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Special Issue

Continuous and Discrete Phase-Space Methods and Their Applications


Edited by

Prof. Dr. Andrei Klimov, Prof. Dr. Luis L. Sánchez-Soto and Prof. Dr. Hubert de Guise



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Coherent Phase States in the Coordinate and Wigner Representations

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Abstract: In this paper, we numerically study the coordinate wave functions and the Wigner functions of the coherent phase states (CPS), paying particular attention to their differences from the standard (Klauder–Glauber–Sudarshan) coherent states, especially in the case of the high mean values of the number operator. In this case, the CPS can possess a strong coordinate (or momentum) squeezing, which is roughly twice weaker than for the vacuum squeezed states. The Robertson–Schrödinger invariant uncertainty product in the CPS logarithmically increases with the mean value of the number operator (whereas it is constant for the standard coherent states). Some measures of the (non)Gaussianity of CPS are considered.

Keywords: Robertson–Schrödinger uncertainty relations; squeezing; wave functions; probability density; Wigner function; super-Gaussian and sub-Gaussian states



Citation: de Freitas, M.C.; Dodonov, V.V. Coherent Phase States in the Coordinate and Wigner Representations. *Quantum Rep.* **2022**, *4*, 509–522. <https://doi.org/10.3390/quantum4040036>

Academic Editors: Andrei Klimov, Luis L. Sánchez-Soto and Hubert de Guise

Received: 30 September 2022

Accepted: 1 November 2022

Published: 8 November 2022

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1. Introduction

The problem of phase in quantum mechanics is as old as quantum mechanics itself [1–24]. One of its main ingredients is the *exponential phase operator*

$$E_- = (\hat{n} + 1)^{-1/2} \hat{a}, \quad \hat{n} = \hat{a}^\dagger \hat{a}, \quad [\hat{a}, \hat{a}^\dagger] = 1. \quad (1)$$

This can be found already in the classical paper by Dirac [25] (although in a slightly incorrect form, corrected by London [26]), but its systematic study began with the paper by Susskind and Glogower [27] (see the history in [28]). Various applications and generalizations of operator (1) have been studied, e.g., in papers [29–35].

Many researchers have studied the properties of eigenstates of the exponential phase operators:

$$|\varepsilon\rangle = \sqrt{1 - |\varepsilon|^2} \sum_{n=0}^{\infty} \varepsilon^n |n\rangle, \quad \varepsilon = |\varepsilon| e^{i\varphi}, \quad |\varepsilon| < 1, \quad \hat{a}^\dagger \hat{a} |n\rangle = n |n\rangle. \quad (2)$$

These states were introduced in papers [36–38] without any special name. The name “coherent phase states” was coined by Shapiro and Shepard [39]. The name “harmonious state” was used by Sudarshan [40]. The name “pseudothermal states” was suggested in Ref. [41], because the state (2) can be considered as a pure-state analog of the mixed equilibrium state of the harmonic oscillator, described by means of the statistical operator

$$\hat{\rho}_{th} = (1 - |\varepsilon|^2) \sum_{n=0}^{\infty} |\varepsilon|^{2n} |n\rangle \langle n|, \quad (3)$$

with the mean number of quanta

$$\langle \hat{n} \rangle_\varepsilon = \langle \hat{n} \rangle_\rho = (1 - |\varepsilon|^2) |\varepsilon|^2 \sum_{n=0}^{\infty} |\varepsilon|^{2n} (n + 1) = \frac{|\varepsilon|^2}{1 - |\varepsilon|^2}, \quad |\varepsilon|^2 = \frac{\langle \hat{n} \rangle}{1 + \langle \hat{n} \rangle}. \quad (4)$$

The states (2) [named sometimes as “phase coherent states”] were also considered from different points of view in Refs. [42–55]. In particular, the physical meaning of these states and their distinguishing property was clarified in paper [56]: “phase-coherent states preserve their coherence under amplification and achieve the best amplifier performance”. More recently, it was shown in paper [57] that “the use of phase coherent states gives better performance in terms of noise tolerance, key rate, and achievable distance than that of common linear coherent states” for continuous variable quantum key distribution protocols. Such states can be used to understand the quantum phase transition in the squeezed Jaynes–Cummings model [58]. Methods of generation of the states (2) were proposed in [59,60]. Further generalizations were studied, e.g., in Refs. [61–66]. In particular, it appears [66–68] that the coherent phase states are the special case of a large family of *nonlinear coherent states* introduced in papers [69,70].

In the papers cited above, the authors studied the properties of state (2) from the point of view of relations between the *phase and number variables*. Our goal is to study the properties of state (2) with respect to the *canonical quadrature operators*

$$\hat{x} = (\hat{a} + \hat{a}^\dagger) / \sqrt{2}, \quad \hat{p} = (\hat{a} - \hat{a}^\dagger) / (i\sqrt{2}),$$

comparing these properties with that of the standard (Klauder–Glauber–Sudarshan) [71–73] coherent states

$$|\alpha\rangle = \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{5}$$

It is well known that the wave function of state (5) in the coordinate representation has the Gaussian form [74] with equal constant variances $\sigma_x^\alpha = \sigma_p^\alpha = 1/2$:

$$\langle x|\alpha\rangle = \pi^{-1/4} \exp\left(-\frac{1}{2}x^2 + \sqrt{2}x\alpha - \frac{1}{2}\alpha^2 - \frac{1}{2}|\alpha|^2\right). \tag{6}$$

We wish to know how the absence of the denominator $\sqrt{n!}$ in the expansion of (2) influences the mean values and variances of the coordinate and momentum operators, as well as the form of the wave function. In view of the equidistant spectrum of the harmonic oscillator, the time evolution of state (2) is equivalent to the linear increase of the phase $\varphi(t) = \varphi(0) + t$ (formally, we consider the situation when $\omega = m = \hbar = 1$).

2. Quadrature Mean Values, Variances, and the Robertson–Schrödinger Uncertainty Product

Mean values of the quadrature operators in the state (2) have the form

$$\langle \hat{x} \rangle = \sqrt{2} \cos(\varphi) S_1, \quad \langle \hat{p} \rangle = \sqrt{2} \sin(\varphi) S_1, \tag{7}$$

$$S_1 = (1 - |\varepsilon|^2) \sum_{n=0}^{\infty} |\varepsilon|^{2n+1} \sqrt{n+1} = (1 + \langle \hat{n} \rangle)^{-1} \sum_{n=0}^{\infty} \left(\frac{\langle \hat{n} \rangle}{1 + \langle \hat{n} \rangle} \right)^{n+1/2} \sqrt{n+1}. \tag{8}$$

The quadrature (co)variances can be written as

$$\left. \begin{matrix} \sigma_x \\ \sigma_p \end{matrix} \right\} = N - S_1^2 \pm (S_2 - S_1^2) \cos(2\varphi), \quad \sigma_{xp} = \sin(2\varphi) (S_2 - S_1^2), \tag{9}$$

$$S_2 = (1 - |\varepsilon|^2) \sum_{n=0}^{\infty} |\varepsilon|^{2n+2} \sqrt{(n+1)(n+2)} = (1 + \langle \hat{n} \rangle)^{-1} \sum_{n=0}^{\infty} \left(\frac{\langle \hat{n} \rangle}{1 + \langle \hat{n} \rangle} \right)^{n+1} \sqrt{(n+1)(n+2)}. \tag{10}$$

Note that for the (mixed) thermal equilibrium states, one has

$$\sigma_x^{therm} = \sigma_p^{therm} = \langle \hat{n} \rangle + 1/2 \equiv N = \frac{1 + |\varepsilon|^2}{2(1 - |\varepsilon|^2)}, \quad \sigma_{xp}^{therm} = 0. \tag{11}$$

If $\varphi = \pi/2$, then $\sigma_x^{\varphi=\pi/2} = N - S_2$. Noticing that

$$\langle \hat{n} \rangle = (1 - |\varepsilon|^2) \sum_{n=1}^{\infty} |\varepsilon|^{2n} n = (1 - |\varepsilon|^2) \sum_{m=0}^{\infty} |\varepsilon|^{2m+2} (m + 1),$$

we arrive at the formula

$$\sigma_x^{\varphi=\pi/2} = \frac{1}{2} - (1 - |\varepsilon|^2) |\varepsilon|^2 \sum_{n=0}^{\infty} \frac{|\varepsilon|^{2n} \sqrt{n+1}}{\sqrt{n+1} + \sqrt{n+2}}. \tag{12}$$

Hence, there is squeezing of the x -quadrature at $\varphi = \pi/2$ (and the same degree of squeezing of the p -quadrature at $\varphi = 0$) for any value of $|\varepsilon|$. The existence of squeezing in the coherent phase states was discovered numerically (presented in the form of tables for $|\varepsilon|^2$ from 0.1 to 0.9999 or $\langle \hat{n} \rangle$ from 0.11 to 9999) in Ref. [75]. For $|\varepsilon| \ll 1$, we have $\sigma_x^{\varphi=\pi/2} \approx 1/2 - |\varepsilon|^2(\sqrt{2} - 1)$. An ideal squeezing ($\sigma_x^{\varphi=\pi/2} \rightarrow 0$) can be obtained for $|\varepsilon| \rightarrow 1$.

Numerical calculations show that the variance $\sigma_x^{\varphi=\pi/2}$ as a function of $|\varepsilon|^2$ is close to the straight line $\frac{1}{2}(1 - |\varepsilon|^2)$. However, the exact function is always above this straight line. A good analytic approximation can be found in the asymptotic region $|\varepsilon|^2 \rightarrow 1$, if one takes into account that the main contribution to the series in Equation (12) is made by terms with large values of the summation index n . Therefore, we make the approximation

$$\frac{\sqrt{n+1}}{\sqrt{n+1} + \sqrt{n+2}} = \left(1 + \sqrt{\frac{n+2}{n+1}}\right)^{-1} \approx \left(2 + \frac{1}{2(n+1)}\right)^{-1} \approx \frac{1}{2} \left(1 - \frac{1}{4(n+1)}\right).$$

Then, using two exact series,

$$\sum_{n=0}^{\infty} x^n = (1 - x)^{-1}, \quad \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \int_0^x dy \sum_{n=0}^{\infty} y^n = -\ln(1 - x),$$

we arrive at the approximate formulas

$$\sigma_x^{\varphi=\pi/2} \approx \frac{1}{2} (1 - |\varepsilon|^2) \left[1 - \frac{1}{4} \ln(1 - |\varepsilon|^2)\right] = \frac{1}{2(1 + \langle \hat{n} \rangle)} \left[1 + \frac{1}{4} \ln(1 + \langle \hat{n} \rangle)\right]. \tag{13}$$

The left-hand part of Figure 1 shows $\sigma_x^{\varphi=\pi/2}$ as function of $|\varepsilon|^2$, comparing the exact (numeric) values with the approximate formula (13). One can see that approximate formula works satisfactorily for all values of $|\varepsilon|^2$. The right-hand part of Figure 1 shows the exact values of $\sigma_x^{\varphi=\pi/2}$ as function of the mean number of quanta $\langle \hat{n} \rangle$. This dependence is compared with the known formula for the pure squeezed vacuum (Gaussian) state with the variance $\sigma_x = (1/2)e^{-2r}$ and $\langle \hat{n} \rangle = \sinh^2(r)$:

$$\sigma_x^{sqz} = \left\{ 2 \left[2\langle \hat{n} \rangle + 1 + 2\sqrt{\langle \hat{n} \rangle (\langle \hat{n} \rangle + 1)} \right] \right\}^{-1}. \tag{14}$$

This quantity behaves as $[4(1 + 2\langle \hat{n} \rangle)]^{-1}$ for $\langle \hat{n} \rangle \gg 1$. Consequently, roughly speaking, the squeezing effect in the squeezed vacuum states is twice as strong as in the coherent phase states.

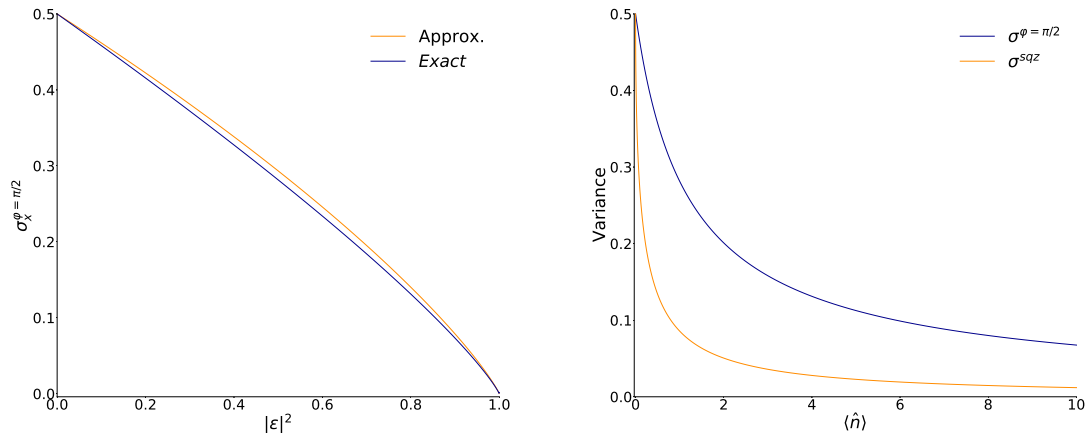


Figure 1. (Left) The variance $\sigma_x^{\phi=\pi/2}$ versus $|\epsilon|^2$. Exact numeric calculations were made taking into account 1000 terms in formula (12). (Right) The exact variance $\sigma_x^{\phi=\pi/2}$ versus the mean number of quanta $\langle \hat{n} \rangle$, compared with the variance (14) in the squeezed vacuum state.

The mean values of the coordinate and momentum in the coherent state (5) satisfy the relation $R \equiv \langle \hat{x} \rangle^2 + \langle \hat{p} \rangle^2 = 2|\alpha|^2 = 2\langle \hat{n} \rangle$. For the coherent phase state (2), we have $R = 2S_1^2$, where S_1 is given by Equation (8). According to this equation, $R \approx 2\langle \hat{n} \rangle$ for $\langle \hat{n} \rangle \ll 1$. However, numeric calculations show that the plot of function $R(\langle \hat{n} \rangle)$ for $\langle \hat{n} \rangle \gg 1$ is very close to the straight line with a smaller inclination coefficient: $R \approx \eta \langle \hat{n} \rangle$ with $\eta \approx 1.59$. A reasonable interpolation formula between the two regimes can be written as

$$R \approx \frac{\langle \hat{n} \rangle (2 + \eta \langle \hat{n} \rangle)}{1 + \langle \hat{n} \rangle}. \tag{15}$$

A comparison of this formula with the exact numerical values is shown in Figure 2.

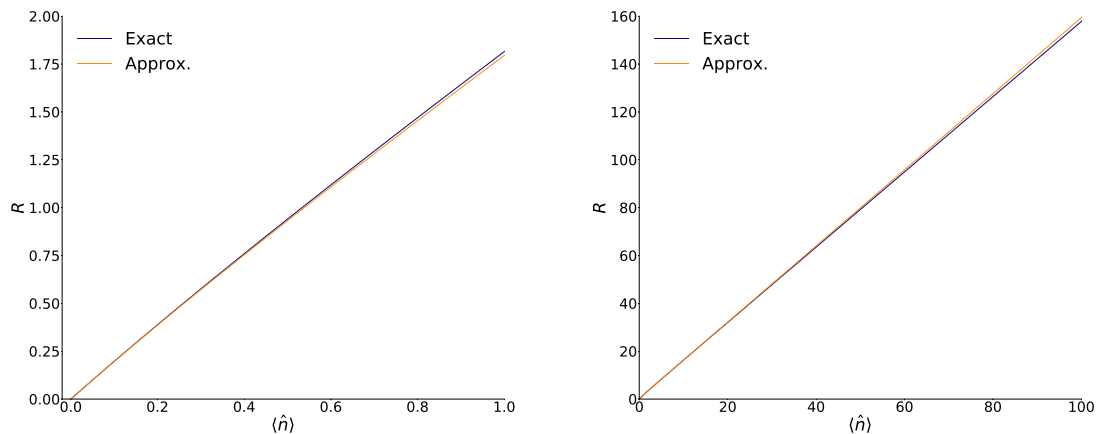


Figure 2. Parameter R versus the mean number of quanta $\langle \hat{n} \rangle$. Left: for $\langle \hat{n} \rangle < 1$. Right: for $\langle \hat{n} \rangle < 100$. The exact numeric calculations took into account 10,000 terms in formula (8).

An interesting quantity is the *Robertson–Schrödinger uncertainty product* [76,77]

$$D \equiv \sigma_x \sigma_p - \sigma_{xp}^2. \tag{16}$$

It cannot be smaller than $\hbar^2/4$ (or $1/4$ in the dimensionless units used in this paper). The equality $D = 1/4$ holds for all Gaussian *pure* states, including the standard coherent states.

On the other hand, $D = (\langle \hat{n} \rangle + 1/2)^2$ for the (mixed Gaussian) thermal states. For the coherent phase state, (2) we have

$$D = (N - S_1^2)^2 - (S_2 - S_1^2)^2 = (N - S_2)(N + S_2 - 2S_1^2). \tag{17}$$

The phase independence of the right-hand side of Equation (17) becomes obvious, if one takes into account the equivalence between the phase change and time evolution. The quantity D is the *quantum universal invariant*, which preserves its value during the time evolution governed by any quadratic Hamiltonian (in one dimension) [78,79].

The right-hand side of Equation (17) can be also written as

$$D = \sigma_x^{\varphi=\pi/2} (2N - \sigma_x^{\varphi=\pi/2} - R). \tag{18}$$

Then, using the approximate Formulas (13) and (15), we arrive at the approximate expression

$$D \approx \frac{1 + (1/4) \ln(1 + \langle \hat{n} \rangle)}{4(1 + \langle \hat{n} \rangle)^2} [2(2 - \eta)\langle \hat{n} \rangle^2 + 1 + 2\langle \hat{n} \rangle - (1/4) \ln(1 + \langle \hat{n} \rangle)]. \tag{19}$$

This formula yields the correct value $D = 1/4$ even for $\langle \hat{n} \rangle = 0$. A simplified version for $\langle \hat{n} \rangle \gg 1$,

$$D \approx \frac{2 - \eta}{2} \left[1 + \frac{1}{4} \ln(1 + \langle \hat{n} \rangle) \right] = \frac{2 - \eta}{2} \left[1 - \frac{1}{4} \ln(1 - |\varepsilon|^2) \right], \tag{20}$$

yields $D \approx 0.677$ for $1 + \langle \hat{n} \rangle = 10^4$ and $\eta = 1.59$. The result of Ref. [75] for $\langle \hat{n} \rangle = 9999$ can be transformed to the form $D = 0.67$. The dependence of D on $\langle \hat{n} \rangle$ is illustrated in Figure 3.

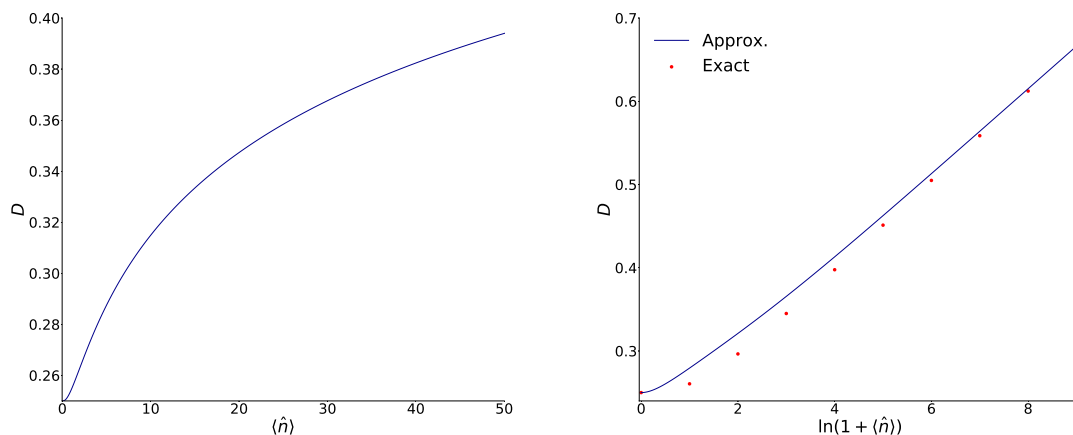


Figure 3. The Robertson–Schrödinger uncertainty product D versus the mean number of quanta $\langle \hat{n} \rangle$ in the coherent phase state (2). Left: D as function of $\langle \hat{n} \rangle$ for relatively small values of $\langle \hat{n} \rangle$ (numeric results). Right: the comparison of the approximate analytical formula (19) with exact numerical values. The best coincidence happens for $\eta = 1.59$. The series S_1 and S_2 were calculated numerically, taking into account up to 640,000 terms for high values of $\langle \hat{n} \rangle$, in order to obtain a good precision.

Using Equations (13), (17), and (19), and remembering that $\sigma_p^{\varphi=0} = \sigma_x^{\varphi=\pi/2}$, we obtain an approximate formula for the coordinate variance for $\varphi = 0$:

$$\sigma_x^{\varphi=0} \approx \frac{2(2 - \eta)\langle \hat{n} \rangle^2 + 1 + 2\langle \hat{n} \rangle - (1/4) \ln(1 + \langle \hat{n} \rangle)}{2(1 + \langle \hat{n} \rangle)}. \tag{21}$$

If $\langle \hat{n} \rangle \gg 1$, then $\sigma_x^{\varphi=0} \approx (2 - \eta)\langle \hat{n} \rangle \approx 0.4\langle \hat{n} \rangle$. The exact numeric values of $\sigma_x^{\varphi=0}$ and $\langle x \rangle^{\varphi=0}$ are plotted in Figure 4. Remember that in the standard coherent state (5) with a real parameter α , we have $\langle x \rangle = \sqrt{2}\langle \hat{n} \rangle$, while $\sigma_x = 1/2 = const.$

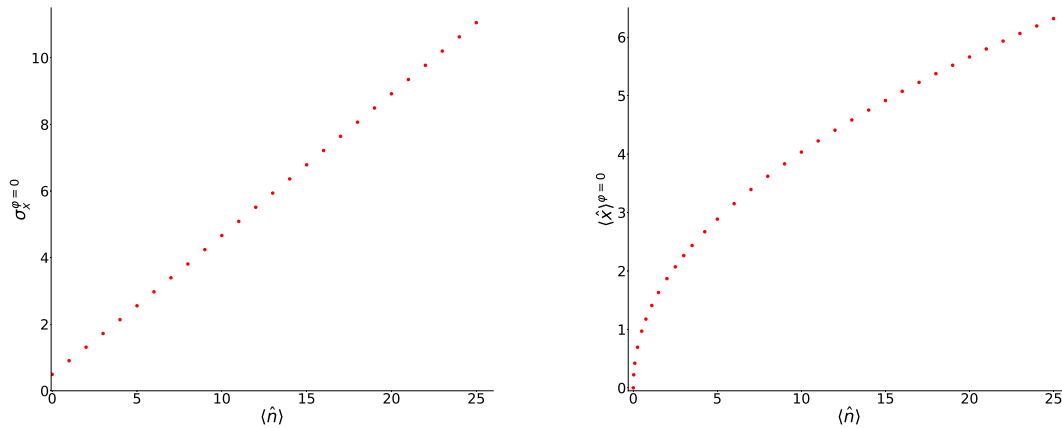


Figure 4. Left: the coordinate variance $\sigma_x^{\varphi=0}$ versus the mean number of quanta $\langle \hat{n} \rangle$ in the coherent phase state (2) (numeric results). Right: The mean value $\langle x \rangle^{\varphi=0}$ in the coherent phase state (2) (numeric results). All numeric results were obtained taking into account 10,000 terms in series S_1 and S_2 .

3. Wave Function and Probability Density

The complex wave function $\psi_\varepsilon(x) \equiv \langle x | \varepsilon \rangle$ is the infinite sum of the Hermite polynomials:

$$\begin{pmatrix} \text{Re}\psi_\varepsilon(x) \\ \text{Im}\psi_\varepsilon(x) \end{pmatrix} = \pi^{-1/4} \sqrt{1 - |\varepsilon|^2} \exp(-x^2/2) \sum_{n=0}^{\infty} \frac{|\varepsilon|^n H_n(x)}{\sqrt{2^n n!}} \times \begin{pmatrix} \cos(n\varphi) \\ \sin(n\varphi) \end{pmatrix}. \quad (22)$$

The most interesting case is $\langle \hat{n} \rangle \gg 1$. Using the results of the preceding section, we can expect that for $\varphi = 0$, the wave function is real and very wide, with the maximum near the point $x_* \approx \sqrt{\eta \langle \hat{n} \rangle}$ and the width of the order of $\sqrt{\sigma_x} \sim \sqrt{D/\sigma_p^{\varphi=0}} \sim \sqrt{\langle \hat{n} \rangle}$ (so that the maximal value of $\psi(x)$ is of the order of $\langle \hat{n} \rangle^{-1/4}$). On the contrary, if $\varphi = \pi/2$, the probability density $|\psi_\varepsilon(x)|^2$ is concentrated nearby the point $x = 0$ in the very narrow region, whose width is of the order of $(2\langle \hat{n} \rangle)^{-1/2}$. Hence, we can expect that the height of the wave function in this case is of the order of $\langle \hat{n} \rangle^{1/4}$.

Illustrations are given in Figures 5 and 6. In the left part of Figure 5, we compare the real wave function $\psi_\varepsilon(x)$ with $\varphi = 0$ and $\langle \hat{n} \rangle = 25$ (i.e., $\varepsilon = \sqrt{25/26}$) with the real coherent state wave function $\psi_\alpha(x)$ (6) with $\alpha = 5$ (i.e., having the same mean number of quanta $\langle \hat{n} \rangle = 25$). The numeric summation in formula (22) was performed for $0 \leq n \leq 1000$. In the right-hand part of the same figure, we compare the probability density $|\psi_\varepsilon(x)|^2$ with the probability density of the Gaussian state,

$$\rho_G(x, x) = (2\pi\sigma_x)^{-1/2} \exp\left[-(x - \langle x \rangle)^2 / (2\sigma_x)\right], \quad (23)$$

having the same mean value of the coordinate $\langle x \rangle = 6.3$ and the same value of the variance $\sigma_x = 10$.

Remember that the width of the coherent state wave function $\psi_\alpha(x)$ does not depend on the shift α (as soon as $\sigma_x \equiv 1/2$ in this case). In addition, the maximum of the coherent state wave function is achieved at the point $x_m = \langle x \rangle = \sqrt{2}\alpha$, and the function is symmetric with respect to this point. On the contrary, the function $\psi_\varepsilon(x)$ shows a strong asymmetry with respect to the point of its maximum $x_m \approx 5 < \langle x \rangle \approx 6.3$, and its width is significantly greater than that of the coherent state.

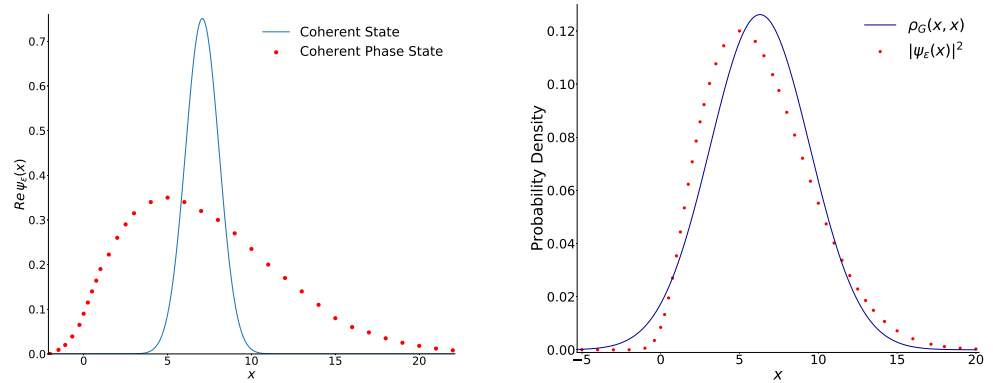


Figure 5. (Left) The real wave function $\psi_\epsilon(x)$ in the case of $\varphi = 0$ and $\langle \hat{n} \rangle = 25$, compared with the coherent state wave function $\psi_\alpha(x)$ with $\alpha = 5$. (Right) The probability density $|\psi_\epsilon(x)|^2$ for $\varphi = 0$ and $\langle \hat{n} \rangle = 25$, compared with the Gaussian probability density (23) with $\langle x \rangle = 6.3$ and $\sigma_x = 10$. Series (22) was calculated with 10,000 terms.

In Figure 6, we plot similar functions for $\langle \hat{n} \rangle = 25$, but now with $\varphi = \pi/2$ and $\alpha = 5i$. The reference Gaussian state in the right-hand side has the values $\langle x \rangle = 0$ and $\sigma_x^{\varphi=\pi/2} = 0.035$. In this figure, the difference between the real parts of the wave functions of the usual coherent state and the phase coherent state is quite impressive.

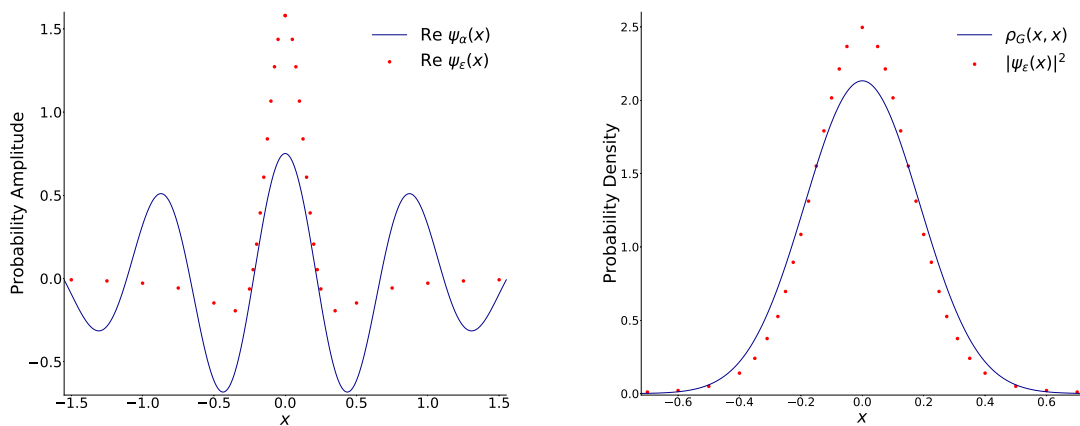


Figure 6. Left: The real part of the wave function $\psi_\epsilon(x)$ in the case of $\varphi = \pi/2$ and $\langle \hat{n} \rangle = 25$, compared with the real part of the coherent state wave function $\psi_\alpha(x)$ with $\alpha = 5i$, i.e., $\psi(x) = \pi^{-1/4} \exp(-x^2/2) \cos(5\sqrt{2}x)$. Right: The probability density $|\psi_\epsilon(x)|^2$ in the case of $\varphi = \pi/2$ and $\langle \hat{n} \rangle = 25$, compared with the probability density $\rho_G(x, x)$ of the Gaussian state with the same values of the coordinate mean value $\langle x \rangle = 0$ and variance $\sigma_x = 0.035$. Series (22) was calculated with 10,000 terms.

On the other hand, the right-hand parts of Figures 5 and 6 show that the *probability density* profiles $|\psi_\epsilon(x)|^2$ seem visually *almost* as Gaussian distributions with the same coordinate mean values and variances. Therefore, it would be useful to have some quantitative measure of the “non-Gaussianity” of the distribution $|\psi_\epsilon(x)|^2$. Several existing measures have been discussed, e.g., in Ref. [80]. For a pure quantum state $|\psi\rangle$, many of those measures are functions of the “fidelity”

$$F_G = \langle \psi | \hat{\rho}_G | \psi \rangle = \int \int dx dy \psi_\epsilon^*(x) \langle x | \hat{\rho}_G | y \rangle \psi_\epsilon(y), \tag{24}$$

where $\hat{\rho}_G$ is the statistical operator of the reference Gaussian state. It is natural to assume that the state $\hat{\rho}_G$ has the same mean values and variances of the quadrature components as the state $|\psi\rangle$ [81]. Unfortunately, the calculation of the double integral (24) meets strong computational difficulties, since the wave function $\psi(x)$ can be calculated only numerically, and the density matrix $\langle x|\hat{\rho}_G|y\rangle$ is not factorized for the mixed Gaussian state. Simple calculations can be performed if one compares the state (2) with its “thermal analog” (3), having the same value of parameter ε . Then, the “thermal fidelity” $F_{th} = \langle \varepsilon|\hat{\rho}_{th}|\varepsilon\rangle$ can be easily found:

$$F_{th} = \frac{1 - |\varepsilon|^2}{1 + |\varepsilon|^2} = (2\langle \hat{n} \rangle + 1)^{-1}. \tag{25}$$

This quantity coincides with the “purity” $\mu = \text{Tr}(\hat{\rho}_{th}^2)$ of the thermal state (3). However, the result (25) seems unsatisfactory, because the probability densities $|\psi_\varepsilon(x)|^2$ and $\langle x|\hat{\rho}_{th}|x\rangle$ are very different for $|\varepsilon| \rightarrow 1$, while the function $|\psi_\varepsilon(x)|^2$ with $\varphi = 0$ or $\varphi = \pi/2$ looks “almost Gaussian” in Figures 5 and 6.

To find a more adequate and simple computable measure of “Gaussianity” we remember that all Gaussian probability distributions in the coordinate space attain their maximal values at the point $x_* = \langle x \rangle$, where $|\psi(x_*)|^2 = (2\pi\sigma_x)^{-1/2}$. Therefore, we use the quantity

$$G = \sqrt{2\pi\sigma_x} |\psi(\langle x \rangle)|^2 \tag{26}$$

as the measure of “Gaussianity” of pure quantum states. According to this definition, distributions can be called as “superGaussian” if $G > 1$ and “subGaussian” if $G < 1$.

The dependence $G(\varepsilon)$ becomes nontrivial. In the limit of $\varepsilon \rightarrow 0$, taking into account only the terms of the order of $|\varepsilon|$ and $|\varepsilon|^2$, we obtain $\langle x \rangle = \sqrt{2} \cos(\varphi)S_1 \approx \sqrt{2} |\varepsilon| \cos(\varphi)$ and the following expressions:

$$|\psi(\langle x \rangle)|^2 \approx \pi^{-1/2} [1 + |\varepsilon|^2 \cos(2\varphi)(1 - \sqrt{2})], \quad \sqrt{2\pi\sigma_x} \approx \pi^{1/2} [1 + |\varepsilon|^2 \cos(2\varphi)(\sqrt{2} - 1)].$$

In this approximation, $G(\varepsilon) \approx 1 + \mathcal{O}(|\varepsilon|^4)$. Then, taking into account the terms of the order of $|\varepsilon|^4$, we obtain the following (probably, unexpected) result (see Appendix A for details):

$$G^{\varphi=\pi/2}(\varepsilon) = G^{\varphi=0}(\varepsilon) \approx 1 + |\varepsilon|^4 (3\sqrt{2} - 3 - \sqrt{3/2}) + \mathcal{O}(|\varepsilon|^6) \approx 1 + 0.018|\varepsilon|^4. \tag{27}$$

Hence, both these extreme states (with $\varphi = 0$ and $\varphi = \pi/2$) are “weakly superGaussian” if $\langle \hat{n} \rangle \ll 1$. However, the behavior of two functions for larger values of $\langle \hat{n} \rangle$ is different, according to the numeric calculations, whose results are shown in Figure 7.

The most interesting behavior is observed for the dependence $G^{\varphi=0}(\langle \hat{n} \rangle)$. This function is slightly larger than 1 if $\langle \hat{n} \rangle < 1$. However, it shows the “subGaussianity” when $\langle \hat{n} \rangle \gg 1$. Unfortunately, we did not succeed in finding good analytic approximations for the functions $G^\varphi(\langle \hat{n} \rangle)$. Therefore, we do not know whether $G^{\varphi=0}(\langle \hat{n} \rangle)$ moves asymptotically to zero or to some finite value when $\langle \hat{n} \rangle \rightarrow \infty$. Moreover, we do not know whether $G^{\varphi=\pi/2}$ grows slowly unlimitedly when $\langle n \rangle \rightarrow \infty$ or whether it approaches some asymptotic finite value.

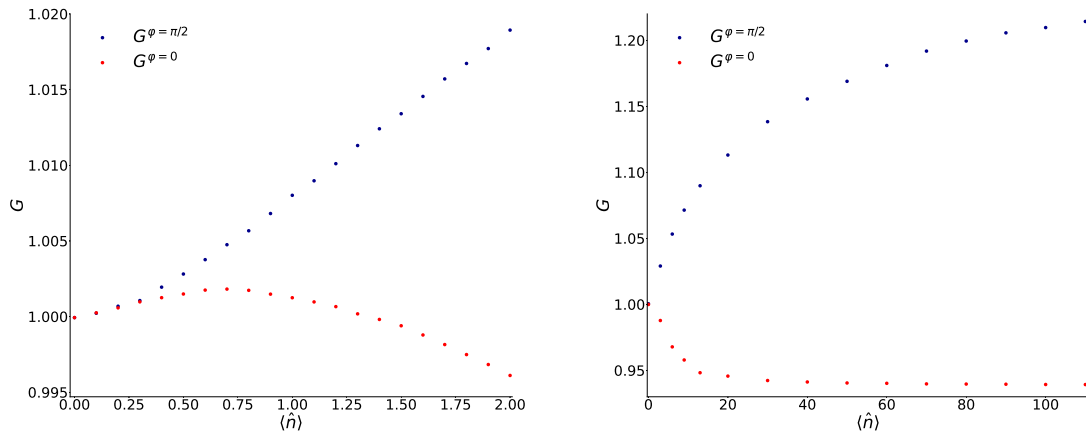


Figure 7. The “Gaussianity” measure (26) as a function of $\langle \hat{n} \rangle$. Numeric calculations were performed with 10,000 terms.

4. Wigner Function

In some cases, the “non-Gaussianity” can be seen distinctly in plots of the Wigner function

$$W(q, p) = \int_{-\infty}^{\infty} dv e^{-ipv} \psi^*(q - v/2) \psi(q + v/2), \tag{28}$$

because $W(q, p)$ must be negative in some regions of the phase space for any pure non-Gaussian quantum state [82]. The Wigner function of the superposition state (2) can be represented as the series over the Weyl–Wigner symbols W_{mn} of the diadic operators $|m\rangle\langle n|$:

$$W = (1 - |\varepsilon|^2) \sum_{m,n=0}^{\infty} |\varepsilon|^{m+n} e^{i(m-n)\varphi} W_{mn}, \tag{29}$$

where [83–86]

$$W_{mn}(q, p) = 2^{1+\lambda/2} (-1)^\mu \sqrt{\frac{\mu!}{\nu!}} b^\lambda e^{-b^2 - i\chi(m-n)} L_\mu^\lambda(2b^2), \tag{30}$$

$$q + ip = be^{i\chi}, \quad b \geq 0, \quad \mu = \min(m, n), \quad \nu = \max(m, n), \quad \lambda = |m - n|.$$

Here, $L_n^\alpha(z)$ is the associated Laguerre polynomial, defined as in [87].

Formulas similar to (29) and (30) were obtained in paper [88] for the “phase state”

$$|\phi\rangle = (2\pi)^{-1/2} \sum_{n=0}^{\infty} e^{in\phi} |n\rangle. \tag{31}$$

However, the state (31) is un-normalizable, and this can lead to some difficulties [2]. In addition, the authors of [88] were interested in the dependence of their Wigner function on phase ϕ only and not in the dependence on the quadrature variables q, p . The 3D plots of the Wigner function of the coherent phase state (2) were shown in paper [50] for $\varepsilon = 0.3$ and $\varepsilon = 0.9$. In the first case, the function looks as a slightly shifted Gaussian hill, without any visible negativity. In the second example, the distribution was clearly non-Gaussian, with one high and three low hills and negative deeps between them. Unfortunately, it is impossible to evaluate the value of negative depth in their 3D figures. Visually, the negativity demonstrated there is not strong.

The combination of Equations (29) and (30) yields the decomposition of the Wigner function in the sum of the ordinary and double series, $W = W_1 + W_2$, where

$$W_1 = 2(1 - |\varepsilon|^2)e^{-b^2} \sum_{n=0}^{\infty} (-|\varepsilon|^2)^n L_n(2b^2), \tag{32}$$

$$W_2 = 4(1 - |\varepsilon|^2)e^{-b^2} \sum_{\mu=0}^{\infty} \sum_{\lambda=1}^{\infty} (-|\varepsilon|^2)^\mu (\sqrt{2}|\varepsilon|b)^\lambda \cos[\lambda(\varphi - \chi)] \sqrt{\frac{\mu!}{(\mu + \lambda)!}} L_\mu^\lambda(2b^2). \tag{33}$$

The “diagonal” sum W_1 is nothing but the Wigner function of the thermal state (3). Indeed, the known generating function of the Laguerre polynomials [87],

$$\sum_{n=0}^{\infty} L_n(x)z^n = (1 - z)^{-1} \exp\left(\frac{xz}{z - 1}\right),$$

results in the Gaussian distribution

$$W_1 = \frac{2}{1 + 2\langle \hat{n} \rangle} \exp\left(-\frac{q^2 + p^2}{1 + 2\langle \hat{n} \rangle}\right). \tag{34}$$

Figure 8 shows the 2D section of the Wigner function, $W(q, 0; \varphi)$, for different values of the phase φ and the mean quantum number $\langle n \rangle$. Plots of other 2D sections, such as $W(0, p)$ for example, look similar, as soon as they can be obtained from $W(q, 0; \varphi)$ by means of some rotations in the phase plane (q, p) . The summation in formula (33) was performed for $\mu \leq 110$ and $\lambda \leq 110$ in the case of $\langle n \rangle = 30$. Increasing the value of $\langle n \rangle$, we see how the initial Gaussian shape becomes more and more deformed, and negative values of the Wigner function become more visible. However, these negative values remain relatively small even for large values of $\langle n \rangle$.

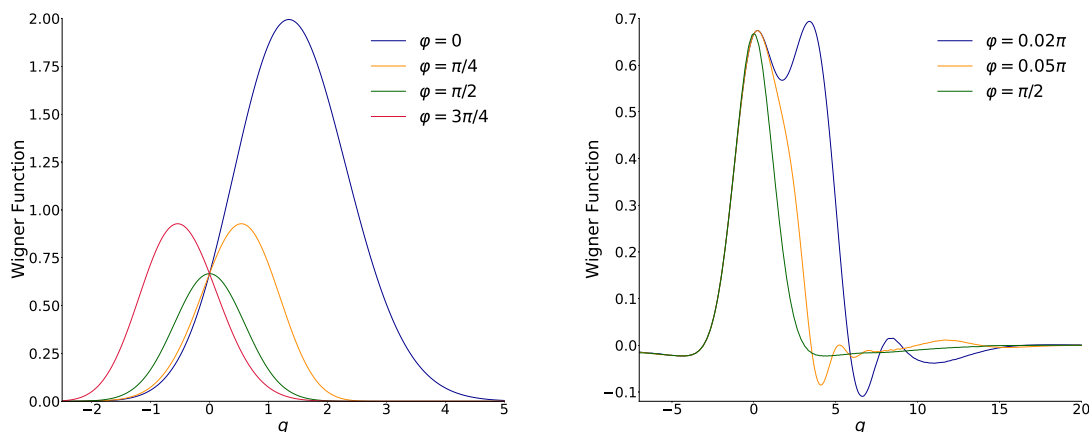


Figure 8. 2D sections of the Wigner function $W(q, 0)$ for some values of the phase φ . Left: for $\langle \hat{n} \rangle = 1$. Right: for $\langle \hat{n} \rangle = 30$.

5. Conclusions

The results of the preceding sections showed that some properties of the coherent phase states (CPS) were very different from the properties of the usual Klauder–Glauber–Sudarshan coherent states (CS), while some other properties were more or less similar. The main difference was in the squeezing properties and the uncertainty products. While the variances of the dimensionless quadrature components did not depend on the parameter α , in the case of standard CS, these variances were very sensitive to the parameter ε , in the case of CPS. Namely, a strong squeezing was observed for certain values of the phase of

complex number ε , when $|\varepsilon|$ was close to unity. This squeezing was only twice weaker than for the ideal vacuum squeezed states with the same mean number of photons.

Another difference was in the value of the Robertson–Schrödinger uncertainty product (16). This product attained the minimal possible value $1/4$ in all the usual coherent states. On the other hand, it grew slowly (but unlimitedly) with an increase in $|\varepsilon|$ (without any dependence on the phase ϕ of the complex number ε), as shown in Equation (20).

In addition, the shapes of the wave functions $\psi_\alpha(x)$ and $\psi_\varepsilon(x)$ were quite different when the mean photon number $\langle n \rangle$ was large enough, as shown in Figures 5 and 6). However, the shapes of the probability density $|\psi_\varepsilon(x)|^2$ or the Wigner function were not very far from the Gaussians, at least for not extremely high values of $\langle n \rangle$, according to Figures 7 and 8. It would be interesting to know whether the non-Gaussianity remains weak in the limit $\langle n \rangle \rightarrow \infty$. The problem is the extremely slow convergence of series (22) and (33) when $|\varepsilon| \rightarrow 1$. We used up to 10,000 terms in the numeric calculations of the “Gaussianity” parameter G with $\langle n \rangle \leq 200$, but we did not succeed in performing reliable calculations for substantially higher values of $\langle n \rangle$. Calculations of the Wigner function were even more involved; we used 110×110 terms in the double sum (33) to obtain a reasonable accuracy for $W(q, 0)$ in the case of $\langle n \rangle = 30$; it was difficult to move to higher values of $\langle n \rangle$.

Author Contributions: M.C.d.F.: analytical and numerical calculations and plotting figures; V.V.D.: conceptualization, methodology, analytical calculations, and writing the paper. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: M.C.d.F. acknowledges the support of the CNPq grant no. 144461/2021-8. V.V.D. acknowledges the partial support of the Brazilian funding agency Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Calculations of the Gaussianity

If $\phi = 0$, then

$$\begin{aligned} \langle x \rangle &= \sqrt{2}S_1 = \sqrt{2}|\varepsilon|(1 - |\varepsilon|^2) \left[1 + \sqrt{2}|\varepsilon|^2 + \sqrt{3}|\varepsilon|^4 + \dots \right] \\ &= \sqrt{2}|\varepsilon| \left[1 + (\sqrt{2} - 1)|\varepsilon|^2 + (\sqrt{3} - \sqrt{2})|\varepsilon|^4 + \dots \right], \end{aligned} \quad (\text{A1})$$

$$S_1^2 = |\varepsilon|^2 \left[1 + 2(\sqrt{2} - 1)|\varepsilon|^2 + (3 + 2\sqrt{3} - 4\sqrt{2})|\varepsilon|^4 + \dots \right], \quad (\text{A2})$$

$$S_1^4 = |\varepsilon|^4 \left[1 + 4(\sqrt{2} - 1)|\varepsilon|^2 + \dots \right], \quad (\text{A3})$$

so that

$$|\psi_\varepsilon(\langle x \rangle)|^2 = \pi^{-1/2} (1 - |\varepsilon|^2) \exp(-2S_1^2) \left(\sum_{n=0}^{\infty} \frac{|\varepsilon|^n H_n(\sqrt{2}S_1)}{\sqrt{2^n n!}} \right)^2 = \pi^{-1/2} E H^2, \quad (\text{A4})$$

$$E = (1 - |\varepsilon|^2) (1 - 2S_1^2 + 2S_1^4 + \dots) = 1 - 3|\varepsilon|^2 + |\varepsilon|^4(8 - 4\sqrt{2}) + \dots, \quad (\text{A5})$$

$$\begin{aligned}
H &= 1 + \frac{|\varepsilon|}{\sqrt{2}}(2\sqrt{2}S_1) + \frac{|\varepsilon|^2}{\sqrt{8}}[4(\sqrt{2}S_1)^2 - 2] + \frac{|\varepsilon|^3(\sqrt{2}S_1)}{\sqrt{48}}[8(\sqrt{2}S_1)^2 - 12] \\
&+ \frac{|\varepsilon|^4}{\sqrt{384}}[16(\sqrt{2}S_1)^4 - 48(\sqrt{2}S_1)^2 + 12] + \dots \\
&= 1 + 2|\varepsilon|S_1 - \frac{|\varepsilon|^2}{\sqrt{2}} + \sqrt{8}(|\varepsilon|S_1)^2 - \sqrt{6}|\varepsilon|^3S_1 + \sqrt{\frac{3}{8}}|\varepsilon|^4 + \dots \\
&= 1 + |\varepsilon|^2\left(2 - \frac{1}{2}\sqrt{2}\right) + |\varepsilon|^4\left(4\sqrt{2} - 2 - \sqrt{6} + \sqrt{\frac{3}{8}}\right) + \dots \tag{A6}
\end{aligned}$$

$$H^2 = 1 + |\varepsilon|^2(4 - \sqrt{2}) + |\varepsilon|^4\left(6\sqrt{2} + \frac{1}{2} - 2\sqrt{6} + \sqrt{\frac{3}{2}}\right) + \dots \tag{A7}$$

$$EH^2 = 1 + |\varepsilon|^2(1 - \sqrt{2}) + |\varepsilon|^4\left(5\sqrt{2} - \frac{7}{2} - 3\sqrt{\frac{3}{2}}\right) + \dots \tag{A8}$$

$$\begin{aligned}
\sigma_x &= \frac{1}{2} + \frac{|\varepsilon|^2}{1 - |\varepsilon|^2} - 2S_1^2 + S_2 \\
&= \frac{1}{2} + |\varepsilon|^2 + |\varepsilon|^4 - 2|\varepsilon|^2[1 + 2(\sqrt{2} - 1)|\varepsilon|^2] + |\varepsilon|^2(1 - |\varepsilon|^2)(\sqrt{2} + \sqrt{6}|\varepsilon|^2) + \dots \\
&= \frac{1}{2} + |\varepsilon|^2(\sqrt{2} - 1) + |\varepsilon|^4(5 - 5\sqrt{2} + \sqrt{6}) + \dots \tag{A9}
\end{aligned}$$

$$\sqrt{2\sigma_x} = 1 + |\varepsilon|^2(\sqrt{2} - 1) + |\varepsilon|^4\left(\frac{7}{2} - 4\sqrt{2} + \sqrt{6}\right) + \dots \tag{A10}$$

$$G^{\phi=0}(\varepsilon) = 1 + |\varepsilon|^4\left(3\sqrt{2} - 3 - \sqrt{\frac{3}{2}}\right) + \dots \approx 1 + 0.018|\varepsilon|^4 + \dots \tag{A11}$$

If $\varphi = \pi/2$,

$$\begin{aligned}
\pi^{1/2}|\psi(0)|^2 &= (1 - |\varepsilon|^2)\left(1 + \frac{|\varepsilon|^2}{\sqrt{2}} + |\varepsilon|^4\sqrt{\frac{3}{8}}\right)^2 + \dots \\
&= 1 + (\sqrt{2} - 1)|\varepsilon|^2 + |\varepsilon|^4\left(\frac{1}{2} + \sqrt{\frac{3}{2}} - \sqrt{2}\right) + \dots \tag{A12}
\end{aligned}$$

$$\sigma_x = N - S_2 = \frac{1}{2} + (1 - \sqrt{2})|\varepsilon|^2 + |\varepsilon|^4(1 + \sqrt{2} - \sqrt{6}) + \dots \tag{A13}$$

$$\sqrt{2\sigma_x} = 1 + (1 - \sqrt{2})|\varepsilon|^2 + |\varepsilon|^4\left(2\sqrt{2} - \frac{1}{2} - \sqrt{6}\right) + \dots \tag{A14}$$

$$G^{\phi=\pi/2}(\varepsilon) = 1 + |\varepsilon|^4\left(3\sqrt{2} - 3 - \sqrt{\frac{3}{2}}\right) + \dots \approx 1 + 0.018|\varepsilon|^4 + \dots \tag{A15}$$

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