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Conformal Bootstrap and Black Holes in AdS/CFT

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Contents

Declaration and coauthorship

I declare that this thesis has not been submitted as an exercise for a degree at this or any other university and it is entirely my own work. I agree to deposit this thesis in the open access institutional repository of the University, or allow the library to do so on my behalf, subject to Irish Copyright Legislation and Trinity College Library conditions of use and acknowledgement. I consent to the examiners retaining a copy of the thesis beyond the examining period, should they so wish. This thesis contain the results of four co-authored papers [1-4] and one single-author paper [5]. Not included in this thesis are the co-authored papers [6,7] which also were an outcome of work performed during my PhD studies. The papers [1,2,6] were co-authored with Manuela Kulaxizi, Andrei Parnachev and Petar Tadić. The paper [7] was co-authored with Andrei Parnachev and Petar Tadić. The paper [3] was co-authored with Manuela Kulaxizi, Gim-Seng Ng, Andrei Parnachev and Petar Tadić. The paper [4] was co-authored with Andrei Parnachev, Valentina Prilepina and Samuel Valach. This thesis further contain the results from the paper [5].

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This thesis is the result of four years of PhD studies. Parts of this thesis follow closely [1-5] which were published in the journal JHEP. The content of [2] can be found in Section 3. The content of [3] can be found in Section 4 and Appendix A. The content of [1] can be found in Section 5 and Appendix B. The content of [5] can be found in Section 6 and Appendix C. The content of [4] can be found in Section 7 and Appendix D. The papers [6,7] published in JHEP were also an outcome of work done during the PhD but will not be presented in this thesis.

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Summary

The conformal bootstrap program has proven to be an effective tool for the study of vacuum correlators and extending this development to finite temperature correlators is of great interest. In particular, for conformal field theories with a holographic dual, finite temperature correlators maps to correlation functions in a black hole background. Probing black hole physics and interpreting its properties in terms of the CFT data, in other words, the spectrum and the OPE coefficients, is an exciting direction of research. This thesis aims to explore some developments in this direction by studying correlation functions in heavy states which are expected to thermalize.

In Section 2, we provide a summary of heavy-heavy-light-light correlators in holographic CFTs that will be the main object of study in this thesis. This section begins with the two-dimensional case where the exchange of the stress tensor and its composites is determined by the Virasoro symmetry. We then consider the same object in space-time dimension $d > 2$ and review the lightcone and the Regge limits.

In Section 3, we study the lightcone limit of a scalar heavy-heavy-light-light correlator in $d = 4$ and following [2]. Imposing crossing symmetry, we determine the contribution due to minimal-twist multi-stress tensor operators, related to the gravitational interaction between a light probe and a black hole in the bulk. This further allows us to extract the OPE coefficients between light scalars and minimal-twist multi-stress tensor operators in holographic CFTs.

In Section 4, we explore the connection between the multi-stress tensor exchanges in $d = 4$ to higher-spin theories in $d = 2$ and following [3]. The four-dimensional results from Section 3 are reminiscent of heavy-heavy-light-light vacuum blocks in $d = 2$, in the latter case this structure is completely fixed by the infinite-dimensional symmetry algebras. This indicates an emergent symmetry algebra in the lightcone limit of the stress tensor sector in four dimensions. Connections to generalized Catalan numbers and diagrammatic rules are explored.

In Section 5, we consider the heavy-heavy-light-light correlator in the Regge limit following [1]; the dual picture is that of a highly energetic particle traveling in a black hole background following [8]. Using the phase shift, related to the Shapiro time delay and the angle deflection of a null geodesic due to the presence of the black hole, we compute, among other things, the anomalous dimensions of heavy-light double-trace operators to next-to-leading order in a perturbative expansion. Whenever the regime of validity overlap, the results agree with the lightcone bootstrap. The phase shift effectively resums an infinite family of multi-stress tensor operators which at each order yields a softer behavior in the Regge limit than any single term in the sum.

In Section 6, we study the leading and next-to-leading singularities in the Regge limit following from exponentiation of the phase shift following [5]. The position space correlator is obtained by a suitable Fourier transform from which we extract the contribution from multi-stress tensor operators. The leading singularity at each order agrees with a light particle propagating in a shockwave background and, when available, the results further agree with expectations from the lightcone bootstrap.

In Section 7, we consider the stress tensor two-point function at finite temperature in holographic CFTs following [4]. In the bulk, this is related to metric fluctuations around a black hole background. We solve the EOM of the metric fluctuations in a near-boundary expansion which partially determines the boundary correlators. In particular, the near-lightcone behavior of the correlators is determined. We further decompose the correlators using the OPE between the stress tensors and read off the anomalous dimensions of double-stress tensors with spin $J = 0, 2, 4$.

Dedicated to my mother Monica and my father Anders.

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1. Introduction

In the 20th century, two main frameworks describing fundamental physics were developed – Einstein's theory of general relativity (GR) describing the effects of gravity as a curved space-time, and quantum field theory (QFT) describing the electromagnetic-, weak- and strong-interactions according to the Standard model. A unification of these forces into a single framework has been a long-standing open problem within the modern high-energy physics community. As in all of physics, the relevant scales of the problems are of fundamental importance. The planetary orbits around stars are accurately described by the theory of general relativity without including quantum effects and likewise, the effects of gravity on scattering experiments taking place at the Large Hadron Collider at CERN are negligible. There are, however, scenarios where the effects of gravity and quantum physics become of comparable order and it is not justified to neglect one over the other. One possible such scenario is when studying very massive objects, such as black holes, where regions of spacetime become singular and our current description breaks down.

An important puzzle where a better understanding of the interplay between gravity and quantum field theory is necessary was introduced by Stephen Hawking in [9]. He considered quantum field theory in a black hole background and showed that black holes emit thermal radiation at a temperature depending only on a few parameters such as the mass, spin, and charge of the black hole. This leads to a conflict with the unitarity of an underlying quantum theory which predicts that information is preserved. Bekenstein and Hawking [9,10] further pointed out that properties of black hole mechanics were reminiscent of the laws of thermodynamics which, in particular, led to the prediction that black holes have an entropy determined by their area

$$S_{BH} = \frac{c^3 A k_B}{4 G_N \hbar}, \tag{1.1}$$

which beautifully contains several constants of nature and where A is the area of the black hole. The fact that entropy scales with the area rather than the volume of the black hole is an indication that a theory of quantum gravity should be "holographic". This is currently best understood in the context of the AdS/CFT correspondence which states that a theory of quantum gravity in $(d+1)$-dimensional Anti de-Sitter space has an equivalent, dual, description as a conformal field theory (CFT) living on the d-dimensional boundary. However, there is currently a surge of developments in holographic descriptions of quantum gravity also for asymptotically flat spacetime as well as for de-Sitter space.

The AdS/CFT correspondence was conjectured in '97 by Maldacena [11] using an explicit string theory setup of string and branes leading to the famous duality between Type IIB string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ SYM in $d = 4$. The dictionary between the bulk theory of gravity and the CFT on the boundary was then further developed by Gubser-Klebanov-Polyakov [12] and Witten [13]. An actual proof of the correspondence is naturally hard to construct due to it being a strong-weak duality. At this point, there is, however, a wealth of evidence for the validity of the correspondence. The property of it being a strong-weak duality is likewise one of the prominent features of the correspondence, it makes it possible to study a class of strongly coupled quantum field theories using semi-classical Einstein gravity. While the dictionary provides a conceptually clear path to obtaining CFT observables from a semi-classical weakly coupled Einstein's theory of gravity, the opposite question of when a CFT is "holographic" is more subtle. An important step in this direction was provided by HPPS in [14] which conjectured that a theory is holographically dual to such a theory of gravity if the central charge C_T is large[1] and the dimension of the lightest (single-trace) operator with spin greater than two is large ($\Delta_{\text{gap}} \gg 1$). This was motivated by showing a one-to-one correspondence between bulk EFT in AdS and solutions to the crossing equations in the dual CFT. Another important development in this direction was provided by CEMZ [15] which argued using bulk causality that corrections to the stress tensor three-point function compared to that of pure GR should be suppressed by powers of Δ_{gap}^{-1}.

Conformal field theories, that is quantum field theories that are also invariant under local rescaling of lengths, are important points in the landscape of quantum field theories. In particular, a typical scenario is to start from a CFT

[1] This is an approximate measure of the number of degrees of freedom.

at high energies (UV) and to add relevant deformations which becomes important as one flows to lower energies (IR). Deep enough in the IR the theory could either have a mass gap, contain massless particles or be a CFT. In the latter case, both the start- and end-point of the RG flow are CFTs connected by a flow through QFTs. Understanding the restricted space of CFTs then provides us with insights about the much larger landscape of QFTs. This is intriguing as our understanding of CFTs in dimension $d > 2$ has developed significantly in the last two decades. This broad development goes under the name of the *conformal bootstrap*. The idea of bootstrapping theories, that is to extract as much information as possible or even solve the theory using symmetries and consistency conditions alone, is not new. The S-matrix bootstrap program began already in the late 50's which, however, at the time only led to partial success. Lately, partly based on the developments of the conformal bootstrap, new ideas have led to another wave of exciting research in this direction.

The goal of the conformal bootstrap program is to fully extract the constraints that come from imposing conformal symmetry and various consistency conditions, such as the associativity of the operator product expansion (OPE). In dimension $d > 2$, seminal work was done in [16] and since then the numerical bootstrap has been used successfully to e.g. "solve the Ising model" [17-21]. The numerical bootstrap has been a prosperous direction leading to many impressive results. It is, however, also of interest to find analytical results when possible. Two important examples when this is feasible consist of weakly coupled theories with a small parameter or when considering a kinematical regime where observables simplify and, in a certain sense described below, become universal.

A pioneering development of the analytic bootstrap was presented in [22,23]. Rather than considering the short-distance expansion $x_2 \to x_1$ (OPE) between two operators at x_1 and x_2 in Euclidean signature, they considered the so-called lightcone limit in Minkowski signature. This is done by letting the space-time distance go to zero when one operator gets close to the lightcone of the other. The major simplification arising in this limit is that the main contribution is due to operators of low twist τ[2]. This led the authors of [22,23]

[2] The twist τ of an operator in a CFT is defined as $\tau = \Delta - J$ where Δ is the dimension and J is the spin of the operator.

to conclude the existence of double-twist operators with large spin in any unitary CFT in $d > 2$, with universal corrections due to the stress tensor exchange which is fixed by conformal symmetry.

Micro-causality, that is commutativity of space-like separated operators, translates into analytic properties of correlators and turns out to be another powerful statement of consistency for CFT correlators. Causality in conjunction with unitarity in the UV was used to prove the ANEC and the conformal collider bounds [24] in [25,26], it was also proven using quantum information theory in [27]. This is closely related to the universality in the lightcone limit in any unitary CFT. Stronger statements are available in holographic CFTs. Essentially, this is due to the fact that the Regge limit (high-energy limit) [28-32] is dominated by operators with high spin, in theories with gravity duals, this will be the stress tensor contribution. This has led to many important results on the role of bulk causality along the lines of [15] from the boundary point of view, see e.g. [26,33-37].

Good behavior in the Regge limit is closely connected to analyticity in spin as shown in [38] who derived the Lorentzian inversion formula, see also [39]. The Lorentzian inversion formula extracts the OPE data in a four-point function from a double-commutator integrated over a Lorentzian region of spacetime. The double-commutator is in many circumstances easier to calculate than the full correlator and further possesses important properties such as being non-negative and bounded. In particular, in holographic CFTs the double-commutator suppresses the contribution from multi-trace operators compared to single-trace operators.

The non-negativity of the ANEC operator, that is the integral of the stress tensor along a null geodesic, leads to important bounds on OPE coefficients which, as mentioned above, is deeply connected to causality. More generally, the role of such non-local operators, light-ray operators, has been emphasized lately and important developments have been made, see e.g. [40-43]. Light-ray operators, and commutativity of such operators, further provide, at least in certain cases, a physical interpretation of dispersive sum rules. Recently there has been significant progress in the understanding of such sum rules [44-50], leading to, among other things, sharp bounds on corrections to theories dual to Einstein gravity in the bulk, further strengthening the work of [14] and [15].

What happens, in the context of holography, if we consider CFT correlators in non-trivial states? According to the AdS/CFT dictionary, vacuum correlators in the CFT are computed by considering fluctuations of fields in a pure AdS background. In particular, stress tensor correlators on the boundary are related to metric perturbations in pure AdS. However, the correspondence further makes prediction about correlation functions in states dual to non-trivial backgrounds in the bulk. The best-known example that will play an important role in this thesis is an asymptotically AdS-Schwarzschild black hole – it is dual to a finite temperature state on the boundary. To fully capture the physics in the bulk, including black holes, one needs to consider a world beyond vacuum correlators of "light" operators. Consistency between finite temperature correlators on the boundary and black hole physics in the bulk impose constraints on the "heavy" sector of the CFT.

Perhaps one of the most interesting results obtained from the AdS/CFT correspondence is in the context of hydrodynamics of strongly coupled QFTs at finite temperature. Hydrodynamics is an effective field theory describing long-wavelength excitations of conserved currents and the dynamical data is contained in transport coefficients. These transport coefficients are further contained in the microscopic finite temperature correlators in the limit of small energy and momenta. Through the duality, this translates to perturbations propagating on a black hole background. This line of work was initiated in [51-55] and led to, among other things, the universality of the shear viscosity η to entropy density s ratio in theories holographically dual to Einstein gravity: $\eta/s = \hbar(4\pi k_B)^{-1}$ [54,55].

Extending recent developments in the conformal bootstrap to finite temperature correlators is, therefore, of great interest, both from the CFT point of view but also due to the interesting application to black hole physics in holographic CFTs[3]. One approach in this direction is to consider correlators of light operators in heavy states, that is high-energy eigenstates. According to the Eigenstate Thermalization Hypothesis [66-70], typical such high-energy states are expected to thermalize in the sense that expectation values of simple observables in a heavy state will be close to the expectation value in the thermal

[3] See e.g. [56-65].

state. It opens the possibility to apply the machinery of the conformal bootstrap to correlators that effectively look thermal. Implementing the consistency conditions mentioned above in this context is of great interest. A step in this direction is what we will explore in this thesis.

1.1. The conformal bootstrap

In this section we give a brief review of some elements of conformal field theory, this will lay the foundation for the rest of thesis. For reviews on CFTs, see [71-74] that inspired this section. A conformal theory is covariant under conformal transformations, that is coordinate transformations $x \to x'(x)$ that leave the metric invariant up to an overall local dilatation $\Omega^2(x)$ (rescaling):

$$\begin{aligned}(ds')^2 &= \delta_{\mu\nu}\frac{dx'^\mu}{dx^\rho}\frac{dx'^\nu}{dx^\sigma}dx^\rho dx^\sigma \\ &= \Omega^2(x)ds^2.\end{aligned} \tag{1.2}$$

The stress tensor operator in a conformal theory is further traceless[4] ${T^\mu}_\mu = 0$. Because of the tracelessness of the stress tensor, one can construct conserved charges from a larger set of vector fields ξ satisfying the conformal Killing equation

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu = \omega(x)\delta_{\mu\nu}. \tag{1.3}$$

The vector fields on R^d satisfying (1.3) are given by

$$\begin{aligned}&p_\mu = \partial_\mu \\ &m_{\mu\nu} = x_\nu\partial_\mu - x_\mu\partial_\nu \\ &d = x^\mu\partial_\mu \\ &k_\mu = 2x_\mu(x\cdot\partial) - x^2\partial_\mu,\end{aligned} \tag{1.4}$$

where p_μ and $m_{\mu\nu}$ corresponds to translations and rotations, respectively, and d and k_μ corresponds to dilatations and special conformal transformations, respectively. Using (1.3) one sees that the divergence of the current $J^\mu_\xi = \xi_\nu T^{\mu\nu}$, when $\xi = d$ or $\xi = k_\mu$, is proportional to the trace ${T^\mu}_\mu = 0$ in conformal theories. From the currents in (1.4), one can construct conserved charges Q_ξ [5]

[4] This is true in R^d but on a general curved spacetime there can be Weyl anomalies.

[5] In the language of [71] these are topological surface operators.

which satisfy the conformal algebra $SO(1, d+1)$, these will be denoted by the same but capitalized, letters as the conformal Killing vectors in (1.4).

An important notion is that of primary and descendant operators. Consider for simplicity a scalar operator $\mathcal{O}(x)$, it is *primary* if

$$\begin{aligned} [D, \mathcal{O}(0)] &= \Delta \mathcal{O}(0) \\ [K_\mu, \mathcal{O}(0)] &= 0, \end{aligned} \tag{1.5}$$

where Δ is the scaling dimension of $\mathcal{O}(x)$, D and K_μ are the generators of dilatations and special conformal transformations, respectively. A corresponding conformal family of $\mathcal{O}(0)$ can then by built by acting with P_μ to construct *descendant* operators $P_{\mu_1} \ldots P_{\mu_l} P^{2n} \mathcal{O}(0)$ with dimension $\Delta + l + 2n$ and spin l. Correlators of descendant operators can then be obtained from correlators of the corresponding primary operators and we will therefore restrict our attention to correlators of primary operators.

In conformally invariant theories on R^d it is natural to foliate the space into spheres with different radii and quantize the theory on S^{d-1}. "Time evolution" then corresponds to radial evolution using the dilatation generator D – this leads to the notion of radial quantization. The states on a sphere S^{d-1} centered around some point $x \in R^d$ are then formally obtained as a path integral over a ball centered around the same point. If there are no operator insertions inside this ball this will be the vacuum state while on the other hand, we can define states $|\mathcal{O}\rangle$ by inserting the operator $\mathcal{O}(x)$ inside the path integral before performing the path integral. Likewise, it is possible to define an operator from an eigenstate of the dilatation operator. The equivalence between these two constructions leads to the state-operator correspondence – the $1-1$ correspondence between local operators and eigenstates of the dilatation operator. See e.g. [71] for a more complete discussion on the state-operator correspondence.

The conformal symmetry imposes strong constraints on correlations functions. E.g., the two- and three-point functions of scalar primary operators are given by

$$\begin{aligned} \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \rangle &= \frac{\delta_{\Delta_1 \Delta_2}}{x_{12}^{2\Delta_1}} \\ \langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \mathcal{O}_3(x_3) \rangle &= \frac{\lambda_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{13}^{\Delta_1+\Delta_3-\Delta_2} x_{23}^{\Delta_2+\Delta_3-\Delta_1}}, \end{aligned} \tag{1.6}$$

where $x_{ij} := |x_i - x_j|$ and λ_{ijk} is a coefficient undetermined by symmetry. It is common to chosen the two-point function of operators to be equal to 1, up to the position-dependent part fixed by conformal symmetry. An important exception to this is the case of the stress tensor operator

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{C_T}{x^{2d}}\Big[\frac{1}{2}(\mathcal{I}_{\mu\rho}(x)\mathcal{I}_{\nu\sigma}(x) + \mathcal{I}_{\mu\sigma}(x)\mathcal{I}_{\nu\rho}(x)) - \frac{1}{d}\delta_{\mu\nu}\delta_{\rho\sigma}\Big], \quad (1.7)$$

where C_T is the *central charge* and $\mathcal{I}_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}$. It plays an important role as it (approximately) counts the number of degrees of freedom in a CFT. In two dimensions[6] this can be made rigorous via the c-theorem which states that $c_1 \geq c_2$ when there is an RG flow from CFT_1 with central charge c_1 to a CFT_2 with central charge c_2.

Before moving on to four-point functions we will introduce the *operator product expansion* (OPE). Consider a sphere S^{d-1} centered at the origin which contains two local operators $\mathcal{O}_1(x)$ and $\mathcal{O}_2(0)$. The path integral over the interior of the sphere defines a state on the boundary which can be evolved inwards to define a state at the center of the sphere. The state-operator correspondence implies that this state can be obtained as a linear combination of local operators acting on the origin. This leads to the OPE:

$$\mathcal{O}_1(x)\mathcal{O}_2(0) = \sum_{\mathcal{O}_k \text{ primary}} \lambda_{12k} C_{12k}(x,\partial)\mathcal{O}_k(0), \quad (1.8)$$

which holds true inside correlation functions given that all other $|x_i| > |x|$[7,8]. Here C_{12k} is a differential operator that is fixed by conformal symmetry and λ_{12k} are the *OPE coefficients* which already appeared in (1.6)[9]. Using the OPE, one can reduce an n-point function to an $(n-1)$-point function and recursively

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6 In two dimensions it is conventional to consider $C_T = \frac{c}{2}$.

7 The origin is not a special point and we are free to perform the OPE around any point as long as we can find a sphere that contains the two operators under consideration and no other appearing in the correlation function.

8 Here $\mathcal{O}_1$ and $\mathcal{O}_2$ are assumed to be scalar and we have suppressed the indices of C_{12k} and $\mathcal{O}_k$.

9 Generally the OPE coefficients and the coefficient appearing in the three-point function might differ by the normalization of $\mathcal{O}_k$ but here we for simplicity assume that all operators are unit-normalized.

reduce the problem to a sum over one-point functions; in particular, in the vacuum, only the identity operator has a non-vanishing one-point function. Defining a CFT in terms of its correlation functions, we are led to the statement that we can abstractly define a CFT as the set of CFT data, that is the spectrum of primary operators (Δ_i, J_i) and the OPE coefficients λ_{ijk} for all operators in the theory.

1.1.1. Four-point functions, conformal blocks, and crossing symmetry

The two- and three-point functions of scalar primary operators are fixed by conformal symmetry up to the choice of normalization and the OPE coefficient. The first non-trivial correlation functions turns out to be four-point functions – they are therefore the main protagonists of the conformal bootstrap program (as well as this thesis). Conformal symmetry still puts constraints of the four-point function which takes the following form

$$\begin{aligned} &\langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle = K_{\Delta_i}(x_i)\mathcal{A}(u,v), \\ &K_{\Delta_i}(x_i) = \frac{1}{(x_{12}^2)^{\frac{\Delta_1+\Delta_2}{2}}(x_{34}^2)^{\frac{\Delta_3+\Delta_4}{2}}}\left(\frac{x_{24}^2}{x_{14}^2}\right)^{\frac{\Delta_1-\Delta_2}{2}}\left(\frac{x_{14}^2}{x_{13}^2}\right)^{\frac{\Delta_3-\Delta_4}{2}}, \end{aligned} \tag{1.9}$$

where (u, v) are the conformally invariant cross-ratios

$$\begin{aligned} u &= z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \\ v &= (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \end{aligned} \tag{1.10}$$

On the other hand, one can also use the OPE repeatedly to decompose the four-point function in terms of *conformal blocks*. Consider for simplicity the case $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{O}$ and $\mathcal{O}_3 = \mathcal{O}_4 = \psi$. We perform two OPE's of $\mathcal{O}(x_1)\mathcal{O}(x_2)$ and $\psi(x_3)\psi(x_4)$, leading to infinite double sums of two-point functions $\langle \mathcal{O}_k\mathcal{O}_{k'}\rangle$, weighted by a product of OPE coefficients and the action of the differential operators appearing in (1.8). Because the two-point functions are diagonal (1.6), this reduces a single infinite sum over exchanged operators $\mathcal{O}_k$ weighted by the product of OPE coefficients and a function g_{Δ_k,J_k} that is fixed by conformal symmetry. Together with the structure (1.9), this leads to the conformal block decomposition of four-point functions[10]

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\psi(x_3)\psi(x_4)\rangle = K(x_i)\sum_{\mathcal{O}_k}\left(-\frac{1}{2}\right)^{J_k}\lambda_{\mathcal{O}\mathcal{O}\mathcal{O}_k}\lambda_{\psi\psi\mathcal{O}_k}g_{\Delta_k,J_k}(z,\bar{z}). \tag{1.11}$$

where $K(x_i) = (x_{12}^2)^{-\Delta_{\mathcal{O}}}(x_{34}^2)^{-\Delta_\psi}$ and $g_{\Delta_k,J_k}(z,\bar{z})$ are conformal blocks capturing the contribution from a primary operator $\mathcal{O}_k$ and all its descendants.

In principle one could use the knowledge of the OPE to try and sum up the contribution from the primary and all the descendants in order to obtain the conformal block. In practice this is difficult and a more convenient method was developed by Dolan and Osborn [75,76]. The idea is to insert in the four-point function a projection operator $P_{\mathcal{O}_k}$ which projects onto the operator $\mathcal{O}_k$ and all its descendants. Now the key point is that the conformal Casimir $C = \frac{1}{2}L^{ab}L_{ba}$ commute with P_μ and the eigenvalue $CP_{\mathcal{O}_k} = c_{\Delta,J}P_{\mathcal{O}_k} = [\Delta(\Delta-d)+J(J+d-2)]P_{\mathcal{O}_k}$ is the same for all the operators in the conformal family of $\mathcal{O}_k$. On the other hand, we can let the generators act on, say, the operators $\mathcal{O}(x_1)$ and $\mathcal{O}(x_2)$ and note that the vacuum is conformally invariant, which leads to a differential equation satisfied by the conformal blocks. This was the strategy taken by Dolan and Osborn which solved this differential equation in terms of $(z,\bar{z})$. More specifically, the conformal blocks satisfy the following differential equation (Here we consider the more general case of four scalar primaries with scaling dimensions Δ_i and $\Delta_{ij} := \Delta_i - \Delta_j$) [75,76]

$$\mathcal{D}g_{\Delta,J}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) = c_{\Delta,J}g_{\Delta,J}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) \tag{1.12}$$

where the differential operator $\mathcal{D}$ is given by

$$\begin{aligned}\mathcal{D} &= \mathcal{D}_z + \mathcal{D}_{\bar{z}} + 2(d-2)\frac{z\bar{z}}{z-\bar{z}}[(1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}],\\ \mathcal{D}_z &= 2z^2(1-z)\partial_z^2 - (2+\Delta_{34}-\Delta_{12})z^2\partial_z + \frac{\Delta_{12}\Delta_{34}}{2}z\end{aligned} \tag{1.13}$$

[10] The conformal blocks and the conformal partial waves differ by the overall factor of $K(x_i)$, below we will mainly consider the former which only depends on the cross-ratios.

and similarly for $\mathcal{D}_{\bar{z}}$. Notice that for $d=2$, the operator $\mathcal{D}$ simplifies and the conformal blocks factorize. The normalization of the blocks[11] can be found by comparing to the OPE limit $x_{12}, x_{34} \to 0$ which is equivalent to $z, \bar{z} \to 0$:

$$g_{\Delta,J}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) \sim \frac{J!}{(\frac{d}{2}-1)_l}(z\bar{z})^{\frac{\Delta}{2}} C_J^{(\frac{d}{2}-1)}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big), \tag{1.14}$$

where $C_J^{(\frac{d}{2}-1)}$ are the Gegenbauer polynomials. An important piece of the conformal blocks in both $d=2$ and $d=4$ are the $SL(2,R)$ blocks

$$k_a(z) = z^a\, {}_2F_1(a - \frac{\Delta_{12}}{2}, a + \frac{\Delta_{34}}{2}; 2a, z), \tag{1.15}$$

where ${}_2F_1$ is the hypergeometric function and k_a is an eigenfunction of the operator $\mathcal{D}_z$ with eigenvalue $\mathcal{D}_z k_a(z) = 2a(a-1)k_a(z)$. A special role will be played by these functions when $\Delta_{12} = \Delta_{34} = 0$ and we therefore define

$$f_a(z) = z^a\, {}_2F_1(a, a; 2a, z). \tag{1.16}$$

These are not only the constituents of the conformal blocks, as will be seen below but will also play a crucial role in the remainder of the thesis since they are also what builds up the so-called stress tensor sector of heavy-heavy-light-light correlators in holographic CFTs. In $d=2$, the conformal blocks are given by

$$g_{\Delta,J}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) = \frac{1}{1+\delta_{J0}}(k_{\frac{\Delta+J}{2}}(z)k_{\frac{\Delta-J}{2}}(\bar{z}) - (z \leftrightarrow \bar{z})) \tag{1.17}$$

while in $d=4$ they are given by

$$g_{\Delta,J}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) = \frac{z\bar{z}}{z-\bar{z}}(k_{\frac{\Delta+J}{2}}(z)k_{\frac{\Delta-J-2}{2}}(\bar{z}) - (z \leftrightarrow \bar{z})). \tag{1.18}$$

The cornerstone of the bootstrap program of four-point functions is the statement of *crossing symmetry.* In its essence, it boils down to the fact that we are free to perform the OPE expansion of the four-point functions between any pair of operators. This leads to the decomposition of the four-point function in three different channels typically denoted as the s-, t-, and u-channel. The convergence of the OPE in different channels depends on the operator insertions

[11] See [74] for a collection of common conventions for the normalization in the literature.

but the crucial point is that there are overlapping regions of convergence where at least two of the conformal block decompositions are simultaneously valid. The equality of these two expansions is the statement of *crossing symmetry.* Importantly, the operators that contribute in one channel can be different from the operators that dominate in another channel. It is therefore often fruitful to consider cases where the expansion in one channel is particularly easy. It is then the bootstrapper's objective to determine how this is reflected in the other channel where the same physics often is not manifest. This is typical in e.g. a weakly coupled theory where there is a small parameter ϵ in which one channel simplifies. In the context of holographic CFTs, this could be the inverse of the central charge $C_T \gg 1$. Another example that has led to important progress in our understanding of CFTs corresponds to a certain kinematical limit where some spacetime distance $|x|$ (this could be e.g. the cross-ratio z or $\bar{z}$) becomes small and the conformal block decomposition can be effectively organized perturbatively in x. This is an incredibly powerful idea when one uses the fact that the spacetime dependence of the contribution of an operator $\mathcal{O}$ can be deduced by looking at (limits of) the conformal block. By a cleverly chosen kinematical limit, one can isolate certain operators in one channel.

An important example of such a kinematical limit is the lightcone limit. Let us consider a pair of identical scalar operators $\psi(x)$ and $\psi(0)$ that become light-like separated by taking $x^+ \to 0$ with x^- fixed where $x^\pm = t \pm x$ and the transverse separation set to zero. Using the OPE one finds the following contribution due to an operator $\mathcal{O}_{\mu_1 \ldots \mu_J}$ with twist-($\tau := \Delta - J$) and spin-J:

$$\psi(x)\psi(0) \sim \frac{(x^+)^{\frac{\tau}{2}}(x^-)^{\frac{\Delta+J}{2}}}{(-x)^{2\Delta_\psi}} \lambda_{\psi\psi\mathcal{O}} \mathcal{O}_{-,\ldots,-}(0) + \ldots, \tag{1.19}$$

where the ellipses denote the contribution from descendants and all operators. From (1.19) one sees that operators with low twist τ dominate in the lightcone limit $x^+ \to 0$. In unitary CFTs in $d > 2$ there is a twist gap and the identity operator with twist $\tau = 0$ gives the leading contribution in the lightcone limit. This was translated to the cross-channel in [22,23] which found that there exist universal "double-twist" operators $[\mathcal{O}_1\mathcal{O}_2]_{n,l}$ with large spin $l \gg 1$. Moreover, assuming there are no light scalars ($\Delta \leq d-2$), the subleading correction is due to conserved currents with twist $\tau = d-2$. Especially, the stress tensor operator is present in any CFT and give further universal corrections to the OPE data of the exchanged double-twist operators. We further note that the scaling (1.19) is reflected in the limit of the conformal blocks. Consider a four-point function $\langle \mathcal{O}_2(\infty)\mathcal{O}_1(1)\mathcal{O}_1(z,\bar{z})\mathcal{O}_2(0)\rangle$ with the cross-ratios given by (1.10). The lightcone limit then corresponds to taking $1-\bar{z} \ll 1$ with z fixed, in this limit the conformal blocks behave as follows

$$g^{(0,0)}_{\Delta,J}(1-z,1-\bar{z}) \sim (1-\bar{z})^{\frac{\tau}{2}} f_{\frac{\tau+2J}{2}}(1-z), \tag{1.20}$$

where f_a is given in (1.16) and encodes the contribution from $\mathcal{O}_{\Delta,J}$ and all its descendants (which contribute in the lightcone limit).

A major role in this thesis will be played by the conformal bootstrap, described above, applied to heavy-heavy-light-light correlators in holographic CFTs.

1.2. Outline

In Section 2, we review the scalar heavy-heavy-light-light correlator in holographic CFTs. We start in dimension $d = 2$ based on [3] that will be used later in Section 4 when studying higher-spin algebras in two dimensions. Then we move on to holographic CFTs in $d > 2$, with the main focus being $d = 4$. This part contains the conformal block expansion in both the direct-channel and the cross-channel together with a summary of which operators will contribute in the lightcone and the Regge limit, respectively.

In Section 3, we study the heavy-heavy-light-light correlator in the lightcone limit. By imposing crossing symmetry, we bootstrap the contribution due to minimal-twist multi-stress tensor in holographic CFTs in $d = 4$. This section is based on the work in [2].

In Section 4, we explore the emergent structure in the lightcone limit of heavy-heavy-light-light correlators in $d = 4$ and higher-spin vacuum blocks in $d = 2$. This further leads us to consider generalized Catalan numbers and diagrammatic rules as possible guiding principles to reconstruct the correlator to all orders. This section is based on [3]. Some related details can be found in Appendix A.

In Section 5, we turn away from the lightcone limit and study the heavy-heavy-light-light correlator in the Regge limit. This is dual to a highly-energetic particle propagating in a black hole background. The corresponding Shapiro time delay and the angle deflection are related to the anomalous dimensions of heavy-light double-trace operators. This section is based on [1]. Some related details can be found in Appendix B.

In Section 6, we continue the study of the Regge limit and determine the consequences due to the exponentiation of the phase shift to higher orders. We find that the leading term at each order agrees with a corresponding shockwave calculation and further read off the subleading term at each order. This section is based on [5]. Some related details can be found in Appendix C.

In Section 7, we move on from the world of scalar correlators to the stress tensor two-point function at finite temperature. First, we study metric fluctuations around a planar AdS-Schwarzschild black hole in semi-classical Einstein gravity and determine the boundary stress tensor two-point function up to sub-sub-leading order in the OPE expansion. In the CFT, we perform the stress tensor OPE and decompose the thermal two-point function into the contribution due to the identity, the stress tensor, and double-trace stress tensor operators with spin $J = 0, 2, 4$. The exchange of the identity is fixed by conformal symmetry and agrees with the bulk calculations. The stress tensor OPE coefficients are further known in theories dual to Einstein gravity and we find agreement with the bulk and the boundary computations. At the next order, we match the bulk computations to the CFT decomposition which allows us to read off the anomalous dimensions of the double-trace stress tensors and, partially, the product of OPE coefficients and thermal one-point functions. We further determine the near-lightcone behavior of the correlators. This section is based on [4]. Some related details can be found in Appendix D.

2. Stress tensor sector of correlators in heavy state

In this section we provide a review of the scalar heavy-heavy-light-light correlator in holographic CFTs since it plays a central role in the rest of this thesis. We begin in Section 2.1 by reviewing the two-dimensional case where the exchange of the stress tensor and composites thereof, is completely determined by the Virasoro symmetry. This will serve as a useful source of intuition before considering CFTs in $d > 2$. In Section 2.2, we move on to holographic CFTs in $d > 2$, mainly considering $d = 4$. This part contains the conformal block expansion in both the direct-channel and the cross-channel together with a summary of which operators will contribute in the lightcone and the Regge limit, respectively.

2.1. Two dimensions and the Virasoro vacuum block

In this section we consider the case of large-c CFTs in $d = 2$ and a heavy-heavy-light-correlator where the scaling weight H of the heavy scalar operator is large, $H \sim c \gg 1$. The Virasoro vacuum block was first computed in [77,78]. Below we will review the explicit mode calculation [79], parts of which was extended for $\mathcal{W}_{N=3,4}$ in [3] that will be reviewed in Section 4.

Conformal field theories in $d = 2$ are different from the higher-dimensional counterparts, this is so because the conformal symmetry enhances to the infinite-dimensional Virasoro algebra. This is especially powerful in constraining the effect of stress tensor dynamics since operators now live in representations of the full Virasoro algebra rather than just the global part. Since much of this thesis focus on the contribution of multi-stress tensors to correlators, this provides a powerful toy model where the analogous quantities can be computed explicitly using the extended symmetry. In the content of heavy-heavy-light correlators, this is captured in the heavy-heavy-light-light Virasoro vacuum block which will be reviewed below. This further lays the foundation for an extension to theories with higher-spin currents, so called $\mathcal{W}_N$ theories. These were shown in [3] to possess interesting similarities with the lightcone limit of the stress tensor sector in higher-dimensional CFTs pointing towards a potential extension of symmetries in this limit for holographic CFTs.

The Virasoro algebra is given by

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \tag{2.1}$$

where L_m are the modes of the stress tensor $T(z)$

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}} \tag{2.2}$$

and can therefore be obtained by

$$L_m = \oint_C \frac{dz}{2\pi i} z^{m+1} T(z). \tag{2.3}$$

The algebra (2.1) corresponds to the following OPE of stress tensors

$$T(z)T(0) = \frac{c}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z} + \ldots, \tag{2.4}$$

where the ellipses denote regular terms in the OPE limit $z \to 0$. Here c is the central charge. Virasoro primary operator have the following OPE with the stress tensor

$$T(z)\mathcal{O}(w) = \Big[\frac{h}{(z-w)^2} + \frac{\partial}{z-w}\Big]\mathcal{O}(w), \tag{2.5}$$

which is equivalent to the following action of the modes

$$[L_m, \mathcal{O}(z)] = z^m\Big[h(m+1) + z\partial\Big]\mathcal{O}(z). \tag{2.6}$$

In this section, we use the Virasoro modes to explicitly calculate the first terms due to Virasoro descendants of the vacuum following [77,78].

We consider a four point function of pair-wise identical operators $\mathcal{O}_H$ and $\mathcal{O}_L$ with conformal weight H and h, respectively, given by $\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle$. We further suppress the anti-holomorphic part and have used conformal symmetry to fix the operators at $0, z, 1, \infty$ and set $\mathcal{O}_H(\infty) = \lim_{z\to\infty} z^{2H}\mathcal{O}_H(z)$. The limit that will be considered is $c \to \infty$ with h and $\frac{H}{c}$ fixed.

We are interested in the contribution due to Virasoro descendants of the vacuum, i.e. states of the schematic form

$$\begin{aligned}\mathcal{G}_2(z) = \langle &\mathcal{O}_H(\infty)\mathcal{O}_H(1) \\ &\times \sum_{\{m_i\},\{n_j\}} \frac{L_{-m_1}L_{-m_2}\ldots L_{-m_i}|0\rangle\langle 0|L_{n_j}\ldots L_{n_2}L_{n_1}}{\mathcal{N}_{\{m_i\},\{n_j\}}}\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle,\end{aligned} \tag{2.7}$$

where $\mathcal{N}_{\{m_i\},\{n_j\}}$ is a normalization factor and $\mathcal{G}_2(z)$ is defined as the HHLL correlator restricted to the contribution of the identity block in the direct channel (the subscript (2) here stands for the Virasoro algebra as opposed to (N) for the $\mathcal{W}_N$). In [79] an orthogonal basis was constructed in the limit $c \to \infty$ and it was shown how to perform this sum using a recursion relation. The correlator organizes into powers of $\frac{H}{c}$ and we will study the first two terms in this expansion. These are due to single and double mode states respectively.

To begin with, consider the contribution from states of the form $L_{-n}|0\rangle$ in (2.7). We can evaluate $\langle 0|L_n \mathcal{O}(z)\mathcal{O}(0)\rangle$ and $\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)L_{-n}|0\rangle$ for $n \geq 2$ with the help of (2.6). We find that

$$\begin{aligned} \langle 0|L_n \mathcal{O}(z)\mathcal{O}(0)\rangle &= z^n[h(n+1)+z\partial]z^{-2h} = h(n-1)z^{n-2h} \\ \langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)L_{-n}|0\rangle &= H(n-1). \end{aligned} \tag{2.8}$$

The norm of these states is given by the central term

$$\mathcal{N}_{n,n} = \langle L_n L_{-n}\rangle = \frac{c}{12}n(n^2-1). \tag{2.9}$$

Combining the above allows one to obtain the single mode state contribution to the vacuum block

$$\mathcal{G}_2(z)|_{\frac{H}{c}} = z^{-2h}\sum_{n=2}^{\infty}\frac{12Hh}{c}\frac{(n-1)}{(n+1)}\frac{z^n}{n} = \frac{2Hh}{c}f_2(z)z^{-2h}, \tag{2.10}$$

where we note the appearance of f_2, which is the conformal block due to the exchange of the quasi-primary $T(z)$ and its global descendants. By quasi-primary we refer to a primary under the global part $\{L_{\pm 1}, L_0\}$ of the Virasoro algebra.

Consider now states of the schematic form $L_{-m}L_{-n}|0\rangle$. These are not orthogonal to the single mode states $L_{-m-n}|0\rangle$ since

$$\langle L_{m+n}L_{-n}L_{-m}\rangle = (2n-m)\frac{c}{12}m(m^2-1) \neq 0. \tag{2.11}$$

Removing this overlap one can construct states $|X_{m,n}\rangle$[12] that are orthogonal to $L_{-m-n}|0\rangle$:

$$|X_{m,n}\rangle = \left[L_{-n}L_{-m} - \frac{\langle L_{m+n}L_{-n}L_{-m}\rangle}{\langle L_{m+n}L_{-m-n}\rangle}L_{-m-n}\right]|0\rangle, \tag{2.12}$$

[12] Note that the states $|X_{m,n}\rangle$ thus defined are not unit normalised.

which contribute at $\mathcal{O}(\frac{H^2}{c^2})$ to $\mathcal{G}(z)$. The contribution of these states are found to be

$$\begin{aligned} \langle 0|L_m L_n \mathcal{O}_L(z)\mathcal{O}_L(0)\rangle &= \left[h^2(n-1)(m-1)+hm(m-1)\right] z^{s-2h}, \\ \langle 0|L_{m+n}\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle &= h(s-1)z^{s-2h}, \end{aligned} \tag{2.13}$$

where $s=m+n$. Using (2.13), we find that

$$\begin{aligned} \langle X_{m,n}|\mathcal{O}_L(z)\mathcal{O}_L(0)|0\rangle = \Big[&h^2(n-1)(m-1)+hm(m-1) \\ &-\frac{(2n-m)\frac{c}{12}m(m^2-1)}{\frac{c}{12}s(s^2-1)}h(s-1)\Big]z^{s-2h} \\ = \Big[&h^2(m-1)(n-1)+h\frac{n(n-1)m(m-1)}{s(s+1)}\Big]z^{s-2h} \end{aligned} \tag{2.14}$$

as in [79]. Furthermore, keeping only the leading term for large H gives

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)|X_{m,n}\rangle = H^2(n-1)(m-1). \tag{2.15}$$

The norm of the states $|X_{m,n}\rangle$ in the large-c limit is given by the square of the central terms, *i.e.*,

$$\mathcal{N}_{X_{m,n}} = \langle L_m L_n L_{-n} L_{-m}\rangle = \left(\frac{c}{12}\right)^2 m(m^2-1)n(n^2-1)+\ldots, \tag{2.16}$$

where the ellipses refer to terms subleading in c. Combining the above one finds the contribution of the states $|X_{m,n}\rangle$ to the vacuum block in (2.7) to be

$$\begin{aligned} \mathcal{G}_2(z)|_{\frac{H^2}{c^2}} = &\frac{z^{-2h}}{2}\Big(\frac{12Hh}{c}\Big)^2\sum_{m,n=2}^{\infty}\frac{(m-1)(n-1)}{(m+1)(n+1)}\frac{z^{m+n}}{mn} \\ &+z^{-2h}\frac{72H^2h}{c^2}\sum_{m,n=2}^{\infty}\frac{(m-1)(n-1)}{(m+1)(n+1)}\frac{z^{m+n}}{(m+n)(m+n+1)}, \end{aligned} \tag{2.17}$$

where we have included a symmetry factor of $\frac{1}{2}$ due to the exchange symmetry $(m \leftrightarrow n)$. The first line in (2.17) comes from the exponentiation of the first term, *i.e.*, it is the square of (2.10) divided by 2

$$\mathcal{G}_2(z)|_{\frac{H^2h^2}{c^2}} = \frac{1}{2}\Big(\frac{2Hh}{c}f_2\Big)^2 z^{-2h}. \tag{2.18}$$

The second line in (2.17) can be written as a sum of products of functions $f_a f_b$ such that $a+b=4$ in the following way

$$\mathcal{G}_2(z)|_{\frac{H^2h}{c^2}} = z^{-2h}\frac{2H^2h}{c^2}\Big[-f_2^2+\frac{6}{5}f_1f_3\Big] \tag{2.19}$$

as was pointed out in [8].

The relative coefficient between the terms in the bracket of (2.19) is precisely such that in the limit $z \to 1$ the coefficient in front of $\log^2(1-z)$ vanishes and (2.19) behaves as

$$\mathcal{G}_2(z)|_{\frac{H^2 h}{c^2}} \underset{z\to 1}{\approx} \#\log(1-z). \tag{2.20}$$

In [79] it was found that this behaviour persists to all orders, *i.e*, the coefficients of all the $\log^p(1-z)$ with $p > 1$ vanish in the limit $z \to 1$ and hence $\mathcal{G}_2(z)$ has a simple logarithmic behavior in this limit. Moreover, the authors of [79] observed that the coefficients in front of the $\log(1-z)$ terms at each order in $\frac{H}{c}$ form the Catalan numbers' sequence. In Section 4, we will see a similar statement being true for $\mathcal{W}_{N=3,4}$ vacuum blocks[13]. In [79] they further showed that the generating function of the Catalan numbers could be uplifted to a differential equation, whose solution gave the Virasaoro vacuum block for any value of z and not only $z \to 1$ as in (2.20). A similar story holds true for the $\mathcal{W}_N$ case while in $d = 4$ it was found in [3] that there is again an interesting sequence appearing, of which much less is known however.

2.2. Stress tensor sector and correlators in heavy states in $d > 2$

Below we review the setup of a heavy-heavy-light-light correlator in holographic CFTs. By holographic CFTs we refer to a family of CFTs with a large central charge $C_T \gg 1$ and a large gap in the spectrum of higher-spin single trace operators $\Delta_{\text{gap}} \gg 1$, where $\Delta_{\text{gap}} = \min \Delta_{J>2}$. This follows the review in [2] which is further based on [8,1,80]. In holographic CFTs with large central charge C_T, there exists multi-trace operators $[\mathcal{O}_1\mathcal{O}_2\ldots\mathcal{O}_k]_{n,l}$. The simplest example are double-trace operators of $[\mathcal{O}_1\mathcal{O}_2]_{n,l}$ which are schematically given by $\mathcal{O}\partial^{2n}\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_l}\mathcal{O}$, appropriately symmetrized to make a primary. Their scaling dimension is given by

$$\Delta_{[\mathcal{O}_1\mathcal{O}_2]_{n,l}} = \Delta_1 + \Delta_2 + 2n + l + \gamma(n,l), \tag{2.21}$$

where $\gamma(n,l)$ are anomalous dimension which are suppressed in the limit $C_T \to \infty$. An important example that will play a significant role in this thesis are

[13] We expect this to be true for arbitrary N.

multi-trace operators made out of stress tensors, which we will call multi-stress tensors and denote schematically by $[T^k]_{n,l}$.

The object that we study is a four-point function of pairwise identical scalars $G(x_i) = \langle \mathcal{O}_H(x_4)\mathcal{O}_L(x_3)\mathcal{O}_L(x_2)\mathcal{O}_H(x_1)\rangle$. Here $\mathcal{O}_H$ and $\mathcal{O}_L$ are scalar operators with scaling dimension $\Delta_H \propto \mathcal{O}(C_T)$ and $\Delta_L \propto \mathcal{O}(1)$, with $C_T \gg 1$ the central charge. Using conformal transformations we define the stress tensor sector of the correlator by

$$\mathcal{G}(z,\bar{z}) = \lim_{x_4\to\infty} x_4^{2\Delta_H} \langle \mathcal{O}_H(x_4)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\Big|_{\text{multi-stress tensors}}, \tag{2.22}$$

where z and $\bar{z}$ are the usual cross-ratios

$$\begin{aligned}(1-z)(1-\bar{z}) &= \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2},\\ z\bar{z} &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}.\end{aligned} \tag{2.23}$$

In (2.22) the"multi-stress tensor" subscript stands to indicate the contribution of the identity and all multi-stress tensor operators, i.e. multi-trace operators made out of the stress tensors, as discussed above, present in holographic CFTs.

The correlator $\mathcal{G}(z,\bar{z})$ can be expanded in the "T-channel" $\mathcal{O}_L(1) \times \mathcal{O}_L(z,\bar{z}) \to \mathcal{O}_{\tau,s}$ as[14]

$$G(z,\bar{z}) = [(1-z)(1-\bar{z})]^{-\Delta_L} \sum_{\mathcal{O}_{\tau,s}} P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} g^{(0,0)}_{\tau,s}(1-z,1-\bar{z}), \tag{2.24}$$

where $\tau = \Delta - s$ and s denote the twist and spin of the exchanged operator, respectively, and $g^{(0,0)}_{\tau,s}(z,\bar{z})$ the conformal block of the primary operator $\mathcal{O}_{\tau,s}$. Moreover, $P^{(HH,LL)}_{\mathcal{O}_{\tau,s}}$ are defined as

$$P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} = \left(-\frac{1}{2}\right)^s \lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}_{\tau,s}} \lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}_{\tau,s}}, \tag{2.25}$$

where $\lambda_{\mathcal{O}_L\mathcal{O}_L\mathcal{O}}$ and $\lambda_{\mathcal{O}_H\mathcal{O}_H\mathcal{O}}$ denote the respective OPE coefficients. The C_T scaling for generic single-trace operators is given by

$$\langle \mathcal{O}_{H,L}\mathcal{O}_{H,L}\mathcal{O}\rangle \sim \frac{1}{\sqrt{C_T}}, \tag{2.26}$$

[14] For reasons of convenience, here and in the rest of the thesis we refer to $\mathcal{G}(z,\bar{z})$ as the correlator; the reader should keep in mind that $\mathcal{G}(z,\bar{z})$ is not the full correlator but only its stress tensor sector, as defined in (2.22).

while for k-trace operators $[\mathcal{O}^k]$

$$\langle \mathcal{O}_{H,L}\mathcal{O}_{H,L}[\mathcal{O}^k]\rangle \sim \frac{1}{C_T^{\frac{k}{2}}}. \tag{2.27}$$

2.2.1. Lightcone limit

The lightcone limit is defined by $\bar{z} \to 1$ with z held fixed. In this limit the T-channel expansion (2.24) is dominated by minimal-twist operators as follows from the behaviour of the conformal blocks

$$\mathcal{G}(u,v) \underset{u\to 0}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\mathcal{O}_{\tau,s}} P^{(HH,LL)}_{\mathcal{O}_{\tau,s}} (1-\bar{z})^{\frac{\tau}{2}} f_{\frac{\tau}{2}+s}(1-z), \tag{2.28}$$

where $\tau = \Delta - s$ is the twist.

For any CFT in $d > 2$ the leading contribution in the lightcone limit comes from the exchange of the identity operator with twist $\tau = 0$. Another operator present in any unitary CFT is the stress tensor with twist $\tau = d-2$. Its contribution to the correlator is completely fixed by a Ward identity and

$$P^{(HH,LL)}_{T_{\mu\nu}} = \mu \frac{\Delta_L}{4} \frac{\Gamma(\frac{d}{2}+1)^2}{\Gamma(d+2)}, \tag{2.29}$$

where

$$\mu := \frac{4\Gamma(d+2)}{(d-1)^2\Gamma(\frac{d}{2})^2} \frac{\Delta_H}{C_T}. \tag{2.30}$$

The correlator admits a natural perturbative expansion in μ

$$\mathcal{G}(z,\bar{z}) = \sum_k \mu^k \mathcal{G}^{(k)}(z,\bar{z}). \tag{2.31}$$

Using (2.28) and (2.29), we find the following contribution due to the stress tensor at $\mathcal{O}(\mu)$

$$\begin{aligned}\mathcal{G}^{(1)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} &\frac{(1-\bar{z})^{\frac{d-2}{2}}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{\Delta_L \Gamma(\frac{d}{2}+1)^2}{4\Gamma(d+2)} (1-z)^{\frac{d+2}{2}} \\ &\times {}_2F_1\Big(\frac{d+2}{2}, \frac{d+2}{2}; d+2; 1-z\Big).\end{aligned} \tag{2.32}$$

Let us study the correlator in powers of μ in the lightcone limit. At k-th order in that expansion we expect contributions from minimal-twist multi-stress tensor operators of the schematic form $[T^k]_{\tau,s} =: T_{\mu_1\nu_1}\ldots\partial_{\lambda_1}\ldots\partial_{\lambda_l}T_{\mu_k\nu_k}$:, where the minimal-twist τ and spin s of these operators are given by

$$\begin{aligned}\tau &= k(d-2)+\mathcal{O}(C_T^{-1}),\\ s &= 2k+l\end{aligned} \tag{2.33}$$

and l an even integer denoting the number of uncontracted derivatives. The scaling dimension, and the twist, is not protected and receives corrections in the C_T^{-1} expansion. We moreover define the product of OPE coefficients for minimal-twist operators at order k as

$$P^{(HH,LL)}_{[T^k]_{\tau,s}} = \mu^k P^{(HH,LL);(k)}_{\tau,s}. \tag{2.34}$$

Compared to the $k=1$ case, there exists an infinite number of minimal-twist multi-stress tensor operators for each value of $k>1$. To obtain their contribution to the correlator in the lightcone limit, we thus have to sum over all these operators.

The correlator can likewise be expanded in the "S-channel" $\mathcal{O}_L(z,\bar z)\times\mathcal{O}_H(0)\to\mathcal{O}_{\tau',s'}$ as

$$\mathcal{G}(z,\bar z) = (z\bar z)^{-\frac{1}{2}(\Delta_H+\Delta_L)}\sum_{\mathcal{O}_{\tau',s'}} P^{(HL,HL)}_{\mathcal{O}_{\tau',s'}} g^{(\Delta_{HL},-\Delta_{HL})}_{\tau',s'}(z,\bar z). \tag{2.35}$$

where $P^{(HL,HL)}_{\mathcal{O}_{\tau',s'}}$ are the products of the corresponding OPE coefficients and $\Delta_{HL}=\Delta_H-\Delta_L$. The operators contributing in the S-channel are "heavy-light double-twist operators" [8,1][15][16] that can be schematically written as $[\mathcal{O}_H\mathcal{O}_L]_{n,l} =: \mathcal{O}_H\partial^{2n}\partial^l\mathcal{O}_L$:, with scaling dimension $\Delta_{n,l}=\Delta_H+\Delta_L+2n+l+\gamma(n,l)$ and spin l. In the $\Delta_H\to\infty$ limit the $d=4$ blocks are given by

$$g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar z) \approx \frac{(z\bar z)^{\frac{1}{2}(\Delta_H+\Delta_L+2n+\gamma)}}{\bar z - z}\left(\bar z^{l+1}-z^{l+1}\right). \tag{2.36}$$

[15] This the naive analogue of light-light double-twist operators for large spin $l\gg 1$ that are present in the cross channel of $\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\mathcal{O}_1\rangle$, with $\mathcal{O}_1$ and $\mathcal{O}_2$ both light, in any CFT [23,22]. See, however, [65] for a recent interpretation in terms of quasi-normal modes.

[16] We expect that generic single-trace operators are not enhanced by factors of $\Delta_H\sim C_T$ and will therefore be subleading in the large C_T expansion.

The anomalous dimensions $\gamma(n,l)$ admit an expansion in μ[17]

$$\gamma(n,l)=\sum_{k=1}^{\infty}\mu^k\gamma_{n,l}^{(k)}. \tag{2.37}$$

Likewise, we expand the product of the OPE coefficients of the double-twist operators as

$$P_{n,l}^{(HL,HL)}=P_{n,l}^{(HL,HL);\mathrm{MFT}}\sum_{k=0}^{\infty}\mu^k P_{n,l}^{(HL,HL);(k)}, \tag{2.38}$$

with $P_{n,l}^{(HL,HL);(0)}=1$. The zeroth order OPE coefficients $P_{n,l}^{(HL,HL);\mathrm{MFT}}$ in the S-channel are those of Mean Field Theory found in [81]

$$\begin{aligned}P_{n,l}^{(HL,HL);\mathrm{MFT}}=&\frac{(\Delta_H+1-d/2)_{\bar h}(\Delta_L+1-d/2)_{\bar h}(\Delta_H)_h(\Delta_L)_h}{\bar h!(h-\bar h)!(\Delta_H+\Delta_L+\bar h+1-d)_{\bar h}(\Delta_H+\Delta_L+h+\bar h-1)_{h-\bar h}}\\&\times\frac{1}{(h-\bar h+d/2)_{\bar h}(\Delta_H+\Delta_L+h-d/2)_{\bar h}},\end{aligned} \tag{2.39}$$

where $\bar h=n$ and $h=n+l$[18] and $(a)_b$ is the Pochhammer symbol. In the limit $\Delta_H\to\infty$ they are given by

$$P_{n,l}^{(HL,HL);\mathrm{MFT}}\approx\frac{(\Delta_L-d/2+1)_n(\Delta_L)_{l+n}}{n!\,l!\,(l+d/2)_n}, \tag{2.40}$$

where $(a)_n$ denotes the Pochhammer symbol. For large l (2.40) simplifies

$$P_{n,l}^{(HL,HL);\mathrm{MFT}}\approx\frac{l^{\Delta_L-1}(\Delta_L-\frac{d}{2}+1)_n}{n!\,\Gamma(\Delta_L)}. \tag{2.41}$$

To reproduce the correct singularities manifest in the T-channel one has to sum over infinitely many heavy-light double-twist operators with $l\gg1$. For such

[17] The exact analytic structure is not known. However, the anomalous dimensions are related to the phase shift which has been calculated holographically to all orders in [8] with a finite radius of convergence. We further expect that there will be non-perturbative corrections which in the bulk are due to tunneling effects recently explored in [65].

[18] We will switch between using (n,l) and $(\bar h,h)$ as it should be clear from the context which is used.

operators the dependence of the OPE data on the spin l for $l \gg 1$ is[19]:

$$\begin{aligned} P_{n,l}^{(HL,HL);(k)} &= \frac{P_n^{(k)}}{l^{\frac{k(d-2)}{2}}}, \\ \gamma_{n,l}^{(k)} &= \frac{\gamma_n^{(k)}}{l^{\frac{k(d-2)}{2}}}. \end{aligned} \tag{2.42}$$

Note that generally the OPE data in the S-channel receives corrections needed to reproduce double-twist operators in the T-channel; however, since we are interested in the stress tensor sector we consider only contributions of the form given in (2.42).

2.2.2. Regge limit

The lightcone limit plays an important role in the analytical CFT bootstrap. It is typically realized inside a four-point function with all operators defined at a fixed time and then one considers the limit where one operator approaches the lightcone of another operator. As discussed above, this singles out operators with low-twist which is a part of the spectrum that is typically well understood [22,23]. Another interesting limit is the so-called Regge limit. It was first studied in detail in the context of AdS/CFT in [28-32] and has further played an important role in our understanding of CFTs, especially in holographic CFTs. For a review of the Regge limit, see e.g. Sec. 5.1 in [40].

The Regge limit of a four-point function is again a limit where operators become lightlike separated, it can be obtained by starting with all operators at a fixed timeslice and two operators in the left Rindler wedge and two in the right Rindler wedge. We now consider a limit where one operator in the left wedge and one in the right become close to lightlike separated while remaining in their respective Rindler wedges. However, in this limit, the operators in the left wedge become timelike separated and also the ones in the right wedge become timelike separated. By carefully keeping track of the ordering of the operators, one finds that in terms of the cross-ratios, we need to perform an analytic continuation before taking an OPE-like limit. This is described in

[19] This behaviour in the large l limit is different from that of the OPE data of light-light double-twist operators [23,22]. Note further that the small μ expansion is closely connected to the large-spin expansion.

detail in e.g. Sec. 5.1 in [40]. Compared to the lightcone limit, this will isolate (in one OPE channel) operators with highest spin rather than lowest twist. In generic CFTs we typically do not have good control over the high spin part of the spectrum, making it difficult to study the Regge limit in practice. However, in holographic CFTs with a large gap for higher-spin operators typically one has more control see e.g. [15,33-37].

The Regge limit for the heavy-heavy-light-light correlator was first explored in [8]. In order to study the Regge limit of the heavy-heavy-light-light correlator, it is convenient to introduce the following coordinates after the analytic continuation ($z \to ze^{-2i\pi}$):

$$\begin{aligned} 1-z &= \sigma e^{\rho} \\ 1-\bar{z} &= \sigma e^{-\rho}. \end{aligned} \tag{2.43}$$

The Regge limit then corresponds to $\sigma \to 0$ with ρ kept fixed[20]. We further refer to the $\rho \to \infty$ limit as the Regge-lightcone limit, it is related to large impact parameter in the bulk.

To approach the Regge limit we analytically continue $z \to \mathrm{e}^{-2\pi i} z$, under which the blocks in the S-channel transform as (see e.g. [82, 35])

$$g_{\Delta,J}(z,\bar{z}) \to \mathrm{e}^{-i\pi(\Delta-J)} g_{\Delta,J}(z,\bar{z}). \tag{2.44}$$

In particular, for double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ with scaling dimension $\Delta = \Delta_H + \Delta_L + 2n + l + \gamma(n,l)$, the blocks transform as

$$g^{\Delta_{HL},-\Delta_{HL}}_{[\mathcal{O}_H\mathcal{O}_L]_{n,l}}(z,\bar{z}) \to e^{-i\pi(\Delta_H+\Delta_L)} e^{-i\pi\gamma(n,l)} g^{\Delta_{HL},-\Delta_{HL}}_{[\mathcal{O}_H\mathcal{O}_L]_{n,l}}(z,\bar{z}). \tag{2.45}$$

In what follows it will be convienent to do a change of variables to $h = n + l$ and $\bar{h} = n$ and to denote the block due to a heavy-light double-trace operator $[\mathcal{O}_H\mathcal{O}_L]_{\bar{h},h-\bar{h}}$ as $g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}$. Substituting the μ expansion (2.37)-(2.38) in the S-channel (2.35) and performing the analytic continuation to $\mathcal{O}(\mu)$ leads to

$$\begin{aligned} G(z,\bar{z})|_{\mu^0} =& (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h} g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}) \\ G(z,\bar{z})|_{\mu^1} =& (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h} \Big(P^{(HL,HL);(1)}_{\bar{h},h} \\ &+ \gamma^{(1)} \Big(\frac{1}{2} (\partial_h + \partial_{\bar{h}}) - i\pi \Big) \Big) \times g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}). \end{aligned} \tag{2.46}$$

[20] Without the analytic continuation this would be an OPE limit with $z, \bar{z} \to 1$ with $(1-z)/(1-\bar{z})$ fixed.

The new single trace operators that can possibly appear here would be subleading in $1/C_T$. Continuing to $\mathcal{O}(\mu^2)$, the imaginary part of the S-channel is given by

$$
\begin{aligned}
\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = & -i\pi(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\times \\
& \times \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h}\Big(\gamma^{(2)}+\gamma^{(1)}P^{(HL,HL);(1)}_{\bar{h},h} \\
& + \frac{(\gamma^{(1)})^2}{2}(\partial_h+\partial_{\bar{h}})\Big)g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z}).
\end{aligned}
\tag{2.47}
$$

Moreover, the real part of the correlator at the same order is given by

$$
\begin{aligned}
\mathrm{Re}(G(z,\bar{z}))|_{\mu^2} = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h}\Big(P^{(HL,HL);(2)}_{\bar{h},h} - \frac{1}{2}\pi^2(\gamma^{(1)})^2+ \\
+\frac{1}{2}(\gamma^{(2)}+P^{(HL,HL);(1)}_{\bar{h},h}\gamma^{(1)})(\partial_h+\partial_{\bar{h}})+\frac{1}{8}(\gamma^{(1)})^2(\partial_h+\partial_{\bar{h}})^2\Big)g^{\Delta_{HL},-\Delta_{HL}}_{h,\bar{h}}(z,\bar{z})\,.
\end{aligned}
\tag{2.48}
$$

The MFT OPE coefficients are given in (2.39). As we will see in Section 5, in the Regge limit the dominant contribution in the S-channel comes from double-trace operators with $h \sim \bar{h} \gg 1$. In this limit the MFT OPE coefficients are given by

$$
P^{(HL,HL);\mathrm{MFT}}_{\bar{h},h} \approx C_{\Delta_L}(h\bar{h})^{\Delta_L-\frac{d}{2}}(h-\bar{h})^{\frac{d}{2}-1}. \tag{2.49}
$$

Let us now change perspective and consider the OPE in the direct-channel $\mathcal{O}_L \times \mathcal{O}_L$. Naively, at $\mathcal{O}(\mu^2)$ there are three infinite families of double-stress tensors of the following schematic form:

$$
\begin{aligned}
[T^2]^{(0)}_{n,l} &= :T_{\mu\nu}(\partial^2)^n\partial_{\mu_1}\ldots\partial_{\mu_l}T_{\rho\kappa}:, \\
[T^2]^{(1)}_{n,l} &= :T_{\mu\rho}(\partial^2)^n\partial_{\mu_1}\ldots\partial_{\mu_l}T^{\rho}{}_{\nu}:, \\
[T^2]^{(2)}_{n,l} &= :T_{\rho\kappa}(\partial^2)^n\partial_{\mu_1}\ldots\partial_{\mu_l}T^{\rho\kappa}:,
\end{aligned}
\tag{2.50}
$$

where the superscript denotes the number of contracted pair of indices between stress tensors and $n = 0, 1, 2, \ldots$ and $l = 0, 2, 4, \ldots$. The double-stress tensors in (2.50) have the following twist $\tau^{(2,i)}_{n,l}$ and spin $s^{(2,i)}_l$:

$$
\begin{aligned}
\tau^{(2,0)}_{n,l} &= 4+2n \qquad s^{(2,0)}_l = 4+l, \\
\tau^{(2,1)}_{n,l} &= 6+2n \qquad s^{(2,1)}_l = 2+l, \\
\tau^{(2,2)}_{n,l} &= 8+2n \qquad s^{(2,2)}_l = l.
\end{aligned}
\tag{2.51}
$$

From (2.51) it is seen that the operators in (2.50) have the same quantum numbers for suitable values of n, l and therefore possibly mix among each other. At $\mathcal{O}(\mu^k)$, for $k > 1$, there again exist different multi-stress tensor operators with overlapping quantum numbers, similar to the double-stress tensor case in (2.50).

We define the leading behaviour of the T-channel block in the Regge limit, that is the analytic continuation $z \to ze^{-2\pi i z}$ and then $\sigma \to 0$, by $g^{\circlearrowright}_{\Delta,J}(\sigma,\rho)$ and it is given by

$$
\begin{aligned}
g^{\circlearrowright,d=2}_{\Delta,J}(\sigma,\rho) &= i\bar{c}_{\Delta,J}\frac{e^{-(\Delta-1)\rho}}{\sigma^{J-1}}+\dots, \\
g^{\circlearrowright,d=4}_{\Delta,J}(\sigma,\rho) &= i\bar{c}_{\Delta,J}\frac{e^{-(\Delta-1)\rho}}{(1-e^{-2\rho})\sigma^{J-1}} \\
&\quad \times \left[1-\frac{\sigma}{4}\Big((\Delta+J-2)e^{\rho}+(2+J-\Delta)e^{-\rho}\Big)+\mathcal{O}(\sigma^2)\right]
\end{aligned}
\tag{2.52}
$$

with

$$
\bar{c}_{\Delta,J} = \frac{4^{\Delta+J-1}\Gamma\left(\frac{\Delta+J-1}{2}\right)\Gamma\left(\frac{\Delta+J+1}{2}\right)}{\Gamma(\frac{\Delta+J}{2})^2}. \tag{2.53}
$$

Here we have included the first subleading correction in $\sigma \to 0$ in four dimensions since this will be needed later on. More generally, the leading behaviour in the Regge limit in any dimension is given by, see e.g. [83,34],

$$
g^{\circlearrowright}_{\Delta,J}(\sigma,\rho) = ic_{\Delta,J}\sigma^{1-J}\Pi_{\Delta-1,d-1}(\rho)+\dots, \tag{2.54}
$$

where $\Pi_{\Delta-1,d-1}(\rho)$ is $(d-1)$-dimensional hyperbolic space propagator of a particle with mass-squared $m^2 = (\Delta-1)^2$

$$
\begin{aligned}
\Pi_{\Delta-1,d-1}(\rho) =& \frac{\pi^{1-\frac{d}{2}}\Gamma(\Delta-1)}{2\Gamma(\Delta-\frac{d-2}{2})}e^{-(\Delta-1)\rho} \\
& \times {}_2F_1\Big(\frac{d-2}{2},\Delta-1;\Delta-\frac{d-2}{2};e^{-2\rho}\Big),
\end{aligned}
\tag{2.55}
$$

and

$$
c_{\Delta,J} = \frac{4^{\Delta+J-1}\Gamma\left(\frac{\Delta+J-1}{2}\right)\Gamma\left(\frac{\Delta+J+1}{2}\right)}{\Gamma(\frac{\Delta+J}{2})^2}\,\frac{2\Gamma\left(\Delta-\frac{d}{2}+1\right)}{\pi^{1-\frac{d}{2}}\Gamma\left(\Delta-1\right)}. \tag{2.56}
$$

3. Leading Multi-Stress Tensors and Conformal Bootstrap

3.1. Introduction and summary of results

In two spacetime dimensions conformal symmetry is described by the infinite-dimensional Virasoro algebra. This symmetry strongly constrains correlators, especially when combined with the $C_T \to \infty$ limit. Of particular interest is the "heavy-heavy-light-light" correlator, which involves two "heavy" operators with conformal dimension $\Delta_H \sim C_T$ and two "light" operators with conformal dimension $\Delta_L \sim \mathcal{O}(1)$. In this case the contribution of the identity operator and all its Virasoro descendants is known as the Virasoro vacuum block and has been calculated in several ways [77-87]. The Virasoro vacuum block (and finite C_T corrections to it) is instrumental in a variety of settings, such as e.g. the problem of information loss [88-93] and properties of the Renyi and entanglement entropies [94-97] (see also [98,99] for the original applications of large C_T correlators in this context).

The heavy-heavy-light-light Virasoro vacuum block exponentiates (see e.g. [79])

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\mathcal{O}_H(0)\rangle \sim e^{\Delta_L \mathcal{F}(\mu;z)}, \tag{3.1}$$

with $\mathcal{F}$ a known function which admits an expansion in powers of $\mu \sim \Delta_H/C_T$

$$\mathcal{F}(\mu;z) = \sum_k \mu^k \mathcal{F}^{(k)}(z). \tag{3.2}$$

The explicit expression can be found in e.g. [79] and the expansion in small μ was studied in detail in [8] and is given by:

$$\mathcal{F}(\mu;z) = -\frac{1}{2}\log z - \log(-2\sinh(\frac{\bar{\alpha}}{2}\log z)) + \log\bar{\alpha}, \tag{3.3}$$

where $\bar{\alpha} = \sqrt{1-\mu}$.

One can consider contributions of various quasi-primaries made out of the stress tensor to $\mathcal{F}^{(k)}$. At $k=1$ the only such quasi-primary is the stress tensor itself, while for $k=2$ one needs to sum an infinite number of quasi-primaries quadratic in the stress tensor (double-stress operators) and labelled by spin. The situation is similar for all other values of k. It is possible to compute the OPE coefficients of the corresponding quasi-primaries, starting from the known result for the Virasoro vacuum block. Interestingly, at each order in μ, $\mathcal{F}^{(k)}$ can be written as a sum of particular terms [8][21]

$$\mathcal{F}^{(k)}(z) = \sum_{\{i_p\}} b_{i_1\dots i_k} f_{i_1} \dots f_{i_k}, \qquad \sum_{p=1}^{k} i_p = 2k, \tag{3.4}$$

where $f_a = f_a(1-z) = (1-z)^a {}_2F_1(a,a,2a,1-z)$.

It is an interesting question whether a similar structure appears when the number of spacetime dimensions d is greater than two. Unlike in two spacetime dimensions, in addition to spin, multi-stress tensor operators are also labelled by their twist. An interesting subset of multi-stress tensor operators is comprised out of those with minimal twist. These operators dominate in the lightcone limit over those of higher twist. In [80] an expression for the OPE coefficients of two scalars and minimal-twist double-stress tensor operators in $d = 4$ was obtained, and the sum was performed to obtain a remarkably simple expression for the near lightcone $\mathcal{O}(\mu^2)$ term in the heavy-heavy-light-light correlator. It was shown to have a similar form to (3.4). One may now wonder if the minimal-twist multi-stress tensor part of the correlator in higher dimensions exponentiates

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\big|_{\text{multi-stress tensors}} \sim e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \tag{3.5}$$

and whether $\mathcal{F}(\mu; z, \bar{z})$ can be expressed as

$$\mathcal{F}(\mu; z, \bar{z}) = \sum_k \mu^k \mathcal{F}^{(k)}(z, \bar{z}), \tag{3.6}$$

with

$$\mathcal{F}^{(k)}(z,\bar{z}) = (1-\bar{z})^{k(\frac{d-2}{2})} \sum_{\{i_p\}} b_{i_1\dots i_k} f_{i_1} \dots f_{i_k}, \qquad \sum_{p=1}^{k} i_p = k\left(\frac{d+2}{2}\right), \tag{3.7}$$

and d an even number.

In the section we investigate this following [2]. We start by assuming that the multi-stress tensor sector of the heavy-heavy-light-light correlator in the near lightcone regime $\bar{z} \to 1$ admits an expansion in μ[22]

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z,\bar{z})\mathcal{O}_H(0)\rangle\big|_{\text{multi-stress tensors}} \sim \sum_k \mu^k \mathcal{G}^{(k)}(z,\bar{z}), \tag{3.8}$$

[21] Similar expressions in a slightly different context appeared in [100].

[22] This is motivated by the fact that in the lightcone limit operators with low-twist dominates and each $[T^k]$ with minimal-twist $k(d-2)$ comes with a factor μ^k.

where each coefficient function $\mathcal{G}^{(k)}(z,\bar{z})$ takes a particular form:

$$\mathcal{G}^{(k)}(z,\bar{z}) = \frac{(1-\bar{z})^{k(\frac{d-2}{2})}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1\ldots i_k} f_{i_1}\ldots f_{i_k}, \qquad \sum_{p=1}^{k} i_p = k\left(\frac{d+2}{2}\right). \tag{3.9}$$

We subsequently use this ansatz to compute the contributions of the multi-stress tensor operators to the near lightcone correlator and extract the corresponding OPE coefficients.

For even d, the hypergeometric functions in (3.9) reduce to terms which contain at most one power of $\log(z)$ each. Their products contain multi-logs whose coefficients turn out to be rational functions of z. We use the conformal bootstrap approach initiated in [16] (for a review and references see eg. [71,74,101]) to relate these functions to the anomalous dimensions and OPE coefficients of the heavy-light double-twist operators in the cross channel. The ansatz (3.9) has just a few coefficients at any finite k which can be determined completely from the cross-channel data derived using the $(k-1)$th term. This is related to the fact that all the $\log^m(z)$ terms with $2 \leq m \leq k$ are completely determined by the anomalous dimensions and OPE coefficients at $\mathcal{O}(\mu^{k-1})$. At each step, we obtain an overconstrained system of equations solved by the same set of $a_{i_1\ldots i_k}$. This provides strong support to the ansatz (3.7). We then proceed to derive the OPE coefficients of the multi-stress tensor operators with two light scalars from our result. In practice, we complete this program to $\mathcal{O}(\mu^3)$ in $d=4$ and to $\mathcal{O}(\mu^2)$ in $d=6$[23]. However the procedure outlined can be easily generalised to arbitrary order in μ and any even d.

In [102] the authors considered holographic CFTs dual to gravitational theories defined by the Einstein-Hilbert Lagrangian plus higher derivative terms and a scalar field minimally coupled to gravity in AdS_{d+1}. Interpreting the scalar propagator in an asymptotically AdS_{d+1} black hole background as a heavy-heavy-light-light four point function, enabled the authors of [102] to extract the OPE coefficients of a few multi-stress tensor operators from holography (see also [103-105] for related work). Ref. [102] also argued that the OPE coefficients of the leading, minimal-twist multi-stress operators are universal – they do not depend on the gravitational higher derivative terms in the Lagrangian. Their results agree with the general expressions obtained in [2], upon substitution of the relevant quantum numbers.

[23] For $d=6$ results, we refer the reader to [2].

Summary of results

In this section we argue that for holographic CFTs in even d, the contribution of minimal-twist multi-stress tensors to the correlator in the lightcone limit can be written as a sum of products of the functions $f_a(z)$.
The stress tensor contribution to the correlator in the lightcone limit is given in any dimension d by

$$\mathcal{G}^{(1)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^{\frac{d-2}{2}}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{\Delta_L \Gamma(\frac{d}{2}+1)^2}{4\Gamma(d+2)} f_{\frac{d+2}{2}}. \tag{3.10}$$

At $\mathcal{O}(\mu^2)$ the contribution from twist-four double-stress tensor operators in $d=4$ is

$$\begin{aligned}\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} & \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L-2)}\right) \times \\ & \Big((\Delta_L-4)(\Delta_L-3) f_3^2 + \frac{15}{7}(\Delta_L-8) f_2 f_4 \\ & + \frac{40}{7}(\Delta_L+1) f_1 f_5\Big) .\end{aligned} \tag{3.11}$$

This result agrees with the expression obtained by different methods in [80].

The contribution from twist-six triple-stress tensors in the lightcone limit in $d=4$ at order $\mathcal{O}(\mu^3)$ is

$$\begin{aligned}\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} & \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(a_{117} f_1^2 f_7 \\ & + a_{126} f_1 f_2 f_6 + a_{135} f_1 f_3 f_5 + a_{225} f_2^2 f_5 \\ & + a_{234} f_2 f_3 f_4 + a_{333} f_3^3\Big),\end{aligned} \tag{3.12}$$

where coefficients a_{ijk} are given by Eq. (3.31).

Furthermore, from (3.12) and (3.31), we find the OPE coefficients of twist-six triple-stress tensor operators as a finite sum (for details see Section 3.2.5). Two such OPE coefficients for twist-6 triple-stress tensors were calculated holographically in [102] and agree with our results.

In general we propose in [2] that the minimal-twist multi-stress tensor contributions to the correlator in even d at $\mathcal{O}(\mu^k)$ in the lightcone limit is given by

$$\begin{aligned}& \mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^{k(\frac{d}{2}-1)}}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1\ldots i_k} f_{i_1} \ldots f_{i_k}, \\ & \sum_{p=1}^{k} i_p = k\Big(\frac{d+2}{2}\Big),\end{aligned} \tag{3.13}$$

where the sum goes over all sets of $\{i_p\}$ with $i_p \leq i_{p+1}$ and $a_{i_1 \ldots i_k}$ coefficients that need to be fixed.[24]

We also check that the stress tensor sector of the near lightcone correlator exponentiates[25]

$$\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \tag{3.14}$$

where $\mathcal{F}(\mu;z,\bar{z})$ is a rational function of Δ_L that remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$. We explicitly verify this up to $\mathcal{O}(\mu^3)$ in $d=4$ and to $\mathcal{O}(\mu^2)$ for $d=6$ in [2]. This section is organized as follows. In Section 3.2, we find the contribution of minimal-twist double- and triple-stress tensor operators in $d=4$ in the lightcone limit. We show that this contribution exponentiates and we write an expression for the OPE coefficients of minimal-twist triple-stress tensors of spin s with scalar operators, in the form of a finite sum. We end with a discussion in Section 3.3.

3.2. Multi-stress tensors in four dimensions

In this section we describe how to use crossing symmetry to fix the contribution of minimal-twist multi-stress tensors to the heavy-heavy-light-light correlator in $d=4$ to $\mathcal{O}(\mu^3)$. The methods described generalize to other even spacetime dimensions, with the six-dimensional case to $\mathcal{O}(\mu^2)$ described in [2]. In principle the same technology can also be used to determine the correlator at higher orders. Moreover, the resulting expression can be decomposed into multi-stress tensor blocks of minimal-twist, allowing us at each order to read off the OPE coefficients of minimal-twist multi-stress tensors.

The idea is to study the S-channel expansion in (2.35) in the limit $1-\bar{z} \ll z \ll 1$. In this limit operators with $l \gg 1$ and low values of n dominate. Expanding the conformal blocks in (2.36) for small $\gamma(n,l)$ and $\bar{z} \to 1$, the blocks in $d=4$ reduce to

$$(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} g^{(\Delta_{HL},-\Delta_{HL})}_{\Delta_H+\Delta_L+2n+\gamma,l}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \bar{z}^l p(\log z, \gamma(n,l)) \frac{z^n}{1-z}, \tag{3.15}$$

[24] One only needs to sum the linearly independent products of functions f_a.

[25] The leading large $\Delta_L \to \infty$ limit can be computed holographically by a geodesic analysis in the AdS BH blackground. Further subleading terms have been obtained in [106].

where $p(\log z, \gamma(n,l))$ is given by

$$p(\log z, \gamma(n,l)) = z^{\frac{1}{2}\gamma(n,l)} = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{\gamma(n,l)\log z}{2}\right)^j . \tag{3.16}$$

Inserting (3.15) into (2.35) and converting the sum into an integral, we have the following expression for the correlator in the limit $\bar{z} \to 1$

$$\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \sum_{n=0}^{\infty} \frac{z^n}{1-z}\int_0^{\infty} dl P_{n,l}^{(HL,HL)} \bar{z}^l p(\log z, \gamma(n,l)). \tag{3.17}$$

In the following we consider an expansion of (3.17) around $z = 0$. The key point is to note that by expanding the anomalous dimensions and OPE coefficients, as in (2.37) and (2.38) respectively, terms proportional to $z^p \log^i z$ with $i = 2, 3, \ldots, k$ and any p at $\mathcal{O}(\mu^k)$, in (3.17) are completely determined in terms of OPE data at $\mathcal{O}(\mu^{k-1})$. Moreover, using (2.42) one sees that the integral over the spin l yields

$$\int_0^{\infty} dl l^{\Delta_L - 1 - k} \bar{z}^l = \frac{\Gamma(\Delta_L - k)}{(-\log \bar{z})^{\Delta_L - k}} \underset{\bar{z}\to 1}{\approx} \frac{\Gamma(\Delta_L - k)}{(1-\bar{z})^{\Delta_L - k}}, \tag{3.18}$$

at $\mathcal{O}(\mu^k)$ in the limit $\bar{z} \to 1$. This correctly reproduces the expected $\bar{z}$ behaviour of minimal-twist multi-stress tensors in the T-channel, thus verifying (2.42).

We now make the following ansatz for the correlator

$$\mathcal{G}^{(k)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^k}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{\{i_p\}} a_{i_1 \ldots i_k} f_{i_1} \ldots f_{i_k}, \tag{3.19}$$

where the sum goes over all sets of $\{i_p\}$ with i_p integers and $i_p \leq i_{p+1}$ such that $\sum_{p=1}^k i_p = 3k$ and $a_{i_1 \ldots i_k}$ coefficients that need to be fixed. Generally $f_a(1-z)$ are given by

$$f_a(z) = q_{1,a}(z) + q_{2,a}(z)\log z, \tag{3.20}$$

where $q_{(1,2),a}(z)$ are rational functions and the ansatz (3.19) at $\mathcal{O}(\mu^k)$ is therefore a polynomial in $\log z$ of degree k. By crossing symmetry terms with $\log^a z$, with $2 \leq a \leq k$, are determined by OPE data at $\mathcal{O}(\mu^{k-1})$. This is what we will use to determine the coefficients $a_{i_1 \ldots i_p}$.

3.2.1. Stress tensor

We start by determining the OPE data at $\mathcal{O}(\mu)$. This is obtained by matching (3.17) at $\mathcal{O}(\mu)$ with the stress tensor contribution (2.32). Explicitly, multiplying both channels by $(1-z)$ we have at $\mathcal{O}(\mu)$

$$\begin{aligned}\frac{\Delta_L f_3(1-z)}{120[(1-z)(1-\bar{z})]^{\Delta_L-1}} =& \frac{1}{(1-\bar{z})^{\Delta_L-1}}\sum_{n=0}^{\infty}\frac{\Gamma(\Delta_L+n-1)z^n}{\Gamma(\Delta_L)n!}\Big(P_n^{(1)} \\ &+\frac{\gamma_n^{(1)}}{2}\log z\Big).\end{aligned} \tag{3.21}$$

Expanding the LHS in (3.21) for $z \ll 1$ we find

$$\begin{aligned}\frac{\Delta_L/120}{[(1-z)(1-\bar{z})]^{\Delta_L-1}}f_3(1-z) = \frac{1}{(1-\bar{z})^{\Delta_L-1}}\Big(&-\frac{\Delta_L}{4}(3+\log z) \\ &- z\frac{\Delta_L}{4}(3(\Delta_L+1)+(\Delta_L+5)\log z) \\ &- z^2\frac{\Delta_L}{8}\Big(3\Delta_L(\Delta_L+3) \\ &+(12+\Delta_L(\Delta_L+11))\Big) \\ &+\mathcal{O}(z^3, z^3\log z)\Big),\end{aligned} \tag{3.22}$$

while the RHS is given by

$$\begin{aligned}\frac{\sum_{n=0}^{\infty}\frac{\Gamma(\Delta_L+n-1)z^n}{\Gamma(\Delta_L)n!}(P_n^{(1)}+\frac{\gamma_n^{(1)}}{2}\log z)}{(1-\bar{z})^{\Delta_L-1}} &= \frac{1}{(1-\bar{z})^{\Delta_L-1}} \\ \times\Big(\frac{P_0^{(1)}+\frac{\gamma_0^{(1)}}{2}\log z}{\Delta_L-1} &+ z(P_1^{(1)}+\frac{\gamma_1^{(1)}}{2}\log z)+ \\ + z^2\frac{\Delta_L}{2}(P_2^{(1)}+\frac{\gamma_2^{(1)}}{2}\log z)+\mathcal{O}(z^3, z^3\log z)\Big)&.\end{aligned} \tag{3.23}$$

Comparing (3.22) and (3.23) order-by-order in z one finds the following OPE data

$$\begin{aligned}\gamma_0^{(1)} &= -\frac{\Delta_L(\Delta_L-1)}{2}, \\ \gamma_1^{(1)} &= -\frac{\Delta_L(\Delta_L+5)}{2}, \\ \gamma_2^{(1)} &= -\frac{12+\Delta_L(\Delta_L+11)}{2},\end{aligned} \tag{3.24}$$

which agrees with eq. (6.10) in [8], and the OPE coefficients

$$
\begin{aligned}
P_0^{(1)} &= -\frac{3\Delta_L(\Delta_L-1)}{4}, \\
P_1^{(1)} &= -\frac{3\Delta_L(\Delta_L+1)}{4}, \\
P_2^{(1)} &= -\frac{3\Delta_L(\Delta_L+3)}{4}.
\end{aligned} \tag{3.25}
$$

It is straightforward to continue and compute the $\mathcal{O}(\mu)$ OPE data in the S-channel for any value of n[26].

3.2.2. Twist-four double-stress tensors

From (3.19) we infer the following expression for the contribution due to twist-four double-stress tensors to the heavy-heavy-light-light correlator in the limit $\bar{z} \to 1$:

$$
\mathcal{G}^{(2)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(a_{15} f_1 f_5 + a_{24} f_2 f_4 + a_{33} f_3^2\Big). \tag{3.26}
$$

By expanding (3.26) further in the limit $z \ll 1$ and collecting terms that goes as $z^p \log^2 z$, we will match with known contributions obtained from (3.17).

Inserting (3.24) and (3.25) in the S-channel (3.17) fixes terms proportional to $z^p \log^2 z$ up to $\mathcal{O}(z^2 \log^2 z)$. Expanding the ansatz (3.26) and matching with the S-channel reproduces the result obtained in [80]:

$$
\begin{aligned}
\mathcal{G}^{(2)}(z,\bar{z}) &\underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^2}{[(1-z)(1-\bar{z})]^{\Delta_L}} \left(\frac{\Delta_L}{28800(\Delta_L-2)}\right) \times \\
&\Big\{(\Delta_L-4)(\Delta_L-3) f_3^2 + \frac{15}{7}(\Delta_L-8) f_2 f_4 + \frac{40}{7}(\Delta_L+1) f_1 f_5\Big\}.
\end{aligned} \tag{3.27}
$$

Using the $\mathcal{O}(\mu)$ OPE data in the S-channel for $n > 2$ in (3.22) and (3.23) one gets an overconstrained system which is still solved by (3.27). This is a strong argument in favor of the validity of our ansatz (3.19).

We can now use (3.27) to derive the $\mathcal{O}(\mu^2)$ OPE data in the S-channel by matching terms proportional to $z^p \log^i z$ as $z \to 0$, with $i = 0, 1$, by comparing

---

[26] One can then do the sum over n and explicitly recover the full light-cone limit of the stress tensor block.

with (3.17). This is done in the same way it was done for $\mathcal{O}(\mu)$ OPE data in the S-channel. For example, one finds the following data for $n = 0, 1, 2, 3$:

$$
\begin{aligned}
\gamma_0^{(2)} &= -\frac{(\Delta_L-1)\Delta_L(4\Delta_L+1)}{8}, \\
\gamma_1^{(2)} &= -\frac{\Delta_L(\Delta_L+1)(4\Delta_L+35)}{8}, \\
\gamma_2^{(2)} &= -\frac{(3+\Delta_L)(68+\Delta_L(69+4\Delta_L))}{8}, \\
\gamma_3^{(2)} &= -\frac{(5+\Delta_L)(204+\Delta_L(4\Delta_L+103))}{8},
\end{aligned} \tag{3.28}
$$

which agrees with Eq. (6.39) in [8], and for the OPE coefficients

$$
\begin{aligned}
P_0^{(2)} &= \frac{(\Delta_L-1)\Delta_L(-28+\Delta_L(-145+27\Delta_L))}{96}, \\
P_1^{(2)} &= \frac{\Delta_L(-596+\Delta_L(-399+\Delta_L(-64+27\Delta_L)))}{96}, \\
P_2^{(2)} &= \frac{-1248+\Delta_L(-2252+\Delta_L(-699+\Delta_L(44+27\Delta_L)))}{96}, \\
P_3^{(2)} &= \frac{-3744+\Delta_L(-4940+\Delta_L(-783+\Delta_L(152+27\Delta_L)))}{96}.
\end{aligned} \tag{3.29}
$$

It is again straightforward to extract the OPE data for any value of n.

3.2.3. Twist-six triple-stress tensors

We now consider the multi-stress tensor sector of the correlator at $\mathcal{O}(\mu^3)$ and proceed similarly to the previous section. From (3.19) we infer the following expression for the contribution due to twist-six triple-stress tensors:

$$
\begin{aligned}
\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \Big(& a_{117} f_1^2 f_7 + a_{126} f_1 f_2 f_6 + a_{135} f_1 f_3 f_5 \\
& + a_{225} f_2^2 f_5 + a_{234} f_2 f_3 f_4 + a_{333} f_3^3 \Big),
\end{aligned} \tag{3.30}
$$

where $f_i = f_i(1-z)$ is given by (1.16).[27] Taking the limit $1-\bar{z} \ll z \ll 1$ of (3.30), we fix the coefficients by matching with terms proportional to $z^p \log^2 z$

[27] Note that we omitted a potential term of the form $f_1 f_4^2$. This can be written in terms of f_3^3, $f_1 f_3 f_5$, $f_2^2 f_5$ and $f_2 f_3 f_4$:

$$
f_3^3 = \frac{20}{21} f_1 f_3 f_5 - \frac{27}{28} f_1 f^2 f_4 - \frac{20}{21} f_2^2 f_5 + \frac{55}{28} f_2 f_3 f_4. \tag{3.31}
$$

and $z^p \log^3 z$, with $p = 0, 1, 2$ from (3.17). This requires using the OPE data of the heavy-light double-twist operators $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ for $n = 0, 1, 2$ and $l \gg 1$ to $\mathcal{O}(\mu^2)$, given in (3.24), (3.25), (3.28) and (3.29).

We find the following solution:

$$
\begin{aligned}
a_{117} &= \frac{5\Delta_L(\Delta_L+1)(\Delta_L+2)}{768768(\Delta_L-2)(\Delta_L-3)}, \\
a_{126} &= \frac{5\Delta_L(5\Delta_L^2-57\Delta_L-50)}{6386688(\Delta_L-2)(\Delta_L-3)}, \\
a_{135} &= \frac{\Delta_L(2\Delta_L^2-11\Delta_L-9)}{1209600(\Delta_L-3)}, \\
a_{225} &= -\frac{\Delta_L(7\Delta_L^2-51\Delta_L-70)}{2903040(\Delta_L-2)(\Delta_L-3)}, \\
a_{234} &= \frac{\Delta_L(\Delta_L-4)(3\Delta_L^2-17\Delta_L+4)}{4838400(\Delta_L-2)(\Delta_L-3)}, \\
a_{333} &= \frac{\Delta_L(\Delta_L-4)(\Delta_L^3-16\Delta_L^2+51\Delta_L+24)}{10368000(\Delta_L-2)(\Delta_L-3)}.
\end{aligned}
\tag{3.32}
$$

We can also consider higher values of p and obtain an overconstrained system of equations, whose solution is still (3.32). Inserting (3.32) into (3.30), we obtain the contribution from minimal-twist triple-stress tensor operators to the heavy-heavy-light-light correlator in the lightcone limit.

Note that for $\Delta_L \to \infty$, the correlator is determined by the exponentiation of the stress tensor, discussed e.g. in [80], i.e.

$$
\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \frac{1}{3!} \left(\frac{\Delta_L}{120}(1-z)^3 {}_2F_1(3,3;6;1-z) \right)^3 + \cdots , \tag{3.33}
$$

which one indeed obtains by taking $\Delta_L \to \infty$ of (3.30) with (3.32). Here ellipses denote terms subleading in Δ_L.

By analytically continuing $z \to e^{-2\pi i} z$ and sending $z \to 1$, one can access the large impact parameter regime of the Regge limit. To do this we use the following property of the hypergeometric function (see e.g. [25]):

$$
{}_2F_1(a,a,2a,1-ze^{-2\pi i}) = {}_2F_1(a,a,2a,1-z) + 2\pi i \frac{\Gamma(2a)}{\Gamma(a)^2} {}_2F_1(a,a,1,z). \tag{3.34}
$$

Using (3.34) the leading term from (3.30) with the coefficients (3.32) in the limit $1-\bar{z} \ll 1-z \ll 1$ is given by

$$\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1, z\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} \times \left(-\frac{9i\pi^3\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{2(\Delta_L-2)(\Delta_L-3)}\left(\frac{1-\bar{z}}{(1-z)^2}\right)^3\right). \tag{3.35}$$

This agrees with the holographic calculation in a shockwave background at $\mathcal{O}(\mu^3)$ given by Eq. (45) in [104] based on techniques developed in [28-32]. The Regge limit will be discussed further in Section 5 and 6.

3.2.4. Exponentiation of leading-twist multi-stress tensors

In $d = 2$ the heavy-heavy-light-light correlator is determined by the heavy-heavy-light-light Virasoro vacuum block. This block contains the exchange of any number of stress tensors and derivatives thereof in the T-channel [77,78,87], and therefore all multi-stress tensor contributions. This block, together with the disconnected part, exponentiates as

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_L(1)\mathcal{O}_L(z)\mathcal{O}_H(0)\rangle = e^{\Delta_L \mathcal{F}(z)}, \tag{3.36}$$

for a known function $\mathcal{F}(z)$ independent of Δ_L. It is interesting to ask if something similar happens for the contribution of the minimal-twist multi-stress tensors in the lightcone limit of the correlator in higher dimensions. By this we mean whether the stress tensor sector of the correlator can be written as

$$\mathcal{G}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}} e^{\Delta_L \mathcal{F}(\mu;z,\bar{z})}, \tag{3.37}$$

for some function $\mathcal{F}(\mu;z,\bar{z})$ which is a rational function of Δ_L and remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$.

The $\bar{z}$ dependence implies the following form of $\mathcal{F}(\mu;z,\bar{z})$:

$$\mathcal{F}(\mu;z,\bar{z}) = \mu(1-\bar{z})\mathcal{F}^{(1)}(z) + \mu^2(1-\bar{z})^2\mathcal{F}^{(2)}(z) + \mu^3(1-\bar{z})^3\mathcal{F}^{(3)}(z) + \mathcal{O}(\mu^4). \tag{3.38}$$

At leading order we observe $\mathcal{F}^{(1)}(z) = \frac{1}{120}f_3(1-z)$, which is just the stress tensor contribution. At second order we find:

$$\mathcal{F}^{(2)}(z) = \frac{(12-5\Delta_L)f_3^2 + \frac{15}{7}(\Delta_L-8)f_2f_4 + \frac{40}{7}(\Delta_L+1)f_1f_5 s}{28800(\Delta_L-2)}. \tag{3.39}$$

Note that $\mathcal{F}^{(2)}(z)$ is independent of Δ_L in the limit $\Delta_L \to \infty$.

To find $\mathcal{F}^{(3)}(z)$ we parametrise it as

$$\mathcal{F}^{(3)}(z) = \Big(b_{117} f_1^2 f_7 + b_{126} f_1 f_2 f_6 + b_{135} f_1 f_3 f_5 + b_{225} f_2^2 f_5 + b_{234} f_2 f_3 f_4 + b_{333} f_3^3\Big). \tag{3.40}$$

It is clear that for terms which do not contain a factor of $f_3(z)$, the coefficients b_{ijk} should satisfy $b_{ijk} = a_{ijk}/\Delta_L$. This is not true for terms which contain a factor of f_3. Inserting $\mathcal{F}^{(1)}$, $\mathcal{F}^{(2)}$ and Eq. (3.40) in (3.37), expanding in μ and matching with (3.30) yields

$$\begin{aligned}
b_{117} &= \frac{a_{117}}{\Delta_L},\\
b_{126} &= \frac{a_{126}}{\Delta_L},\\
b_{225} &= \frac{a_{225}}{\Delta_L},\\
b_{135} &= -\frac{11\Delta_L^2 - 19\Delta_L - 18}{1209600(\Delta_L-2)(\Delta_L-3)},\\
b_{234} &= \frac{(\Delta_L-2)(\Delta_L+2)}{1209600(\Delta_L-2)(\Delta_L-3)},\\
b_{333} &= \frac{7\Delta_L^2 - 18\Delta_L - 24}{2592000(\Delta_L-2)(\Delta_L-3)}.
\end{aligned} \tag{3.41}$$

From (3.39) and (3.41), one finds that the correlator exponentiates to $\mathcal{O}(\mu^3)$ in the sense described above, i.e. $\mathcal{F}(\mu; z, \bar{z})$ is a rational function of Δ_L of $\mathcal{O}(1)$ as $\Delta_L \to \infty$.

To leading order in Δ_L, exponentiation for large Δ_L is a prediction of the AdS/CFT correspondence. The two-point function of the operator $\mathcal{O}_L$ in the state created by the heavy operator $\mathcal{O}_H$ is given in terms of the exponential of the (regularized) geodesic distance between the boundary points in the dual bulk geometry. For details on this, see e.g. [80].

3.2.5. OPE coefficients of triple-stress tensors

In this section we describe how to decompose the correlator (3.30) into an infinite sum of minimal-twist triple-stress tensor operators. In order to do this we use the following multiplication formula for hypergeometric functions [80]:

$$\begin{aligned}
{}_2F_1(a,a;2a;w)\,{}_2F_1(b,b;2b;w) = &\sum_{m=0}^{\infty} p[a,b,m]\, w^{2m}\\
&\times {}_2F_1[a+b+2m, a+b+2m, 2a+2b+4m, w],
\end{aligned} \tag{3.42}$$

where

$$
\begin{aligned}
p[a,b,m] = &\frac{2^{-4m}\Gamma(a+\frac{1}{2})\Gamma(b+\frac{1}{2})}{\sqrt{\pi}\Gamma(a)\Gamma(b)\Gamma(m+1)} \\
\times &\frac{\Gamma(m+\frac{1}{2})\Gamma(a+m)\Gamma(b+m)\Gamma(a+b+m-\frac{1}{2})\Gamma(a+b+2m)}{\Gamma(a+m+\frac{1}{2})\Gamma(b+m+\frac{1}{2})\Gamma(a+b+m)\Gamma(a+b+2m-\frac{1}{2})}.
\end{aligned} \tag{3.43}
$$

It is useful to note that by using (3.42) we can write a similar formula for the functions f_a defined in (1.16):

$$
f_a(z)f_b(z) = \sum_{m=0}^{\infty} p[a,b,m] f_{a+b+2m}(z), \tag{3.44}
$$

where $p[a,b,m]$ is defined in (3.43). It is now clear that the correlator (3.30) can be written as a double sum over functions $f_{9+2(n+m)}$. We can thus write the stress tensor sector of the correlator in the lightcone limit at $\mathcal{O}(\mu^3)$ as

$$
\mathcal{G}^{(3)}(z,\bar{z}) \underset{\bar{z}\to 1}{\approx} \frac{(1-\bar{z})^3}{[(1-z)(1-\bar{z})]^{\Delta_L}} \sum_{n,m=0}^{\infty} c[m,n] f_{9+2(n+m)}(z), \tag{3.45}
$$

with

$$
\begin{aligned}
c[m,n] =&\big(a_{333}p[3,3,m]p[3,6+2m,n] + a_{117}p[1,7,m]p[1,8+2m,n] \\
&+ a_{126}p[2,6,m]p[1,8+2m,n] + a_{135}p[3,5,m]p[1,8+2m,n] \\
&+ a_{225}p[2,5,m]p[2,7+2m,n] + a_{234}p[3,4,m]p[2,7+2m,n]\big),
\end{aligned} \tag{3.46}
$$

where coefficients a_{ijk} are fixed in (3.32).

Comparing (3.45) with (2.28) we see that the contribution at $\mathcal{O}(\mu^3)$ comes from operators of the schematic form $:T_{\alpha\beta}T_{\gamma\delta}\partial_{\rho_1}\dots\partial_{\rho_{2l}}T_{\mu\nu}:$. These operators have $\frac{\tau}{2}+s = 9+2l$, where s is total spin $s = 6+2l$. The corresponding OPE coefficients of such operators will be a sum of all contributions in (3.45) for which $n+m = l$.

Now, one can write OPE coefficients of operators of type $:T_{\alpha\beta}T_{\gamma\delta}\partial_{\rho_1}\dots\partial_{\rho_{2l}}T_{\mu\nu}:$ as

$$
P^{(HH,LL);(3)}_{6,6+2l} = \sum_{n=0}^{l} c[l-n,n]. \tag{3.47}
$$

Let us write a few of the coefficients explicitly here:

$$\begin{aligned}
\mu^3 P_{6,6}^{(HH,LL);(3)} &= \mu^3 \frac{\Delta_L(3024+\Delta_L(7500+\Delta_L(7310+143\Delta_L(25+7\Delta_L))))}{10378368000(\Delta_L-2)(\Delta_L-3)}, \\
\mu^3 P_{6,8}^{(HH,LL);(3)} &= \mu^3 \frac{\Delta_L(2688+\Delta_L(7148+\Delta_L(9029+13\Delta_L(464+231\Delta_L))))}{613476864000(\Delta_L-3)(\Delta_L-2)}, \\
\mu^3 P_{6,10}^{(HH,LL);(3)} &= \mu^3 \frac{\Delta_L(888+\Delta_L(2216+\Delta_L(3742+17\Delta_L(181+143\Delta_L))))}{9468531072000(\Delta_L-3)(\Delta_L-2)}.
\end{aligned} \tag{3.48}$$

We further find that $P_{6,6}^{(HH,LL);(3)}$ and $P_{6,8}^{(HH,LL);(3)}$ agree with the expression obtained holographically in [102].

3.3. Discussion

In this section we considered the minimal-twist multi-stress tensor contributions to the heavy-heavy-light-light correlator of scalars in large C_T CFTs in even spacetime dimensions. We provide strong evidence for the conjecture that all such contributions are described by the ansatz (3.13) and determine the coefficients by performing a bootstrap procedure. In practice this is completed for twist-four double-stress tensors and twist-six triple-stress tensors in four dimensions as well as twist-eight double-stress tensors in six dimensions. In principle it is straightforward to use our technology to determine the coefficients $a_{i_1 \ldots i_k}$ to arbitrarily high order in μ; this must be related to the universality of the minimal-twist OPE coefficients.

In two dimensions the heavy-heavy-light-light Virasoro vacuum block exponentiates [see eq. (3.1)], with $\mathcal{F}(\mu; z)$ independent of Δ_L. In higher dimensions we observe a similar exponentiation with $\mathcal{F}(\mu; z, \bar{z})$ a rational function of Δ_L that remains $\mathcal{O}(1)$ as $\Delta_L \to \infty$. It would be interesting to see whether it is possible to write down a closed-form recursion formula for $\mathcal{F}(\mu; z, \bar{z})$. Solving such a recursion formula would give a higher-dimensional analogue of the two-dimensional Virasoro vacuum block.

An immediate technical question concerns CFTs in odd spacetime dimensions. We could not immediately generalize our results in this context – the ansatz in eq. (3.13) fails in odd dimensions. However, the heavy-light conformal blocks are known [1], so a similar approach should be feasible.

Another interesting direction concerns the study of the bulk scattering phase-shift in the presence of a black hole background. In the context of higher dimensional CFTs, this problem was first considered in [8] where the gravitational expression was given to all orders in μ and the CFT computation was performed to $\mathcal{O}(\mu)$. Subsequently, $\mathcal{O}(\mu^2)$ was discussed in [1]. In [104] the $\mathcal{O}(\mu)$ contribution was exponentiated to yield the scattering phase shift in the presence of a shock-wave geometry. A CFT computation of the phase shift to all orders in μ is still lacking. This would in principle involve understanding Regge theory beyond the leading order. It would be interesting to see whether the results of this section could be helpful in this regard.

4. CFT correlators, $\mathcal{W}$-algebras and Generalized Catalan Numbers

4.1. Introduction and summary of results

The Virasoro algebra induces a natural decomposition of correlation functions into Virasoro conformal blocks, capturing the contribution from a given Virasoro primary and all its Virasoro descendants. With respect to the global conformal algebra, each Virasoro representation contains an infinite number of quasi-primaries – the Virasoro symmetry therefore imposes strong constraints on the theory as seen from the perspective of someone that only knew about its global part. Further, the presence of symmetries in CFTs is deeply connected to universal features. An example is Cardy's formula for the density of high energy of states in two-dimensional CFTs [108]. It follows from the large conformal transformation of the torus and the dominance of the lowest dimension operator in the partition function in the low-temperature limit.

A priori, the multi-stress tensor $[T^k]$ OPE coefficients in the OPE of identical scalar operators, $[T^k]_{\tau,s} \subset \mathcal{O}_\Delta \times \mathcal{O}_\Delta$, are not fixed by symmetries in $d > 2$, in contrast to the two-dimensional case. These operators are, however, ubiquitous in theories with gravity duals since they are related to the exchange of multi-graviton states in the bulk. In order to understand the emergence of gravity in the bulk from the CFT data on the boundary, these operators play a vital role. It is further interesting to ask if there is a notion of universality in the exchanges of multi-stress tensors in holographic CFTs with large C_T and a large gap in the spectrum of higher-spin single trace operators.

An important case where the exchange of these multi-stress tensors is expected to dominate compared to that of generic operators is when considering heavy states. This is so because the OPE coefficients of multi-stress tensors $[T^k]$ in a scalar OPE $\mathcal{O}_\Delta \times \mathcal{O}_\Delta$ scale like Δ^k for large Δ[28]. An extreme example of this is when the heavy states have dimension Δ of order C_T. Such heavy states are expected to thermalize in holographic CFTs and according to the AdS/CFT dictionary, thermal states on the boundary are dual to black holes in the bulk. Correlation functions of light operators in heavy states therefore

[28] This was seen explicitly in the previous section where these contributions were studied in the lightcone limit. Holographically, this is also expected from a geodesic calculation at large scaling dimension (mass).

provide a possible window into one of the most interesting questions in the AdS/CFT correspondence, the physics of black holes.

As explained in Section 3, progress can be made using the conformal bootstrap techniques as well as using the gravitational dual description. In Section 3 based on [2], see also [80], it was argued that the contribution of all minimal-twist operators $[T^k]_{\tau_{\min},s}$, with $\tau_{\min} = 2k$ and spin $s = 2k + l$ for $l = 0, 2, 4, \ldots$, in holographic CFTs, takes a specific form which is reminiscent to that obtained from the Virasoro vacuum block. It repackages an infinite number of minimal-twist multi-stress tensor OPE coefficients in the HHLL correlator and it is natural to ask if this is governed by an underlying emergent symmetry. It would play a role similar to how the Virasoro algebra determines the heavy-heavy-light-light vacuum blocks in $d = 2$.

In this section based on [3], we study the HHLL vacuum blocks in two-dimensional CFTs with $\mathcal{W}_N$ higher-spin symmetry[29], see [109-113] for related work,

$$\mathcal{G}_N(z) := \langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle|_{1_{\mathcal{W}_N}}, \tag{4.1}$$

where the $|_{1_{\mathcal{W}_N}}$ denotes that we restrict to the $\mathcal{W}_N$ vacuum block, i.e. the exchange of all operators that are $\mathcal{W}_N$ descendants of the vacuum[30]. The semi-classical vacuum blocks were found for $N = 3$ in [111,114] and for general N in [112]. In this case, the charges of the "light" operator are large but much smaller than those of the heavy operator which scale with the central charge $c \gg 1$. Expanding the $\mathcal{W}_N$ vacuum blocks in $\frac{q_H^{(i)}}{c}$, where $q_H^{(i)}$ is the spin$-i$ charge of the heavy operator, we find that the result is again similar to the expansion of the Virasoro vacuum block, with a decomposition in terms of composite operators with the correct weight under the global conformal algebra.

[29] We will mainly consider $N = 3, 4$ but the methods used and the structure remains similar for any N.

[30] The precise correlator will be defined below. Note that compared to the $d = 4$ case discussed above, the light operators are inserted at $\mathcal{O}_L(0)$ and $\mathcal{O}_L(z)$ since this simplifies some of the calculations in $d = 2$. This is the same as in the Virasoro case in Section 2.1.

In particular, when $q_H^{(3)} \sim c \gg q_H^{(i\neq 3)}$, the dominant contributions[31] are due to composite quasi-primary operators with the schematic form $[W^k]_{2l}$ made out of the spin-three current $W(z)$. The resulting functions, which are linear combinations of products of hypergeometric functions, are also present in the result for the minimal-twist stress tensor sector of the $d = 4$ HHLL correlator. This is one of the main motivations for our work.

We further explicitly compute the first few terms of the $\mathcal{W}_N$ HHLL vacuum blocks for $N = 3, 4$ in the limit $q_H^{(3)} \sim c \gg q_H^{(i\neq 3)}$ using an explicit mode calculation. This limit has the advantage that the charges of the light operators are kept fixed as $c \to \infty$ and sheds further light on how the resulting structure that appears in the four-dimensional stress tensor sector of the HHLL correlator could appear from an underlying symmetry algebra. The results agree with those obtained from the expansion of the semi-classical vacuum blocks which assumed that the charges of the light operators were large. This gives further evidence that those results remain true also for finite charge. The mode calculation presented in this work can in principle also be used to compute $\frac{1}{c}$ corrections to the HHLL vacuum blocks.

Focusing on the logarithm of the $\mathcal{W}_3$ HHLL vacuum block we further show that it satisfies a non-linear differential equation which, in a certain limit, reduces to a cubic equation for the generating function for the sequence of integers given by A085614 in [115]. The $\mathcal{W}_3$ HHLL vacuum block can also be obtained from a set of diagrammatic rules similar to the Virasoro vacuum block [79]. The story can be generalized in the case of the $\mathcal{W}_4$ HHLL block both in the limit where the spin-4 charge scales with the central charge and is parametrically larger than all other charges and in the limit where the spin-3 charge scales with the central charge and is parametrically larger than the rest of the charges. We expect a similar story to hold for all $\mathcal{W}_N$ blocks. From a mathematician's point of view, the $\mathcal{W}_N$ vacuum blocks provide generating functions

[31] Note that it is only the spin-3 charge of the "heavy" operators that scales with c and, in particular, their scaling dimension is small compared to c. We will still refer to these as heavy. It is possible to extend our results to the case when all the charges of the heavy operators are large but we will not attempt to do so since it is the spin-3 sector that resembles the stress tensor sector in four dimensions.

for several new sequences which can be understood as different generalizations of the Catalan numbers' sequence.

Further, we examine the stress tensor sector of the four-dimensional HHLL correlator when the conformal dimension of the light operator vanishes, $\Delta_L \to 0$[32]. A similar picture emerges with the relevant sequence of numbers given by the number of linear extensions of the one-level grid partially ordered set (poset)[33] $G[(1^{k-1}), (0^{k-2}), (0^{k-2})]$.[34] We observe the same structure appearing in $d = 6, 8$ as well. In this case, the sequences of numbers are related to the linear extensions of the $G[(\frac{d-2}{2})^{k-1}, (0)^{k-2}, (0^{k-2})]$ posets. In the spirit of the two-dimensional cases examined here, one would hope that knowing the algebraic equation satisfied by the generating function of this sequence, would allow the determination of a differential equation satisfied by the all-orders stress-tensor sector of the HHLL correlator in the lightcone limit for $\Delta_L \to 0$. However, to our knowledge, the generating functions of the number of linear extensions of $G[(\frac{d-2}{2})^{k-1}, (0)^{k-2}, (0^{k-2})]$ are not known.

Consider a heavy-heavy-light-light (HHLL) four-point function in a two-dimensional CFT with a large central charge c and a higher-spin $\mathcal{W}_N$ symmetry $\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle$. The operators $\mathcal{O}_H$ and $\mathcal{O}_L$ are $\mathcal{W}_N$ primaries and carry higher-spin charges $q_H^{(i)}$ and $q^{(i)}$, with $i = 2, 3, \ldots, N$, respectively. Such a four-point function can be decomposed into blocks which contain contributions from a $\mathcal{W}_N$ primary $\mathcal{O}$ and all its $\mathcal{W}_N$-descendants. We define $\mathcal{G}_N(z)$ as the holomorphic part of the HHLL correlator restricted to the identity block contribution in the direct channel $\mathcal{O}_L \times \mathcal{O}_L \to 1_{\mathcal{W}_N} \to \mathcal{O}_H \times \mathcal{O}_H$. We specify our discussion to the cases $N = 3, 4$ although it can be generalized to any N.

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[32] Note that this is below the unitarity bound. However, certain observables are independent on Δ_L, such as the phase shift. Obtaining a closed-form expression in this limit might be a step towards obtaining such observables to all orders from the CFT.

[33] Partially ordered sets (posets) have a notion of ordering between some of the elements but not necessarily all of them. A linear extension of a partial ordering is a linear extension to a totally ordered set where all the elements are ordered in such a way that the original partial ordering is preserved.

[34] The Catalan numbers are also the numbers of linear extensions of the one-level grid poset $G[(0^{k-1}), (0^{k-2}), (0^{k-2})]$.

We start by considering the case $N = 3$ where the CFT protagonists are the stress tensor $T(z)$ and a spin-3 field $W(z)$. $\mathcal{G}_3(z)$ contains the exchange of all states schematically denoted by

$$|\{a_i, b_j\}\rangle := W_{a_1} W_{a_2} \dots W_{a_n} L_{b_1} L_{b_2} \dots L_{b_k} |0\rangle - (\dots)|0\rangle, \tag{4.2}$$

where L_b and W_a are the modes of $T(z)$ and $W(z)$, respectively, and the ellipses ensure that these states are mutually orthogonal. In particular, the subsector consisting of only states with modes L_b acting on the vacuum is that of the Virasoro vacuum block and was studied in detail in [79]. We are interested in heavy states with a large spin-3 charge $w_H \equiv q_H^{(3)}$ with[35]

$$\begin{aligned} &h_H \ll w_H \sim c \to \infty, \\ &h, w \ll c. \end{aligned} \tag{4.3}$$

The effect of using (4.3) is that the dominant contribution to $\mathcal{G}_3(z)$ is due to states of the form

$$|\{a_i\}\rangle = W_{a_1} W_{a_2} \dots W_{a_n} |0\rangle - (\dots)|0\rangle \tag{4.4}$$

because each W-mode will to leading order contribute a factor of w_H when acting on the heavy operators. Inserting the projection on the single mode states $W_{-m}|0\rangle$ in the correlator one finds the $\mathcal{O}(\frac{w_H}{c})$ term of the vacuum block

$$\mathcal{G}_3(z)\Big|_{\frac{w_H}{c}} = \frac{3 w w_H}{c} \frac{f_3(z)}{z^{2h}}, \tag{4.5}$$

where z^{-2h} is the disconnected correlator. The result in (4.5) is the conformal block due to the exchange of the quasi-primary $W(z)$ and all its descendants under the global conformal group.

It is useful to recall the behavior of a d-dimensional conformal block, $g_{\tau,s}^{(0,0)}(z,\bar{z})$, in the lightcone limit $\bar{z} \to 0$

$$g_{\tau,s}^{(0,0)}(z,\bar{z}) \sim \bar{z}^{\frac{\tau}{2}} f_{\frac{\tau}{2}+s}(z). \tag{4.6}$$

[35] It is straightforward to extend our results to the case when all the heavy charges are $\mathcal{O}(c)$ but we will not attempt to do so. See however Appendix A.1 and A.2. For notational simplicity, we drop the light subscript on the light operators.

In four dimensions, the stress tensor block with $\tau = s = 2$ has the same z-dependence as (4.5) (as can be seen from (4.6)).

Going back to $d = 2$, we consider the $\mathcal{O}(\frac{w_H^2}{c^2})$ contribution to $\mathcal{G}_3(z)$. This is due to the (unnormalized) states

$$|Y_{m,n}\rangle = \Big[W_{-n}W_{-m} - \frac{(3n+2m)m(m^2-1)(m^2-4)}{30(m+n)((m+n)^2-1)}L_{-m-n}\Big]|0\rangle, \qquad (4.7)$$

where the second term ensures that they are orthogonal to the states $L_{-n-m}|0\rangle$. Projecting onto these states one finds that

$$\mathcal{G}_3(z)\Big|_{\frac{w_H^2}{c^2}} = \Big[\frac{1}{2}\Big(\frac{3ww_H}{c}f_3(z)\Big)^2 - \frac{9w_H^2 h}{70c^2}w_3(z)\Big]z^{-2h}, \qquad (4.8)$$

where $w_3 = -14f_3^2 + 15f_2f_4$. The resulting simple-looking expression can be decomposed into global conformal blocks of $[W^2]_{2l}$, with weights $h = 6, 8, \ldots$, with the use of a product formula for hypergeometric functions found in [80].

Eq. (4.8) shows that the vacuum block contribution to the correlation function at quadratic order in the heavy charge expansion can be written as a sum of products $f_a f_b$ such that $a + b = 6$, where $h = 6$ is the weight of the lightest operator $[W^2]_0$. In higher, even spacetime dimension a similar picture emerges. In particular it was shown in [80,2] that the minimal-twist double-stress tensor contributions to HHLL correlators in four dimensions can be written as $\mathcal{G}_{d=4}|_{\Delta_H^2/C_T^2} \propto a_{15}f_1f_5 + a_{24}f_2f_4 + a_{33}f_3^2$, for some Δ_L dependent coefficients a_{ij}.

Let us now include a spin-4 current $U(z)$. With the four-dimensional results quoted above in mind, we consider the $\mathcal{W}_4$ HHLL vacuum block in the limit where the spin-3 charge is parametrically larger than the rest (this is done in Appendix A). The states (4.7) have a non-vanishing overlap with the single mode states $U_{-m-n}|0\rangle$ and by removing this overlap, one finds that the correction to the $\mathcal{O}(\frac{w_H^2}{c^2})$ term in (4.8) is proportional to the spin-4 charge u of the light operator. The result takes the form

$$\mathcal{G}_4(z)\Big|_{\frac{w_H^2}{c^2}} \propto a_{4,15}f_1f_5 + a_{4,24}f_2f_4 + a_{4,33}f_3^2, \qquad (4.9)$$

with coefficients $a_{4,ij}$ linear in the charges (h, u) of the light operator and quadratic in w due to the first term in (4.8)[36].

The results herein, obtained using explicit mode calculations, are in agreement with those for the $\mathcal{W}_N$ semi-classical vacuum blocks obtained in [112]. While the mode calculation becomes tedious at higher orders in $\frac{w_H}{c}$, the expansion of the semi-classical vacuum block is straightforward. Generally, we find that the expansion of the logarithm of the HHLL vacuum block in powers of $\frac{w_H}{c}$ can be written as a linear combination of products of hypergeometric:

$$\log\left(z^{2h}\mathcal{G}_N(z)\right) = \sum_{k=1}^{\infty}\left(\frac{w_H}{c}\right)^k \sum_{\{i_p\}} b_{N,i_1\ldots i_k} f_{i_1}(z)\ldots f_{i_k}(z), \tag{4.10}$$

where we have normalized the expression by the (holomorphic) part of the disconnected correlator z^{-2h}. i_p are integers such that $i_1 + \ldots + i_k = 3k$ and the coefficients $b_{N,i_1,\ldots i_p}$ are linear in the charges $q^{(i)}$ of the light operator[37].

It is instructive to examine the behavior of the vacuum blocks when $z \to 1$. Similarly to the case of the Virasoro vacuum block, we observe that the logarithm of the $\mathcal{W}_N$ vacuum block, with one of the heavy charges $q_H \sim c \to \infty$ and all other charges fixed and parametrically smaller, has the following behavior in the limit $z \to 1$:

$$\log(\mathcal{G}_N(z)) \sim B_N\left(q^{(i)}, \frac{q_H}{c}\right)\log(1-z), \tag{4.11}$$

where the function B_N is linear in the light charges $q^{(i)}$ and can be perturbatively expanded in $\frac{q_H}{c}$. This behavior is non-trivial since generally a product of k functions f_a is a k-th order polynomial in $\log(1-z)$ with coefficients that are rational functions of z.

For the Virasoro case, the corresponding function B_2 is the generating function of the Catalan numbers. For $\mathcal{W}_3$ in the limit $w_H \sim c \to \infty$, with the other charges parametrically smaller and for certain values of the ratio of the charges

[36] Whilst the form of the $\mathcal{G}_4(z)$ at quadratic order matches that of the four-dimensional result (notice the presence of the $f_1 f_5$-term), there is no choice of the charges of the light operators which would yield an exact match.

[37] Although the form of the $\mathcal{W}_N$ vacuum block expansion resembles that of the four-dimensional one, there is no value of N that would yield an exact match.

of the light operator, we find that B_3 satisfies a cubic equation. Inspired by it, one can construct similarly to the Virasoro case, a cubic differential equation satisfied by $\mathcal{F}_3 \equiv \log \mathcal{G}_3$ with (4.3). We present it below in the case $h = 3w$:

$$\frac{1}{6w}\frac{d^3}{dz^3}\mathcal{F}_3(z) = -\frac{1}{54w^3}\left(\frac{d}{dz}\mathcal{F}_3(z)\right)^3 + \frac{1}{6w^2}\left(\frac{d^2}{dz^2}\mathcal{F}_3(z)\right)\left(\frac{d}{dz}\mathcal{F}_3(z)\right) + \frac{2x}{(1-z)^3}, \tag{4.12}$$

where $x = 6\frac{w_H}{c}$. We also derive diagrammatic rules for the $\mathcal{W}_3$ HHLL vacuum block satisfies.

We also consider the $\mathcal{W}_4$ HHLL vacuum block in Appendix A.2. We study its behavior in the region $z \sim 1$ in two different cases; when the spin-4 charge, $u_H \sim c \gg 1$ while $h_H, w_H \ll c$ and when the spin-3 charge scales with c, $w_H \sim c \gg 1$ but $u_H, h_H \ll c$. In both cases the logarithm of the HHLL vacuum block behaves as $\mathcal{F}_4 \sim \log(1-z)$ in the limit $z \to 1$. In the former case, the generating function B_4 defined according to (4.11), satisfies a quartic equation for four different choices of the ratio h/u. In particular, when $h = 5u$ one can show that $\log \mathcal{G}(z)$ solves a differential equation whose form is inspired by the algebraic equation satisfied by B_4. The situation is similar but slightly more involved when the spin-3 charge, $w_H \sim c$.

Finally, we study the stress tensor sector of the HHLL correlator in d-spacetime dimensions in the limit $z \to 1$. In this case, we further have to take the $\Delta_L \to 0$ limit in order to remove higher log terms and find that the corresponding sequence of numbers are those of the number of linear extensions of posets $G[(\frac{d-2}{2})^{k-1}, (0)^{k-2}, (0^{k-2})]$. These are generalizations of the Catalan numbers which can be obtained as the number of linear extensions of the simpler poset $G[(0^{k-1}), (0^{k-2}), (0^{k-2})]$.

Outline

Section 4.2 is devoted to explicit mode calculations of the HHLL vacuum blocks. In Section 2.1 we reviewed the Virasoro counterpart and in Section 4.2 we generalize this calculation to the case of the $\mathcal{W}_3$ HHLL vacuum block. In Section 4.3, we study the behavior of the HHLL vacuum blocks in the region $z \sim 1$. After a short review of the Virasoro case, we focus on the $\mathcal{W}_3$ vacuum block. We observe the appearance of a generalized Catalan sequence, determine its generating function and the algebraic equation the latter satisfies. Inspired by this algebraic equation, we determine a cubic differential equation satisfied by the logarithm of the $\mathcal{W}_3$ vacuum block for certain ratios of the charges of the light operators. We conclude the discussion of the spin-3 case with new diagrammatic rules for the $\mathcal{W}_3$ vacuum block expansion. We then investigate in a similar manner the stress tensor sector of the four-dimensional HHLL correlator in holographic CFTs. We conclude with a discussion in Section 4.4. In Appendix A.1, one finds further details on the explicit mode calculations for the $\mathcal{W}_3$ HHLL vacuum block. In Appendix A.2, we consider the $\mathcal{W}_4$ HHLL vacuum block. When w_H is the only large charge, we show using the $\mathcal{W}_4$-algebra that one gets an extension of the $\mathcal{W}_3$ result which takes a form similar to that of the stress tensor sector of the HHLL correlator in $d = 4$. When u_H is the only large charge, we show that the HHLL vacuum block and a specific choice of the light charges is again governed by a generalization of the Catalan numbers, and that a corresponding non-linear differential equation can be written down analogous to the $\mathcal{W}_3$ case. A similar albeit more involved story emerges in the $z \to 1$ limit when the only large charge is w_H.

4.2. $\mathcal{W}_3$ HHLL blocks by mode decomposition

In this section we perform a mode calculation of $\mathcal{W}_N$ higher-spin vacuum blocks in two-dimensional CFTs with large central charge. We review the calculation of the Virasoro vacuum block in Section 2.1 following [77,78] and extend this to include higher-spin currents in this section. The semi-classical vacuum block, for large charges, in $\mathcal{W}_N$ theories has been calculated in [111,114] for $N = 3$ and in [112] for general N in the dual bulk theory using a Wilson line prescription. Expanding these known results we find agreement with those obtained from the mode calculation. The calculation of the $\mathcal{W}_N$ vacuum block using an explicit mode expansion can in principle be extended to include finite central charge as well as finite charges of the external operators.

In an effort to elucidate the connection between the structure of the vacuum block in the $\frac{H}{c}$ expansion and the underlying symmetry algebra, we consider now a 2d CFT with a spin-3 current $W(z)$. The spin-3 modes are defined by

$$W(z) = \sum_n W_n z^{-n-3}, \tag{4.13}$$

and satisfy the $\mathcal{W}_3$ algebra

$$\begin{aligned}
[L_m, W_n] &= (2m-n)W_{m+n}, \\
[W_m, W_n] &= \frac{c}{360}m(m^2-1)(m^2-2^2)\delta_{m+n}+ \\
&+ (m-n)\Big[\frac{1}{15}(m+n+3)(m+n+2) - \frac{1}{6}(m+2)(n+2)\Big]L_{m+n} \\
&+ \frac{16}{22+5c}(m-n)\Lambda_{m+n},
\end{aligned} \tag{4.14}$$

where $\Lambda_m = \sum_p : L_{m-p}L_p : -\frac{3}{10}(m+2)(m+3)L_m$. The spin-3 current $W(z)$ is a primary operator normalised so that $\langle W(z)W(0)\rangle = \frac{c}{3z^6}$. Note that the non-linear terms in (4.14) are suppressed in the large-c limit.

We will study the $\mathcal{W}_3$ vacuum block $\mathcal{G}_3$ contribution to the four point function of pairwise identical scalars $\mathcal{O}_H$ and $\mathcal{O}_L$. These are $\mathcal{W}_3$ primaries and have conformal weights H and h, as before, as well spin-3 charges $\pm w_H$ and $\pm w$, respectively, with the following scaling as $c \to \infty$: [38]

$$w_H \gg H, h, w, \qquad \frac{w_H}{c} = \text{fixed}. \tag{4.15}$$

As we will see, the contribution from the pure Virasoro modes considered in the previous section is suppressed compared to that containing the spin-3 charge modes of the "heavy" operator and is due to states of the schematic form $W_{-m_1}\dots W_{-m_i}L_{-n_1}\dots L_{-n_j}|0\rangle$. To evaluate the contribution of such states explicitly, we need to construct an orthogonal basis using the algebra (4.14) and find the commutator $[W_m, \mathcal{O}]$.

Consider first the commutator $[W_m, \mathcal{O}]$. This is determined by the singular terms in the OPE

$$\begin{aligned}
W(z)\mathcal{O}(0)|0\rangle &= z^{-3}W_0|h,w\rangle + z^{-2}W_{-1}|h,w\rangle + z^{-1}W_{-2}|h,w\rangle + \mathcal{O}(z^0) \\
&= z^{-3}w\mathcal{O}|0\rangle + z^{-2}(\mathcal{O}_{h+1} + \frac{3w}{2h}\partial\mathcal{O})|0\rangle \\
&+ z^{-1}(\mathcal{O}_{h+2} + \frac{2}{h+1}\partial\mathcal{O}_{h+1} + \frac{3w}{h(2h+1)}\partial^2\mathcal{O})|0\rangle + \dots,
\end{aligned} \tag{4.16}$$

[38] In [116] it was shown that unitary representations have weight $\tilde{h} \sim c$ and therefore neither the heavy nor the light operators we consider are unitary.

where $\mathcal{O}_{h+1}$ and $\mathcal{O}_{h+2}$ are quasi-primary operators with conformal weight $h+1$ and $h+2$, respectively, and are given by

$$\begin{aligned}\mathcal{O}_{h+1}(0)|0\rangle &:= \Big[W_{-1}\mathcal{O} - \frac{3w}{2h}L_{-1}\mathcal{O}\Big]|0\rangle,\\ \mathcal{O}_{h+2}(0)|0\rangle &:= \Big[W_{-2}\mathcal{O} - \frac{2}{h+1}L_{-1}\mathcal{O}_{h+1} - \frac{3w}{h(2h+1)}L_{-1}^{2}\mathcal{O}\Big]|0\rangle.\end{aligned} \tag{4.17}$$

Being quasi-primaries, they satisfy $[L_1, \mathcal{O}_{h+1}(0)] = [L_1, \mathcal{O}_{h+2}(0)] = 0$ which can be verified using the algebra (4.14). The commutator $[W_n, \mathcal{O}]$ can be found using translation invariance, multiplying with $\int_{\mathcal{C}(z)} \frac{dw}{2\pi i} w^{n+2}$ and using the OPE (4.16) :

$$\begin{aligned}[W_m, \mathcal{O}(z)] =& \frac{w(m+1)(m+2)}{2} z^m \mathcal{O}(z) + (m+2)z^{m+1}(\mathcal{O}_{h+1}(z) + \frac{3w}{2h}\partial\mathcal{O}(z))\\ &+ z^{m+2}(\mathcal{O}_{h+2}(z) + \frac{2}{h+1}\partial\mathcal{O}_{h+1}(z) + \frac{3w}{h(2h+1)}\partial^2\mathcal{O}(z)).\end{aligned} \tag{4.18}$$

Consider now the contribution to $\mathcal{G}(z)$ from states $W_{-n}|0\rangle$. In order to calculate $\langle W_n \mathcal{O}(z)\mathcal{O}(0)\rangle$[39], we note that $\langle \mathcal{O}_{h+1}(z)\mathcal{O}(0)\rangle = \langle \mathcal{O}_{h+2}(z)\mathcal{O}(0)\rangle = 0$ since these and $\mathcal{O}$ are quasi-primaries with different conformal weights. It follows that only $\mathcal{O}$ and its global descendants in (4.18) contribute to $\langle W_m \mathcal{O}(z)\mathcal{O}(0)\rangle$, leading to

$$\begin{aligned}\langle W_n \mathcal{O}(z)\mathcal{O}(0)\rangle &= z^n\Big[\frac{w}{2}(n+1)(n+2) + \frac{3w}{2h}(n+2)z\partial_z + \frac{3w}{h(2h+1)}z^2\partial_z^2\Big]z^{-2h}\\ &= \frac{w}{2}(n-1)(n-2)z^{n-2h},\end{aligned} \tag{4.19}$$

where the operator at z has spin-3 charge w and the operator at 0 has charge $(-w)$. On the other hand, for the heavy part, one finds that

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)W_n\rangle = \frac{w_H}{2}(n-1)(n-2), \tag{4.20}$$

where the operator at $z=1$ carries spin-3 charge $(-w_H)$ and the one at $z\to\infty$, charge w_H. Multiplying (4.19) with (4.20), dividing with the norm given by the central term in (4.14) and summing over $n = 3, 4, \ldots$, one finds the expected result for the $\mathcal{W}_3$ vacuum block due to the exchange of a spin-3 quasi-primary

$$\mathcal{G}_3(z)|_{\frac{w_H w}{c}} = z^{-2h}\frac{90 w_H w}{c}\sum_{n=3}^{\infty}\frac{(n-1)(n-2)}{(n+1)(n+2)}\frac{z^n}{n} = \frac{3w_H w}{c} f_3(z) z^{-2h}. \tag{4.21}$$

[39] We denote $\mathcal{O}_L \equiv \mathcal{O}$ to simplify the notation.

Consider now states of the form $W_{-n}W_{-m}|0\rangle$. These are orthogonal to $W_{-n}|0\rangle$ since $W(0)$ does not appear in the OPE $W(z)W(0)$. On the other hand, the stress tensor appears in this OPE and the overlap $\langle L_{m+n}W_{-n}W_{-m}\rangle$ is non-zero. The overlap can be calculated using the fact that $W(z)$ is a primary field. With the help of the first line in (4.14) one finds

$$\langle L_{m+n}W_{-n}W_{-m}\rangle = \frac{c}{360}(3n+2m)m(m^2-1)(m^2-4). \tag{4.22}$$

Removing this overlap leads to states orthogonal to the single-mode ones

$$|Y_{m,n}\rangle = \Big[W_{-n}W_{-m} - \frac{(3n+2m)m(m^2-1)(m^2-4)}{30(m+n)((m+n)^2-1)}L_{-m-n}\Big]|0\rangle, \tag{4.23}$$

with norm $\mathcal{N}_{Y_{m,n}} = \langle Y_{m,n}|Y_{m,n}\rangle = (\frac{c}{360})^2 m(m^2-1)(m^2-4)n(n^2-1)(n^2-4)$. The overlap with the double-mode states $L_{-m}L_{-n}|0\rangle$ is suppressed in the large-c limit.

The next step is to compute $\langle W_m W_n \mathcal{O}(z)\mathcal{O}(0)\rangle$ using the commutator $[W_n, \mathcal{O}(z)]$ in (4.18). We find that

$$\begin{aligned}\langle W_m W_n \mathcal{O}(z)\mathcal{O}(0)\rangle = z^n\Big[&\frac{w}{2}(n+1)(n+2) + \frac{3w}{2h}(n+2)z\partial_z \\ &+ \frac{3w}{h(2h+1)}z^2\partial_z^2\Big]\langle W_m \mathcal{O}(z)\mathcal{O}(0)\rangle \\ &+ z^{n+1}\Big[(n+2) + \frac{2}{h+1}z\partial\Big]\langle W_m \mathcal{O}_{h+1}(z)\mathcal{O}(0)\rangle \\ &+ z^{n+2}\langle W_m \mathcal{O}_{h+2}(z)\mathcal{O}(0)\rangle.\end{aligned} \tag{4.24}$$

To evaluate (4.24) one may use the commutators $[W_m, \mathcal{O}_{h+1}(z)]$ and $[W_m, \mathcal{O}_{h+2}(z)]$ which are found in Appendix A. Alternatively, recall that the three-point functions $\langle W(z)\mathcal{O}_{h+1}(z)\mathcal{O}(z)\rangle$, and $\langle W(z)\mathcal{O}_{h+2}(z)\mathcal{O}(z)\rangle$, are fixed by conformal symmetry up to the respective OPE coefficients. This gives

$$\begin{aligned}&z^{n+1}\Big[(n+2) + \frac{2}{h+1}z\partial\Big]\int\frac{dz_3}{2\pi i}z_3^{m+2}\langle W(z_3)\mathcal{O}_{h+1}(z)\mathcal{O}(0)\rangle \\ &= \lambda_{W\mathcal{O}_{h+1}\mathcal{O}}\frac{m(m-1)(m-2)(h(n-2)+2m+n)}{6(h+1)}z^{m+n-2h},\end{aligned} \tag{4.25}$$

where $\lambda_{W\mathcal{O}_{h+1}\mathcal{O}}$ is the OPE coefficient of $\mathcal{O}$ in the OPE $W\times\mathcal{O}_{h+1}$. Likewise, $\langle W_m\mathcal{O}_{h+2}(z)\mathcal{O}(0)\rangle$ is given by

$$z^{n+2}\langle W_m \mathcal{O}_{h+2}(z)\mathcal{O}(0)\rangle = \frac{\lambda_{W\mathcal{O}_{h+1}\mathcal{O}}}{24}(m-2)(m-1)m(m+1)z^{m+n-2h}. \tag{4.26}$$

The OPE coefficients are found with the help of the algebra, (4.14), by taking the limit $z \to 0$

$$\begin{aligned}\langle \mathcal{O}(z_3) W(z) \mathcal{O}_{h+1}(0)\rangle &\approx z^{-4}\langle \mathcal{O}(z_3) W_1 (W_{-1} - \frac{3w}{2h} L_{-1})\mathcal{O}(0)\rangle \\ &= z^{-4} z_3^{-2h}\Big[\frac{h(2-c+32h)}{22+5c} - \frac{9w^2}{2h}\Big],\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}&\langle \mathcal{O}(z_3) W(z) \mathcal{O}_{h+2}(0)\rangle \\ &\approx z^{-5}\langle \mathcal{O}(z_3) W_2 (W_{-2} - \frac{2}{h+1} L_{-1} W_{-1} + \frac{3w}{(h+1)(2h+1)} L_{-1}^2)\mathcal{O}(0)\rangle \\ &= z^{-5} z_3^{-2h}\Big[\frac{8h(6+c+8h)}{22+5c} - \frac{2}{h+1}\frac{4h(2-c+32h)}{22+5c} + \frac{36w^2}{(h+1)(2h+1)}\Big].\end{aligned} \tag{4.28}$$

From (4.27) and (4.28) we deduce that for large-c

$$\begin{aligned}\lambda_{W\mathcal{O}_{h+1}\mathcal{O}} &= -\frac{h}{5} - \frac{9w^2}{2h}, \\ \lambda_{W\mathcal{O}_{h+2}\mathcal{O}} &= \frac{8h}{5} + \frac{8h}{5(h+1)} + \frac{36w^2}{(h+1)(2h+1)}.\end{aligned} \tag{4.29}$$

Using (4.25) and (4.26) and the OPE coefficients given in (4.29) to evaluate (4.24), we find that $\langle Y_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle$ is given by

$$\begin{aligned}\langle Y_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle =& \Big[\frac{w^2}{4}(m-1)(m-2)(n-1)(n-2) \\ &- \frac{h}{30}\frac{m(m-1)(m-2)n(n-1)(n-2)}{(m+n)(m+n+1)}\Big] z^{m+n-2h},\end{aligned} \tag{4.30}$$

with $|Y_{m,n}\rangle$ defined in (4.23). The heavy part $\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)|Y_{m,n}\rangle$ can be calculated in a similar manner,

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)|Y_{m,n}\rangle = \frac{w_H^2}{4}(m-1)(m-2)(n-1)(n-2), \tag{4.31}$$

in the limit $w_H \gg H$. Multiplying (4.30) and (4.31), dividing by the norm $(\frac{c}{360})^2 m(m^2-1)(m^2-4)n(n^2-1)(n^2-4)$ and summing over $m, n = 3, 4, \ldots$ we determine the contribution of the states $|Y_{m,n}\rangle$ to the $\mathcal{W}_3$ vacuum block to be:

$$\begin{aligned}\mathcal{G}_3(z)|_{\frac{w_H^2}{c^2}} = \frac{z^{-2h}}{2}\sum_{m,n=3}^{\infty}\Big[&\Big(\frac{90 w_H w}{c}\Big)^2 \frac{(m-1)(m-2)(n-1)(n-2)}{(m+1)(m+2)(n+1)(n+2)}\frac{1}{mn} \\ &- \frac{540 w_H^2 h}{c^2}\frac{(m-1)(m-2)(n-1)(n-2)}{(m+1)(m+2)(n+1)(n+2)}\frac{1}{s(s+1)}\Big] z^s,\end{aligned} \tag{4.32}$$

where $s = m + n$. The first line in (4.32) is the exponentiated term analogous to the Virasoro case:

$$\mathcal{G}_3(z)|_{\frac{w_H^2 w^2}{c^2}} = \frac{1}{2}\Big(\frac{3w_H w}{c} f_3\Big)^2 z^{-2h}, \tag{4.33}$$

while the second line can be summed to

$$\mathcal{G}_3(z)|_{\frac{w_H^2 h}{c^2}} = -\frac{9w_H^2 h}{70c^2} w_3(z) z^{-2h}, \tag{4.34}$$

where $w_3(z)$ is a sum of products $f_a f_b$ with $a + b = 6$:

$$\begin{aligned} w_3(z) \equiv& -14 f_3^2(z) + 15 f_2(z) f_4(z) \\ =& 4200 \sum_{m,n=3}^{\infty} \frac{(m-1)(m-2)(n-1)(n-2)}{(m+1)(m+2)(n+1)(n+2)} \frac{z^s}{s(s+1)}. \end{aligned} \tag{4.35}$$

Similar to the Virasoro case, it is easy to verify that the non-exponentiated term $w_3(z)$ behaves as $\log(1-z)$ when $z \to 1$.

We can also calculate the contribution to the $\mathcal{W}_3$ vacuum block from states of the form $\Big[L_{-m}W_{-n} - \frac{\langle W_{m+n} L_{-m} W_{-n}\rangle}{\langle W_{m+n} W_{-m-n}\rangle} W_{-m-n}\Big]|0\rangle$. This results in a term that contributes to exponentation and takes the form $\propto \frac{w_H H w h}{c^2} f_2 f_3$, as well as a term $\propto \frac{w_H H w}{c^2}(f_1 f_4 - \frac{7}{9} f_2 f_3)$. Such terms are subleading in the limit $w_H \gg H$ (see Appendix A.1 for further details).

4.3. Generalized Catalan numbers and differential equations

In this section we study the logarithm of the correlator defined by $\mathcal{F}_N \equiv \log \mathcal{G}_N$. We start by reviewing the behavior of the logarithm of the Virasoro vacuum block, $\mathcal{F}_2 = \log \mathcal{G}_2$, in the limit $z \to 1$, the appearance of the Catalan numbers's sequence, and the differential equation satisfied by $\mathcal{F}_2$, following [79]. Next, we focus on the case $N = 3$ where a very similar story emerges. Besides a certain generalization of the Catalan sequence, we also find a set of diagrammatic rules governing the expansion of the $\mathcal{W}_3$ vacuum block along with a differential equation satisfied by $\mathcal{F}_3$ for certain ratios of the values of the charges of the light operators. We also consider the logarithm of the stress-tensor sector of the four-dimensional correlator in the lightcone limit, which we denote by $\mathcal{G}_{d=4}$ and $\mathcal{F}_{d=4}$ respectively. We investigate the behavior in the limit $z \to 1$ and observe similarities with the two-dimensional cases when $\Delta_L \to 0$.

4.3.1. The Virasoro vacuum block

In [79] it was shown how one can derive a differential equation satisfied by the logarithm of the Virasoro vacuum block, by studying its behavior in the $z \to 1$ limit. Expanding $\mathcal{F}_2$ in powers of h_H/c the authors of [79] observed that $\mathcal{F}_2$ behaves logarithmically when $z \to 1$. Furthermore they noticed that the sequence of the numerical coefficients multiplying the logarithm at the each order forms the sequence of Catalan numbers given by $c_{2,k}$:

$$c_{2,k} = \frac{\Gamma(2k-1)}{\Gamma(k)\Gamma(k+1)}, \qquad k \geq 1. \tag{4.36}$$

These numbers are generated by the following generating function

$$B_2(x) = \sum_{k=1}^{\infty} c_{2,k} x^k = \frac{1-\sqrt{1-4x}}{2}, \tag{4.37}$$

which satisfies

$$B_2(x) = B_2(x)^2 + x. \tag{4.38}$$

The Catalan numbers $c_{2,k}$ are known to appear in various problems in combinatorics. Here we would like to point out that they can also be understood as the numbers of linear extensions of one-level grid posets[40] $G([0^{k-1}],[0^{k-2}],[0^{k-2}])$, for $k \geq 1$. Generally, one-level grid-like posets $G[\mathbf{v},\mathbf{t},\mathbf{b}]$, where $\mathbf{v} = (v_1,\ldots,v_n)$, $\mathbf{t} = (t_1,\ldots,t_{n-1})$ and $\mathbf{b} = (b_1,\ldots,b_{n-1})$, can be represented with Hasse diagrams of the following type:

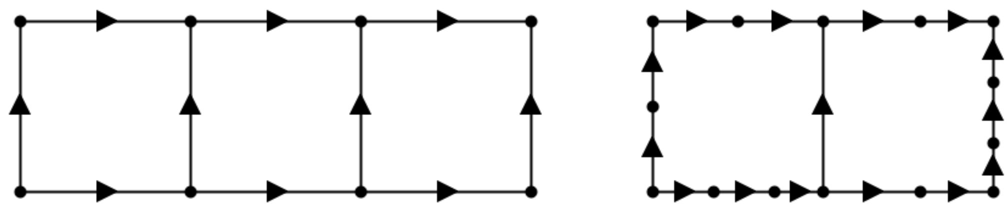

Fig. 1: Posets denoted by $G([0,0,0,0],[0,0,0],[0,0,0])$ and $G([1,0,2],[1,1],[2,1])$, respectively.

[40] Partially ordered sets (posets) have a notion of ordering between some of the elements but not necessarily all of them. A linear extension of a partial ordering is a linear extension to a totally ordered set where all the elements are ordered in such a way that the original partial ordering is preserved.

The numbers v_i denote the number of nodes in the i-th vertical edge, t_i denote the number of nodes in the i-th top edge and b_i denote the number of nodes in the i-th bottom edge, with the endpoints excluded. The Catalan numbers are the numbers of linear extensions of posets of the type depicted in the left Hasse diagram of Fig. 1.

The logarithm of the correlator $\mathcal{F}_2(z) = \log \mathcal{G}_2(z)$ when $z \to 1$ therefore behaves as

$$\mathcal{F}_2(z) \underset{z\to 1}{\approx} -2hB_2(x)\log(1-z), \tag{4.39}$$

with $x = 6\frac{h_H}{c}$. Inspired by (4.38) and (4.39) the authors of [79] find a differential equation satisfied by $\mathcal{F}_2(z)$ for all z:

$$\frac{1}{2h}\frac{d^2}{dz^2}\mathcal{F}_2(z) = \frac{1}{4h^2}\left(\frac{d}{dz}\mathcal{F}_2(z)\right)^2 + \frac{x}{(1-z)^2}. \tag{4.40}$$

4.3.2. The $\mathcal{W}_3$ vacuum block

Here we uncover a similar story for the $\mathcal{W}_3$ vacuum block $\mathcal{G}_3$. Expanding in powers of $\frac{w_H}{c}$,

$$\log \mathcal{G}_3 \equiv \mathcal{F}_3(z) = \sum_{k=0}^{\infty}\left(\frac{w_H}{c}\right)^k \mathcal{F}_3^{(k)}(z), \tag{4.41}$$

with

$$\mathcal{F}_3^{(0)}(z) = -2h\log(z), \tag{4.42}$$

and using the exact expression known for the $\mathcal{W}_3$ vacuum block (see for example eq. (4.24) in [113]) one finds that

$$\begin{aligned}\left\{\lim_{z\to 1}\left(-\frac{\mathcal{F}_3^{(k)}(z)}{6^{k+1}\log(1-z)}\right)\middle| k=1,2,\dots\right\} &= \\ w\times\{1, n, 16, 35n, 768, 2002n, 49152, 138567n, \dots\},&\end{aligned} \tag{4.43}$$

where we set $n \equiv h/w$. $\mathcal{F}_3$ in the limit $z \to 1$ is given by

$$\mathcal{F}_3(z) \underset{z\to 1}{\approx} -6w\log(1-z)B_3(x,n), \tag{4.44}$$

where $B_3(x,n)$ is the generating function of the sequence (4.43)

$$\begin{aligned}B_3(x,n) = \sum_{k=1}^{\infty} c_{3,k}x^k =& \frac{1}{6}\sqrt{3}\sin(\frac{1}{3}\arcsin(6\sqrt{3}x)) \\ &- n\cos(\frac{1}{3}\arcsin(6\sqrt{3}x)) + n).\end{aligned} \tag{4.45}$$

Remarkably, there exist exactly three values of n for which $B_3(x,n)$ satisfies a cubic equation; these are $n=\pm 3$ and $n=0$. For these values of the ratios of the light charges, the $\mathcal{W}_3$ vacuum block simplifies dramatically; it can be expressed in terms of a single function of z raised to a given power[41].

For $n=\pm 3$ the sequence of (4.43) reduces to

$$\left\{\lim_{z\to 1}\left[-\left(\pm\frac{1}{6}\right)^{k+1}\frac{\mathcal{F}_3^{(k)}(z)}{\log(1-z)}\right]\Bigg|k=1,2,\dots\right\}= \\ = w\times\left\{1,\pm 3,16,\pm 105,768,\pm 6006,49152,\pm 415701,\dots\right\}. \tag{4.46}$$

Each term in this sequence can be derived from the following formula

$$c_{3,k}=\frac{(\pm 2)^{k-1}(3k-3)!!}{k!(k-1)!!},\qquad k\geq 1. \tag{4.47}$$

Moreover, one can check that function (4.45) with $n=\pm 3$ satisfies the following relation

$$B_3(x,\pm 3)=-2B_3(x,\pm 3)^3\pm 3B_3(x,\pm 3)^2+x. \tag{4.48}$$

with $x=6\frac{w_H}{c}$. Inspired by (4.48) we search for a cubic differential equation satisfied by $\mathcal{F}_3(z)$. It is easy to see, using the exact expression for the $\mathcal{W}_3$ block given for example in eq. (4.24) of [113], that $\mathcal{F}_3(z,n=3)\equiv\hat{\mathcal{F}}_3(z)$ satisfies the following differential equation

$$\frac{1}{6w}\frac{d^3}{dz^3}\hat{\mathcal{F}}_3(z)=-\frac{1}{54w^3}\left(\frac{d}{dz}\hat{\mathcal{F}}_3(z)\right)^3+\frac{1}{6w^2}\left(\frac{d^2}{dz^2}\hat{\mathcal{F}}_3(z)\right)\left(\frac{d}{dz}\hat{\mathcal{F}}_3(z)\right)+\frac{2x}{(1-z)^3}. \tag{4.49}$$

When $\frac{h}{w}=-3$ a similar equation can be found by taking $w\to -w$ and $1-z\to\frac{1}{1-z}$. The case $n=0$ is special and is discussed in Appendix A.

4.3.3. Diagrammatic rules for the $\mathcal{W}_3$ block

Here we formulate diagrammatic rules for computing the logarithm of $\mathcal{W}_3$ vacuum block $\mathcal{F}_3(z)=\log\mathcal{G}_3(z)$, in the limit where $w_H\sim c\gg 1$ and all other charges are parametrically suppressed. The ratio of the charges of the light operator, n, is left arbitrary . The rules are similar to those in [79] for computing the logarithm of the Virasoro vacuum block.

[41] For other values of n the generating function satisfies a sixth order algebraic equation. As a result writing a differential equation becomes cumbersome.

We now have cubic and quartic vertices and the exchanged states are modes of the stress tensor and spin-3 current, which we refer to collectively as currents. The only relevant diagrams in the limit we consider, are those where a single propagator connects to the light operator $\mathcal{O}_L$. The rules can be stated as follows:

1. Label the k initial currents connected to operator $\mathcal{O}_H$ with integers $a_1, a_2, \ldots, a_k$.

2. Draw all diagrams where the k initial currents combine via 3-pt and 4-pt vertices to become a single current, which connects with the light operators.

3. For each propagator define its momentum p as the sum of the a_i flowing through it. Momentum is conserved at vertices. Each propagator comes with a factor

$$\frac{1}{(p+1)(p+2)}.$$

4. For each vertex coupling a current of momentum a_i to the external operator $\mathcal{O}_H$, include a factor of

$$\frac{w_H}{\sqrt{c}}(a_i-1)(a_i-2).$$

5. For each vertex coupling a current of momentum p to the external operator $\mathcal{O}_L$, include a factor of

$$\frac{1}{6\sqrt{c}}\left((-1)^k(h-3w)+h+3w\right)(p-1)(p-2).$$

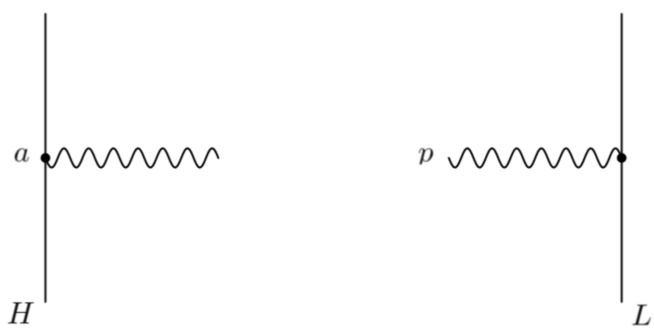

Fig. 2: Vertices denoting the coupling of an exchanged current with the external states $\mathcal{O}_H$ and $\mathcal{O}_L$, respectively.

6. For each 4-current vertex, include a factor of $-2/3c$. For each 3-current vertex, where two currents carry momentum m and n, while the third current carries momentum $m+n$ (see fig. 3), include a factor of

$$\frac{1}{\sqrt{c}}(m+n+2).$$

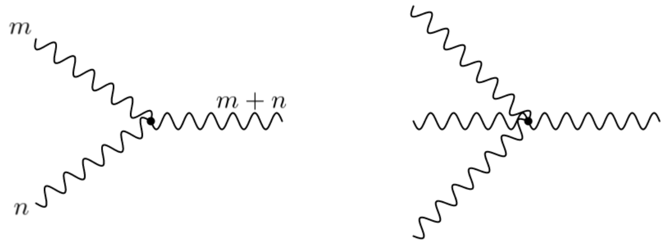

Fig. 3: Vertices denoting 3-pt and 4-pt coupling of currents, respectively.

7. Take the product of the propagators and vertices and then multiply the result by

$$\frac{36^k}{k!}\frac{z^s}{s(s-1)(s-2)},$$

where $s = \sum_{i=1}^{k} a_i$.

8. Sum the resulting tree diagrams over all a_i from 3 to ∞ to obtain the $\frac{w_H^k}{c^k}$ term in $\mathcal{F}(z)|_{\mathcal{W}_3}$.

At orders w_H/c and w_H^2/c^2 there is just one diagram to take into account, while at order w_H^3/c^3 there are two different types of diagrams. This way, one obtains the expansion of the logarithm of $\mathcal{W}_3$ vacuum block, which is given by eq. (4.24) in [113].[42]

[42] We explicitly checked this up to $\mathcal{O}(w_H^4/c^4)$.

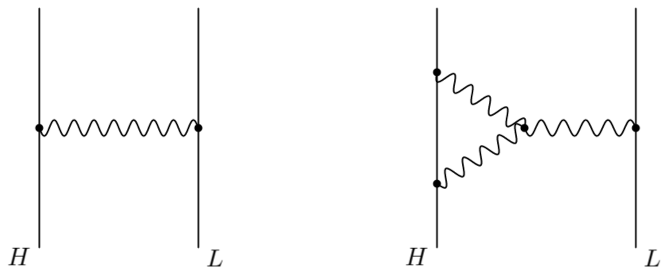

Fig. 4: Diagrams at orders w_H/c and w_H^2/c^2, respectively.

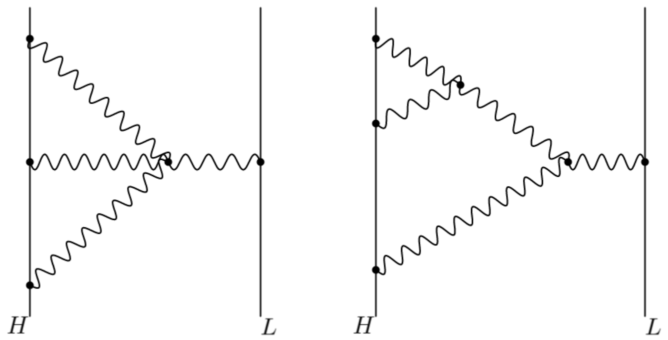

Fig. 5: Diagrams at order w_H^3/c^3.

4.3.4. Stress tensor sector in $d = 4$

The stress tensor sector of the HHLL correlator in four-dimensional spacetime and in the lightcone limit ($\bar{z} \to 0$)[43] is given according to [2] by

$$\mathcal{G}_{d=4}(z, \bar{z}) = \frac{1}{(z\bar{z})^{\Delta_L}} \left(1 + \sum_{k=1}^{\infty} \mu^k \bar{z}^k \mathcal{G}^{(k)}_{d=4}(z) \right), \tag{4.50}$$

$$\mathcal{G}^{(k)}_{d=4}(z) = \sum_{\{i_p\}} a_{i_1 \ldots i_k} f_{i_1}(z) \ldots f_{i_k}(z), \tag{4.51}$$

where the sum goes over all sets of $\{i_p\}$ with $i_p \leq i_{p+1}$ and $a_{i_1 \ldots i_k}$ coefficients that depend on Δ_L, and the expansion parameter μ is given by

$$\mu \equiv \frac{160}{3} \frac{\Delta_H}{C_T}. \tag{4.52}$$

[43] Note that in this section we change conventions and put the light operators at $\mathcal{O}_L(0)$ and $\mathcal{O}_L(z, \bar{z})$ since it is more convenient when comparing to the $d = 2$ case.

Explicit expressions for $\mathcal{G}^{(k)}_{d=4}$ with $k = 1, 2, 3$ are given in [2]. There it was also shown that $\mathcal{G}_{d=4}(z, \bar{z})$ can be written as

$$\mathcal{G}_{d=4}(z, \bar{z}) = e^{\Delta_L \mathcal{F}_{d=4}(z,\bar{z})}, \tag{4.53}$$

$\mathcal{F}_{d=4}(z, \bar{z})$ being of $\mathcal{O}(1)$ in the limit $\Delta_L \to \infty$ and which can be expanded as follows

$$\mathcal{F}_{d=4}(z, \bar{z}) = \mathcal{F}^{(0)}_{d=4}(z, \bar{z}) + \sum_{k=1}^{\infty} \mu^k \bar{z}^k \mathcal{F}^{(k)}_{d=4}(z). \tag{4.54}$$

with $\mathcal{F}^{(k)}_{d=4}$ being schematically of the same form as the $\mathcal{G}^{(k)}_{d=4}$ in (4.51). For $k = 0, 1, 2, 3$ for instance, we have

$$\begin{aligned}
&\mathcal{F}^{(0)}_{d=4}(z, \bar{z}) = -\log(z\bar{z}),\\
&\mathcal{F}^{(1)}_{d=4}(z) = \frac{1}{120} f_3(z),\\
&\mathcal{F}^{(2)}_{d=4}(z) = \frac{(12 - 5\Delta_L) f_3(z)^2 + \frac{15}{7}(\Delta_L - 8) f_2(z) f_4(z) + \frac{40}{7}(\Delta_L + 1) f_1(z) f_5(z)}{28800(\Delta_L - 2)},\\
&\mathcal{F}^{(3)}_{d=4}(z) = b_{117} f_1^2(z) f_7(z) + b_{126} f_1(z) f_2(z) f_6(z) + b_{135} f_1(z) f_3(z) f_5(z)\\
&\quad + b_{225} f_2^2(z) f_5(z) + b_{234} f_2(z) f_3(z) f_4(z) + b_{333} f_3^3(z),
\end{aligned} \tag{4.55}$$

where

$$\begin{aligned}
b_{117} &= \frac{5(\Delta_L + 1)(\Delta_L + 2)}{768768(\Delta_L - 2)(\Delta_L - 3)},\\
b_{126} &= \frac{5(5\Delta_L^2 - 57\Delta_L - 50)}{6386688(\Delta_L - 2)(\Delta_L - 3)},\\
b_{225} &= -\frac{7\Delta_L^2 - 51\Delta_L - 70}{2903040(\Delta_L - 2)(\Delta_L - 3)},\\
b_{135} &= -\frac{11\Delta_L^2 - 19\Delta_L - 18}{1209600(\Delta_L - 2)(\Delta_L - 3)},\\
b_{234} &= \frac{(\Delta_L - 2)(\Delta_L + 2)}{1209600(\Delta_L - 2)(\Delta_L - 3)},\\
b_{333} &= \frac{7\Delta_L^2 - 18\Delta_L - 24}{2592000(\Delta_L - 2)(\Delta_L - 3)}.
\end{aligned} \tag{4.56}$$

Inspired by the two-dimensional case, we consider the $\mathcal{F}^{(k)}_{d=4}(z)$ in the limit $z \to 1$. We observe that all terms proportional to $\log^i(1 - z)$ with $i \geq 2$ vanish in this limit as long as $\Delta_L \to 0$. In this special case, one can show that

$$\left\{ \lim_{z\to 1, \Delta_L \to 0} \frac{(-4)^k (k!) \mathcal{F}^{(k)}_{d=4}(z)}{\log(1 - z)} \middle| k = 1, 2, 3, 4, 5, \ldots \right\} = \{1, 1, 6, 71, 1266, \ldots\}. \tag{4.57}$$

The sequence of numbers in the (4.57) is known as the number of linear extensions of the one-level grid poset $G[(1^{k-1}),(0^{k-2}),(0^{k-2})]$, for $k \geq 1$, given by A274644 in [115]. As an example, the $k=5$ case is represented by the Hasse diagram in Fig. 2.

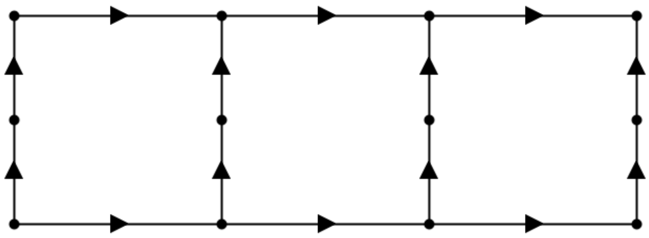

Fig. 6: The poset denoted by $G([1,1,1,1],[0,0,0],[0,0,0])$.

We do not explicitly discuss it here but the relevant posets in even number of dimension d are $G[(\frac{d-2}{2})^{k-1},(0)^{k-2},(0^{k-2})]$. The generating functions and the general formulas for the numbers of linear extensions of posets $G[(\frac{d-2}{2})^{k-1},(0)^{k-2},(0^{k-2})]$ are not (currently) known.

4.4. Discussion

We consider the $\mathcal{W}_N$ vacuum block contributions to heavy-heavy-light-light correlators in two-dimensional CFTs with higher-spin symmetries. We perform explicit mode calculations for $\mathcal{W}_3$ and $\mathcal{W}_4$ blocks and show that they reproduce the semi-classical vacuum blocks whose explicit form can be found in e.g. [112]. We observe that terms in the expansion of these blocks in powers of $(q_H^{(i)}/c)$ satisfy the suitably modified ansatz which was used to compute the stress tensor sector of the $d=4$ HHLL correlator in [2].

The HHLL Virasoro vacuum block is governed by the Catalan numbers whose generating function satisfies a quadratic equation allowing the construction of a non-linear differential equation for the logarithm of the vacuum block [79]. We show that the $\mathcal{W}_3$ and $\mathcal{W}_4$ HHLL vacuum blocks are governed by generalizations of the Catalan numbers; for certain values of the light operator charges, their generating functions satisfy cubic and quartic algebraic equations respectively. We further show that these equations uplift to non-linear differential equations satisfied by the logarithm of the blocks. What's more, the leading twist stress tensor sector of HHLL correlators in even number of spacetime dimensions d has the same structure in the limit $\Delta_L \to 0$. The relevant generalization of the Catalan numbers is now the number of linear extensions of partially ordered sets $G[(\frac{d-2}{2})^{k-1}, (0)^{k-2}, (0)^{k-2}]$. For $d > 2$ the generating functions for these sequences are not known.

The appearance of the generating function $B_N(x)$ comes from the limit $z \to 1$ of the logarithm $\mathcal{F}_N$ of the block, where $\mathcal{F}_N \sim B_N(x)\log(1-z)$. For example, eq. (4.48) defines generalizations of Catalan numbers; this and similar equations were studied in [117]. For the $\mathcal{W}_3$ case, we observe that for generic light charges h and w, the generating function satisfies a polynomial equation of degree 6, rather than 3, which however does not take the form studied in [117]. The numbers relevant for the $d = 4$ result also do not seem to come from equations of this form; it would be interesting to understand this better.

Note that in the $d = 4$ case, the logarithm of the minimal-twist stress tensor sector of HHLL correlators, $\mathcal{F}_{d=4}$, is a rational function of Δ_L which is $\mathcal{O}(1)$ for large Δ_L. An important difference with the $d = 2$ $\mathcal{W}_N$ result is that in the limit $z \to 1$, at k-th order in the $\mu \simeq \frac{\Delta_H}{C_T}$ expansion, $\mathcal{F}^{(k)}_{d=4} \sim g(\Delta_L)\log^k(1-z)$ for some function $g(\Delta_L)$. However, in the limit $\Delta_L \to 0$, we do find that $\mathcal{F}_{d=4} \sim B_{d=4}(\mu)\log(1-z)$ with $B_{d=4}$ being the generating function of the number of linear extensions of the $G[(1^{k-1}), (0^{k-2}), (0^{k-2})]$ posets (this is also the number of Young tableaux with restrictions; similar numbers were recently studied in [118])[44]. If we knew an algebraic equation satisfied by $B_{d=4}$, we could perhaps construct a differential equation whose solution would give the full minimal-twist stress tensor sector in $d = 4$ large-N CFTs in the limit $\Delta_L \to 0$.

Heavy-heavy-light-light $\mathcal{W}_N$ vacuum blocks where the spin-3 charge $q_H^{(3)} \sim c$ and $q_H^{i\neq 3} \ll c$ take a form similar to the minimal-twist stress tensor sector in four spacetime dimensions. In both cases, at order $(\frac{q_H^{(3)}}{c})^k$ in $d = 2$ and order $\mu^k \simeq (\frac{\Delta_H}{C_T})^k$ in $d = 4$, the result is a sum of products $f_{a_1} f_{a_2} \ldots f_{a_k}$ with $a_1 + a_2 + \ldots + a_k = 3k$. In two dimensions, we have shown how at $k = 1, 2$ and $N = 3, 4$, this follows from an explicit mode calculation and the knowledge of the higher-spin algebra. It would be interesting to understand if the $d = 4$

[44] A similar story holds in d dimensions with the relevant poset now being $G[(\frac{d-2}{2})^{k-1}, (0)^{k-2}, (0^{k-2})]$.

minimal-twist stress tensor sector can also be related to an emergent symmetry algebra in the lightcone limit. Recently there have been several works devoted to the lightray operators made out of the stress tensor and to the study of the algebra of such operators [105-49]. It would be interesting to understand if there is a connection to our work.

5. Black Holes and Conformal Regge Bootstrap

5.1. Introduction and summary of results

Holographic CFTs satisfy the following defining properties: (1) large central charge C_T together with factorization of correlation functions and (2) a parametrically large gap in the spectrum of single trace operators above spin-2. As argued in [14], they are dual to theories of quantum gravity in asymptotically AdS spacetimes with local physics below the AdS scale. In holographic CFTs the Regge limit of a four-point function, extensively studied in [28-32][45], is dominated by operators of spin two – the stress tensor and the double-trace operators (this is a consequence of the gap in the spectrum). In gravity, it reproduces a Witten diagram with graviton exchange (see e.g. [137]). The Regge limit corresponds to special kinematics, which on the gravity side is described by the scattering of highly energetic particles whose trajectories in the bulk are approximately null.

Such scattering can be described in the eikonal approximation where particles follow classical trajectories but their wavefunctions acquire a phase shift $\delta(S, L)$. The phase shift is a function of the total energy S and the impact parameter L. In the CFT language, this phase shift can be extracted from the Fourier transform of the four-point function. In [29] the phase shift extracted from the four-point function of the type $\langle \mathcal{O}_1 \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_2 \rangle$ was shown to be equal (up to a factor of $-\pi$) to the anomalous dimension of the double-trace operators $[\mathcal{O}_1 \mathcal{O}_2]_{n,l}$ at leading order in $1/N^2$. The Regge limit implies that the calculation is valid for $n, l \gg 1$. These anomalous dimensions have been subsequently verified in [138,139-147].

Above, the operators $\mathcal{O}_1$ and $\mathcal{O}_2$ were assumed to have conformal dimensions of order one. What happens if one pair of the operators become heavy? As explained in [8], one can define the phase shift as a Fourier transform of the $\langle \mathcal{O}_H \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \rangle$ four-point function. It is related to the time delay and angle deflection of a highly energetic particle traveling along a null geodesic in the background of an asymptotically AdS black hole. The black hole corresponds

[45] See also [123-136] for other recent applications of Regge limit in CFTs.

to the insertion of the heavy operator $\mathcal{O}_H$; its mass in the units of AdS radius is proportional to μ.

The phase shift $\delta(S, L)$ was computed in gravity in [8] as an infinite series expansion in μ, *i.e.*,

$$\delta(S, L) = \sum_{k=1}^{\infty} \delta^{(k)} \mu^k \,, \tag{5.1}$$

with terms subleading in $1/C_T$ suppressed. The anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ admit a similar expansion

$$\gamma(n, l) = \sum_{k=1}^{\infty} \gamma^{(k)} \mu^k. \tag{5.2}$$

In [8] it was also proven that

$$\gamma^{(1)} = -\frac{\delta^{(1)}}{\pi} \tag{5.3}$$

where the following identifications are implied:

$$h = n + l, \qquad \bar{h} = n, \qquad S = 4h\bar{h}, \qquad e^{-2L} = \frac{\bar{h}}{h}. \tag{5.4}$$

However, it was observed that this relation does not hold for higher order terms, i.e. in general $\gamma^{(k)}$ is not proportional to $\delta^{(k)}$. One of the aims of the paper [1] reviewed in this section is to explain how higher order anomalous dimensions are related to higher order terms in the phase shift.

Summary of results

In this section we explain how to compute the anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H \mathcal{O}_L]_{n,l}$ order by order in μ, using the phase shift result of [8]. In particular, we show that the $\mathcal{O}(\mu^2)$ anomalous dimensions in any d are given by

$$\gamma^{(2)} = -\frac{\delta^{(2)}}{\pi} + \frac{\gamma^{(1)}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}, \qquad \Delta_H \gg l, n \gg 1. \tag{5.5}$$

Using known results for $\delta^{(1)}$ and $\delta^{(2)}$ from [8], we find an explicit expression for $\gamma^{(2)}$ and compare it with the known results in the lightcone limit ($\Delta_H \gg l \gg n \gg 1$). We find perfect agreement.

The rest of the section is organized as follows. In Section 5.2, we focus on four-dimensional holographic CFTs. At $\mathcal{O}(\mu)$, we use the crossing equation between the S- and T-channel to solve for the anomalous dimensions of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$. The result is Eq. (5.3), valid for $l, n \gg 1$. We then introduce the impact parameter representation which allows us to rewrite the S-channel expansion as a Fourier transform. We use this to relate the phase shift to the anomalous dimensions of $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ at $\mathcal{O}(\mu^2)$, thereby deriving (5.5). Using a known result for the phase shift $\delta^{(2)}$, we write down an explicit expression for $\gamma^{(2)}$. In the subsequent $l \gg n$ limit, , corresponding to a large impact parameter, it reduces to the result which has been obtained in [8] in a completely different way (by computing corrections to the energies of excited states in the AdS-Schwarzschild background).

In Section 5.3, we generalize these results to any d ($d = 2$ is treated separately in Appendix B). By solving the Casimir equation in the limit $\Delta_H \gg \Delta_L, l, n$, we obtain the conformal blocks for heavy-light double-trace operators in the S-channel. Using the explicit expression for the blocks together with the mean field theory OPE coefficients, we derive an impact parameter representation valid in general dimensions. Just as in the $d = 4$ case, this allows us to write the S-channel sum as a Fourier transform. Hence, we show that (5.5) holds for any d. We compute $\gamma^{(2)}$ in the lightcone limit and find perfect agreement with the results quoted in [8]. In addition, we find an expression for the $\mathcal{O}(\mu^2)$ corrections to the OPE coefficients.

Section 5.5 discusses various observations and mentions some open problems. Appendix B contain additional technical details. The conformal bootstrap calculations are summarized in Appendix B.1, the proof of the impact parameter representation in $d = 4$ in Appendix B.2 and the proof in general dimension d in Appendix B.3. The special case of $d = 2$ is treated in Appendix B.4. Appendix B.5 discusses the fate of some boundary terms. Appendices B.6 and B.7 contain some identities which are used in the main part of this section.

5.2. Anomalous dimensions of heavy-light double-trace operators in $d = 4$

Consider

$$G(z, \bar{z}) = \lim_{x_4 \to \infty} x_4^{2\Delta_H} \langle \mathcal{O}_H(x_4)\mathcal{O}_L(1)\mathcal{O}_L(z, \bar{z})\mathcal{O}_H(0)\rangle. \tag{5.6}$$

Inserting the conformal blocks in (2.46) together with the MFT OPE coefficients in the Regge limit (2.49), we approximate the sums in the S-channel expansion by integrals and find the following expression at $\mathcal{O}(\mu^0)$

$$G(z,\bar{z})|_{\mu^0} = \frac{C_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L - 2}(h-\bar{h}) \left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1} \right). \quad (5.7)$$

The integrals are computed in Appendix B.1; the result is the disconnected correlator in the T-channel $[(1-z)(1-\bar{z})]^{-\Delta_L}$ in the Regge limit $\sigma \to 0$, where $1-z = \sigma e^{\rho}$ and $1-\bar{z} = \sigma e^{-\rho}$.

At $\mathcal{O}(\mu)$ in holographic CFTs the leading corrections in the T-channel come from the exchanges of the stress tensor and double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ ($[\mathcal{O}_H\mathcal{O}_H]_{n,l=2}$ are heavy and therefore decouple). The conformal block for the T-channel exchange of the stress tensor is found after $z \to e^{-2\pi i}z$ and then $\sigma \to 0$ to be given by

$$g_{T_{\mu\nu}} = \frac{360 i\pi e^{-\rho}}{\sigma(e^{2\rho}-1)} + \dots, \quad (5.8)$$

where ... denotes non-singular terms. The contribution from the stress tensor exchange in the T-channel is thus imaginary for real values of σ and ρ. The only imaginary term at order μ in the S-channel expansion (2.46) comes from the term proportional to $-i\pi\gamma$; it must reproduce (5.8).

In the Regge limit, we approximate the sum in the S-channel by an integral and insert the OPE coefficients from (2.49); the imaginary part at $\mathcal{O}(\mu)$ in the S-channel is thus given by

$$\begin{aligned} \mathrm{Im}(G(z,\bar{z}))|_{\mu^1} = & \frac{-i\pi C_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h})\gamma^{(1)}(h,\bar{h}) \\ & \times \left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1} \right). \end{aligned} \quad (5.9)$$

With the ansatz $\gamma^{(1)}(h,\bar{h}) = c_1 h^a \bar{h}^b/(h-\bar{h})$ the integrals in (5.9) can be computed (for more details see Appendix B.1). In order to reproduce the exchange of the stress tensor, the anomalous dimensions at $\mathcal{O}(\mu)$ must be equal to

$$\begin{aligned} \gamma^{(1)} &= -\frac{90\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}}\lambda_{\mathcal{O}_L\mathcal{O}_L T_{\mu\nu}}}{\mu\Delta_L} \frac{\bar{h}^2}{h-\bar{h}} \\ &= -\frac{3\bar{h}^2}{h-\bar{h}}, \end{aligned} \quad (5.10)$$

where in the second line we inserted the OPE coefficients from (2.29). With the form (5.10) not only the stress tensor exchange is reproduced, but also an infinite sum of spin-2 double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ with scaling dimension $\Delta_n = 2\Delta_L + 2 + 2n$. This is similar to what happens in the light-light case [35].

To determine the second order corrections to the anomalous dimensions we use the derivative relationship:

$$P^{(0)}P^{(1)} = \frac{1}{2}(\partial_h + \partial_{\bar{h}})\left(P^{(0)}\gamma^{(1)}\right). \tag{5.11}$$

We will prove below (see Section 5.3.3) that this relationship is true in the limit $h, \bar{h} \gg 1$. The imaginary part at $\mathcal{O}(\mu^2)$ in the S-channel from (2.46) is then given by

$$\begin{aligned}\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = &- i\pi \int_0^\infty dh \int_0^h d\bar{h} P^{(0)} \Big(\gamma^{(2)} + \gamma^{(1)} P^{(1)} \\ &+ \frac{(\gamma^{(1)})^2}{2}(\partial_h + \partial_{\bar{h}})\Big) g_{h,\bar{h}}.\end{aligned} \tag{5.12}$$

With the help of (5.11), one can write (5.12) as

$$\begin{aligned}\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = &- i\pi \int_0^\infty dh \int_0^h d\bar{h} P^{(0)} \left(\gamma^{(2)} - \frac{\gamma^{(1)}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}\right) g_{h,\bar{h}} \\ &+ \text{total derivative},\end{aligned} \tag{5.13}$$

where the total derivate term does not contribute (see Appendix B.5 for details). In order to fix $\gamma^{(2)}$ completely from crossing symmetry, we would need to consider the exchange of infinitely many double-trace operators made out of the stress tensor in the T-channel. Instead, we will use an impact parameter representation to relate $\gamma^{(2)}$ to the bulk phase shift calculated from the gravity dual in [8].

5.2.1. $4d$ impact parameter representation and relation to bulk phase shift

In [29] the anomalous dimensions of light-light double-trace operators in the limit $h, \bar{h} \gg 1$ were shown to be related to the bulk phase shift. An impact parameter representation for the case when one of the operators is heavy was introduced in [8], where it was also shown that the bulk phase shift and the anomalous dimensions are equal at $\mathcal{O}(\mu)$. The goal of this section is to see explicitly how the bulk phase shift and the anomalous dimensions are related to $\mathcal{O}(\mu^2)$.

The correlator (5.6) can be written in an impact parameter representation as

$$G(z,\bar{z}) = \int_0^\infty dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}} f(h,\bar{h}), \tag{5.14}$$

with $\mathcal{I}_{h,\bar{h}}$ given by

$$\mathcal{I}_{h,\bar{h}} = (z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(0)} g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) \tag{5.15}$$

and $f(h,\bar{h})$ some function that generically depends on the anomalous dimension and corrections to the OPE coefficients. In particular, for $f(h,\bar{h}) = 1$, (5.14) is equal to the disconnected correlator. In Appendix B it is shown that $\mathcal{I}_{h,\bar{h}}$ can be equivalently written as

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta_L) \int_{M^+} \frac{d^4p}{(2\pi)^4} (-p^2)^{\Delta_L-2} e^{-ipx} (h-\bar{h}) \delta(p\cdot\bar{e}+h+\bar{h})\, \delta\left(\frac{p^2}{4}+h\bar{h}\right) \tag{5.16}$$

where M^+ is the upper Milne wedge with $\{p^2 \leq 0,\ p^0 \geq 0\}$, $C(\Delta_L)$ given by (with $d=4$)

$$C(\Delta) \equiv \frac{2^{d+1-2\Delta}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta)\Gamma(\Delta-\frac{d}{2}+1)} \tag{5.17}$$

and $\bar{e} = (1,0,0,0)$. Moreover, following [8], we will set $z = e^{ix^+}$ and $\bar{z} = e^{ix^-}$, with $x^+ = t+r$ and $x^- = t-r$ in spherical coordinates.

Using the identity

$$\delta(p\cdot\bar{e}+h+\bar{h})\, \delta\left(\frac{p^2}{4}+h\bar{h}\right) = \frac{1}{|h-\bar{h}|}\left(\delta\left(\frac{p^+}{2}-h\right)\delta\left(\frac{p^-}{2}-\bar{h}\right) + (h\leftrightarrow\bar{h})\right), \tag{5.18}$$

with $p^+ = p^t + p^r$, $p^- = p^t - p^r$, the integrals over $h,\bar{h}$ in (5.14) are easily computed. With the identification $h = \frac{p^+}{2}$ and $\bar{h} = \frac{p^-}{2}$ it follows that a generic term like (5.14) can be written as a Fourier transform

$$\int_0^\infty dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}} f(h,\bar{h}) = C(\Delta_L) \int_{M^+} \frac{d^4p}{(2\pi)^4} (-p^2)^{\Delta_L-2} e^{-ipx} f\left(\frac{p^+}{2},\frac{p^-}{2}\right). \tag{5.19}$$

We thus see that the impact parameter representation allows us to rewrite the S-channel expression as a Fourier transform.

The phase shift $\delta(p)$ for a pair of operators $\mathcal{O}_H$ and $\mathcal{O}_L$, with scaling dimensions $\Delta_H/C_T \propto \mu$ and $\Delta_L/C_T \ll 1$, respectively, was defined in [8] by

$$\mathcal{B}(p) \equiv \int d^4x e^{ipx} G(x) = \mathcal{B}_0(p) e^{i\delta(p)}, \tag{5.20}$$

where $G(x)$ is given in (5.6) and $\mathcal{B}_0(p)$ denotes the Fourier transform of the disconnected correlator. The phase shift admits an expansion in μ:

$$\delta(p) = \mu\delta^{(1)}(p) + \mu^2\delta^{(2)}(p) + \ldots, \tag{5.21}$$

where $\ldots$ denotes higher order terms in the expansion. Expanding the exponential in (5.20) in μ we get

$$\mathcal{B}(p) = \mathcal{B}_0(p)\Big(1 + i\mu\delta^{(1)} + \mu^2(-\frac{(\delta^{(1)})^2}{2} + i\delta^{(2)}) + \ldots\Big). \tag{5.22}$$

With (5.22) the relationship between the anomalous dimensions and the bulk phase shift to $\mathcal{O}(\mu^2)$ can be established using (2.46), (2.47) and (5.19):

$$\begin{aligned} \gamma^{(1)} &= -\frac{\delta^{(1)}}{\pi} \\ \gamma^{(2)} &= -\frac{\delta^{(2)}}{\pi} + \frac{\gamma^{(1)}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}(h, \bar{h}). \end{aligned} \tag{5.23}$$

The phase shift was calculated in closed form to all orders in μ for the four-dimensional case [8], with the first and second order terms given by

$$\begin{aligned} \delta^{(1)} &= \frac{3\pi}{2}\sqrt{-p^2}\frac{e^{-L}}{e^{2L}-1} \\ \delta^{(2)} &= \frac{35\pi}{8}\sqrt{-p^2}\frac{2e^{L}-e^{-L}}{(e^{2L}-1)^3}, \end{aligned} \tag{5.24}$$

where

$$-p^2 = p^+p^-, \qquad \cosh L = \frac{p^+ + p^-}{2\sqrt{-p^2}}. \tag{5.25}$$

Using (5.24) and (5.25), the $\mathcal{O}(\mu)$ corrections to the anomalous dimensions are given by $\gamma^{(1)} = -3n^2/l$, which agrees with (5.10). From (5.24) and (5.23), we deduce the anomalous dimensions at $\mathcal{O}(\mu^2)$:

$$\gamma^{(2)} = -\frac{35}{4}\frac{(2l+n)n^3}{l^3} + 9\frac{n^3}{l^2}. \tag{5.26}$$

Taking the lightcone limit ($l \gg n \gg 1$) in (5.26) we find

$$\gamma^{(2)}_{\text{l.c.}} = -\frac{17}{2}\frac{n^3}{l^2}. \tag{5.27}$$

The anomalous dimensions in the lightcone limit (5.27) agree with eq. (6.40) in [8], which was obtained independently by considering corrections to the energy levels in the AdS-Schwarzschild background.

5.3. OPE data of heavy-light double-trace operators in generic d

In this section we will write the general form of conformal blocks for heavy-light double-trace operators in the limit $\Delta_H \sim C_T \gg 1$ and general $d > 2$. These blocks will be used to confirm the validity of the impact parameter representation in Appendix B.2 and B.3. Using the impact parameter representation the OPE data will be related to the bulk phase shift. In particular, we show that (5.23) remains valid in any number of dimensions and find explicit expressions for the corrections to the OPE coefficients up to $\mathcal{O}(\mu^2)$.

5.3.1. Conformal blocks in the heavy limit

In order to find conformal blocks in general spacetime dimension d in the limit $\Delta_H \gg \Delta_L, h, \bar{h}$, we write them in the following form:

$$g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) = (z\bar{z})^{\frac{\Delta_H+\Delta_L}{2}} F(z,\bar{z}), \tag{5.28}$$

where the function $F(z,\bar{z})$ does not depend on Δ_H and is symmetric with respect to the exchange $z \leftrightarrow \bar{z}$. Let us now insert the expression (5.28) into the Casimir equation and consider the leading $\mathcal{O}(\Delta_H)$ term:

$$z\frac{\partial}{\partial z}F(z,\bar{z}) + \bar{z}\frac{\partial}{\partial \bar{z}}F(z,\bar{z}) - (h+\bar{h})F(z,\bar{z}) = 0. \tag{5.29}$$

The most general solution to eq. (5.29) is:

$$F(z,\bar{z}) = z^{h+\bar{h}} f\Big(\frac{\bar{z}}{z}\Big), \tag{5.30}$$

where f is an arbitrary function that satisfies $f(\frac{1}{x}) = x^{-h-\bar{h}}f(x)$, since conformal blocks must be symmetric with respect to the exchange $z \leftrightarrow \bar{z}$.

The behaviour of the conformal blocks as $z,\bar{z} \to 0$ and $z/\bar{z}$ fixed is given by [75,148]

$$g_{\Delta,l}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) \to \frac{l!}{(\frac{d}{2}-1)_l}(z\bar{z})^{\frac{\Delta}{2}} C_l^{(\frac{d}{2}-1)}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big), \tag{5.31}$$

where $\Delta = \Delta_1 + \Delta_2 + 2n + l$ and $C_q^{(p)}(x)$ are the Gegenbauer polynomials. Using (5.31), we can completely determine the function f:

$$f\Big(\frac{\bar{z}}{z}\Big) = \frac{(h-\bar{h})!}{(\frac{d}{2}-1)_{h-\bar{h}}}\Big(\frac{\bar{z}}{z}\Big)^{\frac{h+\bar{h}}{2}} C_{h-\bar{h}}^{(\frac{d}{2}-1)}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big). \tag{5.32}$$

That is, the conformal blocks in the limit of large Δ_H are given by

$$g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) = \frac{(h-\bar{h})!}{(\frac{d}{2}-1)_{h-\bar{h}}}(z\bar{z})^{\frac{\Delta_H+\Delta_L+h+\bar{h}}{2}} C_{h-\bar{h}}^{(\frac{d}{2}-1)}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big). \tag{5.33}$$

It is easy to explicitly check that this form of the conformal blocks agrees with the one we used in $d=4$ in the previous Section.

5.3.2. Anomalous dimensions

In Appendix B.3, we prove the validity of the impact parameter representation in any d. This means that the derivation of (5.23) goes through for arbitrary d. Using known results for the bulk phase shift from [8], we thus find

$$\gamma^{(1)} = -\frac{\bar{h}^{\frac{d}{2}}}{h^{\frac{d}{2}-1}} \frac{\Gamma(d)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+1)} {}_2F_1(\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}). \tag{5.34}$$

In the lightcone limit ($h = l \gg \bar{h} = n$) this reduces to

$$\gamma^{(1)}_{l.c.} = -\frac{\bar{h}^{\frac{d}{2}}}{h^{\frac{d}{2}-1}} \frac{\Gamma(d)}{\Gamma(\frac{d}{2})\Gamma(\frac{d}{2}+1)}. \tag{5.35}$$

Similarly, using (5.23) together with Eq. (2.29) and Eq. (A.5) from [8], we find the $\mathcal{O}(\mu^2)$ corrections to the anomalous dimensions in the limit $h, \bar{h} \gg 1$:

$$\begin{aligned}
\gamma^{(2)} &= -\frac{\delta^{(2)}}{\pi} + \frac{1}{2}\gamma^{(1)} \left\{ \frac{2}{h+\bar{h}}\gamma^{(1)} - \frac{\Gamma(d)}{\Gamma\left(\frac{d}{2}\right)^2} \bar{h}^{\frac{d}{2}-1} h^{\frac{d}{2}-1} \frac{(h-\bar{h})^{3-d}}{h+\bar{h}} \right\} = \\
&= -\left(\frac{\bar{h}^{d-1}}{h^{d-2}}\right) \frac{2^{2d-4}\Gamma\left(d+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(d)} \; {}_2F_1[2d-3, d-2, d, \frac{\bar{h}}{h}] + \\
&+ \frac{\bar{h}^d h^{2-d}}{(h+\bar{h})} \frac{4\Gamma^2(d)}{d^2\,\Gamma^4\left(\frac{d}{2}\right)} \left({}_2F_1[\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}] \right)^2 + \\
&+ \frac{\bar{h}^{d-1}(h-\bar{h})^{3-d}}{h+\bar{h}} \frac{\Gamma^2(d)}{d\,\Gamma^4\left(\frac{d}{2}\right)} \; {}_2F_1[\frac{d}{2}-1, d-1, \frac{d}{2}+1, \frac{\bar{h}}{h}]
\end{aligned} \tag{5.36}$$

Taking further the lightcone limit ($h \gg \bar{h}$) we find that

$$\gamma^{(2)}_{l.c.} = \frac{\bar{h}^{d-1}}{h^{d-2}} \frac{2^{2d-4}}{\pi} \left(\frac{d\Gamma\left(\frac{d+1}{2}\right)^2}{\Gamma\left(\frac{d+2}{2}\right)^2} - \frac{\sqrt{\pi}\Gamma\left(d+\frac{1}{2}\right)}{\Gamma\left(d\right)} \right). \tag{5.37}$$

The result (5.37) agrees with Eq. (6.42) in [8] which was obtained independently using perturbation theory in the bulk. In order to see this explicitly, one should notice the following expression for the hypergeometric function:

$$
{}_3F_2(1,-\frac{d}{2},-\frac{d}{2};1+\frac{d}{2},1+\frac{d}{2};1)=\frac{1}{2}\Big(1+\frac{\Gamma^4(1+\frac{d}{2})\Gamma(2d+1)}{\Gamma^4(d+1)}\Big). \tag{5.38}
$$

5.3.3. Corrections to the OPE coefficients

So far, we have only considered the imaginary part of the S-channel. The real part at $\mathcal{O}(\mu)$ is given by the following expression:

$$
\begin{aligned}
\mathrm{Re}(G(z,\bar{z}))|_\mu =&(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty}dh\int_0^h d\bar{h}P^{(0)}\Big(P^{(1)}\\
&+\frac{1}{2}\gamma^{(1)}(\partial_h+\partial_{\bar{h}})\Big)g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}),
\end{aligned} \tag{5.39}
$$

which can be rewritten as:

$$
\begin{aligned}
\mathrm{Re}(G(z,\bar{z}))|_\mu =& (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty}dh\int_0^h d\bar{h}g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}\times\\
&\times\Big(P^{(0)}P^{(1)}-\frac{1}{2}(\partial_h+\partial_{\bar{h}})(P^{(0)}\gamma^{(1)})\Big)+\text{total derivative}.
\end{aligned} \tag{5.40}
$$

The total derivative term in (5.40) can be shown to vanish as explained in Appendix B.5.

To derive a relation between the corrections to the OPE coefficients and the anomalous dimensions at $\mathcal{O}(\mu)$, let us consider the limit $h,\bar{h}\gg 1$ and substitute $\bar{h}$ by h everywhere. Using (5.34), one can deduce $\gamma^{(1)}\propto h$. Then, it follows that $(\partial_h+\partial_{\bar{h}})(P^{(0)}\gamma^{(1)})\propto P^{(0)}$ and hence the second term on the right hand side of (5.40) behaves as:

$$
\begin{aligned}
&(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)}\int_0^{+\infty}dh\int_0^h d\bar{h}\Big(-\frac{1}{2}g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(\partial_h+\partial_{\bar{h}})(P^{(0)}\gamma^{(1)})\Big)\\
&\propto\frac{1}{\sigma^{2\Delta_L}}.
\end{aligned} \tag{5.41}
$$

On the other hand, we know that in the Regge limit the leading contribution in the T-channel at $\mathcal{O}(\mu)$ comes from the exchange of the stress tensor. The real part of its conformal block is proportional to σ^d, so the T-channel result behaves as $\frac{1}{\sigma^{2\Delta_L-d}}$. This is way less singular than (5.41). Hence (5.41) must be canceled by the first term on the right hand side of (5.40), at least in the limit $h, \bar{h} \gg 1$. That is:

$$P^{(0)}P^{(1)} = \frac{1}{2}(\partial_h + \partial_{\bar{h}})(P^{(0)}\gamma^{(1)}). \tag{5.42}$$

A similar relation holds for the OPE coefficients of light-light double-trace operators, e.g. see [14,81,149]. In that case it was observed in [138] that the relation is not exact in $(h, \bar{h})$. We expect the same to be true here. Furthermore, the real part at $\mathcal{O}(\mu^2)$ was given in (2.48) as:

$$\begin{aligned}\mathrm{Re}(G(z,\bar{z}))|_{\mu^2} = (z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \sum_{h\geq\bar{h}\geq 0}^{\infty} P^{(0)}\Big(P^{(2)} - \frac{1}{2}(\pi\gamma^{(1)})^2 + \\ + \frac{1}{2}(\gamma^{(2)} + P^{(1)}\gamma^{(1)})(\partial_h + \partial_{\bar{h}}) + \frac{1}{8}(\gamma^{(1)})^2(\partial_h + \partial_{\bar{h}})^2\Big) g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}.\end{aligned} \tag{5.43}$$

Using the impact parameter representation this can be expressed as:

$$\begin{aligned}\mathrm{Re}(G(z,\bar{z}))|_{\mu^2} = \int_0^\infty dh \int_0^h d\bar{h}\mathcal{I}_{h,\bar{h}} \ \Big(P^{(2)} - \frac{\pi^2}{2}(\gamma^{(1)})^2 \\ - \frac{1}{2P^{(0)}}(\partial_h + \partial_{\bar{h}})(P^{(0)}(\gamma^{(2)} + P^{(1)}\gamma^{(1)})) + \frac{1}{8P^{(0)}}(\partial_h + \partial_{\bar{h}})^2(P^{(0)}(\gamma^{(1)})^2)\Big),\end{aligned} \tag{5.44}$$

where we repeatedly integrated by parts. It follows from (5.22) and (5.19), together with $\pi\gamma^{(1)} = -\delta^{(1)}$, that the corrections to the OPE coefficients at $\mathcal{O}(\mu^2)$ satisfy the following relationship:

$$P^{(0)}P^{(2)} = \frac{1}{2}(\partial_h+\partial_{\bar{h}})(P^{(0)}(\gamma^{(2)}+P^{(1)}\gamma^{(1)}))-\frac{1}{8}(\partial_h+\partial_{\bar{h}})^2(P^{(0)}(\gamma^{(1)})^2). \tag{5.45}$$

The arguments above are similar to the ones used in [35,29].

5.3.4. Flat space limit

In the flat space limit the relation between the scattering phase shift and the anomalous dimensions has been previously discussed in [150]. Hence, it is interesting to consider the flat space limit of eq. (5.5). This limit is achieved by taking the apparent impact parameter to be much smaller than the AdS radius. This corresponds to the small L regime or, equivalently, using $e^{-2L} = \bar{h}/h$ to the $1 \ll l \ll n \ll \Delta_H$ limit.

In this limit, according to (5.34), the behavior of $\gamma^{(1)}$ is given by

$$\gamma^{(1)} \propto n \left(\frac{n}{l}\right)^{d-3} . \tag{5.46}$$

Hence, the $\gamma^{(1)}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}$ term in eq. (5.5) behaves as

$$\gamma^{(1)}\partial_n\gamma^{(1)} \propto n \left(\frac{n}{l}\right)^{2d-6} . \tag{5.47}$$

Similarly, using equation (A.5) from [8], one finds that $\delta^{(2)}$ behaves as

$$\delta^{(2)} \propto n \left(\frac{n}{l}\right)^{2d-5} . \tag{5.48}$$

Since (5.47) is subleading to (5.48), in the flat space limit the anomalous dimensions are proportional to the phase shift,

$$\gamma^{(2)} \approx -\frac{\delta^{(2)}}{\pi} \tag{5.49}$$

5.4. Discussion

In this section we studied, following [1], a four-point function of pairwise identical scalar operators, $\mathcal{O}_H$ and $\mathcal{O}_L$, in holographic CFTs in generic dimensions. Scaling Δ_H with the central charge, the CFT data admits an expansion in the ratio $\mu \sim \Delta_H/C_T$ which we keep fixed. Using crossing symmetry and the bulk phase shift calculated in [8], we studied $\mathcal{O}(\mu^2)$ corrections to the OPE data of heavy-light double-trace operators $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$ for large l and n. In particular, the relationship between the bulk phase shift and the OPE data of heavy-light double-trace operators is found using an impact parameter representation. Furthermore, this allows us in principle to determine the OPE data of $[\mathcal{O}_H\mathcal{O}_L]_{n,l}$, for $l, n \gg 1$ to all orders in μ, i.e., to all orders in an expansion in the dual black hole Schwarzschild radius.

It is interesting that each term in the μ-expansion of the bulk phase shift, computed in gravity in [8], can be rewritten as an infinite sum of "Regge conformal blocks" corresponding to operators of dimension $\Delta = k(d-2) + 2n + 2$ and spin $J = 2$. Explicitly,

$$i\,\delta^{(k)}(S,L) = f(k) \sum_{n=0}^{\infty} \lambda_k(n)\ g^R_{k(d-2)+2n+2,\,2}(S,L)\,, \tag{5.50}$$

where the coefficients $(f(k), \lambda_k(n))$ are listed in Appendix B.6 and we set $S \equiv \sqrt{-p^2}$ compared to [8]. Here $g^R_{\Delta,J}(S,L)$ denotes a "Regge conformal block", and is equal to the leading behaviour of the analytically continued T-channel conformal block in the Regge limit [151,34]

$$g^R_{\Delta,J}(S,L) = i\, c_{\Delta,J}\; S^{J-1}\; \Pi_{\Delta-1,d-1}(L) \tag{5.51}$$

defined in terms of

$$1 - z = \frac{e^L}{S}, \quad 1 - \bar{z} = \frac{e^{-L}}{S} \tag{5.52}$$

as $S \to \infty$ and L fixed. Here $c_{\Delta,J}$ are known coefficients which can be found in Appendix B.6 and $\Pi_{\Delta-1,d-1}(L)$ denotes the $(d-1)$-dimensional hyperbolic space propagator for a massive scalar of mass square $m^2 = (\Delta - 1)$.

To understand the implications of (5.50) consider double-stress tensors in the lightcone limit. For these, one expect that the dominant contribution to the bulk phase shift comes from the infinite sum of the minimal twist double-trace operators built from the stress tensor, schematically denoted by $T_{\mu\nu}\partial_{\mu_1}\cdots\partial_{\mu_\ell}T_{\rho\sigma}$. The expression in (5.50) implies that this infinite sum gives rise to a contribution which yields a softer Regge behaviour and that it effectively looks like the exchange of a single "effective" operator of the same twist $\tau = 2(d-2)$, but spin $J = 2$. At finite impact parameter, one would then need to add the contributions of an infinite tower of such effective operators of twist $\tau = 2(d-2) + 2n$ and spin $J = 2$, as in (5.50). From this point of view, the coefficients λ_n in (5.50) can be interpreted as ratios of sums of OPE coefficients of double-trace operators. A similar structure is present for any k and we discuss this further in Section 6.5.

It would be interesting to investigate whether Rindler positivity constrains the Regge behaviour of the bulk phase shift to grow at most linearly with the energy S, similarly to Section 5.2 in [34]. If this were the case, one would perhaps only need to understand the origin of the λ_n to compute the bulk phase shift purely from the boundary point of view.

6. Multi-Stress tensors and next-to-leading singularities in the Regge limit

6.1. Introduction and summary of results

Restricting to CFTs that are holographic, much progress has been made in using the bootstrap approach to constrain the CFT data perturbatively in an expansion in $1/C_T$. Especially interesting are gravitational interactions in the bulk; these are related to the exchange of multi-stress tensor operators in the boundary theory, schematically denoted by $[T^k]_{n,l}$. In this Section we will continue the study of the Regge limit of heavy-heavy-light-light correlators and the exchange of such operators.

Following [5] and Section 5, the Regge limit of a four-point function of pair-wise identical light scalar operators is related to the phase shift of $2 \to 2$ elastic scattering of highly energetic particles at fixed impact parameter in the bulk [28-32][46]. In the heavy-heavy-light-light case, the phase shift [8] was defined in the bulk in terms of the Shapiro time delay and the angle deflection of a highly energetic particle propagating in an AdS-Schwarzschild background. In the CFT the phase shift is related to a Fourier transform of the correlator and the expansion parameter is given by $\mu \sim \frac{\Delta_H}{C_T}$. At k-th order, the phase shift is given by a massive scalar propagator in $(k(d-1)-(k-1))$-dimensional hyperbolic space. On the other hand, the leading Regge behaviour of a conformal block in d dimensions takes the form of a scalar propagator in $(d-1)$-dimensional hyperbolic space. The higher-dimensional propagators appearing in the phase shift can, however, be decomposed into infinite sums of propagators with increasing scaling dimensions in H^{d-1} [1]. This appears to be a more natural representation of the phase shift from the boundary point of view.

In particular, we will study the leading and next-to-leading singularities of the stress tensor sector of the heavy-heavy-light-light correlator in the Regge limit. This is done perturbatively in μ and the stress tensor sector of the

[46] For further discussion about the Regge limit and the phase shift in holographic CFTs, see also [36,35,34,15].

correlator $\mathcal{G}(z,\bar{z})$ is given by (after $z \to e^{-2\pi i}z$, see Section 2.2.2 for a review on the Regge limit.)

$$\mathcal{G}(\sigma,\rho) := \mathcal{G}_0(\sigma)\sum_{k=0}^{\infty}\mu^k\mathcal{G}^{(k)}(\sigma,\rho), \qquad \begin{aligned}\sigma e^{\rho} &= 1-z \\ \sigma e^{-\rho} &= 1-\bar{z},\end{aligned} \tag{6.1}$$

where $\mathcal{G}_0(\sigma) := \sigma^{-2\Delta_L}$ is the disconnected correlator and $\sigma \to 0$ in the Regge limit with ρ fixed. The stress tensor sector $\mathcal{G}$ of the correlator contains the contribution of multi-stress tensor operators in the direct-channel expansion of the correlator $\mathcal{O}_L \times \mathcal{O}_L \to \mu^k[T^k]_{n,l}$. Here it is seen that the contribution at k-th order is due to multi-stress tensors made out of k stress tensors.

At k-th order, the stress tensor sector $\mathcal{G}^{(k)}$ behaves as follows in the Regge limit:

$$\mathcal{G}^{(k)}(\sigma,\rho) = \frac{F_{k,\mathrm{L}}(\rho)}{\sigma^k} + \frac{F_{k,\mathrm{NL}}(\rho)}{\sigma^{k-1}} + \mathcal{O}(\sigma^{-k+2}) \qquad \sigma \to 0, \quad \rho\text{−fixed}, \tag{6.2}$$

for some functions $F_{k,\mathrm{L}}(\rho)$ and $F_{k,\mathrm{NL}}(\rho)$. We define the leading and next-to-leading Regge singularity of the stress tensor sector of the correlator $\mathcal{G}^{(k)}$ at $\mathcal{O}(\mu^k)$ by

$$\begin{aligned} &\text{Leading Regge singularity}: && \frac{F_{k,\mathrm{L}}(\rho)}{\sigma^k}, \\ &\text{Next−to−leading Regge singularity}: && \frac{F_{k,\mathrm{NL}}(\rho)}{\sigma^{k-1}}. \end{aligned} \tag{6.3}$$

The aim of [5], reviewed in this section, is to calculate $F_{k,\mathrm{L}}$ and $F_{k,\mathrm{NL}}$ for any value of k and fixed ρ. This is done by Fourier transforming the momentum space correlator given in terms of the bulk phase shift.

We recall the definition of the Regge limit,

$$\begin{aligned} \text{Regge limit}: \quad & z \to e^{-2\pi i}z \qquad \text{with} \quad \sigma \to 0, \quad \rho\text{−fixed}, \\ & \sigma e^{\rho} = 1-z \qquad \sigma e^{-\rho} = 1-\bar{z}. \end{aligned} \tag{6.4}$$

In this limit, we assume that the momentum space correlator $\mathcal{B}(S,L)$ is given by the exponentiation of the bulk phase shift $\delta(S,L;\mu)$, where S is the energy and L the impact parameter:

$$\mathcal{B}(S,L) = \mathcal{B}_0(S)e^{i\delta(S,L;\mu)}, \tag{6.5}$$

where $\mathcal{B}_0(S)$ is the Fourier transform of the disconnected correlator. The phase shift $\delta(S, L; \mu)$ was calculated in Einstein gravity in [8] to all orders in μ and we denote the k-th term in that expansion $\delta^{(k)}$. In the Regge limit $S \gg 1$, the phase shift is linear in S and the leading ($\sim \sigma^{-k}$) and next-to-leading Regge singularities ($\sim \sigma^{-k+1}$) of $\mathcal{G}^{(k)}(\sigma, \rho)$ are due to terms in (6.5) of the form:

$$\mathcal{B}(S, L)\Big|_{\mu^k} = \mathcal{B}_0(S)\Big[\frac{(i\delta^{(1)})^k}{k!} + i\delta^{(2)}\frac{(i\delta^{(1)})^{k-2}}{(k-2)!} + \dots\Big], \tag{6.6}$$

where the ellipses denote terms that contribute at subleading order in $\sigma \to 0$.

By Fourier transforming the first term in the brackets in (6.6), it is found that the leading Regge singularities of $\mathcal{G}$ are given by

$$\begin{aligned}\mathcal{G}^{(k)}(\sigma, \rho)\Big|_{\sigma^{-k}} = \sum_{n=0}^{\infty}\frac{(3i\pi)^k}{k!}\binom{n+k-2}{n}(\Delta_L)_{2k+n-1}(\Delta_L-1)_{1-k-n} \\ \times \frac{e^{-(3k+2n)\rho}}{\sigma^k(1-e^{-2\rho})},\end{aligned} \tag{6.7}$$

for $k = 1, 2, \dots$. This agrees with the result in [104] obtained by considering a light particle propagating in a shockwave background. In this case, we see that the leading Regge singularities are fully determined by the phase shift at first order in μ. The first-order phase shift is in turn fixed by the exchange of the stress tensor in the CFT and is therefore universal in holographic CFTs (with a large gap).

The next-to-leading Regge singularity $\sim \sigma^{-k+1}$ at $\mathcal{O}(\mu^k)$ gets two contributions, there is a subleading correction in σ coming from $(\delta^{(1)})^k$ in (6.6) as well as a contribution from $\delta^{(2)}(\delta^{(1)})^{k-2}$ in (6.6). The former gives the following contribution:

$$\begin{aligned}\mathcal{G}^{(k)}(\sigma, \rho)\Big|_{(\delta^{(1)})^k, \sigma^{-k+1}} =& \frac{1}{2}\sum_{n=0}^{\infty}\frac{(3i\pi)^k}{k!}\binom{n+k-2}{n}(\Delta_L)_{2k+n-1}(\Delta_L-1)_{1-k-n} \\ &\times \Big[(k+n-1)e^{-\rho} - (2k+n)e^{\rho}\Big]\frac{e^{-(3k+2n)\rho}}{\sigma^{k-1}(1-e^{-2\rho})},\end{aligned} \tag{6.8}$$

while the latter gives:

$$\begin{aligned}\mathcal{G}^{(k)}(\sigma, \rho)\Big|_{\delta^{(2)}(\delta^{(1)})^{k-2}, \sigma^{-k+1}} = \frac{1}{320}\sum_{p=0}^{\infty}\sum_{n=0}^{p}\frac{i^{k-1}(3\pi)^{k-2}}{(k-2)!}\binom{k+p-n-3}{p-n} \\ \times (\Delta_L)_{2k+p-2}(\Delta_L-1)_{1-k-p}\lambda_2(n)\bar{c}_{6+2n,2}\frac{e^{-(3k+2p-1)\rho}}{\sigma^{k-1}(1-e^{-2\rho})},\end{aligned} \tag{6.9}$$

where $\bar{c}_{6+2n,2}$ are constants given in (2.17). The coefficients $\lambda_2(n)$ are related to the decomposition of the second-order phase shift into Regge conformal blocks in (4.3) and are valid assuming there are no higher-derivative corrections in the bulk gravitational action. Adding (6.8) and (6.9) gives the full expression for the next-to-leading singularities $\sim \sigma^{-k+1}$ at $\mathcal{O}(\mu^k)$ due to multi-stress tensor of the schematic form $[T^k]_{n,l}$.

In particular, in the limit $\rho \to \infty$, only the $p = 0$ term in (6.9) contributes

$$\mathcal{G}^{(k)}(\sigma,\rho)\Big|_{\delta^{(2)}(\delta^{(1)})^{k-2},\sigma^{-k+1}} \underset{\rho\to\infty}{\approx} \frac{35}{6}\frac{(3i\pi)^{k-1}}{(k-2)!}(\Delta_L)_{2k-2}(\Delta_L-1)_{1-k}\frac{e^{-(3k-1)\rho}}{\sigma^{k-1}}. \tag{6.10}$$

The result in (6.9) is obtained using the phase shift obtained in Einstein gravity in the bulk. In the limit $\rho \to \infty$ given in (6.10), the result is expected to be universal in theories with large gap since there is by now much evidence of universality in the minimal-twist subsector of multi-stress tensors [102,104,105,2]. In other words, we expect (6.10) to be independent of higher-derivative terms in the gravitational action. We find perfect agreement between (6.7)-(6.9) and known results for minimal-twist double-stress and triple-stress tensors obtained using lightcone bootstrap [80,2].

In Section 6.2, general properties of the heavy-heavy-light-light correlator in CFTs with large central charge is considered as well as its connection to the bulk phase shift. In Section 6.3, the procedure for decomposing products of Regge conformal blocks in $d = 2, 4$ is described. In Section 6.4, the leading and next-to-leading Regge singularities in four dimensions are found from the exponentiation of the phase shift. Section 6.5 is devoted to discussion and the appendices contain some technical details and further matching with results obtained from lightcone bootstrap.

6.2. Heavy-heavy-light-light correlator in holographic CFTs

In the lightcone limit, multi-stress tensors with minimal twist and arbitrary spin dominate. Since the twist is bounded from below, one can study the correlator perturbatively in a kinematical expansion close to the lightcone. On the other hand, in the Regge limit multi-stress tensors of highest spin dominate due to the behaviour σ^{1-J} of the blocks in the Regge limit $\sigma \to 0$, with J being the spin. This limit is therefore difficult to study a priori in CFTs. Instead, we use the bulk phase shift calculated in the dual gravitational theory to extract contributions to the stress tensor sector of the correlator in the Regge limit. Approaching the large impact parameter limit of the Regge limit, we can make contact with results obtained using lightcone bootstrap.

In Section 6.2, the heavy-heavy-light-light correlator in CFTs is reviewed with emphasis on its behaviour in the lightcone- and the Regge limit. We then review known results for the subsector of minimal-twist double- and triple-stress tensors are studied in the large impact parameter regime of the Regge limit. Then the connection between the bulk phase shift and the heavy-heavy-light-light correlator is explained following [8].

6.2.1. The Regge limit of minimal-twist double- and triple-stress tensors

It was argued in [2], as covered in Section 3, that the subsector of minimal-twist multi-stress tensor operators is universally fixed by crossing symmetry in terms of the exchange of the stress tensor. The contribution of minimal-twist multi-stress tensors $[T^k]^{(0)}_{0,l}$ to the heavy-heavy-light-light correlator takes the following particular form[47]:

$$\mathcal{G}^{(k)}_{\rm LC}(z,\bar{z}) = (1-\bar{z})^{k(\frac{d-2}{2})} \sum_{\{i_p\}} a_{i_1 \dots i_k} f_{i_1} \dots f_{i_k}, \qquad \sum_{p=1}^{k} i_p = k\left(\frac{d+2}{2}\right), \quad (6.11)$$

with coefficients that were determined in Section 3. Note that each term $\mathcal{G}^{(k)}_{\rm LC}$ sums an infinite number of multi-stress tensor operators $[T^k]_{0,l}$ with twist $k(d-2)$ and spin $2k+l$, for $l = 0, 2, \dots$. For more details on the minimal-twist multi-stress tensors, see [2] and Section 3 and 4.

Here we assume a large C_T CFT with large $\Delta_{\rm gap}$. Explicitly, we use the bulk phase shift calculated in the gravitational dual to study the CFT correlator in the Regge limit. To make contact between the Regge limit and the lightcone limit, we analytically continue $\mathcal{G}^{(k)}_{\rm LC}(z,\bar{z})$ according to $z \to e^{-2\pi i} z$:

$$\mathcal{G}^{(k),\circlearrowleft}_{\rm LC}(z,\bar{z}) := \mathcal{G}^{(k)}_{\rm LC}(z e^{-2\pi i},\bar{z}). \quad (6.12)$$

Sending also $z \to 1$, we refer to this as the Regge-Lightcone limit.

[47] In this section we refer to the results obtained using the light-cone bootstrap by $\mathcal{G}_{\rm LC}$ to distinguish it from the results obtained from the phase shift in this section.

Using the explicit results (3.11) for $\mathcal{G}_{\rm LC}^{(2)}$ in $d=4$ [80,2], we find the following leading and next-to-leading singularities in the Regge-Lightcone limit due to minimal-twist double-stress tensors

$$
\begin{aligned}
\mathcal{G}_{\rm LC}^{(2),\circlearrowleft}(\sigma,\rho) = & -\frac{9\pi^2\Delta_L(\Delta_L+1)(\Delta_L+2)}{2(\Delta_L-2)}\frac{e^{-6\rho}}{\sigma^2}+ \\
& + \Big[\frac{35i\pi\Delta_L(\Delta_L+1)}{2(\Delta_L-2)}+\frac{18\pi^2\Delta_L(\Delta_L+1)(\Delta_L+2)}{2(\Delta_L-2)}\Big]\frac{e^{-5\rho}}{\sigma}+\dots,
\end{aligned}
\tag{6.13}
$$

where the ellipses denote non-singular terms as $\sigma\to 0$. Likewise, in the Regge-Lightcone limit of $\mathcal{G}_{\rm LC}^{(3),\circlearrowleft}$ due to the exchange of minimal-twist triple-stress tensors [2], one finds the following leading and next-to-leading singularities in the Regge limit from (3.12):

$$
\begin{aligned}
\mathcal{G}_{\rm LC}^{(3),\circlearrowleft}(\sigma,\rho)\Big|_{\sigma^{-3}} &= -\frac{9i\pi^3\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{2(\Delta_L-2)(\Delta_L-3)}\frac{e^{-9\rho}}{\sigma^3}, \\
\mathcal{G}_{\rm LC}^{(3),\circlearrowleft}(\sigma,\rho)\Big|_{\sigma^{-2}} &= \Big[-\frac{105\pi^2\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)}{2(\Delta_L-2)(\Delta_L-3)} \\
&\quad +\frac{27i\pi^3\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{2(\Delta_L-2)(\Delta_L-3)}\Big]\frac{e^{-8\rho}}{\sigma^2}.
\end{aligned}
\tag{6.14}
$$

The results (6.13)-(6.14) from lightcone bootstrap will be compared to the results obtained using the bulk phase shift to study the Regge limit. While we are mainly interested in terms that behave as σ^{-k} and σ^{-k+1} at $\mathcal{O}(\mu^k)$ in the Regge limit, the term proportional to σ^{-1} at $\mathcal{O}(\mu^3)$ is further given by:

$$
\begin{aligned}
\mathcal{G}_{\rm LC}^{(3),\circlearrowleft}(\sigma,\rho)\Big|_{\sigma^{-1}} &= \Big[i\Big(\frac{1155\pi\Delta_L(\Delta_L+1)(\Delta_L+2)}{8(\Delta_L-2)(\Delta_L-3)} \\
&\quad -\frac{9\pi^3\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+4)(19+7\Delta_L)}{4(\Delta_L-2)(\Delta_L-3)}\Big) \\
&\quad +\frac{525\pi^2\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)}{4(\Delta_L-2)(\Delta_L-3)}\Big]\frac{e^{-7\rho}}{\sigma}.
\end{aligned}
\tag{6.15}
$$

6.2.2. Bulk phase shift of a light particle in an AdS black hole background

The relationship between the bulk phase shift and the heavy-heavy-light-light correlator was described in [8] which we briefly review here for completeness. We consider a four-point function defined on the cylinder parameterized by time τ and a unit vector $\hat{n}$ on S^{d-1}:

$$
G(x) \equiv \langle \mathcal{O}_H^{\rm cyl}(\tau_4,\hat{n}_4)\mathcal{O}_L^{\rm cyl}(\tau_3,\hat{n}_3)\mathcal{O}_L^{\rm cyl}(\tau_2,\hat{n}_2)\mathcal{O}_H^{\rm cyl}(\tau_1,\hat{n}_1)\rangle. \tag{6.16}
$$

Inserting the heavy operators at $\tau_{4,1} = \pm\infty$ and going to the plane using the transformation $r = e^{\tau}$, we have

$$G(x) = (r_2 r_3)^{\Delta_L} \frac{\hat{\mathcal{G}}(z,\bar{z})}{x_{32}^{2\Delta_L}}, \tag{6.17}$$

where the cross-ratios are given by

$$\begin{aligned} z\bar{z} &= \frac{x_2^2}{x_3^2} = e^{2(\tau_2-\tau_3)} \\ (1-z)(1-\bar{z}) &= \frac{x_{23}^2}{x_3^2} = 1 + e^{2(\tau_2-\tau_3)} - 2e^{\tau_2-\tau_3}\hat{n}_2\cdot\hat{n}_3. \end{aligned} \tag{6.18}$$

The function $\hat{\mathcal{G}}(z,\bar{z})$ can be expanded in conformal blocks on the plane. Especially, we will be interested in the stress tensor sector $\mathcal{G}$ of $\hat{\mathcal{G}}$:

$$\mathcal{G}(z,\bar{z}) \equiv \hat{\mathcal{G}}(z,\bar{z})\Big|_{\text{multi}-\text{stress tensors}} \tag{6.19}$$

As we will see, when Fourier transforming the phase shift, there are contributions to $\hat{\mathcal{G}}(z,\bar{z})$ coming from double-trace operators that are of the schematic form $[\mathcal{O}_L\mathcal{O}_L]_{n,l}$. By definition, these do not contribute to $\mathcal{G}(z,\bar{z})$

We introduce two points P_2 and P_3 on the cylinder which differ by Lorentzian time π and are diametrically opposite on the sphere[48], i.e. $\hat{n}(P_3) = -\hat{n}(P_2)$. By translational and rotational invariance, the operator $\mathcal{O}_L(x_2)$ is inserted at P_2 and $\mathcal{O}_L(x_3)$ is inserted close to P_3 with $\hat{n}_3\cdot\hat{n}(P_3) = \cos\varphi$. Starting from Euclidean kinematics, we Wick-rotate by $\tau_i \to it_i$ and set $t_3 - t_2 = \pi + x_0 - i\frac{\epsilon}{2}$, where $x^0 \geq 0$ parameterizes the time delay. Using (6.18) one can solve for $z,\bar{z}$ in terms of $x^{\pm} = x^0 \pm \varphi$:

$$\begin{aligned} z &= e^{-ix^+} \\ \bar{z} &= e^{-ix^-}. \end{aligned} \tag{6.20}$$

Note that a highly energetic light particle in pure AdS starting at P_2 will propagate to the point P_3; the (x^0,φ)-coordinates measure the position of $\mathcal{O}_L(x_3)$

48 In pure AdS, a null geodesic starting at P_2 traverse the bulk and ends at P_3. Below we explore the deviation from this due to the presence of the black hole as first explored in [8].

relative to the point P_3. These kinematics are obtained starting with the operators close to P_2, corresponding to $x^+ \approx -2\pi$, and then $\mathcal{O}_L(\tau_3, \hat{n}_3)$ is moved close to P_3 by taking $x^+ \to x^+ + 2\pi$. In terms of the cross-ratios, this corresponds to taking $z \to e^{-2\pi i} z$. With these kinematics, the correlator $G(x)$ in (6.17) is given in the Regge limit $x^\pm \to 0$, with their ratio kept fixed, by

$$G(x) = \frac{\hat{\mathcal{G}}(z, \bar{z})}{(-x^2 - i\epsilon x^0)^{\Delta_L}} \left[1 + \mathcal{O}((x^+)^2)\right], \tag{6.21}$$

with $-x^2 = (x^0)^2 - \varphi^2$.

The phase shift is defined by the following Fourier transform:

$$\mathcal{B}(p) \equiv \mathcal{B}_0(p) e^{i\delta} = \int d^d x G(x) e^{-ipx}, \tag{6.22}$$

where $\mathcal{B}_0(p)$ denotes the Fourier transform of the disconnected correlator and $e^{i\delta}$ contain the (non-trivial) dynamics of the correlator. Explicitly, the Fourier transform of the disconnected correlator is given by

$$\mathcal{B}_0(p) = \int d^d x \frac{e^{-ipx}}{(-x^2 - i\epsilon x^0)^{\Delta_L}} = \theta(p^0)\theta(-p^2) e^{i\pi\Delta_L} C(\Delta_L)(-p^2)^{\Delta_L - \frac{d}{2}} \tag{6.23}$$

where

$$C(\Delta_L) = \frac{2^{d+1-2\Delta_L} \pi^{1+\frac{d}{2}}}{\Gamma(\Delta_L)\Gamma(\Delta_L - \frac{d-2}{2})}. \tag{6.24}$$

Here the combination $\theta(p^0)\theta(-p^2)$ ensures that p lies in the upper Milne wedge M^+.

We further introduce the parametrization $p^\mu = \sqrt{-p^2}\omega^\mu$ in terms of two vectors ω and $\bar{e}$, such that $\omega^2 = \bar{e}^2 = -1$ and $\bar{e}^0 = 1$ with all other components set to 0. Then

$$\begin{aligned} S &= \sqrt{-p^2}, \\ \cosh L &= -\bar{e} \cdot \omega = \frac{p^+ + p^-}{2\sqrt{-p^2}}. \end{aligned} \tag{6.25}$$

Likewise, we define $x^\mu = \sqrt{-x^2} e^\mu$ with $e^2 = -1$ such that

$$\begin{aligned} \sqrt{-x^2} &= \sqrt{-\log z \log \bar{z}}, \\ -e \cdot \bar{e} &= \frac{i \log z\bar{z}}{2\sqrt{-\log z \log \bar{z}}}. \end{aligned} \tag{6.26}$$

Expanding (6.26) in the Regge limit $\sigma \to 0$, one finds

$$\begin{aligned} \sqrt{-x^2} &= -i\sigma\Big(1 + \frac{\sigma}{2}\cosh\rho + \dots\Big), \\ -e \cdot \bar{e} &= \cosh\rho + \frac{e^{-2\rho}(1 - e^\rho)^2(1 + e^\rho)^2}{8}\sigma + \dots, \end{aligned} \tag{6.27}$$

where the ellipses denote subleading corrections in σ.

Following [8], we review how a single Regge conformal block in momentum space can be transformed into position space in any dimension. The leading result in $\sigma \to 0$ can then be identified with the leading Regge behaviour of a conformal block due to an operator exchange in the direct-channel. In the case when the operator appears in the spectrum, its coefficient is related to the product of OPE coefficients[49].

In Section 6.3.1, we show that a product of Regge conformal blocks in two dimensions is again a Regge conformal block. In Section 6.3.2, the four-dimensional case is considered where, on the other hand, it is shown that products of Regge conformal blocks can be decomposed into an infinite sum of Regge conformal blocks of different twist $\Delta - J$. Using this decomposition, one can do the Fourier transform and read off the contribution to the position space correlator. In particular, in the limit $\rho \gg 1$, only the term with minimal twist in the decomposition is important. In this limit one can, therefore, approximate products of Regge conformal blocks in $d = 4$ with a single Regge conformal block. This is reminiscent of what happens in $d = 2$.

A Regge conformal block was defined in [1] by

$$g^{\mathrm{R}}_{\Delta,J}(S,L) = i c_{\Delta,J} S^{J-1} \Pi_{\Delta-1,d-1}(L), \tag{6.28}$$

with $\Pi_{\Delta-1,d-1}(L)$ a $(d-1)$-dimensional hyperbolic space propagator of a particle with mass-squared $m^2 = (\Delta-1)^2$, defined in (2.55)[50], and $c_{\Delta,J}$ given by (2.56). Note that the Regge conformal blocks in (6.28) is identical to the leading Regge behaviour of the analytically continued blocks in (2.54) with the following replacement $S \to \sigma^{-1}$ and $L \to \rho$.

The hyperbolic space propagator in (2.55) can be written in terms of functions $\Omega_{i\nu} = \frac{i\nu}{2\pi}(\Pi_{i\nu+\frac{d}{2}-1} - \Pi_{-i\nu+\frac{d}{2}-1})$ as[51]

$$\Pi_{\Delta-1}(L) = \int_{-\infty}^{\infty} d\nu \frac{\Omega_{i\nu}(L)}{\nu^2 + (\Delta - \frac{d}{2})^2}, \tag{6.29}$$

[49] The term “effective operator” is used below when the Fourier transform of a Regge conformal block can be identified with the leading Regge behaviour of a conformal block even though such an operator does not appear in the t-channel expansion .

[50] See e.g. [152] for further details.

[51] For brevity, we denote $\Pi_{\Delta-1} \equiv \Pi_{\Delta-1,d-1}$ and likewise for $\Omega_{i\nu}$.

which can be shown using (2.55) and deforming the integration contour to compute the integral. The functions $\Omega_{i\nu}$ constitute a basis of regular eigenfunctions of the Laplacian operator on H_{d-1}, for more details, see e.g. [152].

Consider the contribution to the correlator due to a single Regge conformal block of dimension Δ and spin J:

$$\mathcal{B}(S,L)\Big|_{\Delta,J} = \mathcal{B}_0(S)\lambda g^{\mathrm{R}}_{\Delta,J}(S,L), \tag{6.30}$$

where λ is a numerical coefficient and $\mathcal{B}_0(S)$ is the disconnected correlator given in (6.23). The position space result from (6.30) is given by the Fourier transform

$$G(x)\Big|_{\Delta,J} = \lambda \int_{M^+} \frac{d^dp}{(2\pi)^d} e^{ipx}\mathcal{B}_0(S) g^{\mathrm{R}}_{\Delta,J}(S,L), \tag{6.31}$$

which by inserting (6.28) and using (6.29) can be written as

$$G(x)\Big|_{\Delta,J} = ic_{\Delta,J}\lambda \int_{M^+} \frac{d^dp}{(2\pi)^d} e^{ipx}\mathcal{B}_0(S)S^{J-1}\int_{-\infty}^{\infty} d\nu \frac{\Omega_{i\nu}(\omega\cdot\bar{e})}{\nu^2+(\Delta-\frac{d}{2})^2}. \tag{6.32}$$

We then need the following identity derived in [8]:

$$\frac{2^{1-a}e^{\frac{i\pi a}{2}}}{\pi^{\frac{d-2}{2}}}\int_{M^+} d^dp e^{ipx} S^{a-d}\Omega_{i\nu}(\omega\cdot\bar{e}) = \frac{\Gamma(\frac{a-\frac{d-2}{2}+i\nu}{2})\Gamma(\frac{a-\frac{d-2}{2}-i\nu}{2})}{(-x^2)^{\frac{a}{2}}}\Omega_{i\nu}(e\cdot\bar{e}). \tag{6.33}$$

Using this identity with $a = 2\Delta_L + J - 1$ and the disconnected correlator in (6.23), (6.32) gives

$$\begin{aligned} G(x)\Big|_{\Delta,J} =& \lambda ic_{\Delta,J}2^{J-1}e^{\frac{-i\pi(J-1)}{2}}(-x^2)^{\frac{-2\Delta_L-J+1}{2}} \\ &\times \int_{-\infty}^{\infty} d\nu \frac{\Gamma(\frac{2\Delta_L+J-\frac{d}{2}+i\nu}{2})\Gamma(\frac{2\Delta_L+J-\frac{d}{2}-i\nu}{2})}{\nu^2+(\Delta-\frac{d}{2})^2}\Omega_{i\nu}(e\cdot\bar{e}). \end{aligned} \tag{6.34}$$

The integrand in (6.34) has simple poles at $\pm i\nu = \Delta - \frac{d}{2}$ coming from the denominator as well as poles due to the Γ-functions. The latter corresponds to the exchange of the double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l}$; we will not consider these since by definition they do not contribute to the stress tensor sector. One can perform the integral in (6.34) by deforming the contour in the lower half-plane where, in particular, one picks up the pole at $i\nu = \Delta - \frac{d}{2}$. This gives the following contribution to the correlator:

$$G(x)\Big|_{\Delta,J} = (-x^2)^{-\Delta_L}\lambda\, p[\Delta,J]\frac{ic_{\Delta,J}\Pi_{\Delta-1,d-1}(e\cdot\bar{e})}{(e^{\frac{i\pi}{2}}\sqrt{-x^2})^{J-1}} + \dots \tag{6.35}$$

with the ellipses denoting double-trace operators which will not contribute to the stress tensor sector $\mathcal{G}(z, \bar{z})$ and we have defined

$$p[\Delta, J] = 2^{J-1}(\Delta_L)_{\frac{\Delta+J-d}{2}}(\Delta_L - \frac{d-2}{2})_{\frac{-\Delta+J+d-2}{2}}. \tag{6.36}$$

In particular, we see that by Fourier transforming a contribution in momentum space of the form (6.30), i.e. the disconnected correlator times a Regge conformal block, one finds from (6.35) the following contribution to the stress tensor sector of the correlator:

$$\mathcal{G}(\sigma, \rho)\Big|_{\Delta,J} = \lambda\, p[\Delta, J]\, g^R_{\Delta,J}(\sqrt{-x^2}, e\cdot\bar{e}), \tag{6.37}$$

valid to subleading order in $\sigma \to 0$ and we have defined the position space Regge conformal block

$$g^R_{\Delta,J}(\sqrt{-x^2}, e\cdot\bar{e}) = ic_{\Delta,J}\frac{\Pi_{\Delta-1,d-1}(e\cdot\bar{e})}{(e^{\frac{i\pi}{2}}\sqrt{-x^2})^{J-1}}. \tag{6.38}$$

Note that in (6.37), we have used the relation (6.21) between the correlator on the cylinder and $\mathcal{G}(z, \bar{z})$ which is valid to subleading order in the Regge limit.

In particular, we will be interested in $d = 4$ where (6.38) can be written in terms of $(z, \bar{z})$ as

$$g^R_{\Delta,J}(z, \bar{z}) = i\bar{c}_{\Delta,J}e^{\frac{i\pi(1-J)}{2}}\Big(\frac{\log z}{\log \bar{z}}\Big)^{\frac{-(\Delta-1)}{2}}\frac{(-\log z \log \bar{z})^{\frac{1-J}{2}}}{1-\frac{\log \bar{z}}{\log z}}, \tag{6.39}$$

which to subleading order in the Regge limit $\sigma \to 0$ reduces to

$$g^R_{\Delta,J}(\sigma, \rho) = i\bar{c}_{\Delta,J}\frac{e^{-(\Delta-1)\rho}}{\sigma^{J-1}(1-e^{-2\rho})}\Big[1 - \frac{\sigma}{4}\Big((\Delta+J-2)e^{\rho} + (2+J-\Delta)e^{-\rho}\Big) + \mathcal{O}(\sigma^2)\Big]. \tag{6.40}$$

Comparing the position space Regge conformal block in (6.40) with the conformal block in the Regge limit (2.52), it is seen that in four dimensions, the former can to subleading order in $\sigma \to 0$ be identified with a conformal block $g^{\circlearrowleft}_{\Delta,J}(\sigma, \rho)$. To leading order this holds in any dimension, i.e., using the relation between (σ, ρ) and $(z, \bar{z})$ in (6.27) and the known form of the conformal blocks (2.54), the contribution to the stress tensor sector $\mathcal{G}(z,\bar{z})$ in (6.37) can be identified with[52]:

$$\mathcal{G}(\sigma,\rho)\Big|_{\Delta,J} = \lambda 2^{J-1}(\Delta_L)_{\frac{\Delta+J-d}{2}}(\Delta_L - \frac{d-2}{2})_{\frac{-\Delta+J+d-2}{2}} g^{\circlearrowleft}_{\Delta,J}(\sigma,\rho) + \ldots . \quad (6.41)$$

In what follows, we describe how to decompose products of Regge conformal blocks into sums of Regge conformal blocks. As we will see in Section 6.4, this is relevant when one considers the exponentiation of the phase shift which, when expanded into a series, will result in products of Regge conformal blocks. After having decomposed these products into sums of Regge conformal blocks, it is straightforward to use (6.37) to find the contribution to the stress tensor sector of the correlator. We further note that while the phase-shift is only known to leading order in $S \gg 1$, the leading and next-to-leading singularities in the Regge limit $\sigma \to 0$ are not affected by subleading corrections to the phase shift.

6.3.1. Two dimensions

Consider a Regge conformal block in two dimensions:

$$g^{\mathrm{R}}_{\Delta,J} = i\bar{c}_{\Delta,J} S^{J-1} e^{-(\Delta-1)L}, \quad (6.42)$$

where $\bar{c}_{\Delta,J}$ a constant given in (2.53). A product of Regge conformal blocks with (Δ_i, J_i) weighted with constants λ_i is trivially given by:

$$\prod_{i=1}^{p} \lambda_i g^{\mathrm{R}}_{\Delta_i,J_i} = i^{p-1}\lambda g^{\mathrm{R}}_{\Delta,J}(S,L), \quad (6.43)$$

with

$$\begin{aligned} \Delta &= \sum_{i=1}^{p} \Delta_i - (p-1) \\ J &= \sum_{i=1}^{p} J_i - (p-1) \\ \lambda &= \frac{1}{c_{\Delta,J}} \prod_{i=1}^{p} \lambda_i \bar{c}_{\Delta_i,J_i}. \end{aligned} \quad (6.44)$$

[52] We have not checked if this holds also at subleading order in arbitrary dimensions.

From (6.43), it is seen that the product of Regge conformal blocks in $d = 2$ is also a Regge conformal block with (Δ, J, λ) given by (6.44). Assume a contribution in momentum space of the form

$$\frac{\mathcal{B}(p)}{\mathcal{B}_0(p)}\bigg|_{\{\Delta_i, J_i\}} := \prod_{i=1}^{p} \lambda_i g^{\mathrm{R}}_{\Delta_i, J_i}(S, L). \tag{6.45}$$

Using (6.43)-(6.44), it follows from the Fourier transform in (6.37) that the product of Regge conformal blocks in (6.45) gives the following contribution to the stress tensor sector to subleading order in σ:

$$\mathcal{G}(\sqrt{-x^2}, e \cdot \bar{e})\Big|_{\{\Delta_i, J_i\}} = i^{p-1} \lambda 2^{J-1} (\Delta_L)_{\frac{\Delta+J-2}{2}} (\Delta_L)_{\frac{-\Delta+J}{2}} g^{R}_{\Delta, J}(\sqrt{-x^2}, e \cdot \bar{e}). \tag{6.46}$$

Because a product of Regge conformal blocks in two dimensions is again a Regge conformal block, we see that it is trivial to perform the Fourier transform.

6.3.2. Four dimensions

In this section, products of Regge conformal blocks in four dimensions are considered. In particular, the decomposition of such products into a sum of Regge conformal blocks is described. Using this decomposition, one can do the Fourier transform using (6.37).

A Regge conformal block in four dimensions is given by:

$$g^{\mathrm{R}}_{\Delta, J} = i \bar{c}_{\Delta, J} \frac{S^{J-1} e^{-(\Delta-1)L}}{1 - e^{-2L}}. \tag{6.47}$$

Consider a product of p Regge conformal blocks with scaling dimension and spin (Δ_i, J_i), $i = 1, 2, \ldots, p$, together with some weights λ_i:

$$\prod_{i=1}^{p} \lambda_i g^{\mathrm{R}}_{\Delta_i, J_i} = i^{p-1} \frac{S^{J-1} e^{-(\Delta_0-1)L}}{(1 - e^{-2L})^p} \prod_{i=1}^{p} \lambda_i \bar{c}_{\Delta_i, J_i}, \tag{6.48}$$

where

$$\begin{aligned} \Delta_0 &= \sum_{i=1}^{p} \Delta_i - (p-1), \\ J &= \sum_{i=1}^{p} J_i - (p-1). \end{aligned} \tag{6.49}$$

Expanding the factor $(1-e^{-2L})^{-p+1}$ in (6.48) into a sum, the product of Regge conformal blocks in (6.48) can be written as

$$\prod_{i=1}^{p} \lambda_i g^{\mathrm{R}}_{\Delta_i,J_i} = \frac{i^{p-1} S^{J-1} e^{-(\Delta_0-1)L}}{(1-e^{-2L})} \prod_{i=1}^{p} \lambda_i \bar{c}_{\Delta_i,J_i} \sum_{n=0}^{\infty} \binom{n+p-2}{n} e^{-2nL}. \quad (6.50)$$

Compared to the two-dimensional case, it is seen from (6.48)-(6.50) that products of Regge conformal blocks in four dimensions decompose into an infinite sum of Regge conformal blocks with dimensions $\Delta_n = \Delta_0 + 2n$ and spin J. Explicitly, the product of Regge conformal blocks have the following decomposition:

$$\prod_{i=1}^{p} \lambda_i g^{\mathrm{R}}_{\Delta_i,J_i}(S,L) = i^{p-1} \sum_{n=0}^{\infty} \lambda_n g^{\mathrm{R}}_{\Delta_n,J}(S,L), \quad (6.51)$$

with

$$\begin{aligned} \Delta_n &= \sum_{i=1}^{p} \Delta_i + 2n - (p-1), \\ J &= \sum_{i=1}^{p} J_i - (p-1), \\ \lambda_n &= \frac{1}{c_{\Delta_n,J}} \binom{n+p-2}{n} \prod_{i=1}^{p} \lambda_i \bar{c}_{\Delta_i,J_i}. \end{aligned} \quad (6.52)$$

Using the decomposition (6.51), it is straightforward to write down the Fourier transform of products of Regge conformal blocks using (6.37). Explicitly, a term in momentum space of the form (6.51)

$$\left.\frac{\mathcal{B}(p)}{\mathcal{B}_0(p)}\right|_{\{\Delta_i,J_i\}} := \prod_{i=1}^{p} \lambda_i g^{\mathrm{R}}_{\Delta_i,J_i}(S,L), \quad (6.53)$$

with Δ_n, J, λ_n given by (6.52), gives the following contribution to the stress tensor sector:

$$\left.\mathcal{G}(\sqrt{-x^2}, e\cdot\bar{e})\right|_{\{\Delta_i,J_i\}} = i^{p-1} \sum_{n=0}^{\infty} p[\Delta_n, J] \lambda_n g^{R}_{\Delta_n,J}(\sqrt{-x^2}, e\cdot\bar{e}), \quad (6.54)$$

to subleading order in σ. Here $p[\Delta, J]$ is the product of Pochhammer symbols defined in (6.36).

In this section, the heavy-heavy-light-light correlator is studied assuming that the correlator in momentum space is given by

$$\mathcal{B}(p) = \mathcal{B}_0(p) e^{i\delta(S,L;\mu)}, \tag{6.55}$$

where $\delta(S,L;\mu)$ is the bulk phase shift. The phase shift was calculated to all orders in a perturbative expansion in μ in [8]:

$$\delta(S,L;\mu) = \sum_{k=0}^{\infty} \mu^k \delta^{(k)}(S,L). \tag{6.56}$$

It was further shown in [1] that $\delta^{(k)} := \sum_{n=0}^{\infty} \delta_n^{(k)}$ can be decomposed in terms of Regge conformal blocks as

$$\begin{aligned} i\,\delta_n^{(k)}(S,L) &= \; f(k)\;\lambda_k(n)\; g^R_{\tau_0(k)+2n+2,2}(S,L) \\ \lambda_k(n) &= a(n)\,\frac{2^{-4n}\left[\left(\frac{\tau_0(k)+4}{2}\right)_n\right]^2}{\left(\frac{\tau_0(k)+3}{2}\right)_n\left(\frac{\tau_0(k)+5}{2}\right)_n}, \quad \tau_0(k) = k(d-2) \end{aligned} \tag{6.57}$$

with

$$\begin{aligned} f(k) &= \frac{\sqrt{\pi}}{64}\,\frac{1}{2^{k(d-2)}\;k!}\,\frac{\Gamma\left(\frac{kd+1}{2}\right)\Gamma\left(\frac{k(d-2)+4}{2}\right)}{\Gamma\left(\frac{k(d-2)+5}{2}\right)\Gamma\left(\frac{k(d-2)+3}{2}\right)}, \\ a(n) &= \frac{2^{2n}}{n!}\,\frac{\tau_0(k)+2}{\tau_0(k)+2+2n}\,\frac{\left(\frac{\tau_0(k)-d+2}{2}\right)_n\left(\frac{\tau_0(k)+1}{2}\right)_n}{\left(\tau_0(k)+n+2-\frac{d}{2}\right)_n}. \end{aligned} \tag{6.58}$$

Note that $\lambda_1(n) = 0$ for $n = 1, 2, \ldots$ implying that the first-order phase shift reduces to a single term in (6.57). Expanding the exponential in (6.55) results in a sum of products of Regge conformal blocks. Using the decomposition of such products in four dimensions described in Section 6.3, we read off the contribution to the stress tensor sector $\mathcal{G}$ of the correlator from the phase shift.

At k-th order, the stress tensor sector of the correlator behaves as

$$\mathcal{G}^{(k)}(\sigma,\rho) = \frac{F_{k,\mathrm{L}}(\rho)}{\sigma^k} + \frac{F_{k,\mathrm{NL}}(\rho)}{\sigma^{k-1}} + \mathcal{O}(\sigma^{-k+2}) \qquad \sigma \to 0, \quad \rho\text{−fixed}, \tag{6.59}$$

for some functions $F_{k,\mathrm{L}}(\rho)$ and $F_{k,\mathrm{NL}}(\rho)$ in the Regge limit. The leading and next-to-leading Regge singularity of the stress tensor sector of the correlator $\mathcal{G}^{(k)}$ at $\mathcal{O}(\mu^k)$ were defined in (6.3) by

$$\begin{aligned} &\text{Leading Regge singularity :} \qquad \frac{F_{k,\mathrm{L}}(\rho)}{\sigma^k}, \\ &\text{Next}-\text{to}-\text{leading Regge singularity :} \qquad \frac{F_{k,\mathrm{NL}}(\rho)}{\sigma^{k-1}}. \end{aligned} \tag{6.60}$$

By expanding (6.55) and Fourier transforming terms propoportional to S^k and S^{k-1} at $\mathcal{O}(\mu^k)$, the leading- and next-to-leading singularities are found perturbatively in μ[53]. In particular, the leading singularities in the Regge limit comes from the exponentiation of the first-order phase shift. We find perfect agreement with the calculation of a light particle propagating in a shockwave background in [104]. It is then shown, from the exchange of stress tensor, that there is no correction to $\delta^{(1)}$ of $\mathcal{O}(S^0)$ for large $S \gg 1$. Using this knowledge, we calculate the next-to-leading Regge singularities to all orders in μ. Both the leading and next-to-leading order Regge singularities agree in the Regge-Lightcone limit with known results obtained using lightcone bootstrap [80,2].

6.4.1. Leading Regge singularities

In this section, the leading terms in the correlator as $\sigma \to 0$, which were defined in (6.60) as the leading Regge singularities, at each order in μ are studied in four dimensions. Expanding (6.55), these come from the exponentiation of the first-order phase shift $\delta^{(1)}$:

$$\begin{aligned} \mathcal{B}(p) &= \mathcal{B}_0(p) e^{i\mu\delta^{(1)}} + \dots \\ &= \mathcal{B}_0(p) \sum_{k=0}^{\infty} \mu^k \Big[\frac{i^k}{k!} (\delta^{(1)})^k + \mathcal{O}(S^{k-1}) \Big]. \end{aligned} \tag{6.61}$$

A term proportional to S^k will, after Fourier transform to position space, scale as σ^{-k} when $\sigma \to 0$. This will be the leading Regge singularity at $\mathcal{O}(\mu^k)$.

[53] For fixed value of n the sum over k can be performed with a finite radius of convergence.

The first-order phase shift is given by (6.57)

$$\begin{aligned} i\delta^{(1)} &= \frac{1}{240} g_{4,2}(S,L) \\ &= i\frac{3\pi}{2}\frac{Se^{-3L}}{1-e^{-2L}}. \end{aligned} \tag{6.62}$$

The term at $\mathcal{O}(\mu^k)$ in (6.61) is a product of k Regge conformal blocks with dimension $\Delta = 4$ and spin $J = 2$. Using the decomposition of products of Regge conformal blocks (6.51)-(6.52), the expansion of the momentum space correlator in (6.61) can be written in terms of Regge conformal blocks with

$$\begin{aligned} \Delta_{k,n} &= 3k + 2n + 1, \\ J_k &= k + 1, \end{aligned} \tag{6.63}$$

where $n = 0, 1, \ldots$. Using (6.37) to do the Fourier transform of each Regge conformal block, this gives the following contribution to the stress tensor sector of the correlator:

$$\begin{aligned} \mathcal{G}^{(k)}(\sigma,\rho)\Big|_{(\delta^{(1)})^k} &= \sum_{n=0}^{\infty} \frac{1}{k!}\left(\frac{i\bar{c}_{4,2}}{240}\right)^k \binom{n+k-2}{n} (\Delta_L)_{\frac{\Delta_{k,n}+J_k-4}{2}} \\ &\times (\Delta_L - 1)_{\frac{-\Delta_{k,n}+J_k+2}{2}} \frac{(-i)}{c_{\Delta_{k,n},J_k}} g^R_{\Delta_{k,n},J_k}(\sqrt{-x^2}, e\cdot\bar{e}), \end{aligned} \tag{6.64}$$

valid to subleading order in σ with $k = 1, 2, \ldots$.

The leading Regge singularities can be written in terms of (σ, ρ) using $\sqrt{-x^2} \approx -i\sigma$ and $-e\cdot\bar{e} \approx \cosh\rho$. From (6.64) we find:

$$\begin{aligned} \mathcal{G}^{(k)}(\sigma,\rho)\Big|_{(\delta^{(1)})^k} &= \sum_{n=0}^{\infty} \frac{(3i\pi)^k}{k!} \binom{n+k-2}{n} (\Delta_L)_{\frac{\Delta_{k,n}+J_k-4}{2}} \\ &\times (\Delta_L - 1)_{\frac{-\Delta_{k,n}+J_k+2}{2}} \frac{e^{-(\Delta_{k,n}-1)\rho}}{\sigma^{J_k-1}(1-e^{-2\rho})} + \ldots, \end{aligned} \tag{6.65}$$

where the ellipses denote terms subleading in $\sigma \to 0$. Explicitly, inserting the dimensions and spins $(\Delta_{k,n}, J_k)$ given in (6.63), we find

$$\begin{aligned} \mathcal{G}^{(k)}(\sigma,\rho)\Big|_{(\delta^{(1)})^k} &= \sum_{n=0}^{\infty} \frac{(3i\pi)^k}{k!} \binom{n+k-2}{n} (\Delta_L)_{2k+n-1}(\Delta_L - 1)_{1-k-n} \\ &\times \frac{e^{-(3k+2n)\rho}}{\sigma^k(1-e^{-2\rho})} + \ldots. \end{aligned} \tag{6.66}$$

The sum over n can further be written as a hypergeometric function:

$$\begin{aligned}\mathcal{G}^{(k)}(\sigma,\rho)\Big|_{(\delta^{(1)})^k} &= \frac{(3i\pi)^k}{k!\sigma^k(1-e^{-2\rho})}(\Delta_L)_{2k-1}(\Delta_L-1)_{1-k} \\ &\times e^{-3k\rho}{}_2F_1(k-1,\Delta_L+2k-1;-\Delta_L+k+1;-e^{-2\rho})+\dots .\end{aligned} \tag{6.67}$$

These are the leading Regge singularities, i.e., terms that behave as σ^{-k} at $\mathcal{O}(\mu^k)$, to all orders in μ. The result (6.67) agrees with the calculation in a shockwave background in [104], for details, see Appendix C.1. In particular, consider the terms in the sum in (6.67) with $k=2,3$[54]:

$$\begin{aligned}\mathcal{G}^{(2)}(\sigma,\rho)\Big|_{\sigma^{-2}} &\underset{\rho\to\infty}{\approx} -\frac{9\pi^2\Delta_L(\Delta_L+1)(\Delta_L+2)}{2(\Delta_L-2)}\frac{e^{-6\rho}}{\sigma^2}, \\ \mathcal{G}^{(3)}(\sigma,\rho)\Big|_{\sigma^{-3}} &\underset{\rho\to\infty}{\approx} -\frac{9i\pi^3\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{2(\Delta_L-2)(\Delta_L-3)}\frac{e^{-9\rho}}{\sigma^3},\end{aligned} \tag{6.68}$$

where we have further taken the limit $\rho\to\infty$. The leading Regge singularities in (6.68) agree with those in (6.13)-(6.14); the latter were found using lightcone bootstrap [80,2] and are due to minimal-twist double-stress and triple-stress tensors.

We note that the first-order phase shift is to leading order in σ fixed by the exchange of stress tensor in the direct channel in the CFT [8]. It is therefore universally fixed by Ward identities and does not depend on higher derivative corrections to the gravity action.

It is seen that the leading Regge singularities in (6.66), which can be identified with the leading behaviour of a conformal block in the Regge limit with dimension $\Delta_{k,n}=3k+2n+1$ and spin $J_k=k+1$, have poles and zeroes specified by the Pochhammer symbols to be given by:

$$\begin{aligned}\text{Zeroes}: &\quad \Delta_L=-(2k+n-2),-(2k+n-3),\dots,0 \\ \text{Poles}: &\quad \Delta_L=2,3,\dots,k+n.\end{aligned} \tag{6.69}$$

The position of the poles and zeroes are seen to be related to the dimension and spin of the blocks that are present in the decomposition of $(\delta^{(1)})^k$. Possible

[54] Note that we assume that Δ_L is not an integer. For integer Δ_L, there is a mixing problem between multi-stress tensors and double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l}$ for suitable choice of n,l. This is discussed e.g. in [104].

implications of the position of poles and zeroes were discussed in [104]. In particular, it is expected that the OPE coefficients of multi-stress tensors with minimal-twist have the same poles as predicted by (6.69) with $n = 0$. This agrees with the results in [102,80,2]. Moreover, we further expect from (6.69) the OPE coefficients for non-minimal-twist multi-stress tensors to have poles at $\Delta = 2, 3, \ldots, k+n$, with n being related to the twist by $\tau = k(d-2)+2n$. This is expected due to the potential mixing when Δ_L is an integer and there no longer a clean separation between the multi-stress tensors and the double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l}$.

6.4.2. The first-order phase shift and the stress tensor exchange

The phase shift in (6.56) calculated in the bulk is linear in the energy $S \gg 1$. In principle, it could receive corrections in an $\frac{1}{S}$ expansion that will be important when expanding (6.55). On the other hand, from the CFT point of view, the stress tensor is the only operator that appears at $\mathcal{O}(\mu)$ in the stress tensor sector. Using this, we show that there is no correction to $\delta^{(1)}$ in four dimensions of order $\mathcal{O}(S^0)$.

The stress tensor exchange in four dimensions is found using the known OPE coefficients and the conformal block given. Explicitly, one finds the following contribution as $\sigma \to 0$:

$$P^{(HH,LL)}_{T_{\mu\nu}} g^{\circlearrowleft}_{4,2}(\sigma,\rho) = \mu\frac{3\pi i\Delta_L e^{-3\rho}}{(1-e^{-2\rho})}\frac{1}{\sigma} - \mu\frac{3\pi i\Delta_L e^{-2\rho}}{(1-e^{-2\rho})} + \mathcal{O}(\sigma). \tag{6.70}$$

On the other hand, expanding the momentum space correlator in (6.55) one finds at $\mathcal{O}(\mu)$:

$$\mathcal{B}(p) = \mathcal{B}_0(p) i\mu\delta^{(1)} + \mathcal{O}(\mu^2), \tag{6.71}$$

with the first-order phase shift in $d = 4$ given in (6.62). Fourier transforming (6.71) using (6.37) gives the following contribution to the correlator in position space:

$$\mathcal{G}(\sigma,\rho)\Big|_{\delta^{(1)}} = \mu\frac{3\pi i\Delta_L e^{-3\rho}}{(1-e^{-2\rho})}\frac{1}{\sigma} - \mu\frac{3\pi i\Delta_L e^{-2\rho}}{(1-e^{-2\rho})} + \mathcal{O}(\sigma). \tag{6.72}$$

where we used $p[4,2] = 2\Delta_L$. Comparing the contribution from the stress tensor in the Regge limit (6.70), with the contribution from $\delta^{(1)}$ in (6.72), we find that both the leading and next-to-leading terms as $\sigma \to 0$ agree[55]. This shows that there is no $\mathcal{O}(S^0)$ correction to the first-order phase shift.

[55] Since the leading terms were known to agree, this follows immediately from the observation below (6.40).

6.4.3. Next-to-leading Regge singularities

In this section, the next-to-leading Regge singularities are considered, i.e. terms proportional to σ^{1-k} at $\mathcal{O}(\mu^k)$, to all orders in μ. These will be due to terms in (6.55) of the form $(\delta^{(1)})^k$ that were calculated in Section 6.4, and terms of the form $(\delta^{(1)})^{k-2}\delta^{(2)}$ which are of $\mathcal{O}(S^{k-1})$ for $S \gg 1$. The contribution to the next-to-leading Regge singularities from terms of the form $(\delta^{(1)})^k$ are therefore given by (6.64).

Consider terms in (6.55) of the form $(\delta^{(1)})^{k-2}\delta^{(2)}$:

$$\frac{\mathcal{B}(p)}{\mathcal{B}_0(p)}\Big|_{\delta^{(2)}(\delta^{(1)})^{k-2}} = \frac{\mu^k i^{k-1}}{(k-2)!}(\delta^{(1)})^{k-2}\delta^{(2)}, \tag{6.73}$$

with $k = 2, 3, \ldots$. Inserting the decomposition of $\delta^{(2)}$ from (6.57)

$$i\delta^{(2)} = f(2)\sum_{n=0}^{\infty}\lambda_2(n)g^R_{4+2n,2}(S, L) \tag{6.74}$$

and the first-order phase shift (6.62), we rewrite (6.73) as

$$\frac{\mathcal{B}(p)}{\mathcal{B}_0(p)}\Big|_{\delta^{(2)}(\delta^{(1)})^{k-2}} = \frac{\mu^k i^{k-1}}{(k-2)!}\left(\frac{3\pi}{2}\right)^{k-2}\sum_{n=0}^{\infty} f(2)\lambda_2(n)\bar{c}_{6+2n,2} \times \frac{S^{k-1}e^{-(3k+2n-1)L}}{(1-e^{-2L})^{k-1}}. \tag{6.75}$$

Expanding $(1-e^{-2L})^{-k+2}$, we find

$$\frac{\mathcal{B}(p)}{\mathcal{B}_0(p)}\Big|_{\delta^{(2)}(\delta^{(1)})^{k-2}} = f(2)\frac{\mu^k i^{k-1}}{(k-2)!}\left(\frac{3\pi}{2}\right)^{k-2} \times \sum_{n,m=0}^{\infty}\binom{m+k-3}{m}\lambda_2(n)\bar{c}_{6+2n,2} \times \frac{S^{k-1}e^{-(\Delta_{n,m}-1)L}}{1-e^{-2L}}, \tag{6.76}$$

with

$$\Delta_{n,m} = 3k + 2(n+m). \tag{6.77}$$

Comparing the product of Regge conformal blocks in (6.73) with (6.76), it is seen that the latter is a decomposition into Regge conformal blocks with dimensions $3k + 2(n+m)$ and spin k. This can conveniently be organized into blocks with different twists

$$\frac{\mathcal{B}(p)}{\mathcal{B}_0(p)}\Big|_{\delta^{(2)}(\delta^{(1)})^{k-2}} = f(2)\frac{\mu^k i^{k-1}}{(k-2)!}\left(\frac{3\pi}{2}\right)^{k-2}\sum_{p=0}^{\infty}\sum_{n=0}^{p}\binom{k+p-n-3}{p-n} \times\lambda_2(n)\bar{c}_{6+2n,2}\frac{S^{k-1}e^{-(3k+2p-1)L}}{1-e^{-2L}}. \tag{6.78}$$

To get the next-to-leading order Regge singularities from (6.78), it is enough to use the leading order relation $\sqrt{-x^2} = -i\sigma$ and $-e \cdot \bar{e} = \cosh\rho$. This is so since terms in (6.78) are of $\mathcal{O}(S^{k-1})$ and therefore start to contribute at σ^{-k+1} in position space. Using (6.37) to perform the Fourier transform of each term in the sum, one finds that (6.78) gives the following contribution to the next-to-leading order Regge singularities in the stress tensor sector:

$$\begin{aligned}\mathcal{G}^{(k)}(\sigma,\rho)|_{\delta^{(2)}(\delta^{(1)})^{k-2}} &= 2f(2)\sum_{p=0}^{\infty}\sum_{n=0}^{p}\frac{i^{k-1}(3\pi)^{k-2}}{(k-2)!}(\Delta_L)_{2k+p-2} \\ &(\Delta_L-1)_{1-k-p}\binom{k+p-n-3}{p-n}\lambda_2(n)\bar{c}_{6+2n,2}\frac{e^{-(3k+2p-1)\rho}}{\sigma^{k-1}(1-e^{-2\rho})} \\ &+\ldots,\end{aligned} \tag{6.79}$$

where the ellipses denote subleading corrections in σ. To get the full result for the next-to-leading Regge singularities we need to add the contribution from (6.64). This is found using the correction to the position Regge conformal block (6.40) and the leading order expression (6.66)

$$\begin{aligned}\mathcal{G}(\sigma,\rho)^{(k)}\Big|_{(\delta^{(1)})^k,\sigma^{-k+1}} =& \frac{1}{2}\sum_{n=0}^{\infty}\frac{(3i\pi)^k}{k!}\binom{n+k-2}{n}(\Delta_L)_{2k+n-1}(\Delta_L-1)_{1-k-n} \\ &\times\Big[(k+n-1)e^{-\rho}-(2k+n)e^{\rho}\Big]\frac{e^{-(3k+2n)\rho}}{\sigma^{k-1}(1-e^{-2\rho})}.\end{aligned} \tag{6.80}$$

The next-to-leading Regge singularities to all orders in μ is therefore given by the sum of (6.79) and (6.80).

Consider the $\rho\to\infty$ limit in which only the $p=n=0$ term in (6.79) contributes. In this limit, (6.79) reduces to

$$\mathcal{G}(\sigma,\rho)^{(k)}|_{\delta^{(2)}(\delta^{(1)})^{k-2}} \underset{\rho\to\infty}{\approx} \frac{35}{6}\frac{(3i\pi)^{k-1}}{(k-2)!}(\Delta_L)_{2k-2}(\Delta_L-1)_{1-k}\frac{e^{-(3k-1)\rho}}{\sigma^{k-1}}. \tag{6.81}$$

This is the contribution of $\delta^{(2)}(\delta^{(1)})^{k-2}$ to the next-to-leading Regge singularity at k-th order in the Regge-Lightcone limit.

Including the contribution to the next-to-leading Regge singularity from (6.80) due to $(\delta^{(1)})^k$ together with (6.81), we find for $\rho\to\infty$ at $\mathcal{O}(\mu^2)$:

$$\mathcal{G}^{(2)}(\sigma,\rho)\Big|_{\sigma^{-1}} = \Big[\frac{35i\pi\Delta_L(\Delta_L+1)}{2(\Delta_L-2)}+\frac{18\pi^2\Delta_L(\Delta_L+1)(\Delta_L+2)}{2(\Delta_L-2)}\Big]\frac{e^{-5\rho}}{\sigma}. \tag{6.82}$$

Likewise, consider the next-to-leading singularity at $\mathcal{O}(\mu^3)$ which using (6.80) and (6.81) gives

$$\begin{aligned}\mathcal{G}^{(3)}(\sigma,\rho)\Big|_{\sigma^{-2}} = \Big[&- \frac{105\pi^2\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)}{2(\Delta_L-2)(\Delta_L-3)} \\ &+ \frac{27i\pi^3\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)(\Delta_L+4)}{2(\Delta_L-2)(\Delta_L-3)}\Big]\frac{e^{-8\rho}}{\sigma^2}.\end{aligned} \tag{6.83}$$

Comparing the next-to-leading Regge singularities when $\rho \to \infty$, (6.82) and (6.83), with (6.13)-(6.14), respectively, we find agreement between the result obtained here using the phase shift and known results obtained using lightcone bootstrap.

Similarily to the leading Regge singularities, the next-to-leading singularities due to $\delta^{(2)}(\delta^{(1)})^{k-2}$ have a simple dependence on the scaling dimension Δ_L – the poles and zeroes are fixed by the dimension and spin of the Regge conformal blocks appearing in the decomposition (6.78). From (6.79), the poles and zeroes are found to be given by:

$$\begin{aligned} \text{Zeroes}: &\qquad \Delta_L = -(2k+p-3), -(2k+p-3), \ldots, 0 \\ \text{Poles}: &\qquad \Delta_L = 2, 3, \ldots, k+p. \end{aligned} \tag{6.84}$$

Note that the poles are the same as those for the leading Regge singularities in (6.69).

6.5. Discussion

Using the first- and second-order phase shift, we derived the leading and next-to-leading Regge singularities of the stress tensor sector to all orders. The leading Regge singularity at each order was shown to be determined by the first-order phase shift. This is universally fixed by the stress tensor exchange and our results agree with the expression obtained in [104]. The next-to-leading Regge singularity at each order further depends on the second-order phase shift. In general, the second-order phase shift is expected to be non-universal in the sense that it depends on higher derivative corrections to the gravitational action in the bulk. However, it is expected to be universal in the large impact parameter, see [6] for the phase shift calculated in Gauss-Bonnet gravity where this indeed is the case.

It has been argued in [102,104,105,2] that the minimal-twist multi-stress tensor sector of CFTs with large central charge is universal, i.e. independent of higher-derivative corrections to the gravitational action. This was argued from the holographic point of view in [102,104]. There the two-point function of a minimally coupled scalar propagating in an AdS black hole background was studied in higher derivative gravity[56]. In [2] it was shown that the ansatz (6.11) solves the crossing relations and that the minimal-twist subsector of the stress tensor sector is, therefore, determined in terms of the exchange of the stress tensor. Since the stress tensor exchange is fixed by Ward identities, this implies that the minimal-twist subsector is universal. In terms of the phase shift, this would imply that when decomposing the phase shift in terms of Regge conformal blocks, the contribution proportional to the block with the lowest twist at each order is universal. It would be interesting to study explicitly the effect of higher derivative terms on the phase shift and verify this. Universality in the minimal-twist sector would imply that the Regge-Lightcone limit of our results for the next-to-leading Regge singularities is universal.

While we have focused on $d = 4$, it would be interesting to understand how to extend this to general dimensions. In particular, in $d = 6$, the hyperbolic space propagators take a similar form as in $d = 4$ and it would be interesting to find a similar decomposition of products of Regge conformal blocks. Moreover, for large impact parameter L, the hypergeometric function in (2.55) can be set to 1 in any dimension. In this limit the Regge conformal blocks in any dimension resemble the two-dimensional blocks.

Consider the exponentiation of the phase shift in $d = 4$ at $\mathcal{O}(\mu^3)$:

$$\mathcal{B}(p)\Big|_{\mu^3} = \mathcal{B}_0(p)\Big[-i\frac{(\delta^{(1)})^3}{3!} - \delta^{(1)}\delta^{(2)} + i\delta^{(3)}\Big]. \tag{6.85}$$

The leading and next-to-leading Regge singularities obtained from (6.85) were already discussed in Section 6.4. In Appendix C.2, it is shown that including

[56] A non-minimally coupled scalar was considered in [153] and was shown to lead to corrections. However, such corrections are suppressed by inverse powers of the higher-spin gap Δ_{gap} as shown in [37]

the first subleading correction for $\sigma \to 0$ to the Fourier transform of the $\delta^{(1)}\delta^{(2)}$ term in (6.85), one finds for $\rho \to \infty$

$$\mathcal{G}^{(3)}(\sigma,\rho)\Big|_{\delta^{(1)}\delta^{(2)},\sigma^{-1}} \underset{\rho\to\infty}{\approx} \frac{525\pi^2\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)}{4(\Delta_L-2)(\Delta_L-3)}\frac{e^{-7\rho}}{\sigma}. \quad (6.86)$$

This agrees with the third line in (6.15) obtained using lightcone bootstrap. More interesting is the last term in (6.85) given by the phase shift at third order. It gives the following contribution to the stress tensor sector of the correlator:

$$\mathcal{G}^{(3)}(\sigma,\rho)\Big|_{\delta^{(3)}} = f(3)\sum_{n=0}^{\infty}\lambda_3(n)p[\tau_0(3)+2n+2,2]\; g^{\circlearrowleft}_{\tau_0(3)+2n+2,2}(\sigma,\rho), \quad (6.87)$$

to leading order in $\sigma \to 0$ and $\lambda_k(n)$ is given by the decomposition of the phase shift in (6.57). In particular, when $\rho \to \infty$, only the $n = 0$ term contributes:

$$\mathcal{G}^{(3)}(\sigma,\rho)\Big|_{\delta^{(3)}} \underset{\rho\to\infty}{\approx} \frac{1155 i\pi\Delta_L(\Delta_L+1)(\Delta_L+2)}{8(\Delta_L-2)(\Delta_L-3)}\frac{e^{-7\rho}}{\sigma}. \quad (6.88)$$

This agrees with the term in the first line in (6.15) due to minimal-twist triple-stress tensors obtained from lightcone bootstrap. The remaining term in (6.15) presumably comes from subsubleading corrections to $(\delta^{(1)})^3$ as well as possible subleading corrections to the second-order phase shift.

Following the discussion[57] above on the term linear in S at $\mathcal{O}(\mu^3)$, it is interesting to study terms linear in S at any order in μ:

$$\mathcal{B}(p)\Big|_{\mu^k,S} = \mathcal{B}_0(p) i\delta^{(k)}. \quad (6.89)$$

The corresponding contribution to the stress tensor sector to leading order in $\sigma \to 0$ can, in any dimension, be identified with the leading Regge behaviour of operators $\mathcal{O}_{\Delta_{k,n},J=2}$ with scaling dimension and spin given by

$$\begin{aligned} \Delta_{k,n} &= k(d-2)+2n+2, \\ J &= 2. \end{aligned} \quad (6.90)$$

We refer to these operators as effective[58] in the sense that they are not necessarily present in the spectrum, but rather are due to the resummation of

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[57] A similar discussion was previously considered in [1] and we elaborate on it here.

[58] Or poles at $J = 2$ in the complex J-plane.

multi-stress tensor with arbitrary spin. The contribution linear in S in (6.89) is easily Fourier transformed using (6.37) and the decomposition of the phase shift in terms of Regge conformal blocks (6.57). Explicitly, it is found that (6.89) gives the following contribution to the stress tensor sector of the correlator to leading order in $\sigma \to 0$ in any dimension d:

$$\mu^k \mathcal{G}^{(k)}(\sigma,\rho)\Big|_{\delta^{(k)}} = \mu^k f(k) \sum_{n=0}^{\infty} p[k(d-2)+2n+2,2]\lambda_k(n)\; g^{\circlearrowleft}_{k(d-2)+2n+2,2}(\sigma,\rho), \tag{6.91}$$

where $\lambda_k(n)$ and $f(k)$ are given in (6.57), $p[\Delta, J]$ is a combination of Pochhammer symbols defined in (6.36) and $g^{\circlearrowleft}_{\Delta,J}(\sigma,\rho)$ is the leading contribution of a conformal block in the Regge limit. Interpreting each term in (6.91) as due to the exchange of an effective operator $\mathcal{O}_{\Delta_{k,n},2}$, the coefficients in (6.91) are products of the corresponding OPE coefficients for such exchanges[59] $P^{HH,LL}_{\mathcal{O}_{\Delta_{k,n}},J=2} = \mu^k f(k) p[k(d-2)+2n+2,2]\lambda_k(n)$:

$$P^{HH,LL}_{\mathcal{O}_{\Delta_{k,n},2}} = \mu^k \frac{\sqrt{\pi}(k(d-2)+2)(\frac{(k-1)(d-2)}{2})_n \left[(\frac{k(d-2)+4}{2})_n\right]^2 (\frac{k(d-2)+1}{2})_n}{2^{5+k(d-2)+2n} k!n!(k(d-2)+2n+2)(\frac{2k(d-2)-d+2n+4}{2})_n}$$
$$\times \frac{\Gamma(\frac{k(d-2)+4}{2})\Gamma(\frac{dk+1}{2})\Gamma(\Delta_L+\frac{k(d-2)+2n-d+4}{2})\Gamma(\Delta_L-\frac{k(d-2)+2n}{2})}{\Gamma(\Delta_L)\Gamma(\Delta_L-\frac{d-2}{2})\Gamma(\frac{k(d-2)+2n+3}{2})\Gamma(\frac{k(d-2)+2n+5}{2})}. \tag{6.92}$$

In [40] it was shown using conformal Regge theory that when the correlator is dominated by an isolated pole in the J-plane, the corresponding exchange is due to a light-ray operator[60]. It would be interesting to understand if there is an interpretation of the operators $\mathcal{O}_{\Delta_{k,n},2}$ mentioned here, which are directly related to the phase shift, in terms of such light-ray operators. See also [41,42]. Note that $\lambda_k(n)$ from (6.57) are valid assuming Einstein gravity in the bulk. While expected to be non-universal for general n, we expect the $\lambda_k(0)$ coefficient in the phase shift to be universal and therefore (6.92) with $n = 0$ to be universal, i.e. independent of higher-derivative gravitational terms in the action.

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[59] Note that we assume that Δ_L is not an integer.

[60] The simplest light-ray operator is the ANEC operator, which is the stress tensor operator integrated over a light-ray. See [40] for a detailed discussion and definition of light-ray operators.

7. Thermal Stress Tensor Correlators, OPE and Holography

In this section we move on from the study of scalar correlators in heavy states to the study of the stress tensor two-point function at finite temperature (in holographic CFTs) based on [4].

7.1. Introduction and summary of results

Hydrodynamics describes low-energy excitations in matter at finite temperature and density [154]. A lot of interest was attracted to the hydrodynamics of conformal field theories at strong coupling and large central charge C_T, which admit a dual gravitational (holographic) description [11-13]. Transport coefficients can be extracted from the two-point functions of the stress tensor (TT-correlators) at finite temperature and holography maps these correlators to two-point functions of metric perturbations in a black hole background [51-54].

Holographic value of the shear viscosity is much closer to the experimentally observed values for quark-gluon plasma than perturbative calculations (see e.g. [155,156] for reviews). The ratio of the shear viscosity to the entropy density was shown to be universal, $\eta/s = \hbar/4\pi k_B$, in all theories with Einstein gravity duals [53,54,157,55] However, the addition of higher derivative terms to the gravity Lagrangian changes this value [158-160]. What does this imply for the hydrodynamics of strongly interacting field theories?

In a way, gravity provides a minimal model for strongly interacting matter, where the only degrees of freedom are the stress tensor and its composites, multi-stress tensors – they are encoded by the fluctuations of the metric in the dual theory. From a CFT point of view, such a minimal model is defined by the OPE coefficients and the spectrum of anomalous dimensions of multi-trace operators. Consider the OPE coefficients which determine the three-point functions of the stress-tensor, which are specified by the three parameters in $d > 3$ dimensions. They change as the bulk couplings in front of the gravitational higher derivative terms are varied.[61] Presumably these OPE coefficients do not completely determine the theory, but is it possible that some sector of the theory is universal?

[61] Note that we expect consistent holographic models with generic graviton three-point couplings to also contain higher spin fields [15].

We can make progress in answering this question by decomposing the TT correlator using the OPE expansion. In a minimal theory the operators that appear are multi-stress tensor operators[62] and one can in principle deduce the conformal data working order-by-order in the temperature $T = \beta^{-1}$ [in d spacetime dimensions k-stress tensors naturally contribute terms $\mathcal{O}(\beta^{-dk})$]. A similar question was recently asked in a simpler setting where a finite temperature state (dual to a black hole) was probed by scalars [102]. A scalar two-point function has a piece which can be computed near the boundary of asymptotically AdS spacetime – this is precisely the term which encodes the contribution of multi-stress tensors. Another piece, left undetermined in the near-boundary expansion, contains the contributions of multi-trace operators which involve the external scalar operator.

To compute it, one needs to solve the equation of motion in the whole spacetime – a nontrivial task in practice. [63]

What happens when a thermal state (or, in the dual language, a black hole) is probed by the stress-tensor operators? In this work we attempt to decompose this correlator by generalizing the analysis of [102] to the case of external operator being the stress tensor. Here we consider the contributions of the identity operator, the stress tensor and the double stress tensors to the correlator. One immediate technical complication that we face is that the external operator with which we probe the system, namely the stress tensor, has integer conformal dimension. In [102] it was observed that some OPE coefficients have poles for integer values of the conformal dimension of the external scalar

[62] In this work we consider Einstein gravity as a holographic model – it is believed to be a consistent truncation. In other words, in the dual CFT language, couplings to other operators and corrections to the OPE coefficients are suppressed by the (large) gap in the spectrum of the conformal dimensions of higher spin operators – see e.g. [37] for a recent discussion.

[63] In [2] an alternative way of computing the stress-tensor sector of the scalar correlator using conformal bootstrap and an ansatz, motivated by [80], was proposed. The procedure of [2] allows one to compute the OPE coefficients with the leading twist multi-stress tensors. The result has many similarities to the Virasoro HHLL vacuum block (see e.g. [3,8,77,78]) but at the moment the full resummed correlator in $d > 2$ is only known in the $\Delta \to \infty$ limit [106]. (see [1,5,6,7,104,65,105,161-169] for related work).

operator. This feature is related to mixing of double stress and double trace operators. The OPE coefficients for both series have poles which cancel, leaving behind logarithmic terms. One can also observe that the coefficients of these terms cannot be fixed by the near-boundary analysis [102,163]. See Appendix D.1 for a discussion on the appearance of logarithmic terms in the case when the scaling dimension is an integer.

In the case of the stress tensor the double-trace operators made out of the external operator $T_{\mu\nu}$ *are also* double stress tensor operators. One may wonder if their OPE coefficients can be determined from the near boundary analysis. The answer turns out to be no. Another important difference from [102] is related to the leading behavior of the OPE coefficients of two stress tensors and a double stress tensor. This OPE coefficient scales like one, as opposed to $\mathcal{O}(C_T^{-1})$ in the scalar case, and gives rise to the disconnected part of the correlator. This implies that the connected part of the TT correlator contains information about conformal data which is subleading in the $1/C_T$ expansion. This leads to some complications, but in the end, we succeed at extracting the leading $1/C_T$ contributions to the anomalous dimensions of the double trace operators. Other conformal data at this order remains undetermined – it should be thought of as an analog of the double trace operator data in the external scalar case.

Let us mention another technical difficulty that we need to confront in the case of external stress tensors. In [102] the symmetry implies that the bulk-to-boundary propagator depends on the time t, the spatial radial coordinate ρ and the AdS radial coordinate r. This is no longer the case in the stress-tensor case, due to the presence of distinct polarizations. We handle this by computing stress-tensor correlators integrated over two parallel (xy)-planes separated in the transverse spatial direction, which we denote by z. There are three independent choices of polarization, distinguished by the transformational properties with respect to rotations of the plane of integration. A suitable modification of the ansatz used in [102] allows us to solve the stress tensor problem. However, integrating over the xy-plane leads to some divergent contributions and to additional logarithmic terms. Fortunately, this does not affect our ability to extract the anomalous dimensions.

The rest of this section is organized as follows. In Section 7.2, we consider metric perturbations on top of a planar AdS-Schwarzschild black hole and compute the stress tensor two-point function in a near-boundary expansion (OPE limit in the dual CFT). In Section 7.3, we perform the OPE expansion of the stress tensor thermal two-point function in $d = 4$ and by comparison to the bulk calculations in the previous section, we read off the anomalous dimensions of double-stress tensor operators with spin $J = 0, 2, 4$. We conclude with a discussion in Section 7.4. In Appendix D.1, we treat the simpler example of scalar perturbations in the bulk as a toy model for the metric perturbations, focusing on the subtleties that arise for external operators with integer dimensions. In addition, we consider scalar correlators integrated over the xy-plane and show how the correct OPE data is recovered in this case. Appendix D.2 lists some of the results that are too lengthy to present in Section 7.2. In Appendix D.3 we introduce conventions and details on the spinning conformal correlators relevant for the decomposition of thermal stress tensor two-point functions.

7.2. Holographic calculation of thermal TT correlator in $d = 4$

Recently some OPE coefficients of scalars and multi-stress tensors were calculated in the context of holographic models [102,163]. This was accomplished by making a comparison between the CFT conformal block decomposition of HHLL correlators on the CFT side and a near-boundary expansion of the bulk-to-boundary propagator in the AdS-Schwarzschild background on the bulk side.

Our goal in this work is to use an analogous approach to extract the CFT data[64] for the stress tensor two-point function in a thermal state dual to the AdS-Schwarzschild black hole, in this section we will focus on the bulk part of this calculation. In practice we will consider the integrated version of the correlator

$$G_{\mu\nu,\rho\sigma}(t,z) := \int_{R^2} dxdy \langle T_{\mu\nu}(x^\alpha) T_{\rho\sigma}(0)\rangle_\beta. \tag{7.1}$$

To compute the TT correlator, it is necessary to consider the linearized Einstein equations in the black hole background. For technical reasons, we will

[64] By the CFT data we mean products of the OPE coefficients and thermal one-point functions and anomalous dimensions of the double-trace stress tensors. This will be explained in greater detail in the next section.

take the large volume limit, where all conformal descendants decouple and an expansion in terms of conformal blocks becomes the OPE expansion. On the bulk side, this corresponds to considering the planar asymptotically AdS black hole. The corresponding system of PDEs is technically difficult to solve because different polarizations mix with each other[65]. To make the problem tractable, we integrate the correlator over two spatial directions in (7.1). The resulting fluctuation equations simplify to three independent PDEs for the three different polarizations. We show explicitly that an ansatz of [102,163], suitably modified to fit our needs, successfully solves these equations.

As a warm-up exercise, we consider the scalar case, discussed in [102,163], but now integrate over the xy-plane. The details of this calculation are described in D.1, but the summary is as follows. For non-integer values Δ_L of the conformal dimensions of the scalar operator all coefficients in the ansatz are fixed, order-by-order, by imposing the scalar field equations of motion in the bulk. Matching to the conformal block expansion then yields the OPE coefficients of scalars and multi-stress tensors, which reproduce the results of [102]. Note that the integrals are only convergent for large Δ_L, but their analytic continuation to small Δ_L yields the correct results.

For integer Δ_L there is mixing between multi-stress and multi-trace operators, which results in logarithmic terms [102]. This mixing is reflected in the appearance of the $\log r$ terms in the bulk ansatz [163]; a closely related fact is that not all coefficients in the ansatz are now determined by the bulk equations of motion. For example, for $\Delta_L = 4$ there is one undetermined parameter at $\mathcal{O}(\mu^2)$; it corresponds to an undetermined factor in a double-trace OPE coefficient.

As explained in Appendix D.1, the addition of spatial integration leads to an additional undetermined coefficient in the ansatz. This coefficient is, roughly speaking, related to the volume of the xy-plane we are integrating over. In practice, we use dimensional regularization, so instead of the volume, a $1/\epsilon$ pole appears in the expression for this undetermined coefficient. The other

[65] Because of this complication, we have to deal with the set of metric fluctuations that depend on all five bulk coordinates, hence one can not use the approach introduced in [102,163].

undetermined coefficient is related to the logarithmic term, just as in the non-integrated case. In summary, we conclude that in the scalar case, the spatial integration does not affect our ability to read off the OPE data.

In this section we perform the bulk calculations for the case where the external operator is the stress-tensor. In other words, we compute the OPE expansion for the thermal TT correlator in holographic CFTs. This section is organized as follows. First we consider metric perturbations around a planar AdS-Schwarzschild black hole. Then we integrate out two out of five space-time directions and, following [170,171], we utilize the resulting $O(2)$ symmetry together with the bulk gauge freedom to reformulate the problem in terms of the three gauge invariant combinations of the gravitational fluctuations in the AdS-Schwarzschild background. The resulting PDEs can then be solved one by one using the ansatz [102,163], naturally adapted to the integrated case. Finally, using the holographic dictionary, we derive the stress tensor two-point function in a thermal state for various polarizations. In Section 7.3 we compare these results with the CFT conformal block decomposition and extract conformal data.

7.2.1. Linearized Einstein equations

We consider the Einstein-Hilbert action with a cosmological constant[66]

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{g} (\mathcal{R} - 2\Lambda), \tag{7.2}$$

where G_5 is the five-dimensional gravitational constant, $\mathcal{R}$ is the Ricci scalar and Λ is the cosmological constant. Decomposing the metric in the background part plus a small perturbation $h_{\mu\nu}$, one obtains the linearized Einstein equations in the form

$$R^{(1)}_{\mu\nu} + d h_{\mu\nu} = 0, \tag{7.3}$$

where $R^{(1)}_{\mu\nu}$ is the linearized Ricci tensor and d is the dimension of the conformal boundary, i.e. $d = 4$ in our case.

We will be interested in the planar AdS-Schwarzschild black hole as the background spacetime,

$$ds^2 = r^2 f(r) dt^2 + r^2 d\vec{x}^2 + \frac{1}{r^2 f(r)} dr^2, \tag{7.4}$$

[66] We will be using the Euclidean signature throughout.

where $\vec{x} = (x, y, z)$ and $f(r) = 1 - \frac{\mu}{r^4}$.

By solving the linearized Einstein equations (7.3) with the appropriate boundary conditions, we obtain the metric perturbation $h_{\mu\nu}$ and, in principle, the holographic dictionary then precisely determines the correlators in the four-dimensional CFT on the boundary. However, due to the complicated form of these equations, this is difficult to do in practice.

To make this problem tractable, we integrate the bulk-to-boundary propagator over the xy-plane. This will simplify the equations of motion to three independent PDEs, which we will be able to solve using the ansatz [102,163]. As a result, the corresponding CFT correlators, which we obtain via holographic dictionary, will be integrated over the xy directions. This will be studied in Section 7.3 from the CFT point of view.

7.2.2. Polarizations and gauge invariants

Our aim is to solve the linearized Einstein equations (7.3) in the background (7.4), with the solution integrated over two spatial directions, which we can choose to be x and y.

Upon integration, the (linearized) gravitational action will exhibit an $O(2)$ rotational symmetry. This property allows us to divide the components $h_{\mu\nu}$ into three representations (referred to as channels in this context) which can be studied separately:

$$\begin{aligned} &\text{Sound} - \text{channel}: \quad && h_{tt},\, h_{tz},\, h_{zz},\, h_{rr},\, h_{tr},\, h_{zr},\, h_{xx} + h_{yy} \\ &\text{Shear} - \text{channel}: \quad && h_{tx},\, h_{ty},\, h_{zx},\, h_{zy},\, h_{rx},\, h_{ry} \\ &\text{Scalar} - \text{channel}: \quad && h_{\alpha\beta} - \delta_{\alpha\beta}(h_{xx} + h_{yy})/2. \end{aligned} \tag{7.5}$$

The sound channel has spin 0, shear channel has spin 1 and the scalar channel (whose equations of motion will be identical to that of the scalar) has spin 2 under $O(2)$.

In every channel, we can define a quantity Z_i [170,171], that is invariant under the gauge transformations $h_{\mu\nu} \to h_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu$ of the gravitational bulk theory. In the position space these are

$$\begin{aligned} Z_1 &= \partial_z H_{tx} - \partial_t H_{xz} \\ Z_2 &= 2f\partial_z^2 H_{tt} - 4\partial_t\partial_z H_{tz} + 2\partial_t^2 H_{zz} - \left((f + \frac{r}{2}f')\partial_z^2 + \partial_t^2\right)(H_{xx} + H_{yy}) \\ Z_3 &= H_{xy}, \end{aligned} \tag{7.6}$$

where $H_{tt} = h_{tt}/fr^2$, $H_{ti} = h_{ti}/r^2$ and $H_{ij} = h_{ij}/r^2$ for $i, j \in \{x, y, z\}$, $f = f(r)$ is the function appearing in the black hole metric and the prime denotes the derivative with respect to r. As is conventional, we refer to Z_1, Z_2 and Z_3 as the shear channel invariant, the sound channel invariant and the scalar channel invariant, respectively.

We can now choose a particular channel, take the linearized Einstein equations (7.3) and assume the metric perturbation to be of the form $h_{\mu\nu} = h_{\mu\nu}(t, z, r)$. Combining the resulting equations, we get PDEs for the invariants. To express the explicit form of these equations it will be useful to define the following quantities:

$$\begin{aligned}
c_1 &= (3\mu^2 - 8\mu r^4 + 5r^8)/r^5 \\
c_2 &= 2\mu(r^4 - \mu)/r^5 \\
c_3 &= (\mu - r^4)^2/r^4 \\
c_4 &= 16\mu^2(r^4 - \mu)/(3r^{10}) \\
c_5 &= 1 + \mu(\mu - 4r^4)/(3r^8) \\
c_6 &= 2 - 4\mu/(3r^4) \\
c_7 &= (\mu^2 - 6\mu r^4 + 5r^8)/r^5 \\
c_8 &= (r^4 - \mu)(9\mu^2 - 16\mu r^4 + 15r^8)/(3r^9) \\
c_9 &= -(\mu - 3r^4)(\mu - r^4)^2/(3r^8).
\end{aligned} \tag{7.7}$$

The equations of motion for the invariants are then given by[67]

$$\begin{aligned}
0 &= (\partial_t^2 + f\partial_z^2)^2 Z_1 + \left(c_1(\partial_t^2 + f\partial_z^2) + c_2(\partial_t^2 - f\partial_z^2)\right) Z_1' + c_3(\partial_t^2 + f\partial_z^2) Z_1'' \\
0 &= (c_4\partial_z^2 + c_5\partial_z^4 + c_6\partial_t^2\partial_z^2 + \partial_t^4) Z_2 + (c_7\partial_t^2 + c_8\partial_z^2) Z_2' + (c_3\partial_t^2 + c_9\partial_z^2) Z_2'' \\
0 &= (\partial_t^2 + f\partial_z^2) Z_3 + c_7 Z_3' + c_3 Z_3''.
\end{aligned} \tag{7.8}$$

7.2.3. Ansatz and the vacuum propagators

In order to solve (7.8) we need to find the bulk-to-boundary propagators $\mathcal{Z}_i$, which are related to the invariants by

$$Z_i(t, z, r) = \int dt' dz' \, \mathcal{Z}_i(t - t', z - z', r) \hat{Z}_i(t', z'), \tag{7.9}$$

[67] These are the equations one obtains by Wick rotating and Fourier transforming the corresponding PDEs in [170].

where $\hat{Z}_i$ is related to the boundary value (up to derivatives) of Z_i as will be explained below. To solve the equations of motion we use the ansatz [102,163] introduced for the case of a scalar field in a black hole background, suitably modified for our integrated case. Let us briefly review its derivation and the logic behind its construction.

Although in $d = 4$ the bulk equations cannot be solved analytically, one can try to find an expansion of the solution corresponding to the OPE limit on the boundary and extract the CFT data. The intuition behind this limit is the expectation that the bulk solution becomes sensitive only to the near-boundary region as the CFT operators approach each other. It was demonstrated in [102,163] that such a bulk regime is given by[68]

$$r \to \infty \qquad \text{with} \qquad rt,\, rz \;\text{ fixed.} \tag{7.10}$$

In [102,163] it was found by explicit computation that this regime partly determined the correlator in the OPE limit and therefore contains information about the CFT data. In this section we will explore the same near-boundary limit, suitably generalized for the integrated correlator. To realize (7.10), it is useful to introduce new coordinates defined by

$$\begin{aligned} \rho &= rz \\ w^2 &= 1 + r^2 t^2 + r^2 z^2 \,. \end{aligned} \tag{7.11}$$

In these coordinates the limit is $r \to \infty$ with w and ρ held fixed. By explicit calculations, we will again see that this is a relevant near-boundary expansion which will retain interesting CFT data in the OPE limit that we read off.

According to [102], one expects the solution to be of the form of the product of the AdS propagator and an expansion in $1/r$, where at each order we have a polynomial $\sum_i \alpha_i(w)\rho^i$. Substituting this into the equations of motion, we can find analytical solutions for all $\alpha_i(w)$. Imposing regularity in the bulk and demanding the proper boundary behaviour[69], we determine the integration constants and find the coefficients $\alpha_i(w)$ as polynomials in w.

[68] Note that in the original non-integrated case one has $|\vec{x}|$ instead of z.

[69] By the proper boundary behaviour we mean that the boundary limit of the bulk solution should reproduce the form of the boundary correlators expected from the boundary CFT.

If there are logarithmic terms[70] $\mathcal{Z}_i$ takes the form [163]

$$\mathcal{Z}_i = \mathcal{Z}_i^{AdS}\left(1 + \frac{1}{r^4}\left(G_i^{4,1} + G_i^{4,2}\log r\right) + \frac{1}{r^8}\left(G_i^{8,1} + G_i^{8,2}\log r\right) + \dots\right), \tag{7.12}$$

where $\mathcal{Z}_i^{AdS}$ is the vacuum bulk-to-boundary propagator for the invariant Z_i and $G_i^{4,j}$, $G_i^{8,j}$, ..., $j \in \{1,2\}$, $i \in \{1,2,3\}$ are given by[71] (we suppress the channel index for simplicity)

$$\begin{aligned} G^{4,j} &= \sum_{m=0}^{2}\sum_{n=-2}^{4-m}(a_{n,m}^{4,j} + b_{n,m}^{4,j}\log w)w^n\rho^m, \\ G^{8,j} &= \sum_{m=0}^{6}\sum_{n=-6}^{8-m}(a_{n,m}^{8,j} + b_{n,m}^{8,j}\log w)w^n\rho^m. \end{aligned} \tag{7.13}$$

Here $G^{4,j}$ corresponds to the stress tensor contribution ($\propto \mu^1$) and $G^{8,j}$ corresponds to the double-stress tensor contributions. We expect to find $b^{4,1} = 0$ and $G^{4,2} = 0$ in all three channels.

The vacuum propagator $\mathcal{Z}_i^{AdS}$ for the i-th channel can be determined using the AdS bulk-to-boundary propagators for the various components of the metric perturbation. Let us describe this calculation in more detail. The AdS propagator for $H_{\mu\nu}$ was computed in [172] and in the five dimensional bulk case can be expressed as

$$\mathfrak{G}_{\mu\nu,\rho\sigma} = \frac{10r^4}{\pi^2(1+r^2(t^2+\vec{x}^2))^4}J_{\mu\alpha}J_{\nu\beta}P_{\alpha\beta,\rho\sigma}, \tag{7.14}$$

where $J_{\mu\nu}$ and $P_{\mu\nu,\rho\sigma}$ are given by

$$\begin{aligned} J_{\mu\nu} &= \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{\frac{1}{r^2} + t^2 + \vec{x}^2} \\ P_{\mu\nu,\rho\sigma} &= \frac{1}{2}(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\nu\rho}\delta_{\mu\sigma}) - \frac{1}{4}\delta_{\mu\nu}\delta_{\rho\sigma}. \end{aligned} \tag{7.15}$$

[70] Logarithmic terms appear, for example, in the case of a scalar field with integer conformal dimension Δ_L or in the presence of anomalous dimensions as in the case of the stress tensor thermal two-point function. They can also be produced upon integration. We comment more on the origin of these terms in Appendix D.1.

[71] We use the bounds of the sums as they were derived for the case of a scalar field [102,163]. As we will see, this will be valid also for the stress tensor case in the scalar and shear channels. In the sound channel we will need a slight modification of the ansatz.

Integrating over the x and y directions, we get

$$\mathcal{G}_{\mu\nu,\rho\sigma}(t,z,r) \equiv \int_{R^2} dxdy\, \mathfrak{G}_{\mu\nu,\rho\sigma}(t,x,y,z,r), \tag{7.16}$$

and the (integrated) AdS solution for $H_{\mu\nu}$ is given by

$$H_{\mu\nu}(t,z,r) = \int dt'dz' \mathcal{G}_{\mu\nu\rho\sigma}(t-t', z-z', r)\hat{H}_{\rho\sigma}(t',z'), \tag{7.17}$$

where $\hat{H}_{\mu\nu}$ are the sources, i.e. the values of the bulk solution on the conformal boundary.

Substituting (7.17) into the definitions of the invariants (7.6), one can accordingly read off the AdS bulk-to-boundary propagators $\mathcal{Z}_i^{AdS}$.

Here we list the resulting expressions for some particular choices of the sources:

Sources	(t,z,r)-result	(w,ρ,r)-results
$\hat{H}_{xy}$	$\mathcal{Z}_3^{AdS} = \frac{r^2}{\pi(r^2(t^2+z^2)+1)^3}$	$= \frac{r^2}{\pi w^6}$
$\hat{H}_{tx}$	$\mathcal{Z}_1^{AdS} = -\frac{6r^4 z}{\pi(r^2(t^2+z^2)+1)^4}$	$= -\frac{6r^3\rho}{\pi w^8}$
$\hat{H}_{xz}$	$\mathcal{Z}_1^{AdS} = \frac{6r^4 t}{\pi(r^2(t^2+z^2)+1)^4}$	$= \frac{6r^3\sqrt{w^2-1-\rho^2}}{\pi w^8}$
$\hat{H}_{tz}$	$\mathcal{Z}_2^{AdS} = -\frac{192r^6 tz}{\pi(r^2(t^2+z^2)+1)^5}$	$= -\frac{192r^4\rho\sqrt{w^2-1-\rho^2}}{\pi w^{10}}$
$\hat{H}_{tt}$	$\mathcal{Z}_2^{AdS} = -\frac{24\left(r^6\left(t^2-7z^2\right)+r^4\right)}{\pi(r^2(t^2+z^2)+1)^5}$	$= -\frac{24r^4(w^2-8\rho^2)}{\pi w^{10}}$
$\hat{H}_{xx}$	$\mathcal{Z}_2^{AdS} = \frac{24r^4-72r^6\left(t^2+z^2\right)}{\pi(r^2(t^2+z^2)+1)^5}$	$= \frac{24r^4(4-3w^2)}{\pi w^{10}}$
$\hat{H}_{zz}$	$\mathcal{Z}_2^{AdS} = \frac{24r^4\left(r^2\left(7t^2-z^2\right)-1\right)}{\pi(r^2(t^2+z^2)+1)^5}$	$= \frac{24r^4(7w^2-8(1+\rho^2))}{\pi w^{10}}$

At this point, we have all the pieces needed for the ansatz (7.12). Inserting it into equations (7.8), we can determine the coefficients $a^{k,j}_{n,m}$ and $b^{k,j}_{n,m}$. We next proceed to discuss the results channel by channel.

7.2.4. Scalar channel

We begin by considering the scalar channel where the equation of motion (7.8) has the simplest form. We confine our attention to the contributions due to the identity operator (μ^0), the stress tensor (μ^1) and double-stress tensors (μ^2). We are therefore interested in finding $G_i^{4,1}$, $G_i^{4,2}$, $G_i^{8,1}$ and $G_i^{8,2}$ in the ansatz (7.12).

In the scalar channel, we may either turn on the source $\hat{H}_{xy} \neq 0$ or $\hat{H}_{xx} = -\hat{H}_{yy} \neq 0$. Since these differ only by an $O(2)$ rotation, the corresponding bulk solutions, as well as the form of the action will be identical. For this reason, we will restrict our attention to the case where $\hat{H}_{xy} \neq 0$. Hence, the invariant Z_3 is given by

$$Z_3(t,z,r) = \int dt' dz' \mathcal{Z}_3^{(xy)}(t-t', z-z', r)\hat{H}_{xy}, \tag{7.18}$$

where $\mathcal{Z}_3^{(xy)}$ is the bulk-to-boundary propagator[72].

Transforming equation (7.8) into the (w,ρ,r)-coordinates with $\mathcal{Z}_3^{(xy)}$ given by (7.12), we find the solution at $\mathcal{O}(\mu)$,

$$\mathcal{Z}_3^{(xy)}\Big|_{\mu^1} = \frac{\mu\left(w^6+w^4+6w^2-2\rho^2\left(w^4+2w^2+3\right)-12\right)}{10\pi r^2 w^8}. \tag{7.19}$$

As expected, there are no log terms in this case. At $\mathcal{O}(\mu^2)$ we find

$$\begin{aligned}\mathcal{Z}_3^{(xy)}\Big|_{\mu^2} =& \frac{\mu^2}{8400\pi r^6 w^{10}}\Big[120w^{10}\left(-4\rho^2+5w^2-6\right)\left(\log(w)-\log(r)\right)+655w^8 \\ &+448w^6+3136w^4-12656w^2+56\rho^4\left(10w^8+20w^6+35w^4+44w^2+36\right) \\ &-4\rho^2\left(750w^{10}+40w^8+345w^6+476w^4+448w^2-2016\right)+8064\Big] \\ &+\frac{1}{\pi r^6}\left[\left(1-6\rho^2\right)a_{6,0}^{8,1(xy)}+a_{8,0}^{8,1(xy)}\left(w^2-8\rho^2\right)\right],\end{aligned} \tag{7.20}$$

where the coefficients $a_{6,0}^{8,1(xy)}$ and $a_{8,0}^{8,1(xy)}$ are not fixed by the near-boundary analysis. We also see the presence of log terms which are due to the xy-integration and the anomalous dimensions of the double-stress tensors.

7.2.4.1. $G_{xy,xy}$

We now use the holographic dictionary to determine the thermal correlator $G_{xy,xy}$. The action for the scalar invariant Z_3 (and Z_1 and Z_2 below) can be obtained by Fourier transforming and Wick rotating the result obtained in [170]:

$$S_3 = \frac{\pi^2 C_T}{160}\lim_{r\to\infty}\int dtdz r^5\left(1-\frac{\mu}{r^4}\right)\partial_r Z_3(t,z,r) Z_3(t,z,r). \tag{7.21}$$

The invariant $Z_3(t,z,r)$ is fully determined by the bulk-to-boundary propagator $\mathcal{Z}_3^{(xy)}$ via Eq. (7.18). To compute the action (7.21) we expand $\mathcal{Z}_3^{(xy)}$ near $r=\infty$ as

$$\mathcal{Z}_3^{(xy)}(t,z,r) = \frac{1}{2}\delta^{(2)}(t,z) + \frac{1}{r^4}\zeta_3^{(xy)}(t,z) + \ldots, \tag{7.22}$$

[72] The superscript index in the parenthesis specifies the choice of the non-zero sources.

where the dots represent subleading contact terms of $\mathcal{O}(r^{-2})$ of the schematic form $\partial^n\delta/r^n$ as well as contributions analytic in (t,z) that are $\mathcal{O}(r^{-6})$. As we will see, in the scalar channel $G_{xy,xy} \propto \zeta_3^{(xy)}$.

To proceed, we substitute the bulk-to-boundary propagator into the action (7.21):

$$\begin{aligned} S_3 = \frac{\pi^2 C_T}{160} \lim_{r\to\infty} \int d^2x d^2x' d^2x'' (r^5 - \mu r)\partial_r \mathcal{Z}_3^{(xy)}(x-x',r)\mathcal{Z}_3^{(xy)}(x-x'',r)\hat{H}_{xy}(x')\hat{H}_{xy}(x'') \\ = -\frac{\pi^2 C_T}{20}\int d^2x d^2x' \zeta_3^{(xy)}(x-x')\hat{H}_{xy}(x)\hat{H}_{xy}(x'), \end{aligned} \tag{7.23}$$

where in the second line we have integrated the delta function. We have used an abbreviated notation $x=\{t,z\}$, $x'=\{t',z'\}$ and $x''=\{t'',z''\}$ and omitted contact terms (see e.g. [173] for a review on holographic renormalization and the treatment of contact terms).

We can now compute the CFT correlator,

$$G^{(bulk)}_{xy,xy} = \langle T_{xy}(t,z)T_{xy}(0,0)\rangle_\beta = -\frac{\delta^2 S_3}{\delta\hat{H}_{xy}(t,z)\delta\hat{H}_{xy}(0,0)} = \frac{\pi^2 C_T}{20}\zeta_3^{(xy)}(t,z) \tag{7.24}$$

Inserting the explicit bulk solution, we obtain the following results order-by-order in μ:

$$\begin{aligned} G^{(bulk)}_{xy,xy}\Big|_{\mu^0} &= \frac{\pi C_T}{10(t^2+z^2)^3} \\ G^{(bulk)}_{xy,xy}\Big|_{\mu^1} &= \frac{\pi\mu C_T(t^2-z^2)}{100(t^2+z^2)^2} \\ G^{(bulk)}_{xy,xy}\Big|_{\mu^2} &= \frac{\pi\mu^2 C_T}{4200}\left(3\left(5t^2+z^2\right)\log\left(t^2+z^2\right) - \frac{2\left(75t^2z^2+61z^4\right)}{t^2+z^2}\right) \\ &\quad + \frac{1}{10}\pi C_T\left(a^{8,1(xy)}_{8,0}\left(t^2-7z^2\right) - 6z^2 a^{8,1(xy)}_{6,0}\right). \end{aligned} \tag{7.25}$$

We will compare them with the CFT calculations in the next section.

7.2.5. Shear channel

We can repeat the procedure above to solve the shear channel bulk equation (7.8) the sources $\hat{H}_{tx}$ and $\hat{H}_{xz}$ and express the results in terms of w, ρ and r. The explicit expressions are listed in Appendix D.2. We will now use them to determine $G_{tx,tx}$ and $G_{xz,xz}$ using the AdS/CFT dictionary; these calculations are summarized below.

7.2.5.1. $G_{tx,tx}$ and $G_{xz,xz}$

The action for the shear channel invariant is given by[73] [170]

$$\begin{aligned} S_1 =& \frac{\pi^2 C_T}{160}\lim_{r\to\infty}\int dtdz \frac{\left(1-\frac{\mu}{r^4}\right)r^5}{\partial_t^2+\partial_z^2\left(1-\frac{\mu}{r^4}\right)}\partial_r Z_1(t,z,r)Z_1(t,z,r) \\ =& \frac{\pi^2 C_T}{160}\lim_{r\to\infty}\int dtdz\left(\frac{r^5}{\partial_t^2+\partial_z^2}+\mathcal{O}(r^2)\right)\partial_r Z_1(t,z,r)Z_1(t,z,r). \end{aligned} \tag{7.26}$$

We begin by turning on the source $\hat{H}_{tx}$ and follow the same approach as in the previous section. The shear channel invariant is given by

$$Z_1(t,z,r) = \int dt'dz'\mathcal{Z}_1^{(tx)}(t-t',z-z',r)\hat{H}_{tx}, \tag{7.27}$$

where $\mathcal{Z}_1^{(tx)}$ is the bulk-to-boundary propagator corresponding to our choice of source.

The near-boundary expansion of $\mathcal{Z}_1^{(tx)}$ reads

$$\mathcal{Z}_1^{(tx)} = \frac{1}{2}\partial_z\delta^{(2)}(t,z)+\frac{1}{r^4}\zeta_1^{(tx)}+\frac{\log r}{r^4}\zeta_{1,log}^{(tx)}+\ldots, \tag{7.28}$$

where the dots correspond to contact terms which are $\mathcal{O}\left(r^{-2}\right)$ and non-contact terms which are $\mathcal{O}\left(r^{-6}\right)$. Here, however, we encounter $\log r$ terms in the expansion,

$$\zeta_{1,log}^{(tx)} = -\frac{z\left(840a_{8,0}^{8,2(tx)}+41\mu^2\right)}{140\pi}. \tag{7.29}$$

The $\log r$ term in (7.28) will lead to a divergence in the correlator as $r\to\infty$, unless the value of the coefficient $a_{8,0}^{8,2(tx)}$ is fixed to be

$$a_{8,0}^{8,2(tx)} = -\frac{41}{840}\mu^2. \tag{7.30}$$

Using the expansion (7.28) in the action (7.26) and proceeding as in the tensor channel case, we obtain

$$G_{tx,tx}^{bulk} = \frac{\pi^2 C_T}{10}\frac{\partial_z}{\partial_t^2+\partial_z^2}\zeta_1^{(tx)} \tag{7.31}$$

[73] Note the presence of the inverse operator $(\partial_t^2+\partial_z^2)^{-1}$ which is a Fourier transform of $(\omega^2+q^2)^{-1}$ that appears in the action derived in [170].

Thus, we arrive at

$$
\begin{aligned}
G^{(bulk)}_{tx,tx}\Big|_{\mu^0} =& -\frac{1}{\partial_t^2+\partial_z^2}\frac{3\pi C_T\left(t^2-7z^2\right)}{5\left(t^2+z^2\right)^5}\\
G^{(bulk)}_{tx,tx}\Big|_{\mu^1} =& \frac{1}{\partial_t^2+\partial_z^2}\frac{3\pi\mu C_T\left(t^4-6t^2z^2+z^4\right)}{200\left(t^2+z^2\right)^4}\\
G^{(bulk)}_{tx,tx}\Big|_{\mu^2} =& -\frac{1}{\partial_t^2+\partial_z^2}\Bigg[\frac{\pi\mu^2 C_T}{8400}\left(\frac{2\left(669t^4z^2+804t^2z^4+271z^6\right)}{\left(t^2+z^2\right)^3}+123\log\left(t^2+z^2\right)\right)\\
&+\frac{3}{5}\pi a^{8,1(tx)}_{8,0}C_T\Bigg].
\end{aligned}
\tag{7.32}
$$

Here we keep the inverse operator $(\partial_t^2+\partial_z^2)^{-1}$ explicit, as in the later comparison we will act on the corresponding CFT expressions with the operator $\partial_t^2+\partial_z^2$.

The correlator $G^{(bulk)}_{xz,xz}$ can be computed in a similar way and the result is presented order-by-order in μ in Appendix D.2.

7.2.6. Sound channel

We now consider the sound channel. Closer inspection reveals that in the sound channel the form of the ansatz must be modified due to a technical issue present for the diagonal sources. We first explain how it arises and how to treat it and then proceed with the computation of the holographic TT correlators.

7.2.6.1. Modified ansatz

We find that for the source $\hat{H}_{tz}$, we are able to extract the corresponding results in the sound channel using the same ansatz as in the scalar and shear channels. However, we observe that if we turn on any of the diagonal sources $\hat{H}_{tt}$, $\hat{H}_{zz}$, $\hat{H}_{xx}$ or $\hat{H}_{yy}$, then the ansatz of the form (7.12) is no longer valid.

The reason for this stems from the structure of the vacuum solution $\mathcal{Z}_2^{AdS}$ in these cases. Let us take $\hat{H}_{tt}\neq 0$ as an example. In this case the AdS propagator has the form $-\frac{24r^4(w^2-8\rho^2)}{\pi w^{10}}$. From (7.12) it is clear that the ansatz will only be valid if the actual solution of the bulk equations is proportional to $(w^2-8\rho^2)$ to all orders in μ. This condition is too restrictive and, as one can show directly, is not satisfied in the case of the equation (7.8).

To solve this issue for the diagonal terms, we separate the vacuum contribution[74]:

$$\mathcal{Z}_i^{\mathrm{diag}} = \mathcal{Z}_i^{AdS} + \left(G_i^{4,1} + G_i^{4,2}\log r\right) + \frac{1}{r^4}\left(G_i^{8,1} + G_i^{8,2}\log r\right) + \ldots, \quad (7.33)$$

with G^4, G^8, ... defined by

$$\begin{aligned} G^{4,j} &= \sum_{m=0}^{4}\sum_{n=-12}^{-4-m}(a_{n,m}^{4,j} + b_{n,m}^{4,j}\log w)w^n\rho^m, \\ G^{8,j} &= \sum_{m=0}^{8}\sum_{n=-16}^{-m}(a_{n,m}^{8,j} + b_{n,m}^{8,j}\log w)w^n\rho^m. \end{aligned} \quad (7.34)$$

The upper and lower bounds of the sums were determined in the same way as it was done at the beginning of this Section. Ultimately, using the original ansatz (7.12) for the off-diagonal sources and the modified one (7.33) for the diagonal ones, allows us to solve the equation of motion (7.8). The results are presented in Appendix D.2.

7.2.6.2. $G_{tz,tz}$, $G_{tt,tt}$, $G_{zz,zz}$ and $G_{xx,xx}$

The action for the sound invariant has the form [170]

$$\begin{aligned} S_2 &= -\frac{3\pi^2 C_T}{640}\lim_{r\to\infty}\int dtdz\frac{r^5\left(1-\frac{\mu}{r^4}\right)}{\left(3\partial_t^2 + \partial_z^2\left(3-\frac{\mu}{r^4}\right)\right)^2}\partial_r Z_2(t,z,r)Z_2(t,z,r) \\ &= -\frac{\pi^2 C_T}{1920}\lim_{r\to\infty}\int dtdz\left(\frac{r^5}{(\partial_t^2+\partial_z^2)^2} + \mathcal{O}(r^2)\right)\partial_r Z_2(t,z,r)Z_2(t,z,r). \end{aligned} \quad (7.35)$$

Expanding bulk-to-boundary propagators for our choices of the sources, eliminating the non-local divergent $\log r$ term and proceeding as above, we eventually obtain

$$G_{ab,ab}^{bulk} = \frac{1}{(\partial_t^2+\partial_z^2)^2}D_{ab}\zeta_2^{(ab)}, \quad (7.36)$$

[74] The form of this ansatz is deduced from the structure of the expected CFT results, see the next section.

where $\zeta_2^{(ab)}$ is the $1/r^4$ term in the near-boundary expansion of the corresponding bulk-to-boundary propagator $\mathcal{Z}_2^{(ab)}$ for the source $\hat{H}_{ab}$ and the operator D_{ab} is given by

$$\begin{aligned} D_{tz} &= -\frac{\pi^2 C_T}{30}\partial_t\partial_z, \\ D_{tt} &= \frac{\pi^2 C_T}{30}\partial_z^2, \\ D_{zz} &= \frac{\pi^2 C_T}{30}\partial_t^2. \end{aligned} \tag{7.37}$$

Using the explicit form of the bulk-to-boundary solution we find that the correlation function $G_{tz,tz}^{(bulk)}$ is given by

$$\begin{aligned} G_{tz,tz}^{(bulk)}\Big|_{\mu^0} &= -\frac{1}{(\partial_t^2+\partial_z^2)^2}\frac{96\pi C_T\left(3t^4-34t^2z^2+3z^4\right)}{5\left(t^2+z^2\right)^7} \\ G_{tz,tz}^{(bulk)}\Big|_{\mu^1} &= \frac{1}{(\partial_t^2+\partial_z^2)^2}\frac{4\pi\mu C_T\left(-t^6+15t^4z^2-15t^2z^4+z^6\right)}{15\left(t^2+z^2\right)^6} \\ G_{tz,tz}^{(bulk)}\Big|_{\mu^2} &= -\frac{1}{(\partial_t^2+\partial_z^2)^2}\frac{2\pi\mu^2 C_T\left(133t^8-1408t^6z^2-110t^4z^4+88t^2z^6+65z^8\right)}{1575\left(t^2+z^2\right)^5}, \end{aligned} \tag{7.38}$$

and analogously for the $G_{tt,tt}^{(bulk)}$ and $G_{zz,zz}^{(bulk)}$ (see Appendix D.2).

We find that we need to be more careful when analyzing the case of $G_{xx,xx}^{(bulk)}$ (and, similarly, $G_{yy,yy}^{(bulk)}$). If we turn on the source $\hat{H}_{xx}$ we find a contribution not only from the action S_2 but also from S_3; the result is

$$G_{xx,xx}^{bulk} = G_{xy,xy}^{bulk} - \frac{\pi^2 C_T}{60}\frac{1}{(\partial_t^2+\partial_z^2)}\zeta_2^{(xx)}. \tag{7.39}$$

The resulting expression for $G_{xx,xx}^{bulk}$ can be found in Appendix D.2. In the following section we will compare these results to their CFT counterparts.

7.3. Stress tensor thermal two-point function in $d = 4$

In this section we study the stress tensor two-point function on $S_\beta^1 \times R^{d-1}$, where $\beta = T^{-1}$ is the inverse temperature, in holographic CFTs, that is, CFTs with large central charge $C_T \gg 1$ and a large gap in the spectrum of higher-spin single-trace operators $\Delta_{\text{gap}} \gg 1$. The case of the purely scalar correlator is reviewed and extended to the integrated correlator in Appendix D.1, it serves as a useful toy model to study before considering the technically more complicated spinning correlator. Using the stress tensor OPE, we isolate the contribution from multi-stress tensor operators $[T^k]_J$ and read off the CFT data (OPE coefficients, thermal one-point functions and anomalous dimensions) via a comparison to the bulk calculations of metric perturbations around a black hole background in the previous section. In particular, we read off the anomalous dimensions of multi-stress tensor operators of the schematic form $:T_{\mu\nu}T_{\rho\sigma}:$, $:T_\mu{}^\rho T_{\rho\nu}:$ and $:T^{\rho\sigma}T_{\rho\sigma}:$ with spin $J = 0, 2, 4$, respectively.

7.3.1. OPE expansion and multi-stress tensor contributions

The contributions of the multi-stress tensor operators to the thermal two-point function of the stress-tensor in (7.1) can be computed using the OPE, which can be schematically written as

$$T_{\mu\nu}(x) \times T_{\rho\sigma}(0) \sim \frac{1}{x^{2d}}\Big[1 + \sum_{i=1}^{3} x^d \lambda^{(i)}_{TTT} A^{(i),\,\alpha\beta}_{\mu\nu\rho\sigma} T_{\alpha\beta}(0) + x^{2d} \sum_{J=0,2,4} \sum_{i\in i_J} \lambda^{(i)}_{TT[T^2]_J} B^{(i),\,\mu_1\ldots\mu_J}_{\mu\nu\rho\sigma} [T^2]_{\mu_1\ldots\mu_J}(0) + \ldots\Big], \tag{7.40}$$

where $[T^k]_{\mu_1\ldots\mu_J}$ are spin-J multi-stress tensor operators, the ellipses denote higher multi-trace operators and their descendants and $i_0 = \{1\}$, $i_2 = \{1, 2\}$ and $i_4 = \{1, 2, 3\}$. On $S^1_\beta \times R^{d-1}$ only multi-stress tensors $[T^k]_{\mu_1\ldots\mu_J}$ with dimension $\Delta_{k,J} = dk + \mathcal{O}(C_T^{-1})$ contribute since the thermal one-point function of operators with derivatives will vanish due to translational invariance see e.g. [57,4] [75]. Here the label (i) denotes the different structures appearing in the OPE of spinning operators. The structures $A^{(i),\,\alpha\beta}_{\mu\nu\rho\sigma}$ and $B^{(i),\,\mu_1\ldots\mu_J}_{\mu\nu\rho\sigma}$ are further fixed by conformal symmetry and depend on $x^\mu/|x|$. Upon inserting the OPE (7.40) in the thermal two-point function (7.1), we find that each term consists of a product of a kinematical piece and the thermal one-point functions $\langle [T^k]_J\rangle_\beta$, weighted by the OPE coefficients $\lambda^{(i)}_{TT[T^k]_J}$. The thermal one-point functions are fixed by symmetry up to an overall coefficient (see e.g. [56,57])

$$\langle [T^k]_{\mu_1\ldots\mu_J}\rangle_\beta = \frac{b_{[T^k]_J}}{\beta^{\Delta_k}}(e_{\mu_1}\cdots e_{\mu_J} - \text{traces}), \tag{7.41}$$

[75] In other words, only operators $[T^k]_J$ with no derivatives but various contractions of indices survive. We therefore denote these operators by the total spin J and the number of stress tensors k. Note also that descendants do not contribute to the two-point function on $S^1_\beta \times R^{d-1}$.

where e_μ is a unit vector on S^1_β. Rather than using the explicit OPE (7.40) together with the thermal expectation value (7.41), we will use the conformal block expansion in a scalar state and take the OPE limit, see Appendix D.3 and [102,7]. To read off the CFT data we compare this to the bulk computations in a planar black hole background. The bulk result is shown to be consistent with the OPE expansion and we determine the $\mathcal{O}(C_T^{-1})$ anomalous dimensions $\gamma_J^{(1)}$ of the double-stress tensor operators of the schematic form $: T_{\mu\nu}T_{\rho\sigma} :$, $: T_\mu{}^\rho T_{\rho\nu} :$ and $: T^{\rho\sigma}T_{\rho\sigma} :$. We further determine the product of coefficients $\langle [T^2]_{J=0,2,4}\rangle_\beta \lambda^{(i)}_{TT[T^2]_J}$ to leading order in C_T^{-1} and partially at subleading order.

Let us now review the expected scaling with C_T due to multi-stress tensors appearing in the OPE. The central charge C_T is defined by the stress tensor two-point function in the vacuum

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle = \frac{C_T}{x^{2d}}\Big[\frac{1}{2}(I_{\mu\rho}I_{\nu\sigma} + \frac{1}{2}I_{\mu\sigma}I_{\nu\rho}) - \frac{1}{d}\delta_{\mu\nu}\delta_{\rho\sigma}\Big], \tag{7.42}$$

where $I_{\mu\nu} = I_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}$. The CFT data is encoded in a perturbative expansion in C_T^{-1} and a generic k-trace operator $[\mathcal{O}^k]$ with dimension Δ_k gives the following contribution in the OPE limit[76] $|x|/\beta \to 0$:

$$\langle T_{\mu\nu}(x)T_{\rho\sigma}(0)\rangle_\beta|_{[\mathcal{O}^k]} \propto |x|^{\Delta_k - 2d}\frac{\langle T_{\mu\nu}T_{\rho\sigma}[\mathcal{O}^k]\rangle\langle[\mathcal{O}^k]|\rangle_\beta}{\langle[\mathcal{O}^k][\mathcal{O}^k]\rangle}. \tag{7.43}$$

Here we are interested in the case of multi-trace stress tensor operators $[\mathcal{O}^k] = [T^k]_J$ which have a natural normalization

$$\langle [T^k]_J[T^k]_J\rangle \sim C_T^k, \tag{7.44}$$

which follows from the completely factorized contribution. In holographic CFTs dual to semi-classical Einstein gravity, the connected part of correlation functions of stress tensors is proportional to C_T:

$$\langle T_{\mu\nu}T_{\rho\sigma}[T^{k\neq 2}]_J\rangle \sim C_T. \tag{7.45}$$

[76] In general, the OPE expansion is a complicated function of x^μ, below, we just keep the scaling with $|x|$. We further suppress the indices of the operators appearing in the OPE.

An important exception to (7.45) occurs for $k = 2$ where there is a disconnected contribution such that

$$\langle T_{\mu\nu} T_{\rho\sigma} [T^2]_J \rangle \sim C_T^2 + \ldots, \tag{7.46}$$

where the dots refer to subleading corrections in C_T^{-1} which will play an important role later. Lastly, the expectation value of a multi-stress tensor operator in the thermal state has the following scaling with C_T

$$\langle [T^k]_J \rangle_\beta \sim \frac{C_T^k}{\beta^{dk}}, \tag{7.47}$$

where we also included the dependence on β which is fixed on dimensional grounds.

Using (7.44)(7.47), we see that the contribution of multi-stress tensor operators $[T^k]_J$ with dimensions $\Delta_k = dk + \mathcal{O}(C_T^{-1})$ to the stress tensor two-point function in the thermal state has the following scaling with C_T for $k \neq 2$

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle_\beta |_{[T^{k\neq 2}]_J} \propto \frac{1}{x^{2d}} C_T \left(\frac{x}{\beta} \right)^{dk}, \tag{7.48}$$

Meanwhile, for $k = 2$, the double stress tensor contributions $[T^2]_{J=0,2,4}$ to the thermal two-point function give rise to the disconnected part of the correlator due to the fact that the three-point function $\langle T_{\mu\nu} T_{\rho\sigma} [T^2]_J \rangle \sim C_T^2$, compared to the $\mathcal{O}(C_T)$ contribution from the connected part. The contribution at $\mathcal{O}(C_T)$ will therefore contain the first subleading correction to the OPE coefficients $\lambda^{(i)}_{TT[T^2]_J}$, the corrections to the thermal one-point functions, as well as the anomalous dimensions of the double-stress tensor operators.

We define coefficients $\rho_{i,J}$ for the double-stress tensor $[T^2]_J$ with dimensions $\Delta_J := \Delta_{2,J}$ by:

$$\begin{aligned} \hat{G}_{\mu\nu,\rho\sigma}(x)|_{\mu^2} = |x|^{-8} \Big[& \rho_{1,0}\, g_{\Delta_0,0,\mu\nu,\rho\sigma}(x) + \sum_{i=1,2} \rho_{i,2}\, g^{(i)}_{\Delta_2,2,\mu\nu,\rho\sigma}(x) \\ & + \sum_{i=1,2,3} \rho_{i,4}\, g^{(i)}_{\Delta_4,4,\mu\nu,\rho\sigma}(x) \Big], \end{aligned} \tag{7.49}$$

where $\hat{G}_{\mu\nu,\rho\sigma}(x) := \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle_\beta$ is the thermal correlator and $g^{(i)}_{\Delta,J,\mu\nu,\rho\sigma}$ can be obtained by taking the OPE limit of the conformal blocks in the differential basis [174,175], see Appendix D.3. The coefficients $\rho_{i,J}$ are therefore products of OPE coefficients and thermal one-point functions, see (7.43). The coefficients $\rho_{i,J}$ and the anomalous dimensions γ_J have a perturbative expansion in C_T^{-1}

$$\begin{aligned}\rho_{i,J} &= \rho_{i,J}^{(0)}\Big[1 + \frac{\rho_{i,J}^{(1)}}{C_T} + \mathcal{O}(C_T^{-2})\Big],\\ \Delta_J &= 2d + \frac{\gamma_J^{(1)}}{C_T} + \mathcal{O}(C_T^{-2}),\end{aligned} \tag{7.50}$$

and lead to the following schematic contribution to the stress tensor two-point function from $[T^2]_J$:[77]

$$\begin{aligned}\hat{G}_{\mu\nu,\rho\sigma}|_{[T^2]_J} &\propto \sum_i \rho_{i,J}|x|^{\gamma_J^{(1)}}\\ &\propto \sum_i \rho_{i,J}^{(0)}\Big[1 + \frac{1}{C_T}\Big(\rho_{i,J}^{(1)} + \gamma_J^{(1)}\log|x|\Big) + \mathcal{O}(C_T^{-2})\Big].\end{aligned} \tag{7.51}$$

Note that the number of structures for the three point functions $\langle T_{\mu\nu}T_{\rho\sigma}[T^2]_{J=0,2,4}\rangle$ is (in $d \geq 4$) $1, 2, 3$ for $J = 0, 2, 4$, respectively, giving a total of 6 different structures at this order. From now on we will mainly consider $d = 4$.

7.3.2. Thermalization of heavy states

The thermal one-point function of an operator $\mathcal{O}$ with dimension Δ and spin J on $S^1_\beta \times R^{d-1}$ is fixed up to an overall coefficient $b_{\mathcal{O}}$ [56,57]

$$\langle \mathcal{O}_{\mu_1\ldots\mu_J}\rangle_\beta = \frac{b_{\mathcal{O}}}{\beta^\Delta}(e_{\mu_1}\cdots e_{\mu_J} - \text{traces}), \tag{7.52}$$

where e^μ is a unit vector along the thermal circle. To leading order in the C_T^{-1} expansion, we expect multi-stress tensor operators to thermalize in heavy states $|\psi\rangle = |\mathcal{O}_H\rangle$ with scaling dimension $\Delta_H \sim C_T$: (see [7] for a discussion on the thermalization of multi-stress tensors and [58,176] for a discussion on ETH in CFTs.)

$$\langle [T^k]_J\rangle_H \approx \langle [T^k]_J\rangle_\beta, \tag{7.53}$$

where we have suppressed the indices. This statement holds to leading order in C_T^{-1}. In (7.53), the inverse temperature $\beta = T^{-1}$ is fixed by the thermalization

77 We stress that this only contains the scaling with $|x| \to 0$ while the explicit expression have a more complicated dependence on x^μ captured in (7.49).

of the stress tensor. In particular, thermalization of the stress tensor $\langle T_{\mu\nu}\rangle_H = \langle T_{\mu\nu}\rangle_\beta$[78] leads to the following relation between β and the scaling dimension Δ_H in $d=4$

$$\frac{b_{T_{\mu\nu}}}{\beta^4} = -\frac{muC_T S_4}{40}, \tag{7.54}$$

where μ is given by[79]

$$\mu = \frac{4\Gamma(d+2)}{(d-1)^2\Gamma(\frac{d}{2})^2 S_d^2}\frac{\Delta_H}{C_T} \tag{7.55}$$

and $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$.

To leading order in C_T^{-1}, the multi-stress tensor operators are expected to thermalize while the expectation value in the heavy state and the thermal state might differ at subleading order. As evident from (7.50), the $\mathcal{O}(C_T\mu^2)$ part of the correlator contains corrections subleading in C_T^{-1} to the dynamical data. When we compare these results to the corresponding bulk results computed in the black hole background these are therefore understood as corrections to the thermal one-point functions of these operators. More specifically, $\rho^{(1)}_{i,J}$ contain the following terms

$$\rho^{(1)}_{i,J} = \lambda^{(i,1)}_{TT[T^2]_J} + b^{(1)}_{[T^2]_J}, \tag{7.56}$$

where $\lambda^{(i,1)}_{TT[T^2]_J}$ and $b^{(1)}_{[T^2]_J}$ are the subleading C_T^{-1} corrections to the OPE coefficients and the thermal one-point functions, respectively.

7.3.3. Identity contribution

In this section we compare the contribution of the identity operator in the $T_{\mu\nu}\times T_{\rho\sigma}$ OPE on the CFT side using (7.42) to the bulk results. To make a comparison to the bulk calculation, we integrate (7.42) over the (x,y) plane

$$\begin{aligned}
G_{xy,xy}|_{\mu^0} &= \frac{\pi C_T}{10\left(t^2+z^2\right)^3}, \\
G_{tx,tx}|_{\mu^0} &= -\frac{\pi C_T\left(t^2-5z^2\right)}{40\left(t^2+z^2\right)^4}, \\
G_{tz,tz}|_{\mu^0} &= -\frac{\pi C_T\left(5t^4-38t^2z^2+5z^4\right)}{60\left(t^2+z^2\right)^5},
\end{aligned} \tag{7.57}$$

[78] We will take the large volume limit $\frac{\beta}{R}\to 0$ of this equation and further set $R=1$.

[79] Note that the definition of C_T differs by a factor of S_d^2 compared to [8].

where $G_{\mu\nu,\rho\sigma}$ is the integrated correlator defined in (7.1). The $G_{xy,xy}$ correlator in (7.57) agrees with (7.25) obtained in the bulk. In order to compare the remaining two polarizations $G_{tx,tx}$ and $G_{tz,tz}$, we further apply the differential operator $(\partial_t^2+\partial_z^2)^p$ with $p=1,2$, respectively, to match these CFT results with their bulk counterparts. Doing so, we find that

$$\begin{aligned}(\partial_t^2+\partial_z^2)G_{tx,tx}|_{\mu^0} &= -\frac{3\pi C_T\left(t^2-7z^2\right)}{5\left(t^2+z^2\right)^5},\\ (\partial_t^2+\partial_z^2)^2G_{tz,tz}|_{\mu^0} &= -\frac{96\pi C_T\left(3t^4-34t^2z^2+3z^4\right)}{5\left(t^2+z^2\right)^7},\end{aligned} \tag{7.58}$$

which agree with (7.32) and (7.38), respectively.

7.3.4. Stress tensor contribution

In this section we consider the stress tensor contribution. The stress tensor three-point function is fixed up to three coefficients in $d \geq 4$ [177][80]

$$\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)T_{\alpha\beta}(x_3)\rangle = \sum_{i=1,2,3}\lambda^{(i)}_{TTT}\mathcal{I}^{(i)}_{\mu\nu,\rho\sigma,\alpha\beta}, \tag{7.59}$$

for three tensor structures $\mathcal{I}^{(i)}_{\mu\nu,\rho\sigma,\alpha\beta}(x_j)$ determined by conservation and conformal symmetry. One way to parametrize these coefficients is in terms of (C_T, t_2, t_4), for further details and conventions see Appendix D.3. In particular, in holographic CFTs dual to semi-classical Einstein gravity it is known that $t_2 = t_4 = 0$ [24]. This fixes two of the coefficients, with the remaining one being fixed by Ward identities in terms of C_T [177].

Using the explicit form of the stress tensor conformal block in the OPE limit together with $t_2 = t_4 = 0$, we can find the explicit contribution of the stress tensor to $G_{\mu\nu,\rho\sigma}$, see Appendix D.3.1 for details. To compare to the corresponding bulk results we further need to integrate the correlator over the xy-plane. This is done in Appendix D.3.1 and we find:

$$\begin{aligned}G_{xy,xy}|_{\mu} &= \frac{\pi C_T\mu}{100}\frac{t^2-z^2}{(t^2+z^2)^2},\\ G_{tx,tx}|_{\mu} &= \frac{\pi C_T\mu}{800}\frac{-9t^4+6t^2z^2+7z^4}{(t^2+z^2)^3},\\ G_{tz,tz}|_{\mu} &= \frac{\pi C_T\mu}{3600}\frac{-105t^6+3t^4z^2+137t^2z^4+77z^6}{(t^2+z^2)^4}.\end{aligned} \tag{7.60}$$

[80] At zero temperature.

The result for $G_{xy,xy}$ in (7.60) agrees with (7.25). For the remaining polarizations we apply the relevant differential operators to find

$$(\partial_t^2 + \partial_z^2) G_{tx,tx}|_\mu = \frac{3\pi C_T \mu}{200} \frac{t^4 - 6t^2 z^2 + z^4}{(t^2 + z^2)^4} \tag{7.61}$$

and

$$(\partial_t^2 + \partial_z^2)^2 G_{tz,tz}|_\mu = -\frac{4\pi C_T \mu}{15} \frac{t^6 - 15t^4 z^2 + 15t^2 z^4 - z^6}{(t^2 + z^2)^6}. \tag{7.62}$$

Upon comparing (7.61) with (7.32) and (7.62) with (7.38) we find perfect agreement between the bulk and the CFT calculation.

7.3.5. Double stress tensor contributions

In this section we consider the contribution due to the double-stress tensor operators of the schematic form $:T_{\mu\nu}T_{\rho\sigma}:$, $:T_\mu{}^\rho T_{\rho\nu}:$ and $:T^{\rho\sigma}T_{\rho\sigma}:$. These are captured by (7.49) with Δ_J and $\rho_{i,j}$ given by (7.50). Details on the conformal blocks are given in Appendix D.3. At $\mathcal{O}(C_T^2\mu^2)$ we see from (7.51) that there are 6 undetermined coefficients $\rho_{i,J}^{(0)}$ and at $\mathcal{O}(C_T\mu^2)$ there is a total of 9 coefficients, in particular, the 6 coefficients $\rho_{i,J}^{(1)}$ and the 3 anomalous dimensions $\gamma_J^{(1)}$:

$$X = \{\rho_{1,0}^{(1)}, \rho_{1,2}^{(1)}, \rho_{2,2}^{(1)}, \rho_{1,4}^{(1)}, \rho_{2,4}^{(1)}, \rho_{3,4}^{(1)}, \gamma_0^{(1)}, \gamma_2^{(1)}, \gamma_4^{(1)}\}. \tag{7.63}$$

7.3.5.1. Disconnected part

As expected from thermalization, the $\mathcal{O}(C_T^2\mu^2)$ disconnected contribution to the stress tensor two-point function in the thermal states factorizes and is independent of the position x:

$$\hat{G}_{\mu\nu,\rho\sigma} = \langle T_{\mu\nu}\rangle_\beta \langle T_{\rho\sigma}\rangle_\beta (1 + \mathcal{O}(C_T^{-1})), \tag{7.64}$$

where β is the inverse temperature related to μ by (7.54). In particular, only the diagonal terms of $\langle T_{\mu\nu}\rangle_\beta$ are non-zero:

$$\begin{aligned} \hat{G}_{xy,xy} &= 0 + \mathcal{O}(C_T\mu^2), \\ \hat{G}_{tx,tx} &= 0 + \mathcal{O}(C_T\mu^2), \\ \hat{G}_{tz,tz} &= 0 + \mathcal{O}(C_T\mu^2), \end{aligned} \tag{7.65}$$

while

$$\hat{G}_{tt,tt} = \left(\frac{3}{4}\right)^2 \frac{b_{T_{\mu\nu}}^2}{\beta^8}\left[1+\mathcal{O}(C_T^{-1})\right]. \tag{7.66}$$

Comparing the conformal block expansion (7.49) to (7.65), we find that 5 out of 6 of the leading order coefficients $\rho_{i,J}^{(0)}$ are determined in terms of the remaining undetermined coefficient $\rho_{1,0}^{(0)}$:

$$\begin{aligned}
\rho_{1,2}^{(0)} &= \frac{324}{7}\rho_{1,0}^{(0)}, \\
\rho_{2,2}^{(0)} &= \frac{-1728}{7}\rho_{1,0}^{(0)}, \\
\rho_{1,4}^{(0)} &= \frac{160}{7}\rho_{1,0}^{(0)}, \\
\rho_{2,4}^{(0)} &= \frac{-1760}{7}\rho_{1,0}^{(0)}, \\
\rho_{3,4}^{(0)} &= \frac{-480}{7}\rho_{1,0}^{(0)}.
\end{aligned} \tag{7.67}$$

The remaining coefficient is fixed by imposing (7.66) which gives

$$\rho_{1,0}^{(0)} = \frac{\pi^4\mu^2 C_T^2}{480000}. \tag{7.68}$$

7.3.5.2. Corrections to double stress tensor CFT data

At $\mathcal{O}(C_T\mu^2)$ there is a total of 9 coefficients that fix $G_{\mu\nu,\rho\sigma}$. The goal of this section is to (partially) determine the CFT data X by comparing the conformal block decomposition at $\mathcal{O}(C_T\mu^2)$ to the bulk calculations. In particular, our analysis will allow us to extract the anomalous dimensions $\gamma_J^{(1)}$ of double-stress tensors $[T^2]_J$, $J = 0, 2, 4$.

In order to do so we again need to integrate the correlator over the (x, y) plane. This is divergent, as is manifest from dimensional analysis(see also(7.48)). We will tame this divergence by including a factor of $|x|^{-\epsilon}$ in the integrals which produces simple poles as $\epsilon \to 0$[81]. These will then be absorbed in the undetermined bulk coefficients, see Appendix D.3.6.

We will fix the CFT data by comparing the polarizations, $G_{xy,xy}$, $G_{tx,tx}$ and $G_{tz,tz}$, with the corresponding conformal block decomposition given in D.3.5,

[81] Alternatively, one can introduce an IR cutoff in the integrals and the result for the anomalous dimensions and the coefficients $\rho_{i,J}^{(1)}$ will remain the same.

with the bulk results given in (7.25), (7.32) and (7.38), respectively. For the latter two polarizations, we apply the differential operators $(\partial_t^2 + \partial_z^2)^p$, with $p = 1, 2$, on the OPE expansion in order to match against the bulk calculations, just as for the identity and stress tensor operator, which give

$$\begin{aligned}
G^{(CFT)}_{xy,xy} - G^{(bulk)}_{xy,xy}\Big|_{\mu^2 C_T} &= 0, \\
(\partial_t^2 + \partial_z^2)\left[G^{(CFT)}_{tx,tx} - G^{(bulk)}_{tx,tx}\right]\Big|_{\mu^2 C_T} &= 0, \\
(\partial_t^2 + \partial_z^2)^2\left[G^{(CFT)}_{tz,tz} - G^{(bulk)}_{tz,tz}\right]\Big|_{\mu^2 C_T} &= 0.
\end{aligned} \tag{7.69}$$

There is a common solution which unambiguously fixes the anomalous dimensions to the values:

$$\begin{aligned}
\gamma_0^{(1)} &= -\frac{2480}{63\pi^4}, \\
\gamma_2^{(1)} &= -\frac{4210}{189\pi^4}, \\
\gamma_4^{(1)} &= -\frac{1982}{35\pi^4},
\end{aligned} \tag{7.70}$$

where we note that the anomalous dimensions in (7.70) are all negative. Further, we find the following relations among three out of the six coefficients $\rho_{i,J}^{(1)}$

$$\begin{aligned}
\rho_{2,2}^{(1)} &= -\frac{14465}{1296\pi^4} + \rho_{1,2}^{(1)}, \\
\rho_{2,4}^{(1)} &= \frac{379}{210\pi^4} + \rho_{1,4}^{(1)}, \\
\rho_{3,4}^{(1)} &= \frac{3083}{1260\pi^4} + \rho_{1,4}^{(1)},
\end{aligned} \tag{7.71}$$

while the remaining CFT data $\{\rho_{1,0}^{(1)}, \rho_{1,2}^{(1)}, \rho_{1,4}^{(1)}\}$ is undetermined and the bulk coefficients are given in Appendix D.3.6. We have further checked that this solution is consistent with several other polarizations such as $G_{zx,zx}, G_{tx,zx}, G_{zz,zz}$ and $G_{tt,tt}$ by inserting (7.70), (7.71) and the solution for the a-coefficients Appendix D.3.6 in the OPE expansion and comparing to the explicit bulk calculations. Comparing $G_{xx,xx}$ from the CFT to the bulk calculation, one finds one more linearly independent equation [82]. The undetermined coefficients $\{\rho_{1,0}^{(1)}, \rho_{1,2}^{(1)}, \rho_{1,4}^{(1)}\}$ can then be expressed in terms of the undetermined bulk coefficients, see Eqs. (D.3.50) and (D.3.51).

[82] The reason for this can be seen from (7.36), when comparing to the CFT result we only apply a differential operator of degree 2 for the $G_{xx,xx}$ polarization compared to a degree 4 operator for other polarizations in the sound channel.

7.3.6. Lightcone limit

In this section we consider the lightcone limit which is obtained by Wick-rotating $t \to it$ and taking $v \to 0$, with $u = t - z$ and $v = t + z$. Imposing unitarity on the stress tensor contribution lead to the conformal collider bounds, see e.g. [24,178,25,26,179,180]. Consider now the lightcone limit of the double-stress tensor contribution. One finds the following result for the integrated correlators in the lightcone limit $v \to 0$:

$$\begin{aligned}
G^{(CFT)}_{xy,xy}(u,v)|_{\mu^2 C_T} &\underset{v\to 0}{\approx} \pi^5 \mu^2 C_T \frac{2\gamma_4^{(1)} - 41\rho_{1,4}^{(1)} + 11\rho_{2,4}^{(1)} + 30\rho_{3,4}^{(1)}}{48000}\frac{u^3}{v},\\
G^{(CFT)}_{tx,tx}(u,v)|_{\mu^2 C_T} &\underset{v\to 0}{\approx} \pi^5 \mu^2 C_T \frac{-113\gamma_4^{(1)} + 16(188\rho_{1,4}^{(1)} - 77\rho_{2,4}^{(1)} - 111\rho_{3,4}^{(1)})}{10752000}\frac{u^4}{v^2},\\
G^{(CFT)}_{tz,tz}(u,v)|_{\mu^2 C_T} &\underset{v\to 0}{\approx} \pi^5 \mu^2 C_T \frac{29\gamma_4^{(1)} - 740\rho_{1,4}^{(1)} + 308\rho_{2,4}^{(1)} + 432\rho_{3,4}^{(1)}}{16128000}\frac{u^5}{v^3},,
\end{aligned} \tag{7.72}$$

where as expected only the spin-4 operator of the schematic form $: T_{\mu\nu}T_{\rho\sigma} :$ contributes[83]. Inserting the solution (7.70)(7.71) we find

$$\begin{aligned}
G^{(CFT)}_{xy,xy}(u,v)|_{\mu^2 C_T} &\underset{v\to 0}{\approx} -\frac{\pi\mu^2 C_T}{2400}\frac{u^3}{v},\\
G^{(CFT)}_{tx,tx}(u,v)|_{\mu^2 C_T} &\underset{v\to 0}{\approx} -\frac{17\pi\mu^2 C_T}{1075200}\frac{u^4}{v^2},\\
G^{(CFT)}_{tz,tz}(u,v)|_{\mu^2 C_T} &\underset{v\to 0}{\approx} -\frac{11\pi\mu^2 C_T}{6048000}\frac{u^5}{v^3},
\end{aligned} \tag{7.73}$$

where we note that the undetermined coefficient $\rho_{1,4}^{(1)}$ drops out in the lightcone limit. The solution in (7.70) - (7.71) obtained from the bulk computations therefore determines completely the lightcone limit of the correlator to this order.

7.4. Discussion

We have examined the thermal two-point function of stress tensors in holographic CFTs. In the dual picture, this corresponds to studying metric perturbations around a black hole background. The thermal two-point function

[83] We have dropped the divergent terms from the integration since does not contain negative powers of v when $v \to 0$.

can be decomposed into contributions of individual operators using the OPE. Important contributions to the OPE of two stress-tensors include the identity operator, the stress tensor itself, and composite operators made out of the stress tensor (multi-stress tensors).

The holographic contribution of the identity reproduces the vacuum result. We also verify that the stress-tensor contribution to the holographic TT correlator agrees with the CFT result, which is fixed by the three-point functions of the stress-tensor in CFTs dual to Einstein gravity (our CFT result agrees with [178]). The leading contribution from the double-stress tensors corresponds to the disconnected part of the correlator.

The anomalous dimensions and the corrections to the OPE coefficients and thermal one-point functions contribute at next-to-leading order in the C_T^{-1} expansion. Comparing the CFT and holographic calculations, we are able to read off the anomalous dimensions of the double-stress tensors with spin $J = 0, 2, 4$ and obtain partial relations for the subleading corrections to the products of OPE coefficients and thermal one-point functions. It would be interesting to compare our results with the one-loop results of [181,182,183,184].

We are unable to fully determine the double-stress tensor contribution from the near-boundary analysis in the bulk; indeed some OPE coefficients remain unfixed, although the leading lightcone behavior of the TT correlators at this order is completely determined. The situation is reminiscent of the scalar case [102], where the contributions of double-trace operators of external scalars were not determined by the near-boundary analysis. It would be interesting to go beyond the near-boundary expansion to further determine this remaining data. In contrast to the scalar case considered in [102], in our analysis we further integrated the correlator over a plane to account for different polarizations of the stress tensor. This feature introduces some technical complications and it would be interesting to study the correlator without integration.

Holography provides a powerful tool to study hydrodynamics of strongly coupled quantum field theories and transport coefficients can be read off from the stress tensor two-point function at finite temperature[84]. The conformal bootstrap provides another window into strongly coupled phenomena

[84] The expansion in small momenta compared to the temperature is opposite of the OPE limit and interpolating between the two is challenging. See e.g. [185-193] for recent work on the convergence of the hydrodynamic expansion.

when perturbation theory is not applicable. While the bootstrap program for vacuum correlators has led to significant developments in the past decade, the corresponding tools for thermal correlators are still developing, see e.g. [194,56,60,57,59,61,62,7,64,65] for related work. In particular, due to an important role played by the stress tensor thermal two-point function, it would be interesting to better understand the constraints imposed by the conformal bootstrap on this correlator as well as the implications for a gravitational dual description.

By the nature of a duality, there are two sides to the same story. We have used the structure of the stress-tensor two-point functions at finite temperature, imposed by conformal symmetry, in order to read off the CFT data by making a comparison to the corresponding calculations in the bulk. At the same time, it would be very interesting to study properties of black holes in AdS by bootstrapping thermal correlators on the boundary. We expect a major role to be played by the stress tensor operator and its composites which are related to the metric degrees of freedom in the bulk.

8. Conclusions and discussion

In this thesis, we have explored some aspects of the conformal bootstrap program in the context of holographic CFTs. In particular to the study of heavy-heavy-light-light correlators, where the heavy operators create high-energy eigenstates which for many observables are expected to thermalize. This was mainly achieved by studying the correlator in two different kinematical limits common in the bootstrap literature. The first one is the lightcone limit which in one channel isolates operators with low-twist. In our case, this is typically the stress tensor and the multi-stress tensor operators. The second one is the Regge limit which, in this setup, is dual to a highly energetic probe particle propagating in an AdS-Schwarzschild black hole background. The physical data in this limit is captured by the Shapiro time delay and the angle deflection of a null geodesic in this background. This can be calculated and used to extract information about the CFT correlator or, when available, be compared against expectations from the CFT obtained by bootstrap methods. Lastly, we studied the thermal two-point functions of stress tensors in holographic CFTs. This was done by solving the equations of motions, in a near-boundary expansion, for metric fluctuations around the black hole background in the bulk. By decomposing the resulting correlators in terms of (spinning) conformal blocks, we read off the underlying CFT data.

In Section 3, we used the conformal bootstrap in the lightcone limit to obtain the contribution due to minimal-twist multi-stress tensors. This lead to, among other things, the OPE coefficients for multi-stress tensors in the OPE of two light scalar operators. Some of these have been calculated in the bulk and are in agreement with the results obtained from the boundary point of view. One of the key features of the minimal-twist multi-stress tensor exchanges is that it takes a remarkable, somewhat, simple form analogous to the two-dimensional Viraosoro vacuum block. There this form is a consequence of the infinite-dimensional Virasoro algebra. An interesting open problem is to understand if there is a similar, emergent, symmetry algebra in the lightcone limit of heavy-heavy-light-light correlators in higher-dimensional holographic CFTs. In Section 4, this was explored by studying two-dimensional CFTs with a higher-spin symmetry algebra. In the case of an additional spin-3 current, the result is reminiscent of the four-dimensional counter-part discussed above. It would be interesting to understand this connection explicitly from first principles.

In Section 4 and 5, we studied the Regge limit of the heavy-heavy-light-light correlators. Partly using bootstrap techniques that led to agreement with expectations from the bulk, and partly by extracting information about the CFT data using known results from the bulk. The Regge limit plays an important role in the conformal bootstrap and exploring it more in this context is interesting. For example, there is a critical impact parameter for which the probe particle gets trapped by the black hole. A slight change in the impact parameter leads to significant change in the behavior of the correlator. Understanding this from the CFT point of view would be interesting.

The stress tensor correlator at finite temperature plays an important role in CFTs, as well as in holography. By studying metric fluctuations around a black hole background, dual to a finite temperature state in the CFT, we were able in Section 7 to obtain this correlator order-by-order in the OPE expansion. Furthermore, by applying the machinery of spinning conformal blocks, we could read off the underlying CFT data. While the OPE limit is opposite of the hydrodynamical limit, it would be interesting to understand what the conformal bootstrap has to say about hydrodynamics.

The stress tensor OPE occupies a central role in the AdS/CFT correspondence. In conformal field theories dual to semi-classical Einstein gravity, its vacuum correlation functions are determined by fluctuations of the Einstein-Hilbert action with a cosmological constant around pure AdS. The absence of higher-spin fields and causality implies that corrections are suppressed by the gap. On the other hand, the black hole background is a solution to the non-linear Einstein's equations. A holographic CFT in this sense should reproduce not only vacuum correlators, but also correlation functions in states dual to other semi-classical solutions to the bulk equations of motions reflecting the full non-linear structure of the bulk gravity. This is an interesting avenue to explore; in this thesis we hope to have made some steps in this direction that can be further built upon.

Appendix A.1. Some details on the calculation of the $\mathcal{W}_3$ block

We now make explicit the contribution of the operator $\mathcal{O}$ to the commutator $[W_m, \mathcal{O}_{h+j}(z)]$. To this end, consider the OPE between two quasiprimaries $\phi_i(z_1) \times \phi_j(z_2)|_{\phi^k}$:

$$\phi_i(z_1) \times \phi_j(z_2)|_{\phi^k} = \lambda_{ijk} \sum_{p=0}^{\infty} \frac{a_p(h_i, h_j, h_k)}{p!} \frac{\partial_{z_2}^p \phi^k(z_2)}{(z_1 - z_2)^{h_i + h_j - h_k - p}}, \tag{A.1.1}$$

where $a_p(h_i, h_j, h_k) = (h_i - h_j + h_k)_p (2h_k)_p^{-1}$. Setting $\phi_i(z_1) = W(z_1)$, $\phi_j = \mathcal{O}_{h+j}(z_2)$, $\phi^k = \mathcal{O}$ and integrating against $\int_{\mathcal{C}(z_2)} \frac{dz_1}{2\pi i} z_1^{m+2} W(z_1) \mathcal{O}_{h+j}(z_2)$ we find that

$$[W_m, \mathcal{O}_{h+j}(z_2)]|_{\mathcal{O}} = \lambda_{W\mathcal{O}_{h+j}\mathcal{O}} \int_{\mathcal{C}(z_2)} \frac{dz_1}{2\pi i} z_1^{m+2} \sum_{n=0}^{j+2} \frac{a_p(3, h+j, h) \partial_{z_2}^p \mathcal{O}(z_2)}{(z_1 - z_2)^{3+j-p} p!}, \tag{A.1.2}$$

and performing the integral we find that

$$[W_m, \mathcal{O}_{h+j}(z_2)]|_{\mathcal{O}} = \lambda_{W\mathcal{O}_{h+j}\mathcal{O}} \sum_{p=0}^{j+2} \frac{a_p(3, h+j, h)(m+2)!}{(m+p-j)!(j+2-p)!p!} z_2^{m+n+p-j} \partial_{z_2}^p \mathcal{O}(z_2). \tag{A.1.3}$$

A.1.1. Mixed states $W_{-n} L_{-m}|0\rangle$

We now consider the following states

$$|A_{m,n}\rangle = L_{-m} W_{-n}|0\rangle - \frac{\langle W_{m+n} L_{-m} W_{-n}\rangle}{\langle W_{n+m} W_{-n-m}\rangle} W_{-m-n}|0\rangle, \tag{A.1.4}$$

where (for $c \to \infty$)

$$\begin{aligned} \langle W_{m+n} L_{-m} W_{-n}\rangle &= (3m+n) \frac{c}{360} n(n^2-1)(n^2-4), \\ \langle W_{n+m} W_{-n-m}\rangle &= \frac{c}{360}(m+n)((m+n)^2-1)((m+n)^2-4), \\ \langle W_n L_m L_{-m} W_{-n}\rangle &= \frac{c^2}{12 \times 360} n(n^2-1)(n^2-4)m(m^2-1). \end{aligned} \tag{A.1.5}$$

Now, one finds that $\langle A_{m,n}|\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle$

$$\begin{aligned} \langle A_{m,n}|\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle &= \mathcal{D}_{L,m}\mathcal{D}_{W,n}\langle \mathcal{O}_L(z)\mathcal{O}_L(0)\rangle \\ &\quad - \frac{\langle W_{m+n} L_{-m} W_{-n}\rangle}{\langle W_{n+m} W_{-n-m}\rangle} \mathcal{D}_{W,m+n}\langle \mathcal{O}_L(z)\mathcal{O}_L(0)\rangle \\ &= \frac{1}{2}(m-1)(n-1)(n-2) w h z^{m+n-2h} + \\ &\quad + \frac{(m-1)m(n-2)(n-1)n(4+m+3n)}{2(m+n)(m+n+1)(m+n+2)} w z^{m+n-2h}, \end{aligned} \tag{A.1.6}$$

where

$$
\begin{aligned}
[Q_m^{(N)}, \mathcal{O}_{h,q^{(N)}}(z)]|_{\mathcal{O}_{h,q^{(N)}}} &= q^{(N)} \int_{\mathcal{C}(z)} \frac{dz_1}{2\pi i} z_1^{m+N-1} \\
&\quad \times \sum_{p=0}^{N-1} \frac{a_p(N,h,h)}{(z_1-z)^{N-p} p!} \partial_z^p \mathcal{O}_{h,q^{(N)}}(z) \\
&= q^{(N)} \sum_{p=0}^{N-1} \frac{a_p(N,h,h)}{p!} \frac{(m+N-1)!}{(N-p-1)!(m+p)!} z^{m+p} \partial_z^p \mathcal{O}_{h,q^{(N)}}(z) \\
&:= \mathcal{D}_{N,m} \mathcal{O}_{h,q^{(N)}}(z).
\end{aligned}
\tag{A.1.7}
$$

For the heavy part, we keep only the quadratic part in the charges such that

$$
\lim_{z_4 \to \infty} z_4^{2H} \langle \mathcal{O}_H(z_4) \mathcal{O}_H(1) | A_{m,n} \rangle = \frac{1}{2}(m-1)(n-2)(n-1) w_H H. \tag{A.1.8}
$$

Multiplying (A.1.6) with (A.1.8) and dividing by the norm $\langle W_n L_m L_{-m} W_{-n} \rangle$ in (A.1.5), we find that

$$
\begin{aligned}
&\sum_{m,n=2}^{\infty} \lim_{z_4 \to \infty} z_4^{2h_H} \frac{\langle \mathcal{O}_H(z_4) \mathcal{O}_H(1) | A_{m,n} \rangle \langle A_{m,n} | \mathcal{O}_L(z) \mathcal{O}_L(0) \rangle}{\langle W_n L_m L_{-m} W_{-n} \rangle} \Big|_{\frac{w_H H w h}{c^2}} \\
&= \frac{1080 w_H H w h}{c^2} z^{-2h} \sum_{m,n=2}^{\infty} \frac{(m-1)(n-1)(n-2)}{(m+1)(n+1)(n+2)} \frac{z^{m+n}}{mn} \\
&= \frac{6 w_H H w h}{c^2} f_2 f_3,
\end{aligned}
\tag{A.1.9}
$$

which as expected is the “exponentiated term”. On the other hand, consider

$$
\begin{aligned}
&\sum_{m,n=2}^{\infty} \lim_{z_4 \to \infty} z_4^{2H} \frac{\langle \mathcal{O}_H(z_4) \mathcal{O}_H(1) | A_{m,n} \rangle \langle A_{m,n} | \mathcal{O}_L(z) \mathcal{O}_L(0) \rangle}{\langle W_n L_m L_{-m} W_{-n} \rangle} \Big|_{\frac{w_H H w}{c^2}} \\
&= \frac{1080 w_H H w}{c^2} z^{-2h} \\
&\times \sum_{m,n=2}^{\infty} \frac{(m-1)(n-1)(n-2)}{(m+1)(n+1)(n+2)} \frac{(4+m+3n) z^{m+n}}{(m+n)(m+n+1)(m+n+2)} \\
&\propto \frac{w_H H w}{c^2} (f_1 f_4 - \frac{7}{9} f_2 f_3).
\end{aligned}
\tag{A.1.10}
$$

Note that in both sums we have trivially extended the summation from $m \geqslant 3$ to $m \geqslant 2$.

On the other hand, by expanding the vacuum block we find precisely the same structure

$$\begin{aligned}
&\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle\big|_{1_{\mathcal{W}_3},\frac{w_H H w h}{c^2}} \propto f_2 f_3, \\
&\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)\mathcal{O}_L(z)\mathcal{O}_L(0)\rangle\big|_{1_{\mathcal{W}_3},\frac{w_H H w}{c^2}} \propto (f_1 f_4 - \frac{7}{9} f_2 f_3).
\end{aligned} \tag{A.1.11}$$

Appendix A.2. $\mathcal{W}_4$ vacuum block

In this appendix we further include a spin-4 current and consider the $\mathcal{W}_4$ algebra. We will show that including a spin-4 current modifies the term proportional to $\frac{w_H^2}{c^2}$ discussed in Section 4. The result can again be written as a sums of the following combination $f_a(z)f_b(z)$, with $a+b=6$. Compared to the case of $\mathcal{W}_3$, the term proportional to $\frac{w_H^2}{c^2}$ in the vacuum block will now depend also on the spin-4 charge u of the light operator.

We denote the spin-4 current by $U(z)$ and the external operators carry eigenvalues $\pm u_H$ and $\pm u$. The heavy operator again has a spin-3 charge of $\mathcal{O}(c)$ while the conformal weight H and the spin-4 charge are small compared to w_H, i.e. $H, u_H \ll w_H$. In this limit, there are no new contributions due to the states $U_{-m}|0\rangle$ since they will be proportional to $\frac{u_H u}{c} f_4 z^{-2h}$, which is suppressed as $c \to \infty$. The first contribution will appear at $\mathcal{O}(\frac{w_H^2}{c^2})$ and is due to the fact that the modes $|Y_{m,n}\rangle$ are not orthogonal to $U_{-m-n}|0\rangle$. In this section we will therefore study the contribution due to the following states:

$$|\tilde{Y}_{m,n}\rangle = \Big[W_{-n}W_{-m} - \frac{\langle L_{m+n}W_{-n}W_{-m}\rangle}{\langle L_{m+n}L_{-m-n}\rangle} L_{-m-n} - \frac{\langle U_{m+n}W_{-n}W_{-m}\rangle}{\langle U_{m+n}U_{-m-n}\rangle} U_{-m-n}\Big]|0\rangle. \tag{A.2.1}$$

There are two new contributions to $\langle \tilde{Y}_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle$ compared to $\langle Y_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle$, one is simply that we need to include the last term in (A.2.1). The second is a correction to the OPE coefficients $\lambda_{W\mathcal{O}_{h+1}\mathcal{O}}$ and $\lambda_{W\mathcal{O}_{h+2}\mathcal{O}}$, these pick up a contribution that depends on the spin-4 charge u due to the fact that $[W_m, W_{-m}]$ contain the spin-4 zero mode U_0. Note that the heavy part remains unchanged since $w_H \gg u_H$ and is therefore given by (4.31):

$$\langle \mathcal{O}_H(\infty)\mathcal{O}_H(1)|\tilde{Y}_{m,n}\rangle = \frac{w_H^2}{4}(m-1)(m-2)(n-1)(n-2), \tag{A.2.2}$$

and the norm of $|\tilde{Y}_{m,n}\rangle$ is also the same as that of $|Y_{m,n}\rangle$ (to leading order in c):

$$\mathcal{N}_{\tilde{Y}_{m,n}} = \langle \tilde{Y}_{m,n}|\tilde{Y}_{m,n}\rangle = (\frac{c}{360})^2 m(m^2-1)(m^2-4)n(n^2-1)(n^2-4). \quad \text{(A.2.3)}$$

We therefore only need to calculate $\langle \tilde{Y}_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle$.

The modes U_m of $U(z)$ are defined by

$$U(z) = \sum_m U_m z^{-m-4}, \quad \text{(A.2.4)}$$

and since U is primary we know that

$$[L_m, U_n] = (3m-n)U_{m+n}. \quad \text{(A.2.5)}$$

Consider now various OPEs of the spin-3 and spin-4 field [85], in terms of quasi-primaries

$$\begin{aligned} W(z)W(0) &= \frac{c}{3z^6} + \frac{2T(0)}{z^4} + \frac{\lambda_{WWU}U(0)}{z^2} + \dots, \\ W(z)U(0) &= \frac{\lambda_{WUW}W(0)}{z^4} + \dots, \\ U(z)U(0) &= \frac{c}{4z^8} + \frac{2T(0)}{z^6} + \lambda_{UUU}\frac{U(0)}{z^4} + \dots, \end{aligned} \quad \text{(A.2.6)}$$

where $\lambda_{WUW} = \frac{3}{4}\lambda_{WWU} = \frac{3}{4}\frac{4}{\sqrt{3}}\sqrt{\frac{(2+c)(114+7c)}{(7+c)(22+5c)}} \approx \sqrt{\frac{21}{5}}$ and the ellipses denote non-linear terms that will be suppressed when $c \to \infty$. From (A.2.6), we can derive the commutator of the various modes. Especially, we want to consider $[W_n, U_m]$, $[W_n, W_m]$ and $[U_n, U_m]$. The last one is given by

$$\begin{aligned} [U_m, U_n] = &\frac{c}{20160} m(m^2-1)(m^2-4)(m^2-9)\delta_{m+n} \\ &+ \frac{(m-n)}{1680}\Big[3(m^4+n^4) + 4m^2n^2 - (2mn+39)(m^2+n^2) \\ &+ 20mn + 108\Big]L_{m+n} + \dots, \end{aligned} \quad \text{(A.2.7)}$$

while $[W_n, W_m]|_U$ is given by

$$[W_m, W_n]|_U = \lambda_{WWU}\frac{m-n}{2}U_{n+m}, \quad \text{(A.2.8)}$$

[85] See e.g. App A.2 in [195] for the $\mathcal{W}_4$ algebra.

as well as

$$[W_m, U_n]|_W = \frac{\lambda_{WUW}}{84}\Big[5m^3 + 9n - 5m^2 n - n^3 - 17m + 3mn^2)\Big]W_{m+n}. \quad \text{(A.2.9)}$$

Using (A.2.8) and (A.2.9), we find that

$$\begin{aligned}\langle U_{m+n}W_{-n}W_{-m}\rangle = &\frac{\lambda_{WUW}cm(m^2-1)(m^2-4)}{30240}\\ &\times\Big[-9m + m^3 - 26n + 6m^2 n + 14mn^2 + 14n^3\Big]\end{aligned} \quad \text{(A.2.10)}$$

and

$$\langle U_{m+n}U_{-m-n}\rangle = \frac{c}{20160}s(s^2-1)(s^2-4)(s^2-9). \quad \text{(A.2.11)}$$

From the three-point function $\langle U(z_3)\mathcal{O}(z)\mathcal{O}(0)\rangle$ and $\lambda_{U\mathcal{O}\mathcal{O}} = u$ one finds that

$$\langle U_{m+n}\mathcal{O}(z)\mathcal{O}(0)\rangle = \frac{u}{6}(m+n-1)(m+n-2)(m+n-3)z^{m+n-2h}. \quad \text{(A.2.12)}$$

Lastly, we need to compute the corrections to the OPE coefficients $\lambda_{W\mathcal{O}_{h+1}\mathcal{O}}$ and $\lambda_{W\mathcal{O}_{h+2}\mathcal{O}}$. This is similar to the calculation in the $\mathcal{W}_3$ case and one finds that ($c\to\infty$, $z\to 0$)

$$\begin{aligned}\langle\mathcal{O}(z_3)W(z)\mathcal{O}_{h+1}(0)\rangle &\approx z^{-4}\langle\mathcal{O}(z_3)W_1(W_{-1} - \frac{3w}{2h}L_{-1})\mathcal{O}(0)\rangle\\ &= z^{-4}z_3^{-2h}\Big[-\frac{h}{5} + \lambda_{WWU}u - \frac{9w^2}{2h}\Big],\end{aligned} \quad \text{(A.2.13)}$$

where we used $[W_1, W_{-1}] = \ldots + \lambda_{WWU}U_0$ and that $U_0\mathcal{O}(0)|0\rangle = u\mathcal{O}|0\rangle$. Likewise, one finds that

$$\begin{aligned}&\langle\mathcal{O}(z_3)W(z)\mathcal{O}_{h+2}(0)\rangle \approx\\ &z^{-5}\langle\mathcal{O}_h(z_3)W_2(W_{-2} - \frac{2}{h+1}L_{-1}W_{-1} + \Big[\frac{3w}{h(h+1)} - \frac{3w}{h(2h+1)}\Big]L_{-1}^2)\mathcal{O}(0)\rangle\\ &= z^{-5}z_3^{-2h}\Big[\frac{8h}{5} + \frac{8h}{5(h+1)} + \frac{36w^2}{(2h+1)(h+1)} + 2\lambda_{WWU}u - \frac{8u}{h+1}\lambda_{WWU}\Big],\end{aligned} \quad \text{(A.2.14)}$$

to leading order when $c\to\infty$ and using $[W_2, W_{-2}]|_U = \ldots + 2U_0$. Putting this altogether gives

$$\begin{aligned}\langle\tilde{Y}_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle = &\langle Y_{m,n}|\mathcal{O}(z)\mathcal{O}(0)\rangle\\ &+\frac{u\lambda_{WWU}(m-2)(m-1)m(n-2)(n-1)nz^{m+n-2h}}{12s(s+1)(s+2)(s+3)}\\ &\times(17 + 2m^2 + 15n + 2n^2 + 15m + 9mn) + \ldots.\end{aligned} \quad \text{(A.2.15)}$$

Given (A.2.15), (A.2.2) and (A.2.3), we find the contribution to the vacuum block from the states $|\tilde{Y}_{m,n}\rangle$ proportional to u is given by

$$\mathcal{G}(z)|_{\frac{w_H^2 u}{c^2}} = \frac{37800 w_H^2 u \lambda_{WWU} z^{-2h}}{c^2}\Big[25\tilde{w}_4(z) + 3w_3(z)\Big], \tag{A.2.16}$$

where w_3 is given by (4.35) and $\tilde{w}_4$ is a sum of products of functions $f_a f_b$ with $a+b=6$ given by

$$\tilde{w}_4 = 3(-f_2 f_4 + \frac{4}{3} f_1 f_5) = \sum_{m=3}^{\infty}\sum_{n=3}^{\infty} 1260 \frac{(m-2)(n-2)(n-1)n(m^2+6(n+2)(n+3)+m(9+4n))}{m(n+2)(n+3)(n+4)s(s+1)(s+2)(s+3)} z^{m+n}. \tag{A.2.17}$$

A.2.1. Differential equation for the $\mathcal{W}_4$ vacuum block

Here we study the $\mathcal{W}_4$ vacuum block, or rather its logarithm, as $z \to 1$. The $\mathcal{W}_4$ HHLL vacuum block is known exactly. One can find it for instance in eq. (C.1) of [113]. In this case, we can choose to scale the spin-3 charge w_H with the central charge c – as in Appendix A.1 – with the hope of uncovering relations similar to those valid for the stress-tensor sector of the four-dimensional correlator in the light cone limit. However, we may also choose to consider the limit $u_H \sim c \gg 1$, with all other charges parametrically smaller.

Remarkably, $\mathcal{F}_4(z)$ behaves logarithmically in the limit $z \to 1$ in both cases. A sequence of numbers, the numerical coefficients of $\log(1-z)$ in the expansion of the relevant heavy charge can be determined, and a quartic differential equation satisfied by the logarithm of the block for certain ratios of the light charges can be found.

Let us first consider the scaling $u_H \sim c \gg 1$ and expand $\mathcal{F}_4(z) = \log \mathcal{G}_4(z)$ in powers of u_H/c as $\mathcal{F}_4(z) = \sum_{k=0}^{\infty} \left(\frac{u_H}{c}\right)^k \mathcal{F}_4^{(k)}(z)$ to obtain in the limit $z \to 1$:

$$\left\{ \lim_{z\to 1}\left(-\frac{\mathcal{F}^{(k)}(z)}{20\times 6^k \log(1-z)}\right)\middle| k=1,2,\dots \right\} = \\ = u \times \big\{1, n-7, 458-14n, 1001n-13307, 732374-34034n, \\ 1939938n - 31667622, \dots\big\}, \tag{A.2.18}$$

where we set

$$n = \frac{18}{5}\frac{h}{u}. \tag{A.2.19}$$

If $B_4(x,n)$ with $x \equiv 6\frac{u_H}{c}$ is the generating function of (A.2.18), then $\mathcal{F}(z)$ behaves in the limit $z \to 1$ as

$$\mathcal{F}_4(z) \underset{z\to 1}{\approx} -20u \log(1-z) B_4(x,n) \tag{A.2.20}$$

There exist four different values of n for which the generating function $B_4(x,n)$ satisfies a quartic equation. These are: $n = \{18, 3, -2, -12\}$.

When $n = 18$, we find the following quartic order equation for the generating function:

$$B_4(x,18) = 36B_4(x,18)^4 - 36B_4(x,18)^3 + 11B_4(x,18)^2 + x. \tag{A.2.21}$$

Inspired by this relation one finds that $\mathcal{F}_4(z, n = 18) \equiv \widetilde{\mathcal{F}}_4(z)$ satisfies the following differential equation

$$\begin{aligned} \widetilde{\mathcal{F}}''''(z) = 120u\Bigg(& \frac{x}{(1-z)^4} + \frac{9\widetilde{\mathcal{F}}'(z)^4}{40000u^4} - \frac{9\widetilde{\mathcal{F}}'(z)^2\widetilde{\mathcal{F}}''(z)}{2000u^3} \\ & + \frac{3\widetilde{\mathcal{F}}''(z)^2 + 4\widetilde{\mathcal{F}}'''(z)\widetilde{\mathcal{F}}'(z)}{400u^2}\Bigg), \end{aligned} \tag{A.2.22}$$

which reduces to the equation (A.2.21) in the limit $z \to 1$ using (A.2.20).

When $n = -12$ the generating function $B_4(x,-12)$ satisfies

$$B_4(x,-12) = -144B_4(x,-12)^4 - 96B_4(x,-12)^3 - 19B_4(x,-12)^2 + x, \tag{A.2.23}$$

whilst $\mathcal{F}_4(z, n = -12) \equiv \hat{\mathcal{F}}(z)$ is a solution of the following differential equation

$$\begin{aligned} \hat{\mathcal{F}}''''(z) = 120u\Bigg(& \frac{x}{(1-z)^4} - \frac{9\hat{\mathcal{F}}'(z)^4}{10000u^4} - \frac{3\hat{\mathcal{F}}'(z)^2\hat{\mathcal{F}}''(z)}{250u^3} \\ & - \frac{7\hat{\mathcal{F}}''(z)^2 + 6\hat{\mathcal{F}}'''(z)\hat{\mathcal{F}}'(z)}{400u^2}\Bigg). \end{aligned} \tag{A.2.24}$$

For $n = -2, 3$ we find the following quartic order equations for the generating function:

$$\begin{aligned} n = 3, \qquad & B_4(x,3) = -2304B_4(x,3)^4 + 384B_4(x,3)^3 - 4B_4(x,3)^2 + x, \\ n = -2, \qquad & B_4(x,-2) = 2916B_4(x,-2)^4 + 324B_4(x,-2)^3 - 9B_4(x,-2)^2 + x. \end{aligned} \tag{A.2.25}$$

In these cases however, the differential equations similarly constructed do not correctly reproduce the vacuum block beyond $z \to 1$ limit. This is analogous to what happens in the case of the $\mathcal{W}_3$ vacuum block for $h = 0$, where the generating function satisfies

$$n = 0, \qquad B_3(x,0) = 16B_3(x,0)^3 + x. \tag{A.2.26}$$

It is curious that these special cases correspond to values for the ratios of the light charges for which $h < w, u$.

Let us now consider the case with $w_H \sim c \gg 1$ and the other charges parametrically smaller. For notational simplicity, we will use here the same symbol $\mathcal{F}_4(z)$. We hope that this will not create any confusion. In this case, $\mathcal{F}_4(z)$ is expanded as

$$\mathcal{F}_4(z) = \sum_{k=0}^{\infty} \left(\frac{w_H}{c}\right)^k \mathcal{F}^{(k)}(z), \tag{A.2.27}$$

with

$$\mathcal{F}_4^{(0)} = -2h \log(z). \tag{A.2.28}$$

Using the exact expression for the $\mathcal{W}_4$ block one finds that

$$\begin{aligned} &\left\{ \lim_{z\to 1} \left(\frac{(-1)^{k+1}\mathcal{F}_4^{(k)}(z)}{2^{k+1}3^{2k}\log(1-z)} \right) \Big| k = 1, 2, \dots \right\} = \\ &= w \times \Big\{1, \frac{2}{45}(18n + 85m), 10, \frac{2}{81}(882n + 2785m), 318, \\ &\qquad \frac{44}{3645}(67158n + 225635m), 13620, \dots \Big\}, \end{aligned} \tag{A.2.29}$$

where n, m denote the ratios of the light charges $n = \frac{h}{w}$ and $m = \frac{u}{w}$, respectively. Notice that in this case ratios of both charges appear as opposed to the previous scaling for which additional simplifications occurred that eliminated w. This may be related to the fact that a spin-3 current, having odd spin, does not appear in the OPE of two spin-4 currents.

Appendix B.1. Details on the conformal bootstrap

Below we review some of the details of the conformal bootstrap calculations. Explicitly, we will show that exchanges of heavy-light double-trace operators in the S-channel reproduce the disconnected correlator at $\mathcal{O}(\mu^0)$ and the stress tensor exchange at $\mathcal{O}(\mu)$.

B.1.1. Solving the crossing equation to $\mathcal{O}(\mu)$ in $d = 4$

We start with the leading $\mathcal{O}(\mu^0)$ term in the S-channel that should reproduce the disconnected propagator in the T-channel. This is given in $d = 4$ by

$$G(z,\bar{z})|_{\mu^0} = \frac{C_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L - 2}(h-\bar{h})(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}). \quad \text{(B.1.1)}$$

Let us look at the following piece of (B.1.1):

$$\begin{aligned} -\int_0^\infty dh \int_0^h d\bar{h}(h\bar{h})^{\Delta_L-2}(h-\bar{h})z^{\bar{h}}\bar{z}^{h+1} &= -\int_0^\infty d\bar{h}\int_{\bar{h}}^\infty dh (h\bar{h})^{\Delta_L-2}(h-\bar{h})z^{\bar{h}}\bar{z}^{h+1} \\ = \frac{\bar{z}}{z}\int_0^\infty dh \int_h^\infty d\bar{h}(h\bar{h})^{\Delta_L-2}(h-\bar{h})z^{h+1}\bar{z}^{\bar{h}} \,. \end{aligned} \quad \text{(B.1.2)}$$

Setting $\bar{z}/z = 1$ to leading order in the Regge limit, we find that the S-channel expression reproduces the disconnected correlator:

$$\begin{aligned} G(z,\bar{z})|_{\mu^0} &= \frac{zC_{\Delta_L}}{z-\bar{z}} \int_0^\infty dh \int_0^\infty d\bar{h}(h\bar{h})^{\Delta_L-2}(h-\bar{h})z^h z^{\bar{h}} \\ &= \frac{zC_{\Delta_L}}{z-\bar{z}} \frac{(\log\bar{z} - \log z)}{(\log z \log \bar{z})^{\Delta_L}} \Gamma(\Delta_L)\Gamma(\Delta_L - 1) \simeq \frac{1}{(1-z)^{\Delta_L}(1-\bar{z})^{\Delta_L}} \,. \end{aligned} \quad \text{(B.1.3)}$$

Notice that to arrive in the last equality we expanded $(z, \bar{z})$ around unity and substituted $C_{\Delta_L} = (\Gamma(\Delta_L)\Gamma(\Delta_L - 1))^{-1}$.

Consider now the imaginary part at $\mathcal{O}(\mu)$ in the S-channel. For convenience we define

$$I^{(d=4)} \equiv \mathrm{Im}(G(z,\bar{z}))|_\mu \,, \quad \text{(B.1.4)}$$

which is then equal to:

$$\begin{aligned} I^{(d=4)} &= \frac{-i\pi C_{\Delta_L}}{\sigma(e^{-\rho} - e^{\rho})} \times \\ &\times \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta_L-2}(h-\bar{h})\gamma(h,\bar{h})\Big((1-\sigma e^{\rho})^{h+1}(1-\sigma e^{-\rho})^{\bar{h}} - (h \leftrightarrow \bar{h})\Big) \,. \end{aligned} \quad \text{(B.1.5)}$$

Notice that we used the variables (σ, ρ) defined as $z = 1-\sigma e^{\rho}$ and $\bar{z} = 1-\sigma e^{-\rho}$.

Consider the following ansatz for $\gamma = \frac{ch^a\bar{h}^b}{h-\bar{h}}$, where (a, b, c) are numbers tobe determined by the crossing equation. Substituting into (B.1.5) and collecting the leading singularity σ^{-k} as $\sigma \to 0$ with $k = 2\Delta_L + a + b - 1$ leads to

$$
\begin{aligned}
I^{(d=4)}|_{\sigma^{-k}} =& \frac{-ic\pi C_{\Delta_L}}{(e^{-\rho}-e^{\rho})}\Big(\Gamma(\Delta_L+a-1)\Gamma(\Delta_L+b-1)(e^{(b-a)\rho}-e^{(a-b)\rho})+ \\
&+\frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+a-1}e^{-(2\Delta_L+a+b-2)\rho}\times \\
&\times {}_2F_1(\Delta_L+a-1, 2\Delta_L+a+b-2, \Delta_L+a, -e^{-2\rho}) - \frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+a-1} \\
&\times e^{(2\Delta_L+a+b-2)\rho}{}_2F_1(\Delta_L+a-1, 2\Delta_L+a+b-2, \Delta_L+a, -e^{2\rho})\Big).
\end{aligned} \tag{B.1.6}
$$

Note that in order to do these integrals we need $\Delta_L + a > 1$ and $\Delta_L + b > 1$. Using the following identity of the hypergeometric function

$$
\begin{aligned}
{}_2F_1(a,b,c,x) =& \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}{}_2F_1(a, a-c+1, a-b+1, \frac{1}{x}) \\
&+\frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}{}_2F_1(b, b-c+1, -a+b+1, \frac{1}{x}),
\end{aligned} \tag{B.1.7}
$$

the third line in (B.1.6) can be simplified and we are left with

$$
\begin{aligned}
I^{(d=4)}|_{\sigma^{-k}} =& \frac{ic\pi C_{\Delta_L}}{(e^{2\rho}-1)}\Big(-\Gamma(\Delta_L+a-1)\Gamma(\Delta_L+b-1)e^{(a-b+1)\rho} \\
&+\frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+a-1}e^{-(2\Delta_L+a+b-3)\rho}\times \\
&\times {}_2F_1(\Delta_L+a-1, 2\Delta_L+a+b-2, \Delta_L+a, -e^{-2\rho}) + \frac{\Gamma(2\Delta_L+a+b-2)}{\Delta_L+b-1} \\
&\times e^{-(2\Delta_L+a+b-3)\rho}{}_2F_1(\Delta_L+b-1, 2\Delta_L+a+b-2, \Delta_L+b, -e^{-2\rho})\Big).
\end{aligned} \tag{B.1.8}
$$

On the other hand, the Regge limit in the T-channel is dominated by operators of maximal spin. In a holographic CFT, we have $J = 2$. If we further take the lightcone limit, $\rho \gg 1$, the dominant contribution is due to the stress tensor exchange and behaves as $\sigma^{-1}e^{-(d-1)\rho}$. To reproduce this behavior from the S-channel, we must set $a = 0$ and $b = 2$ and make an appropriate choice for the overall constant c. Substituting the designated values of (a, b, c) revals that the first term in (B.1.8) precisely matches the T-channel stress tensor contribution, which in the Regge limit (after analytic continuation) behaves like:

$$g_{\Delta,J} \propto \frac{1}{\sigma^{J-1}} \frac{e^{-(\Delta-3)\rho}}{(e^{2\rho}-1)} + \ldots, \tag{B.1.9}$$

with $\Delta = d$ and $J = 2$. Furthermore, the remaining terms correspond to the exchange of operators with spin 2 and dimension $2\Delta_L + 2 + 2n$; these are the double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$.

B.1.2. Integrating the S-channel result at $\mathcal{O}(\mu^2)$ in $d = 4$

Below we describe how to use the results for the anomalous dimensions at $\mathcal{O}(\mu^2)$ in order to recover the imaginary part of the correlator to the same order. Using the obtained expressions for the anomalous dimensions (5.10) and (5.26), we note that the integrand in (5.13) can be written as

$$\begin{aligned} P^{(0)}\left(\gamma^{(2)} - \frac{\gamma^{(1)}}{2}(\partial_h + \partial_{\bar{h}})\gamma^{(1)}\right) &= -\frac{35\bar{h}^3(2h-\bar{h})}{4(h-\bar{h})^3} P^{(0)} \\ &= -\frac{35 h^{\Delta_L-3}\bar{h}^{\Delta_L+1}}{2\Gamma(\Delta_L-1)\Gamma(\Delta_L)} \sum_{n=0}^{\infty}\left(\frac{\bar{h}}{h}\right)^n (1+\frac{n}{2}). \end{aligned} \tag{B.1.10}$$

Therefore we see that (5.13) can be written as an infinite sum of integrals of the same form that appeared at $\mathcal{O}(\mu)$ in (B.1.5). It then follows that the full S-channel result can be integrated in order to obtain the correlator in position space. Especially, the lightcone result is obtained by setting $k = 0$ in (B.1.10) and taking $\rho \to \infty$ which gives

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = \frac{i35\pi\Delta_L(\Delta_L+1)}{2(\Delta_L-2)} \frac{e^{-3\rho}}{\sigma^{2\Delta_L+1}(e^{2\rho}-1)} + \ldots, \tag{B.1.11}$$

with $\ldots$ denoting terms that are subleading in the lightcone limit. The result (B.1.11) has a form consistent with the contribution of an operator with spin-2 and $\Delta = 6$. The full result (beyond the lightcone limit) further contains an infinite number of operators with spin-2 of dimension $\Delta = 6 + 2n$ and $\Delta = 2\Delta_L + 2n + 2$.

B.1.3. Solving the crossing equation to $\mathcal{O}(\mu)$ in $d = 2$

Here we review the calculations needed for the $d = 2$ case. To $\mathcal{O}(\mu^0)$ the S-channel (2.46) is given by

$$G(z,\bar{z})|_{\mu^0} = \frac{1}{\Gamma(\Delta_L)^2}\int_0^\infty \int_0^h d\bar{h}(h\bar{h})^{\Delta_L-1}(z^h\bar{z}^{\bar{h}} + (z\leftrightarrow\bar{z})). \tag{B.1.12}$$

The integrand in (B.1.12) is symmetric w.r.t. $h \leftrightarrow \bar{h}$ and can thus be rewritten as

$$G(z,\bar{z})|_{\mu^0} = \frac{1}{\Gamma(\Delta_L)^2}\int_0^\infty \int_0^\infty d\bar{h}(h\bar{h})^{\Delta_L-1}z^h\bar{z}^{\bar{h}}, \tag{B.1.13}$$

which can easily be seen to reproduce the disconnected correlator $[(1-z)(1-\bar{z})]^{-\Delta_L}$ in the Regge limit.

As in the previous subsection we proceed to consider the imaginary part of the correlator in the S-channel expansion to $\mathcal{O}(\mu)$. Using a similar notation,

$$I^{(d=2)} \equiv \mathrm{Im}(G(z,\bar{z}))|_\mu\,, \tag{B.1.14}$$

combined with the ansatz $\gamma_1(h,\bar{h}) = c\, h^a\bar{h}^b$, allows us to write:

$$I^{(d=2)} = -\frac{ic\,\pi}{\Gamma(\Delta_L)^2}\int_0^\infty \int_0^h d\bar{h}(h\bar{h})^{\Delta_L-1}h^a\bar{h}^b(z^h\bar{z}^{\bar{h}} + (z\leftrightarrow\bar{z})). \tag{B.1.15}$$

The integrals in (B.1.15) can be easily performed given that $a+\Delta_L > 0$ and $b+\Delta_L > 0$. Changing variables to $z = 1-\sigma e^\rho$, $\bar{z} = 1-\sigma e^{-\rho}$ and collecting the most singular term σ^{-k}, with $k = 2\Delta_L + a + b$, leads to

$$\begin{aligned} I^{(d=2)}|_{\sigma^{-k}} &= \frac{ic\pi}{\Gamma(\Delta_L)^2}\Big(\Gamma(a+\Delta_L)\Gamma(b+\Delta_L)(-e^{\rho(b-a)} - e^{\rho(a-b)}) \\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{-\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L, a+b+2\Delta_L, 1+a+\Delta_L, -e^{-2\rho}) \\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L, a+b+2\Delta_L, 1+a+\Delta_L, -e^{2\rho})\Big). \end{aligned} \tag{B.1.16}$$

Using again (B.1.7) we express (B.1.16) as follows

$$\begin{aligned} I^{(d=2)}|_{\sigma^{-k}} &= \frac{ic\pi}{\Gamma(\Delta_L)^2}\Big(- \Gamma(a+\Delta_L)\Gamma(b+\Delta_L)e^{\rho(a-b)} \\ &+ \frac{\Gamma(a+b+2\Delta_L)e^{-\rho(a+b+2\Delta_L)}}{a+\Delta_L}{}_2F_1(a+\Delta_L, a+b+2\Delta_L, 1+a+\Delta_L, -e^{-2\rho}) \\ &- \frac{\Gamma(a+b+2\Delta_L)e^{-(a+b+2\Delta_L)\rho}}{b+\Delta_L}{}_2F_1(b+\Delta_L, a+b+2\Delta_L, 1+b+\Delta_L, -e^{-2\rho})\Big). \end{aligned} \tag{B.1.17}$$

In matching (B.1.17) with the T-channel expansion, following the same logic as in the previous subsection we deduce that $a = 0$ and $b = 1$ and fix c. The first line in (B.1.17) then reproduces the exchange of the stress tensor in the T-channel. The other two lines match the contribution of double-trace operators $[\mathcal{O}_L\mathcal{O}_L]_{n,l=2}$ with dimension $\Delta = 2\Delta_L + 2n + 2$ and spin 2 in the T-channel expansion.

Appendix B.2. Details on the impact parameter representation in $d = 4$

Here we will see how the impact parameter representation in four dimensions leads to the expression for the disconnected correlator in the Regge limit, in terms of the integral over $h, \bar{h}$.

The objective of this section is to explicitly see that the disconnected contribution of the correlator in the Regge limit

$$\frac{1}{[(1-z)(1-\bar{z})]^{\Delta}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)} \int_0^\infty dh \int_0^h d\bar{h} (h\bar{h})^{\Delta-2} \frac{h-\bar{h}}{z-\bar{z}} \times \left(z^{h+1}\bar{z}^{\bar{h}} - z^{\bar{h}}\bar{z}^{h+1}\right), \tag{B.2.1}$$

can be equivalently written as

$$\int_0^\infty dh \int_0^h d\bar{h}\, \mathcal{I}_{h,\bar{h}}\,, \tag{B.2.2}$$

with

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta) \int_{M^+} \frac{d^4p}{(2\pi)^4} (-p^2)^{\Delta-2} e^{-ipx} (h-\bar{h}) \delta(p\cdot\bar{e} + h + \bar{h})\, \delta\left(\frac{p^2}{4} + h\bar{h}\right). \tag{B.2.3}$$

where M^+ is the upper Milne wedge with $\{p^2 \le 0,\ p^0 \ge 0\}$ and

$$C(\Delta) \equiv \frac{2^{d+1-2\Delta}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta)\Gamma(\Delta - \frac{d}{2} + 1)}\,, \tag{B.2.4}$$

with d the dimensionality of the spacetime, here $d = 4$.

In practice, we need to perform the integral over p in (B.2.3). To do so, we will use spherical polar coordinates and write:

$$\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3} \int_{-\infty}^{\infty} dp^0 \int_0^{\infty} dp^r\,(p^r)^2 \int_{-1}^{1} d(\cos\theta)\;(-p^2)^{\Delta-2}\,\theta(p^0)\theta(-p^2)\times$$
$$e^{ip^0x^0}\,e^{-irp^r\cos\theta}\left[\delta\left(\frac{p^0+p^r}{2}-h\right)\delta\left(\frac{p^0-p^r}{2}-\bar{h}\right)+h\leftrightarrow\bar{h}\right]. \tag{B.2.5}$$

The overall factor of (2π) is simply the result of the integration with respect to the angular variable ϕ. Next we perform the integral over $\cos\theta$:

$$\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3} \int_{-\infty}^{\infty} dp^0 \int_0^{\infty} dp^r\,(p^r)^2\,(-p^2)^{\Delta-2}\,e^{ip^0x^0}$$
$$\times\left(\frac{e^{-irp^r}-e^{irp^r}}{-irp^r}\right)\theta(p^0)\theta(-p^2)\,(\delta\,\delta), \tag{B.2.6}$$

where we set

$$(\delta\,\delta)\equiv\delta\left(\frac{p^0+p^r}{2}-h\right)\delta\left(\frac{p^0-p^r}{2}-\bar{h}\right)+h\leftrightarrow\bar{h}\,. \tag{B.2.7}$$

Notice that

$$\int_0^{\infty} dp^r\,\frac{p^r}{ir}\,(-p^2)^{\Delta-2}\,e^{irp^r}\,(\delta\,\delta) - \int_0^{\infty} dp^r\,\frac{p^r}{ir}\,(-p^2)^{\Delta-2}\,e^{-irp^r}\,(\delta\,\delta) =$$
$$= \int_{-\infty}^{\infty} dp^r\,\frac{p^r}{ir}\,(-p^2)^{\Delta-2}\,e^{irp^r}\,(\delta\,\delta)\,. \tag{B.2.8}$$

Hence we can write (B.2.6) as follows

$$\mathcal{I}_{h,\bar{h}} = \frac{C(\Delta)}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dp^+\,dp^-}{2}\,\frac{p^+-p^-}{i(x^+-x^-)}\,(-p^2)^{\Delta-2}\,e^{\frac{i}{2}(p^+x^-+p^-x^+)}$$
$$\times\;\theta(p^+)\theta(p^-)\,(\delta\,\delta)\,. \tag{B.2.9}$$

Performing the last two integrations is trivial due to the delta-functions. The result is

$$\mathcal{I}_{h,\bar{h}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)}\,\frac{h-\bar{h}}{i(x^+-x^-)}\,(h\bar{h})^{\Delta-2}\,(e^{ihx^+}e^{i\bar{h}x^-}-e^{i\bar{h}x^+}e^{ihx^-})\,, \tag{B.2.10}$$

which allows us to write (B.2.2) as follows:

$$\int_0^{\infty} dh\int_0^{h} d\bar{h}\,\mathcal{I}_{h,\bar{h}} = \frac{1}{\Gamma(\Delta)\Gamma(\Delta-1)}\int_0^{\infty} dh\int_0^{h} d\bar{h}\frac{h-\bar{h}}{i(x^+-x^-)}\,(h\bar{h})^{\Delta-2}$$
$$(z^h\bar{z}^{\bar{h}}-z^{\bar{h}}\bar{z}^h)\,. \tag{B.2.11}$$

Here we also used the identification ($z = e^{ix^+}, \bar{z} = e^{ix^-}$).

Observe that (B.2.11) is equal to (B.2.1) in the Regge limit, where

$$\frac{z}{z-\bar{z}} \simeq \frac{1}{i(x^+ - x^-)}, \quad \frac{\bar{z}}{z-\bar{z}} \simeq \frac{1}{i(x^+ - x^-)}. \tag{B.2.12}$$

However, when considering next order corrections in (x^+, x^-) the impact parameter represention may require corrections. Below we show that these are irrelevant for the questions we are interested in.

B.2.1. Exact Fourier transform

Here we will compute the Fourier transform for the S-channel expression with the identification ($z = e^{ix^+}, \bar{z} = e^{ix^-}$) and show that the leading order results in the Regge limit given in the previous section do not miss any important contributions.

The generic term in the S-channel which we would like to Fourier transform looks like:

$$\int dh\, d\bar{h}\, g(x^+, x^-) \tilde{f}(h, \bar{h}), \tag{B.2.13}$$

where

$$g(x^+, x^-) = \frac{e^{i(1+h)x^+} e^{i\bar{h}x^-} - e^{i\bar{h}x^+} e^{i(h+1)x^-}}{(e^{ix^+} - e^{ix^-})}, \tag{B.2.14}$$

and

$$\tilde{f}(h, \bar{h}) = i\pi (h\bar{h})^{\Delta-2}(h - \bar{h}) f(h, \bar{h}), \tag{B.2.15}$$

where $f(h, \bar{h})$ stands for all the contributions in the S-channel to a given order.

The Fourier transform is:

$$\int d^4x\, e^{ipx} \int dh\, d\bar{h}\, g(x^+, x^-) \tilde{f}(h, \bar{h}) = \int dh\, d\bar{h} \tilde{f}(h, \bar{h}) \int d^4x\, e^{ipx} g(x^+, x^-), \tag{B.2.16}$$

where we simply reversed the order of integration. Our focus in what follows will be the integral:

$$I \equiv \int d^4x\, e^{ipx} g(x^+, x^-). \tag{B.2.17}$$

Since $x^+ = t + r$ and $x^- = t - r$, it is convenient to use spherical polar coordinates to perform the integration. The angular integration over ϕ gives us an overall factor of (2π) as the integrand is independent of ϕ. Next we perform the integration over the other angular variable. Similar to what was discussed in the previous section,

$$\int_{-1}^{1} d(\cos\theta)\, e^{ip^r r\cos\theta} = \frac{e^{irp^r} - e^{-irp^r}}{irp^r} \,. \tag{B.2.18}$$

Combining the above we can write:

$$I = 2\pi \int_{-\infty}^{\infty} dt e^{-itp^t} \int_0^{\infty} drr \, \frac{e^{irp^r} - e^{-irp^r}}{ip^r}\, g(t,r) \,. \tag{B.2.19}$$

It is easy to see that $g(t,r) = g(t,-r)$ and as a result:

$$\int_0^{\infty} dr\, re^{-irp^r}\, g(t,r) = -\int_{-\infty}^{0} dr\, re^{irp^r}\, g(t,r)\,, \tag{B.2.20}$$

which allows us to write the integral as:

$$I = 2\pi \int_{-\infty}^{\infty} \frac{dx^+dx^-}{2} e^{ip\cdot x} \frac{x^+ - x^-}{i(p^+ - p^-)} g(x^+, x^-)\,. \tag{B.2.21}$$

Here $e^{ip\cdot x} = e^{-\frac{i}{2}(p^+x^- + p^-x^+)}$ and the above integral can be thought of as a two-dimensional Fourier transform.

To proceed we need the explicit form of $g(x^+, x^-)$ which we write as

$$g(x^+, x^-) = \frac{e^{ihx^+} e^{i\bar{h}x^-}}{1 - e^{-i(x^+ - x^-)}} + (x^+ \leftrightarrow x^-) \tag{B.2.22}$$

and then expand the denominator in the Regge limit

$$\frac{1}{1 - e^{-i(x^+ - x^-)}} = \frac{1}{i(x^+ - x^-)}\left[1 - \frac{i}{2}(x^+ - x^-) + \cdots\right]\,. \tag{B.2.23}$$

Substituting into (B.2.21) leads to:

$$\begin{aligned} I = 2\pi \frac{1}{(-p^+ + p^-)} \int \frac{dx^+dx^-}{2} e^{ip\cdot x} \{ e^{ihx^+} e^{i\bar{h}x^-} [1 - \frac{i}{2}(x^+ - x^-) + \cdots] \\ + (x^+ \leftrightarrow x^-)\}\,. \end{aligned} \tag{B.2.24}$$

Let us compute the integral term by term. The leading term in the Regge limit yields the standard delta functions:

$$\begin{aligned} I_0 &= 2^2\pi^3\, \frac{1}{p^- - p^+} \delta(\frac{p^+}{2} - \bar{h})\delta(\frac{p^-}{2} - h) + (p^+ \leftrightarrow p^-) = \\ &= 2\pi^3\, \frac{1}{h - \bar{h}} \left\{ \delta(\frac{p^+}{2} - \bar{h})\delta(\frac{p^-}{2} - h) + (p^+ \leftrightarrow p^-) \right\} = \\ &= 2\pi^3\, \frac{1}{h - \bar{h}} \delta(p\cdot \bar{e} + h + \bar{h})\delta(\frac{p^2}{4} + h\bar{h})\,. \end{aligned} \tag{B.2.25}$$

The subleading terms on the other hand produce the same result except that the delta functions are replaced with derivatives of themselves with respect to $p^r = \frac{p^+ - p^-}{2}$.

Let us now consider the full result which up to an overall numerical coefficient can be written as:

$$\int dh\, d\bar{h}\, \tilde{f}(h,\bar{h}) \left(1 - \frac{\partial}{\partial p^r} + \cdots\right) \delta(p\cdot\bar{e} + h + \bar{h})\delta(\frac{p^2}{4} + h\bar{h})\,. \tag{B.2.26}$$

To evaluate the terms with derivatives of the delta function we need to integrate by parts. Now recall that we are interested in the imaginary piece of the S-channel whose leading behaviour is $\sim \sqrt{-p^2}$ (this dependence is hidden in what we called $\tilde{f}$). It is obvious that the derivatives will produce subleading terms which we are not interested in.

What about the other pieces in the S-channel which are not imaginary? To $\mathcal{O}(\mu^2)$ in this case, we know that the leading behaviour grows like $\sim (\sqrt{-p^2})^2$, so by differentiation, a term of the order $\sqrt{-p^2}$ may be produced. However, it is clear that this term will never contribute to the *imaginary* term of the S-channel (note that the coefficient in the first term in the parenthesis in (B.2.26) is real). We thus deduce that the subleading terms in (B.2.24) are irrelevant for our study.

Appendix B.3. Impact parameter representation in general spacetime dimension d

Here we want to prove the following equation for general spacetime dimension d:

$$\mathcal{I}_{h,\bar{h}} = (z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(0)} g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}), \tag{B.3.1}$$

using the form of conformal blocks given in (5.33). We start with the definition of $\mathcal{I}_{h,\bar{h}}$ that is given as:

$$\mathcal{I}_{h,\bar{h}} = C(\Delta_L) \int_{M^+} \frac{d^d p}{(2\pi)^d} (-p^2)^{\Delta_L - \frac{d}{2}} e^{-ipx} (h-\bar{h})\delta(p\cdot\bar{e} + h + \bar{h})\delta(\frac{p^2}{4} + h\bar{h}), \tag{B.3.2}$$

where:

$$C(\Delta_L) \equiv \frac{2^{d+1-2\Delta_L}\pi^{1+\frac{d}{2}}}{\Gamma(\Delta_L)\Gamma(\Delta_L - \frac{d}{2}+1)}\,. \tag{B.3.3}$$

Using spherical coordinates we write (B.3.2) as:

$$\begin{aligned}\mathcal{I}_{h,\bar{h}} = C(\Delta_L)\int_{-\infty}^{\infty} dp^t e^{ip^t t}\int_0^{\infty} dp^r (p^r)^{d-2}\int_{S_{d-2}} \sin^{d-3}\phi_1 d\phi_1\, d\Omega_{d-3} \\ \times e^{-ip^r r\cos\phi_1}(-p^2)^{\Delta_L-\frac{d}{2}}\theta(-p^2)\theta(p^t)\Big\{\delta\left(\frac{p^t+p^r}{2}-h\right)\delta\left(\frac{p^t-p^r}{2}-\bar{h}\right) \\ + (h\leftrightarrow\bar{h})\Big\}\,,\end{aligned} \tag{B.3.4}$$

where $\Omega_{d-3} = \frac{2\pi^{\frac{d-2}{2}}}{\Gamma(\frac{d-2}{2})}$ denotes the area of the unit $(d-3)$-dimensional hypersphere.

Notice now that

$$\int_0^{\pi} \sin^{d-3}\phi_1 e^{-ip^r r\cos\phi_1} d\phi_1 = \sqrt{\pi}\Gamma(\frac{d}{2}-1)_0F_1(\frac{d-1}{2};-\frac{1}{4}(p^r)^2r^2)\,. \tag{B.3.5}$$

Substituting (B.3.5) back in to (B.3.4), one is left with integrals with respect to p^t and p^r only. These integrals are trivial due to the presence of delta functions.[86] When these integrations are done, the expression for $\mathcal{I}_{h,\bar{h}}$ is given as:

$$\begin{aligned}\mathcal{I}_{h,\bar{h}} = \frac{2^{3-d}\sqrt{\pi}}{\Gamma(\Delta_L)\Gamma(\Delta_L-\frac{d}{2}+1)} e^{it(h+\bar{h})}(h-\bar{h})^{d-2}(h\bar{h})^{\Delta_L-\frac{d}{2}} \\ {}_0F_{1R}(\frac{d-1}{2};-\frac{1}{4}(h-\bar{h})^2r^2),\end{aligned} \tag{B.3.6}$$

where ${}_0F_{1R}(a,x) = \Gamma(a)^{-1}{}_0F_1(a,x)$. Relations between coordinates t and r with x^+ and x^- are given as: $x^+ = t+r$ and $x^- = t-r$.

On the other hand, using the explicit form for conformal blocks (5.33) and OPE coefficients in the Regge limit (2.49) one finds that:

$$\begin{aligned}(z\bar{z})^{-\frac{(\Delta_H+\Delta_L)}{2}} P^{(0)} g_{h,\bar{h}}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) = \frac{\Gamma(\frac{d}{2}-1)}{\Gamma(\Delta_L)\Gamma(\Delta_L-\frac{d}{2}+1)} \\ \times (h\bar{h})^{\Delta_L+\frac{d}{2}}(h-\bar{h})(z\bar{z})^{\frac{h+\bar{h}}{2}} C_{h-\bar{h}}^{(\frac{d}{2}-1)}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big).\end{aligned} \tag{B.3.7}$$

[86] One only needs to remember that $h \geq \bar{h} \geq 0$.

Using the relations between coordinates r, t and $z, \bar{z}$ it is easy to see that $(z\bar{z})^{\frac{h+\bar{h}}{2}} = e^{it(h+\bar{h})}$. Next, one can use the relation between Gegenbauer polynomials and hypergeometric functions:

$$C_n^{(\alpha)}(z) = \frac{(2\alpha)_n}{n!}\,{}_2F_1(-n, 2\alpha+n, \alpha+\frac{1}{2}; \frac{1-z}{2}), \tag{B.3.8}$$

which for $h - \bar{h} = l \gg 1$ gives:

$$C_l^{(\frac{d}{2}-1)}\Big(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}}\Big) = \frac{l^{d-3}}{\Gamma(d-2)}\,{}_2F_1(-l, l+d-2, \frac{d-1}{2}; \frac{1}{2} - \frac{1}{2}(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}})). \tag{B.3.9}$$

With the help of the following properties of hypergeometric functions:

$$\begin{aligned} {}_2F_1(a, b, c; z) &= (1-z)^{-b}\,{}_2F_1(c-a, b, c; \frac{z}{z-1}), \\ \lim_{m,n\to\infty} {}_2F_1(m, n, b; \frac{z}{mn}) &= {}_0F_1(b; z). \end{aligned} \tag{B.3.10}$$

Using these, together with the assumption that in the Regge limit the values of $x^+ l$ and $x^- l$ are fixed constants: $x^+ l = a_1$ and $x^- l = a_2$ while $l \to \infty$, one can easily see[87] that (B.3.6) reproduces (B.3.1). This confirms the validity of the impact parameter representation.

Appendix B.4. Anomalous dimensions of heavy-light double-trace operators in $d = 2$

The OPE data of the heavy-light double trace operators in $d = 2$ dimensions can be directly obtained from the heavy-light Virasoro vacuum block [77,78]. For completeness, in this appendix we investigate the anomalous dimension of $[\mathcal{O}_H \mathcal{O}_L]_{\bar{h}, h-\bar{h}}$ in $d = 2$. As in $d = 4$, we introduce an impact parameter representation following [8]. We calculate the anomalous dimensions to $\mathcal{O}(\mu)$ by solving the crossing equation and then use the impact parameter representation to relate them to the bulk phase shift. We find a precise agreement between the two. Using the bulk phase shift we furthermore determine the anomalous dimension to second order in μ. Much of the discussion follows closely the four-dimensional case and will be briefer.

[87] By noting that:

$$\Gamma(x - \frac{1}{2}) = 2^{2-2x}\sqrt{\pi}\frac{\Gamma(2x-1)}{\Gamma(x)}. \tag{B.3.11}$$

B.4.1. Anomalous dimensions in the Regge limit using bootstrap

The conformal blocks in two dimension are given by [75,74]

$$g^{\Delta_{12},\Delta_{34}}_{\Delta,J}(z,\bar{z}) = f_{\Delta+J}(z) f_{\Delta-J}(\bar{z}) + (z \leftrightarrow \bar{z}) \,, \tag{B.4.1}$$

where $f_a(z)$ was defined in (1.16). Similar to the four dimensional case, the blocks for heavy-light double-trace operators simplify in the heavy limit ($\Delta_H \sim C_T$)

$$g^{\Delta_{HL},-\Delta_{HL}}_{[\mathcal{O}_H\mathcal{O}_L]_{h,\bar{h}}}(z,\bar{z}) = (z\bar{z})^{\frac{1}{2}(\Delta_H+\Delta_L)} (z^h \bar{z}^{\bar{h}} + (z \leftrightarrow \bar{z})) \,. \tag{B.4.2}$$

Inserting this form of the conformal blocks in (2.46) together with the OPE coefficients in the Regge limit (2.49) and approximating the sums with integrals, one can due to symmetry extend the region of integration and it is easily found that the disconnected correlator in the T-channel is reproduced.

Similar to the four-dimensional case the stress tensor dominates at order μ in the T-channel. The block of the stress tensor after analytic continuation in the Regge limit is given by

$$g_{T_{\mu\nu}} = \frac{24 i\pi e^{-\rho}}{\sigma} + \dots \,, \tag{B.4.3}$$

where $\dots$ denote non-singular terms. As in the four-dimensional case, this has to be reproduced in the S-channel by the term in (2.46) proportional to $-i\pi\gamma$.

With the conformal blocks (B.4.2), the imaginary part in the S-channel to $\mathcal{O}(\mu)$ is given by

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu} = -i\pi C_{\Delta_L} \int_0^{\infty} dh \int_0^{h} d\bar{h} (h\bar{h})^{\Delta_L-1} \gamma^{(1)}(h,\bar{h}) \left(z^h \bar{z}^{\bar{h}} + z^{\bar{h}} \bar{z}^{h} \right) . \tag{B.4.4}$$

Using the ansatz $\gamma^{(1)}(h,\bar{h}) = c_1 h^a \bar{h}^b$ we find that the T-channel contribution is reproduced for $a = 0$ and $b = 1$ (see Appendix B.1 for details). We thus find using (2.29)

$$\gamma^{(1)} = -\frac{6\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}} \lambda_{\mathcal{O}_L\mathcal{O}_L T_{\mu\nu}}}{\mu\Delta_L} \bar{h} = -\bar{h}. \tag{B.4.5}$$

To $\mathcal{O}(\mu^2)$ we can use (5.13) to find the following contribution to the purely imaginary terms in the S-channel

$$\mathrm{Im}(G(z,\bar{z}))|_{\mu^2} = -i\pi C_{\Delta_L} \int_0^{\infty} dh \int_0^{h} d\bar{h} (h\bar{h})^{\Delta_L-1} \left(\gamma^{(2)} - \frac{c_1^2 \bar{h}}{2} \right) (z^h \bar{z}^{\bar{h}} + z^{\bar{h}} \bar{z}^{h}). \tag{B.4.6}$$

B.4.2. 2d impact parameter representation and relation to bulk phase shift

Similar to the four-dimensional case we introduce an impact parameter representation in order to relate the anomalous dimension with the bulk phase shift. The impact parameter representation in $d = 2$ is given by

$$\mathcal{I}_{h,\bar{h}} \equiv C(\Delta_L) \int_{M^+} d^2p(-p^2)^{\Delta-1} e^{-ipx}(h-\bar{h})\delta(p\cdot\bar{e}+h+\bar{h})\,\delta\left(\frac{p^2}{4}+h\bar{h}\right), \tag{B.4.7}$$

with straightforward generalization of the $d = 4$ case explained above. This is again chosen such that when the impact parameter represetation is integrated over $h, \bar{h}$ the disconnected correlator is reproduced:

$$\int_0^\infty dh \int_0^h \mathcal{I}_{h,\bar{h}} = \frac{1}{[(1-z)(1-\bar{z})]^{\Delta_L}}. \tag{B.4.8}$$

The discussion of the phase shift is completely analogous to the four-dimensional case, as in (5.23) we find the following relation between the bulk phase shift and the anomalous dimension to second order in μ

$$\begin{aligned} \gamma^{(1)} &= -\frac{\delta^{(1)}}{\pi} \\ \tilde{\gamma}^{(2)} - \frac{c_1^2 p^-}{4} &= -\frac{\delta^{(2)}}{\pi}. \end{aligned} \tag{B.4.9}$$

In [8] the phase shift in $d = 2$ was found to be

$$\begin{aligned} \delta^{(1)} &= \frac{\pi}{2}\sqrt{-p^2}e^{-L} \\ \delta^{(2)} &= \frac{3\pi}{8}\sqrt{-p^2}e^{-L}. \end{aligned} \tag{B.4.10}$$

Using the identification $p^+ = 2h$ and $p^- = 2\bar{h}$ together with (5.25) we find for the anomalous dimension in the Regge limit

$$\begin{aligned} \gamma^{(1)} &= -\bar{h} \\ \gamma^{(2)} &= -\frac{1}{4}\bar{h}. \end{aligned} \tag{B.4.11}$$

We thus see that the first order result agrees with that obtained from bootstrap (B.4.5). Furthermore, the second order correction agrees also in $d = 2$ with the result (6.40) in [8].

Appendix B.5. Discussion of the boundary term integrals

There are a few integrals containing total derivative terms that we have ignored throughout Section 5 [1] and we analyze more carefully here. Let us start with a total derivative term which shows up in the real part of the correlator at $\mathcal{O}(\mu)$. It is given by[88]:

$$I_1 = \frac{1}{2}(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \int_0^{+\infty} dl \Big[P^{(0)}\gamma^{(1)} g_{n+l,n}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) \Big]_{n=0}^{n\to\infty}. \tag{B.5.1}$$

Let us focus on the integrand: $\Big[P^{(0)}\gamma^{(1)} g_{n+l,n}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) \Big]_{n=0}^{n\to\infty}$. When $n=0$, the expression within the brackets trivially vanishes. On the other hand, when $n\to\infty$, it takes the form $n^{2\Delta_L-2}(z\bar{z})^n \times f(l)$, where f is some function of l only. We are instructed here to take the limit $n\to\infty$ independently of all other limits (recall that the Regge limit is taken after the integration). For generic values $0<(z,\bar{z})<1$ it is clear that $\lim_{n\to\infty}\Big[P^{(0)}\gamma^{(1)} g_{n+l,n}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) \Big] = \lim_{n\to\infty} n^{2\Delta_L-2}(z\bar{z})^n \times f(l) \to 0$. In other words, the expression $\Big[P^{(0)}\gamma^{(1)} g_{n+l,n}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) \Big]_{n=0}^{n\to\infty} \to 0$, and we conclude that the integral (B.5.1) does not contribute to the S-channel expansion of the correlator.

There are a few more integrals of similar kind that appear at $\mathcal{O}(\mu^2)$. We will analyse one of them here:

$$I_2 = \frac{-i\pi}{2}(z\bar{z})^{-\frac{1}{2}(\Delta_H+\Delta_L)} \int_0^{+\infty} dl \Big[P^{(0)}(\gamma^{(1)})^2 g_{n+l,n}^{\Delta_{HL},-\Delta_{HL}}(z,\bar{z}) \Big]_{n=0}^{n\to\infty}. \tag{B.5.2}$$

The same logic can be applied here. Again, the value of the expression in brackets at $n=0$ is trivially zero, while for large n it behaves like: $n^{2\Delta_L+d-4}(z\bar{z})^n\tilde{f}(l)$. As long as $(z,\bar{z})<1$, this vanishes exponentially in the limit $n\to\infty$. One concludes therefore that the integral (B.5.2) vanishes. The same logic is valid for all other integrals of similar total derivative terms that appear at $\mathcal{O}(\mu^2)$.

88 We are again using variables n and l, one can notice that $n=\bar{h}$ and $l=h-\bar{h}$. It is trivial to prove that $\partial_n = \partial_h + \partial_{\bar{h}}$.

Appendix B.6. An identity for the bulk phase shift.

The aim is to elaborate on the results of [8] for the bulk phase shift in a black hole background as computed in gravity. Firstly, let us note the following identity involving hypergeometric functions:

$$\sum_{n=0}^{\infty} a(n) x^n \, {}_2F_1[\tau_0+2n+1, \frac{d}{2}-1, \tau_0+2n-\frac{d}{2}+3, x] = {}_2F_1[\tau_0+1, \frac{\tau_0}{2}, \frac{\tau_0}{2}+2, x]$$

$$a(n) = \frac{2^{2n}}{n!} \frac{\tau_0+2}{\tau_0+2+2n} \frac{(\frac{\tau_0}{2}+1-\frac{d}{2})_n \left(\frac{\tau_0+1}{2}\right)_n}{(\tau_0+n+2-\frac{d}{2})_n}, \quad \tau_0 \neq 0\,. \tag{B.6.1}$$

Given that both sides of the equality can be expressed as an infinite series expansion around $x=0$, one simply needs to show that the expansion coefficients match to all orders in x. This is proven in Appendix G.

Consider now the case $\tau_0 = k(d-2)$ where $k \in N^\star$. Setting $x \equiv e^{-2L}$ and multiplying both sides with $e^{-[k(d-2)+1]L}$ yields:

$$\Pi_{k(d-2)+1,k(d-2)+1}(L) = \sum_{n=0}^{\infty} \beta_n \Pi_{k(d-2)+2n+1,d-1}(L)$$

$$\beta(n) \equiv \pi^{\frac{(1-k)(d-2)}{2}} \frac{a(n)}{(k(d-2)+1)_n} \frac{\Gamma\left[k(d-2)-\frac{d}{2}+2n+3\right]}{\Gamma\left[\frac{k(d-2)}{2}+2\right]}\,. \tag{B.6.2}$$

The left hand side represents the hyperbolic space propagator for a scalar field of squared mass equal to $k(d-2)+1$ in a hyperbolic space of dimensionality $k(d-2)+1$ and is proportional to the k-th order expression for the bulk phase shift computed in gravity in [8], where

$$\delta^{(k)}(S,L) = \frac{1}{k!} \frac{2\Gamma\left(\frac{dk+1}{2}\right)}{\Gamma\left(\frac{k(d-2)+1}{2}\right)} \frac{\pi^{1+\frac{k(d-2)}{2}}}{\Gamma\left(\frac{k(d-2)}{2}+1\right)} S\, \Pi_{k(d-2)+1,k(d-2)+1}(L)\,. \tag{B.6.3}$$

On the other hand, the right-hand side of (B.6.2) expresses the k-th order term of the bulk phase shift as an infinite sum of $(d-1)$-dimensional hyperbolic space propagators for fields with mass-squared equal to $m^2 = k(d-2)+1+2n$.

It can be shown [151,34] that the analytically continued T-channel scalar conformal block in the Regge limit behaves like:

$$g_{\Delta,J}(\sigma,\rho) = i\, c_{\Delta,J} \frac{\Pi_{\Delta-1,d-1}(\rho)}{\sigma^{J-1}}\,, \tag{B.6.4}$$

where

$$c_{\Delta,J} = \frac{4^{\Delta+J-1}\Gamma\left(\frac{\Delta+J-1}{2}\right)\Gamma\left(\frac{\Delta+J+1}{2}\right)}{\Gamma(\frac{\Delta+J}{2})^2}\,\frac{2\Gamma\left(\Delta-\frac{d}{2}+1\right)}{\pi^{1-\frac{d}{2}}\Gamma\left(\Delta-1\right)}\,. \tag{B.6.5}$$

Here $\Pi_{\Delta-1,d-1}$ denotes as usual the $(d-1)$-dimensional hyperbolic space propagator for a massive scalar of mass-squared $m^2 = (\Delta-1)$.

It follows that the k-th order term in the μ-expansion of the bulk phase shift in a black hole background can be expressed as an infinite sum of conformal blocks corresponding to operators of twist $\tau = \tau_0(k) + 2n = k(d-2) + 2n$ and spin $J = 2$ in the Regge limit. In other words, we can write:

$$\begin{aligned} i\,\delta^{(k)}(S,L) =&\ f(k)\sum_{n=0}^{\infty}\lambda_k(n)\ g^R_{\tau_0(k)+2n+2,2}(S,L) \\ \lambda_k(n) = a(n)&\,\frac{2^{-4n}\left[\left(\frac{\tau_0(k)+4}{2}\right)_n\right]^2}{\left(\frac{\tau_0(k)+3}{2}\right)_n\left(\frac{\tau_0(k)+5}{2}\right)_n}, \quad \tau_0(k) = k(d-2) \end{aligned} \tag{B.6.6}$$

where

$$f(k) \equiv \frac{\sqrt{\pi}}{64}\,\frac{1}{2^{k(d-2)}\,k!}\,\frac{\Gamma\left(\frac{kd+1}{2}\right)\Gamma\left(\frac{k(d-2)+4}{2}\right)}{\Gamma\left(\frac{k(d-2)+5}{2}\right)\Gamma\left(\frac{k(d-2)+3}{2}\right)}\,, \tag{B.6.7}$$

and

$$g^R_{\Delta,J}(S,L) = ic_{\Delta,J}\,S^{J-1}\,\Pi_{\Delta-1,d-1}(L)\,. \tag{B.6.8}$$

Appendix B.7. An identity for hypergeometric functions.

Here we will show that for $q \neq 0$,

$$\begin{aligned} &\sum_{n=0}^{\infty} a(n)x^n\,{}_2F_1[q+2n+1,\frac{d}{2}-1,q+2n-\frac{d}{2}+3,x] = {}_2F_1[q+1,\frac{q}{2},\frac{q}{2}+2,x] \\ &a(n) = \frac{2^{2n}}{n!}\,\frac{q+2}{q+2+2n}\,\frac{(\frac{q}{2}+1-\frac{d}{2})_n\left(\frac{q+1}{2}\right)_n}{(q+n+2-\frac{d}{2})_n}, \quad q\neq 0\,. \end{aligned} \tag{B.7.1}$$

Given that both sides of the equality can be expressed as an infinite series expansion around $x=0$, one simply needs to show that the expansion coefficients match to all orders in x. Let us first set:

$$\begin{aligned} b(n,m) &\equiv \frac{1}{m!}\,\frac{(q+1+2n)_m\left(\frac{d}{2}-1\right)_m}{\left(q-\frac{d}{2}+2n+3\right)_m} \\ c(\ell) &\equiv \frac{1}{\ell!}\,\frac{(q+1)_\ell\left(\frac{q}{2}\right)_\ell}{\left(\frac{q}{2}+2\right)_\ell} = \frac{(q+1)_\ell}{\ell!}\,\frac{q(q+2)}{(q+2\ell)(q+2\ell+2)}\,, \end{aligned} \tag{B.7.2}$$

such that:

$$
\begin{aligned}
{}_2F_1[q+2n+1,\frac{d}{2}-1,q+2n-\frac{d}{2}+3,x] &= \sum_{m=0}^{\infty} b(n,m)x^m, \\
{}_2F_1[q+1,\frac{q}{2},\frac{q}{2}+2,x] &= \sum_{\ell=0}^{\infty} c(\ell)x^\ell.
\end{aligned}
\tag{B.7.3}
$$

It is easy to check that the coefficients of the first few powers of x precisely match. Indeed, e.g.,

$$
\begin{aligned}
a(0)b(0,0)-c(0) &= 0 \\
a(1)b(1,0)+a(0)b(0,1)-c(1) &= 0 \\
a(2)b(2,0)+a(1)b(1,1)+a(0)b(0,2)-c(2) &= 0.
\end{aligned}
\tag{B.7.4}
$$

To show that the above identity is true for all powers of x we must show that:

$$
\sum_{k=0}^{\ell} a(k)b(k,\ell-k) = c(\ell)\,, \tag{B.7.5}
$$

for all $\ell \in N$. The left-hand side of (B.7.5) can be easily summed to yield:

$$
\sum_{k=0}^{\ell} a(k)b(k,\ell-k) = \frac{1}{\ell!}\frac{\Gamma[q+1+\ell]}{\Gamma[q]}\frac{(q+2)}{(q+2\ell)(2+2\ell+q)}\,, \tag{B.7.6}
$$

which can be trivially shown to be equal to $c(\ell)$.

Appendix C.1. Comparing leading Regge singularities with the shockwave calculation

In order to compare the leading Regge singularities in (6.66) with the stress tensor sector calculated in [104], the following identity is useful

$$\begin{aligned}
&\frac{e^{2(\Delta_L+\frac{k}{2})\rho}}{e^{2\rho}-1}\frac{\Gamma(1-k)\Gamma(\Delta_L+2k-1)\Gamma(2\Delta_L+k)}{k!\Gamma(\Delta_L)\Gamma(\Delta_L-1)\Gamma(\Delta_L+k)(2\Delta_L+k-1)}\Bigg[\tilde{F}_{\Delta_L,n,-1}\\
&+\frac{(e^{2\rho}-1)(\Delta_L+2k-1)}{\Delta_L+k}\tilde{F}_{\Delta_L,n,0}-\frac{e^{2\rho}(\Delta_L+2k-1)(\Delta_L+2k)}{(\Delta_L+k)(\Delta_L+k+1)}\tilde{F}_{\Delta_L,n,1}\Bigg]\\
=&\frac{e^{-3k\rho}}{1-e^{-2\rho}}\frac{\Gamma(\Delta_L-k)\Gamma(\Delta_L+2k-1)}{k!\Gamma(\Delta_L-1)\Gamma(\Delta_L)}\\
&\times {}_2F_1(k-1,\Delta_L+2k-1;-\Delta_L+k+1,-e^{-2\rho}),
\end{aligned}\tag{C.1.1}$$

where[89]

$$\begin{aligned}
\tilde{F}_{\Delta_L,n,a}(e^{-2\rho})=&\frac{\Gamma(\Delta-k-a)\Gamma(\Delta_L+k+a+1)}{\Gamma(1-k)\Gamma(2\Delta_L+k)}\\
&\times e^{-2(\Delta_L+2k+a)\rho}{}_2F_1(\Delta_L+2k+a,k;-\Delta_L+k+a+1;-e^{-2\rho}).
\end{aligned}\tag{C.1.2}$$

With (C.1.1) one can check that (6.66) agrees with the contribution from the stress tensor sector for fixed ρ, or η, in [104].

Appendix C.2. Further comparison with lightcone results

In this section, we further compare predictions obtained using the phase shift with known results in the lightcone limit.

C.2.1. Triple-stress tensors in four dimensions

Consider the momentum space correlator (6.55) at $\mathcal{O}(\mu^3)$. In the large impact parameter limit, this is compared with the explicit resummation of minimal-twist triple-stress tensors discussed in Section 6.2.

Consider the correlator (6.55) at $\mathcal{O}(\mu^3)$:

$$\frac{\mathcal{B}(p)}{\mathcal{B}_0(p)}\Big|_{\mu^3}=-i\frac{(\delta^{(1)})^3}{3!}-\delta^{(1)}\delta^{(2)}+i\delta^{(3)}.\tag{C.2.1}$$

89 $\tilde{F}_{\Delta_L,n,a}$ is related to $F_{\Delta_L,n,a}$ in [104] if one uses their identity Eq. (44) and keep only the part relevant to the stress tensor sector and set $(\eta)_{\text{there}}=(e^{-2\rho})_{\text{here}}$.

The leading and next-to-leading singularities are due to the first two terms in (C.2.1) and were discussed in Section 6.4. At $\mathcal{O}(\frac{1}{\sigma})$ there will be a contribution from the last term $i\delta^{(3)}$ in (C.2.1). Using the decomposition of the phase shift in (6.57), it is straightforward to use (6.37) to find the corresponding contribution to the stress tensor sector:

$$\mathcal{G}^{(3)}(\sigma,\rho)|_{\delta^{(3)}} = f(3)\sum_{n=0}^{\infty}\lambda_3(n)p[\tau_0(3)+2n+2,2]\; g^{\circlearrowleft}_{\tau_0(3)+2n+2,2}(\sigma,\rho)+\ldots, \tag{C.2.2}$$

in any dimension d and the ellipses denote subleading corrections in $\sigma\to 0$. Here $\tau_0(k)=k(d-2)$ is the minimal-twist of multi-stress tensors at k-th order. To compare the large impact parameter limit with the contribution from minimal-twist multi-stress tensors, consider the term in (C.2.2) with $n=0$:

$$\mathcal{G}^{(3)}(\sigma,\rho)|_{\delta^{(3)}} \underset{\rho\to\infty}{\approx} \frac{1155 i\pi\Delta_L(\Delta_L+1)(\Delta_L+2)}{8(\Delta_L-2)(\Delta_L-3)}\frac{e^{-7\rho}}{\sigma}. \tag{C.2.3}$$

We thus see that (C.2.3) agree with the first line in (6.15) at $\mathcal{O}(\mu^3)$ due to minimal-twist triple-stress tensors in $d=4$.

There will also be a contribution at $\mathcal{O}(\frac{1}{\sigma})$ due to the first subleading correction to the second term $-\delta^{(1)}\delta^{(2)}$ in (C.2.1). One can include the correction to the position space Regge conformal block in (6.40) to the expression (6.79) found in Section 6.4. Taking the large impact parameter with $k=3$ one finds:

$$\mathcal{G}^{(3)}(\sigma,\rho)|_{\delta^{(1)}\delta^{(2)},\sigma^{-1}} \underset{\rho\to\infty}{\approx} \frac{525\pi^2\Delta_L(\Delta_L+1)(\Delta_L+2)(\Delta_L+3)}{4(\Delta_L-2)(\Delta_L-3)}\frac{e^{-7\rho}}{\sigma}, \tag{C.2.4}$$

which agree with the last line in (6.15) obtained using lightcone bootstrap.

C.2.2. Double-stress tensors in six dimensions

Consider the correlator (6.55) at $\mathcal{O}(\mu^2)$ in $d=6$:

$$\frac{\mathcal{B}(p)}{\mathcal{B}_0(p)}\Big|_{\mu^2} = -(\delta^{(1)})^2 + i\delta^{(2)}. \tag{C.2.5}$$

The last term $i\delta^{(2)}$ in (C.2.5) can be transformed to position space using (6.37). From (6.57), one finds that the lowest-twist contribution to the second-order phase shift in six dimensions is given by

$$\delta_0^{(2),d=6} = \frac{693\pi}{16}\frac{S(4e^{2L}-3)e^{-11L}}{(1-e^{-2L})^3}. \tag{C.2.6}$$

Explicitly, Fourier transforming (C.2.6), we find the following contribution to the stress tensor sector in the limit $\rho \to \infty$

$$\mathcal{G}^{(2),d=6}(\sigma,\rho)|_{\delta^{(2)}} \underset{\rho\to\infty}{\approx} \frac{i\pi 693 p[10,2]}{4}\frac{e^{-9\rho}}{\sigma}. \tag{C.2.7}$$

This agrees with the imaginary term at $\mathcal{O}(\frac{1}{\sigma})$ after analytically continuing the resummation of minimal-twist double-stress tensors given by Eq. (4.8) in [2].

Appendix D.1. Integrated Scalar

As several new features emerge in the case of integrated correlators, we will first discuss a toy model – ($d = 4$) scalar field, that will serve as a consistency check. We will show that one is able to extract the same OPE data when working with correlators integrated over the xy plane, as in the original approach.

This appendix is divided into two parts: first subsection focuses on the case of a scalar field with non-integer scaling dimension, while the second one studies the $\Delta = 4$ case, which is more relevant for the stress tensor calculations.

In both subsections we begin by solving the bulk equations of motion where two spatial dimensions were integrated out. We find the solution using the ansatz introduced recently in [102,163], naturally adapted for the integrated case.

On the CFT side we examine the integrated conformal blocks in the OPE limit. In the integer case we explain the emergence of the log term as a result of mixing of the scalar and stress tensor sectors. We also find that further regularization is needed as a result of the integration.

Finally we extract the OPE coefficients[90] from the comparison of the bulk calculations and the structures expected by CFT and comment on the emergence of the log terms and undetermined coefficients. We conclude that the integrated problem can be equivalently well used for the extraction of the OPE data as was the original one.

D.1.1. Scalar field with non-integer scaling dimension

D.1.1.1. Bulk-side

Our aim is to calculate the bulk-to-boundary propagator satisfying the scalar field equation

$$\begin{gathered}(\partial^2 - m^2)\phi = 0 \\ \Delta(\Delta - 4) - m^2 = 0,\end{gathered} \tag{D.1.1}$$

on the planar Euclidean AdS-Schwarzschild black hole background

$$ds^2 = r^2(1 - \frac{\mu}{r^4})dt^2 + r^2 d\vec{x}^2 + \frac{1}{r^2(1 - \frac{\mu}{r^4})}dr^2, \tag{D.1.2}$$

[90] In the leading order in the large C_T limit.

where $\vec{x} = (x, y, z)$.

According to the AdS/CFT dictionary we obtain the thermal two-point function of the corresponding scalar operator in the limit

$$\langle \mathcal{O}_L(x_1)\mathcal{O}_L(x_2)\rangle_\beta = \lim_{r\to\infty} r^\Delta \phi(r, x_1, x_2). \tag{D.1.3}$$

In this subsection we consider the conformal dimension Δ_L is not an integer.

We now integrate over the xy-plane, hence we work with the integrated bulk-to-boundary propagator

$$\Phi(t, z, r) = \int_{R^2} dxdy\, \phi(t, \vec{x}, r)\,. \tag{D.1.4}$$

Equation (D.1.1) in the background (D.1.2) is then given by

$$\left[\Delta(\Delta-4) - r(4+f)\partial_r - r^2 f\partial_r^2 - \frac{1}{r^2}\partial_z^2 - \frac{1}{r^2 f}\partial_t^2\right]\Phi = 0, \tag{D.1.5}$$

where $f = 1 - \frac{\mu}{r^4}$.

To solve this equation, we first transform coordinates (t, z, r) to (w, ρ, r) defined by

$$\begin{aligned} \rho &\equiv rz \\ w^2 &\equiv 1 + r^2t^2 + r^2z^2. \end{aligned} \tag{D.1.6}$$

These are the natural integrated analogues of the variables introduced in [102]. In these coordinates we have the following equation for Φ:

$$\begin{aligned} &\big[C_1 + C_2\partial_r + C_3\partial_\rho + C_4\partial_w + C_5\partial_r^2 + C_6\partial_\rho^2 \\ &\qquad + C_7\partial_w^2 + C_8\partial_r\partial_\rho + C_9\partial_\rho\partial_w + C_{10}\partial_w\partial_r\big]\,\Phi = 0, \end{aligned} \tag{D.1.7}$$

where

$$\begin{aligned} C_1 &= -r^4w^3(\Delta-4)\Delta(r^4-\mu) \\ C_2 &= rw^3(5r^8 - 6r^4\mu + \mu^2) \\ C_3 &= \rho w^3(5r^8 - 6r^4\mu + \mu^2) \\ C_4 &= w^2(w^2-1)(5r^8 - 6r^4\mu + \mu^2) + r^8(1+\rho^2) \\ &\quad + (r^4-\mu)^2(w^2-1) + r^4(r^4-\mu)(w^2-\rho^2) \\ C_5 &= (r^4-\mu)^2r^2w^3 \\ C_6 &= (r^4-\mu)^2w^3\rho^2 + r^4(r^4-\mu)w^3 \\ C_7 &= r^8w(w^2-\rho^2-1) + (r^4-\mu)^2w(w^2-1)^2 + r^4(r^4-\mu)w\rho^2 \\ C_8 &= 2rw^3\rho(r^4-\mu)^2 \\ C_9 &= 2(r^4-\mu)^2w^2(w^2-1)\rho + 2r^4(r^4-\mu)w^2\rho \\ C_{10} &= 2rw^2(r^4-\mu)^2(w^2-1). \end{aligned} \tag{D.1.8}$$

Here, using the same logic as in [102], we assume the ansatz (focusing only on the solution that corresponds to the stress tensor sector on the CFT side, see [102] for more details) as

$$\Phi = \Phi_{AdS} \left(1 + \frac{G_4}{r^4} + \frac{G_8}{r^8} + \dots\right), \tag{D.1.9}$$

where

$$\begin{aligned} G^4 &= \sum_{m=0}^{2} \sum_{n=-2}^{4-m} a^4_{n,m} w^n \rho^m \\ G^8 &= \sum_{m=0}^{6} \sum_{n=-6}^{8-m} a^8_{n,m} w^n \rho^m . \end{aligned} \tag{D.1.10}$$

The vacuum propagator Φ_{AdS} can be obtained by integrating the known vacuum bulk-to-boundary propagator for the scalar field:

$$\Phi_{AdS}(t,z,r) = \int dxdy \left[\frac{r}{1 + r^2(t^2 + x^2 + y^2 + z^2)}\right]^{\Delta} = \frac{\pi r^{\Delta-2}}{\Delta - 1} \left(1 + r^2(t^2 + z^2)\right)^{1-\Delta} . \tag{D.1.11}$$

Changing the coordinates in this prefactor to (w, ρ, r) we get

$$\Phi_{AdS}(w, \rho, r) \propto \frac{r^{\Delta-2}}{w^{2-2\Delta}}. \tag{D.1.12}$$

Inserting the ansatz into equation (D.1.7) we can determine the coefficients $a^j_{n,m}$ as functions of Δ and μ. In the non-integer case all coefficients $a^4_{n,m}$ and $a^8_{n,m}$ can be found. Here we list the non-zero ones appearing at $\mathcal{O}(\mu^1)$:

$$\begin{aligned} a^4_{-2,0} &= \frac{2\mu(1-\Delta)}{5} \\ a^4_{0,0} &= \frac{\mu(\Delta-1)}{5} \\ a^4_{2,0} &= \frac{3\mu\Delta(\Delta-1)}{20(\Delta-2)} \\ a^4_{4,0} &= \frac{\mu\Delta(\Delta-1)(3\Delta-10)}{120(\Delta-3)(\Delta-2)} \\ a^4_{-2,2} &= -\frac{\mu(\Delta-1)}{5} \\ a^4_{0,2} &= -\frac{\mu\Delta}{10} \\ a^4_{2,2} &= -\frac{\mu\Delta(\Delta-1)}{30(\Delta-2)}. \end{aligned} \tag{D.1.13}$$

D.1.1.2. CFT-side

On the CFT side, the object dual to the scalar field two-point function in the black hole background, is the heavy-heavy-light-light correlator $\langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle$.

Decomposing this four-point function into conformal blocks and integrating, we obtain

$$\mathcal{G}_\Delta \equiv \int dxdy \langle \mathcal{O}_H \mathcal{O}_L \mathcal{O}_L \mathcal{O}_H \rangle = \int dxdy \sum_{\Delta_i, J} C_{\Delta_i, J} \frac{g_{\Delta_i, J}(Z, \overline{Z})}{(Z\overline{Z})^\Delta}, \tag{D.1.14}$$

where Z and $\overline{Z}$[91] are the cross ratios defined in terms of t, x, y and z as:

$$\begin{aligned} Z &= -t - i\sqrt{x^2 + y^2 + z^2} \\ \overline{Z} &= -t + i\sqrt{x^2 + y^2 + z^2}, \end{aligned} \tag{D.1.15}$$

and $C_{\Delta_i, J}$ is the product of OPE coefficients corresponding to the exchange of an operator with conformal dimension Δ_i and spin J.

In the heavy-heavy-light-light correlator, the important set of operators contributing in the T-channel are the multi- stress tensors, which we consider below. The first nontrivial contribution to the correlator (D.1.14) comes from the exchange of the stress tensor. In the OPE limit the corresponding conformal block is

$$g_{4,2}(Z, \overline{Z}) \approx Z\overline{Z}(Z^2 + Z\overline{Z} + \overline{Z}^2). \tag{D.1.16}$$

Hence, at this order we find that (D.1.14) becomes

$$\mathcal{G}_\Delta \Big|_{\mu^1} = -C_{4,2} \frac{\pi (t^2 + z^2)^{2-\Delta}(t^2(10 - 3\Delta) + z^2(\Delta - 2))}{(\Delta - 3)(\Delta - 2)}. \tag{D.1.17}$$

Following the same approach, one gets the corresponding integrated conformal blocks for the double-trace stress tensors.

---

[91] We will temporarily use this unusual notation, as we have to distinguish the cross ratios and the space coordinate z.

D.1.1.3. Comparison

To determine the OPE coefficients, we compare the bulk and the CFT results. We connect the two sides by equation (D.1.3), which is now of the form

$$\mathcal{G}_{\Delta} = \lim_{r\to\infty} r^{\Delta}\Phi_{AdS}\left(1 + G^T + G^{\phi}\right). \tag{D.1.18}$$

Where $G^T = \frac{G^4}{r^4} + \frac{G^8}{r^8} + \ldots$ corresponds to the stress tensor sector and G^{ϕ} corresponds to the double-trace scalars (possibly dressed with $T_{\mu\nu}$). As we mentioned above, these two sectors are decoupled for non-integer Δ_L, hence we can consider only the multi-stress tensors. The stress tensor contribution to (D.1.18) is given by

$$\begin{aligned} \mathcal{G}_{\Delta}\Big|_{\mu^1} &= \lim_{r\to\infty} \frac{\pi r^{2\Delta-6}}{\Delta-1}\frac{G^4(t,z,r)}{(1+r^2(t^2+z^2))^{\Delta-1}} \\ &= \frac{\pi(t^2+z^2)^{2-\Delta}(t^2(3\Delta-10)+z^2(2-\Delta))\Delta\mu}{120(\Delta-3)(\Delta-2)}, \end{aligned} \tag{D.1.19}$$

where in the second equality we have used the bulk results for G^4.

Comparing (D.1.17) and (D.1.19) we extract the OPE coefficient:

$$C_{4,2} = \frac{\Delta\mu}{120}, \tag{D.1.20}$$

which is in agreement with the result (3.65) in [102].

The OPE coefficients at higher orders in μ can be determined in a similar way.

D.1.2. Scalar field with $\Delta = 4$

D.1.2.1. Bulk-side

We now consider $\Delta = 4$. The setup for this case is identical as above, i.e. we have to solve the bulk equation of motion (D.1.7) but now for $\Delta = 4$.

Here, however, the situation becomes more tricky as some of the OPE coefficients are singular for $\Delta = 4$. On the other hand, for integer Δ the multi-stress-tensor sector and double-trace scalar sector are no longer decoupled. We expect the contribution from the $[\mathcal{O}\mathcal{O}]$ to compensate for these divergent parts in the $[T^n]$ OPE coefficients. As a result, log terms will appear in the solution. We will explain this in more details below.

In the bulk this leads to a slightly modified ansatz [163]:

$$\Phi = \Phi_{AdS}\left(1 + \frac{1}{r^4}\left(G^{4,1} + G^{4,2}\log r\right) + \frac{1}{r^8}\left(G^{8,1} + G^{8,2}\log r\right) + \dots\right), \tag{D.1.21}$$

where Φ_{AdS} is the same vacuum propagator as in the previous section and $G^{4,j}$ and $G^{8,j}$ are given by

$$\begin{aligned} G^{4,j} &= \sum_{m=0}^{2}\sum_{n=-2}^{4-m}(a^{4,j}_{n,m} + b^{4,j}_{n,m}\log w)w^n\rho^m \\ G^{8,j} &= \sum_{m=0}^{6}\sum_{n=-6}^{8-m}(a^{8,j}_{n,m} + b^{8,j}_{n,m}\log w)w^n\rho^m. \end{aligned} \tag{D.1.22}$$

Inserting this ansatz into the equation (D.1.7), we can determine the coefficients $a^{k,j}_{n,m}$ and $b^{k,j}_{n,m}$.

The result (in the w, ρ and r coordinates) is

$$\begin{aligned} \Phi = &\frac{\pi}{25200 r^6 w^{10}}\big[8400w^4(r^8 + w^6((1-6\rho^2)a^{8,1}_{6,0} + (w^2 - 8\rho^2)a^{8,1}_{8,0})) \\ &+ 840r^4w^2(-12 + 6w^2 + w^4 + w^6 - 2(3 + 2w^2 + w^4)\rho^2)\mu \\ &+ (8064 - 12656w^2 + 3136w^4 + 448w^6 + 655w^8 - 4(-2016 \\ &+ 448w^2 + 476w^4 + 345w^6 + 40w^8 + 750w^{10})\rho^2 \\ &+ 56(36 + 44w^2 + 35w^4 + 20w^6 + 10w^8)\rho^4 \\ &+ 120w^{10}(-6 + 5w^2 - 4\rho^2)(\log r + \log w))\mu^2\big] + \mathcal{O}(\mu^3). \end{aligned} \tag{D.1.23}$$

For the stress tensor exchange all log terms vanish and we are also able to determine all the coefficients. We get the same results as in the non-integer case, as expected.

For the double stress tensor exchange (μ^2) the coefficients $a^{8,1}_{6,0}$ and $a^{8,1}_{8,0}$ are not fixed by near-boundary analysis.

D.1.2.2. CFT-side

At order $\mathcal{O}(\mu^0)$ and $\mathcal{O}(\mu^1)$ the contribution for $\Delta = 4$ will be the same as was for non-integer Δ. Let us therefore focus on the μ^2 terms.

Here the contributions of the double-trace stress tensors mix with the double-trace scalar $[\mathcal{O}\mathcal{O}]$. We thus have to consider 4 contributions to the correlator at order μ^2 – three from the double stress tensor (we label them by the conformal dimension and the spin: (Δ_i, J)):

$$\begin{aligned} &T_{\mu\nu}T^{\mu\nu} \qquad (8,0) \\ &T_{\mu\nu}T^{\nu}{}_{\alpha} \qquad (8,2) \\ &T_{\mu\nu}T_{\alpha\beta} \qquad (8,4) \end{aligned} \tag{D.1.24}$$

and one contribution from the double-trace scalar:

$$[\mathcal{O}\mathcal{O}] \quad \Longleftrightarrow \quad (8,0), \tag{D.1.25}$$

which will mix with the $(8,0)$ contribution from $[T^2]$. This agrees with the fact that it is only the coefficient $C^{TT}(8,0)$, that is expected to diverge.

Let us have a closer look at the divergencies that appear here. First, as the coefficient $C^{TT}_{8,0}$ has a pole in $\Delta = 4$ [102] we can write it as

$$C_{8,0} = \frac{C_{sing}}{\Delta - 4} + C^{TT}_{reg} \tag{D.1.26}$$

where the term C^{TT}_{reg} is regular in $\Delta = 4$ and C_{sing} is the residue. In order for the singular part to be cancelled, the OPE coefficient of the double-trace scalar must also have a pole at $\Delta = 4$ with the same residue but with an opposite sign [102]:

$$C^{\mathcal{O}\mathcal{O}}_{8,0} = -\frac{C_{sing}}{\Delta - 4} + C^{\mathcal{O}\mathcal{O}}_{reg} \tag{D.1.27}$$

Now, since the conformal block for $J = 0$ in the OPE limit is $g_{\Delta',0} \approx (Z\overline{Z})^{\Delta'}$, we can study what happens if the contributions from $[T^2]$ and $[\mathcal{O}\mathcal{O}]$ mix. Consider first $\Delta = 4 + \delta$, sum the contributions from $[T^2]$ and $[\mathcal{O}\mathcal{O}]$ and then take the limit $\delta \to 0$:

$$\lim_{\delta\to 0}\left[C^{TT}_{8,0}(Z\overline{Z})^{4-(4+\delta)} + C^{\mathcal{O}\mathcal{O}}_{8+2\delta,0}(Z\overline{Z})^{(4+\delta)-(4+\delta)}\right] \tag{D.1.28}$$

Using (D.1.26) and (D.1.27) we see that the singular parts result in a log term:

$$\lim_{\delta\to 0}\frac{C_{sing}}{\delta}\left[(Z\overline{Z})^{-\delta} - 1\right] = \lim_{\delta\to 0}\frac{C_{sing}}{\delta}\left[1 - \delta\,\log Z\overline{Z} + \mathcal{O}(\delta^2) - 1\right] = -C_{sing}\,\log Z\overline{Z} \tag{D.1.29}$$

Thus the complete contribution at $\mathcal{O}(\mu^2)$ is

$$\begin{aligned}\mathcal{G}_4\Big|_{\mu^2} = \int dxdy\Bigg(& C^{TT}_{reg} + C^{\mathcal{O}\mathcal{O}}_{reg} - C_{sing}\log Z\overline{Z} + \\ & C_{8,2}\frac{Z^2+Z\overline{Z}+\overline{Z}^2}{Z\overline{Z}} \\ & + C_{8,4}\frac{Z^4+Z^3\overline{Z}+Z^2\overline{Z}^2+Z\overline{Z}^3+\overline{Z}^4}{Z^2\overline{Z}^2}\Bigg).\end{aligned} \tag{D.1.30}$$

where $g_{8,2}$ and $g_{8,4}$ are composed out of the corresponding conformal blocks in the OPE limit:

$$\begin{aligned} g_{8,2} &= \frac{Z^2+Z\overline{Z}+\overline{Z}^2}{Z\overline{Z}} \\ g_{8,4} &= \frac{Z^4+Z^3\overline{Z}+Z^2\overline{Z}^2+Z\overline{Z}^3+\overline{Z}^4}{Z^2\overline{Z}^2}.\end{aligned} \tag{D.1.31}$$

It is evident, that the integral (D.1.30) is divergent and thus needs to be regulated. In practise we can do this using the dimensional regulatization: we multiply the integrand by a factor $|x|^{-\epsilon} = \left(t^2+x^2+y^2+z^2\right)^{-\frac{\epsilon}{2}}$, integrate and then expand the resulting expression around $\epsilon = 0$. This way we get:

$$\begin{aligned}\mathcal{G}_4\Big|_{\mu^2} =& \frac{8\pi t^2(C_{8,2}-3C_{8,4})}{\epsilon} \\ & + \pi\Bigg[C_{8,4}\frac{\left(15t^4-2t^2z^2-z^4+12t^2\left(t^2+z^2\right)\log\left(t^2+z^2\right)\right)}{t^2+z^2} \\ & + C_{8,2}\big(t^2+z^2-4t^2\log\left(t^2+z^2\right)\big) \\ & + (t^2+z^2)\Big(C_{sing}(\log(t^2+z^2)-1) \\ & - C^{TT}_{reg} - C^{\mathcal{O}\mathcal{O}}_{reg}\Big)\Bigg] + \mathcal{O}(\epsilon^1).\end{aligned} \tag{D.1.32}$$

D.1.2.3. Comparison

To compare the bulk and the CFT results, we can use equation (D.1.18). We are only interested in the double-trace sector

$$\mathcal{G}_4\Big|_{\mu^2} = \lim_{r\to\infty} r^4\Phi_{AdS}\frac{G^{8,1}+G^{8,2}\log r}{r^8}. \tag{D.1.33}$$

The RHS of this relation is obtained by taking the limit of the $\mathcal{O}(\mu^2)$ term in the bulk result (D.1.23), yielding:

$$\begin{aligned}\mathcal{G}_4\Big|_{\mu^2} = \frac{\pi}{1260}\Bigg[&420\Big(-6z^2 a^{8,1}_{6,0} + (t^2-7z^2)a^{8,1}_{8,0}\Big)\\ &+\mu^2\left(-\frac{2(75t^2z^2+61z^4)}{t^2+z^2} + 3(5t^2+z^2)\log\left(t^2+z^2\right)\right)\Bigg].\end{aligned} \tag{D.1.34}$$

Comparing (D.1.32) and (D.1.34) we can extract the coefficients $C_{8,2}$, $C_{8,4}$ and C_{sing}:

$$\begin{aligned}C_{8,2} &= \frac{\mu^2}{560}\\ C_{8,4} &= \frac{\mu^2}{720}\\ C_{sing} &= \frac{\mu^2}{420},\end{aligned} \tag{D.1.35}$$

while for the coefficients C^{TT}_{reg}, $C^{\mathcal{O}\mathcal{O}}_{reg}$ and the parameter ϵ we get the following relations

$$\begin{aligned}C^{\mathcal{O}\mathcal{O}}_{reg} + C^{TT}_{reg} &= 2a^{8,1}_{6,0} + \frac{7}{3}a^{8,1}_{8,0} + \frac{239\mu^2}{2520}\\ \frac{1}{\epsilon} &= -\frac{420(3a^{8,1}_{6,0}+4a^{8,1}_{8,0})+47\mu^2}{12\mu^2}.\end{aligned} \tag{D.1.36}$$

To conclude, our double-trace results for $C_{8,2}$, $C_{8,4}$ and the residual part of $C^{TT}_{8,0}$ are in a perfect agreement with the results for the non-integer case extrapolated to $\Delta = 4$, see [102], while the remaining CFT data is related to the undetermined coefficients on the bulk side by the relations (D.1.36).

Appendix D.2. List of bulk results for Z_1 and Z_2

In this appendix we list some expressions for the invariants in the shear and sound channels.

D.2.1. Results in the shear channel

For the source $\hat{H}_{tx}$ we find the solution of equation (7.8) at $\mathcal{O}(\mu^1)$ as

$$\mathcal{Z}^{(tx)}_1\Big|_{\mu^1} = \frac{\mu\rho\left(96\left(\rho^2+2\right)+3w^6+\left(6-4\rho^2\right)w^4-12\left(\rho^2+8\right)w^2\right)}{10\pi r w^{10}} \tag{D.2.1}$$

and at $\mathcal{O}(\mu^2)$ as

$$
\begin{aligned}
\mathcal{Z}_1^{(tx)}\Big|_{\mu^2} =& \frac{\mu^2\rho}{8400\pi r^5 w^{12}}\Big[-40320\left(\rho^2+2\right)^2-4920w^{12}\log(w)\\
&+\left(6920-7280\rho^2\right)w^{10}+5\left(272\rho^4-2880\rho^2+271\right)w^8\\
&+40\left(136\rho^4-331\rho^2-154\right)w^6\\
&+280\left(33\rho^4+26\rho^2-268\right)w^4+896(\rho^4+140\rho^2\\
&+262)w^2\Big]-\frac{12\rho\left(a_{8,0}^{8,2(tx)}\log(r)+a_{8,0}^{8,1(tx)}\right)}{\pi r^5},
\end{aligned}
\tag{D.2.2}
$$

where $a_{8,0}^{8,1(tx)}$ and $a_{8,0}^{8,2(tx)}$ are undetermined coefficients.

Choosing a source $\hat{H}_{xz}$, we get the bulk result as

$$
\begin{aligned}
\mathcal{Z}_1^{(xz)}\Big|_{\mu^1} = -\frac{f_0\sqrt{-\rho^2+w^2-1}}{10\pi r w^{10}}\Big[&96\left(\rho^2+2\right)+w^6+\left(2-4\rho^2\right)w^4\\
&-12\left(\rho^2+6\right)w^2\Big]
\end{aligned}
\tag{D.2.3}
$$

and

$$
\begin{aligned}
\mathcal{Z}_1^{(xz)}\Big|_{\mu^2} =& \frac{\mu^2\sqrt{-\rho^2+w^2-1}}{8400\pi r^5 w^{12}}\Big[40320\left(\rho^2+2\right)^2-4200w^{12}\log(w)\\
&+120\left(38\rho^2+17\right)w^{10}+5\left(-272\rho^4+1792\rho^2+437\right)w^8\\
&+8\left(-680\rho^4+885\rho^2+448\right)w^6-\\
&-168\left(55\rho^4+46\rho^2-284\right)w^4-896\left(\rho^4+122\rho^2+226\right)w^2\Big]+\\
&+\frac{12\sqrt{-\rho^2+w^2-1}\left(a_{8,0}^{8,2(xz)}\log(r)+a_{8,0}^{8,1(xz)}\right)}{\pi r^5}.
\end{aligned}
\tag{D.2.4}
$$

Using the results for the bulk-to-boundary propagator $\mathcal{Z}_1^{(xz)}$ (D.2.3) and (D.2.4) we obtain the correlator $G_{xz,xz}^{(bulk)}$ as

$$
\begin{aligned}
G_{xz,xz}^{(bulk)}\Big|_{\mu^0} =& -\frac{1}{\partial_t^2+\partial_z^2}\frac{3\pi C_T\left(z^2-7t^2\right)}{5\left(t^2+z^2\right)^5}\\
G_{xz,xz}^{(bulk)}\Big|_{\mu^1} =& -\frac{1}{\partial_t^2+\partial_z^2}\frac{3\pi\mu C_T\left(t^4-6t^2z^2+z^4\right)}{200\left(t^2+z^2\right)^4}\\
G_{xz,xz}^{(bulk)}\Big|_{\mu^2} =& \frac{1}{\partial_t^2+\partial_z^2}\Bigg[\frac{\pi\mu^2 C_T}{8400\left(t^2+z^2\right)^3}\Big(210t^6+648t^4z^2+6t^2z^4-160z^6\\
&+105\left(t^2+z^2\right)^3\log\left(t^2+z^2\right)\Big)-\frac{3}{5}\pi a_{8,0}^{8,1(xz)}C_T\Bigg].
\end{aligned}
\tag{D.2.5}
$$

D.2.2. Results in the sound channel

First we list the solutions of the sound channel equations of motion (7.8) for various polarizations.

For the source $\hat{H}_{tz}$ we get

$$\mathcal{Z}_2^{(tz)}\Big|_{\mu^1} = \frac{16\mu\rho\sqrt{-\rho^2+w^2-1}}{5\pi w^{12}}\left[-w^6-3w^4-96w^2+2\rho^2\left(w^4+4w^2+60\right)+240\right] \tag{D.2.6}$$

and

$$\begin{aligned}\mathcal{Z}_2^{(tz)}\Big|_{\mu^2} =& -\frac{4\mu^2\rho\sqrt{-\rho^2+w^2-1}}{315\pi r^4 w^{14}}\Big[18144\left(\rho^2+2\right)^2+798w^{12}\\ &+\left(1356-584\rho^2\right)w^{10}+\left(176\rho^4-2240\rho^2+1779\right)w^8\\ &+12\left(88\rho^4-384\rho^2+147\right)w^6\\ &+336\left(9\rho^4-17\rho^2+58\right)w^4+672\left(7\rho^4-55\rho^2-131\right)w^2\Big].\end{aligned} \tag{D.2.7}$$

For the source $\hat{H}_{tt}$ we get

$$\begin{aligned}\mathcal{Z}_2^{(tt)}\Big|_{\mu^1} =& -\frac{2\mu}{5\pi w^{12}}\Big[8\rho^4\left(w^4+4w^2+60\right)-8\rho^2\left(w^6+3w^4+66w^2-120\right)\\ &+w^2\left(w^6+2w^4+48w^2-96\right)\Big]\end{aligned} \tag{D.2.8}$$

and

$$\begin{aligned}\mathcal{Z}_2^{(tt)}\Big|_{\mu^2} =& \frac{\mu^2}{3150\pi r^4 w^{14}}\Big[362880\rho^2\left(\rho^2+2\right)^2+15960w^{14}\log(w)\\ &+120\left(279\rho^2-113\right)w^{12}-15\left(1072\rho^4-4048\rho^2+593\right)w^{10}\\ &+20\left(176\rho^6-3120\rho^4+4083\rho^2-294\right)w^8\\ &+120\left(176\rho^6-1083\rho^4+651\rho^2-406\right)w^6\\ &+1344(5\left(9\rho^4-24\rho^2+91\right)\rho^2\\ &+131)w^4+3360\left(28\rho^6-265\rho^4-632\rho^2-36\right)w^2\Big]\\ &+\frac{a_{0,0}^{8,1(tt)}+a_{0,0}^{8,2(tt)}\log(r)}{r^4}\end{aligned} \tag{D.2.9}$$

For the source $\hat{H}_{zz}$ we get

$$\begin{aligned}\mathcal{Z}_2^{(zz)}\Big|_{\mu^1} =& \frac{2\mu}{5\pi w^{12}}\Big[480\left(\rho^4+3\rho^2+2\right)+w^8+\left(2-8\rho^2\right)w^6\\ &+8\left(\rho^4-3\rho^2+31\right)w^4+16\left(2\rho^4-43\rho^2-72\right)w^2\Big]\end{aligned} \tag{D.2.10}$$

and

$$\mathcal{Z}_2^{(zz)}\Big|_{\mu^2} = \frac{\mu^2}{630\pi r^4 w^{14}}\Big[-72576\left(\rho^2+1\right)\left(\rho^2+2\right)^2+1560w^{14}\log(w)$$
$$-24\left(99\rho^2+16\right)w^{12}+3\left(720\rho^4-1776\rho^2-37\right)w^{10}$$
$$-4\left(176\rho^6-2240\rho^4+1791\rho^2+75\right)w^8$$
$$-24\left(176\rho^6-781\rho^4+177\rho^2-1806\right)w^6 \tag{D.2.11}$$
$$-4032\left(3\rho^6-5\rho^4+37\rho^2+81\right)w^4$$
$$-672\left(28\rho^6-255\rho^4-1032\rho^2-848\right)w^2\Big]$$
$$+\frac{a_{0,0}^{8,1(zz)}+a_{0,0}^{8,2(zz)}(0,0)\log(r)}{r^4}.$$

For the source $\hat{H}_{xx}$ we get

$$\mathcal{Z}_2^{(xx)}\Big|_{\mu^1} = -\frac{8\mu}{5\pi w^{12}}\Big[60\left(\rho^2+2\right)+25w^4-4\left(5\rho^2+33\right)w^2\Big] \tag{D.2.12}$$

and

$$\mathcal{Z}_2^{(xx)}\Big|_{\mu^2} = \frac{\mu^2}{3150\pi r^4 w^{14}}\Big[181440\left(\rho^2+2\right)^2-11880w^{14}\log(w)$$
$$+180\left(43-60\rho^2\right)w^{12}+15\left(176\rho^4-1136\rho^2+315\right)w^{10}$$
$$+10\left(880\rho^4-2292\rho^2+369\right)w^8$$
$$+120\left(151\rho^4-237\rho^2-700\right)w^6+672\left(45\rho^4+100\rho^2+1084\right)w^4$$
$$+3360\left(5\rho^2\left(\rho^2-40\right)-406\right)w^2\Big]+\frac{a_{0,0}^{8,1(xx)}+a_{0,0}^{8,2(xx)}\log(r)}{r^4}. \tag{D.2.13}$$

Using the prescription (7.39) for the sound channel and the solutions above, we get the correlator order-by-order in μ for the source $\hat{H}_{tt}$ as

$$G_{tt,tt}^{(bulk)}\Big|_{\mu^0} = \frac{1}{(\partial_t^2+\partial_z^2)^2}\frac{96\pi C_T\left(t^4-18t^2z^2+21z^4\right)}{5\left(t^2+z^2\right)^7}$$
$$G_{tt,tt}^{(bulk)}\Big|_{\mu^1} = \frac{1}{(\partial_t^2+\partial_z^2)^2}\frac{4\pi\mu C_T\left(t^6-15t^4z^2+15t^2z^4-z^6\right)}{15\left(t^2+z^2\right)^6}$$
$$G_{tt,tt}^{(bulk)}\Big|_{\mu^2} = -\frac{2\pi\mu^2 C_T}{(\partial_t^2+\partial_z^2)^2}\frac{\left(-691t^8+1900t^6z^2+1910t^4z^4+860t^2z^6+133z^8\right)}{1575\left(t^2+z^2\right)^5}, \tag{D.2.14}$$

and for the source $\hat{H}_{zz}$ as

$$
\begin{aligned}
G^{(bulk)}_{zz,zz}\Big|_{\mu^0} =& \frac{1}{(\partial_t^2+\partial_z^2)^2}\frac{96\pi C_T\left(21t^4-18t^2z^2+z^4\right)}{5\left(t^2+z^2\right)^7}\\
G^{(bulk)}_{zz,zz}\Big|_{\mu^1} =& \frac{1}{(\partial_t^2+\partial_z^2)^2}\frac{4\pi\mu C_T\left(t^6-15t^4z^2+15t^2z^4-z^6\right)}{15\left(t^2+z^2\right)^6}\\
G^{(bulk)}_{zz,zz}\Big|_{\mu^2} =& \frac{2\pi\mu^2 C_T}{(\partial_t^2+\partial_z^2)^2}\frac{\left(-65t^8-724t^6z^2+810t^4z^4+140t^2z^6+79z^8\right)}{1575\left(t^2+z^2\right)^5}.
\end{aligned}
\tag{D.2.15}
$$

Finally, using the relation (7.39) we get the $G^{bulk}_{xx,xx}$ in the form

$$
\begin{aligned}
G^{(bulk)}_{xx,xx}\Big|_{\mu^0} =& \frac{1}{\partial_t^2+\partial_z^2}\frac{24\pi C_T}{5\left(t^2+z^2\right)^4}\\
G^{(bulk)}_{xx,xx}\Big|_{\mu^1} =& 0\\
G^{(bulk)}_{xx,xx}\Big|_{\mu^2} =& \frac{1}{\partial_t^2+\partial_z^2}\left[\frac{\pi\mu^2 C_T}{3150}\left(126\log\left(t^2+z^2\right)+\frac{-135t^4+90t^2z^2-71z^4}{\left(t^2+z^2\right)^2}\right)\right.\\
&\left. -\frac{1}{60}\pi C_T\left(72\left(a^{8,1(xy)}_{6,0}+a^{8,1(xy)}_{8,0}\right)+\pi a^{8,1(xx)}_{0,0}\right)\right],
\end{aligned}
\tag{D.2.16}
$$

where the undetermined coefficients $a^{8,1(xy)}_{6,0}$ and $a^{8,1(xy)}_{8,0}$ come from the scalar channel contribution and $a^{8,1(xx)}_{0,0}$ originates in the sound channel.

Appendix D.3. Conventions and details on spinning conformal correlators

In this appendix we summarize our conventions and provide some details on spinning conformal correlators in embedding space that is used in the main part of Section 7.4 following [175,174]. The basic building blocks are

$$
\begin{aligned}
V_{i,jk} &= \frac{(Z_i\cdot P_j)(P_i\cdot P_k)-(Z_i\cdot P_k)(P_i\cdot P_j)}{P_j\cdot P_k},\\
H_{ij} &= -2[(Z_i\cdot Z_j)(P_i\cdot P_j)-(Z_i\cdot P_j)(Z_j\cdot P_i)],
\end{aligned}
\tag{D.3.1}
$$

where $V_1\equiv V_{1,23}$, $V_2\equiv V_{2,31}$ and $V_3\equiv V_{3,12}$. Here P_i and Z_i are null vectors in $R^{1,d+1}$.

One possible basis for the three-point function of two stress tensors and a spin-J operator with dimension Δ is given by ($P_{ij} = -2P_i \cdot P_j$)

$$\langle T(P_1,Z_1)T(P_2,Z_2)\mathcal{O}(P_3,Z_3)\rangle = \frac{\sum_{p=1}^{10} x_p^{(TT\mathcal{O})} Q_p}{(P_{12})^{d+2-\frac{\Delta+J}{2}}(P_{23})^{\frac{\Delta+J}{2}}(P_{31})^{\frac{\Delta+J}{2}}}, \quad \text{(D.3.2)}$$

where

$$\begin{aligned}
Q_1 &= V_1^2 V_2^2 V_3^J, \\
Q_2 &= (H_{23}V_1^2V_2 + H_{13}V_2^2V_1)V_3^{J-1}, \\
Q_3 &= H_{12}V_1V_2V_3^J, \\
Q_4 &= (H_{13}V_2 + H_{23}V_1)H_{12}V_3^{J-1}, \\
Q_5 &= H_{13}H_{23}V_1V_2V_3^{J-2}, \\
Q_6 &= H_{12}^2V_3^J, \\
Q_7 &= (H_{13}^2V_2^2 + H_{23}^2V_1^2)V_3^{J-2}, \\
Q_8 &= H_{12}H_{13}H_{23}V_3^{J-2}, \\
Q_9 &= (H_{13}H_{23}^2V_1 + H_{23}H_{13}^2V_2)V_3^{J-3}, \\
Q_{10} &= H_{13}^2H_{23}^2V_3^{J-4}.
\end{aligned} \quad \text{(D.3.3)}$$

Conservation of the stress tensor further reduces the number of independent structures. In particular, when $\mathcal{O} = T$ there are 3 independent structures while for non-conserved operators of dimension Δ and spin $J = 0, 2, 4$, there are $1, 2$ and 3 independent structures, respectively. However, we will mainly consider the differential basis introduced in [175] since this is powerful when considering the four-point conformal blocks. It is based on multiplication by H_{12} as well as the differential operators

$$\begin{aligned}
D_{11} =& (P_1\cdot P_2)(Z_1\cdot \frac{\partial}{\partial P_2}) - (Z_1\cdot P_2)(P_1\cdot\frac{\partial}{\partial P_2}) \\
&- (Z_1\cdot Z_2)(P_1\cdot\frac{\partial}{\partial Z_2}) + (P_1\cdot Z_2)(Z_1\cdot\frac{\partial}{\partial Z_2}), \\
D_{12} =& (P_1\cdot P_2)(Z_1\cdot\frac{\partial}{\partial P_1}) - (Z_1\cdot P_2)(P_1\cdot\frac{\partial}{\partial P_1}) + (Z_1\cdot P_2)(Z_1\cdot\frac{\partial}{\partial Z_1}),
\end{aligned} \quad \text{(D.3.4)}$$

and D_{22} and D_{21} obtained from D_{11} and D_{12} by $1 \leftrightarrow 2$. We further define the following differential operators:

$$
\begin{aligned}
D_1 &= D_{11}^2 D_{22}^2 \Sigma^{2,2},\\
D_2 &= H_{12} D_{11} D_{22} \Sigma^{2,2},\\
D_3 &= D_{21} D_{11}^2 D_{22} \Sigma_L^{3,1} + D_{12} D_{22}^2 D_{11} \Sigma_L^{1,3},\\
D_4 &= H_{12}(D_{21} D_{11} \Sigma_L^{3,1} + D_{12} D_{22} \Sigma_L^{1,3}),\\
D_5 &= D_{12} D_{21} D_{11} D_{22} \Sigma^{2,2},\\
D_6 &= H_{12}^2 \Sigma^{2,2},\\
D_7 &= D_{21}^2 D_{11}^2 \Sigma_L^{4,0} + D_{12}^2 D_{22}^2 \Sigma_L^{0,4},\\
D_8 &= H_{12} D_{12} D_{21} \Sigma^{2,2},\\
D_9 &= D_{12}^2 D_{21}^2 \Sigma^{2,2},\\
D_{10} &= D_{12} D_{21}^2 D_{11} \Sigma^{3,1} + D_{21} D_{12}^2 D_{22} \Sigma^{1,3},
\end{aligned}
\tag{D.3.5}
$$

where $\Sigma_L^{m,n}$ denotes a shift $\Delta_1 \to \Delta_1 + m$ and $\Delta_2 \to \Delta_2 + n$. The three-point functions in the differential basis are then given by

$$
\begin{aligned}
&\langle T(P_1, Z_1) T(P_2, Z_2) \mathcal{O}_{\Delta,J}(P_3, Z_3)\rangle \\
&\qquad = \sum_{i=1}^{10} \lambda^{(i)}_{TT\mathcal{O}_{\Delta,J}} D_i \frac{V_3^J}{P_{12}^{\Delta_1+\Delta_2-\Delta-J} P_{23}^{\Delta+\Delta_2-\Delta_1+J} P_{13}^{\Delta+\Delta_1-\Delta_2+J}},
\end{aligned}
\tag{D.3.6}
$$

where we kept $\Delta_{1,2}$ to keep track of the action of $\Sigma_L^{(\cdot,\cdot)}$ in (D.3.5).

The spinning conformal partial waves can be obtained from the scalar partial waves $W_{\mathcal{O}}$:

$$
W_{\mathcal{O}} = \left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_{12}}{2}} \left(\frac{P_{14}}{P_{13}}\right)^{\frac{\Delta_{34}}{2}} \frac{g_{\Delta,J}^{(\Delta_{12},\Delta_{34})}(z,\bar{z})}{P_{12}^{\frac{\Delta_1+\Delta_2}{2}} P_{34}^{\frac{\Delta_3+\Delta_4}{2}}}
\tag{D.3.7}
$$

with $\Delta_{ij} = \Delta_i - \Delta_j$ and the cross-ratios (u, v) are given by

$$
\begin{aligned}
u &= \frac{P_{12} P_{34}}{P_{13} P_{24}},\\
v &= \frac{P_{14} P_{23}}{P_{13} P_{24}}.
\end{aligned}
\tag{D.3.8}
$$

The scalar blocks are normalized as follows in the limit $u \to 0, v \to 1$:

$$
g_{\Delta,J}^{(\Delta_{12},\Delta_{34})}(z,\bar{z}) \sim \frac{J!}{(-2)^J (\frac{d}{2}-1)_J} (z\bar{z})^{\frac{\Delta}{2}} C_J^{(\frac{d}{2}-1)}\Big(\frac{v-1}{2\sqrt{u}}\Big),
\tag{D.3.9}
$$

where $C_J^{(n)}$ are Gegenbauer polynomials and $(a)_J$ denote the Pochammer symbol. The spinning conformal partial waves are then obtained by

$$W_{\mathcal{O}}^{\{i\}} = \mathcal{D}_L \mathcal{D}_R W_{\mathcal{O}}, \tag{D.3.10}$$

where

$$\mathcal{D}_L = H_{12}^{n_{12}} D_{12}^{n_{10}} D_{21}^{n_{20}} D_{11}^{m_1} D_{22}^{m_2} \Sigma_L^{m_1+n_{20}+n_{12}, m_2+n_{10}+n_{12}}, \tag{D.3.11}$$

where i labels the structure in the scalar partial wave and $\mathcal{D}_R$ is similarly defined with $1 \to 3$ and $2 \to 4$. The integers $n_{ij} \geq 0$ and m_i that labels the structure are determined by the solutions to the following equations ensuring the correct homogeneity under $P \to \alpha P$ and $Z \to \beta Z$:

$$\begin{aligned} m_1 &= J_1 - n_{12} - n_{12} \geq 0, \\ m_2 &= J_2 - n_{12} - n_{20} \geq 0, \\ m_0 &= J_0 - n_{10} - n_{20} \geq 0, \end{aligned} \tag{D.3.12}$$

where $J = J_0$ is the spin of the exchanged operator. In the case of two spin-2 operators at P_1 and P_2 and scalar operators at P_3 and P_4, the possible combinations appearing in (D.3.11) can be taken to be the ones given in (D.3.5).

We are interested in the OPE limit of the contribution of individual blocks to

$$\hat{G} := P_{34}^{\Delta_H} \langle T(P_1, Z_1) T(P_2, Z_2) \mathcal{O}_H(P_3) \mathcal{O}_H(P_4) \rangle \tag{D.3.13}$$

where $\mathcal{O}_H$ is a scalar operator with dimension Δ_H. In this case we have using (D.3.10)

$$\hat{G}(P_i, Z_i)|_{\mathcal{O}_{\Delta,J}} = \sum_{i=1}^{10} \lambda_{TT\mathcal{O}_{\Delta,J}}^{(i)} \lambda_{\mathcal{O}_H \mathcal{O}_H \mathcal{O}_{\Delta,J}} D_i \left(\frac{P_{24}}{P_{14}}\right)^{\frac{\Delta_{12}}{2}} \frac{g_{\Delta,J}^{(\Delta_{12},0)}(u,v)}{P_{12}^{\frac{\Delta_1+\Delta_2}{2}}}, \tag{D.3.14}$$

where the differential operators D_i are given by (D.3.5) and $\Delta_1 = \Delta_2 = d$.

The spinning correlator in embedding space with indices is then obtained using

$$\hat{G}_{MN,PS}(P_i) = \frac{1}{2^2(\frac{d}{2}-1)^2} \hat{D}_M^{(1)} \hat{D}_N^{(1)} \hat{D}_P^{(2)} \hat{D}_S^{(2)} \hat{G}(P_i, Z_i) \tag{D.3.15}$$

where $\hat{D}^{(i)}_M$ is given by

$$\hat{D}^{(i)}_M = \left(\frac{d-2}{2} + Z_i \cdot \frac{\partial}{\partial Z_i}\right)\frac{\partial}{\partial Z_i^M} - \frac{1}{2} Z_{iM}\frac{\partial^2}{\partial Z^2}. \tag{D.3.16}$$

In order to project down to physical space one imposes $P_i^M = (1, x_i^2, x_i^\mu)$ and contract indices in embedding space with $\frac{\partial P_i^M}{\partial x^\nu} = (0, 2x_\nu^{(i)}, \delta_\nu^\mu)$ [175,174]. We then set $x_1^\mu = (1, \vec{0})$, $x_2^\mu = (1+t, \vec{x})$, $x_3^\mu = (0, \vec{0})$ and $x_4 \to \infty$ with $|x_{21}| \ll 1$ in the OPE limit, such that $u \to 0$ and $v \to 1$.

D.3.1. Stress tensor block

The relation between different basis for the stress tensor three-point function can be found in e.g. Appendix C.1 in [180], some of which we summarize here for convenience. In embedding space formalism [175,174] the stress tensor three-point function can be built from (D.3.2)

$$\langle T(P_1, Z_1) T(P_2, Z_2) T(P_3, Z_3)\rangle = \frac{\sum_{p=1}^{8} x_p Q_p}{P_{12}^{\frac{d+2}{2}} P_{23}^{\frac{d+2}{2}} P_{31}^{\frac{d+2}{2}}}, \tag{D.3.17}$$

and the coefficients $x_p \equiv x_p^{(TTT)}$ are constrained due to permutation symmetry and conservation to satisfy

$$\begin{aligned}
x_1 &= 2x_2 + \frac{1}{4}(d^2 + 2d - 8)x_4 - \frac{1}{2}d(d+2)x_7,\\
x_8 &= \frac{1}{\frac{d^2}{2} - 2}\Big[x_2 - \Big(\frac{d}{2} + 1\Big)x_4 + 2dx_7\Big],\\
x_2 &= x_3,\\
x_4 &= x_5,\\
x_6 &= x_7.
\end{aligned} \tag{D.3.18}$$

The stress tensor three-point function can be parameterized in terms of $(\hat{a}, \hat{b}, \hat{c})$ [177] where one of these can further be traded for C_T using the Ward identity

$$C_T = 4S_d \frac{(d-2)(d+3)\hat{a} - 2\hat{b} - (d+1)\hat{c}}{d(d+2)}. \tag{D.3.19}$$

For the relation between the x_p basis, $(\hat{a}, \hat{b}, \hat{c})$ and the (t_2, t_4) coefficients that are natural when considering a conformal collider setup, we refer the reader to

App. C in [180]. However, we recall C.10 in [24] that relates these to t_2 and t_4 in $d=4$:

$$\begin{aligned} t_2 =& \frac{30(13\hat{a}+4\hat{b}-3\hat{c})}{14\hat{a}-2\hat{b}-5\hat{c}} \\ t_4 =& -\frac{15(81\hat{a}+32\hat{b}-20\hat{c})}{2(14\hat{a}-2\hat{b}-5\hat{c})} \end{aligned} \tag{D.3.20}$$

and for $t_2=t_4=0$ one finds $\hat{a}=\frac{4\hat{c}}{23}$ and $\hat{b}=\frac{17\hat{c}}{92}$. On the other hand, the ratio of the anomaly coefficients a, c are given by C.12 in [24]

$$\frac{a}{c} = \frac{9\hat{a}-2\hat{b}-10\hat{c}}{3(14\hat{a}-2\hat{b}-5\hat{c})}, \tag{D.3.21}$$

with $a=c$ when $t_2=t_4=0$. We further need stress tensor three-point function with the two heavy scalar operators

$$\langle \mathcal{O}_H(x_1)\mathcal{O}_H(x_2)T_{\mu\nu}(x_3)\rangle = \lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}} \frac{W_\mu W_\nu - \frac{1}{d}W^2\delta_{\mu\nu}}{x_{12}^{2\Delta_H-2}x_{23}^2 x_{31}^2}, \tag{D.3.22}$$

where $W^\mu = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}$. Conformal Ward identities fixes $\lambda_{\mathcal{O}_H\mathcal{O}_H T}$ to be

$$\lambda_{\mathcal{O}_H\mathcal{O}_H T_{\mu\nu}} = -\frac{d}{d-1}\frac{\Delta_H}{S_d}, \tag{D.3.23}$$

where $S_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}$ and is related to μ and β according to (7.54) and (7.55).

From now on we consider $d=4$. For the stress tensor block we will work with parametrization in terms of $(\hat{a},\hat{b},\hat{c})$. In the channel $\hat{G}_{xy,xy}$, following the procedure described above, one obtains

$$\begin{aligned} &\hat{G}_{xy,xy}|_T = \frac{\Delta_H}{2\pi^4(14\hat{a}-2\hat{b}-5\hat{c})(t^2+\vec{x}^2)^5} \times \\ &\times \Big[4\hat{b}(-5t^4(x^2+y^2)+\vec{x}^2(x^4+6x^2y^2+y^4+(x^2+y^2)z^2)-4t^2(x^4+9x^2y^2+y^4 \\ &+(x^2+y^2)z^2))+\hat{c}(-3t^6+t^4(13(x^2+y^2)-5z^2)-\vec{x}^2(5x^4-6x^2y^2+5y^4 \\ &+4(x^2+y^2)z^2-z^4)+t^2(11x^4+102x^2y^2+11y^4+10(x^2+y^2)z^2-z^4)) \\ &+4\hat{a}(t^6+t^4(-17(x^2+y^2)+3z^2)-t^2(13x^4+106x^2y^2+13y^4+10(x^2+y^2)z^2 \\ &-3z^4)+\vec{x}^2(5x^4-6x^2y^2+5y^4+6(x^2+y^2)z^2+z^4))\Big], \end{aligned} \tag{D.3.24}$$

where $\vec{x} = (x, y, z)$ which after integrating over x and y gives

$$G_{xy,xy}|_T = \int dxdy \hat{G}_{xy,xy}|_T = -\frac{2(7\hat{a} + 2\hat{b} - \hat{c})\Delta_H(t^2 - z^2)}{3\pi^3(14\hat{a} - 2\hat{b} - 5\hat{c})(t^2 + z^2)^2}. \tag{D.3.25}$$

and for $t_2 = t_4 = 0$:

$$G_{xy,xy}|_T = \frac{2\Delta_H(t^2 - z^2)}{15\pi^3(t^2 + z^2)^2}. \tag{D.3.26}$$

The results (D.3.25) and (D.3.26) are in agreement with [178] which used the explicit OPE in order to evalute the stress tensor two-point function in a thermal state.

Consider now $\hat{G}_{tx,tx}|_T$, one finds that it is given by

$$\begin{aligned}\hat{G}_{tx,tx}|_T =& \frac{\Delta_H}{2\pi^4(14\hat{a} - 2\hat{b} - 5\hat{c})(t^2 + \vec{x}^2)^5} \times \\ & \times \Big[4\hat{b}(-t^6 - x^2\vec{x}^4 + t^4(-23x^2 + 4(y^2 + z^2)) \\ & + t^2\vec{x}^2(9x^2 + 5(y^2 + z^2))) + \hat{c}(15t^6 - (5x^2 - y^2 - z^2)\vec{x}^4 + t^4(41x^2 \\ & + 7(y^2 + z^2)) - t^2\vec{x}^2(43x^2 + 7(y^2 + z^2))) \\ & + 4\hat{a}(-13t^6 + (3x^2 - y^2 - z^2)\vec{x}^4 - 3t^4(13x^2 + y^2 + z^2) + t^2\vec{x}^2(41x^2 \\ & + 9(y^2 + z^2)))\Big],\end{aligned} \tag{D.3.27}$$

which after integrating over x and y gives

$$G_{tx,tx}|_T = -\Delta_H \frac{(64\hat{a} + 14\hat{b} - 19\hat{c})t^4 - 12(16\hat{a} + 5\hat{b} - 4\hat{c})t^2z^2 + 3(2\hat{b} + \hat{c})z^4}{12(14\hat{a} - 2\hat{b} - 5\hat{c})\pi^3(t^2 + z^2)^3}. \tag{D.3.28}$$

For $t_2 = t_4 = 0$ this reduces to

$$G_{tx,tx}|_T = \Delta_H \frac{-9t^4 + 6t^2z^2 + 7z^4}{60\pi^3(t^2 + z^2)^3}. \tag{D.3.29}$$

Consider now $\hat{G}_{tz,tz}$. Before integration there is an $SO(3)$ rotational symmetry so $\hat{G}_{tz,tz}$ can be obtained from (D.3.27) by $x \leftrightarrow z$. Integrating over the xy-plane one finds

$$\begin{aligned}G_{tz,tz}|_T =& \frac{\Delta_H}{(14\hat{a} - 2\hat{b} - 5\hat{c})\pi^3(t^2 + z^2)^4}\Big[(-6\hat{a} + \hat{b} + 2\hat{c})t^6 \\ & + (-10\hat{a} - 7\hat{b} + 3\hat{c})t^4z^2 + (30\hat{a} + 7\hat{b} - 8\hat{c})t^2z^4 + (2\hat{a} - \hat{b} - \hat{c})z^6\Big]\end{aligned} \tag{D.3.30}$$

For $t_2 = t_4 = 0$ this reduces to

$$G_{tz,tz}|_T = \Delta_H \frac{-105t^6 + 3t^4z^2 + 137t^2z^4 + 77z^6}{270\pi^3(t^2+z^2)^4}. \tag{D.3.31}$$

D.3.2. Spin-0 double-stress tensor block

The simplest double-stress tensor operator is the scalar operator $[T^2]_{J=0}$ with dimension Δ_0. In the differential basis we need the differential operators D_1, D_2 and D_6 from (D.3.5) and the three-point function is give by (D.3.6) with $\lambda_{i,0} \equiv \lambda^{(i)}_{TT[T^2]_{J=0}}$. In order to impose conservation one demands that $\frac{\partial}{\partial P_M}\hat{D}_M$ acting on (D.3.6) is 0 [174], where $\hat{D}_M$ is given by (D.3.16). This implies that the number of structures reduce

$$\begin{aligned} \lambda_{2,0} &= -\frac{3}{4}(\Delta_0 - 6)(\Delta_0 + 2)\lambda_{1,0}, \\ \lambda_{6,0} &= \frac{3}{32}(\Delta_0 - 6)(\Delta_0 - 4)\Delta_0(\Delta_0 + 2)\lambda_{1,0}, \end{aligned} \tag{D.3.32}$$

and one is left with a single coefficient $\lambda_{1,0}$. The corresponding contribution to the correlator $\hat{G}(P_i, Z_i)$ (in embedding space) is given by

$$\hat{G}(P_i, Z_i)|_{[T^2]_0} = \sum_{i=1,3,6} \rho_{i,0} D_i W_{[T^2]_0}, \tag{D.3.33}$$

where the conformal partial wave $W_{[T^2]_0}$ is given by (D.3.7). Note that the coefficients $\rho_{i,0}$ are related to $\lambda_{i,0}$ by an overall factor of the one-point function in the scalar state, they therefore satisfy the same conservation condition as the λ's in (D.3.32). The projection to the physical space and the relevant kinematics are described in the first part of this appendix.

D.3.3. Spin-2 double-stress tensor block

Because the spin-2 double-stress tensor $[T^2]_{J=2}$ is not conserved there will be only two structures in the three-point function compared to 3 for the stress tensor, even though they both have $J = 2$. In the differential basis these can be labelled $\lambda_{i,2} \equiv \lambda^{(i)}_{TT[T^2]_{J=2}}$ with $i = 1, 2, \ldots 8$ in (D.3.6), which are reduced to two coefficients by imposing conservation:

$$
\begin{aligned}
\lambda_{3,2} &= \frac{(\Delta_2+2)\,(192\lambda_{2,2}-(\Delta_2-4)\,\Delta_2\,((3\Delta_2-16)\,(3\Delta_2+4)\,\lambda_{1,2}+20\lambda_{2,2}))}{6\Delta_2\,(\Delta_2\,(\Delta_2\,((\Delta_2-8)\,\Delta_2+2)+56)+96)},\\
\lambda_{4,2} &= \frac{(\Delta_2-4)\,(\Delta_2+2)}{16\,(\Delta_2\,(\Delta_2\,((\Delta_2-8)\,\Delta_2+2)+56)+96)}\Big[(((\Delta_2-4)\Delta_2(3(\Delta_2-4)\Delta_2\\
&\quad -52)-64)\lambda_{1,2}+4(\Delta_2-8)(\Delta_2+4)\lambda_{2,2})\Big],\\
\lambda_{5,2} &= \frac{(\Delta_2-4)\,\Delta_2\,((15\,(\Delta_2-4)\,\Delta_2+52)\,\lambda_{1,2}+52\lambda_{2,2})-96\lambda_{2,2}}{12\,(\Delta_2\,(\Delta_2\,((\Delta_2-8)\,\Delta_2+2)+56)+96)},\\
\lambda_{6,2} &= \frac{(\Delta_2-4)\,\Delta_2}{128\,(\Delta_2\,(\Delta_2\,((\Delta_2-8)\,\Delta_2+2)+56)+96)}\Big[((256\\
&\quad -(\Delta_2-4)\Delta_2((\Delta_2-4)\Delta_2(3(\Delta_2-4)\Delta_2-56)+688))\lambda_{1,2}\\
&\quad -4((\Delta_2-4)\Delta_2(5(\Delta_2-4)\Delta_2-52)+416)\lambda_{2,2})\Big]\\
&\quad -\frac{48\lambda_{2,2}}{\Delta_2(\Delta_2((\Delta_2-8)\Delta_2+2)+56)+96},\\
\lambda_{7,2} &= \frac{(\Delta_2+2)\,(\Delta_2+4)}{12\,(\Delta_2-2)\,\Delta_2\,(\Delta_2\,(\Delta_2\,((\Delta_2-8)\,\Delta_2+2)+56)+96)}\Big[((\Delta_2-4)\Delta_2\times\\
&\quad \times((3(\Delta_2-4)\Delta_2-44)\lambda_{1,2}+4\lambda_{2,2})-96\lambda_{2,2})\Big],\\
\lambda_{8,2} &= -\frac{(3\,(\Delta_2-4)\,\Delta_2+16)}{48\,(\Delta_2\,(\Delta_2\,((\Delta_2-8)\,\Delta_2+2)+56)+96)}\Big[((\Delta_2-4)\Delta_2((3(\Delta_2-4)\Delta_2\\
&\quad -44)\lambda_{1,2}+4\lambda_{2,2})-96\lambda_{2,2})\Big].
\end{aligned}
\tag{D.3.34}
$$

The corresponding contribution to the correlator $\hat{G}(P_i, Z_i)$ is given by

$$
\hat{G}(P_i, Z_i)|_{[T^2]_2} = \sum_{i=1}^{8} \rho_{i,2} D_i W_{[T^2]_2}, \tag{D.3.35}
$$

where the conformal partial wave $W_{[T^2]_2}$ is given by (D.3.7). Again, the coefficients $\rho_{i,2}$ are related to $\lambda_{i,2}$ by an overall factor of the one-point function in the scalar state, they therefore satisfy the same conservation condition as the λ's in (D.3.34). The projection to the physical space and the relevant kinematics are described in the first part of this appendix.

D.3.4. Spin-4 double-stress tensor block

For the spin-4 double-stress tensor operator $[T^2]_{J=4}$ there are a priori 10 structures labelled by $\lambda_{i,4} \equiv \lambda^{(i)}_{TT[T^2]_{J=4}}$ with $i = 1, 2, \ldots 10$ in (D.3.6). Conservation reduces the number of structures to 3 according as follows

$$
\begin{aligned}
\lambda_{4,4} = {} & \frac{1}{96((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4-44)+192)}\Big[(-((\Delta_4-4)\Delta_4((\Delta_4-4)\times \\
& \times \Delta_4((\Delta_4-4)\Delta_4(3(\Delta_4-4)\Delta_4-200)+5712)-92032))-485376)\lambda_{3,4} \\
& -2(\Delta_4-6)(\Delta_4+4)(((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4(3(\Delta_4-4)\Delta_4-68)-1024) \\
& +13056)\lambda_{1,4}+2((\Delta_4-4)\Delta_4(3(\Delta_4-4)\Delta_4-116)+768)\lambda_{2,4})\Big], \\
\lambda_{5,4} = {} & \frac{1}{8}(2((\Delta_4-4)\Delta_4+16)\lambda_{1,4}+(\Delta_4-8)(\Delta_4+2)\lambda_{3,4}+4\lambda_{2,4}), \\
\lambda_{6,4} = {} & \frac{1}{256(\Delta_4-6)(\Delta_4+4)((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4-44)+192)}\Big[2(\Delta_4-6)\times \\
& \times(\Delta_4+4)(((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4 \\
& -64)+1040)+11392)-262144)+2162688)\lambda_{1,4}+2((\Delta_4-4)\Delta_4\times \\
& \times((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4-88)+2448)-17408) \\
& +86016)\lambda_{2,4})+((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4\times \\
& \times((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4-108)+5104)-131904) \\
& +2009088)-18300928)+81788928)\lambda_{3,4}\Big], \\
\lambda_{7,4} = {} & \frac{1}{24((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4-44)+192)}\Big[(\Delta_4-4)(\Delta_4+6)\times \\
& \times(2(\Delta_4-6)(\Delta_4+4)(((\Delta_4-4)\Delta_4+20)\lambda_{1,4}+2\lambda_{2,4})+((\Delta_4-4)\Delta_4\times \\
& \times((\Delta_4-4)\Delta_4-36)+704)\lambda_{3,4})\Big], \\
\lambda_{8,4} = {} & \frac{1}{32(\Delta_4-6)(\Delta_4+4)((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4-44)+192)}\Big[2(\Delta_4-6)\times \\
& \times(\Delta_4+4)(((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4-20)((\Delta_4-4)\Delta_4 \\
& +24)+4736)+135168)\lambda_{1,4}+2((\Delta_4-4)\Delta_4((\Delta_4-6)(\Delta_4-4)\Delta_4(\Delta_4+2) \\
& -320)+7680)\lambda_{2,4})+(\Delta_4(\Delta_4(\Delta_4(\Delta_4(\Delta_4(((\Delta_4-20)\Delta_4+120)\Delta_4^3-1968\Delta_4 \\
& +2112)+23296)-78848)-327680)+1638400)+5111808)\lambda_{3,4}\Big],
\end{aligned}
\tag{D.3.36}
$$

and

$$\lambda_{9,4} = \frac{1}{3(\Delta_4-6)(\Delta_4+4)((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4-44)+192)}\Big[(\Delta_4-6)\times$$
$$\times(\Delta_4+4)\times(((\Delta_4-4)\Delta_4(9(\Delta_4-4)\Delta_4+68)-768)\lambda_{1,4}$$
$$+16((\Delta_4-4)\Delta_4-6)\lambda_{2,4})+((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4\times$$
$$\times(3(\Delta_4-4)\Delta_4-116)+3136)-15360)\lambda_{3,4}\Big],$$
$$\lambda_{10,4} = \frac{1}{12((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4-44)+192)}\Big[((\Delta_4-4)\Delta_4+12)\times$$
$$\times(-2(\Delta_4-6)\times(\Delta_4+4)(((\Delta_4-4)\Delta_4+20)\lambda_{1,4}+2\lambda_{2,4})$$
$$-((\Delta_4-4)\Delta_4((\Delta_4-4)\Delta_4-36)+704)\lambda_{3,4})\Big]. \tag{D.3.37}$$

The corresponding contribution to the correlator $\hat{G}(P_i, Z_i)$ is given by

$$\hat{G}(P_i, Z_i)|_{[T^2]_4} = \sum_{i=1}^{10} \rho_{i,4} D_i W_{[T^2]_4}, \tag{D.3.38}$$

where the conformal partial wave $W_{[T^2]_4}$ is given by (D.3.7). The coefficients $\rho_{i,4}$ are related to $\lambda_{i,4}$ by an overall factor of the one-point function in the scalar state, they therefore satisfy the same conservation condition as the λ's in (D.3.36) and (D.3.37). The projection to the physical space and the relevant kinematics are described in the first part of this appendix.

D.3.5. Integrated double stress tensor contribution

In this section we list the explicit expression for the integrated $\mathcal{O}(C_T\mu^2)$ part of the conformal block expansion of $G_{xy,xy}$, $G_{tx,tx}$ and $G_{tz,tz}$ obtained using the procedure described above. The integrals over the double-stress tensor blocks are divergent which we regulate by including a factor of $|x|^{-\epsilon}$, this produces simple poles at $\epsilon \to 0$. For $G_{xy,xy}$ one finds as $\epsilon \to 0$:

$$G_{xy,xy}|_{\mu^2 C_T} = p^{(0)}_{xy,xy}(t,z) + p^{(1)}_{xy,xy}(t,z)\log(t^2+z^2) + \frac{c_1 t^2 + c_2 z^2}{\epsilon} \tag{D.3.39}$$

where c_1, c_2 are some constants depending on the CFT data and

$$p^{(0)}_{xy,xy}(t,z) = \frac{\pi^5\mu^2 C_T}{1693440000(t^2+z^2)} \sum_{j=0}^{2} p^{(0,2j)}_{xy,xy} t^{4-2j} z^{2j} \tag{D.3.40}$$

with

$$
\begin{aligned}
p^{(0,0)}_{xy,xy} &= -8(22050\rho^{(1)}_{1,0} - 162243\rho^{(1)}_{1,2} - 11683490\rho^{(1)}_{1,4} + 129168\rho^{(1)}_{2,2} + 4702775\rho^{(1)}_{2,4} \\
&\quad + 6991740\rho^{(1)}_{3,4}) + 4410\gamma^{(1)}_0 + 304479\gamma^{(1)}_2 - 3577875\gamma^{(1)}_4, \\
p^{(0,2)}_{xy,xy} &= 2\Big(- 8(22050\rho^{(1)}_{1,0} - 89343\rho^{(1)}_{1,2} - 3641540\rho^{(1)}_{1,4} + 56268\rho^{(1)}_{2,2} + 1646645\rho^{(1)}_{2,4} \\
&\quad + 2005920\rho^{(1)}_{3,4}) + 4410\gamma^{(1)}_0 + 14364\gamma^{(1)}_2 - 964005\gamma^{(1)}_4\Big), \\
p^{(0,4)}_{xy,xy} &= 7\Big(8(-3150\rho^{(1)}_{1,0} + 2349\rho^{(1)}_{1,2} - 215350\rho^{(1)}_{1,4} + 2376\rho^{(1)}_{2,2} + 90475\rho^{(1)}_{2,4} \\
&\quad + 123300\rho^{(1)}_{3,4}) + 630\gamma^{(1)}_0 - 39393\gamma^{(1)}_2 + 74415\gamma^{(1)}_4\Big),
\end{aligned}
\tag{D.3.41}
$$

and

$$
p^{(1)}_{xy,xy}(t,z) = -\frac{\pi^5\mu^2 C_T}{15680000}\sum_{j=0}^{1} p^{(1,2j)}_{xy,xy} t^{2-2j} z^{2j} \tag{D.3.42}
$$

with

$$
\begin{aligned}
p^{(1,0)}_{xy,xy} &= 3\Big(16(-702\rho^{(1)}_{1,2} - 19565\rho^{(1)}_{1,4} + 702\rho^{(1)}_{2,2} + 8085\rho^{(1)}_{2,4} \\
&\quad + 11480\rho^{(1)}_{3,4}) + 490\gamma^{(1)}_0 - 5607\gamma^{(1)}_2 + 12040\gamma^{(1)}_4\Big), \\
p^{(1,2)}_{xy,xy} &= \Big(16(486\rho^{(1)}_{1,2} - 6055\rho^{(1)}_{1,4} - 486\rho^{(1)}_{2,2} \\
&\quad + 2695\rho^{(1)}_{2,4} + 3360\rho^{(1)}_{3,4} + 280\gamma^{(1)}_4) + 1470\gamma^{(1)}_0 - 189\gamma^{(1)}_2\Big).
\end{aligned}
\tag{D.3.43}
$$

For $(\partial_t^2 + \partial_z^2)G_{tx,tx}$ one finds

$$
(\partial_t^2 + \partial_z^2)G_{tx,tx}|_{\mu^2 C_T} = p^{(0)}_{tx,tx}(t,z) + p^{(1)}_{tx,tx}\log(t^2+z^2) + \frac{c_3}{\epsilon} \tag{D.3.44}
$$

for some constant c_3 and where

$$
p^{(0)}_{xy,xy}(t,z) = -\frac{\pi^5\mu^2 C_T}{423360000\,(t^2+z^2)^3}\sum_{j=0}^{3} p^{(0,2j)}_{tx,tx} t^{6-2j} z^{2j} \tag{D.3.45}
$$

with

$$
\begin{aligned}
p^{(0,0)}_{tx,tx} &= 176400\rho^{(1)}_{1,0} - 265032\rho^{(1)}_{1,2} + 30139760\rho^{(1)}_{1,4} + 529632\rho^{(1)}_{2,2} - 12698840\rho^{(1)}_{2,4} \\
&\quad - 17881920\rho^{(1)}_{3,4} + 97020\gamma^{(1)}_0 - 345492\gamma^{(1)}_2 - 792435\gamma^{(1)}_4, \\
p^{(0,2)}_{tx,tx} &= \frac{1}{48}(25401600\rho^{(1)}_{1,0} + 203700096\rho^{(1)}_{1,2} - 9623496960\rho^{(1)}_{1,4} - 165597696\rho^{(1)}_{2,2} \\
&\quad + 3983253120\rho^{(1)}_{2,4} + 5576739840\rho^{(1)}_{3,4} + 21591360\gamma^{(1)}_0 + 81539136\gamma^{(1)}_2 \\
&\quad + 356907600\gamma^{(1)}_4), \\
p^{(0,4)}_{tx,tx} &= \frac{1}{48}(25401600\rho^{(1)}_{1,0} + 270884736\rho^{(1)}_{1,2} + 270063360\rho^{(1)}_{1,4} - 232782336\rho^{(1)}_{2,2} \\
&\quad - 40360320\rho^{(1)}_{2,4} - 293207040\rho^{(1)}_{3,4} + 29211840\gamma^{(1)}_0 + 125629056\gamma^{(1)}_2 \\
&\quad - 45889200\gamma^{(1)}_4), \\
p^{(0,6)}_{tx,tx} &= 176400\rho^{(1)}_{1,0} + 1134648\rho^{(1)}_{1,2} - 6309520\rho^{(1)}_{1,4} - 870048\rho^{(1)}_{2,2} + 2824360\rho^{(1)}_{2,4} \\
&\quad + 3044160\rho^{(1)}_{3,4} + 255780\gamma^{(1)}_0 + 573048\gamma^{(1)}_2 - 71715\gamma^{(1)}_4,
\end{aligned}
\tag{D.3.46}
$$

and

$$
\begin{aligned}
p^{(1)}_{tx,tx} &= -\frac{3\pi^5\mu^2 C_T}{15680000}\Big[-8\left(4\left(81\rho^{(1)}_{1,2} + 35\rho^{(1)}_{1,4} - 81\rho^{(1)}_{2,2} - 35\rho^{(1)}_{3,4}\right) + 315\gamma^{(1)}_4\right) \\
&\quad + 980\gamma^{(1)}_0 + 63\gamma^{(1)}_2\Big].
\end{aligned}
\tag{D.3.47}
$$

Lastly, for $(\partial_t^2 + \partial_z^2)^2 G_{tz,tz}$ one finds

$$
(\partial_t^2 + \partial_z^2)^2 G_{tz,tz}|_{\mu^2 C_T} = \frac{\pi^5\mu^2 C_T}{2940000\,(t^2+z^2)^5}\sum_{j=0}^{4} p^{(0,2j)}_{tz,tz} t^{8-2j} z^{2j}, \tag{D.3.48}
$$

where

$$
\begin{aligned}
p^{(0,0)}_{tz,tz} &= -7776\rho^{(1)}_{1,2} + 197120\rho^{(1)}_{1,4} + 7776\rho^{(1)}_{2,2} - 86240\rho^{(1)}_{2,4} - 110880\rho^{(1)}_{3,4} \\
&\quad + 1470\gamma^{(1)}_0 + 189\gamma^{(1)}_2 - 1400\gamma^{(1)}_4, \\
p^{(0,2)}_{tz,tz} &= -248832\rho^{(1)}_{1,2} + 19983040\rho^{(1)}_{1,4} + 248832\rho^{(1)}_{2,2} - 8451520\rho^{(1)}_{2,4} \\
&\quad - 11531520\rho^{(1)}_{3,4} - 5880\gamma^{(1)}_0 - 152712\gamma^{(1)}_2 - 845320\gamma^{(1)}_4, \\
p^{(0,4)}_{tz,tz} &= 233280\rho^{(1)}_{1,2} - 82577600\rho^{(1)}_{1,4} - 233280\rho^{(1)}_{2,2} + 34496000\rho^{(1)}_{2,4} \\
&\quad + 48081600\rho^{(1)}_{3,4} - 14700\gamma^{(1)}_0 + 82530\gamma^{(1)}_2 + 3193400\gamma^{(1)}_4, \\
p^{(0,6)}_{tz,tz} &= 435456\rho^{(1)}_{1,2} + 29986880\rho^{(1)}_{1,4} - 435456\rho^{(1)}_{2,2} - 12246080\rho^{(1)}_{2,4} \\
&\quad - 17740800\rho^{(1)}_{3,4} - 5880\gamma^{(1)}_0 + 218736\gamma^{(1)}_2 - 1147160\gamma^{(1)}_4, \\
p^{(0,8)}_{tz,tz} &= -38880\rho^{(1)}_{1,2} - 257600\rho^{(1)}_{1,4} + 38880\rho^{(1)}_{2,2} + 86240\rho^{(1)}_{2,4} + 171360\rho^{(1)}_{3,4} \\
&\quad + 1470\gamma^{(1)}_0 - 16695\gamma^{(1)}_2 + 12320\gamma^{(1)}_4.
\end{aligned}
\tag{D.3.49}
$$

D.3.6. Comparison with the bulk calculations

Solving (7.69) one finds the anomalous dimensions (7.70), the relations (7.71) and the bulk coefficients $(a^{(xy)}_{8,1}(6,0), a^{(xy)}_{8,1}(8,0), a^{(tx)}_{8,1}(8,0))$:

$$
\begin{aligned}
a^{8,1(xy)}_{6,0} &= \frac{\pi^4\mu^2\left(2\rho^{(1)}_{1,0} - 3\rho^{(1)}_{1,2} + \rho^{(1)}_{1,4}\right)}{1440} - \frac{3150449\mu^2}{47628000} + \frac{1441\mu^2}{37800\epsilon}, \\
a^{8,1(xy)}_{8,0} &= -\frac{\pi^4\mu^2\left(2\rho^{(1)}_{1,0} - 3\rho^{(1)}_{1,2} + \rho^{(1)}_{1,4}\right)}{1920} + \frac{1820863\mu^2}{127008000} - \frac{1801\mu^2}{50400\epsilon}, \\
a^{8,1(tx)}_{8,0} &= \frac{\pi^4\mu^2\left(2\rho^{(1)}_{1,0} + 3\rho^{(1)}_{1,2} - 5\rho^{(1)}_{1,4}\right)}{2880} - \frac{132403\mu^2}{1411200} - \frac{47\mu^2}{45360\epsilon},
\end{aligned}
\tag{D.3.50}
$$

which are divergent as $\epsilon \to 0$. Note that by studying also the $G_{xx,xx}$ polarization one finds one more linearly independent equation:

$$
a^{8,1(xx)}_{0,0} = \frac{\pi^3\mu^2\left(\rho^{(1)}_{1,0} + 2\rho^{(1)}_{1,4}\right)}{80} - \frac{6713281\mu^2}{5292000\pi} + \frac{11741\mu^2}{6300\pi\epsilon}. \tag{D.3.51}
$$

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