Eur. Phys. J. Plus (2024) 139:278
https://doi.org/10.1140/epjp/s13360-024-05014-4

THE EUROPEAN PHYSICAL JOURNAL PLUS

Regular Article

On the symmetries of elementary fermions

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Received: 16 January 2024 / Accepted: 16 February 2024

Abstract This paper deals with the symmetries of elementary fermions and their derivation from fundamental physical principles such as the Lorentz invariance and from the spinor-helicity formalism employed to three-vectors. The generators of the Lorentz group are discussed, and the physics of the associated chiral spins is used to establish the chiral symmetry $SU(2)$. The helicity of spin is defined which, when being applied to the hadronic isospin, yields the symmetry group $SU(4)$. The Kronecker product of these two basic symmetries defines a unified symmetry $SU(2) \otimes SU(4)$ of the basic light fermions, namely the chiral doublets of the single lepton and triple quarks of the first family. Breaking this symmetry, according to mechanisms that are used in the electroweak interactions of the Standard Model (SM) of quantum field theory, yields quantum electrodynamics (QED) with symmetry group $U(1)$, weak interactions with symmetry group $SU(2)$, and what we like to name quantum hadrodynamics (QHD, akin to quantum chromodynamics QCD of the SM) with symmetry group $SU(3)$. The gauge bosons associated with QED and QHD remain massless, but the weak bosons and the V bosons, related to the transformation of the quarks into a lepton and vice versa, become massive by the Higgs mechanism. Their masses are defined by the Higgs vacuum and the two coupling constants involved in the unified model. The V-boson mass is predicted to be 35.4 GeV. Furthermore, a possible explanation of hadron confinement is given in terms of the two hadronic charge operators. Moreover, the concept of hypercharge as used in the SM is not needed. Various versions of the extended Dirac equation that include the above symmetry groups are derived.

1 Introduction

Among the important open questions in modern elementary particle physics are the key ones which address the physical origin of such striking properties of the spin-one-half fermions as their colour and flavour. Why do quarks just come in three colours, and why do leptons and quarks occur in three generations or in six flavours? Why are all elementary fermions found empirically to be organized in doublets, like the electron and neutrino, or the up and down quark, as well as their heavier relatives? What are the deeper physical reasons for the appearance of the two gauge groups $SU(2)$ and $SU(3)$? Can they be connected or unified?

Some of these fundamental questions have remained unanswered for decades. They are also raised and competently discussed for the non-expert physicist or educated layman in the three excellent popular books by Veltman [1], Huang [2], and Wilczek [3]. Of course, for the physics student and specialist all issues are dealt with in depth and mathematically rigorously in the textbooks of quantum field theory (QFT) and the Standard Model (SM), like the older ones by Kaku [4] and Peskin and Schroeder [5], and the modern one by Schwartz [6]. In this paper we hope to give convincing physical answers to some of the questions raised above.

The content is organized in the following way. The physics of spin is presented. Its role in defining the generators of the Lorentz group and therewith the chiral symmetry is described. Space–time symmetry and inner symmetries can be combined in the form of the Pauli-Lubański [7–9] operator, which merges them as Kronecker products in consistency with the Coleman–Mandula theorem [10]. Various versions of the Dirac equation can be derived, which include the $SU(2)$ and $SU(4)$ symmetries by use of their Lie groups. In a short interlude section we suggest a possible origin of flavour from permutation symmetry involving the Pauli matrices.

The novel notion of spin helicity is introduced. Then the idea of an hadronic isospin is proposed, which by means of isospin helicity yields the $SU(4)$ Lie group. A unification scheme is developed which suggests the existence of three vector gauge bosons named V that can change leptons into quarks and vice versa. The $SU(3)$ symmetry as known from the SM is obtained by symmetry breaking of the product group $SU(2) \otimes SU(4)$. As a result this procedure defines what we like to call Quantum Hadro Dynamics (QHD), similar to the QCD of the SM. Yet the concept of colour charge is not needed but replaced by the hadronic charge. Moreover, the concept of hypercharge seems obsolete. The eight gluons can be arranged in an interesting new way. Via the Higgs mechanism the masses of the V and W bosons are finally derived. In the end, a summary and our conclusions are given.

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Published online: 21 March 2024

2 Spin and the generators of the Lorentz Group

2.1 Spin

The physical spin is a fundamental quantity in quantum mechanics and a genuine characteristic of elementary particles in quantum field theory. Spin may be considered as a kind of intrinsic rotation, which is described by the complex three-vector operator $\mathbf{S}$ having non-commuting components. It obeys the commutator relation

$$\mathbf{S} \times \mathbf{S} = \mathrm{i}\mathbf{S}. \tag{1}$$

For its physics and mathematics see, for example, the modern textbook of Weinberg [11]. For the historically interested reader we may recommend Tomonaga's [12] enlightening book about the story of spin.

In what follows the spin one-half plays a key role. It was introduced by Pauli [13] into the Schrödinger equation to describe the magnetic moment of the electron. For that spin we have $\mathbf{S} = 1/2\boldsymbol{\sigma}$ in terms of the Pauli matrix vector, which reads explicitly in standard form

$$\boldsymbol{\sigma} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right). \tag{2}$$

It is well known that these three matrices are the generators of the Lie group $SU(2)$, which is isomorphic to the group $SO(3)$ of rotations in real Euclidian space $\mathbb{R}^3$. Let $\mathbf{a}$ be a three-vector, then the quantity $(\boldsymbol{\sigma} \cdot \mathbf{a})$ is called its helicity. Since $(\boldsymbol{\sigma} \cdot \mathbf{a})^2 = 1_2\, \mathbf{a} \cdot \mathbf{a}$, this relation can be used to replace the scalar vector product by the square of the related helicity matrix. This method, sometimes called spinor-helicity formalism, will be exploited subsequently. Here the 1_n denotes the unit matrix of dimension n.

2.2 The generators of the Lorentz Group

The abstract general spin also plays an important role in the space–time symmetries and determines the Lorentz-invariant particle propagation. It defines the generators of the rotation group in the three-dimensional Euclidian space, for $SO(3)$ in the adjoint three-vector representation, and for $SU(2)$ in the fundamental spinor representation. Moreover, the two vector generators, rotation $\mathbf{J}$ and boost $\mathbf{K}$, of the Lorentz Group (LG) [14, 15] in Minkowski space–time, can be combined to define the two spins

$$\mathbf{S}_\pm = \frac{1}{2}(\mathbf{J} \pm \mathrm{i}\mathbf{K}), \quad \mathbf{S}_\pm^2 = \frac{3}{4} 1_4. \tag{3}$$

They express the basic dual property of the chirality of the LG. These two chiral spins form subalgebras of the Lorentz algebra, commute with each other $[\mathbf{S}_\pm, \mathbf{S}_\mp] = 0$, and also obey the commutator (1). In the fundamental spinor representation of the LG one simply has the set, $\mathbf{S}_+ = 1/2\boldsymbol{\sigma}$ and $\mathbf{S}_- = 0$, or the set $\mathbf{S}_- = 1/2\boldsymbol{\sigma}$ and $\mathbf{S}_+ = 0$ in terms of the Pauli matrices [13]. For the four-vector representation of the LG, Marsch and Narita [16] have shown that the right- and left-chiral spin can be expressed as $\mathbf{S}_\pm = 1/2\boldsymbol{\Sigma}_\pm$ in terms of the 4×4 spin matrices

$$\Sigma_{\pm\mathrm{x}} = \begin{pmatrix} 0 & \pm 1 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\mathrm{i} \\ 0 & 0 & \mathrm{i} & 0 \end{pmatrix}, \ \Sigma_{\pm\mathrm{y}} = \begin{pmatrix} 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ \pm 1 & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}, \Sigma_{\pm\mathrm{z}} = \begin{pmatrix} 0 & 0 & 0 & \pm 1 \\ 0 & 0 & -\mathrm{i} & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \end{pmatrix}, \ \Delta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4}$$

These matrices have been used to define an extended Dirac equation [16–19], which also encompasses the up and down components of a fermion in relation to the chiral $SU(2)$ symmetry. In this context, there appears an associated matrix that we call Delta matrix as defined above. Its square $\Delta^2 = 1_4$. Delta corresponds to the metric of the real Minkowski space. Furthermore, one finds that $\boldsymbol{\Sigma}_\pm \Delta = \Delta \boldsymbol{\Sigma}_\mp$. The Sigma matrices have the following properties: Their commutator obeys $[\boldsymbol{\Sigma}_\pm, \boldsymbol{\Sigma}_\mp] = 0$, and $\boldsymbol{\Sigma}_\pm \times \boldsymbol{\Sigma}_\pm = 2\mathrm{i}\boldsymbol{\Sigma}_\pm$. By complex conjugation of the Sigma matrices in (4) one finds that $(\boldsymbol{\Sigma}_\pm)^* = -\boldsymbol{\Sigma}_\mp$. The sigma component matrices squared give unity, and their sum yields $\boldsymbol{\Sigma}_\pm^2 = 3\, 1_4$. Thus they represent non-fundamental representations of $SU(2)$ and of spin one-half.

With the help of these matrices, we can reformulate the four-vector Lorentz transformation [16, 17] and give it a form that manifestly reveals the $SU(2) \otimes SU(2)$ group structure. The four-vector Lorentz transformation is a real 4×4 matrix operator, as it operates on a real four-vector V^μ in Minkowski space $\mathbb{R}^4$. To obtain the associated spinor representation of the LG, we simply let the above Sigmas matrices operate on a four-component complex vector in the Hilbert space $\mathbb{C}^4$, with the row vector $z = (z_1, z_2, z_3, z_4)$ and with z_j being an element of $\mathbb{C}$.

The above chiral Sigma matrices can, by means of a unitary transformation [16], related to a change of the basis in $\mathbb{C}^4$, be expressed as Kronecker products involving the Pauli matrices as follows

$$\boldsymbol{\Sigma}_\mathrm{R} = 1_2 \otimes \boldsymbol{\sigma}, \quad \boldsymbol{\Sigma}_\mathrm{L} = \boldsymbol{\sigma} \otimes 1_2. \tag{5}$$

These two Sigma matrix vectors in the new basis do of course also commute component wise. The associated Delta matrix then reads

$$\tilde{\Delta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{6}$$

One finds again that $\mathbf{\Sigma}_{\rm R,L}\tilde{\Delta} = \tilde{\Delta}\mathbf{\Sigma}_{\rm L,R}$, and that $\tilde{\Delta}^2 = 1_4$. The form of Eq. (5) reveal the tensor-product nature of the LG, indicated by $SU(2) \otimes SU(2)$, and emphasize the central role played by the Pauli matrices.

3 Linarization of the Casimir operators yields the Dirac equation

In their influential work, Wigner [14] and Bargman and Wigner [20] emphasized the important role played by the two Casimir operators of the Poincaré and Lorentz group (LG) [15]. Let us start with the first Casimir operator of the LG, which is the squared four-momentum of a particle (equal to its mass squared if it is massive). The second is the first times the spin squared $\mathbf{S}^2$, a rotational invariant in real space. We will not consider orbital angular momentum here, but just the intrinsic spin of a particle, and obtain in Fourier space for the four-momentum $p^\mu = (E, \mathbf{p})$

$$C_1 = p^\mu p_\mu = m^2, \tag{7}$$

$$C_2 = m^2\mathbf{S}^2 = m^2 s(s+1)1_{2s+1}. \tag{8}$$

The second equation can also be described in manifestly covariant form in terms of the Pauli-Lubański [7–9] operator. We note that, by use of the four-vector p^μ, with energy E and momentum $\mathbf{p}$, the components of which commute, and any spin $\mathbf{S}$, we get from (7) and (8) an equation involving the spin, energy, and momentum explicitly as follows

$$(E^2 - \mathbf{p}^2)\mathbf{S}^2 = \mathbf{S}^2E^2 - (\mathbf{S}\cdot\mathbf{p})(\mathbf{S}\cdot\mathbf{p}) - (\mathbf{S}\times\mathbf{p})\cdot(\mathbf{S}\times\mathbf{p}) \tag{9}$$

This equation is just the second Casimir operator C_2 of the Lorentz group in another form. For spin $s = 1/2$, one finds that $(\boldsymbol{\sigma}\times\mathbf{p})\cdot(\boldsymbol{\sigma}\times\mathbf{p}) = 2(\boldsymbol{\sigma}\cdot\mathbf{p})^2$, a relation that is only valid for the Pauli matrices yet not for any other matrices of larger spins. When exploiting that relation we obtain from (9) the simple equation

$$(E^2 - m^2)1_2 = (\boldsymbol{\sigma}\cdot\mathbf{p})^2, \tag{10}$$

in which now the momentum or kinetic helicity appears, and from which the Dirac equation in the Weyl basis readily follows by operation on Pauli bi-spinors and by a subsequent factorization of the resulting equations to the first-order in p^μ. We use units with $c = \hbar = 1$ and replace the four-momentum p_μ by the related operator $\hat{P}_\mu = \mathrm{i}\partial_\mu = \mathrm{i}(\partial_t, \partial_{\mathbf{x}})$, which acts on a four-component spinor field $\psi(x)$ with x being as an abbreviation for $x^\mu = (t, \mathbf{x})$. This procedure yields the famous Dirac [21] equation

$$\gamma^\mu \mathrm{i}\partial_\mu\psi = m\psi. \tag{11}$$

It describes a fermion with mass m and spin 1/2 by means of the four 4×4 Dirac gamma matrices [4, 6]. They can be written as Kronecker products

$$\gamma^\mu = (\gamma_0, \boldsymbol{\gamma}) = (\sigma_{\rm x}\otimes 1_2, \mathrm{i}\sigma_{\rm y}\otimes\boldsymbol{\sigma}), \tag{12}$$

in the convenient Weyl basis for the Dirac equation. The sigmas on the left side of the Kronecker product describe the particle-antiparticle doublet and on the right side the spin one-half doublet of the fermion.

4 Interlude: six flavours of the Dirac equation

The elementary fermions of the SM [4–6] come in six flavours. Why is this so? This basic physics question has been asked repeatedly yet still remains unanswered. Here we present a possible mathematical explanation in terms of permutation symmetry. The flavour degrees of freedom seem to emerge naturally from the permutation of the Pauli matrices that appear on the left of the Kronecker products in the gamma matrices as written in the above Eq. (12). This plausible idea was suggested by Marsch [22], but it remains to be validated. Inspection of the Dirac gamma matrices shows that by permutation only six such versions are possible:

$$\begin{aligned} \gamma^{\mu}_{\mathrm{W}} &= (\sigma_{\mathrm{x}} \otimes 1_2, \mathrm{i}\sigma_{\mathrm{y}} \otimes \boldsymbol{\sigma}), \\ \gamma^{\mu}_{\mathrm{D}} &= (\sigma_{\mathrm{z}} \otimes 1_2, \mathrm{i}\sigma_{\mathrm{y}} \otimes \boldsymbol{\sigma}), \\ \gamma^{\mu}_{\mathrm{y,x}} &= (\sigma_{\mathrm{y}} \otimes 1_2, \mathrm{i}\sigma_{\mathrm{x}} \otimes \boldsymbol{\sigma}), \\ \gamma^{\mu}_{\mathrm{z,x}} &= (\sigma_{\mathrm{z}} \otimes 1_2, \mathrm{i}\sigma_{\mathrm{x}} \otimes \boldsymbol{\sigma}), \\ \gamma^{\mu}_{\mathrm{x,z}} &= (\sigma_{\mathrm{x}} \otimes 1_2, \mathrm{i}\sigma_{\mathrm{z}} \otimes \boldsymbol{\sigma}), \\ \gamma^{\mu}_{\mathrm{y,z}} &= (\sigma_{\mathrm{y}} \otimes 1_2, \mathrm{i}\sigma_{\mathrm{z}} \otimes \boldsymbol{\sigma}). \end{aligned} \tag{13}$$

The first is known as the Weyl representation and the second as the Dirac representation. The other four bear no name and apparently have not been used. The six versions are obtained by cyclic permutation of the spatial index pairs at the Pauli matrices. The threefold multiplicity of these pairs just reflects the fact that the $SU(2)$ group has three generators corresponding exactly to the dimensions of real space. We suggest that this striking permutation symmetry is reflected by the six possible flavours of a fermion in the SM. By means of similarity transformations [22], these representations are all mutually connected. However, is there a physical meaning to such similarity. Does it correspond to physical degrees of freedom? Presently, we cannot answer this question but will leave it open here.

Considering merely those pairs having the same Pauli matrix located at the $\boldsymbol{\sigma}$ term, one may suggest that the resulting three pairs could just correspond to the three families of the SM. So the number three may simply reflect that real space is three-dimensional. However, the doublet nature (electron, neutrino; up and down quarks) of the fermions of each family can better be explained in more physical terms by the chiral property of the Lorentz transformation, which is done in the next section.

5 Chirality and $SU(2)$ symmetry

The standard Dirac equation has been extended [16, 17, 23] in order to include explicitly the chiral spins (3). This procedure leads to larger gamma matrices, then denoted as capital Gamma, $\Gamma^{\mu} = (\Gamma_0, \boldsymbol{\Gamma})$, and yields the extended Dirac equation

$$\Gamma^{\mu}\mathrm{i}\partial_{\mu}\Psi = m\Psi. \tag{14}$$

This equation describes a fermion and its antiparticle with mass m, the physical spins of 1/2, and the chiral $SU(2)$ up-and-down doublet. The associated four 8×8 Dirac Gamma matrices also obey the Clifford algebra,

$$\Gamma^{\mu}\Gamma^{\nu} + \Gamma^{\nu}\Gamma^{\mu} = 2g^{\mu\nu}1_8. \tag{15}$$

The above Ψ is a Dirac-type spinor but with eight components. In order to derive (14), we describe here two paths different from that taken previously by Marsch and Narita [16, 17] to obtain the extended Dirac equation. Reconsidering the result (10) and choosing the Sigma matrices (4) instead of the Pauli matrices (2), we readily obtain

$$(E^2 - m^2)1_4 = (\boldsymbol{\Sigma}_{\pm} \cdot \mathbf{p})^2, \tag{16}$$

which contains four inner degrees of freedom, two for the up and down spin and two for the right- and left-chiral isospin. However, which of the two Sigmas should be chosen? Are they independent? Obviously not, since they are connected by the above stated relation $\Delta\boldsymbol{\Sigma}_{\pm}\Delta = \boldsymbol{\Sigma}_{\mp}$. We present two ways to solve this problem.

5.1 First way to treat chirality

Here we consider the two Sigmas to be equivalent. Therefore, they should be employed on an equal footing. When using the above relation, we can rewrite it as follows

$$\begin{aligned} (E\Delta + \boldsymbol{\Sigma}_{+} \cdot \mathbf{p}\,\Delta)(E\Delta - \boldsymbol{\Sigma}_{-} \cdot \mathbf{p}\,\Delta) = m^2 1_4, \\ (\Delta E + \Delta\boldsymbol{\Sigma}_{+} \cdot \mathbf{p})(\Delta E - \Delta\boldsymbol{\Sigma}_{-} \cdot \mathbf{p}) = m^2 1_4. \end{aligned} \tag{17}$$

These two equations are mathematically the same as (16), which is obtained by multiplying the Δ through. Thus only one of them needs to be considered now. We use the second. We can make it a second-order wave equation by replacing the Fourier four-vector p_{μ} by the related quantum-mechanical operator $\hat{P}_{\mu} = \mathrm{i}\partial_{\mu}$ that operates on a complex Minkowski spinor $\Phi(x)$, which is a four-component vector in the Hilbert state $\mathbb{C}^4$. The resulting Klein-Gordon-type wave equation can then be decomposed into two linear wave equations reading

$$\begin{aligned} \mathrm{i}(\Delta\tfrac{\partial}{\partial t} - \Delta\boldsymbol{\Sigma}_{+} \cdot \tfrac{\partial}{\partial \mathbf{x}})\Phi_{+}(x) = m\Phi_{-}(x) \\ \mathrm{i}(\Delta\tfrac{\partial}{\partial t} + \Delta\boldsymbol{\Sigma}_{-} \cdot \tfrac{\partial}{\partial \mathbf{x}})\Phi_{-}(x) = m\Phi_{+}(x). \end{aligned} \tag{18}$$

They are the Weyl equations of the extended Dirac equation (14), which involves the eight-component spinor $\Psi^\dagger = (\Phi_+^\dagger, \Phi_-^\dagger)$. The corresponding Gamma matrices read

$$\Gamma_\mu = \left(\begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}, \begin{pmatrix} 0 & \Delta\mathbf{\Sigma}_- \\ -\Delta\mathbf{\Sigma}_+ & 0 \end{pmatrix} \right). \tag{19}$$

There are three additional important Gamma matrices, the matrix $\Gamma_5 = \mathrm{i}\Gamma_0\Gamma_\mathrm{x}\Gamma_\mathrm{y}\Gamma_\mathrm{z}$, the spin matrix vector and the isospin matrix vector. They are given by the subsequent 8×8-matrix operators

$$\Gamma_5 = \begin{pmatrix} -1_4 & 0 \\ 0 & 1_4 \end{pmatrix}, \ \mathbf{S} = \frac{1}{2}\begin{pmatrix} \mathbf{\Sigma}_+ & 0 \\ 0 & \mathbf{\Sigma}_- \end{pmatrix}, \ \mathbf{I} = \frac{1}{2}\begin{pmatrix} \mathbf{\Sigma}_- & 0 \\ 0 & \mathbf{\Sigma}_+ \end{pmatrix}. \tag{20}$$

As the two Sigma matrix vectors commute component-wise, the chiral isospin I commutes with the spin $\mathbf{S}$, and what is more important also with the five Gamma matrices, and therefore it constitutes the $SU(2)$ symmetry as an outcome of the Lorentz group. Therefore the extended Dirac equation describes a spin one-half fermion that is a chiral isospin up-and-down doublet. We note incidentally that there are also six equivalent versions [23] (or flavours?) of these Gamma matrices which correspond to the ones already discussed in Sect. 4 and are all related by appropriate similarity transformations.

5.2 Second way to treat chirality

Going back to Eq. (16) and comparing it with (10) using the Pauli matrices, we see that we just need to replace $\boldsymbol{\sigma}$ by $\mathbf{\Sigma}_\pm$ in the expression (12) for the gamma matrices. Thus in the Weyl basis we obtain

$$\Gamma_\pm^\mu = (\Gamma_0, \mathbf{\Gamma}_\pm) = (\sigma_\mathrm{x} \otimes 1_4, \mathrm{i}\sigma_\mathrm{y} \otimes \mathbf{\Sigma}_\pm). \tag{21}$$

Note that trivially $1_2 \otimes 1_2 = 1_4$. The associated spin and chiral isospin matrix operators then read

$$\mathbf{S}_\pm = \frac{1}{2} 1_2 \otimes \mathbf{\Sigma}_\pm, \ \ \mathbf{I}_\pm = \frac{1}{2} 1_2 \otimes \mathbf{\Sigma}_\mp, \tag{22}$$

which obviously commute as the Sigmas commute. The Γ_5 matrix remains unchanged in both cases. Of course, also $\left[\mathbf{I}_\pm, \Gamma_\pm^\mu\right] = 0$, and furthermore $\left[\mathbf{S}_\pm, \Gamma_0\right] = 0$.

The above sets of matrices are entirely equivalent, because they can be transformed into each other by a unitary transformation reading

$$U = U^\dagger = U^{-1} = 1_2 \otimes \Delta. \tag{23}$$

Then for example, we obtain $U\Gamma_\pm^\mu U = \Gamma_\mp^\mu$. So both Gammas are connected intimately via the Delta matrix (4). One may chose the one appearing to be most appropriate mathematically for the physical purpose in mind.

Using the chiral Sigma matrices as defined in Eq. (5), then we obtain for the above quantities in the right-chiral case

$$\Gamma_\mathrm{R}^\mu = (\sigma_\mathrm{x} \otimes 1_4, \mathrm{i}\sigma_\mathrm{y} \otimes 1_2 \otimes \boldsymbol{\sigma}). \tag{24}$$

The associated spin and isospin matrix operators then read

$$\mathbf{S}_\mathrm{R} = \frac{1}{2} 1_4 \otimes \boldsymbol{\sigma}, \ \ \mathbf{I}_\mathrm{R} = \frac{1}{2} 1_2 \otimes \boldsymbol{\sigma} \otimes 1_2. \tag{25}$$

Similarly, we obtain for the above quantities in the left-chiral case

$$\Gamma_\mathrm{L}^\mu = (\sigma_\mathrm{x} \otimes 1_4, \mathrm{i}\sigma_\mathrm{y} \otimes \boldsymbol{\sigma} \otimes 1_2). \tag{26}$$

The associated spin and isospin matrix operators then read

$$\mathbf{S}_\mathrm{L} = \frac{1}{2} 1_2 \otimes \boldsymbol{\sigma} \otimes 1_2, \ \ \mathbf{I}_\mathrm{L} = \frac{1}{2} 1_4 \otimes \boldsymbol{\sigma}. \tag{27}$$

Using the rule of matrix multiplication for Kronecker products, $(A \otimes B)(C \otimes D) = (AC \otimes BD)$ for quadratic matrices A, B, C, D, one readily finds that $\left[\mathbf{I}_\mathrm{R,L}, \Gamma_\mathrm{R,L}^\mu\right] = 0$. Moreover, one finds $\left[\mathbf{S}_\mathrm{R,L}, \Gamma_{0,\mathrm{R,L}}\right] = 0$, which means that spin and isospin both are well defined in the rest frame of the particle, and thus they can be used to classify its properties. The eight-component spinor just encompasses the three doublets, particle and antiparticle, spin up and down, and isospin up and down.

Like in the previous paragraphs, the two above sets of matrices are entirely equivalent, because they can be transformed into each other by a unitary transformation reading

$$V = V^\dagger = V^{-1} = 1_2 \otimes \tilde{\Delta}. \tag{28}$$

Then for example, we obtain $V\Gamma_\mathrm{R,L}^\mu V = \Gamma_\mathrm{L,R}^\mu$. So both Gammas are connected intimately via the $\tilde{\Delta}$ matrix (6), which transposes the second with the third column or row, respectively.

The gamma matrices can be brought in different forms by a unitary transformation. First, note that the left-chiral case can also be written

$$\Gamma^\mu_\mathrm{L} = (\sigma_\mathrm{x} \otimes 1_2, \mathrm{i}\sigma_\mathrm{y} \otimes \boldsymbol{\sigma}) \otimes 1_2 = \gamma^\mu \otimes 1_2, \tag{29}$$

with the standard Dirac equation as in (12). Second, we follow Marsch and Narita [24] who studied extensively the connections in the Dirac equation between the Clifford algebra of Lorentz invariance and the Lie algebra of the $SU(N)$ gauge symmetry. Thus we introduce the unitary transposition 8×8 matrix W reading

$$W = W^\dagger = W^{-1} = \begin{pmatrix} 1_2 & 0 & 0 & 0 \\ 0 & 0 & 1_2 & 0 \\ 0 & 1_2 & 0 & 0 \\ 0 & 0 & 0 & 1_2 \end{pmatrix} = \tilde{\Delta} \otimes 1_2. \tag{30}$$

By its application we then obtain $\tilde{\Gamma}_\mathrm{R} = W\Gamma^\mu_\mathrm{R} W = 1_2 \otimes \gamma^\mu$. Consequently, the picture that emerges is that with the left-chiral Gammas we describe a Dirac fermion, which yet has two additional intrinsic isospin degrees of freedom related to the original chiral doublet. Whereas with the transposed right-chiral Gammas we describe two standard Dirac fermions, which however are assembled in a two-tupel corresponding to an $SU(2)$ doublet.

6 Spin helicity

In two recent papers Marsch and Narita [17, 19] discussed and applied the spinor helicity formalism. This theoretical approach enables one to replace $\mathbf{a}^2 1_2$ of any three-vector $\mathbf{a}$ by $(\boldsymbol{\sigma} \cdot \mathbf{a})^2$, as was already done to obtain (10), and thus to get rid of the scalar product. This powerful method will here be applied to an arbitrary spin $\mathbf{S}$. For that purpose we use the Pauli matrices and obtain, for any two three vectors $\mathbf{a}$ and $\mathbf{b}$ that do not commute, the special result

$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b})1_2 + \mathrm{i}\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}). \tag{31}$$

Application of this relation to the three-vector spin operator $\mathbf{S}$ yields

$$1_2\mathbf{S}^2 = (\boldsymbol{\sigma} \cdot \mathbf{S})(\boldsymbol{\sigma} \cdot \mathbf{S} + 1_{2(2s+1)}) = 1_{2(2s+1)} s(s+1). \tag{32}$$

This relation was first obtained by Dirac [25] in 1936. At this point it seems natural to define the novel quantity spin helicity [17] as follows

$$H(s) = \boldsymbol{\sigma} \cdot \mathbf{S} = \sigma_\mathrm{x} \otimes S_\mathrm{x} + \sigma_\mathrm{y} \otimes S_\mathrm{y} + \sigma_\mathrm{z} \otimes S_\mathrm{z}. \tag{33}$$

For the matrices representing $\mathbf{S}$, this expression implies Kronecker multiplication indicated again by the symbol $\otimes$ with the Pauli matrices. We can thus extend Eq. (16) in Sigma matrix form, in order to encompass the kinetic helicity as well as the chiral spin helicity. This expanded matrix equation finally reads

$$\begin{aligned}(E^2 1_4 - (\boldsymbol{\Sigma}_\pm \cdot \mathbf{p})^2) \otimes (H(s)(H(s) + 1_{2(2s+1)}) \\ = m^2 s(s+1) 1_{8(2s+1)}.\end{aligned} \tag{34}$$

It has the advantage that there are no scalar products of neither $\mathbf{p}$ nor $\mathbf{S}$ with itself any more, but instead matrix multiplications, yet in addition we had to introduce two new helicity degrees of freedom associated with the Pauli matrices. We emphasize that the above equation connects by Kronecker multiplication the Lorentz group (LG) with the group of the intrinsic spin. This direct multiplication procedure is consistent with the strict requirements of the Coleman–Mandula theorem [10].

The basis of eigenfunctions of the spin helicity $H(s)$ operator was derived and discussed in the reference [17]. It turns out that there are, respectively, $2(2s+1)$ orthogonal basis vectors, i.e. twice as many as for the original spin $\mathbf{S}$ with quantum number s. Therefore the dimension of the space associated with $H(s)$ is doubled in comparison with that of the genuine matrices related to the spin vector $\mathbf{S}$. For the spin one-half the above Kronecker product (33) involving the Pauli matrices the helicity matrix is a real 4×4 matrix. We can by unitary transformation bring the $H(1/2)$ matrix into a diagonal form based on the eigenvectors. Then one obtains the simple matrix

$$H\left(\frac{1}{2}\right) = \frac{1}{2}\mathrm{diag}[1, 1, 1, -3]. \tag{35}$$

Apparently, the spin-helicity space of dimension 4 factorizes into two asymmetric spaces of dimensions 3 and 1. This asymmetry is a result of the basic Eq. (32), in which the spin helicity enters with a linear and quadratic term, owing to the fact that the three components of the spin $\mathbf{S}$ do not commute.

7 Hadronic isospin and $SU(4)$ symmetry

Marsch and Narita [19] have considered what they called the hadronic isospin $\mathbf{I}_{\rm H}$, which corresponds to the simplest possible intrinsic spin being $\mathbf{I}_{\rm H} = 1/2\boldsymbol{\sigma}$. Therewith they wanted to account for the empirical duality found between proton and neutron as the basic building blocks of atomic nuclei, even in modern nuclear physics based in the SM on Quantum Chromodynamics (QCD). They were inspired by the early intuitive ideas of Heisenberg [26], who just considered proton and neutron as being non-composite fundamental fermions forming an isospin doublet. Yet Marsch and Narita [19] used hadronic isospin differently, namely in order to derive the symmetry of the strong interactions from scratch and to describe the empirical fermion duality of quarks and leptons.

According to the previous section, for spin one-half one obtains the 4×4 matrix (35) for the related helicity operator, and therefore one gets a four-dimensional state space. Inspection of the elements of the symmetry group $SU(4)$, describing the general rotations in the four-dimensional complex space $\mathbb{C}^4$, reveals that the spin helicity matrix is identical (aside from a normalization constant) with the fifteenth matrix λ^{15} of the $SU(4)$ group. Returning again to (34), we obtain the result

$$\begin{aligned}&(1_4 E + \boldsymbol{\Sigma}_\pm \cdot \mathbf{p})(1_4 E - \boldsymbol{\Sigma}_\pm \cdot \mathbf{p}) \otimes (H(\tfrac{1}{2})(H(\tfrac{1}{2}) + 1_4)\\ &= \tfrac{3}{4} 1_{16} m^2,\end{aligned} \tag{36}$$

whereby we express the spin helicity operator in terms of its eigenfunctions according to Eq. (35). We now introduce the diagonal matrices $h_0 = 2H(\frac{1}{2})$ and $h_1 = 2/3(h_0/2 + 1_4)$. They commute and their product $h_0 h_1 = 1_4$, with $h_0 = \mathrm{diag}[1, 1, 1, -3]$ and $h_1 = \mathrm{diag}[1, 1, 1, -1/3]$. Using the rules of matrix multiplication for Kronecker products, we can rewrite the above equation as

$$((1_4 E + \boldsymbol{\Sigma}_\pm \cdot \mathbf{p}) \otimes h_0)((1_4 E - \boldsymbol{\Sigma}_\pm \cdot \mathbf{p}) \otimes h_1) = m^2 1_{16}. \tag{37}$$

In analogy to the Weyl equations (18) we thus obtain

$$\begin{aligned}&\mathrm{i}(1_4 \tfrac{\partial}{\partial t} - \boldsymbol{\Sigma}_\pm \cdot \tfrac{\partial}{\partial \mathbf{x}}) \otimes h_0\, \psi_+(x) = m\psi_-(x)\\ &\mathrm{i}(1_4 \tfrac{\partial}{\partial t} + \boldsymbol{\Sigma}_\pm \cdot \tfrac{\partial}{\partial \mathbf{x}}) \otimes h_1\, \psi_-(x) = m\psi_+(x).\end{aligned} \tag{38}$$

Here we introduced the extended spinor $\psi_\pm = \Phi_\pm \otimes \phi_\pm$, whereby $\phi_\pm$ is a four-component complex vector in the Hilbert space of the spin-helicity, and $\Phi_\pm$ is the previously defined Minkowski spinor associated with the chiral spins. We can chose the vectors in the spin-helicity Hilbert space such that $h_0\phi_+ = \phi_-$ and $h_1\phi_- = \phi_+$.

But we recall that the matrices $h_{0,1}$ are diagonal, and when being multiplied give 1_4. So, we can also proceed further instead from Eq. (38) directly from (37), and make use thereby of the Gamma matrices (21) to obtain

$$(\Gamma^\mu_\pm p_\mu)^2 = (E^2 1_4 - (\Sigma_\pm \cdot \mathbf{p})^2) = (E^2 - \mathbf{p}^2) 1_4. \tag{39}$$

The associated extended Dirac spinor is defined as $\Psi^\dagger = (\psi_+^\dagger, \psi_-^\dagger)$, which can then be rewritten as $\Psi^\dagger = (\Phi_+, \Phi_-)^\dagger \otimes \phi$, whereby ϕ is again a complex vector of the Hilbert space $\mathbb{C}^4$. By making use of the above Gamma matrices, or by multiplying up the Weyl equations (38), we can write an extended Klein-Gordon-type equation in the form

$$(\Gamma^\mu_\pm \hat{P}_\mu \otimes 1_4)^2 \Psi_\pm = m^2 \Psi_\pm. \tag{40}$$

This equation can readily be linearized and gives the extended Dirac equation involving as well the four-component helicity state vectors. Note that we get two versions like before, involving either the left- or right-chiral spin,

$$(\Gamma^\mu_\pm \mathrm{i}\partial_\mu \otimes 1_4)\Psi_\pm = m\Psi_\pm. \tag{41}$$

We recall that for both equations, which are equivalent and related by the unitary transformation (23), we still have the chiral isospin $\mathbf{I}_\pm$ which commutes with the respective Gamma matrices. Each of these further extended Dirac equations describes a chiral spin-one-half fermion, which has four new degrees of freedom related to the hadronic isospin helicity.

With this notion we have completed the derivation of the fundamental symmetries of an elementary fermion of finite mass m, which is described by a large 32-component Dirac spinor Ψ encompassing the particle-antiparticle doublet, the two doublets of spin one-half and chiral isospin with $SU(2)$ symmetry stemming from the LG, and the quadruplet of the spin helicity with $SU(4)$ symmetry stemming from the intrinsic hadronic isospin. As we will see below, breaking $SU(4)$ symmetry gives rise to the lepton singlet and the quark triplet. But before addressing this issue we discuss the unification of symmetries.

8 Unification of symmetries

Let us at the outset of this section summarize the above results and continue in the following with an appropriate version of the Dirac equation. For that purpose we can insert in Eq. (41) the Gamma matrix $\Gamma_{\rm L}$ as obtained from the transformed Eq. (29). The final result is a further expanded Dirac equation including the product symmetry $SU(2) \otimes SU(4)$, which reads as follows

$$(\gamma^\mu \otimes 1_2 \otimes 1_4)\mathrm{i}\partial_\mu \hat{\psi} = \hat{\gamma}^\mu \mathrm{i}\partial_\mu \hat{\psi} = m\hat{\psi}. \tag{42}$$

To discriminate the 32-component spinor and the related gamma matrix from the standard Dirac terms we gave both a hat.

We can now define a rotation in the chiral-spin Hilbert space $\mathcal{H}_2$ (associated with the chiral isospin $SU(2)$ symmetry) by the unitary phase operator

$$U_2 = \exp\left(\mathrm{i}g_2 \sum_{a=1}^{3} \alpha^a \kappa^a\right) \tag{43}$$

with the constant rotation angle α^a and the group element 2×2-matrix κ^a. This short note means an a-tupel, with the integer index a running over the number of the group generators, which is 3 in case of $SU(2)$, and again summation convention is assumed. The gauge phase operator U_2 leaves the probability density of the spinor field in $\mathcal{H}_2$ unchanged. Here g_2 is a related coupling constant. Of course, in this case the kappas are just given by the three Pauli matrices of Eq. (2).

Similarly, a rotation in the spin-helicity Hilbert space $\mathcal{H}_4$ (associated with the hadronic isospin $SU(4)$ symmetry) by the unitary phase operator

$$U_4 = \exp\left(\mathrm{i}g_4 \sum_{b=1}^{15} \beta^b \lambda^b\right) \tag{44}$$

with the constant rotation angle β^b and the group element 4×4-matrix λ^b. This short note means a b-tupel, with the integer index b running over the number of the group generators, which is 15 in case of $SU(4)$, and summation convention is assumed. The gauge phase operator U_4 leaves the probability density of the spinor field in $\mathcal{H}_4$ unchanged. Here g_4 is the related coupling constant. We point out that the group $SU(4)$ contains $SU(3)$ as a subgroup. This will become important after symmetry breaking. Finally, the resulting combined unitary transformation reads

$$U = U_0 \otimes U_2 \otimes U_4, \quad U^\dagger = U_4^\dagger \otimes U_2^\dagger \otimes U_0^\dagger, \quad UU^\dagger = 1_{32}, \tag{45}$$

where in addition we admitted a trivial phase operator $U_0 = \exp(\mathrm{i}g_0\varphi 1_4)$. If we define the transformed spinor as $\psi_U = U\psi$, then as a result we obtain $\hat{\psi}_U^\dagger \hat{\psi}_U = \hat{\psi}^\dagger \hat{\psi}$.

We emphasize that the combined symmetry as described by the unitary operator U is not what is usually understood under unification. The full dimension of the inner Hilbert spaces encompassing chirality and hadronic isospin is 8, and thus unification would correspond to the Lie group $SU(8)$. It has 63 generators, which correspond to all possible rotations in the Hilbert space $\mathbb{C}^8$. However, as we have shown before the origin of $SU(2)$ is related to space–time symmetry associated with the Lorentz transformation, but that of $SU(4)$ is connected with the inner symmetry associated with the empirical duality of leptons and quarks, the latter of which form as triples the basic hadronic building blocks of atomic nuclei, i.e. the proton and neutron [6, 27], and other hadrons.

We cannot recognize a cogent physical argument suggesting to merge these symmetries, although it seems mathematically a natural and clean thing to do. For example, Marsch and Narita [28, 29] used, what they called permutational and/or combinatorial, symmetry for a unification scheme, similar to the one suggested above by the operator U. Various attempts to unify the symmetries have been suggested previously by several then leading physicists [30–32] in that field. In particular the $SU(4)$ group, which here emerged naturally from spin-helicity, was already suggested ad hoc by Pati and Salam [33, 34]. However, in nature these key symmetries are not found to be unified, yet on the contrary are empirically found to be broken, a fact which was incorporated ab initio in the SM model by use of the separate symmetry groups $SU(3)$ for quarks [35] and $U(1)$ for leptons, and the group $SU(2)$ for the weak interactions [36, 37], whereby its symmetry was broken through the by today famous Higgs mechanism [38]. Symmetry breaking is the topic of the subsequent section.

Before the discussion of that issue, we quote the covariant derivative associated with the general unitary phase operator (45). The conventional transition to gauge field theory [4, 6, 39] is made by making all the above rotation angles functions of the space–time coordinate $x = (t, \mathbf{x})$ in Minkowski space and by considering the related gauge fields, of which there are nineteen. To ease the notation we use the connections fields as abbreviations for the sums over angles and group generators such that

$$\mathbb{A}_\mu(x) = \sum_{a=1}^{3} A_\mu^a(x)\kappa^a, \quad \mathbb{B}_\mu(x) = \sum_{b=1}^{15} B_\mu^b(x)\lambda^b, \quad P_\mu(x)1_4. \tag{46}$$

Furthermore, in order to define the so-called covariant derivative, we define the following sum of 32×32-matrices involving various Kronecker products

$$\mathbb{G}_\mu = 1_{32} g_0 P_\mu(x) + 1_4 \otimes g_2\mathbb{A}_\mu(x) \otimes 1_4 + 1_4 \otimes 1_2 \otimes g_4\mathbb{B}_\mu(x). \tag{47}$$

For mathematical convenience, we sum up these gauge connection fields in the total field $\mathbb{G}_\mu$, and can thus write the covariant derivative as

$$D_\mu = 1_{32}\partial_\mu - \mathrm{i}\mathbb{G}_\mu(x). \tag{48}$$

The field $\mathtt{G}_\mu$ is a kind of unified gauge field, as it encompasses the $U(1)$, chiral $SU(2)$ and hadronic $SU(4)$ symmetries. The definition (47) is motivated by the fact that an expression like $\mathtt{G}_\mu$ is obtained from $U^\dagger \partial_\mu U$, whereby the correspondence is, $A^a_\mu(x)$ to $\partial_\mu \alpha^a$, $B^b_\mu(x)$ to $\partial_\mu \beta^b$, and $P_\mu(x)$ to $\partial_\mu \varphi$. When we differentiate the transformed spinor $\hat{\psi}_U$, we obtain

$$\partial_\mu \hat{\psi}_{\mathrm{U}} = (\partial_\mu U)\hat{\psi} + U(\partial_\mu \hat{\psi}) = U(U^\dagger \partial_\mu U + \partial_\mu)\hat{\psi}. \tag{49}$$

The troublesome term $\partial_\mu U$ can be canceled [4, 6] by an adequate gauge transformation. If we define a new gauge field by applying standard procedures as in the SM, we obtain

$$\mathtt{G}'_\mu = -\mathrm{i}(\partial_\mu U)U^\dagger + U\mathtt{G}_\mu U^\dagger. \tag{50}$$

Then the covariant derivative of $\hat{\psi}$ transforms under the symmetry transformation U like the spinor-field itself. This finally gives the relation

$$D'_\mu \hat{\psi}_U = (1_{32}\partial_\mu - \mathrm{i}\mathtt{G}'_\mu(x))\hat{\psi}_U = U(D_\mu \hat{\psi}). \tag{51}$$

The commutator of the covariant derivative produces the gauge field combination

$$\mathrm{i}\big[D_\mu, D_\nu\big]\hat{\psi} = (\partial_\mu \mathtt{G}_\nu - \partial_\nu \mathtt{G}_\mu - \mathrm{i}\big[\mathtt{G}_\mu, \mathtt{G}_\nu\big])\hat{\psi} = \mathtt{F}_{\mu\nu}\hat{\psi}. \tag{52}$$

Then by means of Eq. (50) it is found that the gauge field tensor transforms covariantly, which means $\mathtt{F}'_{\mu\nu} = U^\dagger \mathtt{F}_{\mu\nu} U$. Note that this tensor splits into three summands belonging to the three types of gauge fields, which commute with each other yet not among themselves owing to the non-Abelian nature of their gauge-symmetry groups.

Finally we can write the extended Dirac equation including the unified gauge field as

$$\hat{\gamma}^\mu \mathrm{i}(1_{32}\partial_\mu - \mathrm{i}\mathtt{G}_\mu(x))\hat{\psi}(x) = m\hat{\psi}(x). \tag{53}$$

Here $\hat{\gamma}^\mu = \gamma^\mu \otimes 1_8$. This derivation concludes the section on unification of symmetries.

9 Symmetry breaking yields the SM symmetries

In the previous section we have described a unified gauge theory which considers the symmetries connected with the chiral nature and spin-helicity properties of a massive spin one-half fermion. The resulting expanded Dirac equation can be written after (53) as follows

$$\gamma^\mu \otimes \big((\mathrm{i}\partial_\mu + g_0 P_\mu(x))1_8 + \mathtt{S}_\mu\big)\hat{\psi}(x) = m\hat{\psi}(x), \tag{54}$$

whereby the gauge-field related to the combined $SU(2) \otimes SU(4)$ symmetry is denoted by the 8×8 matrix gauge field

$$\mathtt{S}_\mu = g_2 \mathtt{A}_\mu(x) \otimes 1_4 + g_4 1_2 \otimes \mathtt{B}_\mu(x), \tag{55}$$

which involves two coupling constants with independent gauge fields. How can one retrieve the symmetries of the standard model? The key is to combine the terms associated with $\kappa^3 = \sigma_z$ of the chiral symmetry group with the spin-helicity matrix $H(\frac{1}{2})$ of the hadronic isospin symmetry group. Note that λ^{15} is aside from a normalization factor equal to H. For this purpose we write out the related diagonal terms of $\mathtt{S}_\mu$ more explicitly as the matrix

$$\mathtt{M}_\mu = \frac{1}{2}\begin{pmatrix} g_2 A^3_\mu 1_4 + g_4 B^{15}_\mu 2\lambda^{15} & 0 \\ 0 & -g_2 A^3_\mu 1_4 + g_4 B^{15}_\mu 2\lambda^{15} \end{pmatrix}. \tag{56}$$

We recall that $\lambda^{15} = \sqrt{3/8}\,\mathrm{diag}\,[1/3,\ 1/3,\ 1/3,\ -1]$, which is normalized such that the trace of its square is equal to 1/2. We emphasize that we did not introduce what is called in the SM the hypercharge operator. To a certain extent the spin helicity matrix H takes over its role. We then follow here the ideas and procedures of Marsch and Narita [29]. So far, the above rewriting has not violated the $SU(2)$ or $SU(4)$ symmetry. However, it will now be broken by making a Weinberg [37] rotation of the involved gauge fields,

$$\begin{pmatrix} A^3_\mu \\ B^{15}_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} E_\mu \\ Z_\mu \end{pmatrix}. \tag{57}$$

Then the matrix $\mathtt{M}$ attains the form

$$\mathtt{M}_\mu = \mathtt{Q}E_\mu + \mathtt{R}Z_\mu. \tag{58}$$

Here we introduced the corresponding charge operators connected with the new gauge fields E_μ and Z_μ. They read as follows

$$\mathtt{Q} = \frac{1}{2}\begin{pmatrix} g_2\cos\theta 1_4 + g_4 \sin\theta 2\lambda^{15} & 0 \\ 0 & -g_2\cos\theta 1_4 + g_4\sin\theta 2\lambda^{15} \end{pmatrix}, \tag{59}$$

and similarly

$$\mathtt{R} = \frac{1}{2}\begin{pmatrix} -g_2 \sin\theta \mathbb{1}_4 + g_4 \cos\theta 2\lambda^{15} & 0 \\ 0 & g_2 \sin\theta \mathbb{1}_4 + g_4 \cos\theta 2\lambda^{15} \end{pmatrix}. \tag{60}$$

These matrices can be simplified by noticing that we can set the electromagnetic coupling strength e by demanding that $e = g_2 \cos\theta = g_4 \sin\theta\sqrt{3/2}$, whereby the square-root factor stems from the normalization of λ^{15}. This constraint defines the so-called Weinberg angle as $\tan\theta_\mathrm{W} = \sqrt{2/3}\, g_2/g_4$. It is well measured within the SM framework [6] and then attains the value 0.53. This measurement fixes the ratio of the coupling constants at $g_2/g_4 = 0.65$. The electric charge unit can then be written in terms of the coupling constants as

$$e = \frac{g_2 g_4}{\sqrt{g_4^2 + \frac{2}{3} g_2^2}}. \tag{61}$$

The resulting electromagnetic charge operator is diagonal and reads

$$\mathtt{Q} = e\,\mathrm{diag}\,[+\frac{2}{3}, +\frac{2}{3}, +\frac{2}{3}, 0, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, -1]. \tag{62}$$

The trace of the charge operator sums up to zero, which expresses the charge conservation withing the fermion octet. The charges are associated with three positively charged up quarks, the neutral neutrino, and the negatively charged three down quarks and the electron. For the charges related with the electroweak field Z_μ we obtain

$$\mathtt{R} = \frac{e}{2}\,\mathrm{diag}\,[q_-, q_-, q_-, l_-, q_+, q_+, q_+, l_+]. \tag{63}$$

Here the definitions for the quarks are, $q_\pm = \pm\tan\theta_\mathrm{W} + \cot\theta_\mathrm{W}/3$, and for the leptons, $l_\pm = \pm\tan\theta_\mathrm{W} - \cot\theta_\mathrm{W}$. These electroweak charges sum up to zero, i.e. the trace of $\mathtt{R}$ vanishes. This is in full compliance with the fact that the traces of all generators of the $SU(2)$ and $SU(4)$ groups are zero, a property that is not changed by the linear transformation (57).

However, both original symmetries are now broken, since the group elements κ^3 and λ^{15} are affected and missing from the genuine sets of elements. As a consequence the combined gauge-field matrix $\mathtt{S}_\mu$ now reads

$$\mathtt{S}_\mu = \mathtt{Q}E_\mu + \mathtt{R}Z_\mu + g_2 \sum_{a=1}^{2} A_\mu^a(x)\kappa^a \otimes \mathbb{1}_4 + g_4 \sum_{b=1}^{14} B_\mu^b(x)\mathbb{1}_2 \otimes \lambda^b. \tag{64}$$

At this point it is important to note that unlike in the SM here all fermions in the octet interact with each other through the electromagnetic field, with the exception of the neutrino. Moreover, they all interact with each through the electroweak field Z_μ, which is not the case in the SM, as there that field acts only within the $SU(2)$ doublets. The coupling strength is of the order of e for both gauge fields. Like in the SM we can now introduce the charged W bosons, and then by following Marsch and Narita [19, 29] we can rewrite the related terms in (64) as

$$\sum_a^2 A_\mu^a(x)\kappa^a \otimes \mathbb{1}_4 = \mathtt{W}_\mu = \frac{1}{2}\begin{pmatrix} 0 & W_\mu^-(x) \\ W_\mu^+(x) & 0 \end{pmatrix} \otimes \mathbb{1}_4. \tag{65}$$

Here $W_\mu^\pm = (A_\mu^1 \pm \mathrm{i}A_\mu^2)$, which must have the electric charge $\mp e$, as can be inferred from the charge operator (62). Note that $\mathtt{W}_\mu$ is hermitian.

Furthermore, we can split the sum over the hadronic gauge fields B_μ into two contributions. One corresponds to the SM Quantum Chromodynamics (QCD), which runs over the eight Gell-Mann matrices of the subgroup $SU(3)$. The other one encompasses the six matrices that connect the quarks with the leptons. The related gauge fields are in the literature often called leptoquarks, which in our mind is a misnomer as we are dealing here with bosons. We define their sum as

$$\sum_{b=9}^{14} B_\mu^b(x)\lambda^b = V_\mu = \sum_{n=1}^{3}((V_n^+)_\mu(x)\Lambda_n^- + (V_n^-)_\mu(x)\Lambda_n^+). \tag{66}$$

The boson gauge fields are defined as $(V_n^\pm)_\mu = (B_\mu^9 \pm \mathrm{i}B_\mu^{10})$, $(V_n^\pm)_\mu = (B_\mu^{11} \pm \mathrm{i}B_\mu^{12})$, and $(V_n^\pm)_\mu = (B_\mu^{13} \pm \mathrm{i}B_\mu^{14})$. The Lambda matrices have only a single entry 1 and otherwise contain 0. They were derived in [19], but for clarity we quote below V_μ explicitly. The corresponding hermitian 8×8 matrix is defined as $\mathtt{V}_\mu = \mathbb{1}_2 \otimes V_\mu$, with the summed gauge fields reading

$$V_\mu = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & (V_1^-)_\mu \\ 0 & 0 & 0 & (V_2^-)_\mu \\ 0 & 0 & 0 & (V_3^-)_\mu \\ (V_1^+)_\mu & (V_2^+)_\mu & (V_3^+)_\mu & 0 \end{pmatrix}. \tag{67}$$

Note that $(V_n^\pm)_\mu$ must have the electric charge $\mp 2/3\, e$. This can readily be concluded from the charge operator (62) and the fact that they connect the three quarks with the single lepton.

However, as the three bosons $(V_n^{\pm})_\mu$ like the quarks carry fractal charges they should not exist freely as singles, but perhaps only appear in confinement as pairs that carry no hadronic charge. This is the case for the three possible boson binaries made of a particle and its anti-particle having anti-parallel spins. Such boson binary could perhaps form a new kind of spinless and uncharged composite gauge boson of heavy mass, which in this case would be about 70.8 GeV, following from the calculations in Sect. 12 below. They might originate during the $SU(4)$ symmetry breaking, and as boson condensates cease fully their interactions with all the fermions as well as the other $SU(4)$ gauge bosons. This way they could act like heavy dark matter.

Finally we obtain for the quark sector the subsequent expression that is well known from QCD of the SM. In our notation its reads $\mathtt{C}_\mu = 1_2 \otimes C_\mu$ with

$$C_\mu = \sum_{b=1}^{8} B_\mu^b(x)\lambda^b. \tag{68}$$

We recall that the above $SU(4)$ lambda matrices have an empty last column and empty lowest row, and thus they form the subgroup $SU(3)$ of $SU(4)$. Summarizing our results we can concisely write the symmetry-related part of the covariant derivative as

$$\mathtt{S}_\mu = \mathtt{Q}E_\mu + \mathtt{R}Z_\mu + g_2\mathtt{W}_\mu + g_4\mathtt{V}_\mu + g_4\mathtt{C}_\mu. \tag{69}$$

So, we summarize the terms again in words. From left to right we have the electromagnetic field coupled to the electric charge (Q), the electroweak field associated with the weak charge (R), the $W^{\pm}$ boson fields that transfer an electric charge of $\mp e$ between the left- and right-chiral (up and down) component of the $SU(2)$ doublet, the three $V^{\pm}$ bosons which change the three quarks to a lepton and thereby transfer an electric charge of $\mp 2/3\, e$, and finally the seven gluon gauge-fields, which couple the three quarks but only among themselves and lead to the strong interactions of nuclear physics, in the SM known as QCD. All terms are 8×8 matrices, reflecting the unification of the forces. The SM symmetries emerge via symmetry breaking. The concept of hypercharge is not needed in our model.

10 Quantum hadrodynamics

We turn now to the strong or hadronic interaction and analyse it in a new way yielding novel conclusions. For that purpose the related covariant derivative $\mathtt{C}_\mu$ is written in a physically more transparent way. We introduce the following three linear combinations of those vector bosons which in the SM are conventionally called coloured gluons. The result is

$$(G_1^{\pm})_\mu = B_\mu^1 \pm \mathrm{i}B_\mu^2,\ (G_2^{\pm})_\mu = B_\mu^4 \pm \mathrm{i}B_\mu^5,\ (G_3^{\pm})_\mu = B_\mu^6 \pm \mathrm{i}B_\mu^7. \tag{70}$$

The gauge fields associated with the two diagonal matrices λ^3 and λ^8 are renamed as G_μ^2 and G_μ^3, with reference in the superscript to the number of non-zero elements on the traces of their lambdas. Then we obtain the hermitian matrix-operator field

$$C_\mu = Q_2 G_\mu^2 + Q_3 \frac{1}{\sqrt{3}} G_\mu^3 + \frac{1}{2}\begin{pmatrix} 0 & (G_1^-)_\mu & (G_2^-)_\mu & 0 \\ (G_1^+)_\mu & 0 & (G_3^-)_\mu & 0 \\ (G_2^+)_\mu & (G_3^+)_\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{71}$$

The factor 1/2 in front of the matrix stems from the standard normalization, namely $\mathrm{trace}(\lambda^b)^2 = 1/2$. The diagonal matrices $Q_{2,3}$ are a kind of "hadronic charge operators" with charges given in units of g_4, which is the hadronic coupling constant that was introduced for the $SU(4)$ symmetry. The related charge operators read

$$Q_2 = \mathrm{diag}\left[\frac{1}{2}, -\frac{1}{2}, 0, 0\right],\quad Q_3 = \mathrm{diag}\left[\frac{1}{2}, \frac{1}{2}, -1, 0\right], \tag{72}$$

whereby the traces of both charge operators vanish. Q_3 is associated with the group $SU(3)$ and Q_2 with its subgroup $SU(2)$. They commute with each other and can have common eigenvectors. The charge operator Q_3 resembles formally the old Gell-Mann Nishijima [40] formula, which yet in their case describes the electric charge and also includes the hypercharge and strangeness number. They both are not needed here.

When acting with the charge operators on first three standard basis vectors (three quarks) in the Hilbert space $\mathbb{C}^4$, which read $\phi_1^\dagger = (1, 0, 0, 0)$, $\phi_2^\dagger = (0, 1, 0, 0)$, and finally $\phi_3^\dagger = (0, 0, 1, 0)$, we obtain the four charges q_2 as $\frac{1}{2}(1, -1, 0, 0)$ for Q_2 and q_3 as $\frac{1}{2}(1, 1, -2, 0)$ for Q_3. The fourth basis vector is the lepton state that does not matter here, just like in the QCD of the SM. The operation of $Q_{2,3}$ on the lepton state, $\phi_4^\dagger = (0, 0, 0, 1)$, gives zero. So the leptons are "hadronic neutrinos". Of course, we find generally that $\mathtt{C}_\mu \phi_l = 0$, meaning that the leptons do not take part in QCD at all. The other three complex-conjugated gluons pairs in (70) simply provide rotations in opposite directions, respectively, about one of the three axes of the complex three-dimensional Hilbert space $\mathbb{C}^3$ of $SU(3)$.

Apparently, $(G_1^{\pm})_\mu$ somewhat resembles the $W_\mu^{\pm}$ of the chiral $SU(2)$ symmetry. But all three $(G_{1,2,3}^{\pm})_\mu$ vector gauge bosons are genuine hadronic, and they do change the hadronic charge. However, since the charge operator $Q_{2,3}$ is diagonal, the related gauge field $G^{2,3}_\mu$ does not change the hadronic charge of the quarks, and thus it corresponds to the weak gauge field Z_μ of the chiral $SU(2)$. As Q_2 relates to the $SU(2)$ subgroup of $SU(3)$, the gauge field $(G^{\pm}_1)_\mu$ merely acts within that subgroup and thus carries the hadronic charge ∓ 1. The field $(G^{\pm}_{2,3})_\mu$ acts within the whole $SU(3)$ group and carries the hadronic fractal charge $\mp 3/2$.

To be more precise, we quote both charges q_2 and q_3 explicitly. Namely, rotations about the three axis of $\mathbb{C}^3$ correspond to associated changes in the quark charges. We therefore also indicate the resulting charge differences, $\Delta_{2\to 1}q_{2,3}$, $\Delta_{3\to 1}q_{2,3}$, $\Delta_{3\to 2}q_{2,3}$, to characterize the gauge fields with attached arguments like $(G^{\pm}_n)_\mu(\Delta q_2, \Delta q_3)$ as follows,

$$(G^{\pm}_1)_\mu(\mp 1, 0),\ (G^{\pm}_2)_\mu\left(\mp\frac{1}{2}, \mp\frac{3}{2}\right),\ (G^{\pm}_3)_\mu\left(\pm\frac{1}{2}, \mp\frac{3}{2}\right). \tag{73}$$

Moreover, since the two gluons $(G^{\pm}_{2,3})_\mu$ have hadronic fractal charges in units of $g_4/2$, they should not exist freely as singles, but may only emerge in complex bound states or as pairs which carry no hadronic charge. This is possibly the case for the binaries made of a gluon and its anti-gluon. Perhaps, such composite "gluon mesons" would form a new kind of spinless and uncharged hadronic particles.

In the QCD of the SM the ad hoc requirement is that the basic hadronic states made of quarks should be colourless. That corresponds in our model to the constraint that the hadronic charge of quark composites must be zero. This can indeed be the case for multiple product-states of quarks, e.g. made of the pairs (mesonic states) of two or of all three (baryonic states) basis quarks. Yet, we stress that colour does not appear here at all, but is replaced by an intuitively more physical quantity, namely the hadronic charge associated originally with the fundamental $SU(4)$ symmetry and its coupling constant g_4.

Let us consider the fermion sector in more detail, for example, the triple baryonic state $\Phi_3 = \phi_1 \otimes \phi_2 \otimes \phi_3$ yields $q_2 = q_3 = 0$, and thus decouples fully from all hadronic interactions that are mediated by the gauge fields $G^{2,3}_\mu$ and the other three $(G^{\pm}_{1,2,3})_\mu$ that transfer hadronic charges. Such a state should then be stable against further hadronic interactions (as the neutral hydrogen atom in the ground state is stable against electrostatic interaction). For more details on the possible charge states of composites of quarks please see the Appendix.

The double mesonic state $\Phi_2 = \phi_1 \otimes \phi_2$ yields $q_2 = 0$, yet one finds that $q_3 = 1$ does not vanish, and thus the coupling to the field G^3_μ as well as to the three $(G^{\pm}_{1,2,3})_\mu$ remains intact. In the QCD of the SM a coloured quark is conceived to be coupled with its antiquark to give a meson with a net colour charge of zero. In our model we then find also that a state, $\bar{\Phi}_2 = \phi_1 \otimes \bar{\phi}_1$, consisting of a quark and its antiquark can give a meson with zero hadronic charge which decouples from the hadronic interactions.

However, for any other n-multiple state Φ_n with a finite charge, i.e. $Q_{2,3}\Phi_n \neq 0$, the coupling to the gauge field $G^{2,3}_\mu$ always remains active as well to the charge-transfer fields $(G^{\pm}_{1,2,3})_\mu$. This feature appears to be the key characteristic of the non-Abelian, and thus nonlinear hadronic interactions, which are essentially mediated by the three massless charge-transfer gauge fields of $SU(3)$. This fundamental property is after our understanding at the heart of QHD. The masses of the hadronic fields are discussed in the next section.

Finally in this section, we consider the possible electric charges of composites of quarks. Inspection of the charge operator (62) suggest the following charge-states of baryons made of a triple quark state that has vanishing net hadronic charges, $q_2 = q_3 = 0$. We quote them as three-tupel states, a single $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$, with electric charge $q_e = +2$, a triple $(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ by permutation, with electric charge $q_e = +1$, another triple $(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$ by permutation, with electric charge $q_e = 0$, and a single $(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$, with electric charge $q_e = -1$. Note that these eight states are combinations of three quarks having different hadronic charges, and thus they are consistent with the Pauli principle. Therefore, the spins of the quarks can be arbitrary, and one may construct spin one-half and spin three-half baryons this way. These possibilities cover the baryons consisting of the up- and down-quarks triple states (without the strange quark) in the baryon octets and decuplets of the $SU(3)$ global symmetry. Similarly, one can construct the mesonic states as particle-antiparticle quark pairs (without the strange quark) as they occur in the meson octets of the $SU(3)$ global symmetry. The resulting hadronic charges are zero, and the possible three electric charges are $q_e = (+1, 0, -1)$, whereby mesons of spin zero and spin one are permitted. For more details on the possible mesonic states please see the Appendix.

We recall that at the outset of this study we assumed an equal and small mass m of the fermions involved, in order not to violate the employed gauge symmetries. To calculate actually the real masses of the baryons and mesons as composites of quarks requires demanding numerical QCD calculations, which have indeed been carried out successfully on large computers. The impressive results were briefly reviewed and discussed in the paper by Wilczek [41]. The resulting theoretical masses of those hadrons, consisting of the up, down and strange quarks only, were found within their error bars to be in excellent agreement with the experimental values.

11 The masses of the vector bosons

The Higgs mechanism [6, 38] has been well established as part of the SM and used to give vector bosons finite masses. So, we can go right into this subject without much analytical derivations. For the particular case discussed here with the help of the extended Dirac equation, Marsch and Narita [19, 29] have provided the relevant mathematical derivations, to which we refer here. Boson masses originate from the expectation values the squared gauge-field-part $\mathtt{S}_\mu$ of the covariant derivative (69). The boson mass term in the Lagrangian is thus given by

$$\mathcal{L}_{\mathrm{M}} = \Phi_v^\dagger \mathtt{S}^\mu \mathtt{S}_\mu \Phi_v. \tag{74}$$

Here the Higgs vacuum state, with vacuum value v, is given by $\Phi_v^\dagger = v(0, 0, 0, 1, 0, 0, 0, 0)$. Obviously, $\mathtt{Q}\Phi_v = 0$, and thus the Higgs boson carries no electric charge like the neutrino in the SM. Inspection of Eq. (71) then reveals that $\mathtt{C}_\mu \Phi_v = 0$, and thus the seven gluons are massless, as well as the photon related to the field E_μ. Therefore, we obtain

$$\mathcal{L}_{\mathrm{M}} = \Phi_v^\dagger (\mathtt{R}Z^\mu + g_2 \mathtt{W}^\mu + g_4 \mathtt{V}^\mu)(\mathtt{R}Z_\mu + g_2 \mathtt{W}_\mu + g_4 \mathtt{V}_\mu)\Phi_v. \tag{75}$$

When evaluating this expression it turns out that the mixing terms between the three gauge fields do not contribute, and only their individual squares matter. Then we obtain

$$\mathcal{L}_{\mathrm{M}} = M_Z^2 Z^\mu Z_\mu + M_W^2 W^{+\mu} W_\mu^- + M_V^2 \sum_{n=1}^{3} V_n^{+\mu} V_{n\mu}^-, \tag{76}$$

whereby the three V bosons have the same mass. The masses are given by

$$M_Z = \sqrt{\Phi_v^\dagger \mathtt{R}^2 \Phi_v},\ M_W = \frac{g_2}{2} v,\ M_V = \frac{g_4}{2} v. \tag{77}$$

For the Z boson mass we find from Eq. (63) the expectation value of $\mathtt{R}^2$ in the Higgs vacuum. The expectation value is $v^2(\frac{e}{2})^2\,(\tan\theta_{\mathrm{W}} + \cot\theta_{\mathrm{W}})^2$. Insertion of the Weinberg angle then delivers the mass

$$M_Z = \frac{e}{2} v\Big(\frac{g_2}{g_4}\sqrt{\frac{2}{3}} + \frac{g_4}{g_2}\sqrt{\frac{3}{2}}\Big) = \frac{1}{2} v \sqrt{g_2^2 + g_4^2 \frac{3}{2}} \tag{78}$$

which involves both coupling constants. The masses $M_Z = 91.2$ GeV and $M_W = 80.4$ GeV are obtained from the SM precision measurements [6]. In our model we obtain the masses

$$M_V = \frac{g_4}{g_2} M_W,\quad M_Z = M_W \sqrt{1 + \frac{3}{2}\Big(\frac{g_4}{g_2}\Big)^2}, \tag{79}$$

which are functions of the ratio of the coupling constants and given here in units of M_W. Therefore, we can calculate that ratio from the measured boson masses and exploit Eq. (61). Then we obtain the numbers, $g_4 = 0.93e$ and $g_2 = 2.1e$. These inferred values of the coupling constants are close the integer relation, $g_2 = 2e$ and $g_4 = e$, within a ten percent margin. Finally, we find that the mass of all three V bosons is $M_V = 0.44\, M_W = 35.4$ GeV, i.e. about half the weight of the W boson.

12 Results and conclusions

In summarizing briefly the results of this paper, we can say that the symmetries of the elementary fermions can be derived from two physical notions. The first is the fundamental empirical principle of Lorentz invariance of physical processes in space–time. The generators of the Lorentz group yield the chiral $SU(2)$ symmetry describing the weak interaction. The second is the abstract idea of intrinsic spin which yields, by means of the spinor-helicity formalism employed to any three-vector in real space, the concept of spin helicity. That in turn leads for spin one-half to the hadronic $SU(4)$ symmetry describing the hadronic (nuclear) interactions. These two notions are independent, but are connected closely in the spirit of the Coleman–Mandula theorem by the multiplicative construction of the Pauli-Lubanski operator. It links both phenomena and leads to a unified symmetry defined by the Kronecker product group $SU(2) \otimes SU(4)$. An extended Dirac equation incorporating the unified symmetry is presented.

Breaking of this symmetry is achieved by the application of the Weinberg rotation (linear combination) to the two gauge fields, which are associated with the key diagonal matrices κ^3 and λ^{15} and define the charge operators. This procedure results in the three well-known interactions, i.e. quantum electrodynamics, the weak interactions, and hadrodynamics (akin to CD of the SM). Furthermore, the application of the Higgs-mechanism creates heavy masses of those gauge bosons that transfer charges between the particles involved. Their massive fields are merely short-reaching and thus provide no coupling but just contact interactions of the fermions. Their binding, however, can be achieved by the massless and far-reaching electromagnetic photon which carries no electromagnetic charge, as well as more strongly by the three non-Abelian gluons which do also carry hadronic charges.

The vanishing hadronic charge of triple-quark states can lead to the confinement of the participating three quarks in the baryons, which are described as colourless particles in the QCD of the SM. The model requires only two coupling constants related to the two symmetries involved. The concept of hypercharge is not necessary. In this way we have achieved a simplification and unification of the symmetries of the SM and of the interactions between the associated fundamental fermions and bosons.

Finally, we should address briefly the physics of the electroweak gauge field Z^μ and the associated charge operator R given in Eq. (63), which is akin to the electromagnetic field E^μ associated with the electromagnetic charge operator Q quoted in (62).

However, whereas the electromagnetic field is far-reaching (massless photon), the electroweak field is extremely short-reaching (massive boson), only of the order of the Compton wavelength based on the boson mass M_Z. Although all three quarks and the single lepton with either left and right chirality have finite weak charges of the order of e, and thus all interact through the field Z^μ, this interaction is expected to be rather weak. This is because of its short range and even worse due to the very short lifetime of the Z boson. Yet, to scrutinize the subtle physics of this rather general interaction between all sixteen elementary fermions discussed here is way beyond the scope of this paper.

Funding Open Access funding enabled and organized by Projekt DEAL.

Data availability No data are associated with this work.

Declarations

Conflict of interest The authors declare that there is no conflict of interest.

Appendix: Appendix on baryons and mesons as quark composites

In this appendix we consider the possible composite double and triple states in the Hilbert space built by the Kronecker products of the basic complex Hilbert space $\mathbb{C}^4$ that is associated with the $SU(4)$ symmetry. The related standard basis vectors are $\phi_1^\dagger = (1, 0, 0, 0)$, $\phi_2^\dagger = (0, 1, 0, 0)$, and $\phi_3^\dagger = (0, 0, 1, 0)$ for the three quarks, and $\phi_4^\dagger = (0, 0, 0, 1)$ for the single lepton. We repeat the hadronic charge (in units of g_4) operators, which after Eq. (72) read as follows, $Q_3 = \frac{1}{2}(1, 1, -2, 0)$ for the subgroup $SU(3)$, and $Q_2 = \frac{1}{2}(1, -1, 0, 0)$ for the subgroup $SU(2)$. The operation of $Q_{2,3}$ on the lepton state, $\phi_4^\dagger = (0, 0, 0, 1)$, gives zero, and thus we can neglect it and just consider the symmetry group $SU(3)$ of QCD, or as we prefer to name it of QHD. We combine the hadronic charge operators as $Q_h = Q_2 + Q_3 = (1, 0, -1, 0)$, with hadronic charge q_h.

The baryonic state $\Psi = \phi_1 \otimes \phi_2 \otimes \phi_3$ yields vanishing hadronic charges, $q_2 = q_3 = q_h = 0$, and thus decouples completely from further hadronic interactions. We recall that in the unified $SU(2) \otimes SU(4)$ we have up and down quarks with electric charges q_e as given by the operator Q_e quoted in Eq. (62). We then get eight composite baryonic states named Ψ_n, which are presented in Table 1.

These eight states correspond in reality to the proton and neutron with spin one-half, and to the four delta resonances with spin three-halves and charges ranging from -1 in steps of one to +2. This means that the baryons not involving the strange quark in QCD of the SM are retained. As the mass differences between the Delta baryons and the Sigmas, Xis, and Omega minus, respectively, is just about the mass of the pions, these particles can be obtained from the QCD decouplet in triangle arrangement by attaching from the right one, two, or three negatively charged pions, or from the left one, two, or three zero-charge pions. The same procedure roughly applies to the Sigmas and Xis in the baryon octet. In the present language these particles would then be penta- and hepta-quarks, and the strange quark would not be needed.

Considering now composite mesonic states, we obtain three different states. They are named $\varphi_1 = \phi_1 \otimes \phi_2$, $\varphi_2 = \phi_1 \otimes \phi_3$, and $\varphi_3 = \phi_2 \otimes \phi_3$. For φ_1 we get the Table 2 with hadronic charge number $q_h = +1$. For φ_2 we get the same table as Table 2 but with hadronic charge number $q_h = 0$. Finally, if we take φ_3, we just obtain $q_h = -1$. So there is a problem with those three definitions,

Table 1 Electrically charged baryons with zero hadronic charge

Ψ_n	Product state	q_e	q_h
Ψ_1	$u \otimes u \otimes u$	+2	0
Ψ_2	$u \otimes u \otimes d$	+1	0
Ψ_3	$u \otimes d \otimes u$	+1	0
Ψ_4	$d \otimes u \otimes u$	+1	0
Ψ_5	$u \otimes d \otimes d$	0	0
Ψ_6	$d \otimes u \otimes d$	0	0
Ψ_7	$d \otimes d \otimes u$	0	0
Ψ_8	$d \otimes d \otimes d$	-1	0

Table 2 Mesons with fractal electrical charge and hadronic charge of plus one

φ_1	Product state	q_e	q_h
Φ_1	$u \otimes u$	+4/3	+1
Φ_2	$u \otimes d$	+1/3	+1
Φ_2	$d \otimes u$	+1/3	+1
Φ_3	$d \otimes d$	−2/3	+1

Table 3 Electrically charged mesons with zero hadronic charge

φ_1	Product state	q_e	q_h
Φ_1	$u \otimes \bar{d}$	+ 1	0
Φ_2	$u \otimes \bar{u}$	0	0
Φ_2	$d \otimes \bar{d}$	0	0
Φ_3	$\bar{u} \otimes d$	− 1	0

Table 4 Double-mesons or tetraquarks with integer electrical charge and zero hadronic charge

Φ product	Double-product state	q_e	q_h
$\Phi_1 \otimes \bar{\Phi}_3$	$(u \otimes u) \otimes (\bar{d} \otimes \bar{d})$	+ 2	0
$\Phi_2 \otimes \bar{\Phi}_2$	$(u \otimes d) \otimes (\bar{d} \otimes \bar{u})$	0	0
$\Phi_2 \otimes \bar{\Phi}_2$	$(d \otimes u) \otimes (\bar{u} \otimes \bar{d})$	0	0
$\Phi_3 \otimes \bar{\Phi}_1$	$(d \otimes d) \otimes (\bar{u} \otimes \bar{u})$	− 2	0

because the resulting electric charges are fractal, which does not correspond to what is observed for real mesons in nature, and we get for $\varphi_{1,3}$ the finite hadronic charge $\pm$ 1. Only the state φ_2 decouples from the hadronic interactions.

To get rid of the fractal electric charges, we have to chose one state as an antiparticle state (indicated by a bar), and thus we ought to consider finally only the four states that are given in Table 3. The electric charges of the resulting four states correspond then to what is observed in reality for the π and ρ mesons, with anti-parallel and parallel spins of the quark binary. It means that the known mesons not involving a strange quark are obtained by this construction.

Furthermore, one can construct so-called tetraquarks as well, which are exotic but can readily be obtained by combining respectively, particle and anti-particles states of Table 2. This gives another list composed in Table 4. Note that the electric charges have doubled by this construction in comparison with the normal mesons.

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