Gravitational waves from a particle in bound geodesic orbits around a Kerr black hole

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Abstract

We show an efficient numerical method to compute gravitational waves radiated by a particle orbiting around a Kerr black hole. In this work, we compute the homogeneous solutions of the Teukolsky equation using formalism developed by Mano, Suzuki and Takasugi. We also compute various modes of gravitational waves using trapezoidal rule in numerical integration. We check our code in the case of simple orbits and report our current status.

1 Introduction

Gravitational waves radiated from a particle orbiting around a Kerr black hole are one of the main targets of LISA. Such systems are called the extreme mass ratio inspirals(EMRI). Observing gravitational waves from EMRI, we may be able to obtain information of the central black hole's spacetime such as mass, spin and the mass distribution of compact objects in the center of galaxy. In order to obtain information from EMRI, we have to achieve the phase accuracy of theoretical gravitational wave forms within one radian over the total cycle of wave, $\sim 10^5$.

There are a lot of previous works which numerically compute gravitational waves from EMRI. For simple orbits such as circular or equatorial orbits around black hole, they achieved the accuracy 10^{-5} which may be sufficient to detect gravitational waves. For more general orbits, Drasco and Hughes computed gravitational waves from EMRI[3]. However, their computational time and numerical accuracy seem to be insufficient to detect gravitational waves. It might be useful if we can compute gravitational waves more efficiently.

In this paper, we show such an efficient method which may be applicable for LISA data analysis.

2 Gravitational waves from EMRIs

Gravitational waves from EMRI are well approximated by the black hole perturbation theory. The basic equation of the black hole perturbation is the Teukolsky equation[1] which is a master equation for Weyl scalars, Ψ_4. The Weyl scalars are related to the amplitudes of gravitational wave as

$$\Psi_4 \to \frac{1}{2}(\ddot{h}_+ - i\,\ddot{h}_\times), \;\; \text{for} \;\; r \to \infty. \tag{1}$$

Teukolsky showed that the master equation can be separated into a radial and a angular part if we expand Weyl scalars in Fourier-harmonic modes as

$$\Psi_4 = \sum_{\ell m} \int_{-\infty}^{\infty} d\omega e^{-i\omega t + im\varphi} \;\; {}_{-2}S^{a\omega}_{\ell m}(\theta) R_{\ell m\omega}(r). \tag{2}$$

where the angular function ${}_{-2}S^{a\omega}_{\ell m}(\theta)$ is a spin-weighted spheroidal harmonic with spin s. The radial function $R_{\ell m\omega}(r)$ satisfies radial part of the Teukolsky equation as

$$\Delta^2 \frac{d}{dr}\left(\frac{1}{\Delta}\frac{dR_{\ell m\omega}}{dr}\right) - V(r)R_{\ell m\omega} = T_{\ell m\omega}. \tag{3}$$

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The potential is given by

$$V(r) = -\frac{K^2 - 2is(r-M)K}{\Delta} - 4is\omega r + \lambda, \tag{4}$$

where $K = (r^2+a^2)\omega - ma$ and λ is the eigenvalue of ${}_{-2}S^{a\omega}_{\ell m}(\theta)$.

We solve the radial Teukolsky equation using the Green function method. Then, the solutions of the Teukolsky equation are derived as

$$R_{\ell m\omega}(r) = R^{\rm up}_{\ell m\omega}(r) Z^{\infty}_{\ell m\omega}(r) + R^{\rm in}_{\ell m\omega}(r) Z^{\rm H}_{\ell m\omega}(r), \tag{5}$$

where $R^{\rm in/up}_{\ell m\omega}(r)$ satisfy ingoing/outgoing wave conditions at the horizon/infinity and

$$Z^{\infty,{\rm H}}_{lm\omega} = \int_{-\infty}^{\infty} dt\, e^{i[\omega t - m\phi(t)]} I^{\infty,{\rm H}}_{lm\omega}[r(t),\theta(t)]. \tag{6}$$

The function $I^{\infty,{\rm H}}_{lm\omega}[r(t),\theta(t)]$ is constructed from the source term of the Teukolsky equation and depends on the orbital worldline of the star perturbing the black hole spacetime. If the trajectory of the star is eccentric and inclined from equatorial plane around a black hole, it is difficult to evaluate Eq. (6) because integrand is multi periodic function and each mode couples. These problems comes from the fact that radial motion and polar motion of the star couple in observer time. They can be solved by introducing a new parameter of time[2, 3]. Then the geodesic equations of the star are given by

$$\begin{aligned}
\left(\frac{dr}{d\lambda}\right)^2 &= \left[(r^2+a^2)E - aL_z\right]^2 - \Delta[r^2 + (L_z - aE)^2 + C] \equiv R(r), \\
\left(\frac{d\theta}{d\lambda}\right)^2 &= C - \left[(1-E^2)a^2 + \frac{L_z^2}{\sin^2\theta}\right]\cos^2\theta \equiv \Theta(\theta), \\
\frac{d\phi}{d\lambda} &= \frac{a}{\Delta}\left[E(r^2+a^2) - aL_z\right] - aE + \frac{L_z}{\sin^2\theta} \equiv \Phi_{\rm r}(r) + \Phi_\theta(\theta), \\
\frac{dt}{d\lambda} &= \frac{r^2+a^2}{\Delta}\left[E(r^2+a^2) - aL_z\right] - a\left[aE\sin^2\theta - L_z\right] \equiv T_{\rm r}(r) + T_\theta(\theta),
\end{aligned} \tag{7}$$

where E, L_z and C are the energy, the z-component of the angular momentum and the Carter constant of a test particle respectively. Since the equations of radial and polar motion are decouple with time parameter λ, the radial and the polar motion are independently periodic functions. The frequencies are given by

$$\frac{2\pi}{\Upsilon_r} = 2\int_{\rm r_{min}}^{\rm r_{max}} \frac{dr}{\sqrt{R(r)}},\ \frac{2\pi}{\Upsilon_\theta} = 4\int_0^{\theta_{\rm min}} \frac{d\theta}{\sqrt{\Theta(\theta)}}. \tag{8}$$

Here, Υ_r and Υ_θ are the frequency of the radial and the polar motion respectively. The frequencies of the time and the azimuthal motion are also derived respectively as

$$\Gamma \equiv \frac{1}{2\pi}\int_0^{2\pi} dw^\theta\, T_\theta\left[\theta(w^\theta)\right] + \frac{1}{2\pi}\int_0^{2\pi} dw^r\, T_r\left[r(w^r)\right], \tag{9}$$

$$\Upsilon_\phi \equiv \frac{1}{2\pi}\int_0^{2\pi} dw^\theta\, \Phi_\theta\left[\theta(w^\theta)\right] + \frac{1}{2\pi}\int_0^{2\pi} dw^r\, \Phi_r\left[r(w^r)\right], \tag{10}$$

where w^r and w^θ are action-angle variables for r and θ motion respectively.

Using new parameterization of time λ, we can evaluate Eq. (6) as

$$Z^{\infty,{\rm H}}_{lm\omega} \equiv \sum_{kn} \tilde{Z}^{\infty,{\rm H}}_{lmkn}\delta(\omega - \omega_{mkn}), \tag{11}$$

where

$$\tilde{Z}^{\infty,{\rm H}}_{lmkn} = \frac{1}{(2\pi)^2}\int_0^{2\pi} dw^\theta \int_0^{2\pi} dw^r\ e^{i(kw^\theta + nw^r)}\, Z^{\infty,{\rm H}}_{lm\omega_{mkn}}[r(w^r),\theta(w^\theta)], \tag{12}$$

and

$$\omega_{mkn} \equiv \Upsilon_{mkn}/\Gamma \equiv (m\Upsilon_\phi + k\Upsilon_\theta + n\Upsilon_r)/\Gamma \; . \tag{13}$$

Computing Eq. (12), we can obtain gravitational wave forms and energy flux to infinity as

$$h_+ - i\, h_\times = -\frac{2}{r}\sum_{lmkn} \frac{Z^\infty_{lmkn}}{\omega^2_{mkn}} \frac{{}_{-2}S^{a\omega_{mkn}}_{lm}(\theta)}{\sqrt{2\pi}} e^{i\omega_{mkn}(r^*-t)+im\phi}, \; \left\langle \frac{dE}{dt} \right\rangle^\infty_{GW} = \sum_{lmkn} \frac{|Z^\infty_{lmkn}|^2}{4\pi\omega^2_{mkn}}. \tag{14}$$

3 An efficient numerical method and results

As explained in the previous section, we have to compute Eq. (12) in order to evaluate gravitational waves. From the points of views of LISA data analysis, it is important to find efficient methods to compute Eq. (12). In order to compute Eq. (12) efficiently, we compute the homogeneous solutions of the Teukolsky equation with the formalism developed by Mano, Suzuki and Takasugi(MST)[4] and use the trapezoidal rule to compute Eq. (12).

In MST formalism, the homogeneous solutions of the Teukolsky equation are expressed in terms of two kinds of series of special functions. Although the application of this method was previously limited to the analytical evaluation of the homogeneous solutions, Fujita and Tagoshi found that it was also useful for numerical calculation[5] because the convergence of the homogeneous solutions are very rapid and the accuracy of gravitational waves were achieved about machine accuracy. Then, we use trapezoidal rule to compute Eq. (12) because the convergence of numerical integration of periodic function over one cycle is very rapid and the accuracy is also very high if we use trapezoidal rule.

As a demonstration, we compute the energy fluxes of gravitational waves radiated to infinity and compare with past results. We show our numerical results in Table 1, 2 and Fig. 3.

l	$\lvert m\rvert$	Fujita and Tagoshi(2005)	This work	Relative error
2	2	$1.78401657716943627674 \times 10^{-2}$	$1.78401657717214383314 \times 10^{-2}$	1.5×10^{-12}
3	3	$1.07647967139507787920 \times 10^{-2}$	$1.07647967139685510340 \times 10^{-2}$	1.7×10^{-12}
4	4	$6.15270976286638737740 \times 10^{-3}$	$6.15270976287798920801 \times 10^{-3}$	1.9×10^{-12}
5	5	$3.62992493707918690085 \times 10^{-3}$	$3.62992493685125290973 \times 10^{-3}$	6.3×10^{-11}
6	6	$2.20229967982856367908 \times 10^{-3}$	$2.20229967906702657834 \times 10^{-3}$	3.5×10^{-10}
7	7	$1.36347483334188827007 \times 10^{-3}$	$1.36347483181112750887 \times 10^{-3}$	1.1×10^{-9}

Table 1: Energy fluxes radiated to the infinity in the case of a circular and the equatorial orbit around a Kerr black hole. In this table, the orbital radius is $1.55M$ and spin of the black hole is $a = 0.99$. We compare our numerical results with the results of high precision calculation[5] which are about 13–14 significant figures in double precision calculation. In order to check the accuracy of our numerical code, we evaluate the energy fluxes by computing Eq. (**??**) with numerical integration though we can compute it without numerical integration. These results shows that accuracy of our code is limited by the accuracy of the numerical integration. In this case, we set the accuracy of the numerical integration about 10 significant figures.

4 Summary

In this paper, we show an efficient numerical method to compute gravitational waves produced by a particle orbiting around a Kerr black hole. We compute the homogeneous solutions of the Teukolsky equation with the formalism developed by Mano, Suzuki and Takasugi and also evaluate Fourier mode of gravitational waves with trapezoidal rule of numerical integration. Numerical accuracy and computational time derived here are more useful for LISA data analysis than past works.

Future work is to develop our code for more general orbits such that the star move across the north and south poles. At the same time, we evaluate gravitational waves including effects of adiabatic evolution of orbital parameter by computing energy fluxes, angular momentum fluxes and change rate of the Carter constants.

e	inclination	Sago et al.(2006)	This work	Relative error
0.01	3°	$6.211124676 \times 10^{-10}$	$6.205511895774004 \times 10^{-10}$	9.0×10^{-4}
0.01	5°	$6.211194911 \times 10^{-10}$	$6.205582654583145 \times 10^{-10}$	9.0×10^{-4}
0.01	7°	$6.211301135 \times 10^{-10}$	$6.205687847727640 \times 10^{-10}$	9.0×10^{-4}
0.03	3°	$6.268368982 \times 10^{-10}$	$6.213139168623904 \times 10^{-10}$	8.8×10^{-3}
0.03	5°	$6.268440256 \times 10^{-10}$	$6.213210249941237 \times 10^{-10}$	8.8×10^{-3}
0.03	7°	$6.268548068 \times 10^{-10}$	$6.213315925452469 \times 10^{-10}$	8.8×10^{-3}
0.05	3°	$6.382857572 \times 10^{-10}$	$6.228283030987133 \times 10^{-10}$	2.4×10^{-2}
0.05	5°	$6.382930950 \times 10^{-10}$	$6.228354756012462 \times 10^{-10}$	2.4×10^{-2}
0.05	7°	$6.383041917 \times 10^{-10}$	$6.228461394314896 \times 10^{-10}$	2.4×10^{-2}

Table 2: Energy fluxes radiated to the infinity in the case of slightly eccentric and inclined orbit around a Kerr black hole. In this table, the semilatus rectum is $100M$ and spin of the black hole is $a = 0.9$. We compare the numerical results with analytical post-Newtonian expansions of the energy fluxes[6] which truncate Taylor expansion to 2.5PN, v^5, order. These results shows that accuracy of our code in the case of eccentric and inclined orbits are consistent with the results of the analytical post-Newtonian expansion.

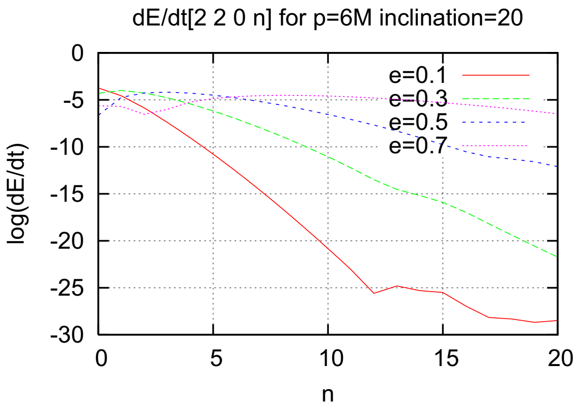

Figure 1: Modal energy fluxes radiated to the infinity in the case of eccentric and inclined orbit around a Kerr black hole. The mode indices are $\ell = M = 2$ and $k = 0$. In this figure, the semilatus rectum is $6M$ and spin of the black hole is $a = 0.9$. Computational time to calculate date for $e = 0.7$ in this figure is about 10 seconds.

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