parallel transport of a vector should not depend on the gauge choice). Hence the Weyl gauge field/symmetry may actually be required! After the breaking of this symmetry, ω_μ decouples, we return to (pseudo)Riemannian geometry and geodesics are then given by extremal proper time condition. Since a dynamical ω_μ also brings in non-metricity, geodesic completeness seems related to non-metricity.

¹⁸ It seems to us this means a negative Σ and therefore a metric signature change in transformation (1).

¹⁹ This Stueckelberg mechanism may apply to more general metric affine theories studied in detail in [80].

²⁰ Actually, a similar situation exists [31, 32] in quadratic gravity in Palatini approach due to Einstein [83].

a mass (above current non-metricity bounds [39, 40]), and the Einstein–Proca action was naturally obtained in the broken phase.

The SMW differs from the SM with conformal symmetry of [84] or [77–79] and from conformal gravity models [85–87] formulated in the (pseudo)Riemannian space and based on $C_{\mu\nu\rho\sigma}^2$ term; these models are metric and do not have a gauged scale symmetry; in our case the $C_{\mu\nu\rho\sigma}^2$ term is largely spectator and may even be absent in a first instance; it was included because its Weyl geometry counterpart contributed a threshold correction to α and it is needed at the quantum level. And unlike the conformal gravity action [88] which is metric, the SMW has a gauge kinetic term for the Weyl field, which 1) makes the geometry non-metric and 2) breaks the special conformal symmetry; this symmetry and non-metricity do not seem compatible.

Concerning the quantum calculations in the SMW, one has two options: one can use the “traditional” dimensional regularization (DR) that breaks explicitly the Weyl gauge symmetry by the presence of the subtraction scale (μ); alternatively, one can use a regularisation similar to [89] that preserves Weyl gauge symmetry at the quantum level. This is possible by using our Stueckelberg field ϕ_0 as a *field-dependent regulator*, to replace the subtraction scale μ generated later by $\mu \sim \langle \phi_0 \rangle$ (after symmetry breaking). This would allow the computation of the quantum corrections without explicitly breaking the Weyl gauge symmetry.²¹

It is interesting to study the renormalizability of the Weyl quadratic gravity and of the SMW. The usual (metric) quadratic gravity theory in the (pseudo-)Riemannian case is known to be renormalizable but not unitary due to the massive spin-2 ghost [96]. Considering now the Weyl quadratic gravity alone, note that for computing the quantum corrections Eq. (8) is not appropriate since this is the (non-renormalizable) *unitary gauge* of Weyl gauge symmetry. Therefore, one should consider computing the necessary quantum corrections in the symmetric phase, for example in \mathcal{L}_0 of Eq. (6). Note that no higher order operators are allowed in (4), (6) by the symmetry since there is no initial mass scale to suppress them, and this is an argument in favour of its renormalizability. Finally, regarding the SMW itself, in a Riemannian notation it simply has an additional (anomaly-free) Weyl gauge field which becomes massive by the Stueckelberg mechanism which cannot affect renormalizability; naively, one then expects the SMW be renormalizable.

²¹ A similar approach exists in the global case [90–95].

4 Conclusions

Since the SM with a vanishing Higgs mass parameter is scale invariant, it is natural to study the effect of this symmetry. This is relevant for physics at high scales or in the early Universe, where this symmetry seems natural. Since a *global* scale symmetry does not survive black-hole physics, we explored the possibility that the SM has a *gauged* scale symmetry. The natural framework is the Weyl geometry where this symmetry is built in. Hence, we considered the SM in Weyl geometry. This embedding is minimal i.e. *no new degrees of freedom* were added beyond those of the SM and of Weyl geometry.

The model has the special feature that both the action *and* its underlying geometry (connection $\tilde{\Gamma}$ and spin connection \tilde{w}_μ^{ab}) are Weyl gauge invariant. This adds consistency to the model and distinguishes it from previous SM-like models with local scale symmetry, built in a (pseudo-)Riemannian geometry whose connection is not local scale invariant.

The SMW model has another attractive feature. In Weyl geometry there exists a (geometric) Stueckelberg mechanism in which this symmetry is spontaneously broken. The Weyl quadratic gravity associated to this geometry is broken spontaneously to the Einstein–Proca action of ω_μ . The Stueckelberg field ϕ_0 has a geometric origin, being “extracted” from \tilde{R}^2 in the Weyl action, and is subsequently eaten by ω_μ . Once the Weyl gauge field decouples, the Weyl connection becomes Levi–Civita and Einstein gravity is recovered. The Planck scale and a positive cosmological constant are both generated by the Stueckelberg field vev. Also, the mass term of the Weyl field is on the Weyl geometry side just a Weyl-covariant kinetic term of the same Stueckelberg field. These aspects relate symmetry breaking and thus mass generation to a geometry change (from Weyl to Riemannian) which is itself related to the non-metricity due to dynamical ω_μ .

The SMW gauge group is a direct product of the SM gauge group and $D(1)$ of the Weyl gauge symmetry, both broken spontaneously. In general, it is only the Higgs field of the SM spectrum that couples to ω_μ , (through the term $\omega_\mu \omega^\mu \sigma^2$). The presence of the Weyl gauge symmetry may have a protective role for the Higgs mass at a quantum level, if broken at low scales. This would need $\alpha \ll 1$ and a light ω_μ , possibly few TeV, ($m_\omega \sim \alpha M_p$). The (ultra)weak couplings ξ , α of Weyl quadratic gravity would play a crucial role in providing a solution to the hierarchy problem. A very small ξ is also necessary for successful inflation, which is interesting.

The fermions can acquire a direct coupling ($Y g' \tan \tilde{\chi}$) to ω_μ only in the case of a (very small) kinetic mixing ($\tilde{\chi}$) of the gauge fields of $U(1)_Y \times D(1)$, allowed by this symmetry and present at classical level (or due to quantum corrections). As a result of this mixing and Higgs coupling to ω_μ , part of Z boson mass is not due to the Higgs mechanism, but to the mixing of Z with the massive Weyl field which has a Stueck-

elberg mass; hence, part of Z mass has a geometric origin, due to a departure from the (pseudo-)Riemannian geometry and Einstein gravity. Since the Z boson mass is accurately measured, one finds strong bounds on the Weyl gauge coupling and the mass of ω_μ , for a given amount of kinetic mixing. We showed how these bounds can be used in other phenomenological studies. If ω_μ is light (few TeV) its effects may be amenable to experimental tests, with consequences for phenomenology e.g. for the dark matter, vacuum birefringence, etc, that can test the model.

The SMW has successful inflation. Intriguingly, in the early Universe the Higgs may be produced via Weyl vector fusion, thus having a geometric origin. With M_p a simple phase transition scale in Weyl gravity, Higgs field values larger than M_p are natural. Note that while the inflationary potential is that of the Higgs, due to the scalar fields mixing it is ultimately a contribution to this potential from the initial scalar mode (ϕ_0) in the \tilde{R}^2 term that is actually responsible for inflation. This explains the close similarities to the Starobinsky R^2 -inflation. With the scalar spectral index n_s fixed to its measured value, the tensor-to-scalar ratio $0.00227 \leq r \leq 0.00303$. Compared to the Starobinsky model, the curve $r(n_s)$ is similar but shifted to smaller r (for same n_s) by the Higgs non-minimal coupling (ξ_h) to Weyl geometry. These interesting results deserve further investigation.

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Appendix

A: Brief guide to Weyl conformal geometry

Weyl conformal geometry is defined by equivalent classes of $(g_{\mu\nu}, \omega_\mu)$ of the metric and Weyl gauge field (ω_μ) related by Weyl gauge transformations:

$$\begin{aligned}\hat{g}_{\mu\nu} &= \Sigma^d g_{\mu\nu}, \quad \sqrt{\hat{g}} = \Sigma^{2d} \sqrt{g}, \\ \hat{\omega}_\mu &= \omega_\mu - \frac{1}{\alpha} \partial_\mu \ln \Sigma, \\ \hat{e}_\mu^a &= \Sigma^{d/2} e_\mu^a, \quad \hat{e}_a^\mu = \Sigma^{-d/2} e_a^\mu\end{aligned}\quad (\text{A-1})$$

where d is the Weyl weight (charge) of $g_{\mu\nu}$ and α is the Weyl gauge coupling. Various conventions exist in the literature for d e.g. $d = 1$ in [7] and $d = 2$ in [8,9]. The latter may be more motivated since from the relation $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ with dx^μ and dx^ν fixed under (A-1) the metric $g_{\mu\nu}$ transforms like ds^2 . In the text we used $d = 1$, but our results can be immediately changed to arbitrary d by simply rescaling the coupling in our results $\alpha \rightarrow \alpha \times d$.

The Weyl gauge field is related to the Weyl connection $(\tilde{\Gamma})$ which is the solution of

$$\tilde{\nabla}_\lambda g_{\mu\nu} = -d \alpha \omega_\lambda g_{\mu\nu} \quad (\text{A-2})$$

where $\tilde{\nabla}_\mu$ is defined by $\tilde{\Gamma}_\mu^\lambda$

$$\tilde{\nabla}_\lambda g_{\mu\nu} = \partial_\lambda g_{\mu\nu} - \tilde{\Gamma}_{\mu\lambda}^\rho g_{\rho\nu} - \tilde{\Gamma}_{\nu\lambda}^\rho g_{\mu\rho}. \quad (\text{A-3})$$

Equation (A-2) says that Weyl geometry is *non-metric*; it may be written as $(\tilde{\nabla}_\lambda + d \alpha \omega_\lambda) g_{\mu\nu} = 0$ as in a metric case, indicating that one can use metric formulae in which replaces the partial derivative ∂_λ acting on a field, metric, etc, by a Weyl-covariant counterpart as in:

$$\partial_\lambda \rightarrow \partial_\lambda + \text{weight} \times \alpha \times \omega_\lambda, \quad (\text{A-4})$$

where 'weight' is the corresponding Weyl charge (of the field, etc). We shall use this later.

The solution to (A-2) is found using cyclic permutations of the indices and combining the equations so obtained, then

$$\tilde{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda + \alpha \frac{d}{2} \left[\delta_\mu^\lambda \omega_\nu + \delta_\nu^\lambda \omega_\mu - g_{\mu\nu} \omega^\lambda \right]. \quad (\text{A-5})$$

where $\Gamma_{\mu\nu}^\lambda$ is the usual Levi–Civita connection

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}). \quad (\text{A-6})$$

$\tilde{\Gamma}$ is invariant under (A-1) as one can easily check. Conversely, one may actually derive the transformation of the Weyl gauge field in (A-1) by imposing that $\tilde{\Gamma}$ be invariant under the metric change in (A-1), since parallel transport should be independent of the gauge choice. Taking the trace in the last equation and denoting $\Gamma_\mu \equiv \Gamma_{\mu\lambda}^\lambda$ and $\tilde{\Gamma}_\mu \equiv \tilde{\Gamma}_{\mu\lambda}^\lambda$

then

$$\tilde{\Gamma}_\mu = \Gamma_\mu + 2d \alpha \omega_\mu. \quad (\text{A-7})$$

Thus, the Weyl gauge field can be thought of as the trace of the departure of the Weyl connection from the Levi–Civita connection. Using $\tilde{\Gamma}$ one computes the scalar and tensor curvatures of Weyl geometry, using formulae similar to those in Riemannian case but with $\tilde{\Gamma}$ instead of Γ . For example

$$\begin{aligned}\tilde{R}_{\mu\nu\sigma}^\lambda &= \partial_\nu \tilde{\Gamma}_{\mu\sigma}^\lambda - \partial_\sigma \tilde{\Gamma}_{\mu\nu}^\lambda + \tilde{\Gamma}_{\nu\rho}^\lambda \tilde{\Gamma}_{\mu\sigma}^\rho - \tilde{\Gamma}_{\sigma\rho}^\lambda \tilde{\Gamma}_{\mu\nu}^\rho, \\ \tilde{R}_{\mu\nu} &= \tilde{R}_{\mu\lambda\sigma}^\lambda, \quad \tilde{R} = g^{\mu\sigma} \tilde{R}_{\mu\nu}.\end{aligned}\quad (\text{A-8})$$

After some algebra one finds

$$\begin{aligned}\tilde{R}_{\mu\nu} &= R_{\mu\nu} + \frac{1}{2} (\alpha d) (\nabla_\mu \omega_\nu - 3 \nabla_\nu \omega_\mu - g_{\mu\nu} \nabla_\lambda \omega^\lambda) \\ &\quad + \frac{1}{2} (\alpha d)^2 (\omega_\mu \omega_\nu - g_{\mu\nu} \omega_\lambda \omega^\lambda),\end{aligned}\quad (\text{A-9})$$

$$\tilde{R}_{\mu\nu} - \tilde{R}_{\nu\mu} = 2d \alpha F_{\mu\nu}, \quad (\text{A-10})$$

$$\tilde{R} = R - 3d \alpha \nabla_\mu \omega^\mu - (3/2) (d \alpha)^2 \omega_\mu \omega^\mu, \quad (\text{A-11})$$

where the rhs is in a Riemannian notation, so ∇_μ is given by the Levi–Civita connection (Γ).

An important property is that \tilde{R} transforms covariantly under (A-1)

$$\hat{\tilde{R}} = (1/\Sigma^d) \tilde{R}, \quad (\text{A-12})$$

which follows from the transformation of $g^{\mu\sigma}$ that enters its definition above and from the fact that $\tilde{R}_{\mu\nu}$ is invariant (since $\tilde{\Gamma}$ is so). Then the term $\sqrt{g} \tilde{R}^2$ is Weyl gauge invariant.

In Weyl geometry one can also define a Weyl tensor $\tilde{C}_{\mu\nu\rho\sigma}$ that is related to that in Riemannian geometry $C_{\mu\nu\rho\sigma}$ as follows

$$\begin{aligned}\tilde{C}_{\mu\nu\rho\sigma} &= C_{\mu\nu\rho\sigma} - \frac{\alpha d}{4} (g_{\mu\rho} F_{\nu\sigma} + g_{\nu\sigma} F_{\mu\rho} - g_{\mu\sigma} F_{\nu\rho} \\ &\quad - g_{\nu\rho} F_{\mu\sigma}) + \frac{\alpha d}{2} F_{\mu\nu} g_{\rho\sigma}\end{aligned}\quad (\text{A-13})$$

which gives [20]

$$\tilde{C}_{\mu\nu\rho\sigma}^2 = C_{\mu\nu\rho\sigma}^2 + \frac{3}{2} (\alpha d)^2 F_{\mu\nu}^2, \quad (\text{A-14})$$

used in the text, Eq. (4). $\sqrt{g} \tilde{C}_{\mu\nu\rho\sigma}^2$ and its above separation are invariant under (A-1).

To introduce the Weyl spin connection, consider first the spin connection in the Riemannian geometry

$$\begin{aligned}s_\mu^{ab} &= \frac{1}{2} \left[e^{\nu a} (\partial_\mu e_\nu^b - \partial_\nu e_\mu^b) - e^{\nu b} (\partial_\mu e_\nu^a - \partial_\nu e_\mu^a) \right. \\ &\quad \left. - e^{\rho a} e^{\sigma b} e_\mu^c (\partial_\rho e_{\sigma c} - \partial_\sigma e_{\rho c}) \right].\end{aligned}\quad (\text{A-15})$$

One verifies that an equivalent form is

$$s_\mu^{ab} = -e^{\lambda b} (\partial_\mu e_\lambda^a - \Gamma_{\mu\lambda}^\nu e_\nu^a). \quad (\text{A-16})$$

Under a transformation of the metric (A-1)

$$\hat{s}_\mu^{ab} = s_\mu^{ab} + (e_\mu^a e^{vb} - e_\mu^b e^{va}) \partial_\mu \ln \Sigma^{d/2}. \quad (\text{A-17})$$

For the Weyl geometry spin connection, one simply replaces the partial derivative in Eq. (A-15) by a Weyl-covariant derivative that takes into account the charge of the field on which it acts (A-4). For the spin connection $\partial_\mu e_v^b \rightarrow [\partial_\mu + (d/2)\alpha \omega_\mu] e_v^b$ since according to (A-1) e_v^b has Weyl weight $d/2$. Using this replacement in (A-15) we find the spin connection \tilde{s}_μ^{ab} in Weyl geometry

$$\tilde{s}_\mu^{ab} = s_\mu^{ab} + (d/2)\alpha (e_\mu^a e^{vb} - e_\mu^b e^{va}) \omega_v. \quad (\text{A-18})$$

Under transformation (A-1) one checks that \tilde{s}_μ^{ab} is invariant, similar to Weyl connection $\tilde{\Gamma}$.

Let us now consider matter fields and find their charges in Weyl geometry by demanding that: a) their Weyl-covariant derivatives transform under (A-1) like the fields themselves and b) that their kinetic terms be invariant. More explicitly, take the kinetic term for a scalar of charge d_ϕ : $\sqrt{g}(\tilde{D}_\mu \phi)^2$ where \tilde{D}_μ is the Weyl-covariant derivative which we demand it transform under (A-1) just like the scalar field itself, i.e. it has same charge d_ϕ . From the invariance of this action under (A-1) one has that $d_\phi = -d/2$. The Weyl covariant derivative is then found according to (A-4) and the kinetic term is

$$L_\phi = \sqrt{g} g^{\mu\nu} \tilde{D}_\mu \phi \tilde{D}_\nu \phi, \quad \tilde{D}_\mu \phi = (\partial_\mu - d/2 \alpha \omega_\mu) \phi. \quad (\text{A-19})$$

with L_ϕ invariant, while ϕ transforms as

$$\hat{\phi} = \Sigma^{-d/2} \phi. \quad (\text{A-20})$$

For a fermion ψ the Weyl charge is found in a similar way, by using (A-4) to write their Weyl covariant derivative, hence the action has the form

$$\begin{aligned} L_\psi &= \frac{i}{2} \sqrt{g} \bar{\psi} \gamma^a e_a^\mu \tilde{\nabla}_\mu \psi + h.c., \\ \tilde{\nabla}_\mu \psi &= \left[\partial_\mu + d_\psi \alpha \omega_\mu + \frac{1}{2} \tilde{s}_\mu^{ab} \sigma_{ab} \right] \psi \end{aligned} \quad (\text{A-21})$$

where $\sigma_{ab} = 1/4[\gamma_a, \gamma_b]$. Since we saw earlier that \tilde{s}_μ^{ab} is Weyl gauge invariant then the above derivative $\tilde{\nabla}_\mu \psi$ transforms covariantly just like a fermion field itself of charge d_ψ . From the structure of the kinetic term and its invariance it follows that $d_\psi = -3d/4$ so, under (A-1)

$$\hat{\psi} = \Sigma^{-3d/4} \psi. \quad (\text{A-22})$$

With this charge and using (A-21), (A-18) one shows that ω_μ cancels out:

$$\gamma^a e_a^\mu \tilde{\nabla}_\mu \psi = \gamma^a e_a^\mu \left[\partial_\mu + \frac{1}{2} s_\mu^{ab} \sigma_{ab} \right] \psi. \quad (\text{A-23})$$

Hence, the fermionic kinetic term has the same form as in the Riemannian geometry

$$\begin{aligned} L_\psi &= \frac{i}{2} \sqrt{g} \bar{\psi} \gamma^a e_a^\mu \nabla_\mu \psi + h.c., \\ \nabla_\mu \psi &= \left[\partial_\mu + \frac{1}{2} s_\mu^{ab} \sigma_{ab} \right] \psi, \end{aligned} \quad (\text{A-24})$$

used in Sect. 2.3. Equations (A-1), (A-20), (A-22) define the Weyl gauge transformation in the presence of matter, as introduced in the text, Eq. (1). For more information see also [8, 9, 20].

B: Weyl quadratic gravity: equations of motion and gauge fixing

Here we present the equations of motion of \mathcal{L}_0 of Eq. (6) and derive some results that were used in the text, Sect. 2.1. Variation of \mathcal{L}_0 with respect to $g^{\mu\nu}$ gives

$$\begin{aligned} \frac{1}{\sqrt{g}} \frac{\delta \mathcal{L}_0}{\delta g^{\mu\nu}} &= -\frac{1}{12} \frac{\phi_0^2}{\xi^2} \left(R_{\mu\nu} \right. \\ &\quad \left. - \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{12} \left(g_{\mu\nu} \square - \nabla_\mu \nabla_\nu \right) \frac{\phi_0^2}{\xi^2} \\ &\quad - \frac{\alpha^2}{16} \frac{\phi_0^2}{\xi^2} \left(g_{\mu\nu} \omega^\rho \omega_\rho - 2\omega_\mu \omega_\nu \right) \\ &\quad + \frac{\alpha}{8} \frac{\phi_0^2}{\xi^2} \left(\nabla_\mu \omega_\nu + \nabla_\nu \omega_\mu - g_{\mu\nu} \nabla_\rho \omega^\rho \right) \\ &\quad + \frac{1}{2} g_{\mu\nu} V + \frac{1}{2} \left(\frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} \right). \end{aligned} \quad (\text{B-1})$$

where we denoted $V \equiv \phi_0^4/(4! \xi^2)$. Taking the trace of (B-1)

$$\begin{aligned} \frac{1}{12} \frac{\phi_0^2}{\xi^2} R + \frac{1}{4} \frac{1}{\xi^2} \square \phi_0^2 - \frac{\alpha^2}{8} \frac{\phi_0^2}{\xi^2} \omega_\rho \omega^\rho \\ - \frac{\alpha}{4} \frac{\phi_0^2}{\xi^2} \nabla_\rho \omega^\rho + 2V = 0. \end{aligned} \quad (\text{B-2})$$

The equation of motion of ϕ_0

$$\frac{1}{12} \frac{\phi_0^2}{\xi^2} R - \frac{\alpha^2}{8} \frac{\phi_0^2}{\xi^2} \omega_\rho \omega^\rho - \frac{\alpha}{4} \frac{\phi_0^2}{\xi^2} \nabla_\rho \omega^\rho + \frac{1}{2} \phi_0 \frac{\partial V}{\partial \phi_0} = 0 \quad (\text{B-3})$$

On the ground state this gives $\langle \phi_0^2 \rangle = -\tilde{R} = -[R - (3/2)\alpha^2 \omega_\rho \omega^\rho]$, which we already know from the equation $\phi_0^2 = -\tilde{R}$ introduced to linearise (4) into (6).

The equation of motion of ω_μ

$$\frac{\alpha^2}{4} \frac{\phi_0^2}{\xi^2} \omega^\rho - \frac{\alpha}{4} \frac{1}{\xi^2} g^{\rho\sigma} \nabla_\sigma \phi_0^2 + \nabla_\sigma F^{\rho\sigma} = 0, \quad (\text{B-4})$$

Subtracting (B-2) from (B-3) then

$$\square \phi_0^2 = 0, \quad \Rightarrow \quad \partial^\mu (\sqrt{g} \partial_\mu \phi_0^2) = 0. \quad (\text{B-5})$$

where $\square = \nabla^\mu \nabla_\mu$. In a FRW universe with $g_{\mu\nu} = (1, -a^2(t), -a^2(t), -a^2(t))$, Eq. (B-5) gives $-3H = (d/dt)[\ln(d\phi_0^2/dt)]$ with $H = \dot{a}/a$, so $(d/dt)\phi_0^2 \sim 1/a^3(t)$; therefore, $\phi_0(t)$ evolves to a constant (vev) at large t , and this is used in the text after Eq. (10).

Further, by applying ∇_σ to (B-4) we find a conserved current

$$\nabla_\mu J^\mu = 0, \quad J^\mu = -\frac{\alpha}{4\xi^2} g^{\mu\nu} (\partial_\nu - \alpha \omega_\nu) \phi_0^2. \quad (\text{B-6})$$

where we used the antisymmetry of $F_{\mu\nu}$. This was used in Eq. (11). Since $\square \phi_0^2 = 0$ then

$$\nabla_\mu J^\mu = \frac{\alpha^2}{4\xi^2} \sqrt{g} \nabla_\mu (\omega^\mu \phi_0^2), \quad (\text{B-7})$$

which vanishes. When ϕ_0 acquires a vev (as discussed above), then $\nabla_\mu (\phi_0^2 \omega^\mu) = 0$ becomes $\nabla_\mu \omega^\mu = 0$. This is the gauge fixing condition for a massive gauge field ω_μ for action (8).

Finally, after the decoupling of massive ω_μ from the Einstein–Proca \mathcal{L}_0 of Eq. (8) (together with (12)), the equation of motion for $g^{\mu\nu}$ gives, after taking the trace

$$R = -4\Lambda. \quad (\text{B-8})$$

This equation is also seen from (B-2) in the absence of ω_μ , by replacing $\phi_0/(6\xi^2) \rightarrow M_p^2$. Equation (B-8) is consistent with the equation $\phi_0^2 = -\tilde{R}$ introduced to linearise (4) into (6). To see this, apply (1) to $\phi_0^2 = -\tilde{R}$, which becomes $\langle \phi_0^2 \rangle = -\tilde{R}$ as already found above; after decoupling of massive ω_μ this gives $\langle \phi_0^2 \rangle = -R$. With notation $\Lambda = \langle \phi_0^2 \rangle/4$, then $4\Lambda = -R$, in agreement with (B-8).

C: Higgs sector: \mathcal{L}_H and the matrix $\mathcal{M}^2(\sigma)$

For convenience, we write here in the Riemannian notation and in the symmetric phase the form of \mathcal{L}_H shown in the text in the Weyl geometry notation Eq. (26) after step (30)

$$\begin{aligned} \mathcal{L}_H = \sqrt{g} \Big\{ & \frac{-1}{2} \left[\frac{1}{6} \theta^2 R + (\partial_\mu \theta)^2 - \frac{\alpha}{2} \nabla_\mu (\theta^2 \omega^\mu) \right] \\ & - \frac{1}{\eta^2} C_{\mu\nu\rho\sigma}^2 + \frac{1}{8} \alpha^2 \theta^2 \left[\omega_\mu - \frac{1}{\alpha} \nabla_\mu \ln \theta^2 \right]^2 - V \\ & + \frac{1}{2} |(\partial_\mu - \alpha/2 \omega_\mu) h|^2 + \frac{1}{2} h^2 \mathcal{A}_\mu \mathcal{A}^\mu \\ & - \frac{1}{4} \left[\frac{1}{\gamma^2} F_{\mu\nu}^2 + 2 \sin \chi F_{\mu\nu} F_y^{\mu\nu} + F_{y\mu\nu}^2 \right] \Big\}, \quad (\text{C-1}) \end{aligned}$$

where $\theta^2 = (1/\xi^2) \phi_0^2 + \xi_h h^2$ denotes the radial direction in the fields space with

$$V = \frac{1}{4!} \left[6\lambda h^4 + \xi^2 (\theta^2 - \xi_h h^2)^2 \right], \quad (\text{C-2})$$

and $\langle \theta \rangle^2 = 6M_p^2$. The first line in \mathcal{L}_H is similar to that of a single field case, see Eq. (7) for $\theta^2 \leftrightarrow (1/\xi^2) \phi_0^2$. Note that \mathcal{L}_H is invariant under the Weyl gauge transformation Eq. (1)

(one checks that the first square bracket is invariant, while for the remaining terms this is easily verified). From this action Eq. (32) then follows, via a Stueckelberg mechanism.

In the formal limit when the radial direction in field space $\langle \theta \rangle \rightarrow 0$ ($M_p \rightarrow 0$) which restores the Weyl gauge symmetry, then from the definition of θ we see that $\phi_0 \rightarrow 0$ and $h \rightarrow 0$ (EW symmetry is also restored) and therefore the potential vanishes $V \rightarrow 0$, as expected due to the Weyl gauge symmetry.

The Higgs-dependent matrix $\mathcal{M}^2(\sigma)$ introduced in Eq. (40) in basis $X = (B'_\mu, A_\mu^3, \omega'_\mu)$ is

$$\begin{aligned} \mathcal{M}^2(\sigma) = & \frac{3M_p^2}{2} \sinh^2 \frac{\sigma}{M_p \sqrt{6}} \\ & \times \begin{pmatrix} g'^2 & -gg' & -g'^2 \tan \tilde{\chi} \\ -gg' & g^2 & gg' \tan \tilde{\chi} \\ -g'^2 \tan \tilde{\chi} & gg' \tan \tilde{\chi} & g'^2 \tan^2 \tilde{\chi} + \alpha^2 \gamma^2 \sec^2 \tilde{\chi} \coth^2 \frac{\sigma}{M_p \sqrt{6}} \end{pmatrix} \quad (\text{C-3}) \end{aligned}$$

This mass matrix is diagonalised by two successive rotations of the fields; first:

$$\begin{pmatrix} A_\mu \\ Z_{1\mu} \\ Z_{2\mu} \end{pmatrix} = \begin{pmatrix} \cos \theta_w & \sin \theta_w & 0 \\ -\sin \theta_w & \cos \theta_w & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B'_\mu \\ A_\mu^3 \\ \omega'_\mu \end{pmatrix} \quad (\text{C-4})$$

After this, $Z_1 - Z_2$ mass mixing usually exists, diagonalized by a final rotation of suitable ζ

$$\begin{pmatrix} A_\mu \\ Z_\mu \\ Z_\mu^\omega \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 \cos \zeta & -\sin \zeta & 0 \\ 0 \sin \zeta & \cos \zeta & 0 \end{pmatrix} \begin{pmatrix} A_\mu \\ Z_{1\mu} \\ Z_{2\mu} \end{pmatrix} \quad (\text{C-5})$$

Combining these two rotations we find a matrix relating the mass eigenstates $(A_\mu, Z_\mu, Z_\mu^\omega)$ to the gauge eigenstates $X_\mu = (B'_\mu, A_\mu^3, \omega'_\mu)$. The inverse of this matrix is shown in Eq. (47).

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