이학박사 학위논문

Kaluza-Klein Compactification of Higher Spin Gauge Theory

고차 스핀 게이지 이론의 칼루자-클라인 콤팩트화

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이 논문을 이학박사 학위논문으로 제출함

2016년 6월

서울대학교 대학원

물리천문학부

김재원

김재원의 박사 학위논문을 인준함

2016년 8월

위 원 장 ________________ (인)

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Abstract

Kaluza-Klein Compactification of Higher Spin Gauge Theory

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A massive higher spin particles are expected to play an important role in describing the quantum theory of gravity. Though its long history and absence of a no-go result, construction of fully consistent interacting massive higher spin theory has not been achieved apart from the String theory due to the technical difficulties related to the number of degrees of freedom and superluminal propagation. In this thesis, we propose novel Kaluza-Klein compactification of massless higher spin theory to achieve interacting massive higher spin theory avoiding technical difficulties. To guarantee the consistency we targeted the Vasiliev theory and that forced us to use anti-de Sitter spacetime as a background. Compactification on anti-de Sitter background causes several interesting problems like higher derivative boundary condition. We analyze it in terms of additional boundary degrees of freedom and succeed to get a lower dimensional massive theory.

Keywords : higher spin, higher spin gauge, massive higher spin, Kaluza-Klein, Kaluza-Klein with boundary, (Anti-)de Sitter space
Student Number : 2009-20405

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Chapter 1

Introduction

Finding a description of quantum gravity is one of the ultimate goals of theoretical physics. Though the String theory provides a toy model for quantum gravity, but formulation apart from the String theory is still far-off. Among many nice properties of the String theory, we pay attention to the existence of infinitely many massive higher spin particles. In field theory point of view, the success of the String theory is a UV completion or resolving the non-renormalizability issue of the Einstein gravity. Massive higher spin particles, whose masses and couplings are tuned properly, may render the UV behavior of gravity theory and can be one of the essential ingredients for such success. In this context, studying interacting massive higher spin theory looks promising.

Also recently, it is shown that the Einstein gravity with higher derivative correction has causality violation and exchange of infinitely mass massive higher spin particle can cure the causality issue [1]. This is exactly what happen to the String theory which contains higher derivative correction to gravity sector and suggests the importance of massive higher spin particles in gravity theory.

Apart from quantum gravity, massive higher spin particles are observed in a laboratory as a hadronic resonance. Though they are considered as composites rather than elementary particles, in IR limit, they should be considered elementary degrees of freedom and there must be an effective theory of them.

In spite of such importance of massive higher spin theory, interacting theory of massive higher spin particles is not understood well and fully consistent theory is not known yet except the String theory. The reason of poor understanding is due to its technical difficulties rather than fundamental obstruction. Actually, there are lots of no-go theorems for interacting massless higher spin theory [2, 3, 4], but for massive higher spin theory, there is none. Instead of the no-go theorem, there is technical obstruction called "Velo-Zwanziger" problem [5, 6, 7]. When one try to turn on the interaction, there might happen 2 serious problems: unphysical new propagating degrees of freedom may appear and superluminal non-causal propagation may appear. These problems can happen even for interacting with background fields, and additional degrees of freedom or non-minimal coupling should be introduced. A perturbative way of constructing interacting massive higher spin theory is known [8, 9], but a non-perturbative resolution is still not known.

Ironically, in spite of no-go theorems, fully consistent interacting massless higher spin theory was discovered by M.A. Vasiliev [10, 11, 12]. Almost of no-go theorems can be evaded by considering (anti-)de Sitter spacetime as a background. De Sitter or anti-de Sitter spacetime is maximally symmetric curved spacetime where S-matrix cannot be defined or measured. Almost of no-go theorems are using S-matrix argument, therefore (anti-)de Sitter spacetime provide a nice background for higher spin theory. Actually, Velo-Zwanziger type argument also can apply to massless theory. However, in contrast to massive theory, a massless theory has a guidance to keep the number of degrees of freedom and prevent superluminal propagation: the gauge symmetry. Gauge symmetry acts as constraints and might be an obstruction for constructing a theory. At the same time, gauge symmetry controls the number of physical degrees of freedom and eliminate unphysical degrees of freedom. Therefore as long as one keep the gauge invariance of the theory, one does not have to consider Velo-Zwanziger problem. Vasiliev first wrote down higher spin gauge algebra and then found a systematic way of constructing gauge invariant theory.

Our idea is to study interacting massive higher spin theory from the Vasiliev theory using Kaluza-Klein compactification. We expect that consistency of the Vasiliev theory is inherited to lower dimensional massive theory and we can by-pass all the technical difficulties. Once we find a proper way of doing compactification, it will provide a nice shortcut to interacting massive higher spin theory. As a first step, we consider Kaluza-Klein compactification of free higher spin theory on anti-de Sitter background. Kaluza-Klein compactification on anti-de Sitter background was studied very little and has lots of interesting features. We claim that circular compactification is not available and boundaries should be introduced. The existence of boundary causes many technical difficulties together with rich structures. One of our main results is that higher spin fields require higher derivative boundary conditions. A higher derivative boundary condition is unusual boundary condition and we find an equivalent description using boundary degrees of freedom. Boundary degrees of freedom gives us a physical understanding of higher derivative boundary condition and we can tell which boundary condition is unitary. We find proper boundary conditions and lower dimensional spectrum depending on the boundary condition, that is the main result of this thesis.

The rest of the thesis is organized as follow. In chapter 2, a brief review of the higher spin theory is given. We introduce a free theory of both massless and massive higher spin. Then the obstruction for constructing interacting theory is discussed and the Vasiliev theory is introduced. This chapter supplements our motivation for doing Kaluza-Klein compactification to circumvent technical difficulties. In chapter 3, compactification of background metric is discussed. We explain the inevitability of introducing boundaries and find a parametrization of the background metric for compactification. Using such parametrization, we give a spin-2 example of compactification and discuss relevant issues which appear in the higher spin case again. In chapter 4, we do Kaluza-Klein compactification of higher spin. First, we find linear combinations of lower dimensional fields which have correct symmetric properties. Then study their gauge transformation to see the lower dimensional spectrum. Various boundary conditions and lower dimensional spectra depending on boundary condition are given. In chapter 5, we analyze higher derivative boundary conditions which are inevitable for the higher spin field. We develop an extended inner product which translates higher derivative boundary condition into corresponding boundary action. With boundary action, we classify boundary conditions into the unitary and non-unitary boundary condition. In chapter 6, we conjecture CFT dual of higher spin theory with boundary based on their spectrum.

Chapter 2

Review on Higher Spin Theory

In this chapter, a brief review of massive and massless higher spin theory is given. Historically, a massive higher spin theory was studied first by Fierz and Pauli [13], and then massless higher spin theory was obtained as a limit of massive theory by Fronsdal and Fang [14, 15]. However, we introduce massless higher spin theory first using gauge symmetry. The free massless higher spin theory is introduced and then the only known interacting massless higher spin theory, the Vasiliev theory, is reviewed. Also, the free massive higher spin theory is introduced and the obstruction for bottom-up construction of interacting massive higher spin theory, so-called "Velo-Zwanziger" problem is reviewed.

2.1 Massless Higher Spin Theory

The most important feature of the massless higher spin theory is that it is a gauge theory. Gauge symmetry gives a guidance or constraints to construct a theory. Especially one does not have to consider an issue of the number of degree of freedom as long as one keep the gauge invariance. For both free and interacting theory, we assume gauge symmetry at the beginning.

2.1.1 Free massless theory

Massless spin-s field of free theory can be considered as a generalization of the Maxwell field and linearized Einstein gravity. It is a symmetric rank s tensor to be a representation of Poincare group. One interesting feature is that it is a reducible representation. Fields of irreducible representation has totally symmetric and traceless indices.

$$\phi^{irr}_{\mu_1\mu_2\cdots\mu_s}\eta^{\mu_1\mu_2} = 0 \tag{2.1}$$

However massless spin-s field, introduced by Fronsdal, is double traceless rather than traceless.

$$\phi_{\mu_1\mu_2\cdots\mu_s}\eta^{\mu_1\mu_2}\,\eta^{\mu_3\mu_4} = 0 \tag{2.2}$$

This field can be understood as a linear combination of traceless rank s tensor and traceless rank $(s-2)$ tensor. Note that fluctuation of Einstein field $h_{\mu\nu}$ is not traceless. Gauge transformation is given as

$$\delta_\xi \phi_{\mu_1\cdots\mu_s} = \partial_{(\mu_1}\xi_{\mu_2\cdots\mu_s)} \tag{2.3}$$

where $\xi_{\mu_1\cdots\mu_{s-1}}$ is a gauge parameter and it is symmetric traceless rank $(s-1)$ tensor. The notation $(\mu_1\cdots\mu_s)$ is total symmetrization of indices with normalization factor $\frac{1}{s!}$. Following action is invariant for the gauge transformation on flat background.

$$\begin{aligned}\mathcal{S} = -\frac{1}{2}\int \mathrm{d}^d x \Big(\partial_\mu\phi_{\nu_1\cdots\nu_s}\partial^\mu\phi^{\nu_1\cdots\nu_s} - \frac{s(s-1)}{2}\partial_\mu\phi_\rho{}^\rho{}_{\nu_1\cdots\nu_{s-2}}\partial^\mu\phi_\sigma{}^{\sigma\nu_1\cdots\nu_{s-2}} \\ -s\,\partial_\rho\phi_{\sigma\nu_1\cdots\nu_{s-1}}\partial^\sigma\phi^{\rho\nu_1\cdots\nu_{s-1}} + s(s-1)\,\partial_\mu\phi_\rho{}^\rho{}_{\nu_1\cdots\nu_{s-2}}\partial_\sigma\phi^{\sigma\mu\nu_1\cdots\nu_{s-2}} \\ -\frac{s(s-1)(s-2)}{4}\,\partial_\mu\phi^\mu{}_\rho{}^\rho{}_{\nu_1\cdots\nu_{s-3}}\partial_\kappa\phi^\kappa{}_\sigma{}^{\sigma\nu_1\cdots\nu_{s-3}}\Big)\end{aligned} \tag{2.4}$$

This action is called Fronsdal action and is fixed up to an overall factor by the gauge transformation (2.3).

For generic background, naive replacement $\partial_\mu \to \nabla_\mu$ does not work. After replacement, ∇_μ's are not commute each other and the action (2.4) is no longer gauge invariant for $s > 2$. Lack of gauge invariance means that unphysical degrees of freedom appears when higher spin fields couple to background graviton minimally. This is very special feature of spin greater than 2 since it is known that lower spin fields can propagate on any background by minimal coupling with background graviton. For dynamical graviton, situation becomes more complecated []. However, for constant curvature background like anti-de Sitter spacetime, gauge invariance can be restored by adding mass-like terms to action []. For d-dimensional anti-de Sitter, AdS$_d$, spacetime which is dominantly used for this thesis, the additional mass-like terms are

$$\begin{aligned}\Delta\mathcal{L} &= m_1\,\phi_{\nu_1\cdots\nu_s}\phi^{\nu_1\cdots\nu_s} + m_2\,\phi_\rho{}^\rho{}_{\nu_1\cdots\nu_{s-2}}\phi_\sigma{}^{\sigma\nu_1\cdots\nu_{s-2}}\,, \\ m_1 &= -\frac{1}{2\,r_{AdS}^2}\Big((s-1)(s-2)+s(d-1)\Big)\,, \\ m_2 &= \frac{s(s-1)}{4r_{AdS}^2}\Big(s(s-3)+(s-1)(d-1)\Big) + \frac{s-1}{r_{AdS}^2}\,,\end{aligned} \tag{2.5}$$

where r_{AdS} is the radius of AdS spacetime. These mass-like terms cancel the effect from curved background and give correct higher spin gauge field analogous to massless field on flat background. Values of m_1 and m_2 also can be found using representation theory of AdS isometry group $so(d-1,2)$ [16].

This is the free massless theory of higher spin which is our starting point of Kaluza-Klein compactification. During compactification, the action (2.4) is rarely used. Instead, the gauge transformation (2.3) on a curved background is used since it dictates the action at the free level. The action only appears when we consider boundary action.

2.1.2 Interacting massless theory

For higher spin field, turnning on interaction even with non-dynamical background field can cause inconsistency. Finding self-interacting higher spin theory is much more evolved and there are lots of no-go theorems. Before introducing only known interacting massless higher spin theory, the Vasiliev theory, imiportant no-go theorems should be reviewed.

Weinberg(1964) [2] Weinberg used factorization property of S-matrix at soft limit to show that coupling constant for massless higher spin should vanish. Consider a S-matrix of a massless spin-s particle and N other particles, $S(p_1, \ldots, p_N; q, \epsilon)$, where q_μ, $\epsilon_{\mu_1 \cdots \mu_s}$ are momentum and polarization tensor of spin-s particle and p_μ's are momenta of other particles. At soft limit of q_μ, S-matrix is factorized into S-matrix of N other particles and soft factor of spin-s particle.

$$\lim_{q \to 0} S(p_i; q, \epsilon) = \sum_{i=1}^{N} g_i \left(\frac{p_i^{\mu_1} \cdots p_i^{\mu_s} \epsilon_{\mu_1 \cdots \mu_s}}{2q \cdot p_i} \right) S(p_i) \tag{2.6}$$

g_i is coupling constant between i-th particle and spin-s field or spin-s charge of i-th particle. Under the Lorentz transformation, polarization tensor is actually not a tensor and gives non-tensor piece due to its unphysical components, therefore Lorentz invariance of S-matrix is not manifest. Instead, following constraint appear.

$$\sum_{i=1}^{N} g_i \, p_i^{\mu_1} \cdots p_i^{\mu_{s-1}} = 0. \tag{2.7}$$

For $s = 1$, it just becomes usual charge conservation of the scattering process, $\sum_i g_i = 0$. For $s = 2$, together with momentum conservation $\sum_i p_i = 0$, it becomes quantum version of equivalence principle, $g_1 = g_2 = \cdots = g$. However for $s > 3$, there is no solution of (2.7) except $g_i = 0$, and this gives no-go theorem of interacting massless higher spin.

Aragone-Deser(1979) [3] Aragon and Deser considered the interaction between high spin particle and graviton. They showed an action of a higher spin particle minimally coupled to gravity is not invariant under higher spin gauge transformation. Such gauge transformation gives terms proportional to the Riemann tensor and unphysical modes cannot be decoupled when the Riemann tensor is non-vanishing. They also showed that any local non-minimal coupling does not change the result. The issue of coupling with background metric which was in 2.1.1 is one of the examples.

Weinberg-Witten(1980) [4] Using S-matrix argument, Weinberg and Witten showed that a particle with $s > 1$ cannot have Lorentz covariant energy-momentum tensor. Suppose there is a such energy-momentum tensor $T_{\mu\nu}$ for spin-s particle. Consider a matrix element of $T_{\mu\nu}$ for initial and final state of spin-s particle with momemtum p_i, p_f and helicity $+s$, $+s$. From the equivalence principle, one can show that

$$\lim_{q\to 0} \langle p_f, +s|\, T_{\mu\nu}\, |p_i, +s\rangle = p_\mu p_\nu \neq 0 \tag{2.8}$$

where $q \equiv p_f - p_i$. However one also can show that $\langle p_f, +s|\, T_{\mu\nu}\, |p_i, +s\rangle = 0$ for any space-like q_μ and this gives contradiction. For simplicity, let's consider 4 dimension example. To show the latter, one should consider specific frame such that $q^\mu = (0, -\vec{q})$, $p_i^\mu = (\frac{1}{2}|\vec{q}|, \frac{1}{2}\vec{q})$, and $p_f^\mu = (\frac{1}{2}|\vec{q}|, -\frac{1}{2}\vec{q})$. Decompose the $T_{\mu\nu}$ as spherical tensor then one get $T_{l,m}$ where $l = 0, 1$ and $m = 0, \pm 1, \cdots, \pm l$ since a symmetirc tracefull tensor $T_{\mu\nu}$ has spin-1 and spin-2 components. Consider a rotation $R(\theta)$ along the $\vec{q}$ direction,

$$\begin{aligned}\langle p_f, +s|\, R^\dagger\, T_{l,m}\, R\, |p_i, +s\rangle &= e^{i\theta m} \langle p_f, +s|\, T_{l,m}\, |p_i, +s\rangle \\ &= e^{\pm 2i\theta s} \langle p_f, +s|\, T_{l,m}\, |p_i, +s\rangle.\end{aligned} \tag{2.9}$$

The first equlity comes from rotational property of the $T_{m,l}$ and the second equlity comes from rotational property of the states. This should hold for generic value of θ, and every component of $T_{m,l}$ vanishes when $s > 1$. Note that this no-go theorem also can be applied to spin-2, however non-covariant transformation of $T_{\mu\nu}$ can be canceled by diffeomorphism which is not the case of $s > 2$.

Except the Aragone-Deser no-go theorem, other no-go theorems are based on S-matrix argument. Those no-go theorems can be evaded by considering (anti-)de Sitter background where S-matrix cannot be defined or observed. Also, the Aragone-Deser theorem can be evaded by introducing non-locality. Vasiliev considered (A)dS background[1] and infinitely many derivatives which are non-local. Evading no-go theorems opens the chance of constructing the theory but does not instruction how to construct. The possibility of appearing unphysical degrees of freedom is still exist. For massless theory, there is very powerful

[1]His formulation is background independent, however, it gives a theory of higher spin only when it is expanded around the curved background.

guidance: gauge symmetry. As long as the gauge invariance of the theory is maintained, the number of degrees of freedom are automatically controlled. Vasiliev considered interacting higher spin gauge algebra which is a generalization of the diffeomorphism and found a systematic way of constructing a theory which keeps the gauge invariance manifestly.

The higher spin algebra depends on dimension of spacetime very much and we only introduce 4 dimensional higher spin algebra, $\mathbf{hs}(4)$. To introduce the higher spin algebra as a generalization of diffeomorphism, we introduce the oscillator realization of $\mathbf{so}(2,3)$ algebra. It is algebra of isometry group of AdS$_4$ spacetime which is natural background for higher spin theory. The generators of $\mathbf{so}(2,3)$ algebra are transvection generators P_a and rotation generators M_{ab}. By spinor notation they are $P_{\alpha\dot{\beta}}$, $M_{\alpha\beta}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}}$. All the $\mathbf{so}(2,3)$ algebra can be realized by

$$P_{\alpha\dot{\beta}} = \hat{y}_\alpha \hat{\bar{y}}_{\dot{\beta}}\,, \quad M_{\alpha\beta} = \frac{1}{2}\{\hat{y}_\alpha, \hat{y}_\beta\}\,, \quad \bar{M}_{\dot{\alpha}\dot{\beta}} = \frac{1}{2}\{\hat{\bar{y}}_{\dot{\alpha}}, \hat{\bar{y}}_{\dot{\beta}}\} \tag{2.10}$$

where

$$[\hat{y}_\alpha, \hat{y}_\beta] = 2i\epsilon_{\alpha\beta}\,, \quad [\hat{\bar{y}}_{\dot{\alpha}}, \hat{\bar{y}}_{\dot{\beta}}] = 2i\epsilon_{\dot{\alpha}\dot{\beta}}\,, \quad [\hat{y}_\alpha, \hat{\bar{y}}_{\dot{\beta}}] = 0\,. \tag{2.11}$$

Using $\hat{y}_\alpha$ and $\hat{\bar{y}}_{\dot{\alpha}}$, one can construct more general operators, for instance $T_{\alpha\beta\dot{\alpha}} = \{\hat{y}_\alpha, \hat{y}_\beta\}\hat{\bar{y}}_{\dot{\alpha}}$. Actually all the irreducible representation of $\mathbf{so}(2,3)$ algebra can be constructed including higher spin generator. Collect every operators which can be made by $\hat{y}_\alpha$ and $\hat{\bar{y}}_{\dot{\alpha}}$ then they form $\mathbf{hs}(4)$ algebra. The fact that $\mathbf{hs}(4)$ comes from AdS$_4$ isometry algebra shows the intrinsic relation between AdS background and higher spin theory. Since the range of the spinor index is 1 to 2, antisymmetric combination of $\hat{y}$'s are not needed. A nice way of considering such generators is using commuting "symbol" y_α, $\bar{y}_{\dot{\alpha}}$ and "star-product" between symbols $y \star y$. Symbols are commuting variable and generators written by symbols have automatically symmetrized indices. Star-product is associative but not commutative and defined to give correct operator product.

$$M_1(\hat{y}) \cdot M_2(\hat{y}) = \sum_i c_{12i}\, M_i(\hat{y}) \quad \Rightarrow \quad M_1(y) \star M_2(y) = \sum_i c_{12i}\, M_i(y) \tag{2.12}$$

Practical definition of star-product is,

$$(P \star Q)(y, \bar{y}) = \frac{1}{(2\pi)^4} \int \mathrm{d}^4u\, \mathrm{d}^4v\, P(y+u, \bar{y}+\bar{u})\, Q(y+v, \bar{y}+\bar{v}) e^{i(u_\alpha v^\alpha + \bar{u}^{\dot{\alpha}} \bar{v}_{\dot{\alpha}})} \tag{2.13}$$

where d^4u is a shorthand for $\mathrm{d}^2u\, \mathrm{d}^2\bar{u}$. One can check that above definition gives correct commutator for y's,

$$[y_\alpha, y_\beta]_\star = 2i\epsilon_{\alpha\beta}\,, \quad [\bar{y}_{\dot{\alpha}}, \bar{y}_{\dot{\beta}}]_\star = 2i\epsilon_{\dot{\alpha}\dot{\beta}}\,, \quad [y_\alpha, \bar{y}_{\dot{\beta}}]_\star = 0 \tag{2.14}$$

where $[A, B]_\star \equiv A \star B - B \star A$.

When the gauge algebra is given, it is useful to consider the "connection 1-form" of algebra. For spin-1, connection 1-form is usual gauge field A_μ. Note that for non-abelian gauge theory, $A_\mu = \sum_a A^a_\mu T^a$ where T^a's are generators of algebra. Also a spin-2 theory can be written by connection 1-form, $W_\mu = e^a_\mu P_a + \frac{1}{2}.\omega^{ab}_\mu M_{ab}$. e^a_μ is gauge field for transvection, called vierbein and ω^{ab}_μ is gauge field for rotation, called spin connection. Field strength 2-form of W_μ is,

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu]_\star \\ &= \mathcal{T}_{\mu\nu}{}^a P_a + \frac{1}{2}\mathcal{R}_{\mu\nu}{}^{ab} M_{ab}\,, \end{aligned} \tag{2.15}$$

or $F = \mathrm{d}W + W \wedge_\star W$ using compact notation. $\mathcal{T}^a$ and $\mathcal{R}^{ab}$ are torsion and curvature 2-form repectively. With these quantities one can write down action or equation of motion. Such formalism is gravity theory as a gauge theory and called "Frame-like" formalism of gravity. What Vasiliev did is generalization of frame-like formalism for higher spin. Connection 1-form of higher spin algebra is

$$\mathcal{W}(y, \bar{y}|x) = \sum_{n,m}^{n+m=\text{even}} \frac{1}{2in!m!}\, \omega^{\alpha_1\cdots\alpha_n\dot{\beta}_1\cdots\dot{\beta}_m}(x)\, y_{\alpha_1}\cdots y_{\alpha_n}\bar{y}_{\dot{\beta}_1}\cdots\bar{y}_{\dot{\beta}_m}\,. \tag{2.16}$$

Only $n + m =$ even generators considered since only bosonic quantities are considered. To construct gauge invariant and consistent system, Vasiliev invented "Unfolded Formulation". Unfolded formulation is consist of p-form fields in general. However system with 1-form and 0-form is already quite general anda able to write down higher spin theory.

$$\mathrm{d}W = W \wedge_\star W \tag{2.17}$$

$$\mathrm{d}B = W \star B - B \star W \tag{2.18}$$

B is 0-form and contains scalar fields and curvatures of higher spin fields which are generalization of the Weyl tensor of spin-2. For a gauge parameter $\epsilon(y, \bar{y}|x)$, above system is gauge invariant,

$$
\begin{aligned}
\delta W &= \mathrm{d}\epsilon - W \star \epsilon + \epsilon \star W \,, \\
\delta B &= \epsilon \star B - B \star \epsilon \,.
\end{aligned}
\tag{2.19}
$$

Now it is time to introduce the Vasiliev equation. To describe interacting higher spin theory, auxiliary oscillator z_α, $\bar{z}_{\dot{\alpha}}$ and 0-form auxiliary field $S(y, z|x)$. Consistent fully non-linear higher spin equation, Vasiliev equations are

$$
\begin{aligned}
\mathrm{d}W &= W \wedge_\star W \,, \\
\mathrm{d}B &= W \star B - B \star \tilde{W} \,, \\
\mathrm{d}S &= W \star S - S \star W \,, \\
S \star B &= B \star \tilde{S} \,, \\
S \star S &= \mathrm{d}z^\alpha \mathrm{d}z_\alpha \,(i + B \star \kappa) + \mathrm{d}\bar{z}^{\dot{\alpha}} \mathrm{d}\bar{z}_{\dot{\alpha}} \,(i + B \star \bar{\kappa}) \,.
\end{aligned}
\tag{2.20}
$$

z and $\bar{z}$ has commutator relation with opposite sign, $[y, y] = -[z, z]$. $\mathrm{d}z$ is anticommuting 1-form. κ is defined as $\kappa = exp(i\, z_\alpha y^\alpha)$. At last, tilde notation is defined as

$$
\tilde{f}(z, \bar{z}, y, \bar{y}) = f(-z, \bar{z}, -y, \bar{z}) \,. \tag{2.21}
$$

Tilde is not necessary for structure like gauge invariance, however without tilde, the whole system become empty and there is no propagating degrees of freedom. One can show that expansion of this systme around AdS background gives Fronsdal fields as perturbative degrees of freedom. This system is invariant under following gauge transformation,

$$
\begin{aligned}
\delta W &= \mathrm{d}\epsilon - W \star \epsilon + \epsilon \star W \,, \\
\delta B &= \epsilon \star B - B \star \tilde{\epsilon} \,, \\
\delta S &= \epsilon \star S - S \star \epsilon \,.
\end{aligned}
\tag{2.22}
$$

and there is no issue about number of degrees of freedom with interaction. For more complete review of this subject, see [17].

2.2 Massive Higher Spin Theory

Contrast to massless higher spin, massive higher spin particles are observed as a hadronic resonance though they considered as a composite. Also, various no-go theorems are not applied to massive higher spin. However fully consistent interacting massive higher spin theory is not known yet except String theory. Study on massive higher spin has long history since Fierz and Pauli [13]. They suggested set of equations which describe free unitary massive higher spin field. After that Singh and Hagen [18, 19] found Lagrangian formulation with auxiliary fields. Auxiliary fields are needed to describe additional constraints which eliminate unphysical degrees of freedom. Such auxiliary fields are fixed by equation of motion and the Lagrangian does not have any gauge invariance. There is an alternative description with more auxiliary fields and gauge symmetry, Stueckelberg formulation. This formulation contains gauge symmetry which can be fixed algebraically. By algebraic gauge fixing, system goes back to that of Singh and Hagen. Stueckelberg formulation has a few advantages and naturally appears as a result of Kaluza-Klein compactification without gauge fixing.

In this section, both Singh-Hagen and Stuckelberg formulation are introduced. Also, the obstruction for interacting massive higher spin theory is introduced.

2.2.1 Free massive theory

Fierz and Pauli realized Wigner classification of mass m and spin s representation as a field theory. Starting from irreducible symmetric rank-s representation of Poincare group $\phi_{\mu_1\cdots\mu_s}$ which is traceless,

$$\phi_{\mu_1\cdots\mu_s}\,\eta^{\mu_1\mu_2} = 0 \tag{2.23}$$

they imposed Klein-Gordom equation,

$$(\Box - m^2)\phi_{\mu_1\cdots\mu_s} = 0. \tag{2.24}$$

To eliminate lower spin component with repect to rotation subgroup and to get positive definite total energy, they further imposed so called Fierz-Pauli condition or transversality condition,

$$\partial^{\mu_1}\phi_{\mu_1\cdots\mu_s} = 0. \tag{2.25}$$

Naive attempt for Lagrangian formulation would fail. One reason is that the number of equation is bigger than the number of degrees of freedom. To get both Klein-Gordon equation and Fierz-Pauli condition from Lagrangian, auxiliary field should be introduced. For instance, consider spin-2 example. The most general Lagrangian is,

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi_{\nu\rho}\partial^\mu\phi^{\nu\rho} - \frac{1}{2}m^2\phi_{\mu\nu}\phi^{\mu\nu} + \frac{1}{2}\alpha\,\partial^\rho\phi_{\rho\mu}\partial_\sigma\phi^{\sigma\mu}\,. \tag{2.26}$$

Equation of motion is,

$$(\Box - m^2)\phi_{\mu\nu} - \frac{1}{2}\alpha\Big(2\partial^\rho\partial_{(\mu}\phi_{\nu)\rho} - \frac{2}{d}\,\eta_{\mu\nu}\partial^\rho\partial^\sigma\phi_{\rho\sigma}\Big) = 0\,, \tag{2.27}$$

where d is dimension of spacetime. If one can get transversality condition $\partial^\mu\phi_{\mu\nu} = 0$ from this equation of motion, then α linear term vanishes and one would recover both Klein-Gordon equation and transversality condition. Try divergence of equation of motion then,

$$\Big((\alpha-2)\Box + 2m^2\Big)\partial^\mu\phi_{\mu\nu} + \alpha(1-\frac{2}{d})\partial_\nu\partial^\rho\partial^\sigma\phi_{\rho\sigma} = 0\,, \tag{2.28}$$

it does not work even for $\alpha = 2$. Resolution is adding auxiliary field ϕ with proper coefficients. Singh-Hagen Lagrangian for spin-2 is,

$$\begin{aligned}\mathcal{L}_{SH} = -\frac{1}{2}\partial_\mu\phi_{\nu\rho}\partial^\mu\phi^{\nu\rho} + \partial^\rho\phi_{\rho\mu}\partial_\sigma\phi^{\sigma\mu} + \frac{d-1}{2(d-2)}\partial_\mu\phi\partial^\mu\phi + \phi\partial^\mu\partial^\nu\phi_{\mu\nu} \\ -\frac{1}{2}m^2\Big(\phi_{\mu\nu}\phi^{\mu\nu} - \frac{d(d-1)}{(d-2)^2}\phi^2\Big)\,.\end{aligned} \tag{2.29}$$

Equations of motion are

$$(\Box - m^2)\phi_{\mu\nu} - 2\partial^\rho\partial_{(\mu}\phi_{\nu)\rho} + \frac{2}{d}\,\eta_{\mu\nu}\partial^\rho\partial^\sigma\phi_{\rho\sigma} + \partial_\mu\partial_\nu\phi - \frac{1}{d}\eta_{\mu\nu}\Box\phi = 0\,, \tag{2.30}$$

$$\Box\phi - 2\frac{d-2}{d-1}\partial^\mu\partial^\nu\phi_{\mu\nu} - m^2\frac{d}{d-2}\phi = 0\,. \tag{2.31}$$

Linear combination $d\,\partial^\mu\partial^\nu(2.30)_{\mu\nu} + \frac{1}{2}\frac{d-1}{d-2}\Big((2-d)\Box - dm^2\Big)(2.31) = \frac{m^4d^2(d-1)}{(d-2)^2}\phi = 0$. Therefore auxiliary field ϕ becomes 0 by equation of motion and equation (2.31) gives $\partial^\mu\partial^\nu\phi_{\mu\nu} = 0$. With this new condition, divergence of equation (2.30) gives correct transversality condition and finally one obtains Klein-Gordon equation for spin-2. Note that $\partial^\mu\partial^\nu\phi_{\mu\nu} = 0$ is just a divergence of transversality condition and does not impose new constraint. From spin s Singh-Hagen Lagrangian formulation, one should derive $\partial^\mu\cdots\partial^{\mu_k}\phi_{\mu_1\cdots\mu_s} = 0$, $k = 2, 3, \cdots, s$. For each k, rank-$(s-k)$ symmetric auxiliary field should be introduced, therefore Singh-Hagen Lagragian for massive spin-s theory consists of rank $0, 1, \cdots, s-2$ and s fields. By taking massless limit of Singh-Hagen Lagrangian, every auxiliary field except rank-$(s-2)$ field decouple and rank-$(s-2)$ and rank-s field form Fronsdal Lagrangian.

Singh-Hagen Lagrangian is standard formulation of massive higher spin theory however its explicit form of Lagrangian is complicated and it is hard to recognize degrees of freedom at Lagrangian level. There is alternative formulation which was introduced by Stueckelberg [20]. The Stueckelberg formalism is usually called "Stueckelberg trick" since it contain lots of auxiliary fields together with gauge symmetry which can be used to eliminate auxiliary fields by gauge fixing. Such gauge fixing is algebraic and almost trivial. After gauge fixing, the system goes back to Singh-Hagen formulation. One can obtain such formalism by writing down every possible quadratic term of Lagrangian and imposing gauge invariance. Or as a short cut, one can obtain from dimensional reduction of Fronsdal action without gauge fixing on flat background [21, 22, 23, 24]. We just introduce the result. One of nice properties of Stueckelberg formalism is the fields which are consist of the system is the same with fields of massless theory: double-traceless symmetric tensors. Also the gauge parameters are the same. Instead, all of spin-$0, 1, \cdots, s$ fields are needed to describe a spin-s particle. Denote the Fronsdal field, guage parameter and Lagrangian of spin-s theory as ϕ^s, ξ^s and $\mathcal{L}_0^s$ respectively. Then the Lagrangian for massive spin-s theory is,

$$\mathcal{L}^s_{Stue} = \sum_{k=0}^{s} \mathcal{L}_0^k + \Delta\mathcal{L}\,, \tag{2.32}$$

$$\begin{aligned}\Delta\mathcal{L} = \sum_{k=0}^{s} & a_k (\phi^{k-1})_{\mu_1\cdots\mu_{k-1}} \partial_\rho (\phi^k)^{\rho\mu_1\cdots\mu_{k-1}} + b_k (\phi^k)^\rho{}_{\rho\mu_1\cdots\mu_{k-2}} \partial_\sigma (\phi^{k-1})^{\sigma\mu_1\cdots\mu_{k-2}} \\ & + c_k \partial^\rho (\phi^k)_\rho{}^\sigma{}_{\sigma\mu_1\cdots\mu_{k-3}} (\phi^{k-1})_\tau{}^{\tau\mu_1\cdots\mu_{k-3}} + d_k (\phi^k)_{\mu_1\cdots\mu_k} (\phi^k)^{\mu_1\cdots\mu_k} \\ & + e_k (\phi^k)^\rho{}_{\rho\mu_1\cdots\mu_{k-2}} (\phi^k)_\sigma{}^{\sigma\mu_1\cdots\mu_{k-2}} + f_k (\phi^k)^\rho{}_{\rho\mu_1\cdots\mu_{k-2}} (\phi^{k-2})^{\mu_1\cdots\mu_{k-2}}\,,\end{aligned} \tag{2.33}$$

where

$$\begin{aligned} & a_k = -k\alpha_{k-1}\,, \quad b_k = -k(k-1)\alpha_{k-1}\,, \quad c_k = -\frac{1}{4}k(k-1)(k-2)\alpha_{k-1}\,, \\ & d_k = \frac{(k+1)(2k+d-3)}{2k+d-4}\alpha_k^2 - \frac{k}{2}\alpha_{k-1}^2 \text{ for } k\geq 1\,, \quad d_0 = \frac{d}{d-2}\alpha_1^2\,, \\ & e_k = -\frac{k(k^2-1)(2k+d)}{8(2k+d-4)}\alpha_k^2 + \frac{k^2(k-1)}{4}\alpha_{k-1}^2\,, \quad f_k = -\frac{k(k-1)}{2}\alpha_{k-1}\alpha_{k-2}\,, \\ & \alpha_k^2 = \frac{(s-k)(s+k+d-3)}{(k+1)(2k+d-2)}m^2\,. \end{aligned} \tag{2.34}$$

$\mathcal{L}^s_{Stue}$ is invariance under the following gauge transformation.

$$\delta(\phi^k)_{\mu_1\cdots\mu_k} = \alpha_k (\xi^{k+1})_{\mu_1\cdots\mu_k} + \partial_{(\mu_1} (\xi^k)_{\mu_2\cdots\mu_k)} + \beta_k \eta_{(\mu_1\mu_2} (\xi^{k-1})_{\mu_3\cdots\mu_k)} \tag{2.35}$$

where $\beta_k = \frac{2}{(k-1)(2k+d-6)}\alpha_{k-1}$. Each ϕ^k is consist of rank-k traceless tensor and rank-$(k-2)$ traceless tensor and ξ^{k+1} can be used to gauge fix rank-k traceless part of ϕ^k. Note that the gauge fixing is algebraic. After fixing every gauge, there remain rank-s and rank-$(s-2), (s-3), \cdots 0$ traceless tensor which consist Singh-Hagen massive theory. Like massless theory, Stueckelberg formalism on AdS background can be achieved by small modifications.

$$
\begin{aligned}
&\partial_\mu \to \nabla_\mu\,, \quad \eta_{\mu\nu} \to g^{AdS}_{\mu\nu}\,,\\
&m^2 \to m^2 + \frac{1}{r^2_{AdS}}(s-k-1)(s+k+d-4)\,,\\
&\Delta d_k = -\frac{1}{r^2_{AdS}}\Big((k-1)(k-4)+(k-2)(d-1)\Big)\,,\\
&\Delta e_k = \frac{k(k-1)}{4r^2_{AdS}}\Big(k(k-3)+(k-1)(d-1)\Big)\,.
\end{aligned}
$$

This is what we use to describe the result of Kaluza-Klein compactification on AdS background. At this level, it may sound too trivial since the everything is done tirivially on flat background. However there are lots of subtleties and rich structure when we consider AdS background.

One most important property of Stueckelberg formalism is that the Lagrangian is determined up to overall normalization factor by the gauge transformation. This property allows us to study Kaluza-Klein compactification of higher spin theory only using gauge transformation which is much simpler than Lagrangian or equation of motion.

2.2.2 Velo-Zwanziger problem

Contrast to massless higher spin theory, massive higher spin theory is not a gauge theory. One might think that it is easier to turn on interaction without gauge symmetry since there is less restriction. There might be more freedom to write down interacting theory but practically, it is more difficult. With gauge symmetry, one could concentrate on gauge symmetry and other issues resolved automatically. Without gauge symmetry, one should handle all the issues manually when one turns on the interaction. The number of propagating degrees of freedom should not depend on coupling constant and causality should be kept. In general, these conditions do not hold and new constraints or new propagating degrees appear. This obstruction for interacting massive higher spin theory is called “Velo-Zwanziger problem” [5, 6, 7]. The most simple example appears for non-zero electromagnetic background. Consider non-self interacting massive spin-s field under electromagnetic background. If we consider just minimal coupling, derivatives should be replaced by covariant derivatives, $\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu$,

$$(D^2 - m^2)\,\phi_{\mu_1\cdots\mu_s} = 0\,, \quad D^{\mu_1}\phi_{\mu_1\cdots\mu_s} = 0\,. \tag{2.36}$$

Since covariant derivatives are not commute, above 2 equations gives,

$$[D^{\mu_1}, D^2 - m^2]\phi_{\mu_1\cdots\mu_s} = ieF^{\mu_1\rho}D_\rho\phi_{\mu_1\cdots\mu_s} = 0\,. \tag{2.37}$$

This constraint only matters for non-zero charge and non-zero electromagnetic background. Therefore the number of degrees of freedom between free and interacting theory do not match. Resolution is considering additional degrees of freedom or non-minimal coupling. Actually, above example can be cured by non-minimal coupling,

$$(D^2 - m^2)\phi_{\mu_1\cdots\mu_2} - 2ie\, s\, F^\rho{}_{(\mu_1}\phi_{\mu_2\cdots\mu_s)\rho} = 0\,, \quad D^{\mu_1}\phi_{\mu_1\cdots\mu_s} = 0\,. \tag{2.38}$$

Commutator of above 2 equations is 0 and there is no pathology. However, there is more serious problem: superluminal propagation. Even for massive spin-2 with electromagnetic background, it is known that there exist superluminal propagation when electromagnetic background is non-zero. This problem is serious since even for infinitesimal background, one always can find a frame where superluminal propagation exists. Again, properly tuned non-minimal coupling and additional degrees of freedom may cure the situation. String theory is unique example of interacting massive higher spin theory whose consistency is check for full order. In general resolving Velo-Zwanziger problem is very challenging and all known systematic resolution is perturbative way [8, 9].

This is our main motivation for studying Kaluza-Klein compactification of higher spin theory. If we do every step of compactification carefully, consistency of the Vasiliev theory may ensure the lower dimensional interacting massive higher spin theory and we may bypass the Velo-Zwanziger problem.

Chapter 3

Compactification of Background Metric

In this chapter, we introduce a background geometry which is used for Kaluza-Klein compactification. One of novel features of our work is that the background is curved and does not have any S^1 fibration. In almost every context of Kaluza-Klein compactification, flat spacetime, $R^{d,1}$, is used as background and one of the spatial directions is considered as an internal space. Then there is no obstruction for circular compactification and internal space become S^1. Even when a curved background is considered, starting point is $M^{d-1,1} \times S^1$ or fibration of S^1 and again, S^1 becomes natural internal space. The benefit of using S^1 as an internal space is clear: one does not have to consider boundary condition which can potentially spoil the consistency of the theory. On the other hand, studying non-trivial boundary condition gives rich structure and one may discover the non-perturbative object of a theory, for instance D-brane of String theory. We claim that compactification with AdS background requires non-trivial boundary together with boundary conditions.

We introduce Poincare coordinate as a starting point and suggest a parametrization for slicing AdS$_{d+2}$ space into AdS$_{d+1}$ space. By doing so, we show that boundary should be introduced. Then, we briefly discuss flat spacetime example with boundary and apply the background metric compactification to a spin-2 example.

3.1 Slicing of Poincare metric

Free massless higher spin theory minimally coupled with background cannot be defined on arbitrary background. The background should be either flat or maximally symmetric. Also free massive higher spin particles cannot couple to arbitrary background because of Velo-Zwanziger problem. Keeping the interacting theory in our mind, we use anti-de Sitter space for both higher and lower dimension of compactification. One of the most convenient coordinate of AdS space is the Poincare coordinate[1]. For AdS$_{d+2}$,

$$\mathrm{d}s^2 = \frac{r_{AdS}^2}{z^2}(-\mathrm{d}t^2 + \mathrm{d}\mathbf{x}_d^2 + \mathrm{d}z^2), \tag{3.1}$$

[1]Actually, Poincare coordinate do not cover whole spacetime therefore called Poincare patch. However, that doesn't affect the procedule and we can get the same result with global coordinate.

where $\mathbf{x_d}$ is vector of R^d. We want to interpret AdS_{d+2} spacetime as a AdS_{d+1} spacetime and internal space. In other word, we want to slice AdS_{d+2} into AdS_{d+1} along an internal direction. There are 2 options with respect to property of internal space. One may choose internal space from one of isometry direction or non-isometry direction. If we can use isometry direction as an internal space, metric of AdS_{d+1} slice is independent of internal coordinate. Then, there is no obstruction to identify end points of internal space and we can do circular compactification which is very simple. We claim that it is impossible and one should use non-isometry direction as an internal space. This is the most important property of our Kaluza-Klein reduction. To see the reason, consider an example of slicing along isometry direction.

$$\begin{aligned} \mathrm{d}s(\mathrm{AdS}_{d+2})^2 &= \frac{r_{AdS}^2}{z^2}\left(-\mathrm{d}t^2 + \mathrm{d}\mathbf{x}_{d-1}^2 + \mathrm{d}z^2\right) + \frac{r_{AdS}^2}{z^2}\mathrm{d}y^2 \\ &= \mathrm{d}s(\mathrm{AdS}_{d+1})^2 + g_{yy}\,\mathrm{d}y^2\,. \end{aligned} \tag{3.2}$$

y is one of $\mathbf{x}_d$ and $\mathbf{x}_{d-1}$ are rest of them. Translation along y is manifest isometry. As a slicing, there is no problem, however we cannot use (3.2) for compactification. The reason is the following. Locally at each y, the isometry of lower dimension $\mathfrak{so}(d,2)$ is part of the original isometry $\mathfrak{so}(d+1,2)$. However, globally, this does not hold, since $\mathfrak{so}(d,2)$ isometry transformation does not commute with translation along y direction. By the same reason, when compactifying along the y-direction, the $(d+2)$-dimensional tensor does not give rise to $(d+1)$-dimensional tensors. Consider, for example, a small fluctuation of the metric. The tensor $\nabla_\mu\, h_{\nu y}$ is dimensionally reduced to $\nabla_\mu\, A_\nu + \delta_{\mu z}\,\frac{1}{z}\,A_\nu$, where $A_\mu \equiv h_{\mu y}$. The second term is a manifestation of non-tensorial transformation in $(d+1)$ dimensions.

Any attempt of compactifying along an isometry direction faces the same difficulties. Instead, we use non-isometry direction as an internal space. Now the metric depends on internal coordinate y. As we require each slice should be AdS_{d+1}, the internal coordinate dependence of each slice must be an overall factor.

$$\begin{aligned} \mathrm{d}s(\mathrm{AdS}_{d+2})^2 &= f(y)\,\mathrm{d}s(\mathrm{AdS}_{d+1})^2 + g(y)\,\mathrm{d}y^2 \\ &= \tilde{f}(\tilde{y})\Big[\mathrm{d}s(\mathrm{AdS}_{d+1})^2 + c\,\mathrm{d}\tilde{y}^2\Big] \end{aligned} \tag{3.3}$$

One can get the second line by proper coordinate change. A constant c can be chosen any value. Put this ansatz to the vacuum Einstein equation with negative cosmological constant then we get a differential equation for $f(y)$.

$$\partial_y\Big(\frac{1}{f(y)}\partial_y f(y)\Big) = \frac{2c}{r_{AdS}^2} f(y) \tag{3.4}$$

Set $c = r_{AdS}^2$, then one can check that $f(y) = (\sec y)^2$ is a solution. We can explicitly construct this slicing. We start from Poincare patch of AdS$_{d+2}$ and change bulk radial coordinate z and another spatial coordinate y to polar coordinates, $z = \rho\cos\theta$, $y = \rho\sin\theta$. With this parametrization, the AdS$_{d+2}$ space can be represented as a fibration of AdS$_{d+1}$ space over the interval, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$:

$$\begin{aligned} \mathrm{d}s_{d+2}{}^2 &= \frac{r_{AdS}^2}{z^2}\left(-\mathrm{d}t^2 + \mathrm{d}\mathbf{x}_{d-1}^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right) = \frac{r_{AdS}^2}{\rho^2\cos^2\theta}\left(-\mathrm{d}t^2 + \mathrm{d}\mathbf{x}_{d-1}^2 + \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2\right) \\ &= \frac{1}{\cos^2\theta}(\mathrm{d}s_{d+1}{}^2 + r_{AdS}^2\,\mathrm{d}\theta^2)\,. \end{aligned} \tag{3.5}$$

This is generic result and any other parametrization is just a coordinate change of this. Contrast to (3.2), $(d+2)$-dimensional tensor can be identified as a $(d+1)$-dimensional tensor. For instance, $\nabla_\mu\, h_{\nu\theta}$ becomes $\nabla_\mu\, A_\nu - \tan\theta\, h_{\mu\nu} + \tan\theta\,\frac{1}{r_{AdS}^2}\, g_{\mu\nu}\,\phi$, where $A_\mu \equiv h_{\mu\theta}$ and $\phi \equiv h_{\theta\theta}$. Therefore for the rest of the thesis, our parametrization of background metric for Kaluza-Klein compactification is,

$$\mathrm{d}s(\mathrm{AdS}_{d+2})^2 = \frac{1}{\cos^2\theta}\left[\mathrm{d}s(\mathrm{AdS}_{d+1})^2 + r_{AdS}^2\,\mathrm{d}\theta^2\right]. \tag{3.6}$$

We use the symbol θ for both coordinate and vector index of internal space.

The most important consequence of using non-isometry direction as an internal space is that the end points of internal space $\theta = \pm\pi/2$ cannot be identified. Any other 2 points of internal space cannot be identified since the value of metric or derivatives of the metric are not continuous at the identified point. Therefore circular compactification is impossible for AdS spacetime and we should introduce boundaries together with proper boundary conditions. For simplicity, we put 2 boundaries at $\theta = \pm\alpha$, $0 < \alpha < \pi/2$. α is correspond to the distance between 2 boundaries and is a tunable parameter of the compactification. $\alpha \to \pi/2$ limit gives original AdS$_{d+2}$ spacetime, and $\alpha \to 0$ limit gives AdS$_{d+1}$ spacetime. The spectrum of Kaluza-Klein compactification depends on α.

3.2 Flat Spacetime Example with Boundaries

To get an idea for compactification with boundaries, let's consider flat spacetime example. What we should deal with is a gauge theory. The structure of gauge theory may give constraints on boundary condition. Also, the relation between boundary condition and spectrum of the lower dimensional theory is what should we find.

3.2.1 Kalauza-Klein mode expansion

As an example, we study the electromagnetic field in $(d+2)$-dimensional flat spacetime with boundaries, paying particular attention to relations between boundary conditions and spectra for fields of different spins. The flat spacetime is $\mathbb{R}^{1,d} \times I_L$, where interval $I_L \equiv \{0 \le z \le L\}$. The $(d+2)$-dimensional coordinates can be decomposed into parallel and perpendicular directions: $x^M = (x^\mu, z)$. The $(d+2)$-dimensional spin-one field is decomposed in $(d+1)$ dimensions to a spin-one field and a spin-zero field: $A_M = (A_\mu, \phi)$. The equations of motions are decomposed as

$$\partial^M F_{M\nu} = \partial^\mu F_{\mu\nu} - \partial_z(\partial_\nu \phi - \partial_z A_\nu) = 0\,, \tag{3.7}$$

$$\partial^M F_{Mz} = \partial^\mu (\partial_\mu \phi - \partial_z A_\mu) = 0\,, \tag{3.8}$$

while the gauge transformations are decomposed as

$$\delta A_\mu = \partial_\mu \Lambda\,, \qquad\qquad \delta \phi = \partial_z \Lambda\,. \tag{3.9}$$

We note that both the equations of motion and the gauge transformations manifest the structure of Stueckelberg system. Recall that the Stueckelberg Lagrangian of massive spin-one vector field is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu \phi\, \partial^\mu \phi + m\, A_\mu \left(\frac{m}{2} A^\mu - \partial^\mu \phi \right)\,, \tag{3.10}$$

which is invariant under Stueckelberg gauge transformations

$$\delta A_\mu = \partial_\mu \lambda \qquad \text{and} \qquad \delta \phi = m\, \lambda\,. \tag{3.11}$$

The field ϕ is referred to as the Stueckelberg spin-zero field. This field is redundant for $m \neq 0$ because it can be eliminated by a suitable gauge transformation. In the massless limit, $m \to 0$, the Stueckelberg system breaks into a spin-one gauge system and a massless spin-zero system.

The $(d+2)$-dimensional spin-one field A_M is excited along the z-direction. The field can be mode-expanded, and expansion coefficients are $(d+1)$-dimensional spin-one and spin-zero fields of varying masses. Importantly, mode functions can be chosen from any complete set of basis functions. It is natural to choose them by the eigenfunctions of $\Box := -(\partial_z)^2$ with a prescribed boundary condition. Mode functions of the gauge parameter Λ should be chosen compatible with the mode function of spin-one field A_M. Combining the two gauge variations Eq.(3.9), we learn that the mode functions ought to be related to each other as

$$\partial_z\,(\text{ mode function of spin-one field } A_\mu(x,z)) \propto (\text{ mode function of spin-zero field } \phi(x,z))\,. \tag{3.12}$$

Being a local relation, this relation must hold at each boundary as well.

It would be instructive to understand, instead of the required Eq.(3.12), what might go wrong if one imposes the same boundary conditions for both A_μ and ϕ, such as zero-derivative (Dirichlet) or one-derivative (Neumann) boundary conditions. Suppose one adopts the zero-derivative (Dirichlet) boundary condition for both fields. From $A_\mu(z)|_{z=0,\,L} = 0$, $\phi(z)|_{z=0,L} = 0$ and from the field equation of ϕ, Eq. (3.8), it follows that

$$\Big(\partial^\mu\,\partial_\mu\,\phi(z) - \partial^\mu\,\partial_z\,A_\mu(z)\Big)\Big|_{z=0,\,L} = -\partial^\mu\,\partial_z\,A_\mu(z)|_{z=0,\,L} = 0\,, \tag{3.13}$$

and hence $\partial_z\,A_\mu(z)|_{z=0,L} = 0$. But A_μ satisfies second-order partial differential equation, so these two sets of boundary conditions — $A_\mu(z)|_{z=0,\,L} = 0$ and $\partial_z\,A_\mu(z)|_{z=0,L} = 0$ — imply that $A_\mu(z)$ must vanish everywhere. Likewise, ϕ satisfies a first-order differential equation Eq.(3.7), so the two sets of boundary conditions imply that $\phi(z)$ vanishes everywhere as well. One concludes that there is no nontrivial field excitations satisfying such boundary conditions. We remind that this conclusion follows from the fact that these boundary conditions do not preserve the relation Eq.(3.12).

The most general boundary conditions compatible with the relation Eq.(3.12) restricts the form of boundary conditions for spin-one and spin-zero fields. For example, if we impose the Robin boundary condition for the spin-zero field, $\mathcal{M}(\partial_z)\phi|_{z=0,L} := (a\partial_z + b)\phi|_{z=0,L} = 0$ where a, b are arbitrary constants, the relation Eq.(3.12) imposes the boundary condition for the spin-one field as $\mathcal{M}\,\partial_z\,A_\mu|_{z=0,\,L} = 0$. Modulo higher-derivative generalizations, we have two possible boundary conditions: $a = 0, b \neq 0$ corresponding to the vector boundary condition and $a \neq 0, b = 0$ corresponding to the scalar boundary condition. Hereafter, we analyze each of them explicitly.

3.2.2 Vector boundary condition

We may impose one-derivative (Neumann) boundary condition on the spin-one field $A_\mu(x,z)$ field and zero-derivative (Dirichlet) boundary condition on spin-zero field $\phi(x,z)$ at $z = 0,\ L$. The corresponding mode expansion for A_μ and ϕ reads

$$A_\mu(z) = \sum_{n=0}^{\infty} A_\mu^{(n)} \cos\left(\frac{n\,\pi}{L}\,z\right) \qquad \text{and} \qquad \phi(z) = \sum_{n=1}^{\infty} \phi^{(n)} \sin\left(\frac{n\,\pi}{L}\,z\right), \tag{3.14}$$

so we mode-expand the field equations Eq.(3.7) and Eq.(3.8) in a suggestive form

$$\sum_{n=0}^{\infty} \cos\left(\frac{n\,\pi}{L}\,z\right)\left[\partial^{\mu}\,F^{(n)}{}_{\mu\nu} - \left(\frac{n\,\pi}{L}\right)\left(\frac{n\pi}{L}\,A_{\nu}^{(n)} + \partial_{\nu}\,\phi^{(n)}\right)\right] = 0\,, \tag{3.15}$$

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\,\pi}{L}\,z\right)\partial^{\mu}\left(\frac{n\,\pi}{L}\,A_{\mu}^{(n)} + \partial_{\mu}\,\phi^{(n)}\right) = 0\,. \tag{3.16}$$

The standing-wave mode functions for $n = 0, 1, \ldots$ form a complete set of the orthogonal basis for square-integrable functions over I_L, so individual coefficient in the above equation ought to vanish. The zero-mode $n = 0$ is special, as only the first equation is nonempty and gives the equation of motion for massless spin-one field. All Kaluza-Klein modes, $n \geq 1$, satisfies the Stueckelberg equation of motion for massive spin-one field with mass $m_n = n\,\pi/L$. The second equation follows from divergence of the first equation, so just confirms consistency of the prescribed boundary conditions. In the limit $L \to 0$, all Stueckelberg fields become infinitely massive. As such, there only remains the massless spin-one field $A_{\mu}^{(0)}$ with associated gauge invariance. Also, there is no spin-zero field $\phi^{(0)}$, an important result that follows from the prescribed boundary conditions. Intuitively, $A_{\mu}^{(0)}$ remains massless and gauge invariant, so Stueckelberg spin-zero field $\phi^{(0)}$ is not needed. Moreover, the spectrum is consistent with the fact that this boundary condition ensures no energy flow across the boundary $z = 0, L$.

The key observation crucial for foregoing discussion is that the same result is obtainable from Kaluza-Klein compactification of gauge transformations Eq.(3.9). The gauge transformations that preserve the vector boundary conditions can be expanded by the Fourier modes:

$$\Lambda = \sum_{n=0}^{\infty} \Lambda^{(n)} \cos\left(\frac{n\,\pi}{L}\,z\right)\,. \tag{3.17}$$

The gauge transformations of $(d+1)$-dimensional fields read

$$\delta\,A_{\mu}^{(n)} = \partial_{\mu}\,\Lambda^{(n)} \quad (n \geq 0) \qquad \text{and} \qquad \delta\,\phi^{(n)} = -\frac{n\,\pi}{L}\,\Lambda^{(n)} \quad (n \geq 1)\,. \tag{3.18}$$

We note that the $n = 0$ mode is present only for the gauge transformation of spin-one field. This is the gauge transformation of a massless gauge vector field. We also note that gauge transformations of all higher $n = 1, 2, \cdots$ modes take precisely the form of Stueckelberg gauge transformations. Importantly, the Stueckelberg gauge invariance fixes quadratic part of action.

3.2.3 Scalar boundary condition

Alternatively, one might impose no-derivative (Dirichlet) boundary condition to the spin-one field A_μ and one-derivative (Neumann) boundary condition to the spin-zero ϕ. In this case, the equations of motion, when mode-expanded, take exactly the same form as above except that the standing-wave mode functions are interchanged:

$$\sum_{n=1}^{\infty} \sin\left(\frac{n\,\pi}{L}\,z\right)\left[\partial^\mu\, F^{(n)}{}_{\mu\nu} - \left(\frac{n\,\pi}{L}\right)\left(\frac{n\pi}{L}\,A^{(n)}_\nu - \partial_\nu\,\phi^{(n)}\right)\right] = 0\,, \tag{3.19}$$

$$\sum_{n=0}^{\infty} \cos\left(\frac{n\,\pi}{L}\,z\right)\partial^\mu\left(\frac{n\,\pi}{L}\;A^{(n)}_\mu - \partial_\mu\,\phi^{(n)}\right) = 0\,. \tag{3.20}$$

Consequently, the zero-mode $n = 0$ consists of massless spin-zero field $\phi^{(0)}$ only ($A^{(0)}_\mu$ is absent from the outset). All Kaluza-Klein modes $n \neq 0$ are again Stueckelberg massive spin-one fields with mass $m_n = n\pi/L$. In the limit $L \to 0$, these Stueckelberg field becomes infinitely massive. Below the Kaluza-Klein scale $1/L$, there only remains the massless spin-zero field $\phi^{(0)}$. Once again, this is consistent with the fact that this boundary condition ensures no energy flow across the boundary.

Once again, the key idea is that the above results are obtainable from the Kaluza-Klein compactification of the gauge transformations. For a gauge transformation that preserves the scalar boundary condition, the gauge function can be expanded as

$$\Lambda(x,z) = \sum_{n=1}^{\infty} \Lambda^{(n)}(x)\,\sin\left(\frac{n\,\pi}{L}\,z\right)\,. \tag{3.21}$$

With these modes, the gauge transformations of fields are

$$\delta\,A^{(n)}_\mu = \partial_\mu\,\Lambda^{(n)} \quad (n \geq 1) \qquad \text{and} \qquad \delta\,\phi^{(n)} = \frac{n\,\pi}{L}\,\Lambda^{(n)} \quad (n \geq 0)\,. \tag{3.22}$$

There is no $n = 0$ zero-mode gauge transformation, and so no massless gauge spin-one field. The spin-zero zero-mode $\phi^{(0)}$ is invariant under the gauge transformations. We also note that the gauge transformations take the form of the Stueckelberg gauge symmetries with masses $m_n = n\,\pi/L$.

Summarizing,

- Kaluza-Klein spectrum is obtainable either from field equations or from gauge transformations.
- Stueckelberg formalism naturally arises from Kaluza-Klein compactification.
- Boundary conditions of lower-dimensional component fields (for example, A_μ and ϕ from A_M) are correlated each other (for example as in Eq.(3.12)).

3.3 Spin-Two Example of Kaluza-Klein Compactification

With the ideas from previous section, we analyze spin-2 example on AdS background. Here, we introduce notations for various fields and their mode function. Fields or operators with bar are $(d+2)$-dimensional field. For instance, $\bar{h}_{\mu\nu}$. Fields or operators without bar are $(d+1)$-dimensional. To denote mode functions of various lower dimensional fields, we use following notation: $\Theta_n^{k|s}(\theta)$. s is spin of original higher dimensional field. In this example, $s = 2$. k denotes spin of corresponding lower dimensional fields, $0 \leq k \leq s$. For example, mode function of $A_\mu = \bar{h}_{\mu\theta}$ is $\Theta_n^{1|2}(\theta)$. We use $M, N, K, \ldots$ for indices of higher dimension and $\mu, \nu, \rho, \ldots$ for indices of lower dimension. h denotes fluctuation of metric, therefore spin-two field and g denotes background AdS metric.

In this example, we use both equations of motion and gauge transformation to ensure that using gauge transformation gives the same result with using equations of motion and is powerful enough to determine the spectrum of lower dimension.

3.3.1 Mode functions of spin-two field

We begin with the method using the equation of motion. The Pauli-Fierz equation of motion for a massive spin-two field in AdS$_{d+2}$ is given by

$$\mathcal{K}_{MN}(\bar{h}) - (d+1)\,(2\,\bar{h}_{MN} - \bar{g}_{MN}\,\bar{h}) - M^2\,(\bar{h}_{MN} - \bar{g}_{MN}\,\bar{h}) = 0\,, \tag{3.23}$$

where M^2 is the mass-squared, $\bar{g}_{MN}$ is the metric of AdS$_{d+2}$ space, and $\mathcal{K}_{MN}(\bar{h})$ is the spin-two Lichnerowicz operator:

$$\begin{aligned}\mathcal{K}_{MN}(\bar{h}) = \Box\,\bar{h}_{MN} - (\overline{\nabla}^L\,\overline{\nabla}_N\,\bar{h}_{ML} + \overline{\nabla}^L\,\overline{\nabla}_M\,\bar{h}_{NL})\\ + \bar{g}_{MN}\,\overline{\nabla}_K\,\overline{\nabla}_L\,\bar{h}^{KL} + \overline{\nabla}_M\,\overline{\nabla}_N\,\bar{h} - \bar{g}_{MN}\,\overline{\Box}\,\bar{h}\,,\end{aligned} \tag{3.24}$$

where ($\bar{h}$ denotes for the trace part, $\bar{g}^{MN}\bar{h}_{MN}$). After the compactification, the $(d+2)$-dimensional spin-two field is decomposed to $(d+1)$-dimensional spin-two, spin-one, and spin-zero component fields:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} + \frac{1}{d-1}\, g_{\mu\nu}\, \bar{h}_{\theta\theta}\,, \qquad \bar{h}_{\mu\theta} = A_\mu\,, \qquad \bar{h}_{\theta\theta} = \phi\,. \tag{3.25}$$

Note that the spin-two field $h_{\mu\nu}$ is defined by the linear combination of $\bar{h}_{\mu\nu}$ and $\bar{h}_{\theta\theta}$[2].

The massless spin-two equation of motion in AdS$_{d+2}$ space decomposes into equations of motion for component fields $(h_{\mu\nu},\, A_\mu,\, \phi)$ in AdS$_{d+1}$ space:

$$\begin{aligned}
&\mathcal{K}_{\mu\nu}(h) - d\,(2\,h_{\mu\nu} - g_{\mu\nu}\,h) + Ł_{d-2}\,Ł_{-2}\,(h_{\mu\nu} - g_{\mu\nu}\,h) \\
&\qquad - Ł_{d-2}\,(\nabla_\mu\,A_\nu + \nabla_\nu\,A_\mu - 2\,g_{\mu\nu}\,\nabla^\rho\,A_\rho) + \frac{d}{d-1}\,g_{\mu\nu}\,Ł_{d-2}\,Ł_{d-3}\,\phi = 0\,, \qquad (3.26)\\
&\nabla^\mu\,F_{\mu\nu} - 2\,d\,A_\nu - Ł_{-2}\,(\nabla^\mu\,h_{\mu\nu} - \nabla_\nu\,h) - \frac{d}{d-1}\,Ł_{d-3}\,\nabla_\nu\,\phi = 0\,, \qquad (3.27)\\
&\Box\,\phi - \left(\frac{d+1}{d-1}\,Ł_{-1}\,Ł_{d-3} + d + 1\right)\phi - 2\,Ł_{-1}\,\nabla^\mu\,A_\mu + Ł_{-1}\,Ł_{-2}\,h = 0\,, \qquad (3.28)
\end{aligned}$$

where h is the trace part, $g^{\mu\nu}h_{\mu\nu}$. The new notation $Ł_m$ is a linear differential operator, $Ł_m \equiv= \partial_\theta + m\,\tan\theta$. The mode expansion of $(d+1)$-dimensional spin-two, spin-one and spin-zero component fields reads

$$h_{\mu\nu} = \sum_n h^{(n)}{}_{\mu\nu}\,\Theta_n^{2|2}(\theta)\,, \quad A_\mu = \sum_n A^{(n)}{}_\mu\,\Theta_n^{1|2}(\theta)\,, \quad \phi = \sum_n \phi^{(n)}\,\Theta_n^{0|2}(\theta)\,. \tag{3.29}$$

Up to now, we know nothing about mode functions. We just assume they exist and try to find their properties. From the equations Eqs.(3.26, 3.27, 3.28), we can expect the relations between mode-functions which can be summarized by following two matrix equations:

$$\begin{pmatrix} 0 & Ł_{d-2} \\ Ł_{-2} & 0 \end{pmatrix}\begin{pmatrix} \Theta_n^{2|2} \\ \Theta_n^{1|2} \end{pmatrix} = \begin{pmatrix} c_n^{12}\Theta_n^{2|2} \\ c_n^{21}\Theta_n^{1|2} \end{pmatrix} \tag{3.30}$$

$$\begin{pmatrix} 0 & Ł_{d-3} \\ Ł_{-1} & 0 \end{pmatrix}\begin{pmatrix} \Theta_n^{1|2} \\ \Theta_n^{0|2} \end{pmatrix} = \begin{pmatrix} c_n^{01}\Theta_n^{1|2} \\ c_n^{10}\Theta_n^{0|2} \end{pmatrix} \tag{3.31}$$

where c_n's are coefficients. We now have two sets of raising and lowering operators, con-

[2] The equations of motion have cross terms between $\bar{h}$ and $\nabla^2\phi$. This linear combination removes these cross terms. This specific combination is also the linear part of diagonalized metric in the original Kaluza-Klein compactification, $\bar{g}_{\mu\nu} = e^{\phi/(d-1)}\,g_{\mu\nu}$.

necting spin-zero and spin-one and spin-one and spin-two, respectively. Accordingly, we have two pairs of Sturm-Liouville problems. The Eq. (3.30) leads to the first set of Sturm-Liouville problems for spin-two and spin-one, respectively:

$$\begin{aligned} \text{Ł}_{d-2}\,\text{Ł}_{-2}\,\Theta_n^{2|2} &= c_n^{21}\,c_n^{12}\,\Theta_n^{2|2} = -M_n^2\,\Theta_n^{1|2}\,, \\ \text{Ł}_{-2}\,\text{Ł}_{d-2}\,\Theta_n^{1|2} &= c_n^{12}\,c_n^{21}\,\Theta_n^{1|2} = -M_n^2\,\Theta_n^{1|2}\,. \end{aligned} \tag{3.32}$$

The Eq. (3.31) leads to the second set of Sturm-Liouville problems for spin-one and spin-zero, respectively:

$$\begin{aligned} \text{Ł}_{d-3}\,\text{Ł}_{-1}\,\Theta_n^{1|2} &= c_n^{10}\,c_n^{01}\,\Theta_n^{1|2}\,, \\ \text{Ł}_{-1}\,\text{Ł}_{d-3}\,\Theta_n^{0|2} &= c_n^{01}\,c_n^{10}\,\Theta_n^{0|2}\,. \end{aligned} \tag{3.33}$$

The two sets of equations appear overdetermined, as the spin-one mode function $\Theta_n^{1|2}$ is the eigenfunction of two separate Sturm-Liouville problems. However, it can be shown that the two Sturm-Liouville problems are actually one and the same problem by using the identity

$$\text{Ł}_m\,\text{Ł}_n - \text{Ł}_{n-1}\,\text{Ł}_{m+1} = (n-m-1)\,. \tag{3.34}$$

This also leads to eigenvalues relations

$$c_n^{10}c_n^{01} = c_n^{21}c_n^{12} - d + 1. \tag{3.35}$$

So, the Sturm-Liouville problems can be summarized by the relations

$$\begin{array}{ccccl} & \Theta_n^{2|2} & & & \\ \text{Ł}_{d-2} & \upharpoonleft\downharpoonright & \text{Ł}_{-2} & : & -M_{n,2|2}^2 = -M_n^2 = c_n^{21}c_n^{12} \\ & \Theta_n^{1|2} & & & \\ \text{Ł}_{d-3} & \upharpoonleft\downharpoonright & \text{Ł}_{-1} & : & -M_{n,1|2}^2 = -(M_n^2 + d - 1) = c_n^{10}c_n^{01} \\ & \Theta_n^{0|2} & & & \end{array} \tag{3.36}$$

We notice that these relations, defined by raising and lowering operators between $(d+1)$-dimensional fields of adjacent spins, is precisely the structure required for Stueckelberg mechanism [3]. If $M_{n,\,2|2}$ and $M_{n,\,1|2}$ were nonzero, the corresponding modes among different spin fields combine and become the Stueckelberg spin-two system. There are two special

[3] Note, however, $M_{n,1|2}$ is not related with mass-like term of spin-one field in Eq.(3.27). $M_n = M_{n,2|2}$ is the mass of the spin-two field in the Eq. (3.26).

cases, vanishing $M_{n,\,2|2}$ or vanishing $M_{n,\,1|2}$. As these are important exceptional situations, leading to so-called partially massless spin-two fields, we will analyze them separately in Section 3.3.2 with examples.

We can also obtain Eq. (3.36) from the method using gauge transformations. The gauge transformations in AdS$_{d+1}$ space, with the gauge parameter $\bar{\xi}_M = \{\xi_\mu,\, \xi_\theta\}$, are decomposed into components

$$\begin{aligned} \delta h_{\mu\nu} &= \nabla_{(\mu}\, \xi_{\nu)} + \frac{1}{d-1}\, g_{\mu\nu}\, Ł_{d-2}\, \xi_\theta\,, \\ \delta A_\mu &= \frac{1}{2}\, \partial_\mu\, \xi_\theta + \frac{1}{2}\, Ł_{-2}\, \xi_\mu\,, \\ \delta\phi &= Ł_{-1}\, \xi_\theta\,. \end{aligned} \tag{3.37}$$

Again, to retain the gauge invariances, the mode functions of gauge parameter are set proportional to mode functions of the fields:

$$\xi_\mu = \sum_n \xi_\mu^{(n)}\, \Theta_n^{2|2}(\theta)\,, \qquad\qquad \xi_\theta = \sum_n \xi_\theta^{(n)}\, \Theta_n^{1|2}(\theta)\,. \tag{3.38}$$

By substituting these to Eq.(3.37) and comparing mode expansion terms in the gauge variations, we see we can recover precisely the same raising and lowering operators as in Eq.(3.36), which was previously derived from the field equations Eqs.(3.26, 3.27, 3.28).

After the mode expansion, the component field equations read

$$\begin{aligned} &\mathcal{K}_{\mu\nu}(h^{(n)}) - d\left[2\, h^{(n)}{}_{\mu\nu} - g_{\mu\nu}\, h^{(n)}\right] + c_n^{21}\, c_n^{12}\left[h^{(n)}{}_{\mu\nu} - g_{\mu\nu} h^{(n)}\right] \\ &\qquad - c_n^{12}\left[\nabla_\mu\, A^{(n)}{}_\nu + \nabla_\nu\, A^{(n)}{}_\mu - 2\, g_{\mu\nu}\, \nabla^\rho\, A^{(n)}{}_\rho\right] + c_n^{01}\, c_n^{12}\, \frac{d}{d-1}\, g_{\mu\nu}\, \phi^{(n)} = 0\,, \end{aligned} \tag{3.39}$$

$$\nabla^\mu\, F^{(n)}{}_{\mu\nu} - 2\, d\, A^{(n)}{}_\nu - c_n^{21}\, \nabla^\mu\left[h^{(n)}{}_{\mu\nu} - g_{\mu\nu}\, h^{(n)}\right] - c_n^{01}\, \frac{d}{d-1}\, \nabla_\nu\, \phi^{(n)} = 0\,, \tag{3.40}$$

$$\Box\, \phi^{(n)} - \left[\frac{d+1}{d-1}\, c_n^{01}\, c_n^{10} + d + 1\right] \phi^{(n)} - 2\, c_n^{10}\, \nabla^\mu\, A^{(n)}{}_\mu + c_n^{21}\, c_n^{10}\, h^{(n)} = 0\,. \tag{3.41}$$

Their gauge transformations read

$$\delta h_{\mu\nu}^{(n)} = \nabla_{(\mu}\, \xi_{\nu)}^{(n)} + \frac{c_n^{12}}{d-1}\, g_{\mu\nu}\, \xi^{(n)}\,, \qquad \delta A_\mu^{(n)} = \frac{1}{2}\, \partial_\mu\, \xi^{(n)} + \frac{c_n^{21}}{2}\, \xi_\mu^{(n)}\,, \qquad \delta\phi^{(n)} = c_n^{10}\, \xi^{(n)}\,. \tag{3.42}$$

We see that this system, Eqs.(3.39, 3.40, 3.41, 3.42), coincides precisely with the spin-two

Stueckelberg system on AdS_{d+1}, once we redefine c_n's as

$$c_n^{12} = -\sqrt{2}M_n, \;\; c_n^{21} = \frac{M_n}{\sqrt{2}},$$
$$c_n^{01} = -\sqrt{\frac{d}{2\,(d-1)}(M_n{}^2 + d - 1)}, \;\; c_n^{10} = \sqrt{\frac{2\,(d-1)}{d}(M_n{}^2 + d - 1)}.$$

It is also known that the Stueckelberg gauge symmetries can uniquely fix free parts in the field equations or equivalently in the action. Therefore, from the knowledge of the gauge transformations Eq.(3.42), we can fully reconstruct the field equations Eqs.(3.39, 3.40, 3.41). In practice, the gauge transformations are much simpler to handle than the field equations. Note that the modes which are neither in the kernel of raising operators nor in the kernel of lowering operators always combine together and undergo the Stueckelberg mechanism for massive spin-two fields.

Before classifying possible boundary conditions, we summarize Stueckelberg spin-two system and Goldstone mode decomposition pattern of it. For general values of the masses, Stueckelberg spin-two system describes the same physical degree of freedom as a massive spin-two field (having maximal number of longitudinal polarizations). This is because spin-one and spin-zero fields can be algebraically removed by the gauge symmetries Eq.(3.42), corresponding to the unitary gauge fixing. However, such gauge fixing is not possible if the masses take special values:

$$M_n^2 = 0 \qquad \text{and} \qquad M_n^2 = -\frac{(d-1)}{r_{AdS}^2}\,. \tag{3.43}$$

At these special values of the mass parameters, the Stueckelberg system breaks into subsystems which can be deduced just from the gauge transformation.

For the situation that $M_n = 0$, the gauge transformations are

$$\delta\, h_{\mu\nu} = \nabla_{(\mu}\,\xi_{\nu)}\,, \qquad \delta\, A_\mu = \frac{1}{2}\,\partial_\mu\,\xi\,, \qquad \delta\,\phi = \frac{1}{r_{AdS}}\sqrt{\frac{2}{d}}(d-1)\,\xi\,. \tag{3.44}$$

We see from the first equation that the spin-two field ought to be massless as it has the spin-two gauge symmetry. We also see that the remaining two equations are precisely the spin-one Stueckelberg system with $m^2 = 2\,d/r_{AdS}^2$. This implies that the Goldstone field of the massive spin-two is given by the massive spin-one system, which in turn was formed by the Stueckelberg system of massless spin-one and massless spin-zero fields. It should be noted that the normalization of each field is not standard.

For the situation that $M_n^2 = -(d-1)/r_{AdS}^2$, a subtlety arises as the coefficients c_n^{12}

and c_n^{21} are pure imaginary. Specifically, the relation Eq.(3.30) implies that one of the two mode functions $\Theta_n^{1|2}$, $\Theta_n^{2|2}$ and corresponding field become pure imaginary. We are thus led to redefine the mode functions $\tilde{\Theta}_n^{1|2} = \pm i\,\Theta_n^{1|2}$ and the fields $\tilde{A}_\mu = \pm i\,A_\mu$ [4]. The gauge transformations now become

$$\delta h_{\mu\nu} = \nabla_{(\mu}\,\xi_{\nu)} + \sqrt{\frac{2}{d-1}}\frac{1}{r_{AdS}}\,g_{\mu\nu}\,\xi\,, \qquad \delta A_\mu = \frac{1}{2}\,\partial_\mu\,\xi + \sqrt{\frac{d-1}{2}}\frac{1}{2\,r_{AdS}}\,\xi_\mu\,, \qquad \delta\phi = 0\,. \tag{3.45}$$

The spin-two gauge transformations and spin-one gauge transformations are coupled each other. In fact, they are precisely the Stueckelberg system of partially massless (PM) spin-two field [27]. We can always gauge-fix the spin-one field to zero, and the remanent gauge symmetry coincides with the partially-massless (PM) spin-two gauge symmetry [28]:

$$\delta h_{\mu\nu} = \nabla_\mu\,\nabla_\nu\,\lambda - \frac{1}{r_{AdS}^2}\,g_{\mu\nu}\,\lambda\,, \qquad \text{where} \qquad \lambda = r_{AdS}\sqrt{\frac{2}{d-1}}\,\xi\,. \tag{3.46}$$

Therefore, when the mass-squared hits the special value $M_n^2 = -(d-1)/r_{AdS}^2$, the Stueckelberg system breaks into a spin-two partially-massless (PM) Stueckelberg system and a massive spin-zero field of mass-squared $m^2 = (d+1)/r_{AdS}^2$, as given above in Eq.(3.41).

This spectrum decomposition pattern perfectly fits to the reducibility structure of the Verma $\mathfrak{so}(d,2)$-module $\mathcal{V}(\Delta,2)$ for spin-two field. For the special values of conformal weights, $\Delta = d$ and $\Delta = d-1$, the Verma module becomes reducible and break into

$$\begin{aligned} \mathcal{V}\,(d,\,2) \quad &= \quad \underbrace{\mathcal{D}\,(d,\,2)}_{\text{massless}\,s=2} \oplus \underbrace{\mathcal{D}\,(d+1,\,1)}_{\text{massive}\,s=1}\,, \\ \mathcal{V}\,(d-1,\,2) \quad &= \quad \underbrace{\mathcal{D}\,(d-1,\,2)}_{\text{PM}\,s=2} \oplus \underbrace{\mathcal{D}\,(d+1,\,0)}_{\text{massive}\,s=0}\,. \end{aligned} \tag{3.47}$$

Here, $\mathcal{D}\,(d,\,2)$ and $\mathcal{D}\,(d-1,\,2)$ are irreducible representations of massless and partially massless states, respectively. Using the relation between the mass-squared and the conformal weights

$$m^2_{\text{spin}-1}\,r^2_{AdS} = \Delta\,(\Delta-d) + (d-1) \qquad \text{and} \qquad m^2_{\text{spin}-0,\,2}\,r^2_{AdS} = \Delta\,(\Delta-d)\,, \tag{3.48}$$

one finds that $\mathcal{D}\,(d+1,\,1)$ corresponds to spin-one field with $m^2 = 2\,d/r_{AdS}^2$, and $\mathcal{D}\,(d+1,\,0)$ corresponds to spin-zero field with $m^2 = (d+1)\,/r_{AdS}^2$. This result exactly matches with the spectrum decomposition patterns we analyzed above.

[4] In the path integral formulation, this amounts to choosing that the integration contour purely imaginary.

Here, we tabulate the four types of fields that appear at special values of masses (the four irreducible representations that appear in Eqs.((3.47)), as they will be shown to arise as the ground modes of the Sturm-Liouville problems with appropriate boundary conditions in section 3.3.2.

type	$\mathcal{D}(\Delta, s)_{\mathfrak{so}(d,2)}$	field	mass-squared
type I	$\mathcal{D}(d+1,1)$	massive Stueckelberg spin-one	$m^2 = 2\,d/r^2_{AdS}$
type II	$\mathcal{D}(d+1,0)$	massive spin-zero field	$m^2 = (d+1)\,/r^2_{AdS}$
type III	$\mathcal{D}(\,d,\,2\,)$	massless spin-two	$m^2 = 0$
type IV	$\mathcal{D}(d-1,2)$	partially-massless Stueckelberg spin-two	$m^2 = -\,(d-1)\,/r^2_{AdS}$

Table 1: *The types of field involved in the inverse Higgs mechanism when spin-two Stueckelberg systems decompose into spin-two gauge field and Goldstone field. Type I and II are Goldstone fields of spin-zero and spin-one. In AdS space, these Goldstone fields are massive. Type III is massless, spin-two gauge field. Type IV is partially massless, spin-two gauge field.*

3.3.2 Boundary conditions for spin-two field

With mode expansions at hand, we now classify possible boundary conditions. In the spin-one on a flat background, boundary conditions of different component fields (spin-one and spin-zero in that case) were related. This property continues to hold for the spin-two situation. For instance, suppose we impose Dirichlet boundary condition for the spin-one component field in AdS_{d+1}, $\Theta^{1|2}|_{\theta=\pm\alpha} = 0$. Then, the spectrum generating complex Eq.(3.36) immediately imposes unique boundary conditions for other component fields:

$$\begin{aligned} &Ł_{-2}\,\Theta^{2|2}_n \sim \Theta^{1|2}_n\,, \qquad && Ł_{-2}\,\Theta^{2|2}|_{\theta=\pm\alpha} = 0\,, \\ &Ł_{d-3}\,\Theta^{0|2}_n \sim \Theta^{1|2}_n\,, \qquad && Ł_{d-3}\,\Theta^{0|2}|_{\theta=\pm\alpha} = 0\,. \end{aligned} \tag{3.49}$$

Likewise, if we impose a boundary condition to a component field, the spectrum generating complex Eq.(3.36) uniquely fixes boundary conditions for all other component fields. The minimal choice is imposing the Dirichlet bounary condition to one of the component fields. As there are $s+1=3$ component fields (spin-two, spin-one and spin-zero), there are three possible minimal boundary conditions:

$$\begin{array}{llccc}
\textbf{B.C. 1:} & \{ & \Theta^{2|2}| = 0\,, & Ł_{d-2}\,\Theta^{1|2}| = 0\,, & Ł_{d-2}\,Ł_{d-3}\,\Theta^{0|2}| = 0 \;\; \} \\
\textbf{B.C. 2:} & \{ & Ł_{-2}\,\Theta^{2|2}| = 0\,, & \Theta^{1|2}| = 0\,, & Ł_{d-3}\,\Theta^{0|2}| = 0 \;\; \} \\
\textbf{B.C. 3:} & \{ & Ł_{-1}\,Ł_{-2}\,\Theta^{2|2}| = 0\,, & Ł_{-1}\,\Theta^{1|2}| = 0\,, & \Theta^{0|2}| = 0 \;\; \}
\end{array} \tag{3.50}$$

where $\Theta|$ is a shorthand notation for the boundary values, $\Theta|_{\theta=\pm\alpha}$. We reiterate that the boundary conditions on each set are automatically fixed by the spectrum generating complex Eq.(3.36). We now examine mass spectra and mode functions for each of the three types of boundary conditions, Eq.(3.50).

To deliver our exposition clear and explicit, we shall perform the analysis for $d = 2$, viz. compactification of AdS_4 to AdS_3 times angular wedge, where the mode solutions of the Strum-Liouville prpblem, Eq. (3.36), are elementary:

$$\Theta^{2|2} = \begin{cases} \sec\theta\,(\tan\theta\,\cos(z_n\theta) - z_n\,\sin(z_n\theta))\,, & \text{odd parity} \\ \sec\theta\,(\tan\theta\,\sin(z_n\theta) + z_n\,\cos(z_n\theta))\,, & \text{even parity} \end{cases} \tag{3.51}$$

$$\Theta^{1|2} = \begin{cases} \sec\theta\,\sin(z_n\theta)\,, & \text{odd parity} \\ \sec\theta\,\cos(z_n\theta)\,, & \text{even parity} \end{cases} \tag{3.52}$$

$$\Theta^{0|2} = \begin{cases} \sec\theta\,\sin(z_n\theta)\,, & \text{odd parity} \\ \sec\theta\,\cos(z_n\theta)\,, & \text{even parity} \end{cases} \tag{3.53}$$

with $z_n^2 = M_n^2 + 1$. Note that the Sturm-Liouville equation and the boundary condition are symmetric under the parity $\theta \to -\theta$, so the solutions are also labelled as either odd or even parity of θ.

We begin our analysis with **B.C. 1**. Substituting the above mode functions to the **B.C. 1**, we get the same expression for spin-two and spin-one component fields except the condition that the parity of the mode functions must take opposite values:

$$\begin{cases} \sec\theta\,(\tan\theta\,\cos(z_n\theta) - z_n\,\sin(z_n\theta))\,|_{\theta=\pm\alpha}\,, & \text{odd } \Theta^{2|2} \text{ and even } \Theta^{1|2} \\ \sec\theta\,(\tan\theta\,\sin(z_n\theta) + z_n\,\cos(z_n\theta))\,|_{\theta=\pm\alpha}\,, & \text{even } \Theta^{2|2} \text{ and odd } \Theta^{1|2} \end{cases}\,. \tag{3.54}$$

We also get the boundary condition for spin-zero component $\Theta^{0|2}$ as

$$\begin{cases} z_n\,\sec\theta\,(\tan\theta\,\cos(z_n\theta) - z_n\,\sin(z_n\theta))\,|_{\theta=\pm\alpha}, & \text{odd } \Theta^{(0|2)} \\ z_n\,\sec\theta\,(\tan\theta\,\sin(z_n\theta) + z_n\,\cos(z_n\theta))\,|_{\theta=\pm\alpha}, & \text{even } \Theta^{(0|2)} \end{cases}\,. \tag{3.55}$$

We note that, modulo the overall spectral factor z_n, this spin-zero boundary condition is the same as the boundary conditions Eq.(3.54). This agreement is not accidental. Once again, they are consequences of the spectrum generating complex Eq.(3.36) and the boundary condition Eq.(3.49).

In general, solutions of each boundary condition, z_n, depend on the domain of angular wedge, α. They are the AdS-counterpart of flat space Kaluza-Klein compactification, and so z_n and M_n blow up as α is sent to zero. They correspond to the “Kaluza-Klein modes”. For these modes, mode functions of each component spin fields combine and form spin-two Stueckelberg system with mass-squared, $M_n^2 = z_n^2 - 1$.

There are, however, two special solutions that are independent of α, $z_n = 1$ and $z_n = 0$. They correspond to “ground modes” and have interesting features that are not shared with the Kaluza-Klein modes. Firstly, masses of the ground modes are equal to the special masses Eq.(3.43) at which the unitary gauge-fixing ceases to work and the Stueckelberg system decomposes into subsystems. Secondly, mode function of some spin components is absent. For $z_n = 1$, the spin-two field is absent as $\Theta^{2|2} = 0$ in this case. The spin-one and spin-zero fields combine and form the Stueckelberg spin-one system of **type I**. For $z_n = 0$, only massive spin-zero field is present because $z_n = 0$ is not a solution of boundary conditions Eq.(3.54) or corresponding mode function is 0. This spin-zero field is of **type II**.

By completing the analysis to other boundary conditions, we find the following spectrums of ground modes:

B.C. 1: **type I** and **type II**
B.C. 2: **type II** and **type III** (3.56)
B.C. 3: **type III** and **type IV**

We see that B.C.1 keeps mostly spin-zero, B.C.3 keeps mostly spin-two, while B.C.2 keeps spin-zero and spin-two even. The complete spectrum of each set of boundary conditions is summarized in Fig. 1.

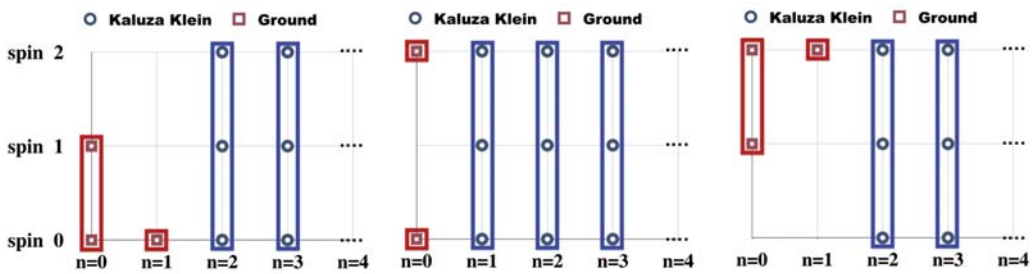

Figure 1: *Spectral pattern for three types of Dirichlet conditions, **B.C.1**, **B.C.2** and **B.C.3** from left to right. Each point represents one mode: squares are from ground modes, while circles are from Kaluza-Klein modes. Points inside the same rectangle form Stueckelberg system.*

The ground mode spectra associated with **B.C. 3** deserves further elaboration, as they in fact describe non-unitary system. Firstly, it is non-unitary because the mass-squared is below the Breitenlohner-Freedman bound of spin-two field in AdS_{d+1} space. In section **??**, we will explain Kaluza-Klein origin of this non-unitarity. Secondly, norms of some mode functions are negative-definite, implying that the Hilbert space has the structure of indefinite metric, leading classically to spectrum unboundedness and instability classically and quantum mechanically to negative probability. Explicitly, for the mode functions

$$\begin{cases} \Theta_1^{2|2} = N_5 \sec^2\theta & \qquad\qquad\qquad\qquad \textbf{type III} \\ \Theta_0^{2|2} = N_3 \sec\theta \tan\theta\,, \qquad \Theta_0^{1|2} = N_4 \tan\theta & \qquad \textbf{type IV}\,, \end{cases} \tag{3.57}$$

the norms of $\Theta_0^{2|2}$ and $\Theta_1^{2|2}$ are $-2\alpha\, N_3{}^2$ and $-\frac{2}{\tan\alpha} N_5{}^2$ and hence negative-definite for all choices of α. Such negative norms indicate that the higher-spin fields associated with these ground modes in **B.C.3** have wrong sign kinetic term.

As the distance between boundaries, α tends to $\pi/2$, the boundaries approach the time-like asymptotic boundary of the AdS_{d+2} space. In other words, our spacetime decompactifies to the AdS_{d+2} space. In this limit, though, the mass spectrum for each boundary conditions does not necessarily get to the spectrum of the massless spin-two field in AdS_{d+2} space. The reason is that some of the boundary conditions we choose are singular in this limit in the sense that mode functions are ill-defined. Take for instance the mass spectrum for **B.C. 2**. It contains the massless spin-two ground mode as well as spin-zero ground mode whose normalized mode functions are

$$\begin{cases} \Theta_0^{0|2} = N_1 \sec\theta\,, & \textbf{type II}\,, \\ \Theta_0^{2|2} = N_2 \sec^2\theta\,, & \textbf{type III}\,, \end{cases} \qquad N_1 = \frac{1}{\sqrt{2\alpha}}, \quad N_2 = \sqrt{\frac{1}{2\tan\alpha}} \quad . \tag{3.58}$$

These ground-mode functions are not normalizable in AdS_{d+2} space: the normalized mode functions vanish as N_2 vanishes in the decompactification limit. This explains why there is no massless spin-two field in the "dimensional degression" method [29]. In the next chapter, we show that, for arbitrary spacetime dimension d and spin s of higher-spin field, the mass spectrum of "dimensional degression" spectrum is the spectrum of **B.C.1** in the decompactification limit.

Summarizing,

- The mode functions of different spins in AdS_{d+1} are related to each other by the spectrum generating complex Eq.(3.36), whose structure is uniquely fixed by consideration of Kaluza-Klein compactification of higher-spin gauge transformations.
- At special values of masses, the Stueckelberg spin-two system decomposes into irreducible representations of massless or partially massless spin-two fields and massive Goldstone fields. The ground modes of Dirichlet boundary conditions Eq.(3.50) are precisely these irreducible representations in Table 1 at the special mass values.

Chapter 4

Compactification of Higher Spin Theory

In this chapter, we do Kaluza-Klein compactification of Higher Spin Theory on the background geometry described in previous chapter. As we learned from spin-two example, the most simple way is using gauge transformation. As far as we consider the free theory, all the information which a theory can have is its spectrum. The spectrum of the theory is dictated by gauge symmetry, therefore, it is enough to determine equation of motion or action to consider reduction of gauge transformation.

4.1 Compactification of Gauge Transformation

We start from gauge transformation of massless spin-s field on AdS$_{d+2}$ background.

$$\delta\hat{\phi}^{(s)}_{M_1\cdots M_s} = \hat{\nabla}_{(M_1}\hat{\xi}^{(s)}_{M_2\cdots M_s)} \tag{4.1}$$

In this section, we show that after compactification, the gauge transform becomes

$$\delta\phi^{(k)}_{\mu_1\cdots\mu_k} = \frac{k}{s}\,\nabla_{(\mu_1}\,\xi^{(k)}_{\mu_2\cdots\mu_k)} + a_1\,\text{Ł}_{-(s+k-1)}\,\xi^{(k+1)}_{\mu_1\cdots\mu_k} + a_2\,\text{Ł}_{d-(s-k)-2}\,g_{(\mu_1\mu_2}\,\xi^{(k-1)}_{\mu_3\cdots\mu_k)}\,, \tag{4.2}$$

where the coefficients are

$$a_1 = \frac{s-k}{s} \qquad \text{and} \qquad a_2 = \frac{k\,(k-1)\,(d+s+k-3)}{s\,(d+2k-5)\,(d+2k-3)}\,.$$

Eq. (4.1) decomposes into

$$\begin{aligned}\delta\hat{\phi}^{(s)}_{M_1\cdots M_s} =& \hat{\nabla}_{(M_1}\hat{\xi}^{(s)}_{M_2\cdots M_s)} = \frac{1}{s}\sum_{i=1}^{s}\hat{\nabla}_{M_i}\hat{\xi}^{(s)}_{M_1\cdots M_{i-1}M_{i+1}\cdots M_s}\\ =& \underbrace{\frac{1}{s}\sum_{i=1}^{s}\partial_{M_i}\hat{\xi}^{(s)}_{M_1\cdots M_s}}_{(1)} \underbrace{-\frac{1}{s}\sum_{\substack{i,j=1\\ i\neq j}}^{s}\hat{\Gamma}^{\lambda}_{M_iM_j}\hat{\xi}^{(s)}_{M_1\cdots M_s\lambda}}_{(2)} \underbrace{-\frac{1}{s}\sum_{\substack{i,j=1\\ i\neq j}}^{s}\hat{\Gamma}^{\theta}_{M_iM_j}\hat{\xi}^{(s)}_{M_1\cdots M_s\theta}}_{(3)}\end{aligned} \tag{4.3}$$

Let's define $\psi^{(s-n)}_{\mu_1\cdots\mu_{s-n}} \equiv \hat{\phi}^{(s)}_{\mu_1\cdots\mu_{s-n}\theta(n)}$, $\zeta^{(s-n)}_{\mu_1\cdots\mu_{s-n-1}} \equiv \hat{\xi}^{(s)}_{\mu_1\cdots\mu_{s-n-1}\theta(n)}$ ($\theta(n)$ denotes n θs), and consider their gauge transform. For $\delta\psi^{(s-n)}_{\mu_1\cdots\mu_{s-n}}$ each part of equation (4.3) become,

$$\begin{aligned}
(1) =& \frac{1}{s}\sum_{i=1}^{s-n}\partial_{\mu_i}\zeta^{(s-n)}_{\mu_1\cdots\mu_{s-n}} + \frac{n}{s}\partial_\theta\zeta^{(s-n+1)}_{\mu_1\cdots\mu_{s-n}} \\
(2) =& \frac{1}{s}\sum_{\substack{i,j=1\\ i\neq j}}^{s-n}\Gamma^\lambda_{\mu_i\mu_j}\zeta^{(s-n)}_{\mu_1\cdots\mu_{s-n}\lambda} + 2\frac{n(s-n)}{s}\tan\theta\,\zeta^{(s-n+1)}_{\mu_1\cdots\mu_{s-n}} \\
(3) =& -\frac{1}{s}\sum_{\substack{i,j=1\\ i\neq j}}^{s-n} g_{\mu_i\mu_j}\tan\theta\zeta^{(s-n-1)}_{\mu_1\cdots\mu_{s-n}} + \frac{n(n-1)}{s}\tan\theta\,\zeta^{(s-n+1)}_{\mu_1\cdots\mu_{s-n}}
\end{aligned} \tag{4.4}$$

Therefore,

$$\begin{aligned}
\delta\psi^{(s-n)}_{\mu_1\cdots\mu_{s-n}} =& \frac{s-n}{s}\nabla_{(\mu_1}\zeta^{(s-n)}_{\mu_2\cdots\mu_{s-n})} + \frac{n}{s}[\partial_\theta - (2s-n-1)\tan\theta]\zeta^{(s-n+1)}_{\mu_1\cdots\mu_{s-n}} \\
&+ \frac{(s-n)(s-n-1)}{s}\tan\theta\, g_{(\mu_1\mu_2}\zeta^{(s-n-1)}_{\mu_3\cdots\mu_{s-n})}
\end{aligned} \tag{4.5}$$

Up to now, there are bare $\tan\theta$ factors rather than $Ł_m$ operators and the form of gauge transform is differ from Eq.(4.2). A reason is that naively defined quantities $\psi^{(k)}$ and $\zeta^{(k)}$ are not proper variables to describe lower dimensional degrees of freedom. We don't fix any gauge during compactification therefore we expect massive fields in Stueckelberg formalism after compactification. As we reviewed in chapter (2), variables describing Stueckelberg massive field must be (double)-traceless and $\psi^{(k)}$, $\zeta^{(k)}$ are not. Therefore we have to consider linear combination of naive quantities to get (double)-traceless quantities.

Due to the (double) traceless conditions of $\hat{\phi}$ and $\hat{\xi}$, ψ and ζ are not (double) traceless.

$$\begin{aligned}
0 = \hat{\phi}^{(s)}_{M_1\cdots M_s}\hat{g}^{M_1M_2}\hat{g}^{M_3M_4} &= \hat{\phi}^{(s)}_{\mu_1\mu_2\mu_3\mu_4\cdots M_s}\hat{g}^{\mu_1\mu_2}\hat{g}^{\mu_3\mu_4} \\
&\quad +2\hat{\phi}^{(s)}_{\mu_1\mu_2\theta\theta\cdots M_s}\hat{g}^{\mu_1\mu_2}\hat{g}^{\theta\theta} + \hat{\phi}^{(s)}_{\theta\theta\theta\theta\cdots M_s}\hat{g}^{\theta\theta}\hat{g}^{\theta\theta} \\
&= \cos^4\theta\Big(\hat{\phi}^{(s)}_{\mu_1\mu_2\mu_3\mu_4\cdots M_s}g^{\mu_1\mu_2}g^{\mu_3\mu_4} \\
&\quad +2\hat{\phi}^{(s)}_{\mu_1\mu_2\theta\theta\cdots M_s}g^{\mu_1\mu_2} + \hat{\phi}^{(s)}_{\theta\theta\theta\theta\cdots M_s}\Big)
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
0 = \hat{\xi}^{(s)}_{M_1\cdots M_{s-1}}\hat{g}^{M_1M_2} &= \hat{\xi}^{(s)}_{\mu_1\mu_2\cdots M_{s-1}}\hat{g}^{\mu_1\mu_2} + \hat{\xi}^{(s)}_{\theta\theta\cdots M_{s-1}}\hat{g}^{\theta\theta} \\
&= \cos^2\theta\Big(\hat{\xi}^{(s)}_{\mu_1\mu_2\cdots M_{s-1}}g^{\mu_1\mu_2} + \hat{\xi}^{(s)}_{\theta\theta\cdots M_{s-1}}\Big)
\end{aligned} \tag{4.7}$$

In terms of $\psi^{(s)}$ and $\zeta^{(s)}$,

$$\psi^{(s)}_{\mu_1\mu_2\mu_3\mu_4\cdots\mu_s} g^{\mu_1\mu_2} g^{\mu_3\mu_4} + 2\psi^{(s-2)}_{\mu_1\mu_2\mu_5\cdots\mu_s} g^{\mu_1\mu_2} + \psi^{(s-4)}_{\mu_5\cdots\mu_s} = 0 \tag{4.8}$$

$$\zeta^{(s)}_{\mu_1\mu_2\cdots\mu_{s-1}} g^{\mu_1\mu_2} + \zeta^{(s-2)}_{\mu_3\cdots\mu_{s-1}} = 0 \tag{4.9}$$

These relations are true for every $s-n$. From now on, spin of original $d+1$ dimensional field will be denoted as s_m and s will denote spin of d dimensional fields.($0 \leq s \leq s_m$)

To get the d dimensional double traceless fields $\phi^{(s)}$, we have to consider linear combination of $\psi^{(s)}$s. Without derivatives, most general ψ first order term with symmetric s indices is $g_{(\mu_1\mu_2} \cdots g_{\mu_{2n-1}\mu_{2n}} \psi^{(s-2n+2k)}_{\mu_{2n+1}\mu_{2n+2}\cdots\mu_s)\nu_1\nu_2\cdots\nu_{2k-1}\nu_{2k}} g^{\nu_1\nu_2} \cdots g^{\nu_{2k-1}\nu_{2k}}$. Let's denote this tensor as $\psi^{(n,s,k)}_{\mu_1\cdots\mu_s}$. In terms of $\psi^{(n,s,k)}$, equation (4.8) becomes

$$\psi^{(n,s,k+2)}_{\mu_1\cdots\mu_s} + 2\psi^{(n,s,k+1)}_{\mu_1\cdots\mu_s} + \psi^{(n,s,k)}_{\mu_1\cdots\mu_s} = 0 \tag{4.10}$$

for any n and k. Up to this relation, the most general linear combination is

$$\phi^{(s)}_{\mu_1\cdots\mu_s} = \sum_{n=0}^{[s/2]} \{\alpha_{n,s}\psi^{(n,s,0)}_{\mu_1\cdots\mu_s} + \beta_{n,s}\psi^{(n,s,1)}_{\mu_1\cdots\mu_s}\} \tag{4.11}$$

However all β_n's should be vanished otherwise the gauge transformation of $\phi^{(s)}$ contain divergence of gauge parameter and that is not we want.[1] To calculate the double trace of $\phi^{(s)}$, one have to know the double trace of $\psi^{(n,s,k)}$. The result is

$$\begin{aligned}
g^{\nu_1\nu_2}\psi^{(n,s,k)}_{\nu_1\nu_2\mu_1\cdots\mu_{s-2}} &= A_{n,s}\psi^{(n-1,s-2,k)}_{\mu_1\cdots\mu_{s-2}} + B_{n,s}\psi^{(n,s-2,k+1)}_{\mu_1\cdots\mu_{s-2}} \\
g^{\nu_1\nu_2}g^{\nu_3\nu_4}\psi^{(n,s,k)}_{\nu_1\nu_2\nu_3\nu_4\mu_1\cdots\mu_{s-4}} &= A_{n,s}A_{n-1,s-2}\psi^{(n-2,s-4,k)}_{\mu_1\cdots\mu_{s-4}} \\
&\quad +(A_{n,s}B_{n-1,s-2} + B_{n,s}A_{n,s-2})\psi^{(n-1,s-4,k+1)}_{\mu_1\cdots\mu_{s-4}} \\
&\quad +B_{n,s}B_{n,s-2}\psi^{(n,s-4,k+2)}_{\mu_1\cdots\mu_{s-4}} \\
&\equiv C^0_{n,s}\psi^{(n-1,s-2,k)}_{\mu_1\cdots\mu_{s-2}} \\
&\quad +C^1_{n,s}\psi^{(n-1,s-4,k+1)}_{\mu_1\cdots\mu_{s-4}} + C^2_{n,s}\psi^{(n,s-4,k+2)}_{\mu_1\cdots\mu_{s-4}}
\end{aligned} \tag{4.12}$$

[1]With $\beta_n = 0$, the number of equations become twice of number of variables α_n's. Therefore existance of solution seems non-trivial. However the double traceless condition of gauge fields are equivalent to traceless condition of gauge parameter and there is a solution.

where $A_{n,s}=\frac{2n(d+1)+4n(s-n-1)}{s(s-1)}$, $B_{n,s}=\frac{(s-2n)(s-2n-1)}{s(s-1)}$ and

$$\begin{aligned} C^0_{n,s} &\equiv A_{n,s}A_{n-1,s-2} \\ C^1_{n,s} &\equiv A_{n,s}B_{n-1,s-2}+B_{n,s}A_{n,s-2} \\ C^2_{n,s} &\equiv B_{n,s}B_{n,s-2} \end{aligned} \tag{4.13}$$

Therefore $\phi^{(s)}_{\nu_1\cdots\mu_s}g^{\nu_1\nu_2}g^{\nu_3\nu_4}$ is

$$\begin{aligned} & \alpha_{0,s}\left[C^2_{0,s}\psi^{(0,s-4,2)}\right] \\ + & \alpha_{1,s}\left[C^2_{1,s}\psi^{(1,s-4,2)}+C^1_{1,s}\psi^{(0,s-4,1)}\right] \\ + & \sum_{n=2}^{[s/2]}\alpha_{n,s}\left[C^2_{n,s}\psi^{(n,s-4,2)}+C^1_{n,s}\psi^{(n-1,s-4,1)}+C^0_{n,s}\psi^{(n-2,s-4,0)}\right] \end{aligned} \tag{4.14}$$

After rearrangement,

$$\sum_{n=0}^{[s/2]-2}\alpha_{n,s}C^2_{n,s}\psi^{(n,s-4,2)}+\alpha_{n+1,s}C^1_{n+1,s}\psi^{(n,s-4,1)}+\alpha_{n+2,s}C^0_{n+2,s}\psi^{(n,s-4,0)} \tag{4.15}$$

By equation (4.10), double traceless condition is

$$\alpha_{n,s}C^2_{n,s}:\alpha_{n+1,s}C^1_{n+1,s}:\alpha_{n+2,s}C^0_{n+2,s}=1:2:1 \tag{4.16}$$

At first sight, it seems there are too many equations: $\frac{\alpha_{n+1,s}}{\alpha_{n,s}}=\frac{2C^2_{n,s}}{C^1_{n+1,s}}$, $\frac{\alpha_{n+1,s}}{\alpha_{n,s}}=\frac{C^1_{n,s}}{2C^0_{n+1,s}}$. However one can check $\frac{2C^2_{n,s}}{C^1_{n+1,s}}=\frac{C^1_{n,s}}{2C^0_{n+1,s}}$ for any n and there exist a solution up to overall normalization. With $\alpha_{0,s}=1$,

$$\begin{aligned} \alpha_{n,s} &= \frac{1}{4^n n!}\frac{s(s-1)\cdots(s-(2n-1))}{(s+(d+1)/2-3)(s+(d+1)/2-4)\cdots(s+(d+1)/2-2-n)} \\ &= \frac{1}{4^n n!}\frac{\Gamma(s+1)\Gamma(s+(d+1)/2-2-n)}{\Gamma(s-2n+1)\Gamma(s+(d+1)/2-2)} \end{aligned} \tag{4.17}$$

From the equation (4.5), gauge transformation of $\psi^{(n,s,0)}$ is

$$\begin{aligned} \delta\psi^{(n,s,0)}_{\mu_1\cdots\mu_s} = \frac{s-2n}{s_m}\nabla_{(\mu_1}\zeta^{(n,s-1,0)}_{\mu_2\cdots\mu_s)} &+ \frac{s_m-s+2n}{s_m}[\partial_\theta-(s_m+s-2n-1)\tan\theta]\zeta^{(n,s,0)}_{\mu_1\cdots\mu_s} \\ &+ \frac{(s-2n)(s-2n-1)}{s_m}\tan\theta\zeta^{(n+1,s,0)}_{\mu_1\cdots\mu_s} \end{aligned} \tag{4.18}$$

where the notation $\zeta^{(n,s,k)}$ is the same notation which already appeared for ψ.(s is the number of indices rather than related spin)

Let's concentrate on part of gauge transformation of $\phi^{(s)}$ which contain gauge parameters with covariant derivative, because this part do not mix with the other.

$$\frac{1}{s_m}\nabla_{(\mu_1}\sum_{n=0}^{[s/2]}\alpha_{n,s}(s-2n)\zeta^{(n,s-1,0)}_{\mu_2\cdots\mu_s)} \tag{4.19}$$

Define $\xi^{(s)}_{\mu_1\cdots\mu_{s-1}} \equiv \sum_{n=0}^{[s/2]}\frac{s-2n}{s}\alpha_{n,s}\zeta^{(n,s-1,0)}_{\mu_1\cdots\mu_{s-1}}$ and calculate trace of $\xi^{(s)}$.

$$\begin{aligned}\xi^{(s)}_{\nu_1\nu_2\mu_1\cdots\mu_{s-3}}g^{\nu_1\nu_2} &= \sum_{n=0}^{[s/2]}\frac{s-2n}{s}\alpha_{n,s}(A_{n,s-1}\zeta^{(n-1,s-3,0)}_{\mu_1\cdots\mu_{s-3}}+B_{n,s-1}\zeta^{(n,s-3,1)}_{\mu_1\cdots\mu_{s-3}}) \qquad (4.20)\\ &= \sum_{n=0}^{[s/2]-1}(\frac{s-2(n+1)}{s}\alpha_{n+1,s}A_{n+1,s-1}\zeta^{(n,s-3,0)}_{\mu_1\cdots\mu_{s-3}}+\frac{s-2n}{s}\alpha_{n,s}B_{n,s-1}\zeta^{(n,s-3,1)}_{\mu_1\cdots\mu_{s-3}})\end{aligned}$$

One can show that $(s-2(n+1))\alpha_{n+1,s}A_{n+1,s-1} = (s-2n)\alpha_{n,s}B_{n,s-1}$ for any n and with equation (4.9), one can recognize that $\xi^{(s)}$ is traceless.

Let's rewrite the other part with this traceless combination $\xi^{(s)}$. Final result should have only $\xi^{(s+1)}_{\mu_1\cdots\mu_s}$ and $g_{(\mu_1\mu_2}\xi^{(s-1)}_{\mu_3\cdots\mu_s)}$, otherwise the gauge transformation will not preserve double tracelessness of field. $g_{(\mu_1\mu_2}\xi^{(s-1)}_{\mu_3\cdots\mu_s)}$ do not contain $\zeta^{(0,s,0)}$. Therefore $\zeta^{(0,s,0)}$ term in equation (20) will fix the coefficient of $\xi^{(s+1)}_{\mu_1\cdots\mu_s}$ and the coefficient is $\frac{s_m-s}{s_m}[\partial_\theta-(s_m+s-1)\tan\theta]$. Now $Ł_m$ operator start to appear. One can guess that final result contain $\frac{s_m-s}{s_m}Ł_{-(s_m+s-1)}\xi^{(s+1)}$ and the remnant is little complicate but after some massage one get the compact expression,

$$\frac{s(s-1)(d+s_m+s-3)}{s_m(d+2s-5)(d+2s-3)}Ł_{d-(s_m-s)-3}\,g_{(\mu_1\mu_2}\xi^{(s-1)}_{\mu_3\cdots\mu_s)} \tag{4.21}$$

In summary, the final expression of gauge transformation of $\phi^{(s)}$ in terms of our previous convention is

$$\begin{aligned}\delta\phi^{(k)}_{\mu_1\cdots\mu_k} &= \frac{k}{s}\nabla_{(\mu_1}\xi^{(k)}_{\mu_2\cdots\mu_k)}+\frac{s-k}{s}Ł_{-(s+k-1)}\xi^{(k+1)}_{\mu_1\cdots\mu_s} \qquad (4.22)\\ &\quad+\frac{k(k-1)(d+s+k-3)}{s(d+2k-5)(d+2k-3)}Ł_{d-(s-k)-2}\,g_{(\mu_1\mu_2}\xi^{(k-1)}_{\mu_3\cdots\mu_k)}\end{aligned}$$

4.2 Boundary Conditions and Spectrum

Having identified the correct gauge transformations Eq.(4.2), we now derive the relations between expansion modes $\Theta^{k|s}_n$s and their differential relations. Requiring each term in the gauge transformations Eq.(4.2) expanded by the same mode functions, we get the relations

$$\begin{pmatrix} 0 & Ł_{d-(s-k)-2} \\ Ł_{-(s+k-2)} & 0 \end{pmatrix} \begin{pmatrix} \Theta_n^{k|s} \\ \Theta_n^{k-1|s} \end{pmatrix} = \begin{pmatrix} c_n^{k-1|k}\, \Theta_n^{k|s} \\ c_n^{k|k-1}\, \Theta_n^{k-1|s} \end{pmatrix}, \quad (4.23)$$

These relations determine the Sturm-Liouville differential equations of $\Theta_n^{k|s}$'s for all $k = 0, \cdots, s$:

$$Ł_{d-(s-k)-2}\, Ł_{-(s+k-2)}\, \Theta_n^{k|s} = c^{k|k-1}\, c^{k-1|k}\, \Theta_n^{k|s}\,, \quad (4.24)$$

$$Ł_{-(s+k-1)}\, Ł_{d-(s-k)-1}\, \Theta_n^{k|s} = c^{k|k+1}\, c^{k+1|k}\, \Theta_n^{k|s}\,. \quad (4.25)$$

Here, $-M_{n,k|s}^2$ is used to represent the n-th characteristic value of the Sturm-Liouville prolems Eqs.(4.24,4.25). One can show that Eq.(4.24) and Eq.(4.25) are equivalent since the identity

$$Ł_m\, Ł_n - Ł_{n-1}\, Ł_{m+1} = (n - m - 1)$$

related the Sturm-Liouville operators are related each other and the characteristic values are related each other by

$$M_{n,k|s}^2 = M_{n,k+1|s}^2 + d + 2k - 3. \quad (4.26)$$

All relations is summarized by the spin-s spectrum generating complex :

$$\begin{array}{ccccl} & \Theta_n^{s|s} & & & \\ Ł_{d-2} & \upharpoonleft\downharpoonright & Ł_{-(2s-2)} & & -M_{n,s|s}^2 = -M_n^2 \\ & \vdots & & & \qquad \vdots \\ Ł_{d-(s-k-1)-2} & \upharpoonleft\downharpoonright & Ł_{-(s+(k+1)-2)} & & -M_{n,k+1|s}^2 = -\big(M_n^2 + (s-k-1)\,(d+s+k-3)\big) \\ & \Theta_n^{k|s} & & & \\ Ł_{d-(s-k)-2} & \upharpoonleft\downharpoonright & Ł_{-(s+k-2)} & & -M_{n,k|s}^2 = -\big(M_n^2 + (s-k)\,(d+s+k-4)\big) \\ & \vdots & & & \qquad \vdots \\ Ł_{d-s-1} & \upharpoonleft\downharpoonright & Ł_{-(s-1)} & & -M_{n,1|s}^2 = -\big(M_n^2 + (s-1)\,(d+s-3)\big) \\ & \Theta_n^{0|s} & & & \end{array} \quad (4.27)$$

Here, $M_{n,s|s}^2$ is the mass-squared of n-th mode of spin-s field. They in turn determine mass-squared of lower spin fields, $k = s-1, s-2, \cdots, 1$. This spectrum-generating complex among mode functions enables us to interpret the gauge transformation Eq.(4.2) as Stueckelberg gauge transformations. Let us choose, for convenience, relative normalizations in Eq.(4.27) as

$$Ł_{-(s+k-2)}\, \Theta_n^{k|s} = -a_{k|s}\, M_{n,k|s}\, \Theta_n^{k-1|s} \quad \text{and} \quad Ł_{d-(s-k)-2}\, \Theta_n^{k-1|s} = \frac{M_{n,k|s}}{a_{k|s}}\, \Theta_n^{k|s} \quad (4.28)$$

where factors independent of mode index n are put together to

$$a_{k|s}^2 = \frac{k\,(d+s+k-3)}{(s-k+1)\,(2k+d-3)}.$$

Then, the gauge transformation Eq.(4.2) precisely gives rise to the Stueckelberg spin-s gauge transformations in AdS$_{d+1}$ space previously derived in [27]:

$$\delta\phi^{(k)}_{\mu_1\cdots\mu_k} = \frac{k}{s}\,\nabla_{(\mu_1}\,\xi^{(k)}_{\mu_2\cdots\mu_k)} + \alpha_k\,\xi^{(k+1)}_{\mu_1\cdots\mu_k} + \beta_k\,g_{(\mu_1\mu_2}\xi^{(k-1)}_{\mu_3\cdots\mu_k)}\,, \tag{4.29}$$

where

$$\begin{aligned}
\alpha_k^2 &= \frac{(k+1)\,(s-k)\,(d+s+k-2)}{s^2\,(d+2k-1)}\,\big(M^2+(s-k-1)\,(d+s+k-3)\big)\,, \qquad (4.30)\\
\beta_k &= -\frac{(k-1)}{(d+2k-5)}\,\alpha_{k-1}.
\end{aligned}$$

Here, the dependence on mode n enters only through the mass-squared $M^2 := M_n^2$. Apart from this, all modes of spin-k fields have the same structure of gauge transformations. Therefore, spin-k gauge transformation of n-th mode is simply the Stueckelberg gauge transformation of spin-k field with mass M_n. In turn, these gauge transformations completely fix the equation of motion for each spin $k = 0, 1, \cdots, s$ and for each mode n. They constitute the Kaluza-Klein modes.

As for lower spin cases, were if M_n^2 is tuned to special negative values, it can happen that $\alpha_k = 0$. These special values are the values at which $M_{n,k+1|s} = 0$ as well. In this case, the Stueckelberg system of spin-s field decompose into two subsystems: the partially massless spin-s system of depth $t = (s-k-1)$ and the Stueckelberg spin-k field. Importantly, the massless spin-s field is also part of the spectrum, since it is nothing but partially massless spin-s field of depth-0 [2]. Together, they constitute the ground modes:

[2]Note that our conventions of the mass-squared of higher-spin field is such that it is zero when the higher-spin fields have gauge symmetries. So, it differs form the mass-squared that appears in the AdS Fierz-Pauli equation, $\left(\nabla^2+\kappa_s^2\right)\phi_{\mu_1\,\mu_2\,\cdot\mu_s} = 0$.

- The upper subsystem consists of $\left(\phi^{(s)},\, \phi^{(s-1)},\, \cdots\, \phi^{(k+1)}\right)$ and forms the Stueckelberg system of partially massless field with depth $t = (s-k-1)$. Their mass spectra are given by

$$M^2 = -t\,(d+2s-t-4)\,/r_{AdS}^2. \tag{4.31}$$

- The lower subsystem consists of $\left(\phi^{(k)},\, \phi^{(k-1)},\, \cdots\, \phi^{(0)}\right)$ and forms the Stueckelberg spin-k field. Their mass spectra are given by

$$M^2 = (s-k+1)(d+s+k-3)/r_{AdS}^2. \tag{4.32}$$

Group theoretically, the decomposition pattern of the ground modes can be understood in terms of the Verma $\mathfrak{so}(d,2)$-modules. At generic conformal weight Δ, the Verma module $\mathcal{V}\,(\Delta,\, s)$ is irreducible. At special values of $\Delta = d+k-1$, however, $\mathcal{V}(\Delta, s)$ decomposes into $\mathfrak{so}(d,2)$ irreducible representations [32, 33]:

$$\mathcal{V}\,(d+k-1,\, s) = \mathcal{D}\,(d+k-1,\, s) \oplus \mathcal{D}\,(d+s-1,\, k)\ . \tag{4.33}$$

In Eq.(4.33), the irreducible representation $\mathcal{D}(d+k-1, s)$ represents the partially massless spin-s field, while $\mathcal{D}\,(d+s-1,\, k)$ represent the massive spin-$(k+1)$ field whose mass-squared is set by the conformal weight Δ

$$m^2\, r_{AdS}^2 = \Delta\,(\Delta-d) - (s-2)\,(d+s-2)\ . \tag{4.34}$$

We next classify all possible boundary conditions and determine the mass spectra. As for the lower-spin fields, we shall only consider boundary conditions derived from Dirichlet conditions on $\Theta^{k|s}|_{\theta=\pm\alpha} = 0$ for some k. As $0 \le k \le s$, there are $(s+1)$ possible Dirichlet conditions. The relations Eq.(4.27) then fix boundary condition for all other fields originating from the same mode functions:

$$\begin{aligned} &Ł_{-(s+k-1)} \cdots\cdots\cdots Ł_{s+\ell-2}\, \Theta^{\ell|s}\Big|_{\theta=\pm\alpha} = 0 \qquad\qquad (\ell = k, k+1, \cdots, s) \\ &Ł_{d-(s-k)-2} \cdots Ł_{d-(s-\ell-1)-2}\, \Theta^{\ell k|s}\Big|_{\theta=\pm\alpha} = 0 \qquad\qquad (\ell = k, k+1, \cdots, s) \end{aligned} \tag{4.35}$$

Below, we show that the pattern of mass spectrum takes the form of Fig.2. First, to counter cluttering indices, we define simplifying notations as $\Theta^\ell := \Theta^{k+\ell|s}$, $M_{n,\ell}^2 := M_{n,\,k+\ell|s}^2$, $U_\ell := Ł_{d-s-2+\ell+k}$ and $D_\ell = Ł_{-s+2-\ell-k}$. Then sub-complex of Eq.(4.27) can be written in the form

$$U_\ell \quad \begin{matrix} \Theta_n^\ell \\ \uparrow\downarrow \\ \Theta_n^{\ell-1} \end{matrix} \quad D_\ell \quad : \qquad -M_{n,\ell}^2 = -\left(M_n^2 + (s-k-\ell)\,(d+s+k+\ell-4)\right) \ . \tag{4.36}$$

By this complex, there is one-to-one map between Θ_n^ℓ and $\Theta_n^{\ell-1}$ for $M_{n,\ell}^2 \neq 0$. If $M_{n,\ell}^2 = 0$, there exists one additional mode Θ_0^ℓ ($\Theta_0^{\ell-1}$) when ℓ is positive (negative). This additional mode satisfies $D_l\,\Theta_0^l = 0$ for positive l, $U_l\,\Theta^{l-1}$ for negative l. After inductively applying this relation from $\ell = 0$, one can show that the structure of mode function is given by

$$\{\Theta^{k+\ell|s}\} = \{K_i^\ell,\, G^l_{a=1,\,2,\,\cdots\,\ell}\}\,, \qquad \text{and} \qquad \{\Theta^{k-\ell|s}\} = \{K_i^{-\ell},\, G^{-\ell}_{a=1,\,2,\,\cdots\,\ell}\}\,. \tag{4.37}$$

Here, K_i^ℓ's are the Kaluza-Klein modes, and G_a^ℓ's are the ground modes which satisfy the equations

$$\begin{cases} D_a\,D_{a+1}\cdots D_\ell\,G_a^\ell = 0 & \text{with } \ D_k\,D_{k+1}\cdots D_\ell\,G_a^\ell \neq 0 \qquad \text{for all } \ a < k \\ U_{-a+1}\,U_{-a}\cdots U_{-\ell+1}\,G_a^{-\ell} = 0 & \text{with } \ U_k\,U_{k+1}\cdots U_{-\ell+1}\,G_a^{-\ell} \neq 0 \qquad \text{for all } \ a < -k+1 \end{cases} . \tag{4.38}$$

The ground modes G_i^ℓ with the same subscript i have the same characteristic value. Their characteristic value can be obtained by the first-order differential equation $D_\ell\,G_\ell^\ell = 0$ and $U_{-\ell+1}G_\ell^{-\ell} = 0$ for positive ℓ. Finally, fields corresponding to G_a^ℓ, $\ell = s-k,\,\cdots\,,a$ form the Stueckelberg system of partially massless spin-s field with depth-$(s-k-a+1)$. We explained this already in Eq.(4.30) and below. Fields corresponding to G_a^ℓ, $\ell = -a,\,\cdots\,,-k$ form the massive spin-$(k-a)$ Stueckelberg field with mass-squared $M^2 = (s-k+a+1)(d+s+k-a-3)/r_{AdS}^2$. These spectra are depicted in Fig. 2.

Summarizing,

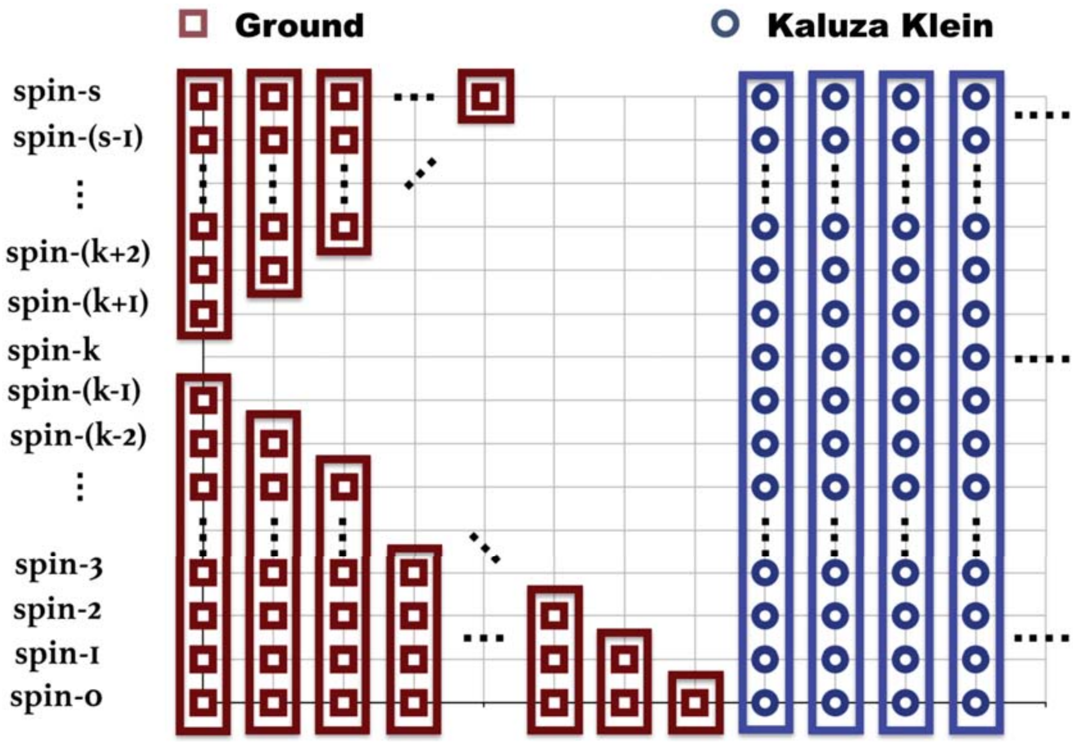

Figure 2: *Mass spectra for all possible boundary conditions characterized by Dirichlet conditions on* $\Theta^{k|s}$*. Each point represents a mode function. Points in the same rectangle form the Stueckelberg system with the highest spin. The the upper triangle consists of the Stueckelberg system of partially massless field, while the lower triangle consists of the Stueckelberg spin* $\ell = 0,\ 1, \cdots k-1$*, as described in Eq.*(4.39).

- Mass spectrum for the boundary condition characterized by Dirichlet condition at $\Theta^{k|s}$ consists of three parts. The first part is the set of massive spin-s Kaluza-Klein tower, whose mass-squared is given by the characteristic value of Sturm-Liouville problem, Eq. (4.24) with $k = s$. The second part is the set of partially massless spin-s field with depth-$(0,\ 1,\ 2, \cdots,\ s-k-1)$. The third part consists of the set of massive Stueckelberg spin $\ell = 0,\ 1, \cdots,\ k-1$ with mass-squared, $M^2 = (s-\ell+1)(d+s+\ell-3)/r_{AdS}^2$.
- The $\mathfrak{so}(d,2)$ representations of ground modes are

$$\begin{cases} \mathcal{D}\,(d+s-t-2,\ s) & \text{for} \quad t = 0,\ 1, \cdots,\ s-k-1 \\ \mathcal{D}\,(d+s-1,\ \ell) & \text{for} \quad \ell = 0,\ 1, \cdots,\ k-1 \end{cases} \tag{4.39}$$

whose masses are set in terms of conformal weights by Eq.(4.34).

4.3 Decompactification Limit

In our setup, the distance between boundaries, α ranges over $[-\alpha, \alpha]$. If the wedge α approaches $\pi/2$, the spacetime decompactifies to the entire AdS_{d+2}. In other words, the $\alpha = \pm\pi/2$ hyperplanes correspond to the AdS_{d+2} boundary. As such, one might anticipate the spectra of compactified theory approach the spectra of AdS_{d+2}. This seems to be in tension with our result as the mass spectra of spin-s field in AdS_{d+1} space arises only for special set of boundary conditions. Here, we discuss subtleties involved in the decompactification limit.

Consider the AdS_{d+2} massless spin-s spectrum from the viewpoint of AdS_{d+1} space. This is just like the $L \to \infty$ limit of flat space we studied in section 2. The $\mathcal{L}^2$ square-integrable modes of massless spin-s field form the $\mathfrak{so}(d+1,2)$-module: $\mathcal{D}(d+s-1, s)_{\mathfrak{so}(\mathrm{d}+1,2)}$. Representation theoretically, we can decompose this module into $\mathfrak{so}(d,2)$-modules, a procedure referred to as the "dimensional digression" in [29]:

$$\mathcal{D}(d+s-1, s)_{\mathfrak{so}(\mathrm{d}+1,2)} = \bigoplus_{n=0}^{\infty} \mathcal{D}(d+n+s-1, s)_{\mathfrak{so}(\mathrm{d},2)} \oplus \bigoplus_{l=0}^{s-1} \mathcal{D}(d+s-1, l)_{\mathfrak{so}(\mathrm{d},2)} \,. \quad (4.40)$$

We can relate these modules with states that arise from the compactifiation of higher-spin field, as the foliation of Fig.**??** precisely matches with the above dimensional digression. There are two kinds of $\mathfrak{so}(d,2)$-modules in the right hand side of Eq.(4.40): the first set of modules have the same spin, spin-s but different conformal dimensions, while the second set of modules have the same conformal dimension but different spins ranging over 0 to $s-1$. We se that the second set of modules in Eq.(4.40) coincide with the set of ground modes for $k = s$ ($\Theta^{s|s}|_{\theta=\pm\alpha} = 0$) in Eq.(4.39). In order to reconstruc the $\mathfrak{s}o(d+1,2)$ module in the left side of Eq.(4.40), we would then need the Kaluza-Klein modes from $k = s$ to match with the first set of modules. Below, we demonstrate this affirmatively.

In the $k = s$ case, the mass spectra of spin-s field are determined by the Sturm-Liouville equation Eq.(4.24) and the Dirichlet condition for mode functions $\Theta^{s|s}$:

$$Ł_{d-2}\, Ł_{-2(s-1)}\, \Theta_n^{s|s} = -M_n^2\, \Theta_n^{s|s} \,, \qquad \text{where} \qquad \Theta^{s|s}|_{\theta=\pm\alpha} = 0 \,. \quad (4.41)$$

The solution is given by

$$\Theta_n^{s|s} = (\cos\theta)^{\mu}\, \left(c_1\, P_\nu^\mu(\sin\theta) + c_2\, Q_\nu^\mu(\sin\theta)\right) \,, \quad (4.42)$$

where P_ν^μ and Q_ν^μ are the associated Legendre functions with arguments, $\mu = \frac{1}{2}(d+2s-3)$ and $\nu(\nu+1) = M_n^2 - \frac{1}{4}\left(1-(d+2s-4)^2\right)$. In the decompactification limit, the boundary conditions $\Theta^{s|s}|_{\theta=\pm\alpha} = 0$ take the form

$$0 = -\frac{\pi}{2}\sin A\Big((P^{\mu}_{\nu})^2 - \frac{4}{\pi^2}(Q^{\mu}_{\nu})^2\Big) - 2\cos A\, P^{\mu}_{\nu}\, Q^{\mu}_{\nu} \tag{4.43}$$

$$\simeq \begin{cases} -\frac{1}{2\pi}\sin A\,(\cos(\mu\,\pi)\,\Gamma(\mu))^2\left(\frac{2}{\epsilon}\right)^{\mu} & \text{for even } d \\ \\ -\frac{\pi}{2}\sin A\left(\frac{1}{\Gamma(1-\mu)}\right)^2\left(\frac{2}{\epsilon}\right)^{\mu} & \text{for odd } d \end{cases} \tag{4.44}$$

where $A = \pi\,(\mu+\nu)$ and $1 \gg \epsilon = 1-\sin\alpha > 0$. Therefore, it must be that $\mu+\nu$ are integer-valued in the decompactification limit. From the relation Eq.(4.34), it immediately follows that the modules that correspond to the Kaluza-Klein modes are precisely $\bigoplus_{n=0}^{\infty}\mathcal{D}(d+n+s-1,s)$.

> Spectrum for the cases of $(k = s)$ goes to the spectrum of "dimensional degression [29]" in the decompactification limit(i.e. $\alpha \to \pi/2$).

All are well so far, so one might anticipate that the spectral match with the dimensional digression continues to hold for $k \neq s$. This, however, is no longer true. The point is that some of the ground modes in Eq.(4.39) contain the modules which are not in massless spin-s modules of AdS$_{d+2}$ space, $\mathcal{D}(d+s-1,s)_{\mathfrak{so}(\mathrm{d}+1,2)}$ in Eq.(4.40). The mode functions that would potentially match are actually singular (equivalently, the normalization factor goes to zero) in the decompactification limit. In particular, the massless spin-s field in the AdS$_{d+2}$ space belongs to one of these singular modes. For spin-two case, this was already shown in Eq.(3.58). Conversely, this explains transparently why the dimensional degression [29] of AdS$_{d+2}$ space does not generate "massless" spin-s fields in AdS$_{d+1}$ space.

Chapter 5

Higher Derivative Boundary Conditions and Boundary Degrees of Freedom

We encountered eigenvalue problems with higher-derivative boundary condition(HD BC). Especially for fields with spin greater than three, HD BC is unavoidable consequence. With HD BC, an Strum-Liouville differential operator is not self-adjoint on L^2 functional space and eigenfunctions are not orthogonal nor complete. Suppose we have an arbitrary function $f(\theta)$ and eigenfunctions $\Theta_n(\theta)$ from eigenvalue problem with HD BC. In previous chapters, we just assumed $f(\theta)$ can be expanded by $\Theta_n(\theta)$ — $f(\theta) = \sum f_n \Theta_n(\theta)$ — and $f(\theta) = 0$ gives $f_n = 0$ for every n. Without orthogonality and completeness of $\Theta_n(\theta)$, each step is unclear and even we cannot get f_n from $f(\theta)$. How can we get $f_n = 0$ which correspond to lower dimensional equation of motions?

There is another issue possibly related with HD BC. As we can see from the spectrum of spin-two example and higher spin compactification, for some boundary conditions, non-unitary partially massless representations appear. Which means some boundary condition gives unitary lower dimensional theory and some boundary condition does not. In terms of HD BC, it is hard to tell which boundary condition is unitary before studying lower dimensional spectrum. How can we see the origin of non-unitarity?

In this chapter, we answer above 2 questions. These questions are closely related and solved by a single resolution. We develop a way of translating HD BC into corresponding boundary action. In terms of boundary action, the physical properties are clear we can tell which boundary condition is unitary. Also, the existence of boundary action or boundary degrees of freedom implies the extension of function space of the field degrees of freedom. We claim that extended function space and inner product gives correct resolution for the mode expansion with HD BC.

In physics literature, HD BC is dealt numerically rather than analytically. In mathematical literature, one can find analytically resolutions and the idea of extended function space [35, 36, 37, 38].

In the first section, we study simple mechanical example of a string where the HD BC naturally appears. From string example, we can learn both mathematical and physical aspects of HD BC. Especially we introduce the extended inner product. Using extended inner product, we naturally translate HD BC into boundary action. Then we apply what we learn from string example to spin-two example with specific 2-derivative boundary conditions. We point out the origin of non-unitarity from HD BC. In the last section, we describe more systematics to find the extended inner product for more HD BCs.

5.1 String Example

As the first case, we study classical field theory of an open string attached to nonrelativistic massive particles at both ends.[1] The motion of open string is subject to boundary conditions. It is intuitively clear that the endpoint particles exert boundary conditions that interpolate between Neumann and Dirichlet types. If the masses are infinite, the string endpoints are pinned to a fixed position. If the masses are zero, the string endpoints move freely. It is less obvious, however, that endpoint particles with finite mass put the open string to higher-derivative boundary conditions.

We explain its physical meaning. We will study this system in three difference ways. At the end, we will get the refined inner product and mechanism for finding boundary action from HD BC.

As the first approach, we shall start with boundary degrees of freedom, integrate them out, and convert their dynamics to HD BC for the open string. Consider an open string of tension T, stretched along x-direction $0 \leq x \leq \ell$ and vibrating with vertical amplitude $y(x,t)$. String's end points are attached to harmonic oscillator particles at $x = 0, \ell$ whose masses, vertical positions and Hooke's constants are $M_1, y_1(t), k_1$ and $M_2, y_2(t), k_2$, respectively. See Fig. 3. The system is described by the action

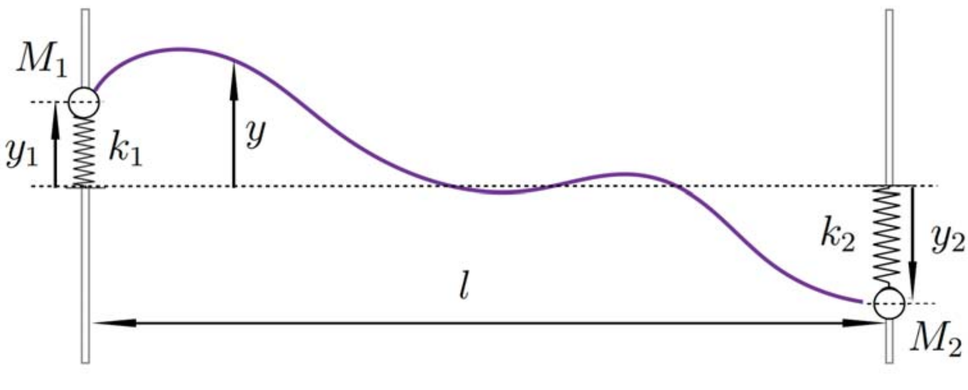

Figure 3: *Open string connected to massive particles in harmonic potential.*

[1]This example was considered in detail at [34].

$$I = \int \mathrm{d}t \big(L_{\text{string}} + L_{\text{particle, 1}} + L_{\text{particle, 2}} \big) \,, \tag{5.1}$$

where the Lagrangians of open string and massive particles are

$$L_{\text{string}} = \frac{T}{2} \int_0^l \mathrm{d}x \, \left((\partial_t y)^2 - (\partial_x y)^2 \right) \qquad \text{and} \qquad L_{\text{particle, a}} = \frac{1}{2} \left(M_a \, \dot{y}_a^2 - k_a \, {y_a}^2 \right) \quad (a = 1, 2). \tag{5.2}$$

The action completely determine dynamics of the variables $(y(x,t), y_1(t), y_2(t))$ without specifying any boundary conditions. As the string is attached to the particles, the string amplitude is related to particle positions by

$$y(x,t)\Big|_{x=0} = y_1(t) \qquad \text{and} \qquad y(x,t)\Big|_{x=\ell} = y_2(t). \tag{5.3}$$

Thus, one can alternatively eliminate the particle variables $y_1(t), y_2(t)$ and express them in terms of just the string amplitude $y(x,t)$. This then replaces the constraints Eq.(5.3) and particle actions by some boundary conditions to the string amplitude $y(x,t)$ at $x = 0, \ell$. Our goal is to derive these boundary conditions, starting from boundary actions $\int \mathrm{d}t L_{\text{particle, 1,2}}$ that are provided by the endpoint particle actions.

So, to figure out the boundary condition, we derive the field equation of the string from the action Eq.(5.1):

$$\delta I = \int \mathrm{d}t \left(-T \int \mathrm{d}x \, \delta y \left[\partial_t^2 y - \partial_x^2 y \right] + T \, \left[\delta y \, \partial_x y \right]_0^l - \sum_{a=1,2} \delta y_a (M_a \ddot{y}_a + k_a y_a) \right) \tag{5.4}$$

Imposing the constraints Eq.(5.3), $\delta y(0,t) = \delta y_1(t)$ and $\delta y(l,t) = \delta y_2(t)$, so we obtain the string field equation of motion

$$\Big(\partial_t^2 - \partial_x^2 \Big) y(x,t) = 0 \qquad (0 \le x \le \ell) \tag{5.5}$$

and equations of motion for particles

$$M_1 \, \ddot{y}_1 + k_1 \, y_1 - T \, \partial_x y\Big|_{x=0} = 0 \qquad \text{and} \qquad M_2 \, \ddot{y}_2 + k_2 \, y_2 + T \, \partial_x y\Big|_{x=\ell} = 0 \,. \tag{5.6}$$

Integrating out the endpoint particles amount to relating $y_1(t), y_2(t)$ to the endpoints of string amplitude by combining Eq.(5.3) with Eq.(5.5). We obtain the sought-for boundary conditions

$$M_1\,\partial_x^2 y - T\,\partial_x y + k_1\,y\Big|_{x=0} = 0 \qquad \text{and} \qquad M_2\,\partial_x^2 y + T\,\partial_x y + k_2\,y\Big|_{x=\ell} = 0\,. \quad (5.7)$$

We see that, for finite M_1 and M_2, the boundary conditions are second order in normal derivatives, so they are indeed HD BCs. Were if M_1, M_2 zero, the boundary conditions are the most general Robin boundary conditions. The Robin boundary conditions are reduced to Neumann and Dirichlet boundary conditions in the limit $k_{1,2}$ are zero and infinite, respectively. Were if M_1, M_2 infinite, regularity of boundary conditions require that $\partial_x^2 y$ vanishes at the boundaries. In turn, $\partial_x y$ is constant at the boundaries, and so the boundary conditions are again reduced to Dirichlet boundary conditions.

Conversely, we can always reinterpret HD BCs on open string as attaching massive particles at the endpoints. Start with an open string whose field equation Eq.(5.5) is subject to HD BCs Eq.(5.7). This is the same situation as we have for the higher-spin field in AdS waveguide. Solving the open string field equation subject to the boundary conditions is the same as extremizing some modified action $\widetilde{I}$ whose variation is given by

$$\begin{aligned}\delta\widetilde{I} = &\int \mathrm{d}t \left(-T\int_0^\ell \mathrm{d}x\ \delta y\big[\partial_t^2 y - \partial_x^2 y\big]\right)\\ &-\int_{x=0}\mathrm{d}t\lambda_1(t)\,\delta y\,\big[M_1\partial_x^2 y - T\partial_x y + k_1 y\big] - \int_{x=\ell}\mathrm{d}t\lambda_2(t)\,\delta y\,\big[M_2\partial_x^2 y + T\partial_x y + k_2 y\big]\\ &- M_1\int_{x=0}\mathrm{d}t\,\delta y\,\big[\partial_t^2 y - \partial_x^2 y\big] - M_2\int_{x=\ell}\mathrm{d}t\,\delta y\,\big[\partial_t^2 y - \partial_x^2 y\big]\,, \end{aligned} \quad (5.8)$$

where $\lambda_{1,2}(t)$ are Lagrange multipliers that imposes the HD BCs. The last line is redundant, since they vanish automatically when the open string field equation from the first line is obeyed. By reparametrization of time t at both boundaries, it is always possible to put them to constant values which we set to unity. To reconstruct the action $\widetilde{I}$, we combine derivative terms that depend on string tension T:

$$T\int \mathrm{d}t\int_0^l \mathrm{d}x\,\delta y\,\partial_x^2 y - T\int \mathrm{d}t\,\big(\delta y\,\partial_x y\big)_0^\ell = -\delta\left(\frac{T}{2}\int \mathrm{d}t\int_0^\ell \mathrm{d}x\,(\partial_x y)^2\right), \quad (5.9)$$

and also combine derivative terms that depend on the mass parameters M_1, M_2:

$$\begin{aligned} &- M_1\int_{x=0}\mathrm{d}t\,\delta y\,\partial_x^2 y - M_1\int_{x=0}\mathrm{d}t\ \delta y\big[\partial_t^2 y - \partial_x^2 y\big] = -M_1\int_{x=0}\mathrm{d}t\,\delta y\,\partial_t^2 y = \delta\left(\frac{M_1}{2}\int \mathrm{d}t\,\dot{y}^2\right)\\ &- M_2\int_{x=\ell}\mathrm{d}t\,\delta y\,\partial_x^2 y - M_2\int_{x=\ell}\mathrm{d}t\ \delta y\big[\partial_t^2 y - \partial_x^2 y\big] = -M_2\int_{x=\ell}\mathrm{d}t\,\delta y\,\partial_t^2 y = \delta\left(\frac{M_2}{2}\int \mathrm{d}t\,\dot{y}^2\right). \end{aligned} \quad (5.10)$$

Combining with other terms in the variation, we get

$$\widetilde{I} = \frac{T}{2}\int \mathrm{d}t \int_0^\ell \mathrm{d}x \, \left[(\partial_t y)^2 - (\partial_x y)^2\right] + \frac{1}{2}\int \mathrm{d}t \, (M_1 \, \dot{y}_1^2 - k_1 y_1^2) + \frac{1}{2}\int \mathrm{d}t \, (M_2 \, \dot{y}_2^2 - k_2 y_2^2) \, . \tag{5.11}$$

By renaming the endpoint positions as in Eq.(5.3), we find that the action $\widetilde{I}$ is precisely the action of open string coupled to dynamical harmonic oscillator particles at each ends, Eq.(5.2).

We still need to understand how the Sturm-Liouville operator $-\partial_x^2$ of open string can be made self-adjoint for HD BC. It is useful to recall implication of self-adjointness for the Robin boundary condition. In this case, we can rewrite the open string action in terms of inner product for square-integrable functions

$$I_{\text{string}} = \frac{T}{2}\int \mathrm{d}t \, \Big(\langle \partial_t y, \partial_t y \rangle - \langle y, (-\partial_x^2) y \rangle \Big) \quad \text{where} \quad \langle f, g \rangle \equiv \int_0^\ell \mathrm{d}x \, f(x) \, g(x) \, . \tag{5.12}$$

Denote the square-integrable normal mode functions of $(-\partial_x^2)$ as X_n $(n = 0, 1, 2, 3, \cdots)$, viz. $(-\partial_x^2) X_n = \lambda_n X_n$. As the Sturm-Liouville operator $(-\partial_x^2)$ is self-adjoint for the Robin boundary condition, the normal mode functions can be made orthonormal and form a complete set of basis of the Hilbert space of square-integrable functions. So, we can decompose the string amplitude $y(x,t)$ as

$$y(x,t) = \sum_n T_n(t) \, X_n(x) \tag{5.13}$$

and the open string action I_{string} as

$$I_{\text{string}} = \sum_n \frac{T}{2} \int \mathrm{d}t \, \Big(\dot{T}_n^2 - \lambda_n T_n^2 \Big) \, . \tag{5.14}$$

Motivated by this line of reasonings, we ask if the combined action of open string with HD BCs can be written in terms of some inner product $\langle\!\langle \, , \, \rangle\!\rangle$:

$$I_{\langle\!\langle,\rangle\!\rangle} = \frac{T}{2}\int \mathrm{d}t \, \Big(\langle\!\langle \partial_t y, \partial_t y \rangle\!\rangle - \langle\!\langle y, (-\partial_x^2) y \rangle\!\rangle \Big). \tag{5.15}$$

We now prove that the inner product $\langle\!\langle \, , \, \rangle\!\rangle$ that renders the Sturm-Liouville opertor $(-\partial_x^2)$ self-adjoint under the HD BC Eq.(5.7) is precisely the extended inner product Eq.(**??**). In the present case, the additional vector space is provided by the positions of two massive particles attached at string endpoints. Therefore, it spans $\mathbb{R} \oplus \mathbb{R}$. The metric of this two-dimensional vector space is given by masses (measured relative to string tension). For a function space $\mathcal{L}^2 \oplus \mathbb{R}^2$, a general element and its inner product with respect to HD BC Eq.(5.7) would take the form

$$\mathbf{f} = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix} \in \mathcal{L}^2 \oplus \mathbb{R}^2, \qquad \mathbf{f} \cdot \mathbf{g} = \int_0^\ell \mathrm{d}x\, f(x)\, g(x) + G_{11} f_1 g_1 + G_{22} f_2 g_2. \tag{5.16}$$

Roughly speaking, two new real numbers $f_{1,2}$ correspond to boundary value of $f(x)$ which are left undetermined by the Sturm-Liouville differential equation and HD BC. The boundary conditions on element of $\mathcal{L}^2 \oplus \mathbb{R}^2$ are HD BC in Eq.(5.7) for $f(x)$, together with $f_1 = f(0)$ and $f_2 = f(l)$. With these boundary conditions, we now define the extended inner product $\langle\!\langle\ ,\ \rangle\!\rangle$ for the open string with HD BC as

$$\langle\!\langle f, g \rangle\!\rangle \equiv \int_0^\ell \mathrm{d}x\, f(x)\, g(x) + \frac{M_1}{T} f(0) g(0) + \frac{M_2}{T} f(l) g(l), \tag{5.17}$$

where the metric of $\mathbb{R}^2$ is chosen by the parameters in the HD BCs, Eq.(5.7). With respect to this extended inner product, we now find that the Sturm-Liouville operator $(-\partial_x^2)$ of open string is indeed self-adjoint:

$$\begin{aligned} &\langle\!\langle f, (-\partial_x^2) g \rangle\!\rangle - \langle\!\langle (-\partial_x^2) f, g \rangle\!\rangle \\ =& -\frac{1}{T} f (M_1 \partial_x^2 g - T \partial_x g + k_1 g)\Big|_{x=0} + \frac{1}{T} (M_1 \partial_x^2 f - T \partial_x f + k_1 f) g\Big|_{x=0} \\ &+ \frac{1}{T} f (M_2 \partial_x^2 g + T \partial_x g + k_2 g)\Big|_{x=\ell} - \frac{1}{T} (M_2 \partial_x^2 f + T \partial_x f + k_2 f) g\Big|_{x=\ell} \\ &= 0\,, \end{aligned} \tag{5.18}$$

where we arranged the harmonic force term (zero derivative terms in the boundary condition) and the HD BC Eq.(5.7) for f and g. With the extended inner product, we shall expand the proposed action Eq.(5.15) in terms of the original inner product over $\mathcal{L}^2$-space and additional inner product over $\mathbb{R}^2$ space. We observe that, after renaming the boundary values of $y(x, t)$ as Eq.(5.3), the proposed action $I_{\langle\!\langle , \rangle\!\rangle}$ in Eq.(5.15) is precisely the action of open string attached to endpoint particles, $I = I_{\text{string}} + I_{\text{boundary}}$. We reiterate the key point here is that extended inner product, HD BCs, and boundary actions are bear the same information and dictate their structures one another.

The extended inner product we introduced poses a new issue originating from the HD BC, equivalently, the endpoint particle dynamics. For some choices of the HD BCs, the extended Hilbert space can be indefinite, viz. the norm $\langle\!\langle y, y\rangle\!\rangle$ can become negative. This happen precisely when the metric components $M_{1,2}/T$ have negative signs. Take, for instance, $M_1 = M_2 = M$ and $k_1 = k_2 = 0$. There then always exists at least one mode

$$X_0(x) = N_0 \sinh\left[m_0 \left(x - \frac{\ell}{2}\right)\right] \quad \text{with} \quad \frac{1}{m_0} = -\frac{M}{T}\tanh\frac{m_0\,\ell}{2}\,, \tag{5.19}$$

whose extended norm is negative for negative value of M

$$\langle\!\langle X_0, X_0\rangle\!\rangle = {N_0}^2\left[-\frac{\ell}{2} + \frac{M}{T}\,\sinh^2\frac{m_0\,\ell}{2}\right] < 0\,. \tag{5.20}$$

This mode is problematic as, upon mode expansion, the corresponding component in the action Eq. (5.14) has the kinetic term with wrong sign,

$$(-)\frac{T}{2}\int \mathrm{d}t\,\left(\dot{T}_0^2 - \lambda_0 T_0^2\right)\,. \tag{5.21}$$

This causes negative energy of the open string at classical level and negative probability (and hence lack of unitarity) at quantum level. Moreover, the mode eigenvalue $\lambda_0 = -m_0^2$ is negative definite (which is again a consequence of negative value of M, as seen from Eq.(5.19)) and so the variable $T_0(t)$ develops an instability to grow exponentially large.

There is another example demonstrating the utility of the boundary degrees of freedom view point. Consider $k_1 = k_2 = k < 0$, $M_1 = M_2 = M > 0$ and $T > 0$ case. In this case, the extended inner product Eq.(5.17) ensures positivity of the norm. However there are some modes with negative eigenvalue. Generic even (with respect to $x = \frac{\ell}{2}$) mode function with negative eigenvalue is $X_e(x) = \cosh\left[\lambda\,\left(x - \frac{\ell}{2}\right)\right]$, $(-\partial_x^2)X_e = -\lambda^2 X_e$. HD BC implies

$$M\lambda^2 + T\,\lambda\,\tanh(\frac{\ell}{2}\lambda) = -k \tag{5.22}$$

and this equation always has solutions because for $\lambda \geq 0$, l.h.s is starting from 0 and monotonically increasing. Also the HD BC of generic odd function $X_o(x) = \sinh\left[\lambda\,\left(x - \frac{\ell}{2}\right)\right]$ implies

$$M\,\lambda^2\,\tanh(\frac{\ell}{2}\lambda) + T\,\lambda = -k\,\tanh(\frac{\ell}{2}\lambda) \tag{5.23}$$

and this equation has solutions when $T < -\frac{\ell}{2}\,k$.[2] Again, these negative eigenvalue modes show instability of the system. In terms of HD BC, it is hard to see the origin of this insta-

[2] In terms of boundary degrees of freedom, this inequality means that repulsive force from spring is bigger than string tension.

bility. However, in terms of boundary degrees of freedom, it is immediate that the origin of instability is the negative spring constant.

So, by relating HD BCs to boundary action of extra degrees of freedom, we gain a better understanding of underlying physics. For M negative, it is hard to recognize the above instability or non-unitarily at the level of the equation of motion and boundary conditions. In contrast, the boundary action clearly shows the origin of instability or non-unitarity and it is simply a consequence of negative mass of the endpoint particles.

5.2 Spin-Two Example

We now apply our understanding of the HD BC in the previous subsection to the spin-two field in AdS waveguide studied in section 3.3.2. Recall that spin-two is the first situation that HD BCs start to appear and, among three possible Dirichlet classes, **B.C. 1** and **B.C. 3** contain two-derivative boundary conditions to some of the component fields. In this subsection, we identify the extended inner product for these boundary conditions and explain the origin of non-unitarity for partially massless representations in AdS$_{d+1}$.

We first construct the extended inner product for spin-two fields in AdS space. The Sturm-Liouville problems with HD BCs that we will consider have the following form:

$$Ł_b\, Ł_a\, \Theta_n = -\lambda_n\, \Theta_n \qquad \text{where} \qquad Ł_c\, Ł_a\, \Theta_n\big|_{\theta=\pm\alpha} = 0 \tag{5.24}$$

for some weights a, b, c. Note that the Sturm-Liouville equation and the boundary condition share the same operator $Ł_a$. From free action of the spin-two field, we get an $\mathcal{L}^2$ inner product

$$\langle \Theta_m\,, \Theta_n \rangle = \int_{-\alpha}^{\alpha} \mathrm{d}\theta\, (\sec\theta)^{d-4}\, \Theta_m(\theta)\, \Theta_n(\theta)\,, \tag{5.25}$$

where the weight factor in the integration measure originates from the conformal factor of the metric Eq.(3.6). As we deal with spin-two, $s = 2$ and so $a+b = d-2s = d-4$. We thus take the weight factor as $(\sec\theta)^{a+b}$. For any conformal factor $(\sec\theta)^c$ with arbitrary weight c, we integrate by part

$$\int_{-\alpha}^{\alpha} \mathrm{d}\theta\, (\sec\theta)^c\, \Theta_m(Ł_a\Theta_n) = -\int_{-\alpha}^{\alpha} \mathrm{d}\theta\, (\sec\theta)^c\, (Ł_{c-a}\Theta_m)\, \Theta_n + (\sec\alpha)^c \big[\Theta_m\, \Theta_n\big]_{-\alpha}^{+\alpha}\,. \tag{5.26}$$

Using this, one finds that the differential operator $Ł_b\, Ł_a$ is not self-adjoint on $\mathcal{L}^2$ functional space,

$$\langle\Theta_m, (Ł_b\, Ł_a\, \Theta_n)\rangle - \langle(Ł_b\, Ł_a\, \Theta_m), \Theta_n\rangle = (\sec\alpha)^{a+b}\big[\Theta_m(Ł_a\Theta_n) - (Ł_a\Theta_m)\Theta_n\big]^{+\alpha}_{-\alpha} \neq 0\,. \tag{5.27}$$

By inspection, we find an extended inner product which renders the Sturm-Liouville operator $Ł_bŁ_a$ self-adjoint,

$$\langle\!\langle\Theta_m, \Theta_n\rangle\!\rangle \equiv \langle\Theta_m, \Theta_n\rangle + \sum_{\sigma=\pm}\mathcal{N}_\sigma\Theta_m(\sigma\alpha)\Theta_n(\sigma\alpha)\,, \tag{5.28}$$

where

$$\mathcal{N}_+ = \mathcal{N}_- = (c-b)^{-1}\cot\alpha(\sec\,\alpha)^{a+b}. \tag{5.29}$$

We can confirm that $Ł_bŁ_a$ is indeed self-adjoint with respect to the extended inner product:

$$\begin{aligned}
&\langle\!\langle\Theta_m, (Ł_b\, Ł_a\Theta_n)\rangle\!\rangle - \langle\!\langle(Ł_b\, Ł_a\Theta_m), \Theta_n\rangle\!\rangle\\
=&(\sec\alpha)^{a+b}\sum_{\sigma=\pm}\Big(\Theta_m(Ł_a\Theta_n) - (Ł_a\Theta_m)\Theta_n + \mathcal{N}_\sigma(\Theta_mŁ_bŁ_a\Theta_n - (Ł_bŁ_a\Theta_m)\,\Theta_n)\Big)(\sigma\alpha)\\
=&(\sec\alpha)^{a+b}\sum_{\sigma=\pm}\mathcal{N}_\sigma\Big(\Theta_m(Ł_cŁ_a\Theta_n)(\sigma\alpha) - (Ł_cŁ_a\Theta_m)\Theta_n(\sigma\alpha)\Big)\,.
\end{aligned} \tag{5.30}$$

The last expression vanishes by the HD BCs in Eq.(5.56).

We apply the extended inner product to the ground modes for the HD BCs, **B.C. 1** and **B.C. 3** in section 3.3.2. In the last subsection, whether a given HD BC lead to non-unitarity or not depends on parameters specifying the boundary conditions. The extended norm-squared is positive definite if unitary, while it is negative definite if non-unitary. For **B.C. 1**, the HD BCs are imposed on spin-zero mode with $a = d-3$, $b = -1$ and $c = d-2$. We see that the normalization constants $\mathcal{N}_\pm$ in Eq.(5.28) are positive-definite, so the norm-squared is positive-definite. In contrast, for **B.C. 3**, HD BCs are imposed on spin-two mode with $a = -2$, $b = d-2$, $c = -1$ and the normalization constants $\mathcal{N}_\pm$ are negative-definite. More explicitly, the ground modes of **B.C. 3** are

$$\begin{cases} \Theta^{2|2}_1 = N_1\,\sec\theta\,\tan\theta\,, \quad \Theta^{1|2}_1 = N_2\,\sec\theta & \textbf{type IV} \text{ in Table 1}\,,\\ \Theta^{2|2}_0 = N_3\,\sec^2\theta & \textbf{type II} \text{ in Table 1}\,,\end{cases} \tag{5.31}$$

which correspond to the PM spin-two and massless spin-two fields, respectively. Boundary condition of spin-one mode function is one-derivative boundary condition and its norm is positive-definite. In contrast, the norms of spin-two modes are

$$\langle\!\langle \Theta_1^{2|2}, \Theta_1^{2|2} \rangle\!\rangle = {N_1}^2 \left(\int_{-\alpha}^{\alpha} d\theta\, \sec^{d-2}\theta\, \tan^2\theta - \frac{2}{d-1} \sec^{d-2}\alpha\, \tan\alpha \right), \tag{5.32}$$

$$\langle\!\langle \Theta_0^{2|2}, \Theta_0^{2|2} \rangle\!\rangle = {N_3}^2 \left(\int_{-\alpha}^{\alpha} d\theta\, \sec^{d}\theta - \frac{2\sec^d\alpha}{(d-1)\tan\alpha} \right). \tag{5.33}$$

It can be shown that the norm Eq.(5.32) which corresponds to PM mode, is always negative[3] by the following estimation.

$$\begin{aligned}\langle\!\langle \Theta_1^{2|2}, \Theta_1^{2|2} \rangle\!\rangle &= {N_1}^2 \left(2\int_{0}^{\alpha} d\theta\, \sec^{d}\theta\, \sin^2\theta - \frac{2}{d-1} \sec^{d-1}\alpha\, \sin\alpha \right) \\ &< {N_1}^2 \left(2\sin\alpha \int_{0}^{\alpha} d\theta\, \sec^{d}\theta\, \sin\theta - \frac{2}{d-1} \sec^{d-1}\alpha\, \sin\alpha \right) \\ &= -{N_1}^2 \frac{2}{d-1}\end{aligned} \tag{5.34}$$

Inequality holds because $\sin\theta < \sin\alpha$ for $0 \le \theta < \alpha < \frac{\pi}{2}$. This negative norm implies that the kinetic term of PM mode has the wrong sign.

With the extended inner product, we can construct boundary action which reveals physical properties of the imposed HD BCs. The action of free massless spin-two field $\bar{h}_{MN}$ on AdS_{d+2} background is

$$\begin{aligned}I_{\mathrm{spin-two}} &= \int \sqrt{\bar{g}}\, \mathrm{d}^{d+2}x\, \mathcal{L}_2\left(\bar{h}_{MN}; \bar{g}_{MN}, d+2\right) \\ &= \int \sqrt{\bar{g}}\, \mathrm{d}^{d+2}x \Big[-\frac{1}{2} \bar{\nabla}^L \bar{h}^{MN} \bar{\nabla}_L \bar{h}_{MN} + \bar{\nabla}^M \bar{h}^{NL} \bar{\nabla}_N \bar{h}_{ML} - \bar{\nabla}^M \bar{h}_{MN} \bar{\nabla}^N \bar{h} \\ &\qquad + \frac{1}{2} \bar{\nabla}^L \bar{h} \bar{\nabla}_L \bar{h} - (d+1)\left(\bar{h}^{MN} \bar{h}_{MN} - \frac{1}{2}\bar{h}^2\right)\Big]\end{aligned} \tag{5.35}$$

After compactification on the AdS waveguide, each term of Eq.(5.35) is decomposed into quadratic terms of component fields, $h_{\mu\nu}$, A_μ and ϕ, which can be expressed as $\mathcal{L}^2$ inner product Eq.(5.25) with $a+b = d-2s = d-4$. For example,

$$\int \mathrm{d}^{d+1}x\, \sqrt{-g} \int_{-\alpha}^{\alpha} \mathrm{d}\theta\, (\sec\theta)^{d-4}\, \nabla^\rho h^{\mu\nu} \nabla_\rho h_{\mu\nu} = \int \mathrm{d}^{d+1}x\, \sqrt{-g}\, \langle \nabla^\rho h^{\mu\nu}, \nabla_\rho h_{\mu\nu} \rangle\, .$$

As in the open string case, we require that each term of the quadratic action expressed by appropriate inner product which ensures orthogonality and completeness of the mode func-

[3] The other norm Eq.(5.33) is negative for $\alpha \sim 0$ and positive for $\alpha \sim \pi/2$. When one of Kaluza-Klein mass hits zero mass, this norm vanishes. In this specific value of α, there is no massless spin-two field, **type III**, in the spectrum and **type II** appear instead.

tions. We now know that, depending on the nature of boundary conditions, some of these terms needs to be the extended inner product which contains the contribution of boundary action. The situation is more involved as there are three component fields each of which obeys different boundary conditions. From the spectrum generating complex, we have three kinds of boundary conditions:

- spin-two Dirichlet, expanded by $\Theta_n^{2|2}$: $\quad h_{\mu\nu}, \quad Ł_{d-2}A_\mu, \quad Ł_{d-2}Ł_{d-3}\phi$
- spin-one Dirichlet, expanded by $\Theta_n^{1|2}$: $\quad Ł_{-2}h_{\mu\nu}, \quad A_\mu, \quad Ł_{d-3}\phi$
- spin-zero Dirichlet, expanded by $\Theta_n^{0|2}$: $\quad Ł_{-1}Ł_{-2}h_{\mu\nu}, \quad Ł_{-1}A_\mu, \quad \phi$.

By straightforward computation, we find that the action is decomposed as

$$
\begin{aligned}
I = & \int \mathrm{d}\theta \, (\sec\theta)^{d-4} \, \mathcal{L}_2 \, (h_{\mu\nu}; g_{\mu\nu}, d+1) \\
& + \Big[-\frac{1}{2} \langle F^{\mu\nu}, F_{\mu\nu} \rangle - 2d \, \langle A^\mu, A_\mu \rangle + \langle Ł_{-2}h^{\mu\nu}, \nabla_\mu A_\nu + \nabla_\nu A_\mu - 2g_{\mu\nu}\nabla^\rho A_\rho \rangle \\
& - \frac{1}{2}\langle Ł_{-2}\, h^{\mu\nu}, Ł_{-2}\, h_{\mu\nu} \rangle + \frac{1}{2}\langle Ł_{-2}\, h, Ł_{-2}\, h \rangle + \frac{d(d+1)}{(d-1)^2}\langle Ł_{d-4}\phi, Ł_{d-4}\phi \rangle - \frac{d}{d-1}\, \langle Ł_{-2}h, Ł_{d-4}\phi \rangle \Big] \\
& + \Big[-\frac{d}{d-1}\Big(\frac{1}{2}\, \langle \nabla^\mu \phi, \nabla_\mu \phi \rangle + \frac{d+1}{2}\, \langle \phi, \phi \rangle\Big) + \frac{2d}{d-1}\, \langle Ł_{-1}A^\mu, \nabla_\mu \phi \rangle \Big] \, . \qquad (5.36)
\end{aligned}
$$

The first line, the second bracket and the third bracket are spin component of modes: $\langle \Theta^{s|2}, \Theta^{s|2} \rangle$ for $s = 2, 1, 0$, respectively [4].

Consider first **B.C. 1**. In this case, the spin-zero component field obeys HD BC:

$$
Ł_{-1}Ł_{d-3}\Theta^{0|2} = -\lambda\Theta^{0|2} \qquad \text{and} \qquad Ł_{d-2}Ł_{d-3}\Theta^{0|2}\big|_{\theta=\pm\alpha} = 0. \qquad (5.37)
$$

So, we need to adopt the extended inner product for terms involving the mode function $\Theta^{0|2}$. They are the terms in the third bracket of Eq.(5.36). Using the extended inner product Eq.(5.28) with $a = d-3$, $b = -1$ and $c = d-2$, we obtain the corresponding boundary action from the difference between extended inner product and original inner product

$$
\int \mathrm{d}^{d+1}x\sqrt{-g}\, \frac{d}{(d-1)^2}\frac{(\sec\alpha)^{d-4}}{\tan\alpha} \sum_{\sigma=\pm} \Big[-\Big(\frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi + \frac{d+1}{2}\phi^2\Big) + \Big(Ł_{-1}A^\mu\nabla_\mu\phi\Big)\Big]_{\theta=\sigma\alpha} . \qquad (5.38)
$$

[4] The classification appears somewhat arbitrary. For instance, $\langle Ł_{d-3}\phi, Ł_{d-3}\phi \rangle$ belongs to $\langle \Theta^{1|2}, \Theta^{1|2} \rangle$,but its another form $\langle \phi, Ł_{-1}Ł_{d-3}\phi \rangle$ obtained by integration by parts belongs to $\langle \Theta^{0|2}, \Theta^{0|2} \rangle$. We will show that the total action is nevertheless the same provided we keep track of boundary terms.

Redefining the boundary values of spin-zero field as

$$\phi^{\sigma} = \left(\frac{d}{(d-1)^2}\frac{(\sec\alpha)^{d-4}}{\tan\alpha}\right)^{1/2} \phi\Big|_{\theta=\sigma\alpha}, \qquad (\sigma = \pm), \tag{5.39}$$

we get the boundary action as

$$I_{\text{boundary,BC1}} = \sum_{\sigma=\pm}\int \mathrm{d}^{d+1}x\sqrt{-g}\,\Big[-\Big(\frac{1}{2}\nabla^{\mu}\phi^{\sigma}\nabla_{\mu}\phi^{\sigma} + \frac{d+1}{2}(\phi^{\sigma})^2\Big) + C\Big(Ł_{-1}A^{\mu}\big|_{\theta=\sigma\alpha}\nabla_{\mu}\phi^{\sigma}\Big)\Big], \tag{5.40}$$

where $C = \left(\frac{d}{(d-1)^2}\cot\alpha(\sec\alpha)^{d-4}\right)^{1/2}$. Note that the sign of kinetic term for boundary spin-zero fields $\phi^{\pm}$ is standard in our convention. This matches precisely with the result of section 5.2 that the waveguide compactification with **B.C. 1** yields only unitary spectrum. For the second term of boundary action, we may interpret it two alternative ways. We can interpret that the bulk field A_{μ} is sourced by the boundary field $\phi^{\pm}$, equivalently, the boundary value of bulk field A_{μ} turns on the boundary field $\phi^{\pm}$. Alternatively, we can eliminate this term by writing the cross term $\langle Ł_{-1}A^{\mu}, \nabla_{\mu}\phi\rangle$ as $\langle A^{\mu}, Ł_{d-3}\nabla_{\mu}\phi\rangle$. This is related to the freedom which is explained in the footnote 4. We will revisit this issue at the end of this section.

Consider next **B.C. 3**. In this case, the spin-two component field is subject to HD BC:

$$Ł_{d-2}Ł_{-2}\Theta^{0|2} = -\lambda\Theta^{2|2} \qquad \text{and} \qquad Ł_{-1}Ł_{-2}\Theta^{2|2}\big|_{\theta=\pm\alpha} = 0. \tag{5.41}$$

We thus need to adopt the extended inner product for terms involving the mode function $\Theta^{2|2}$. They are the first term in Eq.(5.36) that contain the kinetic and mass-like terms of spn-two field $h_{\mu\nu}$. Using the extended inner product Eq.(5.28) with $a = -2$, $b = d-2$, $c = -1$, we now get the boundary action as

$$I_{\text{boundary,BC3}} = -\sum_{\sigma=\pm}\int \mathrm{d}^{d+1}x\sqrt{-g}\,\mathcal{L}_2\left(h^{\sigma}_{\mu\nu}; g_{\mu\nu}, d+1\right) \tag{5.42}$$

where we renamed the boundary value of the bulk spin-two field by

$$h^{\sigma}_{\mu\nu} = \left(\frac{1}{d-1}\frac{(\sec\alpha)^{d-4}}{\tan\alpha}\right)^{1/2} h_{\mu\nu}\Big|_{\theta=\sigma\alpha}, \qquad (\sigma = \pm). \tag{5.43}$$

Most significantly, with extra minus sign in front of the boundary action Eq.(5.42), the boundary spin-two field $h^{\pm}_{\mu\nu}$ has kinetic terms of wrong sign. Again, this fits perfectly with the result of section 5.2 that the waveguide compactification with **B.C. 3** yields non-unitary spectrum for the partially massless spin-two fields. As stressed already, this result was hardly obvious to anticipate just from the HD BCs. With the extended inner product, we now have a firm understanding for the origin of non-unitarity of partially massless spin-two field without ever invoking $\mathfrak{so}(d,2)$ representation theory.

Summarizing,

- For HD BC, we need to extend the functional space from $\mathcal{L}^2$ to $\mathcal{L}^2 \oplus \mathbb{R}^N$ to render Sturm-Liouville operator self-adjoint. We showed that this extension can be physically understood as adding N many boundary degrees of freedom.
- From the extended inner product, we constructed the boundary action for a given HD BC. The boundary action enabled to directly trace the origin of (non)unitarity of waveguide spectrum.
- For **B.C. 3** in section 3.3.2, the boundary action of boundary spin-two fields has kinetic term of the wrong sign. This explained why the partially massless spin-two field is non-unitary.
- For **B.C. 1** in section 3.3.2, the boundary action of boundary spin-zero fields have kinetic term of conventional sign. This explains why the massive spin-zero field is unitary.

Before concluding this subsection, let us revisit the ambiguity mentioned in the footnote 4. Consider **B.C. 1** and the term $\langle Ł_{d-3}\phi, Ł_{d-3}\phi\rangle$. Such term was classified as originating from $\langle\Theta^{1|2},\Theta^{1|2}\rangle$, so appears not to be refined. On the other hand, using Eq.(5.26), this term can also be rewritten as $-\langle\phi, Ł_{-1}Ł_{d-3}\phi\rangle$ with surface term $\left[(\sec\alpha)^{d-4}\,\phi\,Ł_{d-3}\phi\right]\Big|_{-\alpha}^{+\alpha}$. This term belongs to $\langle\Theta^{0|2},\Theta^{0|2}\rangle$, so needs to be refined. Thought this seems to pose ambiguous, it is actually not. From

$$\begin{aligned}\langle\!\langle\, \phi, Ł_{-1}Ł_{d-3}\phi \,\rangle\!\rangle &= \langle\phi, Ł_{-1}Ł_{d-3}\phi\rangle + \sum_{\sigma=\pm}\mathcal{N}_\sigma\Big(\phi\,Ł_{-1}Ł_{d-3}\phi\Big)_{\theta=\sigma\alpha} \\ &= -\langle Ł_{d-3}\phi, Ł_{d-3}\phi\rangle + \sum_{\sigma=\pm}\mathcal{N}_\sigma\Big(\phi\,Ł_{d-2}Ł_{d-3}\phi\Big)_{\theta=\pm\alpha},\end{aligned} \tag{5.44}$$

we see that $\langle\!\langle\phi, Ł_{-1}Ł_{d-3}\phi\rangle\!\rangle$ and $-\langle Ł_{d-3}\phi, Ł_{d-3}\phi\rangle$ are the same up to boundary conditions. One can start with any bulk action, and the extended action is the same. There is no ambiguity.

5.3 Systematics for HDBC

In this section, we describe how to expand function space and inner product for 2, 3, 4 derivatives boundary conditions. See appendix B for 5 and 6 derivatives cases. We concentrate on the HD BC comes from the Dirichlet boundary condition on $\Theta_n^{s|s}$ which means, n-derivative boundary condition for $\Theta_n^{s-n|s}$. For 2 derivative boundary condition, the situation is much simple and we analyze with more generality. Unfortunately, our method cannot apply to generic higher derivative boundary condition since the complexity grows too fast.

5.3.1 Strategy

Since now we deal with more general cases with various spin, let's define more specific notation for inner product.

$$\langle\, f(\theta), g(\theta)\,\rangle_{L^2}^{c} = \frac{1}{(\sec\alpha)^c}\int_{-\alpha}^{+\alpha} \mathrm{d}\theta\,(\sec\theta)^c f(\theta)g(\theta) \tag{5.45}$$

Here c denotes weight factor for L^2 inner product. Under L^2 inner product, difference between $\langle\Theta_m, Ł_bŁ_a\Theta_n\rangle_{L^2}^{a+b}$ and $\langle Ł_bŁ_a\Theta_m, \Theta_n\rangle_{L^2}^{a+b}$ is a surface term $\big[\Theta_m(Ł_a\Theta_n) - (Ł_a\Theta_m)\Theta_n\big]_{-\alpha}^{+\alpha}$. This surface term vanish only for the Dirichlet boundary condition $\Theta_n|_{\pm\alpha} = 0$ or mixed boundary condition. Using eigenvalue equation and derivative of eigenvalue equation at the boundary, one can expand eigenfunction $\Theta_n \in L^2$ to generalized eigenvector $\vec{\Theta}_n \in L^2 \oplus R^N$. With generalized eigenvector $\vec{\Theta}_n$, also the eigenvalue operator is naturally expanded to eigenvalue operator $\mathbb{E}$ acting on $\vec{\Theta}_n$. By defining proper inner product of generalized eigenvector $\langle\vec{\Theta}_n, \vec{\Theta}_m\rangle_{L^2\oplus R^N}$, one can make the difference between $\langle\vec{\Theta}_n, \mathbb{E}\vec{\Theta}_m\rangle_{L^2\oplus R^N}$ and $\langle\mathbb{E}\vec{\Theta}_n, \vec{\Theta}_m\rangle_{L^2\oplus R^N}$ to a surface term which vanish under given higher derivative boundary condition. Then, under this inner product, operator $\mathbb{E}$ is self-adjoint and every nice properties are restored. Note that this procedure at $\theta = +\alpha$ and $\theta = -\alpha$ are totally independent therefore expanded space always should be $L^2 \oplus R^{2N}$. This is the consequence of considering identical parallel 2 boundaries. This system is symmetric under $\theta \to -\theta$, therefore every calculation for $\pm\alpha$ is completely parallel. From the next subsection, we will concentrate on surface terms at $\theta = +\alpha$ and omit the $\theta = -\alpha$ part except for 2 derivative example.

More concretely, eigenfunction $\Theta_n(\theta)$ and eigenvalue equation $Ł_bŁ_a\Theta_n(\theta) = -\lambda\Theta_n(\theta)$

are expanded to

$$
\mathbb{E}\,\vec{\Theta}_n = \mathbb{E}\cdot\begin{pmatrix}\Theta_n(\theta)\\ I_1(\Theta_n)|_{+\alpha}\\ I_1(\Theta_n)|_{-\alpha}\\ I_2(\Theta_n)|_{+\alpha}\\ \vdots\\ I_N(\Theta_n)|_{+\alpha}\\ I_N(\Theta_n)|_{-\alpha}\end{pmatrix} = \begin{pmatrix}Ł_bŁ_a\Theta_n(\theta)\\ J_1(\Theta_n)|_{+\alpha}\\ J_1(\Theta_n)|_{-\alpha}\\ J_2(\Theta_n)|_{+\alpha}\\ \vdots\\ J_N(\Theta_n)|_{+\alpha}\\ J_N(\Theta_n)|_{-\alpha}\end{pmatrix} = -\lambda\begin{pmatrix}\Theta_n(\theta)\\ I_1(\Theta_n)|_{+\alpha}\\ I_1(\Theta_n)|_{-\alpha}\\ I_2(\Theta_n)|_{+\alpha}\\ \vdots\\ I_N(\Theta_n)|_{+\alpha}\\ I_N(\Theta_n)|_{-\alpha}\end{pmatrix} = -\lambda\vec{\Theta}_n
\tag{5.46}
$$

I_i and J_i are the linear functions of Θ_n and satisfy the relation

$$
J_i(\Theta_n(\theta)) = -\lambda I_i(\Theta_n(\theta)) \tag{5.47}
$$

I_i and J_i are constructed from the eigenvalue equation and derivatives of eigenvalue equation. Also I_i, J_i give the definition of $\mathbb{E}$ therefore they are the main object we should calculate. Under these conditions, $\{\vec{\Theta}_n\}$ forms complete basis for $L^2 \oplus R^{2N}$ and arbitrary element $\vec{f} \in L^2 \oplus R^{2N}$ can be expanded by $\{\vec{\Theta}_n\}$.

$$
\vec{f} = \begin{pmatrix} f(\theta)\\ f_1\\ \vdots\\ f_{2N}\end{pmatrix} = \sum_m c_m \vec{\Theta}_m \tag{5.48}
$$

Here, f_i's are just arbitrary real numbers and indepent with $f(\theta)$. Natural inner product of $L^2 \oplus R^{2N}$ is

$$
\langle \vec{f}, \vec{g}\rangle_{L^2\oplus R^{2N}} \equiv \langle f, g\rangle_{L^2}^{a+b} + \sum_{i=1}^{2N} \mathcal{N}_i\, f_i\, g_i \equiv \langle f, g\rangle_{L^2}^{a+b} + B_{2N}(\vec{f}, \vec{g}) \tag{5.49}
$$

$\mathcal{N}_i$ is the normalization factor ($\mathcal{N}_{2k-1} = \mathcal{N}_{2k}$ because of $\theta \to -\theta$ symmetry) and the sign of this factor is the most important information. So what we have to calculate are I_i, J_i and $\mathcal{N}_i$. Guiding principle for determining I_i, J_i and $\mathcal{N}_i$ comes from the previous requirement on the inner product of $\vec{\Theta}_n$.

$$
\langle \vec{\Theta}_n, \mathbb{E}\vec{\Theta}_m\rangle_{L^2\oplus R^{2N}} - \langle \mathbb{E}\vec{\Theta}_n, \vec{\Theta}_m\rangle_{L^2\oplus R^{2N}} = \big[\text{Surface Term}\big]_{-\alpha}^{+\alpha} \tag{5.50}
$$

This surface term should vanish under given higher derivative boundary condition. N is the number of necessary I_i and J_i to achieve this requirement.

Actually, we already know the relation between N and the number of derivatives in the boundary condition. From the direct calculation, we got eigenfunctions for given higher derivative boundary condition, and classified them into "Kaluza-Klein Modes" and "Ground Modes". The eigenvalues of "Kaluza-Klein Modes" depend on the distance between the boundaries, α but the eigenvalues of "Ground Modes" do not depend on α. For M-derivative boundary condition, there are M "Ground Modes". When M is even, "Kaluza-Klein Modes" form a basis for L^2 space and the number of "Ground Modes" is the same with $2N$. When M is odd, "Kaluza-Klein Modes" together with one of "Ground Modes", which is analogous to the ground mode of the Neumann boundary condition, form a basis of L^2 space and the number of the rest of the "Ground Modes", $M-1$, is the same with $2N$. Therefore $N = [M/2]$. This is not a rigorous statement and is related with the uniqueness of expansion to the generalized eigenvector. Up to now, we don't have complete proof and will just take $N = [M/2]$.

5.3.2 Basis of Higher Derivatives

The elements for extended inner product are I_i and J_i and they are linear functional of higher derivative to make a surface term which vanish by HD BC. To deal with arbitrary linear higher derivatives, we should fix the basis for higher derivatives. Also to keep the structure of the eigenfunctions, we should use Ł$_m$ operators. Considering higher derivatives in terms of Ł$_m$ operator is highly ambiguous since one can consider various m's. One nice basis is

$$\text{Ł}^{(m)}_{k|s} \equiv \underbrace{\text{Ł}_{d-s+k-3+m} \cdots \text{Ł}_{d-s+k-1}\text{Ł}_{d-s+k-2}}_{m \ \text{Ł}'s} \tag{5.51}$$

for $m > 0$. This operator is product of successive m raising Ł$_m$ operators, therefore

$$\text{Ł}^{(m)}_{k|s}\Theta^{k|s}_n \sim \Theta^{k+m|s}_n \,. \tag{5.52}$$

We can omit subscript $k|s$ since this operator only action $\Theta^{k|s}$ and there is no ambiguity. Using this basis is notationally and schematically convenient, because boundary condition is given in terms of Ł$^{(m)}$. One can define this operator for $m < 0$ with different set of Ł$_m$'s.

When $Ł_{-(s+k-1)}Ł_{d-s+k-2}\Theta_n^{k|s} = -\lambda\Theta_n^{k|s}$,

$$Ł_{-(s+(k+m)-1)}Ł_{d-s+(k+m)-2}Ł^{(m)}\Theta_n^{k|s} = \left(-\lambda + \sum_{i=1}^{m}(d+2k-4+2i)\right)Ł^{(m)}\Theta_n^{k|s} \tag{5.53}$$

In terms of $Ł^{(m)}$ only,

$$\left(Ł^{(m+2)} - A_m^{k|s}\,Ł^{(m+1)} - B_m^{k|s}\,Ł^{(m)}\right)\Theta_n^{k|s} = -\lambda\,Ł^{(m)}\Theta_n^{k|s} \tag{5.54}$$

$$\begin{aligned} A_m^{k|s} &= \Delta_{m+1}^{k|s}\,\mathcal{T}_\theta = (d+2k-2+2m)\,\tan\theta \\ B_m^{k|s} &= \sum_{i=1}^{m}\Delta_i^{k|s} = m\,(d+2k-3+m) \\ \Delta_i^{k|s} &\equiv d+2k-4+2i\,, \qquad \mathcal{T}_\theta \equiv \tan\theta \end{aligned}$$

for $m \geq 1$, equation (5.54) is the m-derivative of the eigenvalue equation and for $m = 0$, it is the eigenvalue equation itself. ($B_0^{k|s}$ = 0) These equations provide complete building block of equation (5.47) therefore I_i and J_i. Consider an arbitrary $I_i = \sum_{m=0}^{M} b_{i,m}Ł^{(m)}$ then corresponding J_i should be

$$J_i = \sum_{m=1}^{M+2} c_{i,m}Ł^{(m)} = \sum_{m=1}^{M+2}(b_{i,m-2} - A_{m-1}^{k|s}\,b_{i,m-1} - B_m^{k|s}\,b_{i,m})\,Ł^{(m)} \tag{5.55}$$

$b_{i,-1} = b_{i,M+1} = b_{i,M+2} = 0$. Therefore indepent variables are I_i and J_i are $b_{i,m}$'s. Actually, $b_{i.m}$'s contain every information we need including the normalization factor $\mathcal{N}_i$. However, for convenience, we set $b_{i,M} = 1$ and consider $\mathcal{N}_i$ as a free variable.

5.3.3 2 derivative

2 derivative boundary condition is relatively simple to analyze so we can do it with more generality. And we write down both surface terms at $\theta = \pm\alpha$ only for this example. From the next example, we will omit the surface term at $\theta = -\alpha$. Consider following eignevalue equation and boundary condition,

$$Ł_bŁ_a\Theta_n = -\lambda\Theta_n \tag{5.56}$$

$$Ł_cŁ_a\Theta_n\Big|_{\theta=\pm\alpha} = 0 \tag{5.57}$$

Eigenvalue equation can be rewritten as

$$J(\Theta_n) \equiv (Ł_c Ł_a - (c-b)\mathcal{T}_\theta\, Ł_a)\,\Theta_n = -\lambda\Theta_n \equiv -\lambda I(\Theta_n) \tag{5.58}$$

This is all we need.($N = [2/2] = 1$) This system can be expanded to that of $\vec{\Theta}_n \in L^2 \oplus R^2$.

$$\mathbb{E}\cdot\begin{pmatrix}\Theta_n(\theta)\\ I(\Theta_n)|_{+\alpha}\\ I(\Theta_n)|_{-\alpha}\end{pmatrix} = \begin{pmatrix}Ł_b Ł_a\Theta_n(\theta)\\ J(\Theta_n)|_{+\alpha}\\ J(\Theta_n)|_{-\alpha}\end{pmatrix} = -\lambda\begin{pmatrix}\Theta_n(\theta)\\ I(\Theta_n)|_{+\alpha}\\ I(\Theta_n)|_{-\alpha}\end{pmatrix} \tag{5.59}$$

Up to now, there was no free parameter that we can tune. The only free parameter of 2 derivative boundary condition is the normalization factor $\mathcal{N}$.

$$\begin{aligned}\langle\vec{\Theta}_n, \mathbb{E}\vec{\Theta}_m\rangle_{L^2\oplus R^2} =&\langle\Theta_n, Ł_bŁ_a\Theta_m\rangle^{a+b}_{L^2} + \mathcal{N}\Big(I(\Theta_n)J(\Theta_m)|_{+\alpha} + I(\Theta_n)J(\Theta_m)|_{-\alpha}\Big)\\ =&\frac{1}{(\sec\alpha)^{a+b}}\int_{-\alpha}^{\alpha}\mathrm{d}\theta\,(\sec\theta)^{a+b}\,\Theta_nŁ_bŁ_a\Theta_m\\ &+\mathcal{N}\,(\Theta_nŁ_cŁ_a\Theta_m - (c-b)\mathcal{T}_\theta\ \Theta_nŁ_a\Theta_m)\,\big|_{+\alpha}\\ &+\mathcal{N}\,(\Theta_nŁ_cŁ_a\Theta_m - (c-b)\mathcal{T}_\theta\ \Theta_nŁ_a\Theta_m)\,\big|_{-\alpha}\end{aligned} \tag{5.60}$$

Then,

$$\begin{aligned}&\langle\vec{\Theta}_n, \mathbb{E}\vec{\Theta}_m\rangle_{L^2\oplus R^2} - \langle\mathbb{E}\vec{\Theta}_n, \vec{\Theta}_m\rangle_{L^2\oplus R^2}\\ =&\big[\Theta_n(Ł_a\Theta_m) - (Ł_a\Theta_n)\Theta_m\big]^{+\alpha}_{-\alpha}\\ &-(c-b)\mathcal{T}_\alpha\,\mathcal{N}\Big(\Theta_n(Ł_a\Theta_m) - (Ł_a\Theta_n)\Theta_m\Big)\big|_{+\alpha} + (c-b)\mathcal{T}_\alpha\,\mathcal{N}\Big(\Theta_n(Ł_a\Theta_m) - (Ł_a\Theta_n)\Theta_m\Big)\big|_{-\alpha}\\ &+\mathcal{N}\Big(\Theta_n(Ł_cŁ_a\Theta_m) - (Ł_cŁ_a\Theta_n)\Theta_m\Big)\big|_{+\alpha} + \mathcal{N}\Big(\Theta_n(Ł_cŁ_a\Theta_m) - (Ł_cŁ_a\Theta_n)\Theta_m\Big)\big|_{-\alpha}\end{aligned} \tag{5.61}$$

Note that $\mathcal{T}_{-\alpha} = -\mathcal{T}_\alpha$. Set $\mathcal{N}$ to $\frac{1}{(c-b)\mathcal{T}_\alpha}$ and the original surface term from L^2 inner product is canceled. Remaining surface term is

$$\begin{aligned}&\langle\vec{\Theta}_n, \mathbb{E}\vec{\Theta}_m\rangle_{L^2\oplus R^2} - \langle\mathbb{E}\vec{\Theta}_n, \vec{\Theta}_m\rangle_{L^2\oplus R^2}\\ =&\frac{1}{(c-b)\mathcal{T}_\alpha}\Big[\Big(\Theta_n(Ł_cŁ_a\Theta_m) - (Ł_cŁ_a\Theta_n)\Theta_m\Big)\big|_{+\alpha} + \Big(\Theta_n(Ł_cŁ_a\Theta_m) - (Ł_cŁ_a\Theta_n)\Theta_m\Big)\big|_{-\alpha}\Big]\end{aligned} \tag{5.62}$$

and this surface term vanish for given boundary condition $Ł_cŁ_a\Theta_n|_{\pm\alpha} = 0$. Therefore the eigenvalue operator $\mathbb{E}$ is self-adjoint under the inner product (5.60) with $\mathcal{N} = \frac{1}{(c-b)\mathcal{T}_\alpha}$.

Basically what we did is adding additional surface term $B_{2N}(\vec{\Theta}_n, \vec{\Theta}_m)$ to L^2 inner product (5.49) s.t. B_{2N} replace original surface term to another surface term which vanishes under higher derivative boundary condition up to $\Theta_n \leftrightarrow \Theta_m$ symmetric piece. Because the surface term from L^2 inner product is alway in the same form, from the next example, we focus on this B_{2N} especially for $\theta = +\alpha$ part, B_N.

5.3.4 3 derivative

From now on, we stick to the boundary condition which comes from the Dirichlet boundary condition on $\Theta^{s|s}$. In that situation, 3 derivative boundary condition is imposed on $\Theta_n^{s-3|s}$.

$$Ł_{-(2s-4)}Ł_{d-5}\Theta_n^{s-3|s} = -\lambda\Theta_n^{s-3|s} \tag{5.63}$$

$$Ł_{d-3}Ł_{d-4}Ł_{d-5}\Theta_n^{s-3|s}\big|_{\theta=\pm\alpha} = Ł^{(3)}\Theta_n^{s-3|s}\big|_{\theta=\pm\alpha} = 0 \tag{5.64}$$

$N = [3/2] = 1$, so we have to determine one I, J and $\mathcal{N}$. This time, 3 derivative should appear so we have to mix following 2 equations to construct I and J.

$$\Big(Ł^{(3)} - (d+2s-6)\mathcal{T}_\theta\, Ł^{(2)} - (d+2s-8)Ł^{(1)}\Big)\Theta_n^{s-3|s} = -\lambda Ł^{(1)}\Theta_n^{s-3|s} \tag{5.65}$$

$$\Big(Ł^{(2)} - (d+2s-8)\mathcal{T}_\theta Ł^{(1)}\Big)\Theta_n^{s-3|s} = -\lambda\Theta_n^{s-3|s} \tag{5.66}$$

If there is $Ł^{(2)}$ term in J, one cannot cancel it nor make it $\Theta_m \leftrightarrow \Theta_n$ symmetric, because there is no $Ł^{(2)}$ term in I. Therefore the only possibility is removing $Ł^{(2)}$ term in J at the beginning. This condition fix everything. ($s-3|s$ will be omitted)

$$\begin{aligned} J(\Theta_n) &= \Big[Ł^{(3)} - \Big((d+2s-6)(d+2s-8)\mathcal{T}_\theta^2 + (d+2s-8)\Big)Ł^{(1)}\Big]\Theta_n \\ &= -\lambda\Big(Ł^{(1)} + (d+2s-6)\mathcal{T}_\theta\Big)\Theta_n \\ &= -\lambda\, I(\Theta_n) \end{aligned} \tag{5.67}$$

or just $b_1 = 1$, $b_0 = (d+2s-6)\mathcal{T}_\theta$ for compact notation. Then,

$$\begin{aligned} B_1(\vec{\Theta}_n, \mathbb{E}\vec{\Theta}_m) &= \mathcal{N}\, I(\Theta_n)J(\Theta_m)\big|_{+\alpha} \qquad (5.68)\\ &= \mathcal{N}\, I(\Theta_n)\cdot Ł^{(3)}\Theta_m\big|_{+\alpha} + \mathcal{N}\, c_1\, Ł^{(1)}\Theta_n \cdot Ł^{(1)}\Theta_m\big|_{+\alpha} + \mathcal{N}\, b_0 c_1\,\Theta_n\cdot Ł^{(1)}\Theta_m\big|_{+\alpha} \end{aligned}$$

where $c_1 = -\Big((d+2s-6)(d+2s-8)\mathcal{T}_\theta^2 + (d+2s-8)\Big)$. Set $\mathcal{N}$ to $-\frac{1}{b_0 c_1}$ then,

$$\begin{aligned} &\langle\vec{\Theta}_n, \mathbb{E}\vec{\Theta}_m\rangle_{L^2\oplus R} - \langle\mathbb{E}\vec{\Theta}_n, \vec{\Theta}_m\rangle_{L^2\oplus R} \\ =&\langle\Theta_n, Ł_{-(2s-4)}Ł_{d-5}\Theta_m\rangle_{L^2} - \langle Ł_{-(2s-4)}Ł_{d-5}\Theta_n, \Theta_m\rangle_{L^2} + B_1(\vec{\Theta}_n, \mathbb{E}\vec{\Theta}_m) - B_1(\mathbb{E}\vec{\Theta}_n, \vec{\Theta}_m) \\ =&-\frac{1}{b_0 c_1}\Big(I(\Theta_n)\cdot Ł^{(3)}\Theta_m - Ł^{(3)}\Theta_n \cdot I(\Theta_m)\Big)\big|_{+\alpha} \end{aligned} \tag{5.69}$$

Again, remaining surface term vanish for $Ł^{(3)}\Theta_n|_{\pm\alpha} = 0$ and the eigenvalue operator become self-adjoint. Note that $Ł^{(1)}\Theta_n \cdot Ł^{(1)}\Theta_m$ term in B_1 is $n \leftrightarrow m$ symmetric and is canceled at equation (5.69). $\mathcal{N}^{-1} = -b_0 c_1 > 0$ therefore the norm is positive definite.

5.3.5 4 derivative

4 derivatibe boundary condition come from $\Theta_n^{s-4|s}$.

$$Ł_{-(2s-5)}Ł_{d-6}\Theta_n^{s-4|s} = -\lambda\Theta_n^{s-4|s} \tag{5.70}$$

$$Ł^{(4)}\Theta_n^{s-4|s}\big|_{\theta=\pm\alpha} = 0 \tag{5.71}$$

This time, $N = [4/2] = 2$ and we have to determine I_1, I_2, J_1 and J_2. One of J (let's say J_2) should contain 4 derivative term and should not contain 3 derivative term. J_1 contain 2 derivative term and should be come from the eigenvalue equation. (There is no other 2 derivative equation to mix with the eigenvalue equation.) With these conditions, there are 3 parameters to be determined, $\mathcal{N}_1$, $\mathcal{N}_2$ and $b_{2,0}$.

$$\begin{aligned} J_1(\Theta_n) &= \left(Ł^{(2)} - (d+2s-10)\mathcal{T}_\theta\, Ł^{(1)}\right)\Theta_n \\ &= -\lambda\,\Theta_n \\ &= -\lambda\, I_1(\Theta_n) \end{aligned} \tag{5.72}$$

$$\begin{aligned} J_2(\Theta_n) &= \left(Ł^{(4)} + c_{2,2}Ł^{(2)} + c_{2,1}Ł^{(1)}\right)\Theta_n \\ &= -\lambda\left(Ł^{(2)} + (d+2s-6)\mathcal{T}_\theta\, Ł^{(1)} + b_{2,0}\right)\Theta_n \\ &= -\lambda\, I_2(\Theta_n) \end{aligned} \tag{5.73}$$

$$\begin{aligned} c_{2,2} &= b_{2,0} - (d+2s-6)(d+2s-8)\mathcal{T}_\theta^2 - 2(d+2s-9) \\ c_{2,1} &= -(d+2s-10)\mathcal{T}_\theta\, b_{2,0} - (d+2s-6)(d+2s-10)\mathcal{T}_\theta \end{aligned}$$

Now $B_2(\vec{\Theta}_n, \mathbb{E}\vec{\Theta}_m) = \mathcal{N}_1\, I_1(\Theta_n)J_1(\Theta_m)\big|_{+\alpha} + \mathcal{N}_2\, I_2(\Theta_n)J_2(\Theta_m)\big|_{+\alpha}$ is parametrized by $\mathcal{N}_1, \mathcal{N}_2$ and $b_{2,0}$. $B_2(\vec{\Theta}_n, \mathbb{E}\vec{\Theta}_m)$ is sum of product of linear function of Θ_n, therefore always in the form of

$$B_2(\vec{\Theta}_n, \mathbb{E}\vec{\Theta}_m) = \mathcal{B}_{ij}(\mathcal{N}_1, \mathcal{N}_2, b_{2,0})\, Ł^{(i)}\Theta_n \cdot Ł^{(j)}\Theta_m\big|_{+\alpha} \tag{5.74}$$

Let's impose the condition, equation (5.50) on B_2. $\mathcal{B}_{ii}$ is the coefficient of $m \leftrightarrow n$ symmetric term and do not appear in (5.50). $\mathcal{B}_{4i}$ and $\mathcal{B}_{i4}$ can be arbitrary because they are the coefficients of the term which vanish for given boundary condition. From the above expression of

I_i and J_i, we get 3 equations,

$$\begin{aligned}\mathcal{B}_{01}-\mathcal{B}_{10} &= -\mathcal{N}_1\,(d+2s-10)\mathcal{T}_\alpha+\mathcal{N}_2\,b_{2,0}\,c_{2,1}=-1\\ \mathcal{B}_{02}-\mathcal{B}_{20} &= \mathcal{N}_1+\mathcal{N}_2\,b_{2,0}\,c_{2,2}=0\\ \mathcal{B}_{12}-\mathcal{B}_{21} &= \mathcal{N}_2(d+2s-6)\mathcal{T}_\alpha\,c_{2,2}-\mathcal{N}_2c_{2,1}=0\end{aligned} \tag{5.75}$$

The solution is

$$\begin{aligned}b_{2,0} &= \frac{1}{2}(d+2s-6)(1+(d+2s-6)\mathcal{T}_\alpha^2)\\ \mathcal{N}_1^{-1} &= 2(d+2s-8)\mathcal{T}_\alpha\\ \mathcal{N}_2^{-1} &= \frac{1}{2}(d+2s-10)(3+(d+2s-6)\mathcal{T}_\alpha^2)\,\mathcal{N}_1^{-1}b_{2,0}\end{aligned} \tag{5.76}$$

Because $s \geq 4$, this norm is positive definite if $d \geq 3$. For 5 and 6 derivative boundary conditions, see appendix B.

5.4 Conjecture for unitary boundary condition

Unfortunately, our procedure of the previous section cannot apply to arbitrary higher derivative boundary condition since the complexity increase very fastly. For 7 derivatives case, we get 6 order algebraic equation which cannot be solved analytically. Finding a recursive way of getting extended inner product can be a resolution. To say something concrete, we need to know general properties of extended inner product.

However, examples up to 6 derivatives boundary condition are enough to make a conjecture. Especially, 2-derivative example tells us what is non-unitary boundary condition. For spin greater than 3, every set of boundary conditions contains 2-derivative boundary condition. When the Dirichlet boundary condition is imposed on $\Theta^{k|s}$, 2-derivative boundary condition on $\Theta^{k-2|s}$ gives positive definite norm but 2-derivative boundary condition on $\Theta^{k+2|s}$ gives non-positive definite norm. As we see from string and spin-two example, this non-positive definite norm is the origin of non-unitarity. The only way to avoid such non-unitarity is to choose $k \geq s-1$. For $k \geq s-1$, $\Theta^{k+2|s}$ does not exist. Also, we can conclude that for $k < s-1$, there is always non-unitarity. Therefore our conjecture for unitary boundary condition is that k should be s or $s-1$. All the other examples of 3 to 6 derivatives boundary condition support our conjecture. They correspond to $k = s$ case, and all the extended norm is positive definite. It is interesting that among various boundary conditions, only the Dirichlet or 1-derivative boundary condition on spin-s component give a unitary spectrum.

Chapter 6

Discussion

Higher spin gauge theory provides the simplest example of AdS/CFT correspondence. It is conjectured that the Vasiliev theory is dual to free/critical $O(N)$ vector model [39]. It would be interesting if we can find holographic dual of our case. In the first section, we discuss possible holographic dual. In the second section, on-going future works are introduced.

6.1 Holography

Since Klebanov and Polyakov [39], lots of nice works has been done for higher spin holography, for instance, [40, 41, 42, 43, 44]. Compare to holography from the String theory, higher spin theory provides very simple setup. The dual CFT is just free/ciritical $O(N)$ vector model. One of strong evidence for higher spin holography is the fact that one can construct all the conserved higher spin current from $O(N)$ vector model. Group theoretically, it can be represented as [45, 46]

$$\mathcal{D}\left(\frac{d}{2}-1,\,0\right)\otimes\mathcal{D}\left(\frac{d}{2}-1,\,0\right)=\bigoplus_{s=0}^{\infty}\mathcal{D}(d+s-2,s) \tag{6.1}$$

As reviewed in appendix A, $\mathcal{D}(d+s-2,s)$ is correspond to massless spin-s field, and $\mathcal{D}\left(\frac{d}{2}-1,\,0\right)$ is called singleton. In terms of boundary CFT, singleton is just a scalar field which satisfy Klein-Gordon eqution.

Our theory of AdS side consists of bulk higher spin theory and boundary theory from HD BC. Note that these boundaries are different from the asymptotic boundary where CFT lives. At the free level of the Vasiliev theory, the bulk higher spin theory is the $(d+2)$-dimensional Fronsdal theory of all the spin-one to ∞, and a scalar of specific mass. It is natural to think $O(N)$ vector model with boundaries as a dual CFT, the boundary theory of $O(N)$ vector model would be holographic dual of boundary theory of higher spin theory. To guess which theory lives at the boundary of $O(N)$ vector model, we should know the correct boundary degrees of freedom for higher spin theory.

Due to lack of extended inner product for generic case, we cannot specify exact spectrum of boundary degrees of freedom. However with the extended inner product of 2 to 6 derivatives, we could find a pattern for specific boundary condition. If we impose the Dirichlet boundary condition on $\Theta^{k|s}$, $k < s-1$, there appear partially massless fields. For such case, corresponding boundary degrees also contains the partially massless fields. We found an observation about the pattern of the depth of partially massless field. When we consider spin-s massless bulk field, it seems that required boundary degrees of freedom contains spin-s partially massless fields of depth-$0, 2, 4 \cdots 2[(s-k)/2]$. In our convention, depth-0 partially massless means just massless. Interestingly, there is following group theoretical identification,

$$\mathcal{D}\left(\frac{d}{2} - p,\, 0\right) \otimes \mathcal{D}\left(\frac{d}{2} - p,\, 0\right) = \bigoplus_{s=0}^{\infty} \bigoplus_{k=1}^{p} \mathcal{D}(d + s - 2k, s). \tag{6.2}$$

$\mathcal{D}(d+s-2k, s)$ is correspond to partially massless field of depth-$(2k-2)$, and $\mathcal{D}\left(\frac{d}{2} - p,\, 0\right)$ is something called "higher-order Signleton". Therefore the spectrum of r.h.s. exactly match the spectrum of our boundary degrees of freedom if $p = [(s-k)/2] + 1$. Which means if we impose the Dirichlet boundary condition on $k = s - 2p + 2$ for each spin-s, all the boundary partially massless degrees of freedom can be obtained from product higher-order singleton. In terms of boundary CFT, higher-order singleton is a scalar whose equation of motion is

$$\Box^p \phi = 0\,. \tag{6.3}$$

Therefore we conjecture that the holographic dual of our theory is vector $O(N)$ model with boundary higher-order singleton.

We should mention one unclear point of our conjecture. Actually, if we impose the Dirichlet boundary condition on $k = s - 2p + 2$, not only partially massless but also massive fields with various spin appear. Therefore we need something more than higher-order singleton at the boundary of CFT.

6.2 Future works

Through the whole thesis, we discuss free higher spin theory. To achieve our original goal, the Kaluza-Klein compactification of the Vasiliev theory, we should study interacting cases. There are 2 ways of studying them.

One is perturbative approach keeping the metric-like formalism. The metric-like formalism of higher spin is easy to use but is only known perturbatively. However, we expect to learn an important lesson for consistency of our formalism at interacting level. Also, there are 2 immediate applications about the spin-two system. Finding unitary theory of partially massless spin-two coupled to graviton at de Sitter background is a promising problem of cosmology. Contrast to the partially massless field on anti-de Sitter background, it is known to be unitary on de Sitter background but the way of unitary coupling with graviton is still unknown. One of our set of boundary condition produce massless and partially massless spin-two field as ground modes and we expect to get a unitary interacting theory of them. Another problem is revisiting Gibbons-Hawking term of gravity in terms of our formalism. It will be instructive to understand perturbative degrees of freedom for Gibbons-Hawking term and we may find another possible boundary term apart from Gibbons-Hawking term.

The other is non-perturbative approach using frame-like formalism. The Vasiliev theory is the generalization of the frame-like formalism of gravity and every variable is generalization of vierbein and spin-connection. Frame-like formalism is useful when one try to write down interacting theory we may be able to succeed to compactify the Vasiliev theory due to the intrinsic relation between frame-like formalism and interaction. As a first step, we did Kaluza-Klein compactification of free higher spin theory in terms of the frame-like variable. It contains more degrees of freedom and more gauge symmetry. Also, it contains not only totally symmetric representation but also mixed symmetric degrees of freedom. Using this, we hope we can achieve our final goal.

Appendices

Chapter A

Verma module and partially massless field

Here we recall the definition of the Verma $\mathfrak{so}(d,2)$-module. Consider a finite dimensional module $\mathcal{Y}(\Delta,\,Y)$ of sub-algebra $\mathfrak{so}(2)\oplus\mathfrak{so}(d)$. We use Δ to denote conformal dimension and Y to denote Young diagram of $\mathfrak{so}(d)$. For the analisys of symmetric higher spin, we limit ourself to the Young diagram of a single row of length s. The Verma $\mathfrak{so}(d,2)$-module $\mathcal{V}(\Delta,\,s)$ is the space generated by action of the raising operators to the module $\mathcal{Y}(\Delta,\,Y)$. We will also denote $\mathcal{D}(\Delta,s)$ for the irreducible quotient of Verma module $\mathcal{V}(\Delta,\,s)$. For generic value, Verma module $\mathcal{V}(\Delta,\,s)$ is irreducible and therefore coincides with $\mathcal{D}(\Delta,s)$. However, for specific values, it becomes reducible with a non-trivial submodule. For instance, $\Delta = d+k-1$ with an integer $0\le k\le s-1$, there is a submodule $\mathcal{D}\left(d+s-1,\,k\right)$. Therefore, $\mathcal{D}\left(d+k-1,\,s\right)$ is not equal to Verma module but is to the quotient of Verma module:

$$\begin{aligned}\mathcal{V}\left(d+k-1,\,s\right) &\simeq \mathcal{D}\left(d+k-1,\,s\right)\oplus\mathcal{D}\left(d+s-1,\,k\right)\,,\\ \mathcal{D}\left(d+k-1,\,s\right) &\simeq \frac{\mathcal{V}\left(d+k-1,\,s\right)}{\mathcal{D}\left(d+s-1,\,k\right)}\,.\end{aligned}\tag{A.1}$$

For $k=s-1$, $\mathcal{D}(d+(s-1)-1,s)$ is unitary and its field theoretical realization is the massless spin-s field propagating in AdS$_{d+1}$. For $0\le k<s-1$, $\mathcal{D}(d+k-1,s)$ is non-unitary and their field theoretical realizations are partially massless(PM) fields[1] with depth $t=(s-k-1)$. (For more general cases, see [32, **?**].) The action for PM field has the PM gauge symmetry which contains covariant derivatives up to order $t-1$. This can be derived by Stueckelberg form of PM field.

$$\delta\,\phi_{\mu_1\mu_2\cdots\mu_s} = \nabla_{(\mu_1}\cdots\nabla_{\mu_{t+1}}\,\xi_{\mu_{t+2}\cdots\mu_s)}+\cdots \tag{A.2}$$

See paragraph below Eq.(3.45) for PM spin-two case. The followings are properties of PM field:

Field type	Δ_+	m^2	Gauge variation: $\delta\,\phi_{\mu_1\mu_2\cdots\mu_s}$
depth-t PM field	$d+s-t-2$	$-\frac{\sigma}{r^2_{AdS}}\,t\,(d+2s-t-4)$	$\nabla_{(\mu_1}\cdots\nabla_{\mu_{t+1}}\,\xi_{\mu_{t+2}\cdots\mu_s)}+\cdots$

Table 2: Partially massless(PM) field

[1] Extrapolating our convention, the massless higher-spin field is the partially-massless higher-spin field with depth-zero.

In Table 2, m is defined by the following convention. By the mass of a field, we refer to the mass in flat limit. Therefore, it is zero when the higher spin gauge symmetry exist. In this convention, the relation between mass-squared and conformal dimension is given by

$$m^2\, r_{AdS}^2 = \Delta\,(\Delta - d) - (s-2)\,(d+s-2)\ . \tag{A.3}$$

Note that this is different from the mass-squared which appears in Fierz-Pauli equation in AdS [?]: $\left(\nabla^2 + \kappa^2\right)\phi_{\mu_1\,\mu_2\,\cdot\mu_s} = 0$ which is given as $\kappa^2\, r_{AdS}^2 = \Delta\,(\Delta - d) - s$.

Finally, $\mathfrak{so}(d{+}1,2)$-module for massless spin-s can be decomposed into $\mathfrak{so}(d,2)$-modules by the following branching rules [?]:

$$\mathcal{D}(d{+}s{-}1,s)_{\mathfrak{so}(\mathrm{d+1,2})} = \bigoplus_{n=0}^{\infty} \mathcal{D}(d{+}n{+}s{-}1,s)_{\mathfrak{so}(\mathrm{d,2})} \oplus \bigoplus_{l=0}^{s-1} \mathcal{D}(d{+}s{-}1,l)_{\mathfrak{so}(\mathrm{d,2})} \tag{A.4}$$

In main text we open omit subscripts $_{\mathfrak{so}(\mathrm{d,2})}$ for brevity.

Chapter B

5 and 6 derivatives boundary conditions

5 derivative

$$Ł_{-(2s-6)}Ł_{d-7}\Theta_n^{s-5|s} = -\lambda\Theta_n^{s-5|s} \tag{B.1}$$

$$Ł^{(5)}\Theta_n^{s-5|s}\big|_{\theta=\pm\alpha} = 0 \tag{B.2}$$

$$I_1(\Theta_n) = \Big(Ł^{(1)} + b_{1,0}\Big)\Theta_n \tag{B.3}$$

$$\begin{aligned}
J_1(\Theta_n) &= \Big(Ł^{(3)} + c_{1,2}Ł^{(2)} + c_{1,1}Ł^{(1)}\Big)\Theta_n \\
I_2(\Theta_n) &= \Big(Ł^{(3)} + (d+2s-6)\mathcal{T}_\theta\, Ł^{(2)} + b_{2,1}Ł^{(1)} + b_{2,0}\Big)\Theta_n \\
J_2(\Theta_n) &= \Big(Ł^{(5)} + c_{2,3}Ł^{(3)} + c_{2,2}Ł^{(2)} + c_{2,1}Ł^{(1)}\Big)\Theta_n
\end{aligned} \tag{B.4}$$

$$\begin{aligned}
c_{1,2} &= b_{1,0} - (d+2s-10)\mathcal{T}_\theta \\
c_{1,1} &= -(d+2s-12)\mathcal{T}_\theta\, b_{1,0} - (d+2s-12) \\
c_{2,3} &= b_{2,1} - (d+2s-6)(d+2s-8)\mathcal{T}_\theta^2 - 3(d+2s-10) \\
c_{2,2} &= b_{2,0} - (d+2s-10)\mathcal{T}_\theta\, b_{2,1} - 2(d+2s-11)(d+2s-6)\mathcal{T}_\theta \\
c_{2,1} &= -(d+2s-12)\mathcal{T}_\theta\, b_{2,0} - (d+2s-12)b_{2,1}
\end{aligned}$$

We have 5 variables $b_{1,0}$, $b_{2,0}$, $b_{2,1}$, $\mathcal{N}_1$ and $\mathcal{N}_2$. And 6 equations,

$$\begin{aligned}
\mathcal{B}_{01} - \mathcal{B}_{10} &= \mathcal{N}_1\, b_{1,0}\, c_{1,1} + \mathcal{N}_2\, b_{2,0}\, c_{2,1} = -1 \\
\mathcal{B}_{02} - \mathcal{B}_{20} &= \mathcal{N}_1\, b_{1,0}\, c_{1,2} + \mathcal{N}_2\, b_{2,0}\, c_{2,2} = 0 \\
\mathcal{B}_{03} - \mathcal{B}_{30} &= \mathcal{N}_1\, b_{1,0} + \mathcal{N}_2\, b_{2,0}\, c_{2,3} = 0 \\
\mathcal{B}_{12} - \mathcal{B}_{21} &= \mathcal{N}_1\, c_{1,2} + \mathcal{N}_2\, b_{2,1}\, c_{2,2} - \mathcal{N}_2\,(d+2s-6)\mathcal{T}_\alpha\, c_{2,1} = 0 \\
\mathcal{B}_{13} - \mathcal{B}_{31} &= \mathcal{N}_1 + \mathcal{N}_2\, b_{2,1}c_{2,3} - \mathcal{N}_2\, c_{2,1} = 0 \\
\mathcal{B}_{23} - \mathcal{B}_{32} &= \mathcal{N}_2\,(d+2s-6)\mathcal{T}_\alpha\, c_{2,3} - \mathcal{N}_2\, c_{2,2} = 0
\end{aligned} \tag{B.5}$$

The solution is,

$$\begin{aligned} b_{1,0} &= 2\Delta_2\mathcal{T}_\alpha \\ b_{2,0} &= \frac{\Delta_2\Delta_3\mathcal{T}_\alpha(1+\Delta_3\mathcal{T}_\alpha^2)(3+\Delta_3\mathcal{T}_\alpha^2)}{3+(4\Delta_2+\Delta_1)\mathcal{T}_\alpha^2} \\ b_{2,1} &= \frac{3\Delta_3(1+\Delta_2\mathcal{T}_\alpha^2)(1+\Delta_3\mathcal{T}_\alpha^2)}{3+(4\Delta_2+\Delta_1)\mathcal{T}_\alpha^2} \\ \mathcal{N}_1^{-1} &= \Delta_1\Delta_2\mathcal{T}_\alpha\big(3+(4\Delta_2+\Delta_1)\mathcal{T}_\alpha^2\big) \\ \mathcal{N}_2^{-1} &= b_{2,0}\Delta_0\Delta_1(3+6\Delta_2\mathcal{T}_\alpha^2+\Delta_2\Delta_3\mathcal{T}_\alpha^4) \end{aligned} \tag{B.6}$$

where $\Delta_m \equiv \Delta_{m+1}^{s-5|s} = d+2s-12+2m$. Because $s \geq 5$, this norm is positive definite if $d \geq 3$. ($\Delta_n > \Delta_0 > 0$, when $n > 0$) Though there was a unique solution, existance of the solution was non trivial. 5 variables should satisfy 6 conditions. Existance of a solution will be clear with semi-recursive construction.

6 derivative

$$Ł_{-(2s-7)}Ł_{d-8}\Theta_n^{s-6|s} = -\lambda\Theta_n^{s-6|s} \tag{B.7}$$

$$Ł^{(6)}\Theta_n^{s-6|s}\big|_{\theta=\pm\alpha} = 0 \tag{B.8}$$

$$I_1(\Theta_n) = \Theta_n \tag{B.9}$$

$$\begin{aligned} J_1(\Theta_n) &= \Big(Ł^{(2)} - (d+2s-14)\mathcal{T}_\theta\, Ł^{(1)}\Big)\Theta_n \\ I_2(\Theta_n) &= \Big(Ł^{(2)} + b_{2,1}Ł^{(1)} + b_{2,0}\Big)\Theta_n \end{aligned} \tag{B.10}$$

$$\begin{aligned} J_2(\Theta_n) &= \Big(Ł^{(4)} + c_{2,3}Ł^{(3)} + c_{2,2}Ł^{(2)} + c_{2,1}Ł^{(1)}\Big)\Theta_n \\ I_3(\Theta_n) &= \Big(Ł^{(4)} + (d+2s-6)\mathcal{T}_\theta\, Ł^{(3)} + b_{3,2}Ł^{(2)} + b_{3,1}Ł^{(1)} + b_{3,0}\Big)\Theta_n \\ J_3(\Theta_n) &= \Big(Ł^{(6)} + c_{3,4}Ł^{(4)} + c_{3,3}Ł^{(3)} + c_{3,2}Ł^{(2)} + c_{3,1}Ł^{(1)}\Big)\Theta_n \end{aligned} \tag{B.11}$$

$c_{i,m}$'s can be described by $b_{i,m}$'s from equation (5.55). We have 5 $b_{i,m}$'s and 3 $\mathcal{N}_i$'s to be determined. And now we have 10 conditions to be satisfied.

$$\begin{aligned} \mathcal{B}_{01} - \mathcal{B}_{10} &= -1 \\ \mathcal{B}_{ij} - \mathcal{B}_{ji} &= 0 \quad , \ 4 \geq j > i \geq 0 \end{aligned} \tag{B.12}$$

Again, there is a unique solution.

$$
\begin{aligned}
b_{2,0} &= \frac{1}{3}\Delta_3\big(3+(4\Delta_3+\Delta_2)\mathcal{T}_\alpha^2\big)\\
b_{2,1} &= 2\Delta_3\mathcal{T}_\alpha\\
b_{3,2} &= \frac{3}{2}\Delta_4\frac{(1+\Delta_4\mathcal{T}_\alpha^2)(9+10\Delta_3\mathcal{T}_\alpha^2+\Delta_3(2\Delta_3+\Delta_2)\mathcal{T}_\alpha^4)}{9+6(2\Delta_3+\Delta_2)\mathcal{T}_\alpha^2+\Delta_3(6\Delta_2+\Delta_4)\mathcal{T}_\alpha^4}\\
b_{3,1} &= 2\Delta_3\Delta_4\mathcal{T}_\alpha\frac{(1+\Delta_4\mathcal{T}_\alpha^2)(3+\Delta_3\mathcal{T}_\alpha^2)(3+\Delta_4\mathcal{T}_\alpha^2)}{9+6(2\Delta_3+\Delta_2)\mathcal{T}_\alpha^2+\Delta_3(6\Delta_2+\Delta_4)\mathcal{T}_\alpha^4}\\
b_{3,0} &= \frac{1}{2}\Delta_3\Delta_4\frac{(1+\Delta_4\mathcal{T}_\alpha^2)(3+\Delta_4\mathcal{T}_\alpha^2)(3+6\Delta_3\mathcal{T}_\alpha^2+\Delta_3\Delta_4\mathcal{T}_\alpha^4)}{9+6(2\Delta_3+\Delta_2)\mathcal{T}_\alpha^2+\Delta_3(6\Delta_2+\Delta_4)\mathcal{T}_\alpha^4}\\
\mathcal{N}_1^{-1} &= 3\Delta_2\mathcal{T}_\alpha\\
\mathcal{N}_2^{-1} &= \frac{2}{3}\Delta_1\Delta_2\Delta_3\mathcal{T}_\alpha\big(9+6(2\Delta_3+\Delta_2)\mathcal{T}_\alpha^2+\Delta_3(6\Delta_2+\Delta_4)\mathcal{T}_\alpha^4\big)\\
\mathcal{N}_3^{-1} &= b_{3,0}\Delta_0\Delta_1\Delta_2\mathcal{T}_\alpha(15+10\Delta_3\mathcal{T}_\alpha^2+\Delta_3\Delta_4\mathcal{T}_\alpha^4)
\end{aligned}
$$

where $\Delta_m \equiv \Delta_{m+1}^{s-6|s} = d+2s-14+2m$. Because $s \geq 6$, this norm is positive definite if $d \geq 3$. ($\Delta_n > \Delta_0 > 0$, when $n > 0$) Using Δ_m is a little bit ambiguous. For example, $\Delta_a + \Delta_b = 2\Delta_{\frac{a+b}{2}}$. However it is very useful to check that $\mathcal{N}$'s are positive. Note that the definition of Δ_m is different from that of 5 derivative case. This can be potentially annoying when we do semi-recursive construction so we should be careful.

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초록

질량을 가지는 고차 스핀 입자들은 양자 중력을 기술하는데 중요한 역할을 한다. 그럼에도 질량을 가지는 고차스핀 입자들의 상호작용하는 이론은 오직 끈이론만이 알려져 있다. 이는 이론을 구성할때 인과율, 자유도의 개수와 관련된 기술적인 문제를 해결하지 못했기 때문이다. 우리는 그러한 기술적 문제들을 회피하고자, 칼루자-클라인 콤팩트화를 이용하여 질량이 없는 고차 스핀 이론으로부터 질량이 있는 이론을 구성하는 방법을 제안하였다. 올바른 이론을 얻기 위하여 유일하게 알려진 고차 스핀 게이지 이론인 Vasiliev 이론을 목표로 하였고 그것을 위해 반 드지터 공간을 사용하여야 했다. 그 과정에서 고차 미분 경계조건과 같은 여러 흥미로운 기술적인 문제들이 발견되었고, 고차 미분 경계조건을 경계에 존재하는 추가적인 자유도로 해석할수 있었다. 경계의 자유도 관점에서 우리는 올바른 경계조건을 찾을수 있었고 낮은 차원의 질량을 가지는 이론을 성공적으로 얻을수 있었다.

주요어 : 고차 스핀, 고차 스핀 게이지, 칼루자 클라인, 경계가 있는 칼루자 클라인, 반 드지터 공간
학번 : 2009-20405