Null shells and double layers in quadratic gravity

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Abstract. For a singular hypersurface of arbitrary type in quadratic gravity motion equations were obtained using only the least action principle. It turned out that the coefficients in the motion equations are zeroed with a combination corresponding to the Gauss-Bonnet term. Therefore it does not create neither double layers nor thin shells. It has been demonstrated that there is no "external pressure" for any type of null singular hypersurface. It turned out that null spherically symmetric singular hupersurfaces in quadratic gravity cannot be a double layer, and only thin shells are possible. The system of motion equations in this case is reduced to one which is expressed through the invariants of spherical geometry along with the Lichnerowicz conditions. Spherically symmetric null thin shells were investigated for spherically symmetric solutions of conformal gravity as applications, in particular, for various vacua and Vaidya-type solutions.

1. Introduction

Quadratic gravity belongs to the class of higher derivative theories analyzed by Ostrogradsky [1], who proved that their Hamiltonian is unbounded from below. Such theories obey Ostrogradsky's instability theorem, which classifies all nondegenerate theories of higher derivatives as Lyapunov unstable. This is a serious problem because such unstable theories usually have negative energy states that are emphatically excluded from quantum field theories. Despite this, recently there has been a surge of interest in theories of higher derivatives for a number of reasons; the main ones are listed below.

Firstly, unlike general relativity higher derivative theories are renormalizable [2]–[5] thus, examination of these theories can provide important clues as to how to quantize gravity. Secondly, several groups of researchers independently discovered [6] – [16] that corrections to the gravitational Lagrangian arising in the one-loop approximation upon renormalization in quantum field theory in curved space include terms that are quadratic in the curvature tensor Riemann and its contractions. Lastly, A.A. Starobinsky [17] used these inevitable corrections to the action of gravity and noticed that a cosmological solution without a singularity, which belongs to the de Sitter type of universe, can be obtained by taking them into account. This led to the creation of an innovative inflation model. Considering all of the above, along with many other reasons, it is clear that the study of quadratic gravity is a topical problem.

Singular hypersurfaces occur in both general relativity and quadratic gravity. These are hypersurfaces on which the curvature tensor has singular components in particular terms proportional to the θ-function or δ-function.

In any theory of gravity singular hypersurfaces are important idealized objects designed to describe the local concentration of matter or energy on a given hypersurface, for example, domain walls, thin layers of matter or gravitational fields, propagation of lightlike matter, gravitational shock waves, matter-vacuum boundaries, caustics, phase transitions in vacuum, etc.

The role of exact solutions in understanding physical phenomena is difficult to overestimate. Since field equations of any theory of gravity are highly nonlinear, search for solutions becomes a very difficult task, therefore, study of singular distributions of matter fields is extremely important.

Einstein's equations on a singular hypersurface were first derived by W. Israel [18]–[20] . In the case of quadratic gravity, the corresponding equations were obtained by J.M.M. Senovilla [21]–[26] . They differ significantly from Israel's equations, primarily in that they contain not only the δ-function, but also its derivative. Thus, they describe not only thin shells, but also so-called double layers.

In the article [27] it was shown that the equations of motion for a singular hypersurface in quadratic gravity can be derived using only the principle of least action. The main advantage of this method is that δ-function derivative does not appear explicitly during computation during the calculations. This paper generalizes the results presented in [27] to lightlike hypersurfaces. In addition, an important special case of spherical symmetry is analyzed for this type of hypersurfaces.

As for applications, null singular hypersurfaces have been investigated for spherically symmetric solutions of conformal gravity, which is a special case of quadratic gravity. The main types of spherically symmetric solutions of conformal gravity were described in article [28]. Here vacuum solutions and Vaidya-type solutions are used.

2. Preliminaries

Adding higher-order curvature terms is a natural generalization of Einstein's theory of gravity. In the first approximation, these corrections give quadratic terms, therefore the action of quadratic gravity in the general case is

$$S_q = -\frac{1}{16\pi}\int_\Omega \sqrt{-g} L_q\, d^4x = \\ = -\frac{1}{16\pi}\int_\Omega \sqrt{-g}\left(\alpha_1 R_{abcd}R^{abcd} + \alpha_2 R_{ab}R^{ab} + \alpha_3 R^2 + \alpha_4 R + \alpha_5 \Lambda\right) d^4x. \quad (1)$$

Here g is a determinant of the metric, whose components serve as dynamic variables, α_i are arbitrary constants, $R^a_{\ bcd}$ is a Riemann tensor:

$$R^a_{\ bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce}\Gamma^e_{bd} - \Gamma^a_{de}\Gamma^e_{bc}.$$

Ricci tensor and scalar curvature are defined as:

$$R_{ab} = R^c_{\ acb},\ R = R^a_a.$$

In what follows the signature $(+,-,-,-)$ is adopted and geometric units where $c = G = 1$ are used.

Consider the four-dimensional spacetime Ω divided into two regions Ω^+ and Ω^- with different geometry by a singular hypersurface Σ_0. A singular hypersurface is a hypersurface on which Riemann curvature tensor has a singular components.

In a coordinate system $\{x^a\}$ continuous in the neighborhood of Σ_0, where the function defining Σ_0 is selected as one of the coordinates, the hypersurface equation is reduced to $n = 0$. In addition, without loss of generality, it can be assumed, that

$$(N_a N^a)|_{\Sigma_0} = (\partial_a n \partial^a n)|_{\Sigma_0} = g^{nn}|_{\Sigma_0} = \varepsilon.$$

Here N^a is a normal to the hypersurface, $\varepsilon = -1$ for the timelike hypersurface, and 1 for the spacelike one. For a lightlike surface:

$$(N_a N^a)|_{\Sigma_0} = (\partial_a n \partial^a n)|_{\Sigma_0} = g^{nn}|_{\Sigma_0} = 0.$$

Taking into account the accepted notation, the metric in the entire spacetime Ω can be formally represented as a sum:

$$g_{ab} = g^+_{ab}(x)\ \theta(n) + g^-_{ab}(x)\ \theta(-n). \tag{2}$$

When differentiating the expression (2), a term with a δ-function appears:

$$\partial_c g_{ab} = \partial_c g^+_{ab}\theta(n) + \partial_c g^-_{ab}\theta(-n) + \delta^n_c \delta(n)[g_{ab}].$$

Hereinafter, δ^n_c is the Kronecker symbol, the brackets $[g_{ab}] = g^+_{ab} - g^-_{ab}$ denote a jump of the corresponding variable on the surface Σ_0.

If $[g_{ab}] \neq 0$ then the δ-function occurs in Christoffel symbols. In this case, δ^2, which is undefined in the theory of generalized functions, appears in the Riemann curvature tensor and in the Lagrangian for both general relativity and quadratic gravity.

Due to coordinate transformations that are different for $\Omega^{\pm}$, it is always possible to make the tensor g_{ab} continuous on Σ_0, this fact was demonstrated, in particular, in the article [29]. Thus for both general relativity and quadratic gravity it is necessary either to work in such a special coordinate system or explicitly require $[g_{ab}] = 0$ in order to avoid the appearance of δ^2 in Lagrangian.

If the metric does not have a jump on Σ_0, from (2) the following expression for the Christoffel symbols is obtained:

$$\Gamma^a_{bc} = \Gamma^{+a}_{bc}\theta(n) + \Gamma^{-a}_{bc}\theta(-n). \tag{3}$$

From which it follows that:

$$R^a_{bcd} = R^{+a}{}_{bcd}\,\theta(n) + R^{-a}{}_{bcd}\,\theta(-n) + (\delta^n_c[\Gamma^a_{bd}] - \delta^n_d[\Gamma^a_{bc}])\,\delta(n). \tag{4}$$

The expression (4) for the Riemann tensor containing the δ-function would be acceptable for the Einstein-Gilbert action and the corresponding equations of motion, because in this case the singular part of the Riemann tensor can be directly related to the surface part of the energy-momentum tensor. This fact was first demonstrated in the papers of W. Israel [18]–[20]. In quadratic gravity, in order to avoid the appearance of undefined functions δ^2 and $\delta(n)\theta(n)$ in the Lagrangian, it is necessary to impose additional restrictions, namely, the Lichnerowicz conditions [30] :

$$[\Gamma^a_{bc}] = 0. \tag{5}$$

Consider the situation when the energy-momentum tensor of matter fields has different behavior in the regions $\Omega^{\ pm}$ and a singular component directly on Σ_0:

$$T^{ab} = S^{ab}\ \delta(n) + T^{+ab}\ \theta(n) + T^{-ab}\ \theta(-n), \tag{6}$$

where the tensor S^{ab} is defined as the surface energy-momentum tensor. In this formulation of the problem, it is assumed that T^{ab} does not contain other singular terms like derivatives of δ-function.

Unlike general relativity the components S^{nn} and S^{ni} of the surface energy-momentum tensor are generally nonzero in quadratic gravity. This fact was first noted in the papers of J.M.M. Senovilla [21]–[26], where S^{nn} and S^{ni} are defined as “external pressure” and “external flow” respectively.

In general relativity the field equations are of the second order in the derivatives of the metric tensor. The appearance of a δ-function in T^{ab} leads to its appearance in the Riemann tensor, then $[\Gamma^a_{bc}] \neq 0$, in this case Σ_0 is called thin shell. Israel's equations describe the relationship between the jumps of the Christoffel symbols and their derivatives with the tensor S^{ab}. If there is only a jump in T^{ab}, then the corresponding jump in curvature describes a shock gravitational wave accompanied by a shock wave in matter.

In quadratic gravity the field equations are of the fourth order in the derivatives of the metric tensor. If certain components of the curvature tensor are continuous on Σ_0, then their second derivatives can contain at most δ-function, which corresponds to the δ-function in T_{ab}, and Σ_0 is a thin shell. If these components of the curvature tensor experience a jump on Σ_0, then their second derivatives contain the δ'-function, and Σ_0 is a double layer. The equations of motion of a double layer in quadratic gravity, as noted above, were first obtained by J.M.M.Senovilla. A jump in curvature describing a gravitational shock wave may or may not be accompanied by a shock wave in the distribution of matter, i.e. in quadratic gravity, a purely gravitational shock wave can exist.

3. Junction conditions in quadratic gravity

The main goal of this study is to find the equations of motion for a singular lightlike hypersurface in quadratic gravity. In order to do this, it is necessary to separate the surface part in the system of equations obtained by varying the action (1) with respect to the inverse metric in the case when the Riemann tensor has the structure (4) and with (6) as the source.

Using the fact that $\theta^2(n) = \theta(n)$ and $\theta(n)\,\theta(-n) = 0$ and substituting the Riemann tensor (4) and its contractions into action (1), the following expression is obtained:

$$S_q = -\frac{1}{16\pi}\int_{\Omega^+}\sqrt{|g|}L_q^+\,d^4x - \frac{1}{16\pi}\int_{\Omega^-}\sqrt{|g|}L_q^-\,d^4x. \tag{7}$$

Here $L_q^+ = \alpha_1 R^+_{abcd}R^{+abcd} + \alpha_2 R^+_{ab}R^{+ab} + \alpha_3 (R^+)^2 + \alpha_4 R^+ + \alpha_5\Lambda$, L_q^- is defined analogically to L_q^+.

From the variation of the action (7) with respect to the inverse metric, one can get [31]:

$$\delta S_q = -\frac{1}{16\pi}\int_{\Omega^+}\sqrt{|g|}\left(H^+_{ab}\delta g^{ab} + \nabla_c V^{+c}\right)d^4x - \frac{1}{16\pi}\int_{\Omega^-}\sqrt{|g|}\left(H^-_{ab}\delta g^{ab} + \nabla_c (V^{-c})\right)d^4x. \tag{8}$$

The following designations are accepted here:

$$H^+_{ab} = 2\alpha_1 R^+_{amlp}R_b^{+mlp} - 2(2\alpha_1+\alpha_2)R_d^{+c}R^{+d}_{\ \ bac} - 4\alpha_1 R_a^{+c}R^+_{bc} + 2\alpha_3 R^+ R^+_{ab} + \alpha_4 R^+_{ab} -$$
$$- \frac{1}{2}g_{ab}L_q^+ + (\alpha_2 + 4\alpha_1)\Box R^+_{ab} + \frac{1}{2}(4\alpha_3+\alpha_2)g_{ab}\Box R^+ - (2\alpha_1+\alpha_2+2\alpha_3)\nabla_a\nabla_b R^+,$$

$$V^{+c} = \left\{(4\alpha_1+\alpha_2)\nabla^c R^{+bd} + \frac{1}{2}(\alpha_2+4\alpha_3)g^{bd}\nabla^c R^+\right\}\delta g_{bd} -$$
$$- 2\nabla^b\left\{(2\alpha_1+\alpha_2)R^{+cd} + \alpha_3\, g^{cd}R^+\right\}\delta g_{bd} + (\alpha_4 + 2\alpha_3 R^+)(g^{ab}g^{cd} - g^{ac}g^{bd})\,\nabla_a\delta g_{bd} +$$
$$+ \left\{\alpha_2(2g^{cd}R^{+ab} - g^{ac}R^{+bd} - g^{bd}R^{+ac}) - 4\alpha_1 R^{+abcd}\right\}\nabla_a\delta g_{bd}\,.$$

The vector V^{-c} and the tensor H^-_{ab} are defined in a similar way.

By definition, to derive the equations of motion using the least action principle a variation with fixed ends should be used, i.e. the values of dynamical variables, metric components in this case, are fixed at the boundary of the entire volume of integration: $\delta g_{ab} = 0$ on $\partial\Omega$. Using the Gauss-Stokes theorem we obtain the following expression for δS_q:

$$\delta S_q = -\frac{1}{16\pi}\int_{\Omega^+}\sqrt{|g|}\left(H^+_{ab}\delta g^{ab}\right)d^4x - \frac{1}{16\pi}\int_{\Omega^-}\sqrt{|g|}\left(H^-_{ab}\delta g^{ab}\right)d^4x + \frac{1}{16\pi}\int_{\Sigma_0}[V^c]\,dS_c,$$

where dS_c is a directed surface element.

In this case it is convenient to use the definition of dS_c, which is applicable to both timelike (spacelike) and lightlike hypersurfaces:

$$dS_c = \epsilon N_c\sqrt{|g(x^a(y^i))|}d^3y,$$

where $|g(x^a(y^i))|$ is a modulus of restriction of the metric determinant in the entire spacetime Ω to the Σ_0, $\left\{y^i\right\}$ are arbitrary internal coordinates on the hypersurface. As before, $\epsilon = -1$ for a timelike hypersurface and 1 for a spacelike hypersurface, but in addition ϵ is also 1 for a lightlike hypersurface.

Similarly, the variation of the matter action with the energy-momentum tensor (6) is divided into volume and surface parts:

$$\delta S_m = \frac{1}{2}\int_{\Omega^+}\sqrt{|g|}\left(T^+_{ab}\delta g^{ab}\right)d^4x + \frac{1}{2}\int_{\Omega^-}\sqrt{|g|}\left(T^-_{ab}\delta g^{ab}\right)d^4x - \\ -\frac{1}{2}\int_{\Sigma_0}\sqrt{|g(x^a(y^i))|}\left(S^{ab}\delta g_{ab}\right)d^3y.$$

According to the least action principle:

$$\delta S = \delta S_q + \delta S_m = 0.$$

From that the system of motion equations is derived:

$$H^\pm_{ab} = 8\pi T^\pm_{ab}, \tag{9}$$

$$\epsilon[V^c]N_c = 8\pi S^{ab}\delta g_{ab}. \tag{10}$$

The analogue of Israel's equation for quadratic gravity can be obtained directly from (10). The cases of timelike and spacelike hypersurfaces are analyzed, in particular, in the papers [24], [27]. As mentioned above, here the focus is on the lightlike case. However, before moving on to a particular type of hypersurface, the expression (10) can be greatly simplified.

Since the variation of the gravitational part of the action δS_q contains divergence from the vector V^c, it can be argued that the vector V^c itself is determined up to the addition of some divergence-free vector multiplied by an arbitrary constant - $2CU^c$. As demonstrated in the article [31], it is convenient to choose the vector U^c as follows:

$$U^c = -\nabla^b\left(R^{cd} - \frac{1}{2}g^{cd}R\right)\delta g_{bd} + \left(R^{ab}g^{cd} - R^{bc}g^{ad}\right)\nabla_a\delta g_{bd}.$$

In this case, let's put $C = 2\alpha_3$, then:

$$\widetilde{V}^{+c} = V^{+c} + 4\alpha_3 U^{+c} = \left\{(4\alpha_1+\alpha_2)\nabla^c R^{+bd} + \frac{1}{2}(\alpha_2+4\alpha_3)g^{bd}\nabla^c R^+\right\}\delta g_{bd} - \\ -2(2\alpha_1+\alpha_2+2\alpha_3)\nabla^b R^{+cd}\delta g_{bd} + (\alpha_4 + 2\alpha_3 R^+)(g^{ab}g^{cd} - g^{ac}g^{bd})\nabla_a\delta g_{bd} \\ + \left\{\alpha_2(2g^{cd}R^{+ab} - g^{ac}R^{+bd} - g^{bd}R^{+ac}) - 4\alpha_1 R^{+abcd}\right\}\nabla_a\delta g_{bd} + \\ + 4\alpha_3(R^{+ab}g^{cd} - R^{+bc}g^{ad})\nabla_a\delta g_{bd}\,.$$

As before, the vectors U^{-c}, $\widetilde{V}^{-c}$ are defined similar to U^{+c}, $\widetilde{V}^{+c}$.

In coordinates $\{x^a\}$: $N_c = \delta^n_c$, therefore, replacing $V^{\pm c}$ with $\widetilde{V}^{\pm c}$ in (10) one can get:

$$\epsilon[\widetilde{V}^n] = 8\pi S^{ab}\delta g_{ab}. \tag{11}$$

If the Lichnerowicz conditions are satisfied, then as a consequence: $[\nabla_a \delta g_{bd}] = 0$, so this factor can be taken out of the bracket when calculating the jump $[\widetilde{V}^n]$:

$$[\widetilde{V}^n] = \left\{(4\alpha_1+\alpha_2)[\nabla^n R^{bd}] + \frac{1}{2}(\alpha_2+4\alpha_3)g^{bd}[\nabla^n R] - 2(2\alpha_1+\alpha_2+2\alpha_3)[\nabla^b R^{nd}]\right\}\delta g_{bd} +$$
$$+\left\{2\alpha_3[R](g^{ab}g^{nd} - g^{an}g^{bd}) + \alpha_2(2g^{nd}[R^{ab}] - g^{an}[R^{bd}] - g^{bd}[R^{an}])\right\}\nabla_a\delta g_{bd} +$$
$$+\left\{4\alpha_3([R^{ab}]g^{nd} - [R^{bn}]g^{ad}) - 4\alpha_1[R^{abnd}]\right\}\nabla_a\delta g_{bd}. \tag{12}$$

In order to transform the vector $[\widetilde{V}^n]$, let's write out expressions for the jumps of the Riemann tensor components and its contractions, bearing in mind that jumps are present only in those quantities that contain derivatives of the metric with respect to n, starting with the second-order derivatives, since the jumps of the first derivatives of the metric are zero due to the Lichnerowicz conditions.

$$[R_{iklm}] = \frac{1}{2}\left(\delta^n_k\delta^n_l[\partial^2_{nn}g_{im}] + \delta^n_i\delta^n_m[\partial^2_{nn}g_{kl}] - \delta^n_i\delta^n_l[\partial^2_{nn}g_{km}] - \delta^n_k\delta^n_m[\partial^2_{nn}g_{il}]\right), \tag{13}$$

$$R^{ab} = \frac{1}{2}\left(g^{an}g^{bn}g^{cd} + g^{nn}g^{ac}g^{bd} - g^{an}g^{bd}g^{cn} - g^{bn}g^{ad}g^{cn}\right)[\partial^2_{nn}g_{cd}], \tag{14}$$

$$[R] = \left(g^{nn}g^{cd} - g^{cn}g^{dn}\right)[\partial^2_{nn}g_{cd}]. \tag{15}$$

The factor in $[\widetilde{V}^n]$ for $\nabla_a\delta g_{bd}$ (let's denote it A^{abd}) requires more detailed consideration, taking into account the relations (13 - 15), one can get:

$$A^{abd} = -(\alpha_2+4\alpha_3)g^{an}g^{bd}[R^{n\ \ n}_{\ nn}] + (\alpha_2+4\alpha_3)g^{an}(g^{bn}g^{dn} - g^{bd}g^{nn})[R_{nn}] +$$
$$+2(2\alpha_1+2\alpha_3+\alpha_2)g^{nb}g^{nd}[R^{n\ \ a}_{\ nn}] + 4(\alpha_3-\alpha_1)g^{an}g^{dn}[R^{\ bn}_{n\ \ n}] +$$
$$+2(2\alpha_1+2\alpha_3+\alpha_2)g^{nn}g^{nd}[R^{\ b\ a}_{n\ n}] - (\alpha_2+4\alpha_1)g^{nn}g^{na}[R^{\ b\ d}_{n\ n}] =$$
$$= -\frac{1}{2}(\alpha_2+4\alpha_3)g^{an}g^{bd}[R] + 2(2\alpha_1+2\alpha_3+\alpha_2)g^{nd}[R^{ab}] - (\alpha_2+4\alpha_1)g^{an}[R^{bd}]. \tag{16}$$

In order to perform the following operation with A^{abd}, it is necessary to return to the integration of A^{abd} over the hypersurface Σ_0 as a part of $[\widetilde{V}^c]$ present in δS_q:

$$\int_{\Sigma_0}\left(A^{abd}\nabla_a\delta g_{bd}\right)\sqrt{|g(x^a(y^i))|}d^3y = \int_{\Sigma_0}\left(-\nabla_i A^{ibd}\delta g_{bd} + A^{nbd}\nabla_n\delta g_{bd}\right)\sqrt{|g(x^a(y^i))|}d^3y +$$
$$+\int_{\Sigma_0}\nabla_i\left(A^{ibd}\delta g_{bd}\right)\sqrt{|g(x^a(y^i))|}d^3y = \int_{\Sigma_0}\left((-\nabla_a A^{abd} + \nabla_n A^{nbd})\delta g_{bd}\right)\sqrt{|g(x^a(y^i))|}d^3y +$$
$$+\int_{\Sigma_0}\left(A^{nbd}\nabla_n\delta g_{bd}\right)\sqrt{|g(x^a(y^i))|}d^3y + \int_{\Sigma_0}\nabla_i\left(A^{ibd}\delta g_{bd}\right)\sqrt{|g(x^a(y^i))|}d^3y, \quad i \neq n.$$

Let's show that the integral $\int_{\Sigma_0}\nabla_i\left(A^{ibd}\delta g_{bd}\right)\sqrt{|g(x^a(y^i))|}d^3y$ is zero. Since the integrand is a complete divergence of some vector field on the hypersurface Σ_0, by virtue of the Gauss-Stokes theorem, this integral reduces to an integral over the boundary of the hypersurface Σ_0, which is a part of the boundary of the entire space-time under consideration $\partial\Omega$, where $\delta g_{bd} = 0$.

With all of the above equation (11) is reduced to:

$$\begin{aligned}(\alpha_2 + 4\alpha_1)\left\{2[\nabla^n R^{bd}] - g^{nn}[\nabla_n R^{bd}]\right\}\delta g_{bd} + \frac{1}{2}(\alpha_2 + 4\alpha_3)g^{bd}\left\{2[\nabla^n R] - g^{nn}[\nabla_n R]\right\}\delta g_{bd} - \\ -(\alpha_2 + 2\alpha_1 + 2\alpha_3)\left\{2g^{ib}[\nabla_i R^{nd}] + g^{nd}[\nabla^b R]\right\}\delta g_{bd} + \\ + \left(-\frac{1}{2}(\alpha_2 + 4\alpha_3)g^{nn}g^{bd}[R] + (2\alpha_1 + 2\alpha_3 + \alpha_2)g^{nd}g^{bn}[R] - (\alpha_2 + 4\alpha_1)g^{nn}[R^{bd}]\right)\nabla_n \delta g_{bd} = \\ = 8\pi\epsilon\, S^{ab}\delta g_{ab}, \quad i \neq n. \quad (17)\end{aligned}$$

When deriving the equation (17), among other things, the following expression was used:

$$[R^{nb}] = \frac{1}{2}g^{nb}[R], \quad (18)$$

which is a consequence of (13-15).

Let's consider in more detail the multiplier for $\nabla_n \delta g_{bd}$; it can be shown that it is zero if at least one of the indices b or d is equal to n.

$$\begin{aligned}-\frac{1}{2}(\alpha_2 + 4\alpha_3)g^{nn}g^{bn}[R] + (2\alpha_1 + 2\alpha_3 + \alpha_2)g^{nn}g^{bn}[R] - (\alpha_2 + 4\alpha_1)g^{nn}[R^{bn}] = \\ = \frac{1}{2}(\alpha_2 + 4\alpha_3)g^{nn}(g^{bn}[R] - 2[R^{bn}]) = 0.\end{aligned}$$

Here the last equality is satisfied due to (18).

As noted in the article [27], on the one hand, the variation of the derivatives of the metric components $\delta(\partial_n g_{ij})$ on Σ_0 is not independent with respect to the variation of the metric components δg_{ij} on the hypersurface, on the other hand, in a sense, the relation between them is arbitrary, since the equations of motion in quadratic gravity are of the fourth order with respect to the derivatives of the metric tensor; therefore, their solutions are not uniquely determined by specifying the initial conditions on the metric components and their first derivatives on some Cauchy hypersurface. Thus, it is necessary to demand:

$$\delta(\partial_n g_{ij}) = B^{kl}_{ij}(y)\delta g_{kl}, \quad i, j, l, k \neq n,$$

where $B^{kl}_{ij}(y)$ are arbitrary functions.

The appearance of arbitrary functions in the equations of motion is due to the implicit presence of δ' and is a marker of a double layer.

Taking into account all of the above, the equations of motion for a double layer of an arbitrary type in quadratic gravity are:

$$2(\alpha_3 - \alpha_1)\left(g^{nn}[\partial^n R] - 2[\nabla^n R^{nn}]\right) + \frac{1}{2}(\alpha_2 + 4\alpha_3)g^{nn}\left(2[\nabla_n R^{nn}] - g^{nn}[\partial_n R]\right) = 8\pi\epsilon S^{nn}, \quad (19)$$

$$\begin{aligned}(\alpha_2 + 6\alpha_1 - 2\alpha_3)[\nabla^n R^{ni}] + 2(\alpha_3 - \alpha_1)g^{nn}[\nabla_n R^{ni}] + \frac{1}{2}(\alpha_2 + 6\alpha_3 - 2\alpha_1)g^{ni}[\partial^n R] - \\ -(\alpha_2 + 2\alpha_1 + 2\alpha_3)\left([\nabla^i R^{nn}] - g^{ni}[\nabla_n R^{nn}] + \frac{1}{2}g^{nn}[\partial^i R] + g^{ni}g^{nj}\Gamma^n_{nj}[R]\right) + \\ + \frac{1}{2}(\alpha_2 + 4\alpha_3)g^{nn}\left(g^{ij}\Gamma^n_{nj}[R] - g^{ni}[\partial_n R]\right) + (\alpha_2 + 4\alpha_1)g^{nn}\Gamma^n_{nj}[R^{ij}] = 8\pi\epsilon S^{ni}, \quad (20)\end{aligned}$$

$$(\alpha_2+4\alpha_1)\left\{2[\nabla^n R^{ij}]-g^{nn}[\nabla_n R^{ij}]+g^{nn}\Gamma^j_{nk}[R^{ik}]+g^{nn}\Gamma^i_{nk}[R^{jk}]-g^{nn}B^{ij}_{kl}(y)[R^{kl}]\right\}+$$
$$+\frac{1}{2}(\alpha_2+4\alpha_3)\left\{2g^{ij}[\partial^n R]-g^{nn}g^{ij}[\partial_n R]+g^{nn}g^{ik}\Gamma^j_{nk}[R]+g^{nn}g^{jk}\Gamma^i_{nk}[R]-g^{nn}g^{kl}B^{ij}_{kl}(y)[R]\right\}+$$
$$+(\alpha_2+2\alpha_1+2\alpha_3)\left\{g^{ni}[\nabla_n R^{nj}]+g^{nj}[\nabla_n R^{ni}]-[\nabla^i R^{nj}]-[\nabla^j R^{ni}]-\frac{1}{2}g^{ni}[\partial^j R]-\frac{1}{2}g^{nj}[\partial^i R]\right\}+$$
$$+(\alpha_2+2\alpha_1+2\alpha_3)[R]\left\{-g^{ni}g^{nk}\Gamma^j_{nk}-g^{nj}g^{nk}\Gamma^i_{nk}+g^{nk}g^{nl}B^{ij}_{kl}(y)\right\}=8\pi\epsilon S^{ij},\ i,j,k,l\neq n. \tag{21}$$

For a null double layer, the system of equations (19 - 21) can be greatly simplified:

$$-4(\alpha_3-\alpha_1)g^{ni}[\nabla_i R^{nn}]=-4(\alpha_3-\alpha_1)g^{ni}[\nabla_i(N_aN_bR^{ab})]=$$
$$=-4(\alpha_3-\alpha_1)g^{ni}\partial_i[R^{nn}]=0=8\pi S^{nn}, \tag{22}$$

$$(\alpha_2+6\alpha_1-2\alpha_3)g^{nj}[\nabla_j R^{ni}]+\frac{1}{2}(\alpha_2+6\alpha_3-2\alpha_1)g^{ni}g^{nj}[\partial_j R]-$$
$$-(\alpha_2+2\alpha_1+2\alpha_3)g^{ni}g^{nj}\Gamma^n_{nj}[R]=8\pi S^{ni}, \tag{23}$$

$$2(\alpha_2+4\alpha_1)g^{nk}[\nabla_k R^{ij}]+(\alpha_2+4\alpha_3)g^{nk}g^{ij}[\partial_k R]-$$
$$-(\alpha_2+2\alpha_1+2\alpha_3)\left\{g^{ik}[\nabla_k R^{nj}]+g^{jk}[\nabla_k R^{ni}]+\frac{1}{2}g^{ni}[\partial^j R]+\frac{1}{2}g^{nj}[\partial^i R]\right\}+$$
$$+(\alpha_2+2\alpha_1+2\alpha_3)[R]\left\{-g^{ni}g^{nk}\Gamma^j_{nk}-g^{nj}g^{nk}\Gamma^i_{nk}+g^{nk}g^{nl}B^{ij}_{kl}(y)\right\}=8\pi S^{ij},$$
$$i,j,k,l\neq n. \tag{24}$$

It should be noted that from the equation (22) it follows that S^{nn} is zero in any coordinate system, since it is a scalar:$S^{nn}=N_aN_bS^{ab}$.

During the derivation of motion equations the hypersurface Σ_0 was assumed to be given a priori, but in applications an inverse problem arises when the solutions of the motion equations in the domains $\Omega^{\pm}$ - $g^{\pm}_{ab}$ are known and it is required to find the equation of the hypersurface on which the matching takes place.

Since only equations with S^{ij} on the right-hand side contain arbitrary functions in the case of a double layer they serve to determine $B^{ij}_{kl}(y)$, while equations with S^{nn} and S^{ni} on the right-hand side together with Lichnerowicz conditions are necessary to determine Σ_0 itself.

The picture is different for thin shells. The main criterion that Σ_0 is a thin shell and not a double layer is the absence of arbitrary functions in the motion equations. For the timelike and spacelike cases this criterion is reduced to the absence of jumps in the Ricci tensor: $[R^{bd}]=0$. For the lightlike hypersurface this is the absence of jumps in the scalar curvature $[R]=0$. In this case, the jump $[R_{nn}]$ can be nonzero, since $g^{nn}|_{\Sigma_0}=0$ for a lightlike hypersurface. It can be shown that the absence of the corresponding jumps for any type of thin shell in addition to the absence of arbitrary functions also implies that $S^{nn}=S^{ni}=0$.

Thus the equations of motion for timelike (spacelike) thin shells are a special case of the system (19 - 21) with the additional conditions $[R^{bd}]=0$:

$$S^{nn}=S^{ni}=0,$$
$$(\alpha_2+4\alpha_1)g^{nn}[\nabla_n R^{ij}]+\frac{1}{2}(\alpha_2+4\alpha_3)g^{nn}g^{ij}[\partial_n R]-$$
$$-(\alpha_2+2\alpha_1+2\alpha_3)g^{ni}g^{nj}[\partial_n R]=8\pi\epsilon S^{ij},\quad i,j\neq n. \tag{25}$$

Similarly for null thin shells we get:

$$S^{nn} = S^{ni} = 0,$$

$$2(\alpha_2 + 4\alpha_1) g^{nk} [\nabla_k R^{ij}] -$$
$$- (\alpha_2 + 2\alpha_1 + 2\alpha_3) \left\{ g^{ik} [\nabla_k R^{nj}] + g^{jk} [\nabla_k R^{ni}] + g^{nj} g^{ni} [\partial_n R] \right\} = 8\pi S^{ij},$$
$$i, j, k \neq n. \quad (26)$$

It will also be shown below that for a null hypersurface the condition $[R] = 0$ is satisfied in particular when the normal vector field to Σ_0 is null not only on the hypersurface but also in some neighborhood of the Σ_0.

4. Null case

As noted earlier, in order to avoid the appearance of undefined functions in the Lagrangian, it is necessary to ensure the continuity of the metric components on the hypersurface. This can always be achieved by choosing the appropriate coordinates. For timelike and spacelike hypersurfaces it is usually convenient to use Gaussian normal coordinates. In the paper [29] it was demonstrated that for these types of singular hypersurfaces it is always possible to dock two different Gaussian normal coordinate systems from the $\Omega^{\pm}$ domains on the hypersurface in such a way that the jumps of the metric components are zero.

For lightlike hypersurfaces there are some analogs of Gaussian normal coordinates, such as Gaussian null coordinates (GNC) [32] – [34] or coordinates for foliation of manifold by null hypersurfaces (NSFC) [35]–[37]. Nevertheless, in the present work a more minimalistic approach was used, the one that does not imply additional information regarding the behavior of the normal vector field of the hypersurface outside Σ_0. Using the formalism described in the book of E. Poisson [38], a coordinate system with the necessary properties namely continuity of the metric components on Σ_0 and coordinates themselves in some neighborhood of the hypersurface was constructed. The following is a brief description of this process.

Consider the lightlike hypersurface Σ_0 defined in the domains $\Omega^{\pm}$ with different coordinates $\{x^{\pm}\}$ by the equations: $n^{\pm}(x^{\pm}) = 0$. The normal vector field to a hypersurface is defined as: $N_{\pm a} = \partial_a n^{\pm}(x^{\pm})$. For the sake of convenience, let's temporarily omit the notations $\pm$ and write all the equations in one of the domains.

In the case of a null hypersurface, the norm $N^a N_a$ is zero on Σ_0, so the normal vector is defined up to multiplication by an arbitrary scalar function. Since the vector N^a is null on Σ_0, it is simultaneously tangent to the hypersurface.

Let's show that vector field N^a on the hypersurface is tangent to the family of null geodesics that lie on Σ_0:

$$\nabla_b N_a N^b = \nabla_b \nabla_a n \, \partial^b n = \nabla_a \partial_b n \, \partial^b n = \frac{1}{2} \nabla_a \left(\partial_b n \, \partial^b n \right).$$

The scalar $N^b N_b$ is zero on Σ_0 therefore its gradient is directed along N_a so:

$$\nabla_b N_a N^b = \kappa N_a,$$

for some scalar κ.

Thus, N^a on the hypersurface is a tangent to lightlike geodesics that lie on Σ_0 and are actually its generators.

Let's choose an arbitrary parameter λ on these generators and two additional coordinates - $\theta^A, A = \{2, 3\}$ for marking geodesics, together they form the internal coordinate system $\{y^i\} = \{\lambda, \theta^A\}$ on Σ_0. The parameter λ is affine if the equation $n(x) = const$ defines a whole family of null hypersurfaces, in this case the scalar $N^a N_a$ is zero not only on the hypersurface, but at least in some neighborhood of it.

Let's calculate the induced metric on the hypersurface:

$$ds^2_{\Sigma_0} = g_{ab}\, dx^a(y^i)\, dx^b(y^j) = g_{ab}\, e^a_i\, e^b_j\, dy^i dy^j, \quad e^a_i = \frac{\partial x^a}{\partial y^i}.$$

The vector fields e^a_i are tangent to the curves lying on the hypersurface, so they are orthogonal to the normal on Σ_0:

$$N_a e^a_A = 0, \quad N_a e^a_1 = N_a\, \frac{\partial x^a}{\partial \lambda} = N_a N^a = 0.$$

These relations also imply that the induced metric on the Σ_0 is effectively two-dimensional:

$$ds^2_{\Sigma_0} = g_{ab}\, e^a_A\, e^b_B\, d\theta^A d\theta^B = \sigma_{AB}\, d\theta^A d\theta^B. \tag{27}$$

The system of vector fields $\{N^a, e^a_A\}$ can be completed to a basis in the restriction of the tangent bundle of the manifold Ω to the hypersurface - $T\Omega|_{\Sigma_0}$ if an auxiliary null vector field l^a with the following properties is found:

$$l^a N_a = 1, \quad l_a e^a_A = 0, \quad l^a l_a = 0.$$

The existence of such a vector, as well as the completeness of the system of vector fields $\{l^a, N^a, e^a_A\}$ in $T\Omega|_{\Sigma_0}$ are demonstrated in the aforementioned book [38], as well as in the article [39], which introduces a generalization for such type of vectors (rigging vector) for singular hypersurfaces of mixed causal character.

Since the vector fields $\{l^a, N^a, e^a_A\}$ form a basis in $T\Omega|_{\Sigma_0}$, it is possible to write completeness relations for the inverse metric:

$$g^{ab} = l^a N^b + l^b N^a + \sigma^{AB} e^a_A e^b_B. \tag{28}$$

Let's choose $\{n, \lambda, \theta^A\}$ as local coordinates in a neighborhood of the hypersurface. In these coordinates on Σ_0 the following relations hold:

$$N_a = \delta^n_a, \quad N^a = \frac{\partial x^a(y^i)}{\partial \lambda} = \delta^a_\lambda, \quad l_\lambda = l^n = 1, \quad l_n = -l^\lambda, \quad l_A = 0. \tag{29}$$

Taking into account (28), in coordinates $\{n, \lambda, \theta^A\}$ the inverse metric has the following structure on the hypersurface:

$$g^{n\lambda} = 1, \quad g^{nn} = g^{nA} = 0, \quad g^{\lambda\lambda} = 2\, l^\lambda, \quad g^{\lambda A} = l^A, \quad g^{AB} = \sigma^{AB}. \tag{30}$$

Then for the tensor g_{ab} inverse to g^{ab} one can get:

$$g_{n\lambda} = 1, \quad g_{\lambda\lambda} = g_{\lambda A} = 0, \quad g_{nn} = -2\, l^\lambda + \sigma_{AB} l^B l^A, \quad g_{nA} = -\sigma_{AB} l^B, \quad g_{AB} = \sigma_{AB}. \tag{31}$$

It should be noted, that the relations (30, 31) are true only on Σ_0, i.e. for $n = 0$. This means, in particular, that the second derivatives with respect to n of $g_{n\lambda}, g_{\lambda\lambda}, g_{\lambda A}$ and their jumps on Σ_0 are generally non-zero. The jumps of the first derivatives with respect to n are taken to be zero by virtue of the Lichnerowicz conditions. A similar kind of metric applies to the neighborhood Σ_0 only if the vector field N^a is null also in some neighborhood of Σ_0, but even in this case $\partial^2_{nn}\, g_{\lambda n}$ is not necessarily zero.

A coordinate system common for $\Omega^\pm$ with the necessary properties is constructed as follows: n^+ and n^- are continuously combined into one coordinate n and y^{+i} or y^{-i} are chosen as the remaining coordinates. The continuity of the metric on Σ_0 is ensured by the relations (30) and the proper selection of functions $y^{+i}(y^{-j})$.

Next, let's proceed to the analysis of the equations (22 - 24, 26) in the coordinates $\{n, \lambda, \theta^A\}$. Let's calculate the jumps present in these equations:

$$[R] = 2\,[R^{n\ \ n}_{\ nn}] = [\partial^2_{nn} g_{\lambda\lambda}], \quad [\partial_i R] = [\partial_i \partial^2_{nn} g_{\lambda\lambda}], \tag{32}$$

$$[R^{ab}] = \frac{1}{2}\left(\delta^a_\lambda\, g^{bc}\,[\partial^2_{nn} g_{c\lambda}] + \delta^b_\lambda\, g^{ac}\,[\partial^2_{nn} g_{c\lambda}] - \delta^a_\lambda\,\delta^b_\lambda\, g^{cd}\,[\partial^2_{nn} g_{cd}]\right), \tag{33}$$

$$[\nabla_\lambda R^{bd}] = \partial_\lambda [R^{bd}] + \delta^b_\lambda\left(\Gamma^\lambda_{\lambda\lambda}[R^{\lambda d}] + \Gamma^d_{\lambda A}[R^{\lambda A}] + \frac{1}{2}\,\Gamma^d_{\lambda n}\,[R]\right) + \\ + \delta^d_\lambda\left(\Gamma^\lambda_{\lambda\lambda}[R^{\lambda b}] + \Gamma^b_{\lambda A}[R^{\lambda A}] + \frac{1}{2}\,\Gamma^b_{\lambda n}\,[R]\right), \tag{34}$$

$$[\nabla_i R^{nd}] = \frac{1}{2}\,\delta^d_\lambda\left(\partial_i [R] + \Gamma^n_{in}\,[R] + 2\,\Gamma^n_{iA}\,[R^{\lambda A}]\right) + \frac{1}{2}\,\Gamma^d_{i\lambda}\,[R]. \tag{35}$$

Taking into account all of the above, the equations for a null double layer in the coordinates $\{n, \lambda, \theta^A\}$ are:

$$S^{nn} = 0, \tag{36}$$

$$2(\alpha_2 + 4\alpha_1)[\nabla_\lambda R^{\lambda\lambda}] + (\alpha_2 + 4\alpha_3) g^{\lambda\lambda}[\partial_\lambda R] - \\ - (\alpha_2 + 2\alpha_1 + 2\alpha_3)\left(2g^{\lambda i}[\nabla_i R^{\lambda n}] + [\partial^\lambda R] + 2\Gamma^\lambda_{\lambda n}[R] - B^{\lambda\lambda}[R]\right) = 8\pi\, S^{\lambda\lambda}, \tag{37}$$

$$(\alpha_2 - 2\alpha_3 + 6\alpha_1)[\nabla_\lambda R^{\lambda n}] + \frac{1}{2}(\alpha_2 - 2\alpha_1 + 6\alpha_3)[\partial_\lambda R] - (\alpha_2 + 2\alpha_3 + 2\alpha_1)\Gamma^n_{\lambda n}[R] = 8\pi\, S^{n\lambda}, \tag{38}$$

$$2(\alpha_2 + 4\alpha_1)[\nabla_\lambda R^{\lambda A}] + (\alpha_2 + 4\alpha_3) g^{\lambda A}[\partial_\lambda R] - \\ - \frac{1}{2}(\alpha_2 + 2\alpha_1 + 2\alpha_3)\left(2g^{iA}[\nabla_i R^{\lambda n}] + [\partial^A R] + g^{\lambda B}\Gamma^A_{\lambda B}[R] + 2\Gamma^A_{\lambda n}[R] - 2B^{\lambda A}[R]\right) = 8\pi\, S^{\lambda A}, \tag{39}$$

$$(\alpha_2 + 4\alpha_3)\sigma^{AB}[\partial_\lambda R] - (\alpha_2 + 2\alpha_1 + 2\alpha_3)[R](\sigma^{AC}\Gamma^B_{C\lambda} - B^{AB}) = 8\pi\, S^{AB}. \tag{40}$$

Let's also consider an important special case when the vector field N^a is null in some neighborhood of Σ_0. Then it follows from (32 - 35) that:

$$[R] = [R^{\lambda A}] = [\partial_i R] = [\nabla_i R^{nd}] = 0, \tag{41}$$

$$[\nabla_\lambda R^{bd}] = \delta^b_\lambda\,\delta^d_\lambda\,[\partial_\lambda R^{\lambda\lambda}]. \tag{42}$$

Thus, it corresponds to the scenario of a thin shell. In the coordinates $\{n, \lambda, \theta^A\}$ for a null thin shell only one equation of the system (26) remains with a nonzero right-hand side:

$$2(\alpha_2 + 4\alpha_1)[\partial_\lambda R^{\lambda\lambda}] - (\alpha_2 + 2\alpha_1 + 2\alpha_3)[\partial_n R] = 8\pi\, S^{\lambda\lambda}. \tag{43}$$

5. Spherically symmetric case

Let's investigate a special case of null singular hypersurfaces namely spherically symmetric null hypersurfaces separating two spherically symmetric space-times. It will be demonstrated below that such singular hypersurfaces can only be thin shells.

Consider a spherically symmetric metric of the most general form:

$$ds^2 = \gamma_{\alpha\beta}(x)\, dx^\alpha dx^\beta - r^2(x)\, \left(d\theta^2 + \sin^2\theta\, d\phi^2\right) = r^2(x) \left(\widetilde{\gamma}_{\alpha\beta}(x)\, dx^\alpha dx^\beta - d\Omega^2\right) = \\ = r^2(x) \left(d\widetilde{s}_2^2 - d\Omega^2\right), \quad \alpha, \beta = 0, 1. \quad (44)$$

Hereinafter $2+2$ decomposition of the metric in the case of spherical symmetry is used.

For this type of geometry, the following relationships hold:

$$R_{\theta\theta} = 1 + r\sigma + \Delta, \quad R_{\phi\phi} = R_{\theta\theta}\, sin^2\theta, \quad (45)$$

$$R_{\alpha\beta} = \frac{\gamma_{\alpha\beta}}{r^2} \left(\frac{1}{2}\widetilde{R} + \Delta - r\,\sigma\right) - \frac{2}{r}\nabla_\alpha\nabla_\beta\, r, \quad (46)$$

$$R = \frac{1}{r^2}\left(\widetilde{R} - 2 - 6\, r\,\sigma\right), \quad (47)$$

where Δ, σ, $\widetilde{R}$ are invariants of spherical geometry.

$$\Delta = \gamma^{\alpha\beta}\, \partial_\alpha r\, \partial_\beta r, \quad \sigma = \gamma^{\alpha\beta}\, \nabla_\alpha\nabla_\beta\, r\,, \quad (48)$$

$$\widetilde{R} = \frac{1}{\widetilde{\gamma}}\left(-\partial^2_{11}\widetilde{\gamma}_{00} + 2\partial^2_{01}\widetilde{\gamma}_{01} - \partial^2_{00}\widetilde{\gamma}_{11}\right) + \frac{1}{2\widetilde{\gamma}^2}\left(\widetilde{\gamma}_{11}\,(\partial_1\widetilde{\gamma}_{00})^2 + \widetilde{\gamma}_{00}\,(\partial_0\widetilde{\gamma}_{11})^2\right) + \\ + \frac{1}{2\widetilde{\gamma}^2}\left(\partial_1\widetilde{\gamma}_{00}\left(-\partial_0\widetilde{\gamma}_{11}\,\widetilde{\gamma}_{10} + \partial_1\widetilde{\gamma}_{11}\,\widetilde{\gamma}_{00} - 2\partial_1\widetilde{\gamma}_{01}\,\widetilde{\gamma}_{10}\right) + \partial_0\widetilde{\gamma}_{11}\left(\partial_0\widetilde{\gamma}_{00}\,\widetilde{\gamma}_{11} - 2\,\partial_0\widetilde{\gamma}_{10}\,\widetilde{\gamma}_{10}\right)\right) + \\ + \frac{1}{2\widetilde{\gamma}^2}\left(\partial_0\widetilde{\gamma}_{00}\left(\partial_1\widetilde{\gamma}_{11}\,\widetilde{\gamma}_{01} - 2\,\partial_1\widetilde{\gamma}_{10}\,\widetilde{\gamma}_{11}\right) - 2\,\partial_0\widetilde{\gamma}_{10}\left(\partial_1\widetilde{\gamma}_{11}\,\widetilde{\gamma}_{00} - 2\,\partial_1\widetilde{\gamma}_{10}\,\widetilde{\gamma}_{10}\right)\right). \quad (49)$$

Here $\widetilde{R}$ and $\widetilde{\gamma}$ are the determinant and the scalar curvature of the two-dimensional "metric" $\widetilde{\gamma}_{\alpha\beta}$ respectively.

For any spherically symmetric lightlike hypersurface in geometry (44) given by the equation $n(x) = 0$ the following is true:

$$\gamma_{00}\left(\partial^0 n(x)\right)^2 + 2\gamma_{01}\,\partial^0 n(x)\,\partial^0 n(x) + \gamma_{11}\left(\partial^1 n(x)\right)^2 = 0.$$

If this equation can be solved with respect to the variable $\frac{\partial^0 n(x)}{\partial^1 n(x)}$ or the reciprocal of it at any values of x, then the function $n(x)$ potentially defines a whole family of lightlike hypersurfaces $n(x) = const$. Since this is a quadratic equation with respect to the indicated variable, in order for it to always have at least one solution, $-\gamma$ must be nonnegative, but this condition is always satisfied for any Lorentzian spherically symmetric manifold wherever the determinant γ exists. It means that the hypersurface is at least locally a leaf of some null foliation in each of the regions $\Omega^\pm$, and this, as shown in the previous section, is equivalent to the condition $[R] = 0$.

Thus, it has been demonstrated that a lightlike spherically symmetric singular hypersurface in a spherically symmetric geometry can only be a thin shell.

The coordinates $\{n, \lambda, \theta, \phi\}$ defined in the previous section is used to describe a spherically symmetric null thin shell. In these coordinates the metric in the neighborhood of Σ_0 has the form:

$$ds^2 = g_{nn}(n,\lambda)\, dn^2 + 2g_{n\lambda}(n,\lambda)\, dn\, d\lambda - r^2(n,\lambda)\, d\Omega^2. \quad (50)$$

The equation (43) describing the dynamics of a null thin shell in this case can be rewritten using previously defined invariants and their derivatives with respect to n:

$$2(\alpha_2 + 4\alpha_1)[\partial_n \Delta] - (\alpha_2 + 2\alpha_1 + 2\alpha_3)[\partial_n \widetilde{R}] + 4r\,(\alpha_2 + 3\alpha_3 + \alpha_1)[\partial_n \sigma] = 8\pi\, r^2\, S^\lambda_n. \quad (51)$$

The Lichnerowicz conditions can also be expressed in terms of invariants:

$$[\Delta] = 0, \quad [\widetilde{R}] = 0. \quad (52)$$

It needs to be clarified that the conditions (52) ensure the absence of jumps $[\partial_n g_{n\lambda}], [\partial_n r]$, while for a metric of the form (50) it is also necessary to require: $[\partial_n g_{nn}] = 0$. This condition could be satisfied by adding $[\sigma] = 0$ to the Lichnerowicz conditions but there is no need for this, since according to (47) it automatically follows from the continuity of R, r and $\widetilde{R}$ on Σ_0.

6. Conformal gravity

The action of the Weyl conformal gravity is a special case of the action (1) with $\alpha_2 = -2\alpha_1, \quad \alpha_3 = \frac{1}{3}\alpha_1, \quad \alpha_4 = \alpha_5 = 0$:

$$S_q = -\frac{1}{16\pi}\int_\Omega \sqrt{-g}\alpha_1 \left(R_{abcd}\, R^{abcd} - 2R_{ab}\, R^{ab} + \frac{1}{3}R^2\right)\, d^4x = \\ = -\frac{1}{16\pi}\int_\Omega \sqrt{-g}\alpha_1 C_{abcd}\, C^{abcd}\, d^4x, \quad (53)$$

where C_{abcd} is the Weyl tensor, which is defined as:

$$C_{abcd} = R_{abcd} + \frac{1}{2}\left(R_{ad}\, g_{bc} + R_{bc}\, g_{ad} - R_{ac}\, g_{bd} - R_{bd}\, g_{ac}\right) + \frac{1}{6}R\left(g_{ac}\, g_{bd} - g_{ad}\, g_{bc}\right).$$

Therefore, the corresponding equations of motion first obtained in the paper of R. Bach [40] are a special case of (9).

$$\nabla_c \nabla_d\, C^{acbd} + \frac{1}{2}\, C^{acbd}\, R_{cd} = \frac{2\pi}{\alpha_1}\, T^{ab}. \quad (54)$$

Let's consider spherically symmetric null thin shells separating two spherically symmetric solutions of (54) as applications. As stated earlier, these solutions were presented in the article [28]. Their general form coincides with (44) provided that $r^2(x)$ are arbitrary functions.

Since null thin shells are considered, the general form of the hypersurface equations in the domains $\Omega^\pm$ is known and is determined from the condition of zero interval. The equation (51) together with the Lichnerowicz conditions (52) do not completely fix the equations of the hypersurface, since there are three equations and four unknown functions: $n^\pm(x^\pm), r^\pm(x^\pm)$. Therefore, instead of looking for the exact equations of the hypersurface, let's focus on determining the possibility of matching different spherically symmetric solutions of conformal gravity and the restrictions imposed on them by the Lichnerowicz conditions. It should also be noted that there are cases in wich it is inconvenient to use coordinates $\{n, \lambda, \theta, \phi\}$, thus it is important that conditions (52) are expressed using the invariants of spherical geometry and look the same in any coordinates.

6.1. Vacuum solutions

Let's analyze the case when $\Omega^\pm$ are two spherically symmetric vacuums of conformal gravity. In the aforementioned paper [28] it was demonstrated that in double null coordinates, where:

$$d\tilde{s}^2_2 = 2\widetilde{H}(u, v)\, du dv, \quad (55)$$

the Bach equations for vacuum are:

$$\partial^2_{uv}\widetilde{R} = -\frac{\widetilde{H}}{4}\left(\widetilde{R}^2 - 4\right), \tag{56}$$

$$\partial_u\left(ln(\partial_u\widetilde{R})\right) = \partial_u\left(ln(\widetilde{H})\right), \tag{57}$$

$$\partial_v\left(ln(\partial_v\widetilde{R})\right) = \partial_v\left(ln(\widetilde{H})\right). \tag{58}$$

The coordinates $\{n,\lambda\}$ are a special case of double null coordinates, therefore $[\partial^2_{n\lambda}\widetilde{R}] = [-\frac{\widetilde{H}}{4}\left(\widetilde{R}^2-4\right)] = 0$ and $[\partial_n\widetilde{R}] = 0$. Therefore a version of the equation (51) for a thin shell separating two vacuums of conformal gravity is:

$$[\partial_n\Delta] = \frac{2\pi r^2}{\alpha_1}S^{\lambda\lambda}. \tag{59}$$

As noted in the article [28] there are three types of spherically symmetric conformal gravity vacua: two with constant two-dimensional scalar curvature $\widetilde{R} = \mp 2$ and one with variable $\widetilde{R}$.

$$\widetilde{H} = \frac{2}{(u \pm v)^2}, \quad \widetilde{R} = \mp 2. \tag{60}$$

$$\begin{aligned} \widetilde{H} = \frac{1}{2}|A(\widetilde{R}(u,v))|, \quad A(\widetilde{R}) = \frac{1}{6}\left(\widetilde{R}^3 - 12\widetilde{R} + C_0\right), \\ \widetilde{R} = \widetilde{R}(v-u): \int\frac{d\widetilde{R}}{A(\widetilde{R})} = \frac{1}{2}(v-u),\ A(\widetilde{R}) > 0, \\ \widetilde{R} = \widetilde{R}(v+u): \int\frac{d\widetilde{R}}{A(\widetilde{R})} = \frac{1}{2}(v+u),\ A(\widetilde{R}) < 0, \end{aligned} \tag{61}$$

where C_0 is some constant.

Since $[\widetilde{R}] = 0$ the following combinations are possible for two vacua: matching a vacuum with a constant $\widetilde{R} = \pm 2$ and a vacuum with a variable $\widetilde{R}$, matching two vacua with variable $\widetilde{R}$, matching two vacua with coinciding constant $\widetilde{R}$.

Let's analyze the situation when Ω^- is a vacuum with a constant $\widetilde{R} = \pm 2$ and Ω^+ is a vacuum with a variable $\widetilde{R}$.

From the condition $[\widetilde{R}] = 0$ it follows that the hypersurface equation in the Ω^+ domain is $n^+(x^+) = \widetilde{R}^+ \mp 2 = 0$. A hypersurface of this type is null only if $C_0 = \pm 16$ when $n^+ = 0$ is an analogue of the double horizon with respect to the variable $\widetilde{R}^+$. In the coordinates $\{n^+, u^+\}$ the two-dimensional "metric" in the Ω^+ domain looks like:

$$d\widetilde{s}_2^{+2} = \frac{1}{6}(n^+)^2(n^+ \pm 6)du^{+2} + 2du^+dn^+. \tag{62}$$

In the domain Ω^- the hypersurface equation is $n^-(x^-) = F^-(v^-) = 0$, where $F^-(v^-)$ is an arbitrary function of v^- (the case when F^- is an arbitrary function of u^- is similar to this one). In the coordinates $\{n^-, u^-\}$ the two-dimensional "metric" in the Ω^- has the following form:

$$d\widetilde{s}_2^{-2} = \frac{4}{(u^- \mp f^-(n^-))^2}\frac{df^-}{dn^-}dn^-du^-, \tag{63}$$

where $f^-(n^-)$ is a function inverse to $F^-(v^-)$.

The continuity of the metric on the hypersurface defines the following connection between coordinates $u^\pm$:

$$r^+(0,u^+) = r^-(0,u^-).$$

The remaining Lichnerowicz condition $[\Delta] = 0$ imposes a restriction on the functions $r^\pm(x^\pm)$:

$$\partial_{n^+} r^+|_{\Sigma_0} = \partial_{\widetilde{R}^+} r^+(2,u^+) = \partial_{n^-} r^-|_{\Sigma_0} = \partial_{n^-} f^-(0)\,\partial_{v^-} r^-(f^-(0),u^-). \tag{64}$$

Let's move on to the analysis of the matching of two vacua with variable $\widetilde{R}$. Without loss of generality the hypersurface Σ_0 can be considered given by the equations: $n^\pm(x^\pm) = F^\pm(u^\pm) = 0$ in $\Omega^\pm$. Here $F^\pm(u^\pm)$ are arbitrary functions of variables $u^\pm$. In this case, it is convenient to choose the coordinates $n^\pm, \widetilde{R}^\pm$:

$$d\widetilde{s}_2^{\pm 2} = A^\pm \left(\frac{df^\pm}{dn^\pm}\right)^2 dn^{\pm 2} + 2sign(A^\pm)\frac{df^\pm}{dn^\pm} dn^+ d\widetilde{R}^\pm. \tag{65}$$

From the continuity of the metric and $\widetilde{R}$ on Σ_0 it follows that $\frac{df^+}{dn^+}(0) = \pm\frac{df^-}{dn^-}(0)$ when $C_0^+ = C_0^-$ .This means that the two given vacua can only differ in functions $r^\pm(n^\pm, \widetilde{R}^\pm)$, but they must also coincide up to the second order in $n^\pm$ on Σ_0:

$$r^+(0,\widetilde{R}^+) = r^-(0,\widetilde{R}^-), \quad \partial_{n^+} r^+(0,\widetilde{R}^+) = \partial_{n^-} r^-(0,\widetilde{R}^-). \tag{66}$$

A similar result, namely a coincidence up to the second derivatives with respect to n of the conformal factor, is obtained by matching two vacua with the same constant $\widetilde{R}$.

6.2. Vaidya-type solution

In this section let's consider the situation when at least one of the domains, namely Ω^+, is a Vaidya-type solution for spherically symmetric conformal gravity also described in the article [28].

In the coordinates $\{\widetilde{R}^+, u^+\}$ the two-dimensional "metric" in the Ω^+ has the following form:

$$d\widetilde{s}_2^{+2} = A^+(\widetilde{R}^+, u^+) du^{+2} + 2du^+ d\widetilde{R}^+, \tag{67}$$

where $A^+(\widetilde{R}^+, u^+) = \frac{1}{6}(\widetilde{R}^{+3} - 12\widetilde{R}^+ + C_0^+(u^+))$, $C_0^+(u^+)$ is an arbitrary function of u^+.

There are two types of null hypersurfaces for this metric. The first type is a hypersurface given by an arbitrary function of the variable u^+, the second type is a hypersurface given by the function $F^+(u^+, \widetilde{R}^+)$ which satisfies the equation:

$$\partial_{u^+} F^+ = \frac{1}{12}(\widetilde{R}^{+3} - 12\widetilde{R}^+ + C_0^+(u^+))\partial_{\widetilde{R}^+} F^+. \tag{68}$$

Let's use the second type of hypersurfaces, since it potentially corresponds to a thin shell with radiation transversal to the hypersurface.

If Ω^- is a vacuum with constant $\widetilde{R}^- = \pm 2$ since $[\widetilde{R}] = 0$ when $n^+(x^+) = \widetilde{R}^+ \mp 2$. It follows from (68) that a hypersurface of this type can be null only if $C_0^+(u^+) = \pm 16$. This corresponds to the degenerate case of the metric (67), which is essentially a vacuum with variable $\widetilde{R}$.

Next let the Ω^- be a vacuum with variable $\widetilde{R}$. Suppose Σ_0 is given by the equation $n^-(x^-) = F^-(u^-) = 0$ in Ω^- and $F^-(u^-)$ is an arbitrary function of u^-. In coordinates $\{n^\pm, \widetilde{R}^\pm\}$ the two-dimensional "metrics" in the $\Omega^\pm$ are:

$$d\widetilde{s}_2^{-2} = A^- \left(\frac{df^-}{dn^-}\right)^2 dn^{-2} + 2sign(A^-)\frac{df^-}{dn^-} dn^- d\widetilde{R}^-, \tag{69}$$

$$d\widetilde{s}_2^{+2} = A^+ \left(\partial_{u+} F^+\right)^{-2} dn^{+2} - 2 \left(\partial_{u+} F^+\right)^{-1} dn^+ d\widetilde{R}^+, \tag{70}$$

where $f^-(n^-)$ is a function inverse to $F^-(u^-)$, $F^+(u^+, \widetilde{R}^+)$ is given by the equation (68).

From the continuity of the metric and $\widetilde{R}$ on Σ_0 it follows that $\frac{df^-}{dn^-}(0) = \pm\partial_{u+}F^+(\widetilde{R}^+, u^+)|_{\Sigma_0}$ and $A^+(\widetilde{R}^+, u^+)|_{\Sigma_0} = A^-(\widetilde{R}^-)|_{\Sigma_0}$. As a result we have: $C_0^+(u^+)|_{\Sigma_0} = C_0^-|_{\Sigma_0} = const$. It means that the hypersurface in Ω^+ is defined by a single variable function $F^+(u^+)$, but this type of null hypersurfaces corresponds to the absence of radiation.

Finally, let's analyze the matching of two Vaidya type solutions. In this case $n^\pm(x^\pm) = F^\pm(u^\pm, \widetilde{R}^\pm)$ are functions defined by the (68). In coordinates $\{n^\pm, \widetilde{R}^\pm\}$ the two-dimensional "metrics" in the $\Omega^\pm$ are:

$$ds_2^{\pm 2} = A^\pm \left(\partial_{u\pm} F^\pm\right)^{-2} dn^{\pm 2} - 2 \left(\partial_{u\pm} F^\pm\right)^{-1} dn^\pm d\widetilde{R}^\pm. \tag{71}$$

From the continuity of the metric and $\widetilde{R}$ on Σ_0 it follows that $\partial_{u-}F^-(\widetilde{R}^-, u^-)|_{\Sigma_0} = \partial_{u+}F^+(\widetilde{R}^+, u^+)|_{\Sigma_0}$ and $A^+(\widetilde{R}^+, u^+)|_{\Sigma_0} = A^-(\widetilde{R}^-, u^-)|_{\Sigma_0}$ hence $C_0^+(u^+)|_{\Sigma_0} = C_0^-(u^-)|_{\Sigma_0}$. It means that the two given solutions can only differ in functions $r^\pm(n^\pm, \widetilde{R}^\pm)$, but they must also coincide up to the second order in $n^\pm$ on Σ_0:

$$r^+(0, \widetilde{R}^+) = r^-(0, \widetilde{R}^-), \quad \partial_{n^+} r^+(0, \widetilde{R}^+) = \partial_{n^-} r^-(0, \widetilde{R}^-). \tag{72}$$

In this case, despite the fact that the hypersurface may be of the second type, it still cannot be interpreted as a shell with radiation.

7. Conclusions

In this concluding section let's discuss the obtained results. Continuing the work started in the article [27], it was shown that for a singular hupersurface of arbitrary type in quadratic gravity equations of motion can be obtained using only the least action principle. It follows from the equations of motion (19-21) that for the ratio of the coefficients corresponding to the Gauss-Bonnet quadratic term: $\alpha_1 = \alpha_3 = -\frac{1}{4}\alpha_2, \quad \alpha_4 = \alpha_5 = 0$ all components of the surface energy-momentum tensor are zero. Thus, even if Riemann curvature tensor or its derivatives undergo a jump on the hypersurface the Gauss-Bonnet term does not create neither double layers nor thin shells given that the Lichnerowicz conditions are satisfied.

Analysis of the equations of motion for the null double layer shows that in contrast to the timelike and spacelike cases $S^{nn} = N_a N_b S^{ab} = 0$ in any coordinate system, which means the absence of the so-called "external pressure" for the null double layer in quadratic gravity.

As for thin shells, the main difference between the null case and the others is the condition for the hypersurface to be a thin shell. For a null thin shell is the absence of a jump in scalar curvature, while for the rest the condition is stronger, namely, the absence of jumps in the Ricci tensor.

It was demonstrated that null spherically symmetric singular hupersurfaces in quadratic gravity cannot be a double layer, and only thin shells are possible. In this case the system of equations of motion (26) is reduced to one which is expressed through the invariants of spherical geometry along with the Lichnerowicz conditions.

Spherically symmetric null thin shells were investigated for spherically symmetric solutions of conformal gravity as an applications, in particular, for various vacua and Vaidya-type solutions.

By virtue of Lichnerowicz conditions $[\widetilde{R}] = 0$ therefore, for two vacua, the following combinations are possible: matching a vacuum with a constant $\widetilde{R}$ and a vacuum with a variable $\widetilde{R}$, matching two vacuums with a variable $\widetilde{R}$, matching two vacua with a coinciding constant $\widetilde{R}$. In the first case, the hypersurface is an analogue of the double horizon in the metric with the variable $\widetilde{R}$; in other cases, matching is possible only if the metrics coincide up to a conformal factor.

With the addition of the Vaidya-type solution new possible matchings appear: Vaidya-type metrics with vacuum with variable $\widetilde{R}$, two Vaidya-type metrics. In the first version, the null shell is actually the singular part of the Vaidya-type solution, in the second, they must coincide up to the conformal factor. Moreover the null shell does not emit in both cases presumably due to the fact that the "external flow" is zero.

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