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## The general solution of Bianchi type $VII_h$ vacuum cosmology

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**Abstract** The theory of symmetries of systems of coupled, ordinary differential equations (ODE) is used to develop a concise algorithm in order to obtain the entire space of solutions to vacuum Bianchi Einsteins field equations (EFEs). The symmetries used are the well known automorphisms of the Lie algebra for the corresponding isometry group of each Bianchi Type, as well as the scaling and the time re-parametrization symmetry. The application of the method to Type  $VII_h$  results in (a) obtaining the general solution of Type  $VII_0$  with the aid of the third Painlevé transcendental  $P_{III}$ ; (b) obtaining the general solution of Type  $VII_h$  with the aid of the sixth Painlevé transcendental  $P_{VI}$ ; (c) the recovery of all known solutions (six in total) without a prior assumption of any extra symmetry; (d) The discovery of a *new* solution (the line element given in closed form) with a  $G_3$  isometry group acting on  $T_3$ , i.e., on time-like hyper-surfaces, along with the emergence of the line element describing the flat vacuum Type  $VII_0$  Bianchi Cosmology.

**Keywords** Bianchi types, Lie symmetries, Painlevé, transcendentalExact solutions

### 1 Introduction

The idea of using the group of automorphisms in order to have a unified development of Bianchi Cosmologies has a long history [1]. In that direction Harvey [2] was the first who found the automorphisms of all three-dimensional Lie Algebras, while the corresponding results for the four-dimensional Lie Algebras have been

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reported in [3]. Jantzen's tangent space approach sees the automorphism matrices as the means for achieving a convenient parametrization of a full scale factor matrix in terms of a desired, diagonal matrix [4; 5; 6]. Siklos used these time-dependent automorphisms as a tool to correctly choose variables with the aim to simplify the ensuing equations [7]. Samuel and Ashtekar were the first to look upon automorphisms from a space viewpoint [8]. The notion of *Time-Dependent Automorphism Inducing Diffeomorphisms* (A.I.D.'s), i.e., coordinate transformations mixing space and time in the new spatial coordinates and inducing automorphic motions on the scale-factor matrix, the lapse, and the shift has been developed in [9]. The use of these covariances enables one to set the shift vector to zero without destroying manifest spatial homogeneity. At this stage one can use the "rigid" automorphisms, i.e., the remaining "gauge" symmetry, as Lie-point symmetries of the EFE's in order to reduce the order of these equations and ultimately completely integrate them [10]. The present work of ours consists in the application of this method to the case of vacuum Bianchi Type  $VII_h$  Cosmology. The method is recapitulated in Sect. 2 while its application to the above mentioned type, resulting in the exhaustive discovery of the entire solution space, is given in Sect. 3. In Sect. 4 we discuss our results and give a brief description of the solution space in the form of two tables.

## 2 The method

As it is well known, for spatially homogeneous space-times with a simply transitive action of the corresponding isometry group [11; 12], the line element, assumes the form

$$ds^2 = (N^\alpha N_\alpha - N^2) dt^2 + 2N_\alpha \sigma_i^\alpha dx^i dt + \gamma_{\alpha\beta} \sigma_i^\alpha \sigma_j^\beta dx^i dx^j \quad (2.1)$$

where the 1-forms  $\sigma_i^\alpha$ , are defined from:

$$d\sigma^\alpha = C_{\beta\gamma}^\alpha \sigma^\beta \wedge \sigma^\gamma \Leftrightarrow \sigma_{i,j}^\alpha - \sigma_{j,i}^\alpha = 2C_{\beta\gamma}^\alpha \sigma_i^\beta \sigma_j^\gamma. \quad (2.2)$$

Then the field equations are (see, e.g., [9]):

$$E_o \doteq K^{\alpha\beta} K_{\alpha\beta} - K^2 - \mathbf{R} = 0 \quad (2.3)$$

$$E_\alpha \doteq K^\mu_\alpha C^\varepsilon_{\mu\varepsilon} - K^\mu_\varepsilon C^\varepsilon_{\alpha\mu} = 0 \quad (2.4)$$

$$E_{\alpha\beta} \doteq \dot{K}_{\alpha\beta} + N (2K^\tau_\alpha K_{\tau\beta} - K K_{\alpha\beta}) + 2N^\rho (K_{\alpha\nu} C^\nu_{\beta\rho} + K_{\beta\nu} C^\nu_{\alpha\rho}) - N \mathbf{R}_{\alpha\beta} = 0 \quad (2.5)$$

where

$$K_{\alpha\beta} = -\frac{1}{2N} (\dot{\gamma}_{\alpha\beta} + 2\gamma_{\alpha\nu} C^\nu_{\beta\rho} N^\rho + 2\gamma_{\beta\nu} C^\nu_{\alpha\rho} N^\rho) \quad (2.6)$$

is the extrinsic curvature and

$$\begin{aligned} \mathbf{R}_{\alpha\beta} = & C_{\sigma\tau}^{\kappa} C_{\mu\nu}^{\lambda} \gamma_{\alpha\kappa} \gamma_{\beta\lambda} \gamma^{\sigma\nu} \gamma^{\tau\mu} + 2C_{\beta\lambda}^{\kappa} C_{\alpha\kappa}^{\lambda} + 2C_{\alpha\kappa}^{\mu} C_{\beta\lambda}^{\nu} \gamma_{\mu\nu} \gamma^{\kappa\lambda} \\ & + 2C_{\beta\kappa}^{\lambda} C_{\mu\nu}^{\mu} \gamma_{\alpha\lambda} \gamma^{\kappa\nu} + 2C_{\alpha\kappa}^{\lambda} C_{\mu\nu}^{\mu} \gamma_{\beta\lambda} \gamma^{\kappa\nu} \end{aligned} \quad (2.7)$$

the Ricci tensor of the hyper-surface.

In [9] particular space-time coordinate transformations have been found, which reveal as symmetries of (2.3), (2.4), (2.5) the following transformations of the dependent variables  $N, N_{\alpha}, \gamma_{\alpha\beta}$  :

$$\tilde{N} = N, \quad \tilde{N}_{\alpha} = \Lambda^{\rho}_{\alpha} (N_{\rho} + \gamma_{\rho\sigma} P^{\sigma}), \quad \tilde{\gamma}_{\mu\nu} = \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} \gamma_{\alpha\beta} \quad (2.8)$$

where the matrix  $\Lambda$  and the triplet  $P^{\alpha}$  must satisfy:

$$\Lambda^{\alpha}_{\rho} C^{\rho}_{\beta\gamma} = C^{\alpha}_{\mu\nu} \Lambda^{\mu}_{\beta} \Lambda^{\nu}_{\gamma} \quad (2.9)$$

$$2P^{\mu} C^{\alpha}_{\mu\nu} \Lambda^{\nu}_{\beta} = \dot{\Lambda}^{\alpha}_{\beta} \quad (2.10)$$

These transformations were first given in the first of [4; 5; 6], see the discussion on p. 3586 of [9].

For all Bianchi Types, this system of equations admits solutions which contain three arbitrary functions of time plus several constants depending on the Automorphism group of each type. The three functions of time, are distributed among  $\Lambda$  and  $P$  (which also contains derivatives of these functions). So one can use this freedom either to simplify the form of the scale factor matrix or to set the shift vector to zero. The second action can always be taken, since, for every Bianchi type, all three functions appear in  $P^{\alpha}$ .

In this work we adopt the latter point of view. When the shift has been set to zero, there is still a remaining “gauge” freedom consisting of all constant  $\Lambda^{\alpha}_{\beta}$  (Automorphism group matrices). Indeed the system (2.9), (2.10) accepts the solution  $\Lambda^{\alpha}_{\beta} = \text{constant}, P^{\alpha} = \mathbf{0}$ . The generators of the corresponding motions  $\tilde{\gamma}_{\mu\nu} = \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} \gamma_{\alpha\beta}$ , induced in the space of dependent variables spanned by  $\gamma_{\alpha\beta}$ ’s (the lapse is given in terms of  $\gamma_{\alpha\beta}, \dot{\gamma}_{\alpha\beta}$  by algebraically solving the quadratic constraint equation), are [13]:

$$X_{(I)} = \lambda_{(I)\alpha}^{\rho} \gamma_{\rho\beta} \frac{\partial}{\partial \gamma_{\alpha\beta}} \quad (2.11)$$

with  $\lambda$  satisfying:

$$\lambda_{(I)\rho}^{\alpha} C^{\rho}_{\beta\gamma} = \lambda_{(I)\beta}^{\rho} C^{\alpha}_{\rho\gamma} + \lambda_{(I)\gamma}^{\rho} C^{\alpha}_{\beta\rho}. \quad (2.12)$$

Now, these generators define a Lie algebra and each one of them induces, through its integral curves, a transformation on the configuration space spanned by the  $\gamma_{\alpha\beta}$ ’s. If a generator is brought to its normal form (e.g.,  $\frac{\partial}{\partial z_i}$ ), then the Einstein equations, written in terms of the new dependent variables, will not explicitly involve  $z_i$ . They thus become a *first order* system in the function  $\dot{z}_i$  [14]. If the above Lie algebra happens to be abelian, then all generators can be brought,

to their normal form simultaneously. If this is not the case, we can diagonalize in one step the generators corresponding to any eventual abelian subgroup. The rest of the generators (not brought in their normal form) continue to define a symmetry of the reduced system of EFE's if the algebra of the  $X_{(I)}$ 's is solvable [15]. One can thus repeat the previous step, by choosing one of these remaining generators. This choice will of course depend upon the simplifications brought to the system at the previous level. Finally if the algebra does not contain any abelian subgroup, one can always choose one of the generators, bring it to its normal form, reduce the system and search for its symmetries (if there are any). Lastly, two further symmetries of (2.3), (2.4), (2.5) are also present and can be used in conjunction with the constant automorphisms: The time reparameterization  $t \rightarrow f(t) + \alpha$ , owing to the non-explicit appearance of time in these equations, and the scaling by a constant  $\gamma_{\alpha\beta} \rightarrow \mu \gamma_{\alpha\beta}$  as can be straightforwardly verified. Their corresponding generators are:

$$Y_1 = \frac{1}{\dot{f}} \frac{\partial}{\partial t} \quad (2.13)$$

$$Y_2 = \gamma_{\alpha\beta} \frac{\partial}{\partial \gamma_{\alpha\beta}} \quad (2.14)$$

These generators commute among themselves, as well as with the  $X_{(I)}$ 's, as it can be easily checked.

### 3 Application to Bianchi type $VII_h$

We are now going to apply the method, previously discussed, to the case of Bianchi Type  $VII_h$ . For this type the structure constants are

$$\begin{aligned} C^1_{13} &= -C^1_{31} = C^2_{23} = -C^2_{32} = -h \\ C^1_{32} &= -C^1_{23} = C^2_{13} = -C^2_{31} = 1 \\ C^\alpha_{\beta\gamma} &= 0 \quad \text{for all other values of } \alpha, \beta, \gamma \end{aligned} \quad (3.1)$$

Using these values in the defining relation (2.2) of the 1-forms  $\sigma_i^\alpha$  we obtain

$$\sigma_i^\alpha = \begin{pmatrix} 0 & e^{hx} \sin x & e^{hx} \cos x \\ 0 & e^{hx} \cos x & -e^{hx} \sin x \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \quad (3.2)$$

The corresponding vector fields  $\xi_\alpha^i$  (satisfying  $[\xi_\alpha, \xi_\beta] = \frac{1}{2} C^\gamma_{\alpha\beta} \xi_\gamma$ ) with respect to which the lie derivative of the above 1-forms is zero are:

$$\xi_1 = \partial_y \quad \xi_2 = \partial_z \quad \xi_3 = \partial_x + (z - hy)\partial_y - (y + hz)\partial_z \quad (3.3)$$

The time depended A.I.D.'s are described by

$$\Lambda_{\beta}^{\alpha} = \begin{pmatrix} c e^{hP(t)} \cos P(t) & c e^{hP(t)} \sin P(t) & x(t) \\ -c e^{hP(t)} \sin P(t) & c e^{hP(t)} \cos P(t) & y(t) \\ 0 & 0 & 1 \end{pmatrix} \quad (3.4)$$

and

$$P^{\alpha} = \left( \frac{x(t)\dot{P}(t) + h^2 x(t)\dot{P}(t) - h\dot{x}(t) + \dot{y}(t)}{2(1+h^2)}, \right. \quad (3.5)$$

$$\left. \frac{y(t)\dot{P}(t) + h^2 y(t)\dot{P}(t) - h\dot{y}(t) - \dot{x}(t)}{2(1+h^2)}, \frac{\dot{P}(t)}{2} \right) \quad (3.6)$$

where  $P(t), x(t)$  and  $y(t)$  are arbitrary functions of time. As we have already remarked the three arbitrary functions appear in  $P^{\alpha}$  and thus can be used to set the shift vector to zero.

The remaining symmetry of the EFE's is, consequently, described by the constant matrix:

$$M = \begin{pmatrix} e^{s_1} & s_2 & s_3 \\ -s_2 & e^{s_1} & s_4 \\ 0 & 0 & 1 \end{pmatrix} \quad (3.7)$$

where the parametrization has been chosen so that the matrix becomes identity for the zero value of all parameters.

Thus the induced transformation on the scale factor matrix is  $\tilde{\gamma}_{\alpha\beta} = M_{\alpha}^{\mu} M_{\beta}^{\nu} \gamma_{\mu\nu}$ , which explicitly reads:

$$\left\{ \begin{array}{l} \tilde{\gamma}_{11} = e^{2s_1} \gamma_{11} - 2e^{s_1} s_2 \gamma_{12} + s_2^2 \gamma_{22} \\ \tilde{\gamma}_{12} = e^{2s_1} \gamma_{12} - s_2^2 \gamma_{12} + e^{s_1} s_2 (\gamma_{11} - \gamma_{22}) \\ \tilde{\gamma}_{13} = e^{s_1} (s_3 \gamma_{11} + s_4 \gamma_{12} + \gamma_{13}) - s_2 (s_3 \gamma_{12} + s_4 \gamma_{22} + \gamma_{23}) \\ \tilde{\gamma}_{22} = e^{2s_1} \gamma_{22} + 2e^{s_1} s_2 \gamma_{12} + s_2^2 \gamma_{11} \\ \tilde{\gamma}_{23} = e^{s_1} (s_3 \gamma_{12} + s_4 \gamma_{22} + \gamma_{23}) + s_2 (s_3 \gamma_{11} + s_4 \gamma_{12} + \gamma_{13}) \\ \tilde{\gamma}_{33} = s_3^2 \gamma_{11} + 2s_3 (s_4 \gamma_{12} + \gamma_{13}) + s_4^2 \gamma_{22} + 2s_4 \gamma_{23} + \gamma_{33} \end{array} \right. \quad (3.8)$$

The previous equations, define a group of transformations  $G_r$  of dimension  $r = \dim(\text{Aut}(VII_h)) = 4$ . The four generators of the group, can be evaluated from the relation:

$$X_A = \left( \frac{\partial \tilde{\gamma}_{\alpha\beta}}{\partial s_A} \right)_{s=0} \frac{\partial}{\partial \gamma_{\alpha\beta}} \quad (3.9)$$

where  $A = \{1, 2, 3, 4\}$ . Applying this definition to (3.8) we have the generators:

$$X_1 = 2\gamma_{11} \frac{\partial}{\partial \gamma_{11}} + 2\gamma_{12} \frac{\partial}{\partial \gamma_{12}} + \gamma_{13} \frac{\partial}{\partial \gamma_{13}} + 2\gamma_{22} \frac{\partial}{\partial \gamma_{22}} + \gamma_{23} \frac{\partial}{\partial \gamma_{23}} \quad (3.10)$$

$$X_2 = -2\gamma_{12} \frac{\partial}{\partial \gamma_{11}} + (\gamma_{11} - \gamma_{22}) \frac{\partial}{\partial \gamma_{12}} - \gamma_{23} \frac{\partial}{\partial \gamma_{13}} + 2\gamma_{12} \frac{\partial}{\partial \gamma_{22}} + \gamma_{13} \frac{\partial}{\partial \gamma_{23}} \quad (3.11)$$

$$X_3 = \gamma_{11} \frac{\partial}{\partial \gamma_{13}} + \gamma_{12} \frac{\partial}{\partial \gamma_{23}} + 2\gamma_{13} \frac{\partial}{\partial \gamma_{33}} \quad (3.12)$$

$$X_4 = \gamma_{12} \frac{\partial}{\partial \gamma_{13}} + \gamma_{22} \frac{\partial}{\partial \gamma_{23}} + 2\gamma_{23} \frac{\partial}{\partial \gamma_{33}} \quad (3.13)$$

The algebra  $g_r$  that corresponds to the group  $G_r$  has the following table of commutators:

$$\begin{aligned} [X_1, X_2] &= 0, & [X_1, X_3] &= X_3, & [X_1, X_4] &= X_4, \\ [X_2, X_3] &= -X_4, & [X_2, X_4] &= X_3, & [X_3, X_4] &= 0 \end{aligned} \quad (3.14)$$

As it is evident from the above commutators (3.14) the group is non-abelian, so we cannot diagonalize at the same time all the generators. However, if we calculate the derived algebra of  $g_r$ , we have

$$g_{r'} = \{[X_A, X_B] : X_A, X_B \in g_r\} \Rightarrow g_{r'} = \{X_3, X_4\} \quad (3.15)$$

and furthermore, it's second derived algebra reads:

$$g_{r''} = \{[X_A, X_B] : X_A, X_B \in g_{r'}\} \Rightarrow g_{r''} = \{0\} \quad (3.16)$$

Thus, the group  $G_r$  is solvable since the  $g_{r''}$  is zero. As it is evident  $X_3, X_4, Y_2$  generate an Abelian subgroup, and we can, therefore, bring them to their normal form simultaneously. The appropriate transformation of the dependent variables is:

$$\begin{cases} \gamma_{11} = e^{u_1 - u_6} \\ \gamma_{12} = e^{u_1 - u_6} u_2 \\ \gamma_{13} = e^{u_1 - u_6} (u_3 + u_2 u_5) \\ \gamma_{22} = e^{u_1 - u_6} u_4 \\ \gamma_{23} = e^{u_1 - u_6} (u_2 u_3 + u_4 u_5) \\ \gamma_{33} = e^{u_1 - u_6} (e^{u_6} + u_3^2 + 2u_2 u_3 u_5 + u_4 u_5^2) \end{cases} \quad (3.17)$$

In these coordinates the generators  $Y_2, X_A$  assume the form:

$$\begin{aligned} Y_2 &= \frac{\partial}{\partial u_1} \quad X_4 = \frac{\partial}{\partial u_5} \quad X_3 = \frac{\partial}{\partial u_3} \\ X_2 &= (1 + 2u_2^2 - u_4) \frac{\partial}{\partial u_2} - u_5 \frac{\partial}{\partial u_3} + 2(u_2 + u_2 u_4) \frac{\partial}{\partial u_4} + u_3 \frac{\partial}{\partial u_5} + 2u_2 \frac{\partial}{\partial u_6} \\ X_1 &= -u_3 \frac{\partial}{\partial u_3} - u_5 \frac{\partial}{\partial u_5} - 2 \frac{\partial}{\partial u_6} \end{aligned} \quad (3.18)$$

Evidently, a first look at (3.17) gives the feeling that it would be hopeless even to write down the Einstein equations. However, the simple form of the first three of the generators (3.18) ensures us that these equations will be of first order in the functions  $\dot{u}_1, \dot{u}_3$  and  $\dot{u}_5$ .

### 3.1 Description of the solution space

Before we begin solving the Einstein equations, a few comments on the allowable range of values for the functions  $u_i, i = 1, \dots, 6$  will prove very useful.

The determinant of  $\gamma_{\alpha\beta}$ , is

$$\det[\gamma_{\alpha\beta}] = e^{3u_1 - 2u_6} (-u_2^2 + u_4) \quad (3.19)$$

so we must have  $u_4 > u_2^2$ .

The two linear constraint equations, written in the new variables (3.17), give

$$E_1 = 0 \Rightarrow \frac{1}{2} e^{-u_6} ((3h - u_2) \dot{u}_3 + (3hu_2 - u_4) \dot{u}_5) = 0 \quad (3.20)$$

$$E_2 = 0 \Rightarrow \frac{1}{2} e^{-u_6} ((1 + 3hu_2) \dot{u}_3 + (u_2 + 3hu_4) \dot{u}_5) = 0 \quad (3.21)$$

This system admits only the trivial solution, since the determinant of the  $2 \times 2$  matrix formed by the coefficients of  $\dot{u}_3, \dot{u}_5$  becomes zero only for the forbidden value  $u_4 = u_2^2$ . We thus have

$$u_3 = k_3, \quad u_5 = k_5 \quad (3.22)$$

Now, these values of  $u_3, u_5$  make  $\gamma_{13}, \gamma_{23}$  functionally dependent upon  $\gamma_{11}, \gamma_{12}, \gamma_{22}$  (see (3.17)). It is thus possible to set these two components to zero by means of an appropriate constant automorphism.

We therefore can, without loss of generality, start our investigation of the solution space for Type  $VII_h$  vacuum Bianchi Cosmology from a block-diagonal form of the scale-factor matrix (and, of course, zero shift)

$$\gamma_{\alpha\beta} = \begin{pmatrix} \gamma_{11} & \gamma_{12} & 0 \\ \gamma_{12} & \gamma_{22} & 0 \\ 0 & 0 & \gamma_{33} \end{pmatrix} \quad (3.23)$$

These unknown functions of time have to satisfy the quadratic and the third linear constraint, as well as the spatial EFE's. As we have earlier remarked, since the algebra (3.14) is solvable, the remaining (reduced) generators  $X_1, X_2$  (corresponding to block-diagonal constant automorphisms) as well as  $Y_2$  continue to define a Lie-point symmetry of the reduced EFE's and can thus be used for further integration of this system of equations.

The remaining (reduced) automorphism generators are

$$\begin{aligned} X_1 &= 2\gamma_{11} \frac{\partial}{\partial \gamma_{11}} + 2\gamma_{12} \frac{\partial}{\partial \gamma_{12}} + 2\gamma_{22} \frac{\partial}{\partial \gamma_{22}} \\ X_2 &= -2\gamma_{12} \frac{\partial}{\partial \gamma_{11}} + (\gamma_{11} - \gamma_{22}) \frac{\partial}{\partial \gamma_{12}} + 2\gamma_{12} \frac{\partial}{\partial \gamma_{22}} \end{aligned} \quad (3.24)$$

The appropriate change of dependent variables which brings these generators - along with  $Y_2$ - into normal form, is described by the following scale-factor matrix:

$$\gamma_{\alpha\beta} = \begin{pmatrix} \frac{1}{2} e^{u_1+2u_6} (1 - 2u_2 \sin 2u_4) & e^{u_1+2u_6} u_2 \cos 2u_4 & 0 \\ e^{u_1+2u_6} u_2 \cos 2u_4 & \frac{1}{2} e^{u_1+2u_6} (1 + 2u_2 \sin 2u_4) & 0 \\ 0 & 0 & e^{u_1} \end{pmatrix} \quad (3.25)$$

The generators are now reduced to

$$Y_2 = -\frac{\partial}{\partial u_1}, \quad X_2 = \frac{\partial}{\partial u_4}, \quad X_1 = -\frac{\partial}{\partial u_6} \quad (3.26)$$

indicating that the system will be of first order in the derivatives of these variables. The remaining variable  $u_2$  will enter, (along with  $\dot{u}_2, \ddot{u}_2$ ) explicitly in the system and is therefore advisable (if not mandatory) to be used as the time parameter, i.e., to effect the change of time coordinate

$$t \rightarrow u_2(t) = s, \quad u_1(t) \rightarrow u_1(t(s)), \quad u_4(t) \rightarrow u_4(t(s)), \quad u_6(t) \rightarrow u_6(t(s)). \quad (3.27)$$

This choice of time will of course be valid only if  $u_2$  is not a constant. We are thus led to consider two cases according to the constancy or non-constancy of this dependent variable.

Until now, we have not commented upon the range of values that the parameter  $h$  can attain. As it is well known, for the value  $h = 0$  we come across the *Class A* model, which admits a Lagrangian description, whereas for  $h \neq 0$  we have the *Class B* model which lacks such a Lagrangian description. So we are forced to examine two further possibilities, as to whether  $h$  is equal to, or different from, zero.

### 3.1.1 Case I: $h = 0$ and $u_2(t) = k_2$

In the parametrization (3.25) the determinant of  $\gamma_{\alpha\beta}$ , is

$$\det[\gamma_{\alpha\beta}] = \frac{1}{4} e^{3u_1+4u_6} (1 - 4k_2^2)$$

so we must have  $-\frac{1}{2} < k_2 < \frac{1}{2}$ . The third linear constraint reads

$$E_3 = 0 \Rightarrow \frac{8k_2^2 \dot{u}_4}{-1 + 4k_2^2} = 0 \Rightarrow u_4 = k_4 \quad \text{or} \quad k_2 = 0 \quad (3.28)$$

The case  $u_4 = k_4$  leads, through equation  $E_{34} = 0$  to  $k_2 = 0$ . Thus, the only possibility is  $k_2 = 0$ . Substituting this value into the quadratic constraint equation  $E_0$  we obtain

$$-\frac{1}{2} (3\dot{u}_1^2 + 8\dot{u}_1\dot{u}_6 + 4\dot{u}_6^2) = 0 \quad (3.29)$$

which has the following two solutions

$$u_1 = k_1 - 2u_6 \quad (3.30a)$$

$$u_1 = k_1 - \frac{2}{3}u_6 \quad (3.30b)$$

For the first of (3.30) all the spatial EFE's are equivalent to the equation

$$2\dot{u}_6 N\dot{N} + 2N^2 (\dot{u}_6^2 - \ddot{u}_6) = 0 \quad (3.31)$$

from which we have for the lapse function

$$N^2 = k e^{-2u_6} \dot{u}_6^2 \quad (3.32)$$

Choosing a time parametrization  $u_6 = -\frac{1}{2} \ln(\frac{\tau^2}{k})$ , and using the automorphism matrix (3.7) with entries  $s_1 = \frac{1}{2} (\ln 2 - k_1)$ ,  $s_2 = s_3 = s_4 = 0$  we arrive at the line element

$$ds^2 = -d\tau^2 + \tau^2 dx^2 + dy^2 + dz^2 \quad (3.33)$$

which describes a *flat* space admitting a manifest  $VII_0$  symmetry [16]. To the best of our knowledge, it is the first time that this line element emerges in the course of investigation of the solution space to this Bianchi Type. Of course, this was to be expected, since *flat* space is clearly LRS Bianchi *I*, hence  $VII_0$  (see Table 3 of [11]).

For the second of (3.30) all the spatial EFE's are equivalent to the equation

$$2\dot{u}_6N\dot{N}-2N^2(\dot{u}_6^2+\ddot{u}_6)=0 \quad (3.34)$$

which gives the lapse function

$$N^2 = k e^{2u_6} \dot{u}_6^2 \quad (3.35)$$

Choosing a time parametrization  $u_6 = \frac{1}{2} \ln(\frac{4\tau^3}{9k})$ , redefining the constant  $k_1 = \frac{1}{3} \ln \frac{256}{9k}$  and using the automorphism matrix (3.7) with entries  $s_1 = \frac{1}{2} \ln \frac{9k}{8}$ ,  $s_2 = s_3 = s_4 = 0$  we arrive at the line element

$$ds^2 = -\tau d\tau^2 + \frac{1}{\tau} dx^2 + \tau^2 dy^2 + \tau^2 dz^2 \quad (3.36)$$

This line element is an axisymmetric geometry belonging to the Kasner family [17]. It was first produced as a Type  $VII_h$  metric by Ellis [18; 19] (see also Table 3 of [11]) and admits, besides the three killing fields (3.3) (with  $h = 0$ ), a fourth symmetry generator

$$\xi_4 = \partial_x \quad (3.37)$$

along with a homothetic vector field

$$\eta = 2\tau \partial_\tau + 4x \partial_x + y \partial_y + z \partial_z \quad (3.38)$$

There is thus a  $G_4$  symmetry group acting (of course, multiply transitively) on each  $V_3$  of this metric. However, it is interesting to note that we have not imposed the extra symmetry from the beginning, but rather it emerged as a result of the investigation process.

### 3.1.2 Case II: $h = 0$ and $u_2(t) = t$

With this choice of time gauge the third linear constraint reads

$$E_3 = 0 \Rightarrow \frac{8t^2}{4t^2-1} \dot{u}_4 = 0 \Rightarrow u_4 = k_4 \quad (3.39)$$

With this information at hand, the quadratic constraint  $E_o$  yields the lapse function

$$N^2 = -\frac{e^{u_1}}{64t^2} \left( (4t^2-1)(\dot{u}_1+2\dot{u}_6)(3\dot{u}_1+2\dot{u}_6) + 16t(\dot{u}_1+\dot{u}_6) + 4 \right) \quad (3.40)$$

We now turn to the spatial equations of motion and substitute the above lapse. The simplest is  $E_{33} = 0$  and the coefficient of  $\ddot{u}_1$  in this equation is proportional to the quantity

$$((4t^2 - 1)\dot{u}_6 + 2t)\dot{u}_1 + (4t^2 - 1)\dot{u}_6^2 + 4t\dot{u}_6 + 1$$

which can be safely regarded different from zero, since by setting this quantity equal to zero and solving for  $\dot{u}_1$  we end up with zero lapse (with the help of the rest of the equations of motion). We can thus solve  $E_{33} = 0$  for  $\dot{u}_1$  and substitute into  $E_{11} = 0$ . In this transformed equation  $E_{11} = 0$ , the coefficient of  $\dot{u}_6$  is proportional to

$$(-2t + \sin(2k_4))(\dot{u}_1 + \dot{u}_6) - 1$$

a quantity which is different from zero, since it's nihilism leads again to zero lapse. From the transformed  $E_{11} = 0$  we have the expression for  $\dot{u}_6$ , so we finally arrive at the following polynomial system of first order in  $\dot{u}_1, \dot{u}_6$

$$\ddot{u}_1 = \langle \dot{u}_1 | A_1 | \dot{u}_6 \rangle, \quad \ddot{u}_6 = \langle \dot{u}_1 | A_2 | \dot{u}_6 \rangle \quad (3.41)$$

where we have introduced the notation  $\langle \dot{u}_i | = (1 \dot{u}_i \dot{u}_i^2 \dot{u}_i^3)$  and  $| \dot{u}_i \rangle = \langle \dot{u}_i |^t$  with the  $4 \times 4$  matrices  $A_1, A_2$  given by

$$A_1 = \begin{pmatrix} \frac{4}{4t^2-1} & \frac{16t}{4t^2-1} & 4 & 0 \\ \frac{4t^2+1}{t(4t^2-1)} & 4 & \frac{-4t^2-1}{t} & 0 \\ -1 & \frac{2(-4t^2+1)}{t} & 0 & 0 \\ \frac{3(1-4t^2)}{4t} & 0 & 0 & 0 \end{pmatrix}, \quad (3.42)$$

$$A_2 = \begin{pmatrix} \frac{6}{-4t^2+1} & \frac{-28t^2+1}{t(4t^2-1)} & -8 & \frac{-4t^2+1}{t} \\ \frac{16t}{-4t^2+1} & -12 & \frac{-8t^2+2}{t} & 0 \\ -3 & \frac{-12t^2+3}{4t} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Due to the form of  $A_1, A_2$  (their components are rational functions of the time  $t$ ), system (3.41) can be partially integrated with the help of the following Lie-Bäcklund transformation

$$\dot{u}_1(t) = \frac{-16r^2(t) + 1}{4(4t^2 - 1)r(t)} - 4t\dot{r}(t) \quad (3.43)$$

$$\dot{u}_6(t) = \frac{-16r^2(t) + 16tr(t) + 3}{8(-4t^2 + 1)r(t)} + 2t\dot{r}(t)$$

resulting in the single, second order ODE for the function  $r(t)$

$$\ddot{r} = -\frac{1}{r}\dot{r}^2 + \frac{(12t^2 + 1)r + t}{(-4t^2 + 1)r} \dot{r} \quad (3.44)$$

At this stage, in order to solve (3.44) we apply the contact transformation:

$$r(t) = \frac{\xi w'(\xi)}{4w(\xi)}, \quad t = \frac{1}{2} - \frac{1}{1+w(\xi)}, \quad \dot{r}(t) = -\frac{(w(\xi)-1)(w(\xi)+1)^3}{8w'(\xi)w(\xi)} \quad (3.45)$$

which reduces it to

$$w''(\xi) = \frac{w'(\xi)^2}{w(\xi)} - \frac{w'(\xi)}{\xi} - \frac{1}{2} \frac{w(\xi)^2 - 1}{\xi} \quad (3.46)$$

which is nothing else but the third Painlevé transcendent  $w := \mathbf{P}_{\text{III}}(\alpha, \beta, \gamma, \delta)$  with entries  $(\alpha, \beta, \gamma, \delta) = (-1/2, 1/2, 0, 0)$ . For completeness we give the general form of the equation that the third Painlevé transcendent satisfies:

$$w''(\xi) = \frac{w'(\xi)^2}{w(\xi)} - \frac{w'(\xi)}{\xi} + \frac{\alpha w(\xi)^2 + \beta}{\xi} + \gamma w(\xi)^3 + \frac{\delta}{w(\xi)} \quad (3.47)$$

Using the final equation (3.46), the contact transformation (3.45) and the Lie-Bäcklund transformation (3.43) we find that the functions  $u_6$  and  $u'_1$  are given by

$$u_6(\xi) = \frac{1}{4} \ln \left( \left| \frac{\xi}{w(\xi)} \right| (w(\xi) + 1)^2 \right) - \frac{1}{2} u_1(\xi) \quad (3.48a)$$

$$u'_1(\xi) = \frac{\xi w'(\xi)^2}{4w(\xi)^2} + \frac{1}{4} w(\xi) + \frac{1}{4w(\xi)} - \frac{1}{4\xi} - \frac{1}{2} \quad (3.48b)$$

and the lapse function has the form

$$N^2 = \frac{1}{16\xi} e^{u_1} \quad (3.49)$$

The scale-factor matrix  $\gamma_{\alpha\beta}$  is thus

$$\gamma_{\alpha\beta} = \begin{pmatrix} \frac{1}{2} \sqrt{\left| \frac{\xi}{w(\xi)} \right|} (w(\xi) + 1) & \frac{1}{2} \sqrt{\left| \frac{\xi}{w(\xi)} \right|} (w(\xi) - 1) & 0 \\ \frac{1}{2} \sqrt{\left| \frac{\xi}{w(\xi)} \right|} (w(\xi) - 1) & \frac{1}{2} \sqrt{\left| \frac{\xi}{w(\xi)} \right|} (w(\xi) + 1) & 0 \\ 0 & 0 & e^{u_1(\xi)} \end{pmatrix} \quad (3.50)$$

which can be brought to diagonal form with the aid of the automorphism matrix (3.7) with entries  $s_1 = -\frac{1}{2} \ln 2, s_2 = -\frac{1}{\sqrt{2}}$

$$\gamma_{\alpha\beta} = \begin{pmatrix} \sqrt{\left| \frac{\xi}{w(\xi)} \right|} w(\xi) & 0 & 0 \\ 0 & \sqrt{\left| \frac{\xi}{w(\xi)} \right|} & 0 \\ 0 & 0 & e^{u_1(\xi)} \end{pmatrix} \quad (3.51)$$

Gathering all the pieces we arrive at the final form of the line element

$$ds^2 = \kappa^2 \left( -\frac{1}{16\xi} e^{u_1} d\xi^2 + \frac{1}{4} e^{u_1} dx^2 + \sqrt{\left| \frac{\xi}{w} \right|} (w \sin^2 x + \cos^2 x) dy^2 + \sqrt{\left| \frac{\xi}{w} \right|} \sin(2x) (w-1) dy dz + \sqrt{\left| \frac{\xi}{w} \right|} (w \cos^2 x + \sin^2 x) dz^2 \right) \quad (3.52)$$

which represents the *general* solution of Bianchi Type  $VII_0$  non-flat Vacuum Cosmology, since it contains the expected number of three essential constants (two implicit in the third Painlevé transcendent plus the overall  $\kappa$ ). The above line element was first given by Lorenz-Petzold [20], but it was not then pointed out that it represented the general solution.

### Particular solutions

In order for the contact transformation (3.45) to be well defined, it is obvious that the function  $w(\xi)$  must not be constant. However, remarkably enough, the resulting line element (3.52) *does not* inherit this restriction. Thus, if there is some constant solution to equation (3.46), it could produce a particular solution through (3.52). By inspection it is obvious that (3.46) admits the solutions  $w(\xi) = \pm 1$ , so we could use them to obtain two particular solutions.

- **Subcase  $w(\xi) = 1$**

With this value of  $w(\xi)$  (3.52) indicates that  $\xi > 0$ , so plugging this value to (3.48) we have

$$u_1(\xi) = k_1 - \frac{1}{4} \ln \xi \quad u_6(\xi) = -\frac{1}{2} k_1 + \frac{1}{8} \ln(16\xi^3) \quad (3.53)$$

which, after using the usual simplifications brought by the automorphism matrix (3.7) and redefining the variable  $\xi$  to  $\xi = \tau^4$ , results in

$$ds^2 = -\tau d\tau^2 + \frac{1}{\tau} dx^2 + \tau^2 dy^2 + \tau^2 dz^2 \quad (3.54)$$

which is the line element (3.36).

- **Subcase**  $w(\xi) = -1$

Now from (3.52) we must have  $\xi < 0$  and from (3.48b) we obtain

$$u_1(\xi) = k_1 - \xi - \frac{1}{4} \ln |\xi| \quad (3.55)$$

while from (3.48a)  $u_6$  remains undefined. The line element (3.52) with the help of the automorphism matrix (3.7) and the definition  $\xi = -\tau^4$  becomes

$$ds^2 = \kappa^2 \left( e^{\tau^4} \tau d\tau^2 + \frac{1}{4\tau} e^{\tau^4} dx^2 + \tau^2 \cos(2x) dy^2 - 2\tau^2 \sin(2x) dy dz - \tau^2 \cos(2x) dz^2 \right) \quad (3.56)$$

which, though physically acceptable, corresponds to Bianchi Type  $VII_0$  symmetry on  $T_3$ , and was first given by Barnes [21].

### 3.1.3 Case III: $h \neq 0$ and $u_2(t) = k_2$

In this case the determinant of the scale factor matrix  $\gamma_{\alpha\beta}$  is

$$\det(\gamma_{\alpha\beta}) = \frac{1}{4} e^{3u_1+4u_6} (1 - 4k_2^2) \quad (3.57)$$

so we must have  $-\frac{1}{2} < k_2 < \frac{1}{2}$  in order for it to be positively defined.

The third linear constraint  $E_3 = 0$  reads

$$\frac{1}{4k_2^2 - 1} (8k_2^2 \dot{u}_4 + 2h(1 - 4k_2^2) \dot{u}_6) = 0 \Rightarrow u_6 = k_6 + \frac{4k_2^2}{h(4k_2^2 - 1)} u_4 \quad (3.58)$$

and the quadratic constraint  $E_o = 0$  gives for the lapse function  $N^2$

$$N^2 = \frac{e^{u_1}}{16h^2(4k_2^2 - 1)(3h^2(4k_2^2 - 1) - 4k_2^2)} \left( 3h^2(4k_2^2 - 1)^2 \dot{u}_1^2 + 32hk_2^2(4k_2^2 - 1) \dot{u}_1 \dot{u}_4 + 16k_2^2(h^2(4k_2^2 - 1) + 4k_2^2) \dot{u}_4^2 \right) \quad (3.59)$$

Now we are ready to attack the spatial equations of motion after substituting in them the above lapse.  $E_{33} = 0$  is again the simplest one. In this equation, the coefficient of  $\ddot{u}_4$  is proportional to

$$k_2^2 \dot{u}_1 (h(4k_2^2 - 1) \dot{u}_1 + (4k_2^2 + h^2(4k_2^2 - 1)) \dot{u}_4) \quad (3.60)$$

so in order to solve  $E_{33} = 0$  for  $\ddot{u}_4$  we must ensure that the above quantity is different from zero. Setting this quantity equal to zero we get

$$u_1 = k_1 - \frac{4k_2^2 + h^2(4k_2^2 - 1)}{h(4k_2^2 - 1)} u_4 \quad (3.61a)$$

$$u_1 = k_1 \quad (3.61b)$$

$$k_2 = 0 \quad (3.61c)$$

- The solution (3.61a) leads to inconsistency.
- The solution (3.61b) forces equation  $E_{33} = 0$  to give either  $k_2 = \pm \frac{h}{2\sqrt{h^2+1}}$  which leads to zero lapse or  $k_2 = \pm \frac{h}{2\sqrt{h^2-1}}$ , which makes the determinant of  $\gamma_{\alpha\beta}$  negative so is unacceptable.
- The solution (3.61c) satisfies all the spatial equations and leads, after the usual simplifications achieved by the automorphism matrix (3.7) and the choice of time gauge  $u_1 = 2 \ln(2h\tau)$ , to the line element

$$ds^2 = -d\tau^2 + h^2 \tau^2 dx^2 + e^{2h\tau} \tau^2 dy^2 + e^{2h\tau} \tau^2 dz^2 \quad (3.62)$$

which describes a *flat* space admitting a manifest  $VII_h$  symmetry, a line-element first presented in [22] (p. 78, relation (51)) and later reproduced by Doroshkevich et al. [23] and Siklos [24].

Having ensured that the term (3.60) is not equal to zero we can solve  $E_{33} = 0$  for  $\dot{u}_4$  and substitute into the other equations of motion. From  $E_{11} = 0$  we have

$$\begin{aligned} & (\sin(2u_4) - 2k_2) \left( \dot{u}_1 + \frac{2(3h^2(4k_2^2 - 1) + 4k_2^2)}{3h(4k_2^2 - 1)} \dot{u}_4 \right) \\ & \times \left( \dot{u}_1 - \frac{2(4k_2^2(h^2 - 1) - h^2)}{h(4k_2^2 - 1)} \dot{u}_4 \right) = 0 \end{aligned} \quad (3.63)$$

which leads to the following possibilities

$$u_4 = \frac{1}{2} \arcsin(2k_2) \quad (3.64a)$$

$$u_1 = k_1 - \frac{2(3h^2(4k_2^2 - 1) + 4k_2^2)}{3h(4k_2^2 - 1)} u_4 \quad (3.64b)$$

$$u_1 = k_1 + \frac{2(4k_2^2(h^2 - 1) - h^2)}{h(4k_2^2 - 1)} u_4 \quad (3.64c)$$

- The solution (3.64a) leads to zero lapse.
- The solution (3.64b) leads to inconsistency.

- The solution (3.64c) satisfies all the spatial equations and leads, after the usual simplifications with the automorphism matrix (3.7) and the choice of time gauge  $u_1 = \tau$ , to the line element

$$\begin{aligned} ds^2 = & \frac{1}{4} e^{\frac{-2\lambda^2+2h^2(\lambda^2-1)}{h(\lambda^2-1)}\tau} (-d\tau^2 + dx^2) - 2\lambda e^{2h(\tau+x)} \sin 2(\tau+x) dy dz \\ & + e^{2h(\tau+x)} (1 + \lambda \cos 2(\tau+x)) dy^2 \\ & + e^{2h(\tau+x)} (1 - \lambda \cos 2(\tau+x)) dz^2 \end{aligned} \quad (3.65)$$

which was presented in [23; 24]. This line element represents a plane-wave solution and admits, besides the three killing fields (3.3), three more, namely

$$\xi_4 = e^{\frac{-\lambda^2+h^2(\lambda^2-1)}{h(\lambda^2-1)}(x-\tau)} \partial_\tau - e^{\frac{-\lambda^2+h^2(\lambda^2-1)}{h(\lambda^2-1)}(x-\tau)} \partial_x \quad (3.66a)$$

$$\begin{aligned} \xi_5 = & y e^{\frac{-\lambda^2+h^2(\lambda^2-1)}{h(\lambda^2-1)}(x-\tau)} \partial_\tau - y e^{\frac{-\lambda^2+h^2(\lambda^2-1)}{h(\lambda^2-1)}(x-\tau)} \partial_x \\ & + e^{\frac{-\lambda^2-h^2(\lambda^2-1)}{h(\lambda^2-1)}(x+\tau)} (c_1 \cos 2(\tau+x) + c_2 \sin 2(\tau+x) + c_3) \partial_y \\ & + e^{\frac{-\lambda^2-h^2(\lambda^2-1)}{h(\lambda^2-1)}(x+\tau)} (c_2 \cos 2(\tau+x) - c_1 \sin 2(\tau+x)) \partial_z \end{aligned} \quad (3.66b)$$

$$\begin{aligned} \xi_6 = & z e^{\frac{-\lambda^2+h^2(\lambda^2-1)}{h(\lambda^2-1)}(x-\tau)} \partial_\tau - z e^{\frac{-\lambda^2+h^2(\lambda^2-1)}{h(\lambda^2-1)}(x-\tau)} \partial_x \\ & + e^{\frac{-\lambda^2-h^2(\lambda^2-1)}{h(\lambda^2-1)}(x+\tau)} (c_2 \cos 2(\tau+x) - c_1 \sin 2(\tau+x)) \partial_y \\ & - e^{\frac{-\lambda^2-h^2(\lambda^2-1)}{h(\lambda^2-1)}(x+\tau)} (c_1 \cos 2(\tau+x) + c_2 \sin 2(\tau+x) - c_3) \partial_z \end{aligned} \quad (3.66c)$$

where the constants  $(c_1, c_2, c_3)$  are given by

$$c_1 = -\frac{\lambda h (\lambda^2 + h^2 (\lambda^2 - 1))}{4 (\lambda^4 + h^4 (\lambda^2 - 1)^2 + 2 h^2 (\lambda^2 - 1) (3 \lambda^2 - 2))} \quad (3.67a)$$

$$c_2 = \frac{\lambda h^2 (\lambda^2 - 1)}{2 (\lambda^4 + h^4 (\lambda^2 - 1)^2 + 2 h^2 (\lambda^2 - 1) (3 \lambda^2 - 2))} \quad (3.67b)$$

$$c_3 = \frac{h}{4 (\lambda^2 + h^2 (\lambda^2 - 1))} \quad (3.67c)$$

Again it is worth mentioning that this  $G_6$  symmetry was not imposed from the beginning but emerged during the seeking of the solution space. The non-vanishing commutators are

$$\begin{aligned}
[\xi_1, \xi_5] &= \xi_4 & [\xi_2, \xi_6] &= \xi_4 \\
[\xi_3, \xi_4] &= 2 \left( \frac{c_1}{c_2} + h \right) \xi_4 & [\xi_3, \xi_5] &= 2 \left( \frac{c_1}{c_2} + \frac{h}{2} \right) \xi_5 + \xi_6 \\
[\xi_3, \xi_6] &= -\xi_5 + 2 \left( \frac{c_1}{c_2} + \frac{h}{2} \right) \xi_6
\end{aligned} \tag{3.68}$$

with  $(c_1, c_2)$  given by (3.67). Finally the line element (3.65) admits a homothetic vector field

$$\eta = \frac{h(\lambda^2 - 1)}{-\lambda^2 + h^2(\lambda^2 - 1)} \partial_\tau - \frac{h(\lambda^2 - 1)}{-\lambda^2 + h^2(\lambda^2 - 1)} \partial_x + y \partial_y + z \partial_z \tag{3.69}$$

### 3.1.4 Case IV: $h \neq 0$ and $u_2(t) = t$

In this case the determinant of the sale factor matrix becomes

$$\det[\gamma_{\alpha\beta}] = \frac{1}{4} e^{3u_1+4u_6} (1 - 4t^2)$$

so we must demand that  $|t| \leq \frac{1}{2}$  in order for  $\gamma_{\alpha\beta}$  to be positive definite.

The third linear constraint  $E_3 = 0$  can be used to define the function  $u_6$

$$E_3 = 0 \Rightarrow \dot{u}_6 = \frac{2t(2t\dot{u}_4 - h)}{h(4t^2 - 1)} \Rightarrow u_6 = k_6 + \int \frac{2t(2t\dot{u}_4 - h)}{h(4t^2 - 1)} dt \tag{3.70}$$

The quadratic constraint  $E_o = 0$  defines the lapse function  $N^2$

$$\begin{aligned}
N^2 &= \frac{e^{u_1}}{16h^2(4t^2 - 1)(3h^2(4t^2 - 1) - 4t^2)} \left( 3h^2(4t^2 - 1)^2 \dot{u}_1^2 \right. \\
&\quad \left. + 32ht^2(4t^2 - 1) \dot{u}_1 \dot{u}_4 + 16t^2(h^2(4t^2 - 1) + 4t^2) \dot{u}_4^2 - 4h^2 \right) \tag{3.71}
\end{aligned}$$

Substituting the above values of the lapse  $N^2$  and the function  $u_6$  in equation  $E_{33} = 0$  we find the coefficient of  $\ddot{u}_1$  to be proportional to

$$4t^2(4t^2 + h^2(4t^2 - 1)) \dot{u}_4^2 + 4ht^2(4t^2 - 1) \dot{u}_1 \dot{u}_4 - h^2$$

a quantity that can be safely regarded different from zero, since it's nihilism leads either to zero lapse or to inconsistency. Thus we can solve  $E_{33} = 0$  for  $\ddot{u}_1$  and substitute it to  $E_{11} = 0$ . In order to solve this equation for  $\ddot{u}_4$  we must be assured that it's coefficient does not vanish. Setting this coefficient equal to zero we arrive at the following equation

$$\dot{u}_1 = \frac{h \cos 2u_4 + 2t}{2t(\sin 2u_4 - 2t)} - \frac{(4t^2(h^2 + 1) - h^2) \dot{u}_4 - 2ht}{h(4t^2 - 1)}$$

which is unacceptable because it leads to inconsistency. After solving equation  $E_{11} = 0$  for  $\ddot{u}_4$  we finally arrive to the following polynomial system of first order in  $\dot{u}_1, \dot{u}_4$

$$\ddot{u}_1 = \langle \dot{u}_1 | B_1 | \dot{u}_4 \rangle, \quad \ddot{u}_4 = \langle \dot{u}_1 | B_2 | \dot{u}_4 \rangle \tag{3.72}$$

where we have used again the notation  $\langle \dot{u}_i \rangle = (1 \ \dot{u}_i \ \dot{u}_i^2 \ \dot{u}_i^3)$  and  $|\dot{u}_i\rangle = \langle \dot{u}_i|^t$  with the  $4 \times 4$  matrices  $B_1, B_2$  given by

$$B_1 = f \begin{pmatrix} \frac{4(h^2 - 4(-1+h^2)t^2)}{(1-4t^2)^4} & 0 & \frac{16t^2(-16t^4+h^4(1-4t^2)^2)}{h^2(1-4t^2)^4} & 0 \\ \frac{-4t(1-8t^2+6h^2(-1+4t^2))}{(-1+4t^2)^3} & \frac{32t^2(-h^2+4(-1+h^2)t^2)}{h(-1+4t^2)^3} & \frac{-4t(-h^2+4(-1+h^2)t^2)(4t^2+3h^2(-1+4t^2))}{h^2(-1+4t^2)^3} & 0 \\ \frac{-3h^2+12(-1+h^2)t^2}{(1-4t^2)^2} & \frac{32t^3}{h(1-4t^2)^2} & 0 & 0 \\ \frac{3t}{-1+4t^2} & 0 & 0 & 0 \end{pmatrix}$$

$$B_2 = g \begin{pmatrix} \frac{2h}{4t^2+h^2(1-4t^2)} & \frac{-6h^2(1-4t^2)^2+4t^2(-3+8t^2)}{-4t^3+h^2t(-1+4t^2)} & \frac{8(4t^4+h^2t^2(-1+4t^2))}{-4ht^2+h^3(-1+4t^2)} & \frac{-4t(-3h^2+4t^2+12h^2t^2)}{h^2} \\ 0 & \frac{16t^2(-1+4t^2)}{-h^2+4(-1+h^2)t^2} & \frac{32t^3(-1+4t^2)}{-4ht^2+h^3(-1+4t^2)} & 0 \\ \frac{3h(1-4t^2)^2}{-8t^2+h^2(-2+8t^2)} & \frac{3t(1-4t^2)^2}{-h^2+4(-1+h^2)t^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$f = \frac{3h^2(4t^2 - 1) - 4t^2}{(4t^2 - 1)^2}, \quad g = \frac{(4t^2 - 1)(3h^2(4t^2 - 1) - 4t^2)}{4(h^2 - 1)t^2 - h^2} \quad (3.73)$$

Due to the form of  $B_1, B_2$  (their components are rational functions of the time  $t$ ), system (3.72) can be partially integrated with the help of the following Lie–Bäcklund transformation

$$\dot{u}_1 = \frac{h^2(3h^2(4t^2 - 1) + 4t^2) \tan r(t)}{2ht\sqrt{1-4t^2}(3h^2(4t^2 - 1) - 4t^2)} - \frac{4(h^2 - 1)t^2 - h^2}{2h\sqrt{1-4t^2}} \dot{r}(t) \quad (3.74a)$$

$$\dot{u}_4 = \frac{3h^2\sqrt{1-4t^2} \tan r(t)}{4t(3h^2(4t^2 - 1) - 4t^2)} + \frac{1}{4} \sqrt{1-4t^2} \dot{r}(t) \quad (3.74b)$$

yielding the single second order ODE for the function  $r(t)$

$$\begin{aligned}
\ddot{r} = & \left( \frac{h}{2} \sqrt{1-4t^2} + \tan r \right) \dot{r}^2 \\
& + \left( -\frac{3(4t^2(3h^2-1)-h^2)}{t(4(3h^2-1)t^2-3h^2)} + \frac{h(3h^2(4t^2-1)-8t^2)}{t(4(3h^2-1)t^2-3h^2)\sqrt{1-4t^2}} \tan r \right) \dot{r} \\
& + \frac{9h^4(\sin 2r + h\sqrt{1-4t^2})}{2t^2(4t^2+3h^2(1-4t^2))^2} \sec^2 r - \frac{h(3h^2(1-4t^2)+8t^2)^2}{2t^2(3h^2(1-4t^2)+4t^2)^2\sqrt{(1-4t^2)^3}}
\end{aligned} \tag{3.75}$$

This equation contains all the information concerning the unknown part of the solution space of the Type  $VII_h$  vacuum Cosmology ( $h \neq 0$ ). Unfortunately, it does not posses any Lie-point symmetries that can be used to reduce its order and ultimately solve it. However, its form can be substantially simplified through the use of new dependent and independent variable  $(\rho, u(\rho))$  according to  $r(s) = \pm \arcsin \frac{u(\rho)}{\sqrt{\rho^2-1}}$ ,  
 $s = \sqrt{\frac{3h^2(\rho-1)}{12h^2(\rho-1)+8}}$  thereby obtaining the equation

$$\begin{aligned}
\ddot{u} = & \pm \frac{h(1-\dot{u}^2)}{\sqrt{(6h^2\rho+4-6h^2)(\rho^2-u^2-1)}} \\
\Rightarrow \dot{u}^2 = & \frac{h^2(1-\dot{u}^2)^2}{(6h^2\rho+4-6h^2)(\rho^2-u^2-1)}
\end{aligned} \tag{3.76}$$

This equation is a special case of the general equation

$$\dot{u}^2 = \frac{(1-\dot{u}^2)^2}{(\kappa+\lambda\rho)(\rho^2-u^2-1)} \tag{3.77}$$

with the values  $\kappa = -6 + \frac{4}{h^2}$ ,  $\lambda = 6$ . The general solution of (3.77) was first given in [25] and can be obtained as follows: First we apply the contact transformation:

$$\begin{aligned}
u(\rho) = & -\frac{8}{\lambda} y(\xi) + \frac{4(2\xi-1)}{\lambda} y'(\xi) & \rho = & -\frac{\kappa}{\lambda} + \frac{4}{\lambda} y'(\xi) \\
\dot{u}(\rho) = & 2\xi - 1 & \dot{u}(\rho) = & \frac{\lambda}{2y''(\xi)}
\end{aligned} \tag{3.78}$$

which reduces it to

$$\xi^2(\xi-1)^2 y''^2 = -4y'(\xi y' - y)^2 + 4y'^2(\xi y' - y) - \frac{\kappa}{2} y'^2 + \frac{\kappa^2 - \lambda^2}{16} y' \tag{3.79}$$

This equation is a special form of the equation SD-Ia, appearing in [26], where a classification of second order second degree ordinary differential equations was

performed. The general solution of (3.79) is obtained with the help of the sixth Painlevé transcendent  $w := \mathbf{P}_{\text{VI}}(\alpha, \beta, \gamma, \delta)$  and reads:

$$y = \frac{\xi^2(\xi-1)^2}{4w(w-1)(w-\xi)} \left( w' - \frac{w(w-1)}{\xi(\xi-1)} \right)^2 + \frac{1}{8} (1 \pm \sqrt{2\alpha})^2 (1-2w) - \frac{\beta}{4} \left( 1 - \frac{2\xi}{w} \right) - \frac{\gamma}{4} \left( 1 - \frac{2(\xi-1)}{w-1} \right) + \left( \frac{1}{8} - \frac{\delta}{4} \right) \left( 1 - \frac{2\xi(w-1)}{w-\xi} \right) \quad (3.80)$$

where the sixth Painlevé transcendent  $w := \mathbf{P}_{\text{VI}}(\alpha, \beta, \gamma, \delta)$  is defined by the ODE:

$$w'' = \frac{1}{2} \left( \frac{1}{w-1} + \frac{1}{w} + \frac{1}{w-\xi} \right) w'^2 - \left( \frac{1}{\xi-1} + \frac{1}{\xi} + \frac{1}{w-\xi} \right) w' + \frac{w(w-1)(w-\xi)}{\xi^2(\xi-1)^2} \left( \alpha + \beta \frac{\xi}{w^2} + \gamma \frac{(\xi-1)}{(w-1)^2} + \delta \frac{\xi(\xi-1)}{(w-\xi)^2} \right) \quad (3.81)$$

The values of the parameters  $(\alpha, \beta, \gamma, \delta)$  of the Painlevé transcendent, can be obtained from the solution of the following system:

$$\alpha - \beta + \gamma - \delta \pm \sqrt{2\alpha} + 1 = -\frac{\kappa}{2} \quad (3.82a)$$

$$(\beta + \gamma)(\alpha + \delta \pm \sqrt{2\alpha}) = 0 \quad (3.82b)$$

$$(\gamma - \beta)(\alpha - \delta \pm \sqrt{2\alpha} + 1) + \frac{1}{4}(\alpha - \beta - \gamma + \delta \pm \sqrt{2\alpha})^2 = \frac{\kappa^2 - \lambda^2}{16} \quad (3.82c)$$

$$\frac{1}{4}(\gamma - \beta)(\alpha + \delta \pm \sqrt{2\alpha})^2 + \frac{1}{4}(\beta + \gamma)^2(\alpha - \delta \pm \sqrt{2\alpha} + 1) = 0 \quad (3.82d)$$

Plugging in (3.82) the values of  $\kappa = -6 + \frac{4}{h^2}$ ,  $\lambda = 6$  for Type  $VII_h$ , we have twenty-four solutions (counting multiplicities) of this system. In order for the parameters  $(\alpha, \beta, \gamma, \delta)$  to be real numbers we end up only with four possibilities

$$(\alpha, \beta, \gamma, \delta) = \left( \frac{4h^2 - 1}{2h^2} - \sqrt{3 - \frac{1}{h^2}}, \frac{1}{2h^2}, -\frac{1}{2h^2}, \frac{1 - 2h^2}{2h^2} \right)$$

$$(\alpha, \beta, \gamma, \delta) = \left( \frac{4h^2 - 1}{2h^2} + \sqrt{3 - \frac{1}{h^2}}, \frac{1}{2h^2}, -\frac{1}{2h^2}, \frac{1 - 2h^2}{2h^2} \right), \quad |h| \geq \frac{1}{\sqrt{3}} \quad (3.83a)$$

and

$$\begin{aligned} (\alpha, \beta, \gamma, \delta) &= \left( \frac{1}{2}, \frac{2-3h^2}{2h^2} + \frac{\sqrt{1-3h^2}}{h^2}, \frac{3h^2-2}{2h^2} + \frac{\sqrt{1-3h^2}}{h^2}, \frac{1}{2} \right) \\ (\alpha, \beta, \gamma, \delta) &= \left( \frac{1}{2}, \frac{2-3h^2}{2h^2} - \frac{\sqrt{1-3h^2}}{h^2}, \frac{3h^2-2}{2h^2} - \frac{\sqrt{1-3h^2}}{h^2}, \frac{1}{2} \right), \quad |h| \leq \frac{1}{\sqrt{3}} \end{aligned} \quad (3.83b)$$

For the values  $h = \pm \frac{1}{\sqrt{3}}$  the above relations coincide and as we will show these values of  $h$  give rise to a particular solution.

Gathering all the pieces the final form of the *general* line element describing Bianchi type  $VII_h$  vacuum Cosmology is

$$\begin{aligned} ds^2 &= \kappa^2 \left( -\frac{e^{u_1(\xi)}}{16h^2\xi(\xi-1)} (d\xi)^2 + \sqrt{\xi(\xi-1)} \right. \\ &\quad \times \left( \sqrt{y'(\xi)} - \sin(2u_4(\xi)) \sqrt{y'(\xi) - \frac{1}{h^2}} \right) (\sigma^1)^2 \\ &\quad + 2 \cos(2u_4(\xi)) \sqrt{\xi(\xi-1)(y'(\xi) - \frac{1}{h^2})} \sigma^1 \sigma^2 \\ &\quad \left. + \sqrt{\xi(\xi-1)} \left( \sqrt{y'(\xi)} + \sin(2u_4(\xi)) \sqrt{y'(\xi) - \frac{1}{h^2}} \right) (\sigma^2)^2 + e^{u_1(\xi)} (\sigma^3)^2 \right) \end{aligned} \quad (3.84)$$

where

$$u'_1(\xi) = \frac{(-1+h^2)(-1+2\xi) + 2h^2y(\xi)}{2h^2(-1+\xi)\xi} \quad (3.85a)$$

$$u'_4(\xi) = \frac{1-2\xi+2h^2y(\xi)}{4h(-1+\xi)\xi(-1+h^2y'(\xi))} \quad (3.85b)$$

and  $y(\xi)$  is given by (3.80). Again, this line element contains three essential constants, thus representing the general solution to the EFE's for the Class B  $VII_h$  case.

### Particular solutions

Even though the line element (3.84) represents the general solution of Bianchi Type  $VII_h$  vacuum Cosmology, it does not come into a manageable form due to

the appearance of the sixth Painlevé transcendent. To partially remedy this inconvenience, we give, in the following, some closed form line-elements arising from particular solutions to (3.80) and (3.81).

- **Subcase**  $y(\xi) = c$  and  $|h| \leq \frac{1}{\sqrt{3}}$

One way to obtain a particular solution from the above line element (3.84) is to follow the reasoning of Case II, i.e. to observe that, although the form of the contact transformation (3.78) implies that the function  $y(\xi)$  cannot be constant, the line element (3.84) is free of this restriction; the difficulty with the negative argument in the square root is circumvented by using the hyperbolic sine/cosine (see (3.86) below). We can thus check if the assumption  $y(\xi) \equiv c$  leads to a particular solution.

Skipping the calculational details, we find that for  $y(\xi) \equiv c = \frac{\sqrt{1-3h^2}}{2h^2}$  all the Einstein's field equations are satisfied and we end up with the **new** line element

$$ds^2 = \kappa^2 \sin(4h\tau) \left( f(\tau) (d\tau)^2 + \sin(h \ln f(\tau)) (\sigma^1)^2 - \sin(h \ln f(\tau)) (\sigma^2)^2 + 2 \cos(h \ln f(\tau)) \sigma^1 \sigma^2 + f(\tau) (\sigma^3)^2 \right) \quad (3.86)$$

$$f(\tau) = \sin^{-\frac{1}{h^2}}(4h\tau) \tan^{-\frac{\sqrt{1-3h^2}}{h^2}}(2h\tau), \quad |h| \leq \frac{1}{\sqrt{3}}$$

which even though is physically acceptable it corresponds to Bianchi Type  $VII_h$  symmetry on  $T_3$ . Since the above line element admits only the three killing fields (3.3) and no homothetic vector field we can conclude that the constant  $\kappa$  is essential.

An interesting property of the line element (3.86) is that, for the value  $h^2 = \frac{1}{3}$ , i.e.,

$$ds^2 = \kappa^2 \left( \csc^2 \frac{4\tau}{\sqrt{3}} (d\tau)^2 - \sin \frac{4\tau}{\sqrt{3}} \sin \left( \sqrt{3} \ln \sin \frac{4\tau}{\sqrt{3}} \right) (\sigma^1)^2 + \sin \frac{4\tau}{\sqrt{3}} \sin \left( \sqrt{3} \ln \sin \frac{4\tau}{\sqrt{3}} \right) (\sigma^2)^2 + 2 \sin \frac{4\tau}{\sqrt{3}} \cos \left( \sqrt{3} \ln \sin \frac{4\tau}{\sqrt{3}} \right) \sigma^1 \sigma^2 + \csc^2 \frac{4\tau}{\sqrt{3}} (\sigma^3)^2 \right) \quad (3.87)$$

admits a fourth killing field, namely

$$\eta = e^{\frac{-2x}{\sqrt{3}}} \sin \frac{4\tau}{\sqrt{3}} \partial_\tau - 2e^{\frac{-2x}{\sqrt{3}}} \cos \frac{4\tau}{\sqrt{3}} \partial_x \quad (3.88)$$

The geometry (3.87) was first given by Petrov [27] and it is the only vacuum solution admitting a simply transitive  $G_4$  as its maximal group of motions. This group of motions has two subgroups  $G_3$  of Bianchi Types I and  $VII_{h^2=\frac{1}{3}}$  acting in time-like hyper-surfaces.

- **Elementary solution of Painlevé transcendent**

As it is well known, although for generic values of the parameters  $(\alpha, \beta, \gamma, \delta)$  the Painlevé functions are transcendental, there exist a lot of elementary solutions for special values of these parameters [28; 29]. In the case at hand the following Lemma is applicable

**Lemma** *The function  $w$  satisfying  $w(\xi)^2 - 2\xi w(\xi) + \xi = 0$  is a solution of (3.81) when the parameters  $(\alpha, \beta, \gamma, \delta)$  obey the relations  $\alpha + \delta = \frac{1}{2}$ ,  $\beta = -\gamma$ .*

*Proof* Direct computation.  $\square$

Using (3.83a), the conditions of the above Lemma are fulfilled for  $h = \pm \frac{2}{\sqrt{11}}$ . Then from the first of (3.83a) we have  $(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, \frac{11}{8}, -\frac{11}{8}, \frac{3}{8})$ . Choosing now the parametrization

$$w(\xi) = \frac{1}{4h^2} e^{4h\tau}, \quad \xi = \cosh^2(2h\tau) \quad (3.89)$$

we can compute  $y(\tau)$  from (3.80) (with the minus sign) and  $u_1(\tau), u_4(\tau)$  from (3.85), thereby arriving at the following line element

$$\begin{aligned} ds^2 = \kappa^2 & \left( -e^{\frac{2\tau}{h}} \sinh^{-\frac{3}{8}}(4h\tau) (d\tau)^2 + e^{-2h\tau} \sinh^{\frac{1}{2}}(4h\tau) (e^{4h\tau} + \sin(4\tau)) (\sigma^1)^2 \right. \\ & + 2e^{-2h\tau} \cos(4\tau) \sigma^1 \sigma^2 + e^{-2h\tau} \sinh^{\frac{1}{2}}(4h\tau) (e^{4h\tau} - \sin(4\tau)) (\sigma^2)^2 \\ & \left. + e^{\frac{2\tau}{h}} \sinh^{-\frac{3}{8}}(4h\tau) (\sigma^3)^2 \right) \end{aligned} \quad (3.90)$$

This geometry was first given by Lukash [30] and, like (3.86), admits only the three killing fields (3.3) and no homothetic vector field. Therefore, the constant  $\kappa$  is essential.

#### 4 Discussion

In [10; 25] a systematic approach for investigating the solution space of Bianchi type cosmologies was developed by the use of automorphisms and the theory of symmetries of ordinary, coupled differential equations. The result was the comprehensive recovery of all known closed form Type *III* solutions, as well as the presentation of the general solution in terms of the sixth Painlevé transcendent. In the present work we have applied the method to the case of Bianchi Type  $VII_h$  family of vacuum geometries. Again, the general solution is implicitly given in terms of the third (3.52) Painlevé transcendent or the sixth Painlevé transcendent (3.84) for the Class A ( $h = 0$ ) and the Class B ( $h \neq 0$ ) case respectively. One could expect on general grounds (since Bianchi *VII* can be obtained via a reduction of the Einstein field equations with respect to an Abelian  $G_2$  and therefore has an Ernst-like formalism) to find a Painlevé function solution when there is a further reduction with an extra Killing vector (see note 7 on p. 109 of [31]). But finding the solutions is far from simple, and our method guarantees that the entire solution space is swept.

Through the investigation of either Particular or Elementary solutions of the Painlevé transients we are able to concisely recover, in a systematic fashion, all six known solutions (3.36), (3.56), (3.62), (3.65), (3.87), (3.90). All these metrics have originally been obtained in a time scale of 20 years or so, by prior assumption of symmetry and/or other physical requirements; e.g., Petrov's solution [27] was derived with the use of automorphisms seeking  $G_4$  homogeneous metrics while Lukash's solution [30] was derived based on a physical interpretation of Type  $VII_h$  cosmological models, in terms of circularly polarized gravitational waves of arbitrary wavelength in a space having constant negative curvature. Their reacquisition single-handed, proves, we believe, the value of our method. A very important result is, of course, the discovery of the new family of solutions (3.86) for the

**Table 1** Bianchi Type  $VII_0$  metrics

Line element	Isometry type	Comments
$ds^2 = -(dt)^2 + (\sigma^1)^2 + (\sigma^2)^2 + \tau^2 (\sigma^3)^2$	$G_{10}$ on $V_4$	Flat space
$ds^2 = -\tau(d\tau)^2 + \tau^2 (\sigma^1)^2 + \tau^2 (\sigma^2)^2 + \frac{1}{\tau} (\sigma^3)^2$	$G_4$ on $V_3$ , $\tau > 0$ $G_4$ on $T_3$ , $\tau < 0$	LRS
$ds^2 = \kappa^2 \left( e^{\tau^4} \tau (d\tau)^2 - \tau^2 (\sigma^1)^2 + \tau^2 (\sigma^2)^2 + \frac{e^{\tau^4}}{\tau} (\sigma^3)^2 \right)$	$G_3$ on $T_3$ , $\tau > 0$	Non-homothetic
$ds^2 = \kappa^2 \left( -\frac{e^{u_1(\xi)}}{16\xi} (d\xi)^2 + \sqrt{ \xi w(\xi) } (\sigma^1)^2 + \sqrt{\left  \frac{\xi}{w(\xi)} \right } (\sigma^2)^2 + e^{u_1(\xi)} (\sigma^3)^2 \right)$	$G_3$ on $V_3$	General solution

range of the group parameter  $h^2 \leq \frac{1}{3}$ . Besides of the obvious value of a new family of solutions to the EFE's it also points to the unexpected existence of a sector with particular behavior for this Bianchi Type. It is known that Type  $VI_h$  model has an exceptional sector corresponding to the value  $h^2 = \frac{1}{9}$  but, for Type  $VII_h$  such a behavior is first observed. It is also very interesting to point out that this family of solutions is related, through a complex transformation of  $t, x$  to the  $VI_h$  solution of Ellis and MacCallum ((13.57) of [12]). The fact may be taken as a further strengthening evidence of the widespread belief that the two Types are very much similar. In fact, it seems highly probable that all the  $VI_0/VI_h$  results, except those for class Bbii of [11], can be similarly related to the solutions here presented. We hope that the application of the method to Type  $VI_h$  will bear analogous fruits and show how far this similarity can reach.

As for Types  $VIII, IX$ , the recent discovery that some particular configurations are described by the third Painlevé transcendent [32] strengthens our belief that their solution space will also be attained by our method. We plan to return to these issues in the immediate future. Finally, we deem it useful to end this discussion by briefly describing the investigated solution space through the following Tables 1 and 2:

Where the 1-forms  $\sigma^\alpha$  are given by

$$\sigma^1 = \sin x dy + \cos x dz, \quad \sigma^2 = \cos x dy - \sin x dz, \quad \sigma^3 = \frac{1}{2} dx \quad (4.1)$$

and  $u_1(\xi)$  is defined by equation (3.48b)

$$u'_1(\xi) = \frac{\xi w'(\xi)^2}{4w(\xi)^2} + \frac{1}{4} w(\xi) + \frac{1}{4w(\xi)} - \frac{1}{4\xi} - \frac{1}{2}$$

with  $w(\xi)$  standing for the third Painlevé transcendent  $w := \mathbf{P}_{\text{III}}(-\frac{1}{2}, \frac{1}{2}, 0, 0)$ , defined by (3.46).

**Table 2** Bianchi Type  $VI_h$  metrics

Line element	Isometry type	Comments
$\frac{ds^2}{d\tau^2} = -(d\tau)^2 + h^2 \tau^2 (\sigma^1)^2 + 4h^2 \tau^2 (\sigma^3)^2$	$G_{10}$ on $V_4$	Flat space
$ds^2 = \frac{1}{4} \exp\left(\frac{-2\lambda^2 + 2h^2(\lambda^2 - 1)}{h(\lambda^2 - 1)} \tau\right) (-d\tau^2 + 4(\sigma^3)^2) + \frac{1}{2} e^{2h\tau} (1 + \lambda \sin 2\tau) (\sigma^2)^2$	$G_6$ on $V_4$	Homothetic
$ds^2 = \kappa^2 \sin(4h\tau) \left( f(\tau) d\tau^2 + \sin(h \ln f(\tau)) (\sigma^1)^2 - \sin(h \ln f(\tau)) (\sigma^2)^2 + 2 \cos(h \ln f(\tau)) \sigma^1 \sigma^2 + f(\tau) (\sigma^3)^2 \right)$	$G_3$ on $T_3$	Non-homothetic $ h  \leq \frac{1}{\sqrt{3}}$
$ds^2 = \kappa^2 \left( \csc^2 \frac{4\tau}{\sqrt{3}} (d\tau)^2 - \sin \frac{4\tau}{\sqrt{3}} \sin \left( \sqrt{3} \ln \sin \frac{4\tau}{\sqrt{3}} \right) (\sigma^1)^2 + \sin \frac{4\tau}{\sqrt{3}} \sin \left( \sqrt{3} \ln \sin \frac{4\tau}{\sqrt{3}} \right) (\sigma^2)^2 + 2 \sin \frac{4\tau}{\sqrt{3}} \cos \left( \sqrt{3} \ln \sin \frac{4\tau}{\sqrt{3}} \right) \sigma^1 \sigma^2 + \csc^2 \frac{4\tau}{\sqrt{3}} (\sigma^3)^2 \right)$	$G_4$ on $T_3$	Non-homothetic Maximal $G_4$ $h^2 = \frac{1}{3}$
$ds^2 = \kappa^2 \left( -e^{2\frac{\tau}{h}} \sinh^{-\frac{3}{8}}(4h\tau) (d\tau)^2 + e^{-2h\tau} \sinh^{\frac{1}{2}}(4h\tau) (e^{4h\tau} + \sin(4\tau)) (\sigma^1)^2 + 2e^{-2h\tau} \cos(4\tau) \sigma^1 \sigma^2 + e^{-2h\tau} \sinh^{\frac{1}{2}}(4h\tau) (e^{4h\tau} - \sin(4\tau)) (\sigma^2)^2 + e^{\frac{2\tau}{h}} \sinh^{-\frac{3}{8}}(4h\tau) (\sigma^3)^2 \right)$	$G_3$ on $V_3$	Non-homothetic $h^2 = \frac{4}{11}$
$ds^2 = \kappa^2 \left( -\frac{e^{t_1(\xi)}}{16h^2 \xi (\xi - 1)} (d\xi)^2 + \sqrt{\xi(\xi - 1)} \left( \sqrt{y'(\xi)} - \sin(2u_4(\xi)) \sqrt{y'(\xi) - \frac{1}{h^2}} \right) (\sigma^1)^2 + 2 \cos(2u_4(\xi)) \sqrt{\xi(\xi - 1)} (y'(\xi) - \frac{1}{h^2}) \sigma^1 \sigma^2 + \sqrt{\xi(\xi - 1)} \left( \sqrt{y'(\xi)} + \sin(2u_4(\xi)) \sqrt{y'(\xi) - \frac{1}{h^2}} \right) (\sigma^2)^2 + e^{u_1(\xi)} (\sigma^3)^2 \right)$	$G_3$ on $V_3$	General solution

Where the 1-forms  $\sigma^\alpha$  are given by

$$\sigma^1 = e^{hx} (\sin x dy + \cos x dz), \sigma^2 = e^{hx} (\cos x dy - \sin x dz), \sigma^3 = \frac{1}{2} dx \quad (4.2)$$

the function  $f(\tau)$  stands for

$$f(\tau) = \sin^{-\frac{1}{h^2}} (4h\tau) \tan^{-\frac{\sqrt{1-3h^2}}{h^2}} (2h\tau) \quad (4.3)$$

and the functions  $u_1(\xi)$ ,  $u_4(\xi)$  are defined by (3.85)

$$u'_1(\xi) = \frac{(-1+h^2)(-1+2\xi)+2h^2y(\xi)}{2h^2(-1+\xi)\xi} \quad (4.4a)$$

$$u'_4(\xi) = \frac{1-2\xi+2h^2y(\xi)}{4h(-1+\xi)\xi(-1+h^2y'(\xi))} \quad (4.4b)$$

with  $y(\xi)$  defined by (3.80).

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