ABSTRACT

U(1) $\times$ U(1) GAUGE THEORY
APPLIED TO MAGNETIC
MONOPOLES

In this paper we build up a U(1) (*i.e.* one dimensional unitary) gauge theory beginning with a standard complex scalar field Lagrangian, then by introducing a local gauge transformation and subsequently redefining the derivative, we will be able to obtain Maxwell's equations with the Noether current. This will show that electromagnetic fields arise in this form as the result of the invariance of the complex scalar field Lagrangian under local gauge transformations. From there we will extend to a U(1) $\times$ U(1) theory by introducing a second vector potential which will result in Maxwell's equations allowing magnetic charge and a Lagrangian with an extra massless photon inconsistent with the Standard Model. Using the Higgs' mechanism this photon will gain mass and represent a new "magnetic photon." We will then extend the theory by introducing a second scalar potential which will result in conditions on magnetic and electric charge.

John Mayo Lucien Scott
December 2016

$U(1) \times U(1)$ GAUGE THEORY APPLIED TO MAGNETIC MONOPOLES

by

John Mayo Lucien Scott

A thesis

submitted in partial

fulfillment of the requirements for the degree of

Master of Science in Physics

in the College of Science and Mathematics

California State University, Fresno

December 2016

APPROVED

For the Department of Physics

We, the undersigned, certify that the thesis of the following student meets the required standards of scholarship, format, and style of the university and the student's graduate degree program for the awarding of the master's degree.

John Mayo Lucien Scott

Thesis Author

Douglas Singleton(Chair) Physics

Gerardo Munoz Physics

Karl Runde Physics

For the University Graduate Committee:

Dean, Division of Graduate Studies

AUTHORIZATION FOR REPRODUCTION

OF MASTER'S THESIS

____X____ I grant permission for the reproduction of this thesis in part or in its entirety without further authorization from me, on the condition that the person or agency requesting reproduction absorbs the cost and provides proper acknowledgment of authorship.

________ Permission to reproduce this thesis in part or in its entirety must be obtained from me.

Signature of thesis author: ______________________________

ACKNOWLEDGMENTS

In the course of my studies I have become distant from many of my friends from both childhood and my schooling at UCLA. That said, a handful have kept in contact and supported my studies. I would like to thank my good friends B. Gardner, M. Diaz and A. Hogan for the quality long distance entertainment via group text messages that kept my spirits lifted during hard times of study. I would also like to thank the Physics Department at Fresno State and the welcoming staff and faculty members. I would like to thank Dr. Singleton for allowing me to study under his work and complete this thesis. I would also like to thank Dr. Runde and Dr. Munoz for being accepting of my thesis so late in the game. And I would like to thank Dr. Munoz again for allowing me to start my degree here by sampling some of his graduate classes before I had applied to the program.

Last but not least, I would like to thank both of my parents, Larry and Rosemary Scott, for their continued support over the course of my schooling. They both sacrificed a lot both socially and economically so that I could go to school and complete a degree. Although we more often than not disagreed on everything, they were still there for me. My father was the driving competitive force keeping me on my toes throughout this masters program and my mother was the back support necessary to deal with the often tough competition. Without them I might not have survived this ordeal.

TABLE OF CONTENTS

NOTATION

The notation used in the main section of this paper will follow that of Lewis Ryder' Quantum Field Theory, 2nd Edition. There are sections where the units are those used in J.D Jackson's Classical Electrodynamics 2nd Edition. Therefore the reader might notice missing or erroneous units of c and $\hbar$ from one chapter to the next, but this is of no great concern since they are simply constants. The 4-vector notation used in later chapters is standard Einstein notation.

I dedicate this thesis to my father, who I hope will eventually read it, and my mother for her support

CHAPTER 1: INTRODUCTION

The magnetic monopole has been a sought after particle for well over a century and a half. And while it has not been found, outside of a few monopole-like simulations, the concept has led to new and interesting discoveries in physics. Despite having not been found, physicists have constructed a handful of physical descriptions of magnetic monopole particles including, but not limited to, the structures that will be discussed in this paper. As a precursor to discussions of magnetic monopoles, one must first address the fact that the Maxwell equations, as we well know, do not allow for magnetic charge. The magnetic field, written as the curl of a vector potential, cannot have divergence due to the mathematical caveat that the divergence of the curl of a well behaved function is simply zero. This leads to Maxwell equations that are strikingly symmetric with the exception that the divergence of the electric field results in a charge density, while the divergence of the magnetic field gives zero. Were it not for this lack of divergence in the magnetic field, the theory would be symmetric with the interchange of the electric and magnetic fields. That is, a duality transformation or a rotation through some arbitrary angle of the electric and magnetic fields could be performed and the Maxwell equations would still be satisfied. Moreover, observed particles would have the same ratio of magnetic charge to electric charge. This lack of magnetic charge density in Maxwell's equations was not enough to discourage physicists from throwing out the concept of a magnetic monopole.

Maxwell chose not to include magnetic charge in his equations due to the fact that experiments at the time showed no profit in considering magnetic monopoles. While this may have been the fruitful convention, notable physicists such as Pierre Curie pointed out that monopoles were not strictly eliminated by

Maxwell's choice. The first suggestive monopole theory came from Paul Dirac [1] in the early 1930s. Dirac envisioned an infinitely long and infinitely thin, solenoid-like construction which would produce a magnetic monopole-like field at its end. Choosing not so well behaved vector potentials, one could construct a magnetic field that had a non-zero divergence, and as long as one posited that the resulting nonphysical singularity could be "hidden" then the construction would appear physical. Furthermore, the construction led to quantized electric charge which excited many in the field since experiments at the time suggested electric charge came in integer values. Today we observe quantized electric charge from quarks and non-Abelian symmetry groups [2] rather than from the existence of magnetic monopoles and the Dirac quantization condition.

In 1974, Gerard t'Hooft and Alexander Polyakov [3] found that when the gauge symmetry is extended from the Abelian gauge group U(1) to a non-Abelian group and symmetry breaking is performed the field equations produce a topological magnetic monopole term [2]. Interestingly, they found the Dirac condition multiplied by a factor of two. In addition, new research in Grand Unified theories suggest the presence of t'Hooft-Polyakov monopoles, although such monopoles have mass orders of magnitude larger than anything the Large Hadron Collider could spit out [4]. While t'Hooft-Polyakov monopoles may be impractical to consider in a collider event, it is still suggested that elementary magnetic monopole-like particles could be found in collisions and experiments such as ATLAS and MoEDAL at the LHC continue to look for such specimens [4].

In this paper, we would like to formalize a new construction of the magnetic monopole. Standard unitary $U(1)$ gauge theory introduces a single vector potential, usually denoted $\overrightarrow{A}$, which is a necessity to keep the Lagrangian invariant under local gauge transformations. Classically we understand the vector potential as the result of $\nabla \cdot \overrightarrow{B} = 0$ which means that the field B must be the curl of some vector potential $\overrightarrow{A}$ such that $\overrightarrow{B} = \nabla \times \overrightarrow{A}$. Beginning with Maxwell's equations and allowing magnetic charge density, we will show that Maxwell's equations can be uniquely and efficiently described with the addition of a second vector potential. This construction will allow us to treat magnetic and electric charge as gauge symmetries. We will also show that such a construction leads to a second massless gauge boson which is unwanted. This unwanted gauge boson will be resolved into a new mass carrying photon-like particle through the introduction of the Higgs' mechanism. The resulting particle will be what is labeled as a "magnetic photon" which carries mass and arises from the introduction of magnetic charge [5].

CHAPTER 2: EARLY MONOPOLE FORMULATIONS

The Dirac Monopole

For 150+ years, physicists have questioned whether or not a magnetic equivalent of the electron exists. This idea of a particle of charge that is not electric in nature but rather something consistent with an isolated north or south magnetic pole has yet to be discovered. The idea of a magnetic monopole is generally first credited to Pierre Curie from his 1894 paper, despite this publication being long after the Maxwell equations (∼1860) were formally written and posited that magnetic charge could not exist as $\nabla \cdot \overrightarrow{B} = 0$.

It is worth noting here that this formulation for Maxwell's equations dismisses the possibility of a magnetic monopole as the result of pure math. If the magnetic field, $\overrightarrow{B}$ is the curl of a vector potential, $\overrightarrow{A}$ such that $\overrightarrow{B} = \nabla \times \overrightarrow{A}$, then we must have that

$$\nabla \cdot \overrightarrow{B} = \nabla \cdot \left(\nabla \times \overrightarrow{A}\right) = 0$$

since the divergence of the curl of a vector must be zero. Notice though that this "proof" makes two subtle requirements; the vector potential must be twice-differentiable, and it also must be "nice" in the sense that it doesn't have any singularities. If one were to consider a vector potential that is not "nice" then it might be that $\nabla \cdot \overrightarrow{B} \neq 0$ and there would be a monopole-like term.

One such vector potential that has singularities and can generate a radial magnetic field term is as follows;

$$\overrightarrow{A} = g\frac{1-\cos\theta}{r\sin\theta}\hat{\phi}\ . \tag{2.1}$$

If one considers an angle in θ such that $\theta \to \pi$, then the denominator goes to zero and the vector potential blows up to infinity. Furthermore, if we calculate the curl of the vector potential we have

$$\overrightarrow{B} = \nabla \times \overrightarrow{A} =$$

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right]\hat{\theta} +$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi} \ .$$

All terms above compute to zero except for ϕ components,

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A_\phi \sin\theta\right)\right]\hat{r} + \frac{1}{r}\left[-\frac{\partial}{\partial r}\left(rA_\phi\right)\right]\hat{\theta} \ .$$

If we then plug in the vector potential and canceling terms we will have,

$$\frac{g}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{1-\cos\theta}{r}\right)\right]\hat{r} - \frac{g}{r}\left[\frac{\partial}{\partial r}\left(\frac{1-\cos\theta}{\sin\theta}\right)\right]\hat{\theta} \ .$$

Notice that the term containing the partial derivative with respect to r has no r dependence; therefore, this term goes to zero leaving,

$$\overrightarrow{B} = \frac{g}{r^2\sin\theta}\left[-\frac{\partial}{\partial\theta}\cos\theta\right]\hat{r} = \frac{g}{r^2}\hat{r} \ . \tag{2.2}$$

As we anticipated we are left with a magnetic field term that is radial in $\hat{r}$ direction and has a magnetic charge coefficient, g. The singularity of the vector potential (2.1) extends along the z axis to infinity and is commonly known as the "Dirac String." Magnetic monopole vector potentials of this kind require a singularity and as a result there will be a characteristic "string" that travels out of the particle to infinity. Often this string is envisioned as an infinitely long and infinitely thin solenoid with some current value. A singularity of this sort is not very physical and hinders the prospect of a magnetic monopole. In standard solenoid examples, the produced field is dipole in nature but if one considers the theoretical nonphysical construction then the magnetic field that would be observed at one end would be that of a monopole. As we will see, one can force conditions that make the un-physical string "disappear." The process will be to consider Aharonov-Bohm effects and force conditions that make the string invisible to an incident particle. Let us extend this example further by rewriting the magnetic field equation from (2.2) as,

$$\overrightarrow{B} = \frac{g}{r^2}\hat{r} = \frac{g}{r^3}\overrightarrow{r} = -g\nabla\left(\frac{1}{r}\right) \ .$$

Taking the divergence of this field we obtain

$$\nabla \cdot \overrightarrow{B} = 4\pi\delta^3(r)$$

Calculating the flux of the magnetic field we obtain

$$\Phi = \int\int \overrightarrow{B} \cdot \hat{n} dS = B \int\int r^2 \sin\theta d\theta d\phi = 4\pi r^2 B = 4\pi g \tag{2.3}$$

Let us now take a free particle with electric charge, e, and place it in the monopole field. The wave function for a free particle is

$$\psi = |\psi| e^{\frac{i}{\hbar}(\overrightarrow{p}\cdot\overrightarrow{r} - Et)}$$

When the free particle is subjected to an electromagnetic field it undergoes a phase change from the potential through the Aharonov-Bohm effect. The Aharonov-Bohm effect describes how the wave function for a free particle will pick up a phase difference due to the potential of the field. That is, $\overrightarrow{p} \to \overrightarrow{p} - \left(\frac{e}{c}\right)\overrightarrow{A}$, giving us,

$$\psi \to \psi e^{-\frac{ie}{\hbar c}\overrightarrow{A}\cdot\overrightarrow{r}}$$

therefore if the phase is written as α, the change is $\alpha \to \alpha - \frac{e}{\hbar c}\overrightarrow{A} \cdot \overrightarrow{r}$ [2]. Now if one computes the total phase change along a closed path with fixed radial direction and θ angular component but allow ϕ angle to rotate through 2π radians then one finds that

$$\delta\alpha = \frac{e}{\hbar c}\oint \overrightarrow{A} \cdot dl$$

An application of Stokes' Theorem allows us to write this as

$$\delta\alpha = \frac{e}{\hbar c}\int \left(\nabla \times \overrightarrow{A}\right) \cdot dS = \frac{e}{\hbar c}\int \overrightarrow{B} \cdot dS$$

One notices that the change in phase is just constant terms times the magnetic flux we calculated in (2.3). That is, $\delta\alpha = \frac{4\pi e}{\hbar c}g$. The free particle we introduced picks up a phase change that depends on the vector potential which gave us our magnetic monopole term. The vector potential considered in (2.1) had singularities which will make the wave function vanish on the axis of the singularity making the phase indeterminate [2]. We must require that the wave function remain single valued so we must require that the change in phase $\delta\alpha = 2\pi n$. Therefore, from our phase change calculation we end with the famous Dirac quantization condition,
$2\pi n = \frac{4\pi e}{\hbar c}g$ or

$$eg = \frac{1}{2}n\hbar c \ . \tag{2.4}$$

This condition suggests that both magnetic charge and electric charge are quantized, assuming magnetic monopoles exist. Furthermore, the condition suggests that magnetic charge is a massive quantity in comparison to electric charge. The electric charge strength is proportional to the square root of the fine structure constant, α_e the dimensionless fine structure constant which is defined as
$\alpha_e = \frac{e^2}{\hbar c} \approx \frac{1}{137}$. The fine structure constant tells one about the strength of the electric interaction. Since $\alpha_e << 1$ this indicates that the electric interaction is small. One can define a magnetic fine structure constant as $\alpha_m = \frac{g^2}{\hbar c}$. Due to the Dirac quantization condition of (2.4) one finds that (for $n = 1$ in equation 2.4)

$\alpha_m \approx \frac{137}{4}$. That is, the strength of the magnetic interaction of magnetic charges are about 10^4 times stronger than their electric charge cousins.

The Wu-Yang Monopole

The method of finding the Dirac quantization condition for magnetic and electric charges is not unique. The magnetic monopole that will be discussed is a construction by Tai Wu and Chen Yang in 1975 [6], and while the construction is physically different, we will find that the condition found in the previous section is preserved. In fact, one might consider this formulation more physical than the standard Dirac string formulation since the singularity in the vector potential is stripped away. As the reader will see, using two vector potentials similar to the Dirac string example, one can hide the nonphysical singular points of the potentials. From the previous section, the Dirac monopole construction considered a vector potential of the form,

$$\overrightarrow{A_1} = g\frac{1-\cos\theta}{r\sin\theta}\hat{\phi} \tag{2.5}$$

This vector potential leads to a singularity along the $r = -z$ axial direction. We can also consider a mirrored potential of the form

$$\overrightarrow{A_2} = -g\frac{1+\cos\theta}{r\sin\theta}\hat{\phi} \tag{2.6}$$

where now the singular point lies along the $r = +z$ direction. Using these two potentials, we would like to stitch together a vector potential that is free of the singularities on the $\pm z$ axis. More precisely, we would like to consider two overlapping regions dividing the space around the monopole in which the vector potential is defined by the two different potentials given above. We will have a northern hemisphere where the vector potential is defined by (2.5) and a southern hemisphere defined by the potential (2.6). In the regions of overlap the two vector potentials will have different values. For this construction to be physical, our total vector potential needs to be single valued. The trick then is to devise a way to write one vector potential as a gauge transformation of the other [2].

If we consider just the components of these vector potentials,

$$A^1_\phi = \frac{g}{r\sin\theta} - \frac{g\cos\theta}{r\sin\theta} \qquad \textit{and} \qquad A^2_\phi = -\frac{g}{r\sin\theta} - \frac{g\cos\theta}{r\sin\theta}$$

then one notices that we can write the vector potentials as,

$$A^2_\phi = A^1_\phi - \frac{2g}{r\sin\theta}\hat{\phi}\ . \tag{2.7}$$

If computed, both of these potentials would give us the same magnetic field equation. If we were to take the curl of both sides of equation (2.7) then we would have the curl of the difference equal to zero. In fact, we force this condition by requiring that the difference between the two vector potentials be curl-less. That is, the factor $-\frac{2g}{r\sin\theta}$ must be the gradient of some other variable, call it α. Then, our gradient term is $\nabla\alpha$. If we turn to look at spherical coordinates, we notice that the gradient of the azimuthal angle is,

$$\nabla\phi = \left[\hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\hat{\phi}\frac{\partial}{\partial\phi}\right]\phi = \frac{1}{r\sin\theta}\hat{\phi}$$

therefore we can write the difference between the two potentials as

$$\frac{2g}{r\sin\theta}\hat{\phi} = 2g\nabla\phi \tag{2.8}$$

Now consider a gauge transformation of the form, $S = e^{2ige\phi}$. We can use this gauge to re-write (2.8) as follows,

$$-\frac{i}{e}S\nabla_\phi S^{-1} = -\frac{i}{e}e^{2ige\phi}\left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}e^{-2ige\phi}\right] = -\frac{i}{e}e^{2ige\phi}\left[\frac{-2ige}{r\sin\theta}e^{-2ige\phi}\right]\hat{\phi} =$$

$$-\frac{2g}{r\sin\theta}\hat{\phi}$$

This allows us to write the transformation from one vector potential to the other as,

$$A_\phi^2 = A_\phi^1 - 2g\nabla\phi = A_\phi^1 - \frac{i}{e}S\nabla_\phi S^{-1} \ . \tag{2.9}$$

We now have have found our full gauge transformation. The requirement that our vector potentials be single-valued means that our gauge transformation function, S, must also be single-valued. Therefore, we require that as $\phi \to \phi + 2\pi$, we obtain the same phase. If we allow $\phi \to \phi + 2\pi$, then the phase becomes $2ige\phi + 4\pi ige$. For single valued-ness we must then have $4\pi ige = 2\pi n$. This equates to,

$$ge = \frac{1}{2}n \tag{2.10}$$

which we recognize (pending units of $\hbar$ and c) to be the Dirac quantization condition (2.4) from the previous section. Indeed, the quantization condition is not unique to the physical construction of the magnetic monopole.

Furthermore, we can show that this construction does indeed give us a monopole by calculating the total magnetic flux. The flux is given by,

$$\Phi = \int F_{\mu\nu}dx^{\mu\nu} = \oint \left(\nabla \times \overrightarrow{A}\right) \cdot d\overrightarrow{S} \ .$$

If we split the integral into two regions, R_1 and R_2 defined by the two vector potentials with the union at $\theta = \pi/2$, we can write

$$\int_{R_1} \left(\nabla \times \overrightarrow{A_1}\right) \cdot dS + \int_{R_2} \left(\nabla \times \overrightarrow{A_2}\right) \cdot d\overrightarrow{S} \ .$$

We then apply Stokes' theorem, noting the orientation of the boundary at $\theta = \pi/2$,

$$\oint_{\theta=\pi/2} \overrightarrow{A_1} \cdot d\overrightarrow{l_1} - \oint_{\theta=\pi/2} \overrightarrow{A_2} \cdot d\overrightarrow{l_2} \ .$$

Now using the gauge transformation,

$$\oint_{\theta=\pi/2} \overrightarrow{A_1} \cdot d\overrightarrow{l_1} - \oint_{\theta=\pi/2} \left[\overrightarrow{A_1} - \frac{i}{e} S \nabla_\phi S^{-1} \hat{\phi}\right] \cdot d\overrightarrow{l_2}$$

we can re-write the last term in the second integral as a full derivative of the natural log of the inverse of S giving,

$$\Phi = \frac{i}{e} \oint \frac{d}{d\phi} \left(\ln S^{-1}\right) d\phi = \frac{i}{e} \oint (-2ige) d\phi = 4\pi g$$

which we note is consistent with the flux calculated in the Dirac string monopole case (2.3). Thus we have obtained both the original Dirac condition and magnetic flux of the Dirac string via the Wu-Yang monopole construction. One will be quick to point out that the conditions were obtained in similar but different fashions. The Dirac string formulation required single valued-ness of the wave equation for an incident electron which would be effected by passing near the infinitesimal solenoid due to the Aharonov-Bohm effect. The Wu-Yang monopole required single valued-ness of the gauge factor in the gauge transformation relating the two vector potentials with opposing singularities. This condition again forced the Dirac condition. Amazingly, two different monopole defining structures have led to the same condition between electric and magnetic charge.

At this point in time we would like to introduce a new formulation for the magnetic monopole. Unlike the Dirac string and the Wu-Yang monopole cases, the formulation we will discuss in a later section will not give an explicit vector potential. We still require an non-zero divergence of the magnetic field of course. With this assumption in Maxwell's equations, we will introduce a second unspecified vector potential and do a Lagrangian formulation in the realm of quantum field theory. Before we jump into that, we must first introduce some basic mathematical methods in the context of field theory. Then we will construct a Lagrangian that contains Maxwell's equations with a non-zero divergence of the magnetic field.

CHAPTER 3: GAUGE SYMMETRY

Complex Scalar Fields with a Global Gauge Transformation

We begin our discussion with a Lagrangian for a complex scalar field containing a kinetic energy and a potential term,

$$L = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi \ . \tag{3.1}$$

We consider a complex scalar field because we want to consider symmetries other than those under translation, rotation or Lorentz transformations. Such transformations can be achieved with simply a single real scalar field Lagrangian [2]. Using this complex scalar field Lagrangian, we hope that once a local gauge transformation is performed, we can show implicitly that the electromagnetic fields arise from the gauge invariance of the scalar fields. To begin, let us first show that the Lagrangian in (3.1) gives us the appropriate equations of motion describing the two real scalar fields. We can write the complex scalar fields in terms of two real scalar fields as

$$\phi = \frac{1}{\sqrt{2}} \left[\phi_1 + i\phi_2 \right] \tag{3.2}$$

and

$$\phi^* = \frac{1}{\sqrt{2}} \left[\phi_1 - i\phi_2 \right] \ . \tag{3.3}$$

Using metric terms, we can rewrite the complex scalar field Lagrangian as,

$$L = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi^* - m^2 \phi^* \phi$$

and then plug in the fields defined by (3.2) and (3.3). This expansion will give us the Lagrangian in terms of two real scalar fields as shown below;

$$L = \frac{1}{2} g^{\mu\nu} \partial_\mu \left(\phi_1 + i\phi_2\right) \partial_\nu \left(\phi_1 - i\phi_2\right) - m^2 \phi^* \phi =$$

$$\frac{1}{2} g^{\mu\nu} \left(\partial_\mu \phi_1 + i\partial_\mu \phi_2\right) \left(\partial_\nu \phi_1 - i\partial_\nu \phi_2\right) - m^2 \phi^* \phi =$$

$$\frac{1}{2} g^{\mu\nu} \left[\partial_\mu \phi_1 \partial_\nu \phi_1 - i\partial_\mu \phi_1 \partial_\nu \phi_2 + i\partial_\mu \phi_2 \partial_\nu \phi_1 + \partial_\mu \phi_2 \partial_\nu \phi_2\right] - \frac{m^2}{2} \phi^* \phi$$

With inspection of the final line, one notices that the two terms with imaginary coefficients are identical and cancel leaving a Lagrangian with two scalar fields;

$$\frac{1}{2} g^{\mu\nu} \left[\partial_\mu \phi_1 \partial_\nu \phi_1 + \partial_\mu \phi_2 \partial_\nu \phi_2\right] - \frac{m^2}{2} \left[\phi_1^2 + \phi_2^2\right] \tag{3.4}$$

The Euler-Lagrange equations, in 4-vector notation, applied to this Lagrangian are,

$$\frac{\partial L}{\partial_\lambda \phi_1} - \partial_\lambda \frac{\partial L}{\partial(\partial_\lambda \phi_1)} = 0 \tag{3.5}$$

and

$$\frac{\partial L}{\partial_\lambda \phi_2} - \partial_\lambda \frac{\partial L}{\partial(\partial_\lambda \phi_2)} = 0 \tag{3.6}$$

These two Euler-Lagrange equations will give us relationships for the two real scalar fields individually. Direct computation shows us that the equations of motion ϕ_1 and ϕ_2 respectively are

$$m^2 \phi_2 + \partial_\lambda \left[\frac{1}{2} g^{\mu\nu} \delta^\lambda_\mu \partial_\nu \phi_2 + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi_2 \delta^\lambda_\nu\right] =$$

$$m^2 \phi_2 + \partial_\lambda \partial^\lambda \phi_2 = 0 \tag{3.7}$$

and similarly,

$$m^2\phi_1 + \partial_\lambda\partial^\lambda\phi_1 = 0 \ . \tag{3.8}$$

We immediately recognize that these are just the Klein-Gordon equations for ϕ_1 and ϕ_2 which describe the two spinless scalar fields. We can reobtain expressions for ϕ and ϕ^* with the relations, $\frac{1}{\sqrt{2}}\left[(3.2) + i(3.3)\right]$ and $\frac{1}{\sqrt{2}}\left[(3.2) - i(3.3)\right]$. Using these we obtain

$$\left[\partial_\lambda\partial^\lambda + m^2\right]\phi = 0 \quad \mathit{and} \quad \left[\partial_\lambda\partial^\lambda + m^2\right]\phi^* = 0$$

We could have alternatively just performed the derivatives in terms of ϕ and ϕ^* and arrived here without using the real parts but it is nice to see the de-construction into real components. What we have achieved here is to show that this complex scalar Lagrangian formulation gives equations of motion for two complex scalar fields. Note that these are fields and not particles. In fact, the Klein-Gordon equations, unlike the Schrodinger equation, allow a probability density to take on negative values. Furthermore, the Klein-Gordon equations allow for negative energy terms which seems problematic. The quantization of the scalar fields resolves these issues [2].

To lead us into the next section, let us now consider a *global* gauge transformation. The global gauge transformation has us multiply the scalar fields by an exponential with some phase written as,

$$\phi \rightarrow e^{-i\Lambda}\phi \quad \mathit{and} \quad \phi^* \rightarrow e^{i\Lambda}\phi^* \tag{3.9}$$

where $\Lambda\epsilon\mathbf{R}$. It is simple enough to show that the equations are unchanged by such a transformation. Direct application gives,

$$L = g^{\mu\nu}\partial_\mu\left[e^{-i\Lambda}\phi\right]\partial_\nu\left[e^{i\Lambda}\phi^*\right] - m^2 e^{-i\Lambda}e^{i\Lambda}\phi^*\phi \ .$$

Since Λ for a global gauge transformation is a scalar, the exponentials are unaffected by the derivatives and cancel. We immediately see that we are left with the original Lagrangian. This transformation is a unitary transformation of dimension one or $U(1)$ and is isomorphic to the rotation group $S0(2)$ [2]. We can see this parallel to the rotation group by writing the transformation of the fields in terms of its real expansion

$$\phi' = \phi_1' + i\phi_2' \to e^{-i\Lambda}\phi = e^{-i\Lambda}\left[\phi_1 + i\phi_2\right]$$

$$\phi'^{*} = \phi_1' - i\phi_2' \to e^{i\Lambda}\phi = e^{i\Lambda}\left[\phi_1 - i\phi_2\right]$$

or

$$\phi_1' + i\phi_2' \to (\cos\Lambda - i\sin\Lambda)\left[\phi_1 + i\phi_2\right] \tag{3.10}$$

$$\phi_1' - i\phi_2' \to (\cos\Lambda + i\sin\Lambda)\left[\phi_1 - i\phi_2\right] \ . \tag{3.11}$$

The addition of (3.10) and (3.11) gives

$$\phi_1' = \cos\Lambda\phi_1 + \sin\Lambda\phi_2$$

while the subtraction of (3.10) and (3.11) gives

$$\phi_2' = -\sin\Lambda\phi_1 + \cos\Lambda\phi_2 \ .$$

Writing these relationships in vector and matrix notation gives,

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\Lambda & \sin\Lambda \\ -\sin\Lambda & \cos\Lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{3.12}$$

which is essentially the rotation group $S0(2)$. Arbitrary constant valued Λ allows the matrix in (3.12) to act as any rotation in the space defined by the two scalar fields.

The global gauge transformation above is not sufficient. While we see that a complex scalar Lagrangian is invariant under a global gauge transformation, we would like to consider gauge parameters that are not constant in space-time. We want to concern ourselves now with a gauge transformation in which Λ is explicitly defined as a function of position in space-time. In the next section, since our phase term will have space-time components, we will write it as $\Lambda(x^\mu)$.

The Local Gauge Transformation

When we considered the global gauge transformation we were restricted to performing the rotation in ϕ-space at all points in space at the same time. We want to consider a transformation that is space-time dependent. Restrictions from requiring local gauge symmetry has, as is seen in Yang-Mills theory, led to fundamental descriptions of particle interactions. Such things as gauge bosons and their interactions are described by local gauge symmetry. We must then consider a gauge transformation with a space-time dependent phase. As we will see, this spoils the symmetry we found in the global gauge transformation case. For the Lagrangian to remain invariant we ultimately redefine our derivative and introduce a vector potential. To start, consider the transformation,

$$\phi \to e^{-i\Lambda(x^\mu)}\phi \qquad \phi^* \to e^{i\Lambda(x^\mu)}\phi^*$$

and its derivatives

$$\partial_\mu \phi \to \partial_\mu \left[e^{-i\Lambda}\phi\right] = -ie^{-i\Lambda}\phi\partial_\mu\Lambda + e^{-i\Lambda}\partial_\mu\phi$$

$$\partial_\nu \phi^* \to ie^{i\Lambda}\phi^*\partial_\nu\Lambda + e^{i\Lambda}\partial_\nu\phi^*$$

Applying this transformation to the Lagrangian,

$$L \to g^{\mu\nu}\left[-ie^{-i\Lambda}\phi\partial_\mu\Lambda + e^{-i\Lambda}\partial_\mu\phi\right]\left[ie^{i\Lambda}\phi^*\partial_\nu\Lambda + e^{i\Lambda}\partial_\nu\phi^*\right] - m^2\phi\phi^* \tag{3.13}$$

where as in the global case, the mass term has no change since the exponentials simply cancel. If we then multiply out the first term in (3.13) and ignore for now the pieces that were in the original Lagrangian, $(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi^*)$, we are left with only the change in the Lagrangian. Labelling this difference, δL, we find that

$$\delta L = g^{\mu\nu}\left[\phi\phi^*\partial_\mu\Lambda\partial_\nu\Lambda - i\phi\partial_\nu\phi^*\partial_\mu\Lambda + i\phi^*\partial_\mu\phi\partial_\nu\Lambda\right] =$$

$$\phi\phi^*\partial_\mu\Lambda\partial^\mu\Lambda + i\left(\phi^*\partial_\mu\phi\partial^\mu\Lambda - \phi\partial^\mu\phi^*\partial_\mu\Lambda\right) \ . \tag{3.14}$$

Looking at the term in (3.14) with the imaginary coefficient, we can raise and lower the indices over the second term leaving the Noether Current, $J^\mu = i\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right)$. Specifically, we can write,

$$\delta L = \phi\phi^*\partial_\mu\Lambda\partial^\mu\Lambda + i\partial_\mu\Lambda\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) = \phi\phi^*\partial_\mu\Lambda\partial^\mu\Lambda + \left(\partial^\mu\Lambda\right)J^\mu \ . \tag{3.15}$$

Due to this extra term, our Lagrangian is not invariant under our global transformation of the fields. We are forced to introduce a new 4-vector term coupled to the 4-current to restore invariance. We also require that our new 4-vector term, A_μ, transforms as $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\Lambda$.[2] We will first try adding a Lagrangian term $L_1 = -eJ^\mu A_\mu$, where e is the coupling strength. It will also become the charge of the scalar field, ϕ. Under the transformation of both the scalar fields and the 4-vector, this term becomes

$$L_1 \to -ie\left[e^{i\Lambda}\phi^*\left(-ie^{-i\Lambda}\phi\partial^\mu\Lambda + e^{-i\Lambda}\partial^\mu\phi\right) - e^{-i\Lambda}\phi\left(ie^{i\Lambda}\phi^*\partial^\mu\Lambda + e^{i\Lambda}\partial^\mu\phi^*\right)\right]\left[A_\mu + \frac{1}{e}\partial_\mu\Lambda\right] =$$

$$-ie\left[-i\phi^*\phi\partial^\mu\Lambda + \phi^*\partial^\mu\phi - i\phi\phi^*\partial^\mu\Lambda - \phi\partial^\mu\phi^*\right]\left[A_\mu + \frac{1}{e}\partial_\mu\Lambda\right] \ .$$

Simplifying this, we have

$$L_1 \to -eJ^\mu A_\mu - J^\mu\partial_\mu\Lambda - 2e\phi^*\phi\partial^\mu\Lambda A_\mu - 2\phi^*\phi\partial^\mu\Lambda\partial_\mu\Lambda \ .$$

Once again if we ignore the terms originally in L_1, we have the leftovers δL_1 as,

$$\delta L_1 = -J^\mu \partial_\mu \Lambda - 2e\phi^* \phi \partial^\mu \Lambda A_\mu - 2\phi^* \phi \partial^\mu \Lambda \partial_\mu \Lambda \ . \tag{3.16}$$

There is now some cancellation between (3.15) and (3.16), but we are still left with extra terms. We must consider a third additional term to our Lagrangian, squared in the scalar field as well as the 4-vector term. This new term is $L_2 = e^2 A_\mu A^\mu \phi\phi^*$ and transforms as,

$$L_2 \to e^2 \left[A_\mu + \frac{1}{e}\partial_\mu \Lambda\right]\left[A^\mu + \frac{1}{e}\partial^\mu \Lambda\right]\phi\phi^*$$

(where I have left out the exponentials from the transformed scalar field since they trivially cancel). Multiplied out we get,

$$L_2 \to \left[e^2 A_\mu A^\mu + eA_\mu \partial^\mu \Lambda + e\partial_\mu \Lambda A^\mu + \partial_\mu \Lambda \partial^\mu \Lambda\right]\phi\phi^*$$

Notice that the two terms with a vector potential and its derivative are equivalent upon raising and lowering of indices. The leftover terms from the transformation can be written then as,

$$\delta L_2 = 2eA_\mu \partial^\mu \Lambda \phi\phi^* + \phi\phi^* \partial_\mu \Lambda \partial^\mu \Lambda \ . \tag{3.17}$$

We now have a complete cancellation of leftover terms. The addition of (3.15),(3.16), and (3.17) gives,

$$\delta L + \delta L_2 + \delta L_2 =$$

$$\phi\phi^* \partial_\mu \Lambda \partial^\mu \Lambda + (\partial^\mu \Lambda) J^\mu - J^\mu \partial_\mu \Lambda - 2e\phi^* \phi \partial^\mu \Lambda A_\mu -$$

$$2\phi^* \phi \partial^\mu \Lambda \partial_\mu \Lambda + 2eA_\mu \partial^\mu \Lambda \phi\phi^* + \phi\phi^* \partial_\mu \Lambda \partial^\mu \Lambda = 0$$

If one now adds $L_{tot} = L + L_1 + L_2$, the new total Lagrangian reads,

$$L_{tot} = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi - eJ^\mu A_\mu + e^2 A_\mu A^\mu \phi \phi^* \ . \tag{3.18}$$

Once again we have a Lagrangian that is invariant under gauge transformation. Moreover, this Lagrangian is invariant under a local gauge transformation. To restore invariance we had to introduce a vector field potential term coupled to the current as well as a Lagrangian term that was squared in the coupling constant, vector field and scalar field terms. While invariance has been re-established, this is not the end of the story. Since we have introduced a vector field, we have to assume it will contribute its own derivative terms to the Lagrangian [2]. The derivative terms we add will also have to be invariant under the local gauge transformation otherwise we will have gained nothing.

Local Gauge Invariance Implicitly Introduces Electromagnetic Fields

We have formulated a total Lagrangian that is invariant under local gauge transformations with primarily the introduction of a 4-vector potential term. Presumably, this term will also contribute its derivatives to the Lagrangian and the Euler-Lagrange equations. We know of such a derivative term from relativistic electrodynamics. The Field Strength tensor, $F_{\mu\nu}$ contains the derivatives of a vector field and futhermore, this term is invariant under the transformation of the vector field due to the local gauge transformation. Direct computation shows,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu$$

$$F_{\mu\nu} \to \partial_\mu \left[A_\nu + \frac{1}{e} \partial_\nu \Lambda \right] - \partial_\nu \left[A_\mu + \frac{1}{e} \partial_\mu \Lambda \right] =$$

$$\partial_\mu A_\nu + \frac{1}{e} \partial_\mu \partial_\nu \Lambda - \partial_\nu A_\mu - \frac{1}{e} \partial_\nu \partial_\mu \Lambda$$

where reordering the derivatives, we have a cancellation of the new terms and are thus left with the same $F_{\mu\nu}$. The standard Lagrangian term for electromagnetic fields is $L_3 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$. If $F_{\mu\nu}$ is invariant, so will be L_3 and we add this to our full Lagrangian,

$$L = \partial_\mu \phi \partial^\mu \phi^* - m^2 \phi^* \phi - eJ^\mu A_\mu + e^2 A_\mu A^\mu \phi \phi^* - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \ . \tag{3.19}$$

This Lagrangian can be made more succinct if we describe a covariant derivative, $D_\mu \phi$. This derivative is analogous to the covariant derivative in relativity, $\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho$, which uses the Christoffel symbols to describe the derivatives of the basis vectors through a transport over a non-Euclidean surface. As a parallel, our covariant derivative will have additional terms which can be viewed as a consequence of transport through the vector field A_μ that has a local gauge transformation. Thus we have $D_\mu \phi = (\partial_\mu + ieA_\mu)\, \phi$ and the conjugate, $(D_\mu \phi)^* = (\partial_\mu - ieA_\mu)\, \phi^*$. We do not have to do any work to retrieve these terms from the Lagrangian as they are already imbedded. Reordering the Lagrangian in (3.19) and looking strictly at the first two terms,

$$L = \partial_\mu \phi \partial^\mu \phi^* + e^2 A_\mu A^\mu \phi \phi^* - m^2 \phi^* \phi - eJ^\mu A_\mu - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

we have exactly,

$$L = (D_\mu \phi)\, (D^\mu \phi)^* - m^2 \phi^* \phi - eJ^\mu A_\mu - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{3.20}$$

We can further show that these derivatives are in fact the covariant derivative by transformation,

$$D_\mu \phi \rightarrow \left[\partial_\mu + ie\left(A_\mu + \frac{1}{e}\partial_\mu \Lambda\right)\right] e^{-i\Lambda}\phi =$$

$$-ie^{-i\Lambda}\phi \partial_\mu \Lambda + e^{-i\Lambda}\left(\partial_\mu \phi + ieA_\mu \phi\right) + ie^{-i\Lambda}\phi \partial_\mu \Lambda = e^{-i\Lambda}\left(\partial_\mu + ieA_\mu\right)\phi$$

and so

$$D_\mu\phi \to e^{-i\Lambda} D_\mu\phi \ .$$

Similar computation gives

$$(D_\mu\phi)^* \to e^{i\Lambda} D_\mu\phi^* \ .$$

We see that the derivative does transform covariantly under the local gauge transformation. Furthermore, we can now associate the scalar fields, ϕ and ϕ^* with the charges e and $-e$ respectively.

So far we have only introduced a 4-vector potential and the field strength tensor terms, but we have not actually shown how this implies the electromagnetic field per se. Let us then show directly that the Maxwell equations arise from this Lagrangian by taking the partials with respect to A_μ in the Euler-Lagrange equations,

$$\frac{\partial L}{\partial A_\mu} - \partial_\eta \left[\frac{\partial L}{\partial\left(\partial_\eta A_\mu\right)}\right] = 0 \ .$$

Let us use metric terms to get the indices the way we want,

$$\frac{\partial}{\partial A_\mu}\left[g^{\mu\gamma} e^2 A_\mu A_\gamma \phi\phi^* - eJ^\mu A_\mu\right] - \partial_\eta \left[\frac{\partial}{\partial\left(\partial_\eta A_\mu\right)}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right)\right] = 0 \ .$$

Application of Euler-Lagrange gives a stand alone current term and the partials on the field strength tensor,

$$\left(e^2 g^{\mu\gamma} A_\gamma + e^2 g^{\mu\gamma} A_\mu \delta^\mu_\gamma\right)\phi\phi^* - eJ^\mu - \partial_\eta \left[\frac{\partial}{\partial\left(\partial_\eta A_\sigma\right)}\left(-\frac{1}{4}g^{\mu\gamma}g^{\nu\rho}F_{\gamma\rho}F_{\mu\nu}\right)\right] = 0 \ .$$

Combining the first two terms and performing the derivatives that are left over gives,

$$2e^2 A^\mu\phi\phi^* - eJ^\mu + \frac{1}{4}g^{\mu\gamma}g^{\nu\rho}\partial_\eta \left[\frac{\partial}{\partial\left(\partial_\eta A_\sigma\right)}\left(\partial_\gamma A_\rho - \partial_\rho A_\gamma\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right] = 0 \ .$$

Reorganizing,

$$2e^2 A^\mu \phi\phi^* - eJ^\mu +$$

$$\frac{1}{4}g^{\mu\gamma}g^{\nu\rho}\partial_\eta\left[\left(\delta^\eta_\gamma\delta^\sigma_\rho - \delta^\eta_\rho\delta^\sigma_\gamma\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)+\left(\partial_\gamma A_\rho - \partial_\rho A_\gamma\right)\left(\delta^\eta_\mu\delta^\sigma_\nu - \delta^\eta_\nu\delta^\sigma_\mu\right)\right] = 0$$

and multiplying metric terms through the deltas what is left is,

$$2e^2A^\mu\phi\phi^* - eJ^\mu +$$

$$\frac{1}{4}\partial_\eta\left[\left(g^{\mu\eta}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\eta}\right)\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)+\left(\partial_\gamma A_\rho - \partial_\rho A_\gamma\right)\left(g^{\eta\gamma}g^{\sigma\rho} - g^{\sigma\gamma}g^{\eta\rho}\right)\right] = 0\ .$$

Or, written more efficiently,

$$2e^2A^\mu\phi\phi^* - eJ^\mu + \frac{1}{4}\partial_\eta\left[\partial^\eta A^\sigma - \partial^\sigma A^\eta - \partial^\sigma A^\eta + \partial^\eta A^\sigma + \partial^\eta A^\sigma - \partial^\sigma A^\eta - \partial^\sigma A^\eta + \partial^\eta A^\sigma\right] = 0\ .$$

Recombining the identical field strength tensor terms and factoring out a minus sign gives,

$$2e^2A^\mu\phi\phi^* - eJ^\mu - \partial_\eta\left[\partial^\sigma A^\eta - \partial^\eta A^\sigma\right] = 0$$

or finally,

$$2e^2A^\mu\phi\phi^* - eJ^\mu - \partial_\eta F^{\sigma\eta} = 0\ .$$

What is left is the derivative of the field strength tensor equal to the current plus an extra term in $2e^2A^\mu\phi\phi^*$. But if we remember, the current J^μ was not invariant under our local gauge transformation. Remembering that the current was defined as,

$$\partial_\eta F^{\sigma\eta} = 2e^2A^\mu\phi\phi^* - ie\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) \tag{3.21}$$

we can rewrite the two terms on the right hand side using the covariant derivative, $D_\mu\phi = \left(\partial_\mu + ieA_\mu\right)\phi$, as

$$\partial_\eta F^{\sigma\eta} = -ie\left(\phi^*D^\mu\phi - \phi D^\mu\phi^*\right)\ .$$

Rewriting this term as, $\mathcal{J}^\mu = i\left(\phi^*D^\mu\phi - \phi D^\mu\phi^*\right)$, as our conserved covariant current [2] shows its explicit invariance under the local gauge transformation. We can then write,

$$\partial_\eta F^{\sigma\eta} = -e\mathcal{J}^\mu$$

and note that the anti-symmetric property of $F^{\mu\nu}$ gives $\partial_\mu \mathcal{J}^\mu = 0$. This derivative on the covariant current gives Maxwell's equations.

CHAPTER 4: U(1)xU(1) GAUGE THEORY

Introducing a Second Vector Potential

We would now like to consider the introduction of a new 4-vector potential, $C^\mu = (\phi_m, \overrightarrow{C})$, with the intention of describing magnetic charge. First consider Maxwell's equations in the presence of a magnetic and electric charge [7],

$$\nabla \cdot \overrightarrow{E} = \rho_e \qquad \nabla \times \overrightarrow{B} = \frac{\partial \overrightarrow{E}}{\partial t} + \frac{1}{c}\overrightarrow{J}_e \tag{4.1}$$

$$\nabla \cdot \overrightarrow{B} = \rho_m \qquad -\nabla \times \overrightarrow{E} = \frac{\partial \overrightarrow{B}}{\partial t} + \frac{1}{c}\overrightarrow{J}_m \tag{4.2}$$

Consider now the electric and magnetic fields as described by the standard 4-vector potential, $A^\mu = (\phi_e, \overrightarrow{A})$, and the new 4-vector potential C^μ,

$$\overrightarrow{E} = -\nabla\phi_e - \frac{1}{c}\frac{\partial \overrightarrow{A}}{\partial t} - \nabla \times \overrightarrow{C} \tag{4.3}$$

$$\overrightarrow{B} = -\nabla\phi_m - \frac{1}{c}\frac{\partial \overrightarrow{C}}{\partial t} - \nabla \times \overrightarrow{A} \ . \tag{4.4}$$

If these 4-vector potentials are chosen to satisfy the Lorentz gauge condition then one can write,

$$\partial_\mu A^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \cdot \left(\phi_e, \overrightarrow{A}\right) = \frac{1}{c}\frac{\partial \phi_e}{\partial t} + \nabla \cdot \overrightarrow{A} = 0 \tag{4.5}$$

$$\partial_\mu C^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right) \cdot \left(\phi_m, \overrightarrow{C}\right) = \frac{1}{c}\frac{\partial \phi_m}{\partial t} + \nabla \cdot \overrightarrow{C} = 0 \ . \tag{4.6}$$

If these definitions of the fields are used in conjunction with Maxwell's equations with magnetic charge, we will obtain what is typically referred to as the inhomogeneous Maxwell equations. The difference here is, instead of having an equation relating a scalar potential to a charge density and an equation relating the vector potential to a current density, we will have two sets of these equations: one for each scalar potential and one for each vector potential. Let us begin with electric charge density. Direct computation gives,

$$\rho_e = \nabla \cdot \overrightarrow{E} = \nabla \cdot \left(-\nabla\phi_e - \frac{1}{c}\frac{\partial \overrightarrow{A}}{\partial t} - \nabla \times \overrightarrow{C} \right) =$$

$$-\nabla^2\phi_e - \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \overrightarrow{A} - \nabla \cdot \left(\nabla \times \overrightarrow{C} \right) \ .$$

If we assume the vector potential $\overrightarrow{C}$ is well behaved, then the last term is the divergence of a curl which is zero. This leaves,

$$\rho_e = -\nabla^2\phi_e - \frac{1}{c}\frac{\partial}{\partial t}\nabla \cdot \overrightarrow{A} \ .$$

If we use the Lorentz gauge condition for the 4-vector A^μ (4.5), and substitute directly into the equation for the electric charge density, this gives

$$\rho_e = -\nabla^2\phi_e + \frac{1}{c^2}\frac{\partial^2\phi_e}{\partial t^2} \ . \tag{4.7}$$

This can now be understood as one of the original inhomogeneous Maxwell equations relating the electric charge density and the scalar field associated to the electric charge. In a similar fashion we can compute an equivalent equation that related the magnetic charge density to the scalar field associated to the magnetic charge. This does not appear in the original Maxwell equations. As before, beginning with the magnetic charge density,

$$\rho_m = \nabla \cdot \overrightarrow{B} = \nabla \cdot \left(-\nabla\phi_m - \frac{1}{c}\frac{\partial \overrightarrow{C}}{\partial t} - \nabla \times \overrightarrow{A} \right) =$$

$$-\nabla^2\phi_m - \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\overrightarrow{C} + \nabla\cdot\left(\nabla\times\overrightarrow{A}\right) \ .$$

Once more, if we assume the vector potential $\overrightarrow{A}$ is well behaved then we can throw away the divergence of a curl which gives,

$$\rho_m = -\nabla^2\phi_m - \frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\overrightarrow{C} \ .$$

Using the Lorentz gauge condition imposed on the vector potential C^μ (4.6), this becomes

$$\rho_m = -\nabla^2\phi_m + \frac{1}{c^2}\frac{\partial^2\phi_m}{\partial t^2} \ . \tag{4.8}$$

What we have constructed so far are the scalar potential inhomogeneous Maxwell's equations. We have one for electric charge as is standard, but now we also have one describing a magnetic charge. Now what is needed are the vector potential formulations. Starting with the second equation in (4.1),

$$\nabla\times\overrightarrow{B} = \frac{1}{c}\left(\frac{\partial\overrightarrow{E}}{\partial t} + \overrightarrow{J}_e\right)$$

substitute in the new electric field definition (4.3),

$$\nabla\times\overrightarrow{B} = \nabla\times\left(-\nabla\phi_m - \frac{1}{c}\frac{\partial\overrightarrow{C}}{\partial t} - \nabla\times\overrightarrow{A}\right) =$$

$$-\nabla\times(\nabla\phi_m) - \frac{1}{c}\frac{\partial}{\partial t}\nabla\times\overrightarrow{C} + \nabla\times\left(\nabla\times\overrightarrow{A}\right) \ .$$

If the vector identities, $[\nabla\times(\nabla\phi)=0]$ and $[\nabla\times(\nabla\times\overrightarrow{v}) = \nabla(\nabla\cdot\overrightarrow{v}) - \nabla^2\overrightarrow{v}]$, are used then this can be written as,

$$\nabla\times\overrightarrow{B} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\times\overrightarrow{C} + \nabla\left(\nabla\cdot\overrightarrow{A}\right) - \nabla^2\overrightarrow{A} \ . \tag{4.9}$$

Also, from the second equation (4.2), and substitution for E with the new definition (4.3), the following relation can be written,

$$\nabla \times \overrightarrow{B} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi_e - \frac{1}{c}\frac{\partial \overrightarrow{A}}{\partial t} - \nabla \times \overrightarrow{C}\right) + \frac{1}{c}\overrightarrow{J}_e \ . \tag{4.10}$$

Equating (4.9) and (4.10) and immediately cancelling the repetitive time derivative of the curl of the 4-vector $\overrightarrow{C}$ term gives,

$$\nabla\left(\nabla \cdot \overrightarrow{A}\right) - \nabla^2\overrightarrow{A} = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\phi_e - \frac{1}{c^2}\frac{\partial^2 \overrightarrow{A}}{\partial t^2} + \frac{1}{c}\overrightarrow{J}_e \ .$$

Rewriting this as,

$$\nabla^2\overrightarrow{A} - \frac{1}{c^2}\frac{\partial^2 \overrightarrow{A}}{\partial t^2} + \frac{1}{c}\overrightarrow{J}_e = \nabla\left[\nabla \cdot \overrightarrow{A} + \frac{1}{c}\frac{\partial}{\partial t}\phi_e\right]$$

one immediately notices that the right hand side is simply the Lorentz gauge condition, $\partial_\mu A^\mu = 0$. Throwing this term away leaves,

$$\nabla^2\overrightarrow{A} - \frac{1}{c^2}\frac{\partial^2 \overrightarrow{A}}{\partial t^2} = -\frac{1}{c}\overrightarrow{J}_e \ . \tag{4.11}$$

This is the standard inhomogeneous Maxwell equation relating the electric charge current and the vector potential. Again, we have introduced a second vector potential and we can expect a similar relationship relating the new vector potential and the corresponding magnetic charge current. To obtain the final equation, we repeat the same procedure but instead begin with the second equation in (4.2). That is, starting with,

$$-\nabla \times \overrightarrow{E} = -\nabla \times \left(-\nabla\phi_e - \frac{1}{c}\frac{\partial \overrightarrow{A}}{\partial t} - \nabla \times \overrightarrow{C}\right)$$

and using vector identities we have,

$$-\nabla \times \overrightarrow{E} = \frac{1}{c}\frac{\partial}{\partial t}\nabla \times \overrightarrow{A} + \nabla \times \left(\nabla \cdot \overrightarrow{C}\right) - \nabla^2\overrightarrow{C} \ .$$

Computation of the left hand side of this equation gives,

$$-\nabla \times \overrightarrow{E} = \frac{1}{c}\frac{\partial}{\partial t}\left(-\nabla\phi_m - \frac{1}{c}\frac{\partial \overrightarrow{C}}{\partial t} + \nabla \times \overrightarrow{A}\right) + \frac{1}{c}\overrightarrow{J}_m \ .$$

Once more, equating the two sides and cancelling the time derivatives of the curl of $\overrightarrow{A}$ gives,

$$-\frac{1}{c}\frac{\partial}{\partial t}\nabla\phi_m - \frac{1}{c^2}\frac{\partial^2 \overrightarrow{C}}{\partial t^2} + \frac{1}{c}\overrightarrow{J}_m = \nabla\left(\nabla \cdot \overrightarrow{C}\right) - \nabla^2\overrightarrow{C}$$

and finally using the Lorentz gauge, $\partial_\mu C^\mu$, we obtain

$$\frac{1}{c}\frac{\partial}{\partial t}\nabla\phi_m + \frac{1}{c^2}\frac{\partial^2 \overrightarrow{C}}{\partial t^2} = \frac{1}{c}\overrightarrow{J}_m \ . \tag{4.12}$$

Summarizing, what has been constructed are the inhomogeneous Maxwell equations describing the charge densities and current densities for magnetic and electric charge. We see that the introduction of a new vector potential and the supplement of magnetic charge in to Maxwell's original equations allows for a new set of inhomogeneous Maxwell equations with magnetic charge and magnetic current density that was not previously present. Writing (4.7), (4.8), (4.11) and (4.12) together in succinct fashion

$$\nabla^2\phi_e - \frac{1}{c^2}\frac{\partial^2\phi_e}{\partial t^2} = -\rho_e \ , \qquad \nabla^2\overrightarrow{A} - \frac{1}{c^2}\frac{\partial^2\overrightarrow{A}}{\partial t^2} = -\frac{1}{c}\overrightarrow{J}_e$$

$$\nabla^2\phi_m - \frac{1}{c^2}\frac{\partial^2\phi_m}{\partial t^2} = -\rho_m \ , \qquad \frac{1}{c}\frac{\partial}{\partial t}\nabla\phi_m + \frac{1}{c^2}\frac{\partial^2\overrightarrow{C}}{\partial t^2} = \frac{1}{c}\overrightarrow{J}_m$$

So far the analysis has been done primarily in 3-vector notation. We now convert these equations to 4-vector notation to coincide with the notation that will be used later in the paper. To write these equations in 4-vector notation, we begin by describing two field strength tensors;

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{4.13}$$

describing the Lagrangian contributions for the derivatives of the vector potential $\overrightarrow{A}$ and

$$G^{\mu\nu} = \partial^\mu C^\nu - \partial^\nu C^\mu \tag{4.14}$$

describing the Lagrangian contributions for the derivatives of the vector potential $\overrightarrow{C}$. Notice that standard Maxwell's equations typically only have the one field strength tensor, (4.13), but since we have added a second vector potential we also need to describe its field strength tensor. An application of 4-derivatives on the field strength tensors allows us to reobtain the new inhomogeneous Maxwell equations we just derived. Computation gives,

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu$$

reordering the derivatives in the second term gives $\partial^\nu \partial_\mu A^\mu$, which is zero by the Lorentz Gauge. We are left with

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu =$$

$$\partial_\mu \partial^\mu A^\nu = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) A^\nu = \frac{1}{c} J_e^\nu$$

Similarly,

$$\partial_\mu G^{\mu\nu} = \partial_\mu \partial^\mu C^\nu - \partial_\mu \partial^\nu C^\mu$$

again reordering the derivatives in the second term gives $\partial^\nu \partial_\mu C^\mu$, which is zero by Lorentz Gauge. We have,

$$\partial_\mu G^{\mu\nu} = \partial_\mu \partial^\mu C^\nu =$$

$$\partial_\mu \partial^\mu C^\nu = \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) C^\nu = \frac{1}{c} J_m^\nu$$

Here, the notation used for the 4-currents respective to their charges are $J_e^\mu = \left(\rho_e, \overrightarrow{J}_e\right)$ and $J_m^\mu = \left(\rho_m, \overrightarrow{J}_m\right)$. Inspection shows immediately that these are exactly descriptions of the inhomogeneous Maxwell equations. The use of 4-vector notation here allows us to write the four inhomogeneous Maxwell equations more efficiently with just two equations. Now that we have built up the Maxwell equations using a second vector potential to describe magnetic charge, we would like to turn our attention to the Lagrangian. We are going to build a complex scalar field Lagrangian with the contributions from the vector potentials as we have done in the previous chapter. Since we now have two vector potentials and two field strength tensors, we expect both terms to appear in the Lagrangian.

The Scalar Lagrangian Containing Magnetic Charge

Consider a complex scalar field Lagrangian, as we have in previous sections, but now with an additional contribution from the derivatives of the second vector potential in terms of the second field strength tensor. The Lagrangian is,

$$L_S = D_\mu \Phi^* D^\mu \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - V(\Phi^2) \tag{4.15}$$

where the full gauge-covariant derivative here is defined as, $D^\mu = \partial^\mu - iq_e A^\mu - iq_m C^\mu$ and the potential term is $V(\Phi^2) = m^2(\Phi^*\Phi) + \lambda(\Phi^*\Phi)^2$. This Lagrangian has been constructed to remain invariant under local gauge transformations as shown in the previous chapter. Notice the potential has a λ term which is a self interaction. This is typically referred to as as Lambda-Phi fourth term and is crucial to the symmetry breaking we will perform in a moment. If we write out the covariant derivatives explicitly, we have,

$$L_S = \left(\partial_\mu + iq_e A_\mu + iq_m C_\mu\right) \Phi^* \left(\partial^\mu - iq_e A^\mu - iq_m C^\mu\right) \Phi -$$

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - V(\Phi^2) =$$

$$\partial_\mu\Phi^*\partial^\mu\Phi - iq_e\partial_\mu\Phi^* A^\mu\Phi - iq_m\partial_\mu\Phi^* C^\mu\Phi + iq_e A_\mu\Phi^*\partial^\mu\Phi + q_e^2 A_\mu A^\mu\Phi^*\Phi +$$

$$q_e q_m A_\mu C^\mu\Phi^*\Phi + iq_m C_\mu\Phi^*\partial^\mu\Phi + q_e q_m C_\mu A^\mu\Phi^*\Phi + q_m^2 C_\mu C^\mu\Phi^*\Phi -$$

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - V(\Phi^2)$$

One notices the standard kinetic energy term for the scalar fields. What appears to be a nasty mess of terms can be made efficient if we define the currents as,

$$J_e^\mu = iq_e\left[\Phi^*\left(\partial^\mu\Phi\right) - \Phi\left(\partial^\mu\Phi\right)^*\right] \tag{4.16}$$

$$J_m^\mu = iq_m\left[\Phi^*\left(\partial^\mu\Phi\right) - \Phi\left(\partial^\mu\Phi\right)^*\right]\ . \tag{4.17}$$

These currents are written following the formulation of the Noether current. We now have two currents, one describing electric charge current and one for magnetic charge current. The subscripts e and m as well as the associated charges differentiate the two. Using (4.16) and (4.17), we can simplify the Lagrangian as

$$L_S = \partial_\mu\Phi^*\partial^\mu\Phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + J_e^\mu A_\mu + J_m^\mu C_\mu - V(\Phi^2)$$

$$+q_e^2 A_\mu A^\mu\Phi^*\Phi + q_m^2 C_\mu C^\mu\Phi^*\Phi + 2q_e q_m A_\mu C^\mu\Phi^*\Phi$$

Now we notice some higher order interaction terms left over from this Lagrangian definition. These terms can be cleaned up by writing,

$$L_S = \partial_\mu\Phi^*\partial^\mu\Phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + J_e^\mu A_\mu + J_m^\mu C_\mu -$$

$$V(\Phi^2) + \left(q_e A_\mu + q_m C_\mu\right)\left(q_e A^\mu + q_m C^\mu\right)\Phi^*\Phi\ . \tag{4.18}$$

At this point, the Lagrangian really does not tell us much. What we would like to do is apply spontaneous symmetry breaking and give the scalar field a negative mass squared value. In doing so, the vacuum expectation value for the scalar field will drop below zero and we will have to re-parameterize to the new minimum. This same process was used by Jeffrey Goldstone to find the Goldstone boson and later in the Higgs' model as well as the Weinberg-Salam model for electroweak interactions. The requirement that the mass squared be negative seems unphysical since it implies imaginary mass. The "mass" at this point is really just considered a parameter and not a mass per se. So let us consider $m^2 < 0$, then the minimum of the potential $V(\Phi^2)$ is

$$\frac{\partial V}{\partial \phi} = m^2 + 2\lambda\Phi^*\Phi = 0 \ .$$

This leads to a minimum of

$$|\Phi| = \Phi^*\Phi = \left[\frac{-m^2}{2\lambda}\right]^{1/2} \equiv \frac{v}{\sqrt{2}} \ .$$

When the fields are quantized, Φ becomes an operator and the magnitude of Φ refers to the vacuum expectation value of the field, written as,

$$\langle\Phi\rangle = |\langle 0|\Phi|0\rangle|^2 = \left[\frac{-m^2}{2\lambda}\right]^{1/2} \equiv \frac{v}{\sqrt{2}} \ .$$

where $|0\rangle$ is the ground state of the wave equation. If the potential is now plotted against the two real scalar fields, one would see the standard "Mexican hat" shaped potential graph. The minimum value corresponds to a circle of points that are all related through rotation about the potential of the real scalar fields. That is to say, the ground state is degenerate and the symmetry of the Lagrangian is no longer shared by the ground state solution. To resolve this, we use our new minimum for the vacuum expectation value to parameterize our complex scalar field Φ so that the vacuum expectation value lies along the real component. This gives

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(v + \eta(x) + i\zeta(x)\right)$$

Consider now the real components of the complex scalar field to be small oscillations, then and exponential term $e^{i\zeta/v}$ can be expanded as

$$\frac{1}{\sqrt{2}}\left(v + \eta(x)\right)e^{i\zeta(x)/v} \approx \frac{1}{\sqrt{2}}\left(v + \eta(x)\right)\left(1 + i\frac{\zeta(x)}{v} - \frac{\zeta(x)^2}{2v^2}\right) =$$

$$\frac{1}{\sqrt{2}}\left(v + i\zeta(x) - \frac{\zeta(x)^2}{2v} + \eta(x) + i\eta(x)\frac{\zeta(x)}{v} - \eta(x)\frac{\zeta(x)^2}{2v^2}\right) \ .$$

Since we are considering small oscillation real scalar fields, we can ignore of order greater than two in the scalar fields which makes this statement approximately

$$\frac{1}{\sqrt{2}}\left(v + i\zeta(x) + \eta(x)\right)$$

We therefore have the approximation that the complex scalar field acts as,

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(v + \eta(x) + i\zeta(x)\right) \approx \frac{1}{\sqrt{2}}\left[v + \eta(x)\right]e^{i\zeta(x)/v} \ . \tag{4.19}$$

Writing the complex scalar field in this way has a nice feature in that we can apply a gauge transformation and lose the exponential term. Consider a gauge transformation, $e^{-i\zeta/v}$, to the unitary gauge. This gauge transformation will effectively eliminate the real $\zeta(x)$ component of our complex scalar field. The transformation is computed as follows

$$\Phi'(x) = e^{-i\zeta(x)/v}\Phi(x) \approx \frac{1}{\sqrt{2}}\left(v + \eta(x)\right) \tag{4.20}$$

$$B_\mu \equiv A'_\mu(x) = A_\mu(x) - \frac{1}{2q_e v}\partial_\mu\zeta(x) \tag{4.21}$$

$$E_\mu \equiv C'_\mu(x) = C_\mu - \frac{1}{2q_m v}\partial_\mu \zeta(x) \tag{4.22}$$

For notational simplicity, I have redefined the transformed vector potentials under new variables. When we write the gauge transformed Lagrangian later, instead of writing out the explicit transformation of the vector potentials, the new notation will be substituted. For now, to show the invariance of terms, I will continue to use the primed notation. Notice that this gauge transformation transforms the covariant derivative as it should,

$$D^\mu \Phi' = \left[\partial^\mu - iq_e\left(A^\mu(x) - \frac{1}{2q_e v}\partial^\mu\zeta(x)\right) - iq_m\left(C^\mu - \frac{1}{2q_m v}\partial^\mu\zeta(x)\right)\right]e^{-i\zeta(x)/v}\Phi(x) =$$

$$e^{-i\zeta(x)/v}\left[\partial^\mu\Phi(x) - iq_e A^\mu(x)\Phi(x) - iq_m C^\mu\Phi(x)\right] -$$

$$ie^{-i\zeta(x)/v}\left[\frac{\partial^\mu\zeta(x)}{v}\Phi(x) - \frac{1}{2v}\partial^\mu\zeta(x)\Phi(x) - \frac{1}{2v}\partial^\mu\zeta(x)\Phi(x)\right] .$$

The last two terms cancel and we are left with

$$D_\mu\Phi' = e^{-i\zeta(x)/v}D_\mu\Phi$$

In addition, the field strength tensor terms are invariant under this transformation,

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu\left[A_\nu(x) - \frac{1}{2q_e v}\partial_\nu\zeta(x)\right] - \partial_\nu\left[A_\mu(x) - \frac{1}{2q_e v}\partial_\mu\zeta(x)\right] =$$

$$\partial_\mu A_\nu(x) - \frac{1}{2q_e v}\partial_\mu\partial_\nu\zeta(x) - \partial_\nu A_\mu(x) + \frac{1}{2q_e v}\partial_\nu\partial_\mu\zeta(x) = F_{\mu\nu}$$

similarly,

$$G'_{\mu\nu} = \partial_\mu C'_\nu - \partial_\nu C'_\mu = \partial_\mu\left[C_\nu(x) - \frac{1}{2q_m v}\partial_\nu\zeta(x)\right] - \partial_\nu\left[C_\mu(x) - \frac{1}{2q_m v}\partial_\mu\zeta(x)\right] = G_{\mu\nu} .$$

The complex scalar Lagrangian (4.15), written with the potential terms,

$$L_S = D_\mu\Phi^* D^\mu\Phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - m^2(\Phi^*\Phi) - \lambda(\Phi^*\Phi)^2$$

has now been shown to contain only terms that are invariant under the gauge transformation. Expanding out the covariant derivatives using our new notation for the vector potentials gives

$$L_S = [\partial_\mu + iq_e B_\mu + iq_m E^\mu]\,\Phi^* \,[\partial^\mu - iq_e B^\mu - iq_m E^\mu]\,\Phi -$$

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - m^2(\Phi^*\Phi) - \lambda(\Phi^*\Phi)^2 \ .$$

Now using the approximation of the complex scalar field (4.20), we have

$$\frac{1}{2}\left[\partial_\mu + iq_e B_\mu + iq_m E^\mu\right](v+\eta(x))\left[\partial^\mu - iq_e B^\mu - iq_m D^\mu\right](v+\eta(x)) \tag{4.23}$$

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - V\left[\frac{1}{2}(v+\eta(x))^2\right]$$

Let us first deal with the potential term, $V(\Phi^2)$. Using the approximation (4.20), this term becomes

$$\frac{m^2}{2}(v+\eta(x))^2 + \frac{\lambda}{4}(v+\eta(x))^4 =$$

$$\frac{m^2}{2}v^2 + m^2 v\eta(x) + \frac{m^2}{2}\eta^2(x) + \frac{\lambda}{4}(v^4 + 4v^3\eta(x) + 6v^2\eta^2(x) + 4v\eta^3(x) + \eta^4(x))$$

Adding constants to a Lagrangian will not affect the equations of motion, therefore we can reduce this term by ignoring pure constants leaving,

$$m^2 v\eta(x) + \frac{1}{2}m^2\eta^2(x) + \lambda v^3\eta(x) + \frac{3}{2}\lambda v^2\eta^2(x) + \lambda v\eta^3(x) + \frac{1}{4}\lambda\eta^4(x)$$

Now if we recall that the vacuum expectation value was found to be $\left(\frac{-m^2}{2\lambda}\right)^{1/2} = \frac{v}{\sqrt{2}}$ or $\lambda = -\frac{m^2}{v^2}$, then we can write this as,

$$m^2 v\eta(x) + \frac{1}{2}m^2\eta^2(x) - \frac{m^2}{v^2}v^3\eta(x) - \frac{3}{2}\frac{m^2}{v^2}v^2\eta^2(x) + \lambda v\eta^3(x) + \frac{1}{4}\lambda\eta^4(x) =$$

$$-m^2\eta^2(x) + \lambda v\eta^3(x) + \frac{1}{4}\lambda\eta^4(x) \tag{4.24}$$

The overall contribution then, labeling the term L_v, is $L_v = m^2\eta^2(x) - \lambda v\eta^3(x) - \frac{1}{4}\lambda\eta^4(x)$. Now referring back the derivative terms in Lagrangian (4.15), we can write this explicitly as,

$$\frac{1}{2}\left[\partial_\mu\eta(x) + iq_eB_\mu\left(v+\eta(x)\right) + iq_mE^\mu\left(v+\eta(x)\right)\right]\times$$

$$\left[\partial^\mu\eta(x) - iq_eB^\mu\left(v+\eta(x)\right) - iq_mE^\mu\left(v+\eta(x)\right)\right]$$

(Note: the use of the multiplication symbol here is simply that and not a cross product. The equation is too long for one line.) Now, expanding out further we see a kinetic energy term, $\frac{1}{2}\partial_\mu\eta(x)\partial^\mu\eta(x)$ and the following

$$-q_eq_mB_\mu E^\mu(v+\eta(x))^2 - \frac{q_e^2}{2}B_\mu B^\mu(v+\eta(x))^2 - \frac{q_m^2}{2}E_\mu E^\mu(v+\eta(x))^2 =$$

$$-\frac{1}{2}\left(v^2 + 2v\eta(x) + \eta^2(x)\right)\left[2q_eq_mB_\mu E^\mu + q_e^2B_\mu B^\mu + q_m^2E_\mu E^\mu\right]$$

The cross terms we notice between the vector potentials are unsatisfactory since they complicate the process of finding the mass spectrum of the gauge bosons. We can write this term in the form of a mass-mixing matrix in order to "de-couple" the vector potentials. The process, as we will see, actually removes one vector potential from the Lagrangian with exception of the terms of the Field-Strength Tensor.

Let us first define a variable, K, that can have values $K = \{\frac{v^2}{2}, \frac{v}{2}\eta(x), \frac{1}{2}\eta^2(x)\}$ then we notice that this last term can be written in the form of matrix multiplication. More precisely, if we index K and sum only over that index, we have

$$K_n\left(B_\mu, E_\mu\right)\begin{pmatrix} q_e^2 & q_eq_m \\ q_eq_m & q_m^2 \end{pmatrix}\begin{pmatrix} B^\mu \\ E^\mu \end{pmatrix} =$$

$$\left(\frac{1}{2}v^2 + v\eta(x) + \frac{1}{2}\eta^2(x)\right)\left[2q_eq_mB_\mu E^\mu + q_e^2B_\mu B^\mu + q_m^2E_\mu E^\mu\right]$$

In this notation we can do a duality transformation and chose the phase angle appropriately so that we can diagonalize. It is worth pointing out that a similar diagonalization of the mass matrix using the properties of the duality transformation was performed by Glashow, Weinberg and Salam to find the mass spectrum of the electroweak gauge bosons in the Standard Model. The duality transformation we use is as follows,

$$B^\mu \to B^\mu \cos\theta + E^\mu \sin\theta$$

$$E^\mu \to -B^\mu \sin\theta + E^\mu \cos\theta$$

Notice that these are just rotations of the vector potentials about some angle. If we apply these transformations to our interaction matrix we obtain the following,

$$(B_\mu, E_\mu)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} q_e^2 & q_e q_m \\ q_e q_m & q_m^2 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} B^\mu \\ E^\mu \end{pmatrix}$$

the computed matrix terms for the charge matrix and the rotations are,

$$(1,1) = q_e^2\cos^2\theta - 2q_e q_m \sin\theta\cos\theta + q_m^2\sin^2\theta$$

$$(1,2) = \left(q_e^2 - q_m^2\right)\sin\theta\cos\theta + q_e q_m\left(\cos^2\theta - \sin^2\theta\right)$$

$$(2,1) = \left(q_e^2 - q_m^2\right)\sin\theta\cos\theta + q_e q_m\left(\cos^2\theta - \sin^2\theta\right)$$

$$(2,2) = q_e^2\sin^2\theta + 2q_e q_m \sin\theta\cos\theta + q_m^2\cos^2\theta$$

The notation used here (i, j) simply represents the row-column element in the matrix. The off diagonal terms in this matrix are the same, so to diagonalize this matrix we simply make a choice of angle that forces,

$$\left(q_e^2 - q_m^2\right)\sin\theta\cos\theta + q_e q_m\left(\cos^2\theta - \sin^2\theta\right) = 0$$

if we make the choice that $\cos\theta = \frac{q_m}{\sqrt{q_r^2+q_m^2}}$ and $\sin\theta = \frac{q_e}{\sqrt{q_e^2+q_m^2}}$ then,

$$\left(q_e^2 - q_m^2\right)\frac{q_e q_m}{q_e^2+q_m^2} + q_e q_m\left(\frac{q_m^2}{q_e^2+q_m^2} - \frac{q_e^2}{q_e^2+q_m^2}\right) = 0$$

One might think that the choices for $\cos\theta$ and $\sin\theta$ force a relationship between q_e and q_m since the angle of rotation of the duality transformation, θ is both $\cos^{-1}\left[\frac{q_m}{\sqrt{q_e^2+q_m^2}}\right]$ and $\sin^{-1}\left[\frac{q_e}{\sqrt{q_e^2+q_m^2}}\right]$. This is not the case. In fact, we could have only given the $\cos\theta$ value and the $\sin\theta$ value would have followed. Furthermore, with a little trigonometry one can show that the two conditions on θ do not force a condition between the electric and magnetic charges. Using the identity that $\sin^{-1}(z) = \cos^{-1}(-z) - \frac{1}{2}\pi$ we find,

$$\cos^{-1}\left[\frac{q_m}{sqrt q_e^2+q_m^2}\right] = \sin^{-1}\left[\frac{q_e}{\sqrt{q_e^2+q_m^2}}\right]$$

$$\cos^{-1}\left[\frac{q_m}{sqrt q_e^2+q_m^2}\right] = \cos^{-1}\left[-\frac{q_e}{\sqrt{q_e^2+q_m^2}}\right] - \frac{1}{2}\pi$$

or

$$\frac{q_m}{\sqrt{q_e^2+q_m^2}} = \cos\left[\cos^{-1}\left[-\frac{q_e}{\sqrt{q_e^2+q_m^2}}\right] - \frac{1}{2}\pi\right]$$

using the identity, $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$,

$$\frac{q_m}{\sqrt{q_e^2+q_m^2}} = -\frac{q_e}{\sqrt{q_e^2+q_m^2}}\cos\left(\frac{1}{2}\pi\right) - \sin\left[\cos^{-1}\left(-\frac{q_e}{\sqrt{q_e^2+q_m^2}}\right)\right]\sin\left(-\frac{1}{2}\pi\right)$$

finally using the identity, $\sin\left(\cos^{-1}(x)\right) = \sqrt{1-x^2}$, we obtain,

$$\frac{q_m}{\sqrt{q_e^2+q_m^2}} = \sqrt{1 - \frac{q_e^2}{q_e^2+q_m^2}}$$

and so they are equal as they should be and there is no condition relating q_e and q_m.

Returning to the max mixing matrix, we have set the off diagonal terms to zero by making conditions on the angle of rotation. Now we show what happens to the diagonal components.

$$\begin{pmatrix} q_e^2\cos^2\theta - 2q_eq_m\sin\theta\cos\theta + q_m^2\sin^2\theta & 0 \\ 0 & q_e^2\sin^2\theta + 2q_eq_m\sin\theta\cos\theta + q_m^2\cos^2\theta \end{pmatrix}$$

Making substitutions, the 1-1 component of the matrix becomes,

$$q_e^2\frac{q_m^2}{q_e^2+q_m^2} - 2q_eq_m\frac{q_eq_m}{q_e^2+q_m^2} + q_m^2\frac{q_e^2}{q_e^2+q_m^2} = 0$$

and the 2-2 component becomes,

$$q_e^2\frac{q_e^2}{q_e^2+q_m^2} + 2q_eq_m\frac{q_eq_m}{q_e^2+q_m^2} + q_m^2\frac{q_m^2}{q_e^2+q_m^2} = \frac{q_e^4+2q_e^2q_m^2+q_m^4}{q_e^2+q_m^2} =$$

$$\frac{(q_e^2+q_m^2)^2}{q_e^2+q_m^2} = q_e^2+q_m^2$$

so we finally see that our Duality transformation with appropriate choice for angle diagonalizes the interaction matrix leaving the following relationships,

$$K_n\left(B_\mu, E_\mu\right)\begin{pmatrix} 0 & 0 \\ 0 & q_e^2+q_m^2 \end{pmatrix}\begin{pmatrix} B^\mu \\ E^\mu \end{pmatrix} \tag{4.25}$$

Writing out these terms we can see the Lagrangian contribution,

$$\left(\frac{1}{2}v^2 + \frac{1}{2}v\eta(x)\frac{1}{2}\eta^2(x)\right)\left(q_e^2+q_m^2\right)E_\mu E^\mu$$

Now piecing together all of our transformed Lagrangian components and the field strength tensor terms we have,

$$L_S = \frac{1}{2}\partial_\mu\eta(x)\partial^\mu\eta(x) + \frac{1}{2}(2m^2)\eta^2(x) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} +$$

$$\left(\frac{1}{2}v^2 + \frac{1}{2}v\eta(x)\frac{1}{2}\eta^2(x)\right)\left(q_e^2+q_m^2\right)E_\mu E^\mu - \lambda v\eta^3(x) - \frac{1}{4}\lambda\eta^4(x)$$

The initial set of particles in the Lagrangian were two massless Gauge bosons and two scalar fields. Via the Higgs' mechanism these have become one massive scalar field, one massless gauge field and one massive gauge field. The coupling strength of the scalar field to the gauge boson is $g = (q_e^2 + q_m^2)^{1/2}$. That is to say the scalar field carries a magnetic charge of strength g.

Addition of a Second Scalar Potential

Let us begin with the Lagrangian considered in Scalar Lagrangian section (4.15). We have,

$$L_S = D_\mu \Phi^* D^\mu \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - V(\Phi^2)$$

which when written out became,

$$L_S = \partial_\mu \Phi^* \partial^\mu \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} + J_e^\mu A_\mu + J_m^\mu C_\mu - V(\Phi^2)$$

$$+q_e^2 A_\mu A^\mu \Phi^* \Phi + q_m^2 C_\mu C^\mu \Phi^* \Phi + 2 q_e q_m A_\mu C^\mu \Phi^* \Phi$$

We would now like to add a second scalar term. The hopes here are to reobtain the A_μ vector potential interaction terms that we lost via the process of the Higgs mechanism in previous sections. Let us consider then,

$$L_{S2} = D_\mu \Phi_1^* D^\mu \Phi_1 + D_\mu \Phi_2^* D^\mu \Phi_2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} - V(\Phi_1^2)$$

We do not add any new potential terms for the second scalar Lagrangian but it should be noted that the covariant derivatives operating on Φ_1 and Φ_2 are different. More precisely they are,

$$D^\mu \Phi_1 = (\partial^\mu - i q_e A^\mu - i q_m C^\mu)\, \Phi_1$$

$$D^\mu \Phi_2 = (\partial^\mu - i q_e' A^\mu - i q_m' C^\mu)\, \Phi_2$$

We will obtain the same expanded Lagrangian terms as before except we will have new terms for Φ_2 contribution,

$$\partial_\mu\Phi_1^*\partial^\mu\Phi_1 - iq_e\partial_\mu\Phi_1^*A^\mu\Phi_1 - iq_m\partial_\mu\Phi_1^*C^\mu\Phi_1 + iq_eA_\mu\Phi_1^*\partial^\mu\Phi_1 + q_e^2A_\mu A^\mu\Phi_1^*\Phi_1 +$$

$$q_eq_mA_\mu C^\mu\Phi_1^*\Phi_1 + iq_mC_\mu\Phi_1^*\partial^\mu\Phi_1 + q_eq_mC_\mu A^\mu\Phi_1^*\Phi_1 + q_m^2C_\mu C^\mu\Phi_1^*\Phi_1 -$$

$$\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - V(\Phi_1^2) +$$

$$\partial_\mu\Phi_2^*\partial^\mu\Phi_2 - iq_e'\partial_\mu\Phi_2^*A^\mu\Phi_2 - iq_m'\partial_\mu\Phi_2^*C^\mu\Phi_2 + iq_e'A_\mu\Phi_2^*\partial^\mu\Phi_2 + (q_e')^2A_\mu A^\mu\Phi_2^*\Phi_2 +$$

$$q_e'q_m'A_\mu C^\mu\Phi_2^*\Phi_2 + iq_m'C_\mu\Phi_2^*\partial^\mu\Phi_2 + q_e'q_m'C_\mu A^\mu\Phi_2^*\Phi_2 + (q_m')^2C_\mu C^\mu\Phi_2^*\Phi_2$$

Now if we consider current terms as

$$J_e^\mu = iq_e\left[\Phi_1^*\left(\partial^\mu\Phi_1\right) - \Phi_1\left(\partial^\mu\Phi_1\right)^*\right]$$

$$J_{e\prime}^\mu = iq_e'\left[\Phi_2^*\left(\partial^\mu\Phi_2\right) - \Phi_2\left(\partial^\mu\Phi_2\right)^*\right]$$

$$J_m^\mu = iq_m\left[\Phi_1^*\left(\partial^\mu\Phi_1\right) - \Phi_1\left(\partial^\mu\Phi_1\right)^*\right]$$

$$J_{m\prime}^\mu = iq_m'\left[\Phi_2^*\left(\partial^\mu\Phi_2\right) - \Phi_2\left(\partial^\mu\Phi_2\right)^*\right]$$

then we can simplify the Lagrangian,

$$\partial_\mu\Phi_1^*\partial^\mu\Phi_1 + \partial_\mu\Phi_2^*\partial^\mu\Phi_2 + J_e^\mu A_\mu + J_{e\prime}^\mu A_\mu + J_m^\mu C_\mu + J_{m\prime}^\mu C_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} -$$

$$V(\Phi_1^2) + q_e^2A_\mu A^\mu\Phi_1^*\Phi_1 + q_m^2C_\mu C^\mu\Phi_1^*\Phi_1 + 2q_eq_mA_\mu C^\mu\Phi_1^*\Phi_1 +$$

$$(q_e')^2A_\mu A^\mu\Phi_2^*\Phi_2 + (q_m')^2C_\mu C^\mu\Phi_2^*\Phi_2 + 2q_e'q_m'A_\mu C^\mu\Phi_2^*\Phi_2$$

or combining terms as we have done before,

$$\partial_\mu\Phi_1^*\partial^\mu\Phi_1 + \partial_\mu\Phi_2^*\partial^\mu\Phi_2 + J_e^\mu A_\mu + J_{e\prime}^\mu A_\mu + J_m^\mu C_\mu + J_{m\prime}^\mu C_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} -$$

$$V(\Phi_1^2) + \left(q_eA_\mu + q_mC_\mu\right)\left(q_eA^\mu + q_mC^\mu\right)\Phi_1^*\Phi_1 + \left(q_e'A_\mu + q_m'C_\mu\right)\left(q_e'A^\mu + q_m'C^\mu\right)\Phi_2^*\Phi_2$$

At this point we would like to consider the gauge transformation done in the previous section. That is

$$\Phi_1'(x) = e^{-i\zeta(x)/v}\Phi_1(x) = \frac{1}{\sqrt{2}}\left(v + \eta(x)\right)$$

$$A_\mu'(x) = A_\mu(x) - \frac{1}{2q_e v}\partial_\mu\zeta(x)$$

$$C_\mu'(x) = C_\mu - \frac{1}{2q_m v}\partial_\mu\zeta(x)$$

which in one recalls gave the covariant derivative transformation

$$D^\mu\Phi_1 \rightarrow e^{-i\zeta(x)/v}D^\mu\Phi_1$$

We also must require that the covariant derivative that operates on Φ_2 remain invariant under this same gauge transformation. The covariant derivative that operates on Φ_2 contains the vector potentials A_μ and C_μ which transform. The requirement that this second derivative remain invariant will force a condition between the charges q_e, q_e', q_m, q_m'. We see this by transforming the derivative,

$$D^\mu\Phi_2 \rightarrow \left(\partial^\mu - iq_e'A^\mu - iq_m'C^\mu\right)\Phi_2$$

Notice that the scalar potential Φ_2 does not transform. That is one avenue that will be considered in another section. For now we will consider a scalar potential that doesn't transform. Continuing we have,

$$D^\mu\Phi_2 \rightarrow \left[\partial^\mu - iq_e'\left(A^\mu(x) - \frac{1}{2q_e v}\partial^\mu\zeta(x)\right) - iq_m'\left(C^\mu - \frac{1}{2q_m v}\partial^\mu\zeta(x)\right)\right]\Phi_2 =$$

$$\left[\partial^\mu - iq_e'A^\mu(x) + i\frac{q_e'}{2q_e v}\partial^\mu\zeta(x) - iq_m'C^\mu + i\frac{q_m'}{2q_m v}\partial^\mu\zeta(x)\right]\Phi_2 =$$

$$D^\mu\Phi_2 + \left[i\frac{q_e'}{2q_e v}\partial^\mu\zeta(x) + i\frac{q_m'}{2q_m v}\partial^\mu\zeta(x)\right]\Phi_2$$

So we see that the term on the right is in addition to the original covariant derivative operating on Φ_2. We require this term to be zero,

$$i\frac{q_e'}{2q_e v}\partial^\mu\zeta(x) + i\frac{q_m'}{2q_m v}\partial^\mu\zeta(x) = 0$$

simplifying we are left with a condition on the charges that,

$$\frac{q_e'}{q_e} = -\frac{q_m'}{q_m}$$

that is, the ratio of electric charges is the negative of the ratio of magnetic charges. We note that this condition is similar to the original condition from Dirac's work that the ratio of electric to magnetic charges be equal. In fact, in the condition above, one can make somewhat arbitrary conditions on the relationship of the charges. That is, one may write a condition that $q_m' = kq_m$ where k is some arbitrary constant. This condition would also then force $q_e' = -kq_e$. Making such a substitution into our Lagrangian gives,

$$\partial_\mu\Phi_1^*\partial^\mu\Phi_1 + \partial_\mu\Phi_2^*\partial^\mu\Phi_2 + J_e^\mu A_\mu + J_{e\prime}^\mu A_\mu + J_m^\mu C_\mu + J_{m\prime}^\mu C_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} -$$

$$V(\Phi_1^2) + \left(q_e A_\mu + q_m C_\mu\right)\left(q_e A^\mu + q_m C^\mu\right)\Phi_1^*\Phi_1 -$$

$$k^2\left(q_e A_\mu - q_m C_\mu\right)\left(q_e A^\mu - q_m C^\mu\right)\Phi_2^*\Phi_2$$

Let us return back to the unexpanded Lagrangian and perform the gauge transformation as we did in the previous section. Upon visual inspection one immediately sees that the process will return the same Lagrangian we had in the previous section with the exception of the new terms involving Φ_2 and its conjugate. The condition we forced on the covariant derivative acting on Φ_2 means that the gauge transformation will not have an overall effect on the term $D_\mu\Phi_2^*D^\mu\Phi_2$. The Lagrangian that will remain if we reuse the approximation,
$e^{-i\zeta(x)/v}\Phi_1 \approx \frac{1}{\sqrt{2}}(v + \eta(x))$, is

$$L_{s2} = \frac{1}{2}\partial_\mu \eta(x)\partial^\mu \eta(x) + D_\mu \Phi_2^* D^\mu \Phi_2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} +$$
$$-\frac{1}{2}\left(v^2 + 2v\eta(x) + \eta^2(x)\right)\left[2q_e q_m B_\mu E^\mu + q_e^2 B_\mu B^\mu + q_m^2 E_\mu E^\mu\right] +$$
$$m^2\eta^2(x) - \lambda v\eta^3(x) - \frac{1}{4}\lambda\eta^4(x)$$

If we then expand the derivatives on the second scalar field and use the notation above for the primed currents we will pick up the extra terms,

$$D_\mu \Phi_2^* D^\mu \Phi_2 = \partial_\mu \Phi_2^* \partial^\mu \Phi_2 + J_{e\prime}^\mu B_\mu + J_{m\prime}^\mu E_\mu + (q_e' B_\mu + q_m' E_\mu)(q_e' B^\mu + q_m' E^\mu)\Phi_2^* \Phi_2$$

Now if we use the charge condition mentioned above we can write this term as,

$$\partial_\mu \Phi_2^* \partial^\mu \Phi_2 + kJ_e^\mu B_\mu + kJ_m^\mu E_\mu + k^2(q_e B_\mu + q_m E_\mu)(q_e B^\mu + q_m E^\mu)\Phi_2^* \Phi_2$$

The last term above is the same mixed vector potentials term that we already have in our Lagrangian. If we make the substitution,

$$L_{s2} = \frac{1}{2}\partial_\mu \eta(x)\partial^\mu \eta(x) + \partial_\mu \Phi_2^* \partial^\mu \Phi_2 + kJ_e^\mu B_\mu + kJ_m^\mu E_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} +$$
$$-\frac{1}{2}\left(v^2 + 2v\eta(x) + \eta^2(x) + \frac{k^2}{2}\Phi_2^*\Phi_2\right)\left[2q_e q_m B_\mu E^\mu + q_e^2 B_\mu B^\mu + q_m^2 E_\mu E^\mu\right] +$$
$$m^2\eta^2(x) - \lambda v\eta^3(x) - \frac{1}{4}\lambda\eta^4(x)$$

Now our Lagrangian is in a form where we can once again use matrix notation and diagonalize to remove mixed vector potential terms. Using the same results as in the previous section we will obtain,

$$L_{s2} = \frac{1}{2}\partial_\mu \eta(x)\partial^\mu \eta(x) - \frac{1}{2}\left(v^2 + 2v\eta(x) + \eta^2(x) + \frac{k^2}{2}\Phi_2^*\Phi_2\right)\left(q_e^2 + q_m^2\right)E_\mu E^\mu +$$
$$\partial_\mu \Phi_2^* \partial^\mu \Phi_2 + kJ_e^\mu B_\mu + kJ_m^\mu E_\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + m^2\eta^2(x) - \lambda v\eta^3(x) - \frac{1}{4}\lambda\eta^4(x)$$

Suppose we consider a Φ_2 scalar field that also transforms under a gauge transformation. If we assume, $\Phi_2' = e^{-ik\zeta(x)/v}\Phi_2$, so that the transformation is just a constant coefficient k multiple of the gauge for the Φ_1 field then the covariant derivative transforms as,

$$D^\mu \Phi_2 \rightarrow (\partial^\mu - iq_e' A^\mu - iq_m' C^\mu)\, e^{-ik\zeta(x)/v}\Phi_2 =$$

$$\left[\partial^\mu - iq_e'\left(A^\mu - \frac{1}{2q_e v}\partial^\mu\zeta(x)\right) - iq_m'\left(C^\mu - \frac{1}{2q_m v}\partial^\mu\zeta(x)\right)\right] e^{-ik\zeta(x)/v}\Phi_2$$

If we compute this through and consider only the left over terms, then for the covariant derivative to transform properly we must have the condition that these extra terms go to zero. What is left then is,

$$\left(-i\frac{k}{v}\partial^\mu\zeta(x) + i\frac{q_e'}{2q_e v}\partial^\mu\zeta(x) + i\frac{q_m'}{2q_m v}\partial^\mu\zeta(x)\right) e^{-ik\zeta(x)/v}\Phi_2 = 0$$

canceling mutual terms leaves,

$$-k + \frac{q_e'}{2q_e} + \frac{q_m'}{2q_m} = 0$$

Since the term k is constant, we absorb the fraction on one half and consider a gauge coefficient term w that must be

$$w = 2k = \frac{q_e'}{q_e} + \frac{q_m'}{q_m}$$

It is interesting to note that if we choose the gauge constant to be zero then (as would be expected) we simply reobtain the charge condition found previously when no gauge transformation was performed on Φ_2. This new charge condition says that the ratio of magnetic charge must be an arbitrary constant value plus the ratio of electric charge. In the Lagrangian, the only term that will differ from the previous example will be the covariant derivative term,

$$D_\mu e^{ik\zeta(x)/v}\Phi_2 D^\mu e^{-ik\zeta(x)/v}\Phi_2 =$$

$$(\partial_\mu + iq_e' B_\mu + iq_m' E_\mu)\, e^{ik\zeta(x)/v}\Phi_2\, (\partial^\mu - iq_e' B^\mu - iq_m' E^\mu)\, e^{-ik\zeta(x)/v}\Phi_2$$

Inspecting this, one can see that it will yield the same current terms, kinetic energy terms and mixed terms as before with the exception that there will be the following additional terms,

$$\frac{k^2}{v^2}\Phi_2^*\Phi_2 + i\frac{k}{v}\left(\Phi_2^*\partial^\mu\Phi_2 - \Phi_2\partial_\mu\Phi_2^*\right) +$$
$$\frac{kq_e'}{v}\left(B^\mu + B_\mu\right)\Phi_2^*\Phi_2 + \frac{kq_m'}{v}\left(E^\mu + E_\mu\right)\Phi_2^*\Phi_2$$

Notice that there is a particle in Φ_2 now that has a mass value of $\frac{k}{v}$ and there is a current associated with this particle, call it J_k^μ such that $J_k^\mu = i\frac{k}{v}\left(\Phi_2^*\partial^\mu\Phi_2 - \Phi_2\partial_\mu\Phi_2^*\right)$. We can rewrite these new terms as,

$$\frac{k^2}{v^2}\Phi_2^*\Phi_2 + J_k^\mu + \frac{kq_e'}{v}\left(B^\mu + B_\mu\right)\Phi_2^*\Phi_2 + \frac{kq_m'}{v}\left(E^\mu + E_\mu\right)\Phi_2^*\Phi_2$$

The last two terms are somewhat unsatisfactory. This leads us to believe that performing a gauge transformation on the second scalar potential is not ideal. From this point, the next option would be to give the second scalar field a negative mass squared and potential term. This would make its vacuum expectation value negative as we find the first scalar field. This more complicated example is where this paper will meet its end. The examination of a second scalar potential with a non zero expectation value will be left for further research.

CHAPTER 5: CONCLUSIONS

In this paper we have discussed a variety of magnetic monopole formulations and laid the ground work to develop more sophisticated models. Starting with the Dirac string monopole from PAM Dirac in 1931 and working up to the Wu-Yang fiber bundle model of 1975, one could show a definitive quantization condition and forced relationship between magnetic and electric charges. The original argument was that if magnetic monopoles exist, such a quantized condition would occur and more importantly, the quantization of electric charge was consistent with experimentation at the time. The apparent quantization of electric charge is today credited to isospin properties and [2] conditions rather than the still missing magnetic monopole particle.

After discussion of monopole examples, the reader was thrust in to a quantum field theory formulation of the monopole. Upon forcing the Maxwell equations to contain magnetic charge, and without explicitly describing a vector potential, an electromagnetic Lagrangian was written and a scalar vector potential term was introduced. While the scalar potential was arbitrarily described, the condition that it had a potential and a negative mass squared term allowed one to apply the Higgs' mechanism to the Lagrangian. Prior to the use of the Higgs' mechanism, the Lagrangian with this scalar potential term would have yielded a second massless photon which is undesirable in a physical model as only one massless photon has been found in nature. The model would have been inconsistent with the standard model. Application of the Higgs' mechanism allowed one to "shift" the mass to one massless photon making it massive, not unlike a W-boson.

With the introduction of a new vector potential term, no condition was found on the magnetic-like particle. That is, no conclusions could be drawn as to its mass and no condition on electric and magnetic charge was found. The theory up to that point started with one scalar potential and two vector potentials. Since scalar and vector potentials are not so mathematically independent in electromagnetic physics, it seemed natural to introduce a second scalar potential so that the theory had two scalar potentials and two vector potentials. Two directions were taken with the extra scalar field. First was scalar field with no gauge transformation. This formulation led to interesting conditions on the electric and magnetic charges. In fact, what was observed was the Dirac condition that magnetic charge and electric charge must have the same ratio, except we observed that condition with a negative sign. The second route was to perform a gauge transformation of a constant value multiple of the gauge transformation on our original scalar field. This resulted in a constant multiple added to the condition we obtained in the first method.

At this point, the next mathematical step would be to consider a second scalar potential that had a negative mass squared value and a potential term in the Lagrangian. With this arrangement we could find the vacuum expectation value of the second scalar field and parameterize as we have done already. This next step will be left for further research and hopefully publication.

REFERENCES

[1] P.A.M. Dirac, "Quantised Singularities in the Electromagnetic Field" , Proc. Roy. Soc. **A 133**, 60-72 (1931) ; P.A.M. Dirac, "The Theory of Magnetic Poles" , Phys. Rev. **74**, 817-830 (1948).

[2] L. H. Ryder, *Quantum Field Theory* 2^{nd} Edition, (Cambridge University Press, 1996) p. 20-90, 120-200

[3] G. 't Hooft, "Magnetic Monopoles in Unified Gauge Theories", Nucl. Phys. **B79**, 276-284 (1974); A.M. Polyakov, "Particle Spectrum in Quantum Field Theory" JETP Letters **20**, 194-195 (1974)

[4] A. Rajantie, "The Search for Magnetic Monopoles" , Physics Today **69**, 41-46 (2016).

[5] D. Singleton, "Magnetic Charge as a Hidden Gauge Symmetry", Int. J. Theo. Phys., **34**, 37-46 (1995)

[6] T.T. Wu and C.N. Yang, "Concept of Nonintegrable Phase Factor and Global Formulation of Gauge Fields", Phys. Rev. **D12**, 3845-3857 (1975)

[7] J.D. Jackson, *Classical Electrodynamics* 2^{nd} Edition, (John Wiley & Sons, 1975) p. 251

[8] D. Singleton, "Electromagnetism with Magnetic Charge and Two Photons", Am. J. Phys. **64**, 452-458 (1996).

[9] Ta-Pei Cheng and Ling-Fong-Li, *Gauge Theory of Elementary Particle Physics* (Oxford University Press, New York, 1984) p.402-406

[10] N. Cabibbo and E. Ferrari, "Quantum Electrodynamics with Dirac Monopoles" , Nuovo Cimento **23**, 1147-1154 (1962)

[11] C.R. Hagen, "Noncovariance of Dirac Monopole" , Phys. Rev. **140**, B804-B810 (1965)

[12] D. Zwanzinger, "Local-Lagrangian Field Theory of Electric and Magnetic Charges ", Phys. Rev. **D3**, 880-891 (1971)

[13] W. Barker and F. Graziani, "Quantum Mechanical Formulation of Electron-Monopole Interaction without Dirac Strings " , Phys. Rev. **D18**, 3849-3857 (1978) ; W. Barker and F. Graziani, "A Heuristic Potential Theory of Electric and Magnetic Monopoles without Strings ", Am J. Phys. **46**, 1111-1115 (1978)

[14] P.W. Higgs, "Broken Symmetries, Massless Particles and Gauge Fields", Phys. Letts. **12**, 132-133 (1964); "Broken Symmetries and the Masses of Gauge Bosons" , Phys. Rev. Letts. **13**, 508-509 (1964); "Spontaneous Symmetry Breaking without Massless Bosons", Phys. Rev. **145**, 1156-1163 (1966)

[15] F. Rohrlich, "Classical Theory of Magnetic Monopoles" , Phys. Rev. **150**, 1104-1111 (1966)

[16] M.N. Saha, "On the Origin of Mass in Neutrons and Proton's " , Ind. J. Phys. **10**, 145-151 (1936); M.N. Saha, "Note on Dirac's Theory of Magnetic Poles" , Phys. Rev. **75**, 1968 (1949); H.A. Wilson, "Note on Dirac's Theory of Magnetic Poles " , Phys. Rev. **75**, 309 (1949)

[17] J. Arafune, P.G.O. Freund, and C.J. Goebel, "Topology of Higgs Field", Jour. Math Phys. **16**, 433-437 (1975)

[18] E.F. Carter and H.A. Cohen, "Classical Problem of Charge and Pole", Am. J. Phys. **41**, 994-1005 (1974)

[19] Kerson Huang, *Quarks, Leptons and Gauge Fields*, (World Scientific Publishing Co., 1982) p. 50

[20] K. Wilson, "Confinement of Quarks" , Phys. Rev. **D10**, 2445-2459 (1974)

[21] A. Guth, "Existence Proof of a Nonconfining Phase in Four Dimensional U(1) Lattice Gauge Theory" , Phys. Rev. **D21**, 2291-2307 (1980)

[22] J. Schwinger, "Gauge Invariance and Mass. II" , Phys. Rev. **128**, 2425-2429 (1962)

[23] B. Holstein, "Anomalies for Pedestrians" , Am. J. Phys. **61**, 142-147 (1993)

[24] W.A. Rodrigues, *et. al.*, "The Classical Problem of the Charge and Pole Motion. A Satisfactory Formalism by Clifford Algebras " , Phys. Letts. **B220**, 195-199 (1989)

Fresno State

<u>Non-Exclusive Distribution License</u>
(to archive your thesis/dissertation electronically via the library's eCollections database)

By submitting this license, you (the author or copyright holder) grant to Fresno State Digital Scholar the non-exclusive right to reproduce, translate (as defined in the next paragraph), and/or distribute your submission (including the abstract) worldwide in print and electronic format and in any medium, including but not limited to audio or video.

You agree that Fresno State may, without changing the content, translate the submission to any medium or format for the purpose of preservation.

You also agree that the submission is your original work, and that you have the right to grant the rights contained in this license. You also represent that your submission does not, to the best of your knowledge, infringe upon anyone's copyright.

If the submission reproduces material for which you do not hold copyright and that would not be considered fair use outside the copyright law, you represent that you have obtained the unrestricted permission of the copyright owner to grant Fresno State the rights required by this license, and that such third-party material is clearly identified and acknowledged within the text or content of the submission.

If the submission is based upon work that has been sponsored or supported by an agency or organization other than Fresno State, you represent that you have fulfilled any right of review or other obligations required by such contract or agreement.

Fresno State will clearly identify your name as the author or owner of the submission and will not make any alteration, other than as allowed by this license, to your submission. **By typing your name and date in the fields below, you indicate your agreement to the terms of this distribution license.**

Embargo options (fill box with an X).

[x] **Make my thesis or dissertation available to eCollections immediately upon submission.**

[ ] Embargo my thesis or dissertation for a period of 2 years from date of graduation. After two years, I understand that my work will automatically become part of the university's public institutional repository.

John Scott

Type full name as it appears on submission

December 2, 2016

Date