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On squashed spheres and warm strings

Applications of supersymmetric localization and integrability in gauge and string theory

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Abstract
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Non-perturbative aspects of quantum field theories are notoriously hard to explore. In this thesis we study applications of two different techniques that give non-perturbative results for supersymmetric quantum field theories.

The first exact technique we use is supersymmetric localization which allows for the exact computation of partition functions on compact manifolds and squashed spheres of various dimensions are the manifolds of our choice. In three dimensions we use the squashed sphere partition functions to test dualities of N=4 gauge theories. For a squashed sphere preserving six supercharges we lift analytic results from the round sphere. On a squashed sphere preserving 4 supercharges we numerically evaluate the ABJM and N=8 super Yang-Mills (SYM) partition functions at low rank and find equality within estimated error margins. In four dimensions we present a framework to obtain partially integrated correlators of 4d N=2 gauge theories from their localized partition functions. Moreover we discuss the general form of the free energy of N=2 superconformal field theories on deformed four-dimensional spheres and use localization in N=4 SYM for an explicit example. In seven dimensions we study the super Yang-Mills on a sphere and propose a contribution of three-dimensional membrane instantons to its localized partition function. We then outline an approach to study the weak negative coupling limit of the SYM theory on the seven-sphere.

As the second approach to exact results we use the integrability of N=4 SYM and ABJM theory in the planar limit. Using the quantum spectral curve we compute the Hagedorn temperature for finite coupling both in N=4 SYM and ABJM theory. On the dual AdS side we use an effective model to compute subleading terms in the curvature expansion of the Hagedorn temperature. We use the numeric CFT calculation to conjecture the analytic form of an unfixed coefficient in the effective model.

Keywords: Supersymmetry, Localization, Gauge theory, Integrability, AdS/CFT, String theory

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I J. Minahan, U. Naseer, C. Thull, *Squashing and supersymmetry enhancement in three dimensions*, SciPost Phys. 12, 025 (2022), arXiv: 2107.07151 [hep-th].

II C. Thull, *Dualities and loops on squashed* S^3, SciPost Phys. 15, 156 (2023), arXiv: 2212.06813 [hep-th].
Erratum: *Figure 8 (c) was not correctly reproduced in the published version. The correct figure can be found in the preprint version and in this thesis at the end of chapter 3 as Figure 3.4b.*

III J. Minahan, U. Naseer, C. Thull, *Conformal field theories on deformed spheres, anomalies, and supersymmetry*, SciPost Phys. 10, 063 (2021), arXiv: 2012.01781 [hep-th].

IV U. Naseer, C. Thull, *Flavor deformations and supersymmetry enhancement in* $4d$ $\mathcal{N}=2$ *theories*, SciPost Phys. 13, 058 (2022), arXiv: 2110.09329 [hep-th].

V J. Minahan, U. Naseer, C. Thull, *Seven-dimensional super Yang-Mills at negative coupling*, SciPost Phys. 14, 028 (2023), arXiv: 2208.01115 [hep-th].

VI S. Ekhammar, J. Minahan, C. Thull, *The asymptotic form of the Hagedorn temperature in planar* $\mathcal{N}=4$ *super Yang-Mills*, J. Phys. A: Math. Theor. 56 435401, arXiv: 2306.09883 [hep-th].

VII S. Ekhammar, J. Minahan, C. Thull, *The ABJM Hagedorn Temperature from Integrability*, J. High Energ. Phys. 2023, 66 (2023), arXiv: 2307.02350 [hep-th].

Contents

1. Introduction

Quantum field theory forms the backbone of our current understanding of particle physics. The most prominent example of a quantum field theory, the Standard Model of particle physics, has been immensely successful in describing the fundamental particles and their interactions. The Standard Model describes with great precision the processes in experiments such as the Large Hadron Collider at CERN. Crucial for the quantum field theory's success with collider experiments is that the interactions at the center of these experiments are weak. This allows us to use perturbative quantum field theory techniques for the description. However beyond the perturbative regime explicit computations in a generic quantum field theory are difficult. One example for this comes from gauge theory and more specifically Yang-Mills theory. The mass gap in Yang-Mills theory is one of the unsolved Clay Millenium Prize problems [1].

One path towards non-perturbative results in quantum field theory starts by imposing supersymmetry. Supersymmetric quantum field theories were first considered in [2,3] and have since flourished into a productive playground for studying quantum field theories. The symmetry between bosons and fermions imposed by supersymmetry is very constraining for a quantum field theory and therefore makes explicit computations more accessible. A series of exact techniques for non-perturbative calculations in supersymmetric quantum field theories have been developed. In this thesis we will discuss applications of two of the approaches.

Supersymmetric localization has made the computation of partition functions in supersymmetric gauge theories accessible. It started with seminal papers by Nekrasov [4] and Pestun [5]. Pestun showed that the partition function and the Wilson loop expectation values in the mass deformed $\mathscr{N}=4$ super Yang-Mills (SYM) theory on the four-sphere could be written as a matrix model. This matrix model is the reduction of the path integral to an integral over a BPS locus. Since the original papers by Nekrasov and Pestun supersymmetric localization has been extended to a large number of supersymmetric gauge theories on compact Euclidean manifolds of various dimensions. In this thesis we use the results for the localized partition functions of supersymmetric gauge theories on squashed spheres of dimension three, four and seven.

For $\mathscr{N}=4$ SYM on the four-sphere the localized partition function is a simple Gaussian integral. However, in general supersymmetric localization gives a complicated matrix model. Depending on the background manifold, the determinant of the quadratic fluctuations around the saddle point gives products of special functions in the integrand of the matrix model. For example, for gauge theories on odd-dimensional squashed spheres one finds multiple sine functions [6–8]. In some cases the fine tuning of parameters of a supersymmetric gauge theory leads to a simplification of the localized partition function. Cancellations in the one-loop determinant are the consequence of relations between fermionic and bosonic modes and thus they hint at enhanced supersymmetry. In Paper III we show that fine tuning the mass parameter in $\mathcal{N} = 2^*$ SYM on the squashed four-sphere leads to a partition function independent of the squashing parameter. With Paper IV we explore the underlying supersymmetry enhancement and extended the result to more general manifolds. Similarly Papers I and II explore squashing independent partition functions and loop operator expectation values of mass deformed $\mathcal{N} = 4$ gauge theories on the squashed three-sphere and find the underlying supersymmetry enhancement for a fine tuned mass parameter. This squashing independence was first observed by [9] in ABJM theory.

When transformations or evaluations of the matrix model are not available we can resort to approximations to access some of the information contained in the localized partition function. After gauge fixing a matrix model it becomes a finite dimensional integral over the eigenvalues of the matrix and for the localized partition function the dimension of the integral equals the rank of the gauge group. One approach viable at low rank is to use brute force and numerically approximate this integral. In Paper II we do this for the partition functions of the ABJM and $\mathcal{N} = 8$ SYM theories at ranks two and three with sufficient precision to plausibly claim their equality. An alternative is to consider the opposite limit where the rank N of the gauge group becomes large. In this large N limit the localized partition function is usually dominated by a saddle point where the eigenvalues of the matrix model are all far apart from each other. In Paper III we compute the partition function of $\mathcal{N} = 4$ SYM on the squashed four-sphere in the large N limit and use it as an example for the general form of the free energy of $\mathcal{N} = 2$ superconformal field theories on deformed four-dimensional spheres.

One often overlooked theory we can study with supersymmetric localization is the 7d super Yang-Mills on the seven-sphere. Usually in holography one expects that supergravity, *i.e.* weakly coupled string theory or M-theory, matches with strongly coupled gauge theory. However it has been suggested that the spherical D6 brane solution of supergravity matches the weak negative coupling limit of SYM on the seven-sphere [10]. With Paper V we try to shed some more light on the nature of this negative coupling limit by studying a saddle point approximation to the localized partition function. Results from lower dimensions suggest that instanton contributions play a crucial role in this limit but only the perturbative part of the partition function has been previously studied in detail. Taking inspiration from lower dimensions and from string theory we suggest in Paper V a contribution of three-dimensional membrane instantons.

The second approach to non-perturbative results that we apply is integrability in the context of the AdS/CFT correspondence. $\mathcal{N}=4$ SYM and ABJM theory are said to be integrable in their respective 't Hooft limits. This has most prominently been used to develop techniques to solve their operator spectra for all values of their 't Hooft couplings. The most powerful of these techniques is the quantum spectral curve and approaches to solve it perturbatively at weak coupling and numerically for finite coupling are known. An adaptation of the quantum spectral curve to compute the Hagedorn temperature in $\mathcal{N}=4$ SYM was developed in [11]. In Paper VII we find the quantum spectral curve for the ABJM Hagedorn temperature. The AdS/CFT correspondence lets us compare the integrability results in $\mathcal{N}=4$ SYM and ABJM to the dual string theory on anti-de Sitter space. Not much is known about the curvature corrections to the Hagedorn temperature in AdS, although the first correction has been found from a supergravity computation [12, 13]. In Papers VI and VII we use an effective model and the integrability results in the dual CFT to compute the AdS Hagedorn temperature to the third subleading term.

1.1 Outline of this thesis

This thesis consists of four parts. The first three parts contain applications of supersymmetric localization and each of these parts focuses on a different dimension. The fourth part of this thesis is about an application of integrability in the AdS/CFT correspondence.

In Part I we discuss supersymmetric gauge theories on the squashed three-sphere. We start chapter 2 by giving an introduction to rigid supersymmetry on the squashed three-sphere, supersymmetric gauge theories on this background and the localized partition function. We relate the squashing independence of the localized partition functions of $\mathcal{N}=4$ supersymmetric gauge theories at a fine-tuned value of a mass parameter to a supersymmetry enhancement from 4 to 6 supercharges. We do the same for loop operator expectation values. In chapter 3 we test mirror dualities by comparing the partition functions for pairs of dual $\mathcal{N}=4$ theories. Here we first lift analytic results from the round sphere to the squashed sphere preserving 6 supercharges. Second, for the duality of ABJM and $\mathcal{N}=8$ SYM we provide at low ranks numerical evidence that the equality of the partition functions also holds for only four preserved supercharges. This part is based on Papers I and II.

Part II is concernced with supersymmetric theories on the squashed four-sphere. We start chapter 4 by discussing supersymmetry on the squashed four-sphere and the localized partition function of $\mathcal{N}=2$ gauge theories on this background. For $\mathcal{N}=2^*$ super Yang-Mills theory we discuss a local supersymmetry enhancement at the poles of the squashed sphere that occurs upon fine-tuning of the mass parameter. The enhanced supersymmetry leads to squashing independence of the partition function and gives constraints on integrated correlators in $\mathcal{N} = 4$ SYM. For $\mathcal{N} = 2$ supersymmetric gauge theories we propose a framework for studying partially integrated correlators. In chapter 5 we discuss the general form of the free energy of $\mathcal{N} = 2$ superconformal field theories on deformed spheres. We show that the logarithm of the localized partition function of $\mathcal{N} = 4$ super Yang-Mills has this general form. This part is based on Papers III and IV.

In Part III we discuss super Yang-Mills on the seven-sphere. In chapter 6 we first review supersymmetry on the seven-sphere and the localization of the SYM partition function. We then describe in detail our proposal for the contribution of three-dimensional membrane instantons to the localized partition function. In chapter 7 we start by motivating the existence of the negative coupling limit of the SYM theory on S^7. We then outline our approach to studying this limit based on the localized partition function and the membrane instanton contributions to this partition function. This part is based on Paper V.

Part IV of this thesis discusses the Hagedorn temperature in AdS/CFT using integrability techniques. In chapter 8 we present the AdS side of the duality and outline an effective model for the computation of the Hagedorn temperature using a scalar field in a Euclidean AdS background. In chapter 9 we present the dual CFT perspective where we use integrability to compute the Hagedorn temperature for finite coupling. We give a detailed description of the quantum spectral curve for both $\mathcal{N} = 4$ SYM and ABJM. Curve fitting the numerical CFT results we conjecture the analytic expression for a parameter in the effective model on the AdS side of the duality. This part is based on Papers VI and VII.

Part I:
Supersymmetric gauge theories on the squashed three-sphere

2. Supersymmetric localization on the squashed three-sphere

This chapter focuses on supersymmetric gauge theories on the squashed three-sphere and introduces many concepts that reoccur in Part II and Part III of this thesis. This includes rigid supersymmetry on curved manifolds and supersymmetric localization.

Two different geometries have been called a "squashed three-sphere" [6]. The one we will be considering has metric

$$\mathrm{d}s^2 = \frac{r^2}{4}\left(\frac{b+b^{-1}}{2}\right)^2 (\mathrm{d}\psi + \cos\theta \mathrm{d}\phi)^2 + \frac{r^2}{4}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2), \tag{2.1}$$

with $b \in \mathbb{R}$ the "squashing parameter" and $(\psi, \theta, \phi) \in [0, 2\pi) \times [0, \pi] \times [0, 2\pi)$. This geometry is obtained by stretching the Hopf fiber of the round S^3 which we return to when $b = 1$. The other geometry is an ellipsoid and is mentioned in paper I but will not play a role in the rest of this thesis.

We start this chapter with a review of the supersymmetric squashed sphere and gauge theories coupled to this background. As part of this review we summarize the squashed sphere preserving the six supercharges found in Paper I. Thereafter we give an introduction to supersymmetric localization and the resulting gauge theory partition functions. We highlight the simplifications due to enhanced supersymmetry observed in Paper I. We close the chapter with a brief discussion of loop operators in gauge theories on the squashed three-sphere pertaining to the results in Paper II.

2.1 Rigid supersymmetry on the squashed three-sphere

An efficient way to define supersymmetric theories on curved spaces was laid out by Festuccia and Seiberg [14]. They noted that in the limit of vanishing Newton constant the fields in a supergravity multiplet are non-dynamical and can be set to a background value. The requirement that this background configuration of supergravity fields preserve some supersymmetries then leads to the Killing spinor equations. Each solution of these equations corresponds to one preserved supercharge. The Killing spinors are the supersymmetry analogue of Killing vectors in Riemannian geometry which describe the isometries of a Riemannian manifold. The supersymmetric coupling of gauge and matter multiplets to the background supergravity multiplet then insures that the gauge theory preserves these same supercharges determined by the Killing spinor equations.

2.1.1 Four preserved supercharges

To present the supersymmetric squashed three-sphere we start with the three-dimensional new minimal supergravity [15]. The bosonic fields of the corresponding supergravity multiplet are the dreibein e^a_μ, a $\mathrm{U}(1)$ R-symmetry gauge field $A^{R_0}_\mu$, the dual graviphoton field strength V_μ and a scalar field H. All the fermionic fields in the background supergravity multiplet are set to zero. Then the Killing spinor equations come from the vanishing of the supersymmetry transformations of the fermionic fields of the multiplet

$$\begin{aligned}\nabla_\mu\zeta - \mathrm{i}\left(A^{R_0}_\mu + V_\mu\right)\zeta - H\gamma_\mu\zeta + \frac{1}{2}\varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta &= 0,\\ \nabla_\mu\widetilde{\zeta} + \mathrm{i}\left(A^{R_0}_\mu + V_\mu\right)\widetilde{\zeta} - H\gamma_\mu\widetilde{\zeta} - \frac{1}{2}\varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\widetilde{\zeta} &= 0.\end{aligned} \tag{2.2}$$

Here the γ-matrices are the Pauli matrices. On the squashed sphere we choose the frame

$$\begin{aligned}e^1 &= -\frac{r_3}{2}\left(\sin\psi \mathrm{d}\theta - \sin\theta\cos\psi \mathrm{d}\phi\right),\\ e^2 &= \frac{r_3}{2}\left(\cos\psi \mathrm{d}\theta + \sin\theta\sin\psi \mathrm{d}\phi\right),\\ e^3 &= -\frac{r_3}{2}\left(\frac{b+b^{-1}}{2}\right)\left(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi\right).\end{aligned} \tag{2.3}$$

Setting the other bosonic fields in the supergravity multiplet to

$$H = \frac{\mathrm{i}\left(b+b^{-1}\right)}{4r_3}, \qquad V = -A^{R_0} = \frac{1}{4}\left(b^2 - b^{-2}\right)\left(\mathrm{d}\psi + \cos\theta \mathrm{d}\phi\right), \tag{2.4}$$

one can check that for any choice of constant spinors $\zeta_0, \widetilde{\zeta}_0$ the spinors

$$\zeta = e^{\frac{\mathrm{i}}{2}\Theta\sigma_3}\cdot g^{-1}\cdot\zeta_0, \qquad \widetilde{\zeta} = e^{-\frac{\mathrm{i}}{2}\Theta\sigma_3}\cdot g^{-1}\cdot\widetilde{\zeta}_0 \tag{2.5}$$

solve the Killing spinor equations 2.2 [16]. We have set

$$e^{\mathrm{i}\Theta} = -b, \qquad g = \begin{pmatrix}\cos\frac{\theta}{2}e^{\frac{\mathrm{i}}{2}(\phi+\psi)} & \sin\frac{\theta}{2}e^{\frac{\mathrm{i}}{2}(\phi-\psi)}\\ -\sin\frac{\theta}{2}e^{-\frac{\mathrm{i}}{2}(\phi-\psi)} & \cos\frac{\theta}{2}e^{\frac{-\mathrm{i}}{2}(\phi+\psi)}\end{pmatrix}.$$

With these four Killing spinors, the supergroup preserved on the squashed three-sphere is $\mathrm{SU}(2|1)\ltimes\mathrm{U}(1)$ [16].

2.1.2 $\mathscr{N}=2$ gauge theories on S^3_b

The 3d new minimal supergravity background from the previous subsection can be coupled to $\mathscr{N}=2$ supersymmetric gauge theories while preserving all four supercharges of the $\mathscr{N}=2$ theory [15]. In a supersymmetric theory, the gauge field is part of a vector multiplet. For $\mathcal{N}=2$ supersymmetry this multiplet contains the gauge field A, a pair of fermions $\lambda, \widetilde{\lambda}$, a real scalar σ and an auxiliary field D, all in the adjoint representation of the gauge group. There are several terms for the Lagrangian of a vector multiplet. The supersymmetrized Chern-Simons term takes the form [15]

$$\mathscr{L}_{CS} = \frac{k}{4\pi}\mathrm{Tr}\left(\mathrm{i}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - 2D\sigma + 2\mathrm{i}\widetilde{\lambda}\lambda\right), \tag{2.6}$$

with $k \in \mathbb{Z}$ the Chern-Simons level. The explicit form of the super Yang-Mills term is known but we refrain from showing it here because it is supersymmetrically exact [16], *i.e.* it takes the form $Q\mathscr{V}$ where Q is one of the preserved supercharges and $\mathscr{V}$ is a function of the fields in the vector multiplet. For each Abelian factor of the gauge group we can also write a Fayet-Iliopoulos term [15]

$$\mathscr{L}_{FI} = \xi(D - A_\mu V^\mu - \sigma H). \tag{2.7}$$

The matter of 3d $\mathcal{N}=2$ supersymmetric gauge theories is in chiral multiplets. A chiral multiplet contains a complex scalar ϕ, a fermion ψ and an auxiliary field F. The conjugate multiplet $(\widetilde{\phi}, \widetilde{\psi}, \widetilde{F})$ is called anti-chiral. The (anti-)chiral multiplet is charged under the R-symmetry $\mathrm{U}(1)_R \subset \mathrm{SU}(2|1)$. This R-charge q of the multiplet gets assigned to the scalars $(\phi, \widetilde{\phi})$ as $(-q, +q)$. As for the super Yang-Mills term of the vector multiplet, the Lagrangian of the chiral multiplet is supersymmetrically exact and we do not reproduce it here.

A theory with one or more chiral multiplets generically has a global "flavor" symmetry. To round off this subsection we consider coupling this flavor symmetry to a non-dynamical background vector multiplet with non-trivial values for the bosonic fields (A, σ, D). To preserve the same supercharges as the background supergravity one can choose

$$\partial_\mu\sigma = 0, \qquad D = -\sigma H, \qquad A = \frac{2\mathrm{i}\sigma r_3}{b+b^{-1}}V. \tag{2.8}$$

Assuming a flavor charge 1 and setting $\sigma = m$ we get that the hypermultiplet Lagrangian contains the terms [15]

$$\mathscr{L}_{hyp} \supset m^2\widetilde{\phi}\phi - \mathrm{i}m\widetilde{\psi}\psi \tag{2.9}$$

and thus we can understand such a flavor background vectormultiplet as a mass term for the hypermultiplet.

2.1.3 Six preserved supercharges

To preserve more supercharges on the squashed three-sphere, we have to start from a larger supergravity multiplet. In paper I we got such a multiplet and the corresponding Killing spinor equations by a twisted dimensional reduction of the 4d $\mathcal{N}=2$ conformal supergravity from [17, 18]. An interesting solution to these extended Killing spinor equations gives in addition to the fields of subsection 2.1.1 a non-zero value to the bosonic fields of a $\mathrm{U}(1)$ vector field (A^F, σ, D). For the fields

$$\sigma = \pm \mathrm{i}\frac{b-b^{-1}}{2r_3}, \qquad D = \mp \mathrm{i}\frac{b-b^{-1}}{2r_3}H, \qquad A^F = \mp \frac{b-b^{-1}}{b+b^{-1}}V, \tag{2.10}$$

we showed in Paper I that there are two additional preserved constant Killing spinors, enhancing the supersymmetry to $\mathrm{SU}(2|1) \ltimes \mathrm{SU}(1|1)$. Note that the signs in (2.10) are correlated and one should choose either the upper or lower sign for all three.

The supergravity multiplet from subsection 2.1.1 has combined with the vector multiplet (A^F, σ, D) to form a bigger supergravity multiplet. The vector multiplet couples to a $\mathrm{U}(1)$ subgroup of the $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_C \times \mathrm{SU}(2)_H$ R-symmetry of flat space 3d $\mathcal{N}=4$ supersymmetry. Checking the couplings of A^F as it enters our construction of the enhanced supersymmetry background we find that it couples to a $U(1)$ of the axial subgroup $(\mathrm{SU}(2)_C \times \mathrm{SU}(2)_H)_{ax}$.

Comparing to (2.8), we note that (2.10) is the result of fine tuning a distinguished mass parameter m_* to $m_* = \pm \mathrm{i}\frac{b-b^{-1}}{2r_3}$. Away from this fine tuned value for m_* only four supercharges are preserved. In subsection 2.2.3 we start from a generic value of the special mass parameter m_* and observe the effect of fine tuning and enhanced supersymmetery on the partition function.

2.1.4 $\mathcal{N}=4$ gauge theories

For a gauge theory to preserve the six supercharges in the previous subsection, the gauge field and the matter must be part of $\mathcal{N}=4$ multiplets. These multiplets can be put together from the $\mathcal{N}=2$ multiplets we discussed in subsection 2.1.2. Besides coupling the constituent $\mathcal{N}=2$ fields to the background fields of subsection 2.1.1 we then couple them to the additional background vector field (2.10) with charges dictated by $\mathcal{N}=4$ supersymmetry to get a gauge theory on the squashed three-sphere preserving six supercharges.

The $\mathcal{N}=4$ vector multiplet is formed from an $\mathcal{N}=2$ vector multiplet and a chiral multiplet of R-charge 1 in the adjoint representation of the gauge group. In terms of the $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_C \times \mathrm{SU}(2)_H$ R-symmetry group, the three scalars of the vector multiplet form an $\mathrm{SU}(2)_C$ triplet. The auxiliary fields are a triplet of $\mathrm{SU}(2)_H$ and the fermions are in the $(2,2)$ representation of the full group.

A hypermultiplet is made from two chiral multiplets of R-charge $\frac{1}{2}$ and in conjugate representations $\mathscr{R}, \overline{\mathscr{R}}$ of the gauge group. The hypermultiplet scalars organize into an $\mathrm{SU}(2)_C$ doublet transforming in representation $\mathscr{R}$ of the gauge group and an $\mathrm{SU}(2)_H$ doublet in $\overline{\mathscr{R}}$. To give a mass to a hypermultiplet while preserving all supercharges, we couple it to a background $\mathcal{N}=4$ vector multiplet for a $\mathrm{U}(1)$ flavor symmetry.

Note that neither the vector- nor the hypermultiplet are invariant under the exchange of $\mathrm{SU}(2)_C$ and $SU(2)_H$. The resulting multiplets under exchange of the roles for the two R-symmetry subgroups are called "twisted" multiplets.

2.2 Supersymmetric localization

After introducing supersymmetric gauge theories on the squashed three-sphere we discuss the derivation of their partition functions using supersymmetric localization. We give a short heuristic introduction to this technique, skipping many of the technical details that were worked out by other authors [6, 16]. We then go straight to the results for the partition function of $\mathcal{N}=2$ gauge theories on the squashed three-sphere. We end the section with a brief account of the results from Paper I concerning simplifications of the localized partition function of $\mathcal{N}=4$ gauge theories on the background with six preserved supercharges.

2.2.1 The supersymmetric localization argument

Supersymmetric localization is a saddle point approximation of the path integral of a supersymmetric gauge theory. However, unlike most other saddle point approximations, in this case the saddle point gives the exact result. The example which sparked rapid developments in supersymmetric localization was for $\mathcal{N}=4$ super Yang-Mills and its $\mathcal{N}=2^*$ deformation on the four-sphere [5]. Since this seminal work, supersymmetric localization has been applied to many other contexts (see [19] for a review). Let us sketch in the following how the localization argument works.

For a supersymmetric gauge theory the partition function,

$$Z=\int D[\text{fields}]e^{-S}, \tag{2.11}$$

is invariant under a supercharge Q. Deform Z to

$$Z(t)=\int D[\text{fields}]e^{-S-tQV}, \tag{2.12}$$

where V is some functional of the fields. If $Q^2=B$ generates a symmetry of the theory, then $Z(t)$ is independent of the parameter t. Choosing QV to be positive semi-definite we can take the limit $t\to\infty$. The only contributions to the partition function $Z=Z(\infty)$ then come from the locus where $QV=0$. Thus one only has to compute this locus as well as the determinant of the second order fluctuations around it.

The most common choice of V has a localization locus where the scalars in the vector multiplet are constants. This is called Coulomb branch localization and the partition function becomes a matrix model, *i.e.* an integral over the Lie algebra for the gauge group. In the next subsection we present the results of this computation for gauge theories on the squashed three-sphere.

2.2.2 The localized partition function

The localized partition function on the squashed three-sphere is an integral over the Cartan subalgebra for the gauge Lie algebra [16]

$$Z = \int \mathrm{d}^{r_G}\sigma \left(\prod_{\alpha\in\Delta_+} \alpha(\sigma)^2 \right) e^{-\mathrm{S}_{\mathrm{cl}}} Z_{1-\mathrm{loop}}(b,\sigma)\,, \tag{2.13}$$

where r_G is the rank of the gauge group and Δ_+ is the set of positive roots of the gauge Lie algebra. In this expression the classical action S_{cl} is the action of the gauge theory evaluated on the localization locus. The name "classical action" is misleading. The action we evaluate is off-shell supersymmetric and we are not refering to any solutions of the classical field equations. As the super Yang-Mills and hypermultiplet actions are Q-exact in three dimensions, they vanish on the localization locus and do not contribute to S_{cl}. However the classical action gets contributions from Chern-Simons and Fayet-Iliopoulos terms. The expressions (2.6) and (2.7) evaluated on the localization locus give the terms

$$\mathrm{S}_{\mathrm{cl}}^{\mathrm{CS}} = \mathrm{i}\frac{k\pi}{2}\mathrm{Tr}(\sigma^2), \qquad\qquad \mathrm{S}_{\mathrm{cl}}^{\mathrm{FI}} = -2\pi\mathrm{i}\xi\,\mathrm{Tr}(\sigma). \tag{2.14}$$

The determinant of the fluctuations around the localization locus Z_{1-loop}, also called the 1-loop determinant, gets a contribution from every vector- and hypermultiplet of the theory. The one-loop determinant of the vector multiplet takes the form

$$Z^{\mathrm{vec}}_{\mathcal{N}=2} = \prod_{\alpha\in\Delta_+} \frac{\sinh(\pi b\alpha(\sigma))\sinh\left(\pi b^{-1}\alpha(\sigma)\right)}{\alpha(\sigma)^2}. \tag{2.15}$$

For chiral multiplets, the 1-loop determinant depends on the R-charge q, the flavor charge F and the mass m and can be written as a product over the weights of the multiplet's representation $\mathscr{R}$ of the gauge group

$$Z^{\mathrm{ch}}(q,F,m) = \prod_{\rho\in\mathscr{R}} S_2\left(\frac{Q}{2}(2-q) - \mathrm{i}Fm + \mathrm{i}\rho(\sigma); b, b^{-1}\right), \tag{2.16}$$

where $Q = b + b^{-1}$. $S_2(x;\omega_1,\omega_2)$ is the double sine function defined as the zeta-function regularization of the infinite product [20]

$$\frac{\prod_{m,n=0}^{\infty}(m\omega_1 + n\omega_2 + x)}{\prod_{k,l=1}^{\infty}(k\omega_1 + l\omega_2 - x)}. \tag{2.17}$$

2.2.3 Enhanced supersymmetry and squashing independence

To get the one-loop determinants for the $\mathscr{N}=4$ multiplets we simply take the product of the determinants for the constituent $\mathscr{N}=2$ multiplets. The $\mathscr{N}=4$ vector multiplet consists of an $\mathscr{N}=2$ vector multiplet and an adjoint chiral multiplet with R-charge $q=1$ and a flavor charge 1. Thus it has the one-loop determinant

$$Z^{\text{vec}}_{\mathscr{N}=4} = Z^{\text{vec}}_{\mathscr{N}=2} Z^{\text{ch}}(1,1,m_*). \tag{2.18}$$

Similarly, the hypermultiplet consists of a pair of R-charge $\frac{1}{2}$ chiral multiplets in conjugate representations of the gauge group

$$Z^{\text{hyp}}(m,m_*) = Z^{\text{ch},\mathscr{R}}(\frac{1}{2},-\frac{1}{2},m_*+2m)Z^{\text{ch},\overline{\mathscr{R}}}(\frac{1}{2},-\frac{1}{2},m_*-2m). \tag{2.19}$$

We have left the value of the special mass parameter m_* from subsection 2.1.3 generic. Thus the background only preserves four supercharges. For the vector and the hypermultiplet we have also made explicit the flavor charges for the coupling to the special mass parameter. Any additional mass parameter m coupling with charge 1 to a flavor symmetry for the hypermultiplet then enters as indicated in (2.19).

To obtain the one-loop determinants for an $\mathscr{N}=4$ gauge theory preserving six supercharges, we set the value of the special mass paramter m_* to $m_* = \pm\mathrm{i}\frac{b-b^{-1}}{2}$. Using the periodicity property of the double sine function

$$S_2(x+\omega_1;\omega_1,\omega_2) = \frac{S_2(x;\omega_1,\omega_2)}{2\sin(\frac{\pi x}{\omega_2})}, \tag{2.20}$$

we find that for this special value of m_* an infinite number of factors cancel in the one-loop determinants of the vector- and hypermultiplet and they simplify to

$$Z^{\text{vec}}_{\mathscr{N}=4} = b^{\mp r_G} \prod_{\alpha\in\Delta_+} \frac{\sinh(\pi b^{\mp 1}\langle\sigma,\alpha\rangle)^2}{\langle\sigma,\alpha\rangle^2}, \tag{2.21}$$

$$Z^{\text{hyp}} = \prod_{\rho\in\mathscr{R}} \frac{1}{\cosh(\pi b^{\mp 1}(\langle\sigma,\rho\rangle + m))}. \tag{2.22}$$

These cancellations in the determinants are a direct consequence of the additional preserved supercharges.

Plugging the simplified expressions into the integral expression (2.13) one observes that by rescaling the integration variable in the matrix model the dependence on the squashing parameter drops out of the partition function. Thus, for $\mathcal{N}=4$ gauge theories preserving six supercharges the partition function is squashing independent

$$Z\left(b, m_* = \pm \mathrm{i}\frac{b-b^{-1}}{2}\right) = Z(b=1, m_*=0). \tag{2.23}$$

Understanding the origin of this squashing independence requires the technical results of [21]. Here they show that the partition function of a gauge theory on a three manifold does not depend on the metric but on the so-called transversally holomorphic foliation (THF). In paper I, we checked that indeed the THF for the additional two-supercharges preserved on the squashed three-sphere is the same as on the round three-sphere.

2.3 Line operators

An important observable in gauge theories are line operators. Compactifying the theory on a sphere, infinite lines get mapped to loops. In this section we discuss some loop operators of 3d supersymmetric gauge theories on the squashed three-sphere and their localized expectation values.

The squashed three-sphere background we presented in subsection 2.1.1 preserves an $\mathrm{SU}(2)\times\mathrm{U}(1)$ rotation group. Inserting a loop operator into this setting preserves at most a $\mathrm{U}(1)$ subgroup of the $\mathrm{SU}(2)$ symmetry. This implies that loop operators preserve at most two of the four supercharges of the squashed three-sphere. For the maximal case the loop operators are called supersymmetric or $\frac{1}{2}$-BPS. To preserve the two supercharges the supersymmetric loop operators must live on a closed orbit of the residual $\mathrm{U}(1)\subset\mathrm{SU}(2)$ rotations. For generic squashing parameters this restricts the loops to either of the two circles at $\theta=0,\pi$. Following the same reasoning for the $\mathrm{SU}(2|1)\ltimes\mathrm{SU}(1|1)$ supergroup preserved by the background in subsection 2.1.3, we see that the $\mathcal{N}=4$ loop operators can be $\frac{1}{3}$ or $\frac{2}{3}$-BPS and are restricted to the same great circles.

2.3.1 Wilson loops

For 3d $\mathcal{N}=2$ gauge theories, a Wilson loop operator can be defined by the insertion into the path integral of a path ordered exponential

$$\mathrm{Tr}_{\mathscr{R}}\mathscr{P}\exp\left(\mathrm{i}\oint_\gamma (A - \mathrm{i}x\sigma|\mathrm{d}\gamma|)\right) \tag{2.24}$$

which depends on the representation $\mathscr{R}$ of the gauge group and the closed path γ. Requiring the Wilson loop on the squashed three-sphere to preserve two supercharges fixes the value of x to ± 1 and the path γ to one of the circles at $\theta = 0, \pi$.

The supersymmetric localization argument we sketched in 2.2.1 also follows through in the presence of $\frac{1}{2}$-BPS Wilson loops. For example, with $x = 1$ the expectation value for the Wilson loop at $\theta = 0$ is equal to the matrix model (2.13) with an insertion of [22, 23]

$$\mathrm{Tr}_{\mathscr{R}} \exp(2\pi b\sigma). \tag{2.25}$$

For $\mathscr{N} = 4$ supersymmetric gauge theories Wilson loops can be defined in the same way. If the background preserves six supercharges Paper I shows that for any choice of $x = \pm 1$ and of sign in (2.10) only one of the two circles at $\theta = 0, \pi$ supports a Wilson loop preserving four supercharges while the Wilson loop on the other circle only preserves two supercharges. The expectation value of the $\frac{2}{3}$-BPS Wilson loop is independent of the squashing parameter b in the same way as the $\mathscr{N} = 4$ partition functions discussed at the end of subsection 2.1.4.

2.3.2 Vortex loops

The effect of a vortex line in a three dimensional gauge theory is that all fields charged under the corresponding symmetry pick up a fixed monodromy on a loop around the line. For abelian symmetries the explicit construction of $\frac{1}{2}$-BPS vortex loops in $\mathscr{N} = 2$ theories was discussed by [24, 23] and on the squashed three-sphere they are allowed on both circles $\theta = 0, \pi$. For an abelian gauge vortex loop the localized expectation value is a prefactor to the partition function while for an abelian flavor symmetry it shifts the mass parameter.

For vortex loops of non-abelian flavor symmetries in $\mathscr{N} = 4$ theories a UV construction was given in [25] and it couples the 3d theory to a 1d theory living on the loop. In Paper II we argued that this coupling is also possible on the squashed three-sphere and can preserve four of the six supercharges. Notably we find that a Wilson and a vortex loop can preserve the same four supercharges but they live on different circles, one at $\theta = 0$, the other at $\theta = \pi$. Localizing the partition function of the coupled 3d/1d system, the vortex loop expectation value is found to be an insertion of the index of the 1d theory into the matrix model for the 3d theory. As for the Wilson loops, the $\frac{2}{3}$-BPS vortex loop expectation value is squashing independent.

3. Testing dualities on the squashed three-sphere

There is a vast zoo of three dimensional $\mathcal{N} \geq 4$ supersymmetric theories in the UV. However, once we flow to the IR many of these theories are related by dualities. The main type of dualities considered in this thesis is mirror symmetry [26]. 3d $\mathcal{N} = 4$ gauge theories allow for two types of vacua. On the Coulomb branch the scalars in the vectormultiplet take non-trivial vacuum expectation values while on the Higgs branch the hypermultiplet scalars have non-trivial vacuum expectation values. Two theories are mirror symmetric if they are related by the exchange of their Coulomb and Higgs branches.

In the first section of this chapter we present some examples of 3d $\mathcal{N} \geq 4$ gauge theories and the dualities they satisfy. In the second section we give an overview of Paper II's results testing dualities using supersymmetric localization.

3.1 Some 3d $\mathcal{N} \geq 4$ gauge theories

3.1.1 Hanany-Witten brane construction and mirror symmetry

One large class of 3d $\mathcal{N} = 4$ gauge theories can be constructed from branes following work by Hanany and Witten [27]. Consider NS5 branes in type IIB string theory spanning the $0,1,2,4,5,6$ directions and at distinct positions in a compact 3 direction. Between neighboring pairs of NS5 branes we suspend stacks of D3 branes extended along the $0,1,2,3$ directions. The resulting effective theory on the D3 branes is a three dimensional $\mathcal{N} = 4$ $\mathrm{U}(N_i)$ super Yang-Mills theory where N_i is the number of D3 branes suspended between the i-th and $(i+1)$-st NS5 branes. Strings spanning between D3 branes on either side of an NS5 brane correspond to a hypermultiplet in the $(N_i, \overline{N}_{i+1})$ representation[1]. In addition we can intersect the D3 branes with D5 branes spanning the $0,1,2,7,8,9$ directions. A D5 brane intersecting the i-th stack of D3 branes corresponds to having a hypermultiplet in the fundamental representation of the $\mathrm{U}(N_i)$ gauge group. The double covers of the rotations in the $4,5,6$ and $7,8,9$ directions are the $\mathrm{SU}(2)_C$ and $\mathrm{SU}(2)_H$ symmetries of Coulomb and Higgs branch respectively. In Figure 3.1 we show a simple example of this brane setup.

[1] As the hypermultiplet is made from two chiral multiplets in conjugate representations, we could as well use the conjugate representation $(\overline{N}_i, N_{i+1})$ to designate the hypermultiplet.

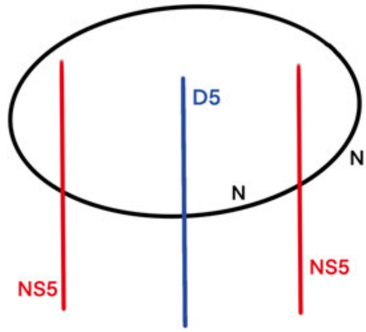


Figure 3.1. We show a simple example of the Hanany-Witten brane setup. A stack of N D3 branes (black) spanning the compact direction is intersected by two NS5 branes (red) and one D5 brane (blue). The corresponding gauge theory has gauge group $\mathrm{U}(N) \times \mathrm{U}(N)$, two hypermultiplets in the $(N, \overline{N})$ representation and a single hypermultiplet in the fundamental representation of only one of the gauge group factors.

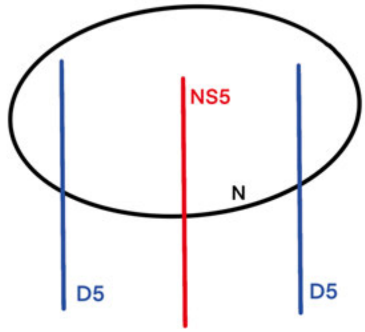


Figure 3.2. We show the brane setup for the mirror theory dual to the example in Figure 3.1. The corresponding gauge theory has gauge group $U(N)$, a hypermultiplet in the adjoint representation and a pair of hypermultiplets in the fundamental representation.

To get a gauge theory duality for this class of gauge theories one may start from the S-duality of Type IIB string theory. This maps the D5 branes into NS5 branes and vice-versa while the D3 branes are mapped to themselves. Combining this with a rotation that exchanges the $4,5,6$ directions with the $7,8,9$ directions, we end up with a brane configuration of the same type as we started from. For the gauge theories on both sides of this duality it follows then from the construction that their IR fixed points are related by the exchange of the Coulomb and Higgs branches. This type of duality is called mirror symmetry [26, 28, 27]. In Figure 3.2 we show the mirror dual of the example in Figure 3.1. Keeping track of the positions of the branes in the $4,5,6$ and $7,8,9$ directions it follows that under mirror symmetry masses of fundamental hypermultiplets get exchanged with FI parameters.

3.1.2 ABJM and maximally supersymmetric Yang-Mills theory

In three dimensions, the Chern-Simons action has conformal symmetry. However, for a single $U(N)_k$ gauge field, where k is the level, it admits a supersymmetric extension only with at most $\mathcal{N} = 3$ supersymmetry. In [29] it was found that with a gauge group $U(N)_k \times U(N)_{-k}$ and with opposite Chern Simons levels k and $-k$ respectively for the two factors of the gauge group more supersymmetry can be preserved. To explicitly construct a three-dimensional $\mathcal{N} = 6$ superconformal gauge theory they coupled the vectormultiplets to a pair of hypermultiplets in the bi-fundamental representation. This theory is commonly known as ABJM theory. For $k = 1,2$ the supersymmetry enhances further to the maximal $\mathcal{N} = 8$ superconformal symmetry. Notably for $k = 1$ it describes the near horizon limit of N M2 branes in flat space.

Another theory that is supposed to describe M2 branes in flat space and thus should be dual to ABJM theory is the IR limit of the 3d $\mathcal{N} = 8$ SYM. This maximal super Yang-Mills theory however is not amenable to the localization method described in the previous chapter [30] as the R-symmetry in the infrared is accidental and does not coincide with the manifest R-symmetry of the UV theory. To circumvent this problem, we can use the mirror symmetry of the previous subsection. 3d $\mathcal{N} = 8$ SYM is a gauge theory with a single adjoint hypermultiplet meaning that as a Hanany-Witten brane configuration it is a stack of D3 branes intersected by a single NS5 brane. Applying mirror symmetry, the dual description has a single D5 brane intersecting the D3 branes, *i.e.* it is the gauge theory with one ajoint and one fundamental hypermultiplet. In the following all computations for the $\mathcal{N} = 8$ theory will be performed in this dual theory.

The duality of ABJM theory and $\mathcal{N} = 8$ SYM is also an example of mirror symmetry. It belongs to a more general class than the previous subsection and we will not discuss any brane realization [31].

3.2 Tests of dualities

Duality of two quantum theories implies the existence of a map between expectation values of observables in both theories. For three dimensional $\mathcal{N} = 4$ gauge theories supersymmetric localization facilitates the computation of partition functions and loop operator observables. The first to use this to test the mirror symmetry outlined in the previous section were [30, 24]. Starting with [32] it was also applied to another type of dualities called Seiberg-like dualities. Examples of this type were first found in [33, 34]. Based on van de Bult's results for transformations of hyperbolic hypergeometric integrals [35], a very large class of these Seiberg-like dualities were tested on the squashed sphere [36]. By contrast the mirror symmetries have largely resisted study on the squashed sphere. Here we describe our results on mirror symmetry dualities.

3.2.1 With six preserved supercharges

Adding mass parameters $\vec{m}$ and FI parameters $\vec{\eta}$ the squashing independence of partition functions from subsection 2.2.3 takes the form

$$Z\left(1;\vec{m};\vec{\eta};m_* = 0\right) = Z\left(b;b\vec{m};\frac{1}{b}\vec{\eta};m_* = \mathrm{i}\frac{b-b^{-1}}{2}\right) \tag{3.1}$$

$$= Z\left(b;\frac{1}{b}\vec{m};b\vec{\eta};m_* = -\mathrm{i}\frac{b-b^{-1}}{2}\right), \tag{3.2}$$

generalizing equation (2.23). In Paper II we observed that this map of partition functions allows us to lift results from the round sphere to the squashed sphere with six supercharges.

For a mirror dual pair of gauge theories from the Hanany-Witten brane setup it was found on the round sphere that the parition functions of the "electric" and "magnetic" theories match [30],

$$Z_{el}(1;\vec{m};\vec{\eta};m_* = 0) = Z_{mag}(1;\vec{\eta};\vec{m};m_* = 0), \tag{3.3}$$

if the mass parameters of the electric theory equal the FI parameters in the magnetic theory and *vice versa*. Using equation (3.1) this lifts to the same statement on the squashed sphere

$$Z_{el}\left(b;\vec{m};\vec{\eta};m_* = \mathrm{i}\frac{b-b^{-1}}{2}\right) = Z_{mag}\left(b;\vec{\eta};\vec{m};m_* = -\mathrm{i}\frac{b-b^{-1}}{2}\right) \tag{3.4}$$

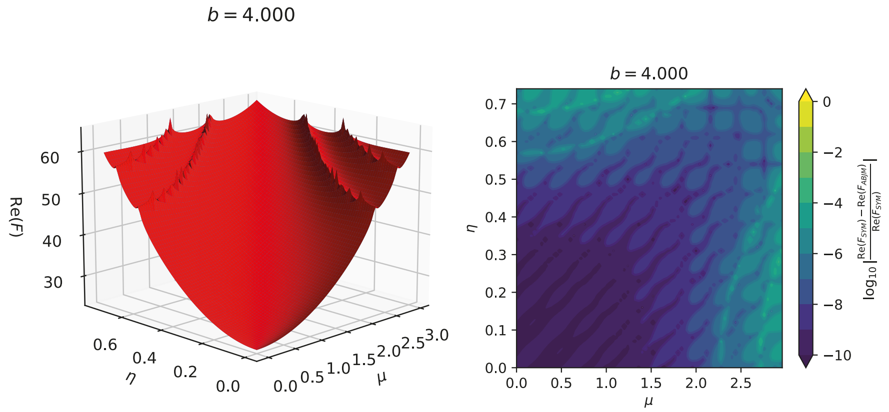
with the additional sign flip for the distinguished mass parameter m_*. This sign flip comes directly from the exchange of the Coulomb and Higgs branches as the parameter m_* couples to the axial subgroup of the $\mathrm{SU}(2)_C \times \mathrm{SU}(2)_H$ symmetry group of the two branches.

For ABJM theory the map from the squashed sphere to the round sphere is more complicated [9], as is the round sphere mapping of the mass and FI parameters in the duality to $\mathcal{N} = 8$ SYM [30]. However the lifting of matching partition functions still works and the parameter mapping is the same as on the round sphere.

Even for loop operators the results of [24,25] can be lifted. The expectation values of a $\frac{2}{3}$-BPS Wilson loop matches to the expectation value of a $\frac{2}{3}$-BPS vortex loop in the mirror dual theory.

3.2.2 Numerical tests with four supercharges

To test the ABJM-$\mathcal{N} = 8$ SYM duality on the squashed sphere with less than six supercharges we numerically evaluate both partition functions for low rank of the gauge group for the special mass parameter m_* set to zero. Localization reduces the evaluation of the partition function to an integral over $\mathbb{R}^{\mathrm{rank}G}$. For



(a) Real part of the free energy for ABJM (blue) and $\mathscr{N} = 8$ SYM (red).

(b) Relative difference of the two numerically evaluated partition functions.

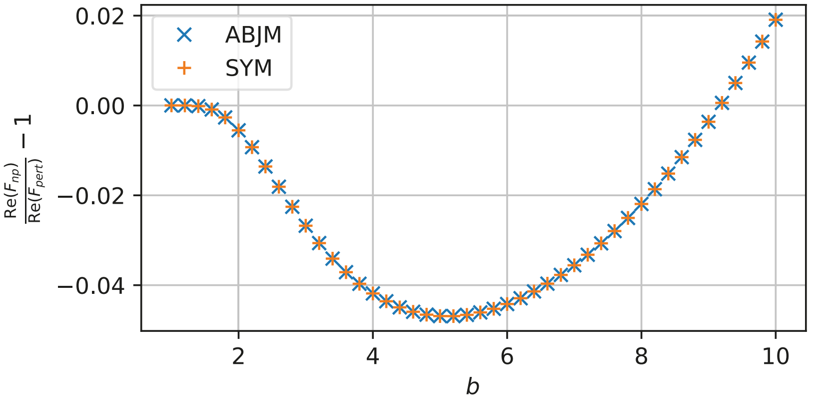
Figure 3.3. Comparing the ABJM and $\mathscr{N} = 8$ SYM partition functions at rank 2 at $b = 4$ for a range of FI parameters η and masses μ of the ABJM theory. Note that in in subfigure *(a)* the ABJM result is not visible as it agrees with the SYM result within the precision of the plotting. The plots are taken from Paper II.

the numerical evaluation we simply replace this integral by a sum over a finite sized equally spaced grid. While this is a very simple algorithm, it is fast enough for evaluations at low rank. Away from the zeros of the integral we find that it has sufficient precision to give a reliable result.

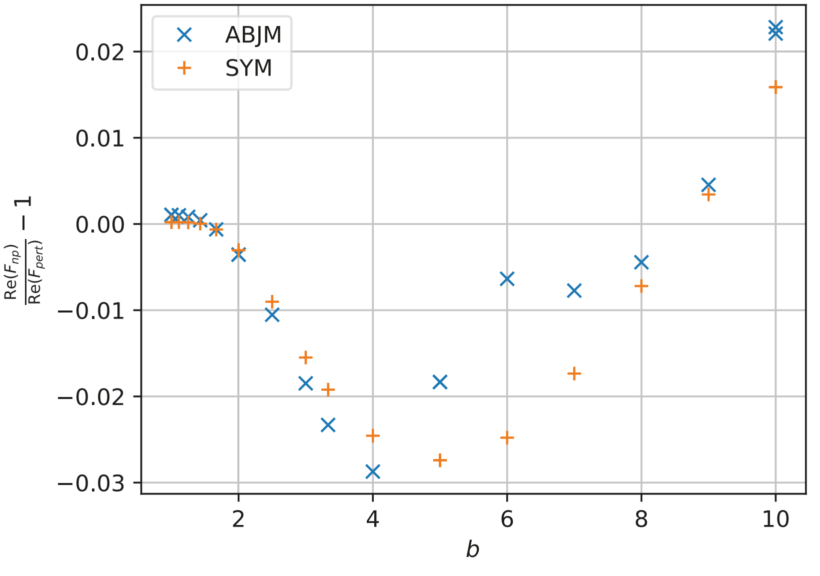
For $\mathscr{N} = 8$ SYM with gauge group $\mathrm{U}(2)$ and ABJM $\mathrm{U}(2)_1 \times \mathrm{U}(2)_{-1}$ we show in Figure 3.3 the results of the numeric evaluation at $b = 4$ for the bi-fundamental mass $\mu \in [0, 2.98]$ and the FI parameter $\eta \in [0, 0.69]$. We compile further results for b in the range $[1, 6]$ into videos showing the evolution of the partition functions under variation of the squashing parameter, see [37]. Away from what appears as ridges in the real part of the free energy $F = -\log(Z)$ we find agreement within the expected precision.

The ridges in the plot of the real part of the free energy correspond to zeros of the partition function, also called Lee-Yang zeros. For the round sphere these zeros were already observed by [38, 39]. We observe that under squashing the Lee-Yang zeros move further from the origin in the (μ, η) parameter space.

For large rank, [40] conjectured an expression for the ABJM partition function dependent on the rank N and the squashing parameter b at parameter values $\eta = \mu = 0$. For rank two and three we compare this against our numeric results, see Figure 3.4. The difference of the results is expected to measure non-perturbative contributions to the ABJM free energy from instantons. *In the published version of Paper II, Figure 3.4a was mistakenly reproduced twice. Figure 3.4b should have been reproduced as Figure 8(c) of Paper II.*



(a) Rank 2



(b) Rank 3

Figure 3.4. Comparing the numerical results for the ABJM and $\mathcal{N} = 8$ SYM partition functions to the conjectured large N expression. We plot the relative difference of the non-perturbative numeric result to the perturbative large N conjecture.

Part II:
Four dimensions

4. Supersymmetric gauge theories on the squashed four-sphere

In this chapter we discuss supersymmetric gauge theories on the squashed four-sphere. We first give a brief description of $\mathcal{N}=2$ gauge theories on curved manifolds using the example of the squashed sphere and their localized partition functions. After that we discuss some of the results from Papers III and IV.

4.1 Supersymmetry on the squashed four-sphere

Unlike in three dimensions, the squashed four-sphere is defined as an ellipsoid embedded into $\mathbb{R}^5$ as desribed by the equation

$$\frac{x_1^2+x_2^2}{l^2}+\frac{x_3^2+x_4^2}{\widetilde{l}^2}+\frac{x_5^2}{r^2}=1. \tag{4.1}$$

We recover the round sphere by setting $l=\widetilde{l}=r$. The metric of squashed S^4 can be written in the form [41]

$$\begin{aligned} \mathrm{d}s^2 &= (g^2+h^2)\mathrm{d}\rho^2+2\sin\rho\, fh\mathrm{d}\theta\mathrm{d}\rho \\ &\quad +\sin^2\rho\left(f^2\mathrm{d}\theta^2+l^2\cos^2\theta\mathrm{d}\phi^2+\widetilde{l}^2\sin^2\theta\mathrm{d}\chi^2\right), \end{aligned} \tag{4.2}$$

$$\begin{aligned} f &= \sqrt{l^2\sin^2(\theta)+\widetilde{l}^2\cos^2(\theta)}, \\ g &= \sqrt{r^2\sin^2\rho+l^2\widetilde{l}^2 f^{-2}\cos^2\rho}, \\ h &= \frac{\widetilde{l}^2-l^2}{f}\cos\rho\sin\theta\cos\theta. \end{aligned}$$

Supersymmetry on this background was first discussed by [41] but we will present it in the language of [17]. We start with the Killing vector

$$v=\frac{1}{l}\partial_\phi+\frac{1}{\widetilde{l}}\partial_\chi \tag{4.3}$$

and choose the two functions $s=2\sin^2\frac{\rho}{2}$, $\widetilde{s}=2\cos^2\frac{\rho}{2}$ such that $s\widetilde{s}=\|v\|^2$. On the southern hemisphere we can then define the spinors

$$\zeta_\alpha^i=\frac{\sqrt{s}}{2}\delta_\alpha^i, \qquad \overline{\chi}_i=\frac{1}{s}v^\mu\overline{\sigma}_\mu\zeta_i, \tag{4.4}$$

with α an $\mathrm{SU}(2)_l$ Lorentz index, i an $\mathrm{SU}(2)_R$ R-symmetry index and the matrices $\sigma_\mu = (-\mathrm{i}\tau^a, 1)$ and $\overline{\sigma}_\mu = (\mathrm{i}\tau^a, 1)$ defined in terms of the Pauli matrices τ^a $(a = 1,2,3)$. To get globally well defined spinors we then apply the $\mathrm{SU}(2)_R$ transformation

$$U_i^j = \mathrm{i}\frac{v^\mu}{\|v\|}\sigma_{\mu i}{}^j \tag{4.5}$$

to get for the northern hemisphere the expressions

$$\overline{\chi}_i^{\dot\alpha} = -\mathrm{i}\frac{\sqrt{\widetilde{s}}}{2}\delta_i^{\dot\alpha}, \qquad \zeta^i = -\frac{1}{\widetilde{s}}v^\mu\sigma_\mu\overline{\chi}^i, \tag{4.6}$$

with $\dot\alpha$ an $\mathrm{SU}(2)_r$ Lorentz index.

Besides the graviton, the bosonic fields in the $\mathscr{N} = 2$ Poincaré supergravity multiplet [42] are an $\mathrm{SU}(2)_R$ connection $V_\mu{}^i{}_j$, a scalar N, a one-form G_μ, a two-form $W_{\mu\nu}$, a scalar $\mathrm{SU}(2)_R$ triplet S_{ij} and a closed two-form $\mathscr{F}_{\mu\nu}$. The corresponding Killing spinor equations admit the globally defined spinors $\zeta, \overline{\chi}$ as a solution. This fixes the fields in the background Poincaré supergravity multiplet. The explicit expressions for these background fields in terms of the functions $s, \widetilde{s}$ and the spinors $\zeta, \overline{\chi}$ are given in [17] but will not be of importance for the rest of this thesis.

Near the poles of the sphere the preserved supersymmetry simplifies. From (4.6) we see that at the north pole $\rho = 0$ of the sphere the Killing spinors become

$$\overline{\chi}_i^{\dot\alpha} = -\frac{\mathrm{i}}{\sqrt{2}}\delta_i^{\dot\alpha}, \qquad \zeta^i = 0.$$

From the diagonal form of $\overline{\chi}$ we see that the preserved supercharge is that of topologically twisted $\mathscr{N} = 2$ theory [43]. Of the rotations $\mathrm{SU}(2)_l \times \mathrm{SU}(2)_r$ the right factor has been identified with the R-symmetry $\mathrm{SU}(2)_R$. Similarly, we see from (4.4) that at the south pole $\rho = \pi$ of the sphere the $\mathrm{SU}(2)_l$ gets twisted with the R-symmetry. Taking also the background fields into consideration, the supersymmetry near the poles approaches that of the Ω-background [4,44].

4.1.1 Gauge theories and cohomological variables

The vector multiplet of four dimensional $\mathscr{N} = 2$ supersymmetry contains the gauge field A_μ, a complex scalar X, gauginos $\lambda_{i\alpha}, \widetilde{\lambda}^i_{\dot\alpha}$ in the (anti-)fundamental of $\mathrm{SU}(2)_R$ and an auxiliary field D_{ij} in the triplet representation of $\mathrm{SU}(2)_R$. Given that on the squashed sphere we are only preserving the Killing spinors $\zeta^i, \overline{\chi}_i$, it is of interest to rewrite the multiplet using cohomological fields [17]

$$\begin{aligned} \psi_\mu &= \zeta_i\sigma_\mu\overline{\lambda}^i + \overline{\chi}^i\overline{\sigma}_\mu\lambda_i, &\qquad \eta &= \zeta_i\lambda^i + \overline{\chi}^i\overline{\lambda}_i, \\ \phi &= \widetilde{s}X + s\overline{X}, & \varphi &= -\mathrm{i}(X - \overline{X}), \end{aligned} \tag{4.7}$$

as well as a pair of two-forms $\chi_{\mu\nu}, H_{\mu\nu}$. The preserved supersymmetry then acts as

$$\begin{aligned} &\delta A = \mathrm{i}\psi, && \delta\psi = \iota_v F + \mathrm{i} d_A \phi, && \delta\phi = \iota_v \psi, \\ &\delta\varphi = \mathrm{i}\eta, && \delta\eta = \mathscr{L}_v^A \varphi - [\phi, \varphi], && \\ &\delta\chi = H, && \delta H = \mathrm{i}\mathscr{L}_v^A \chi - \mathrm{i}[\phi, \chi], && \end{aligned} \tag{4.8}$$

where $F = \mathrm{d}_A A$ and we use the covariant version of the Lie derivative $\mathscr{L}_v^A = \mathscr{L}_v - \mathrm{i}[\iota_v A, \bullet]$ with $\iota_v A$ the contraction of the vector v with the one-form A. In these cohomological variables the Yang-Mills action takes the form

$$S_{YM} = \frac{1}{g_{YM}^2} \int_M \Omega \wedge \mathrm{Tr}(\phi + \psi + F)^2, \tag{4.9}$$

up to δ-exact terms. The multiform Ω is a sum of a zero-, two- and four-form, and closed under the action of the equivariant differential $\mathrm{i}d + \iota_v$. By definition, the integration is over the four-form component of the integrand. Up to δ-exact terms the integrand is equivariantly closed

$$(\mathrm{i}d + \iota_v)\Omega \wedge \mathrm{Tr}(\phi + \psi + F)^2 = \delta(\mathrm{Tr}(\phi + \psi + F)^2). \tag{4.10}$$

We can thus apply the Atiyah-Bott-Berline-Verne formula to localize the action to the fixed points of the Killing vector v [45–47]. The fixed points are the north and south poles and thus up to δ-exact terms the action is [48]

$$S_{YM} = -4\pi\mathrm{i}\tau l\tilde{l}\,\mathrm{Tr}(X^2)(N) + 4\pi\mathrm{i}\overline{\tau} l\tilde{l}\,\mathrm{Tr}(\overline{X}^2)(S), \tag{4.11}$$

where we have introduced the complexified coupling $\tau = \frac{\theta}{2\pi} + \frac{4\pi\mathrm{i}}{g_{YM}^2}$.

The 4d hypermultiplet consists of an $\mathrm{SU}(2)_R$ doublet of bosons q_A, a pair of fermions $\psi, \overline{\psi}$ and a doublet of auxiliary fields F_A. The hypermultiplet also admits a rewriting in cohomological variables [49]. The Lagrangian for the hypermultiplet is

$$L_{hyp} = \delta V_{hyp}, \tag{4.12}$$

for a specific function V_{hyp} of the hyper- and vectormultiplet fields, and is thus supersymmetrically exact for the action of the preserved supercharges [41,49].

4.2 The localized partition function

The partition function of supersymmetric gauge theories on the squashed four-sphere was first localized by [41], generalizing the round sphere computation of Pestun [5]. The localized partition function takes the form of an integral over the Cartan subalgebra of the gauge group

$$Z = \int d^{r_G}\sigma \left(\prod_{\alpha\in\Delta_+} |\alpha(\sigma)|^2 \right) Z_{\text{classical}}(\sigma) Z_{\text{Nek}}(\sigma, m, q, \overline{q}) \times Z^{\text{vec}}_{1-\text{loop}}(\sigma) Z^{\text{hyp}}_{1-\text{loop}}(\sigma, m). \quad (4.13)$$

The only parameter of the squashed sphere this partition function depends on explicitly is $b = \sqrt{l/\widetilde{l}}$. For the coupling dependence of the non-perturbative contributions Z_{Nek} we use $q = e^{2i\pi\tau}$. The bare hypermultiplet mass parameters have been rescaled to $m = \sqrt{l\widetilde{l}}m_{\text{bare}}$.

As noted in the previous section, the Yang-Mills action is not exact but gets contributions from the scalar fields X, $\overline{X}$ which on the localization locus are real and equal to the constant $\frac{\sigma}{2\sqrt{l\widetilde{l}}}$. Plugging this into equation (4.11) we find the value of the classical action,

$$\log Z_{\text{classical}} = -\frac{8\pi^2}{g^2_{\text{YM}}}\text{Tr}(\sigma^2). \quad (4.14)$$

The BPS equations of 4d SYM on the squashed sphere allow for instantons, *i.e.* anti-self-dual gauge fields $F^+ = 0$, at the north pole and anti-instantons, *i.e.* self-dual gauge fields $F^- = 0$, at the south pole. The contributions of these saddles to the partition function we put together into Z_{Nek}. They factor into an instanton contribution $Z^{\text{inst}}(\sigma, m, q)$ from the north pole and an anti-instanton contribution $Z^{\text{anti}-\text{inst}}(\sigma, m, \overline{q}) = \overline{Z^{\text{inst}}}$ from the south pole. As these (anti-)instantons are localized at the poles of the sphere their partition function is determined by the supersymmetry near the poles. Therefore Z^{inst} is equal to the SYM partition function on the four dimensional Ω-background, also called the Nekrasov partition function [4, 44].

To write the 1-loop contributions to the localized partition function, we define the Upsilon function $\Upsilon(x;b)$ [50]. This function is given by the regularized infinite product

$$\Upsilon(x;b) = \prod_{n_1,n_2\geq 0} \frac{\left((n_1+1)b + (n_2+1)b^{-1} - x\right)\left(n_1 b + n_2 b^{-1} + x\right)}{\left((n_1+\frac{1}{2})b + (n_2+\frac{1}{2})b^{-1}\right)^2}. \quad (4.15)$$

Notably it satisfies $\Upsilon(\frac{b+b^{-1}}{2}) = 1$. For a hypermultiplet in a representation $\mathscr{R}$ of the gauge group and for a vector multiplet the one-loop contributions to the partition function are respectively

$$Z^{\text{hyp}}_{1-\text{loop}}(\sigma, m) = \prod_{\rho\in\mathscr{R}} \frac{1}{\Upsilon\left(\frac{b+b^{-1}}{2} + i\rho(\sigma) + im; b\right)}, \quad (4.16)$$

$$Z^{vec}_{1-\text{loop}}(\sigma) = \left(\Upsilon'(0;b)\right)^{r_G} \prod_{\alpha\in\Delta_+} \frac{\Upsilon(i\alpha(\sigma);b)\Upsilon(-i\alpha(\sigma);b)}{\alpha(\sigma)^2}. \quad (4.17)$$

The product for the hypermultiplet is over the weights of the representation $\mathscr{R}$ and for the vectormultiplet the product is over the positive roots of the gauge Lie algebra. As for the instanton contributions, the 1-loop determinants only depend on the supersymmetry near the poles of the sphere.

4.2.1 Local supersymmetry enhancement in $\mathscr{N}=2^*$

In four dimensions the maximal super Yang-Mills theory has $\mathscr{N}=4$ supersymmetry. Its massive deformation is the so-called $\mathscr{N}=2^*$ theory, *i.e.* the $\mathscr{N}=2$ supersymmetric gauge theory with a massive adjoint hypermultiplet. It can be obtained from the $\mathscr{N}=4$ theory by introducing a background $\mathscr{N}=2$ vector multiplet for a $\mathrm{U}(1)$ subgroup of the $\mathrm{SU}(4)_R$ R-symmetry. The mass parameter corresponds to the scalar field in this background vector multiplet. To preserve rigid $\mathscr{N}=2$ supersymmetry on the squashed sphere the auxiliary field in the background vector multiplet has to be proportional to the mass.

From the perspective of $\mathscr{N}=4$ supersymmetry, the preserved supersymmetry at the poles of the squashed four-sphere corresponds to the so-called half-twist. The $\mathrm{SU}(4)_R$ R-symmetry group of $\mathscr{N}=4$ has a $\mathrm{SU}(2)^L_R\times\mathrm{SU}(2)^R_R$ subgroup. Focusing on the symmetry at the north pole of the sphere, the half-twist identifies the $\mathrm{SU}(2)^L_R$ subgroup with the $\mathrm{SU}(2)_r$ subgroup of the Lorentz group $\mathrm{SO}(4)\simeq\mathrm{SU}(2)_l\times\mathrm{SU}(2)_r$. Adding a background field for a mass coupling to $\mathrm{SU}(2)^R_R$, one finds enhanced supersymmetry at fine-tuned values of the mass parameter, *cf.* Paper IV and [51,52]. The interesting point is $m=\pm\mathrm{i}\frac{b-b^{-1}}{2}$. Then the $\mathrm{SU}(2)^R_R$ symmetry gets identified with the $\mathrm{SU}(2)_l$ Lorentz symmetry. This is the Marcus twist of $\mathscr{N}=4$ [53]. At the same values $m=\pm\mathrm{i}\frac{b-b^{-1}}{2}$ for the adjoint hypermultiplet mass a similar local enhancement of the supersymmetry also happens at the south pole of the squashed four-sphere.

Although the supersymmetry enhancement upon fine-tuning of the $\mathscr{N}=2^*$ mass parameter is only local on the squashed four-sphere, it still implies simplifications of the partition function. For one, the instanton partition function becomes trivial $Z_{\mathrm{inst}}=1$ [54] as a direct cause of the Marcus twist [53]. Second, the 1-loop determinants of the vector- and hypermultiplet cancel for this special value of the mass parameter. With the 1-loop and instanton contributions trivial, the $\mathscr{N}=2^*$ partition function at the fine-tuned value of mass is equal to the $\mathscr{N}=4$ partition function on the round sphere,

$$Z^{\mathscr{N}=2^*}\left(m=\pm\mathrm{i}\frac{b-b^{-1}}{2};b\right)=Z^{\mathscr{N}=4}(m=0;b=1)=\left(\frac{1}{2\mathrm{Im}(\tau)}\right)^{\frac{r_G}{2}}.$$

4.3 (Partially) integrated correlators

For quantum field theories on a Euclidean manifold correlators are computed as functional derivatives of the partition function with respect to background fields. However, if we would like to use the supersymmetric localization results to study correlators we have to be careful. A generic background field breaks supersymmetry and thus cannot be used in conjunction with localization. Following [55, 56] a way out of this is to think of couplings and mass parameters as background fields. Derivatives with respect to these parameters will then give correlation functions where the operator insertions have been integrated over all of space. For gauge theories on the (squashed) four-sphere coupling derivatives stand out from this general discussion. Following equation (4.11) the integrated operator insertions for coupling derivatives localize to operators inserted at the poles of the sphere [57]. In the rest of this section we will highlight the results of Papers III and IV on integrated correlators.

Constraints in $\mathcal{N} = 4$ SYM

The partition function of the $\mathcal{N} = 2^*$ SYM theory on the squashed sphere depends on the coupling τ, the squashing parameter b and the hypermultiplet mass m. Its derivatives evaluated at $(b,m) = (1,0)$ give integrated correlators of $\mathcal{N} = 4$ SYM on the round sphere. In Paper III we observed that the squashing independence for the fine tuned mass $m = \pm \mathrm{i}\frac{b-b^{-1}}{2}$ implies an infinite tower of constraints on the integrated correlators in $\mathcal{N} = 4$. Notably at four-points we showed that this implies the exact relations

$$0 = (\partial_\tau \partial_{\bar{\tau}} \partial_m^2 - \partial_\tau \partial_{\bar{\tau}} \partial_b^2) \log Z\big|_{m=0,b=1}, \tag{4.18}$$

$$0 = (-6\partial_b^2 \partial_m^2 + \partial_m^4 + \partial_b^4 - 15\partial_b^2) \log Z\big|_{m=0,b=1}. \tag{4.19}$$

These relations were earlier observed in [58] by explicitly evaluating the localized partition function.

Partially integrated correlators

In $\mathcal{N} = 2$ gauge theories with matter, flavor symmetry background fields preserving all the supercharges of the squashed four-sphere background do not have to be constant. In Paper IV we showed that if the scalars $X, \overline{X}$ in the background vector multiplet are constant along the flow of the Killing vector v one can find a background auxiliary field D_{ij} and a background gauge field A such that the supercharges are preserved. The localization computation of [17, 49] still goes through the same way as with constant background fields. The resulting partition function only depends on the value of the scalar fields at the poles of the sphere, *i.e.* $X(N)$ and $\overline{X}(S)$. The functional derivatives with respect to these position dependent background fields give us correlators integrated only along the orbits of the Killing vector v.

5. Superconformal field theories on deformed spheres

The classical action of conformal field theories in flat space is indepentent of any scales. However, this scale independence is broken when we consider the quantum theory on a compact manifold. On a compact manifold UV divergences require the introduction of a cutoff scale Λ_{UV}. On a compact four-dimensional manifold $\mathscr{M}$, this implies that the free energy F of the CFT takes the form

$$-F = \log Z_{\mathscr{M}} = A_4 \left(\mathrm{vol}_{\mathscr{M}} \Lambda_{UV}^4\right) + A_2 \left(\mathrm{vol}_{\mathscr{M}} \Lambda_{UV}^4\right)^{1/2} + A_0 \log \left(\mathrm{vol}_{\mathscr{M}} \Lambda_{UV}^4\right) + \text{finite}. \quad (5.1)$$

The coefficients A_4, A_2, A_0 as well as the finite term may depend on all the parameters of the theory, including the number of degrees of freedom and marginal couplings. Notably, this expression for the free energy is ambiguous. A cosmological constant counter-term will contribute to the quartic term, shifting the coefficient A_4. Similarly, the quadratic term corresponds to an Einstein-Hilbert counter-term. In addition, due to the logarithmic term, the finite part of the free energy depends on the regularization scheme. Only the coefficient A_0 is unambiguous.

In Paper III we looked at the free energy of $\mathscr{N} = 2$ superconformal field theories on curved four dimensional manifolds. In the following sections we will review our results on the coefficient A_0 of the logarithmic term and the form of the finite term. We use the localized partition function of $\mathscr{N} = 4$ super Yang-Mills as an example and compare it to our results for the general form of the SCFT free energy.

5.1 Deformations and the Weyl anomaly

The coefficient A_0 of the logarithmic term in the free energy (5.1) is unambiguous in the sense that it cannot be changed by adding counterterms and that it is independent of the regularization scheme we choose. For a conformal field theory on a compact four-manifold it has been related to the conformal anomalies [59–62]

$$A_0 = \frac{1}{64\pi^2} \int \mathrm{d}^4 x \sqrt{g} (-a E_4 + c C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}). \quad (5.2)$$

The two terms contributing to A_0 are examples of the two distinct types of anomalies [63]. The integral of the Euler density E_4 is a topological invariant and therefore we classify the a anomaly as a type-A anomaly. By contrast, a type-B anomaly like the c anomaly term which couples to the square of the Weyl tensor C depends also on smooth deformations of the manifold.

Consider the example of the four-sphere and smoothly deform the metric to $\mathrm{d}^2 s = \mathrm{d}s^2_{\text{round}} + h_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. Then at leading order the change of A_0 takes the form

$$\delta A_0 = \frac{1}{32}\delta_\sigma \int \mathrm{d}^4x\sqrt{g(x)} \int \mathrm{d}^4y\sqrt{g(y)} h^{\mu\nu}(x) h^{\rho\sigma}(y) \langle T_{\mu\nu}(x) T_{\rho\sigma}(y)\rangle, \quad (5.3)$$

where we used that the c anomaly is proportional to the normalization of the stress tensor $T_{\mu\nu}$ [64, 65]. The notation δ_σ represents a scale variation, also called a "Weyl" variation, and is explicitly given by the derivative $\frac{\mathrm{d}}{\mathrm{d}\log\Lambda_{UV}}$. Thus it picks out the logarithmically divergent piece of the integrated two-point correlator coming from the coincident limit $x \to y$.

5.1.1 The case of $\mathcal{N} = 2$ SCFTs

For superconformal field theories the logarithmic term in the free energy gets contributions from all the fields in the background supergravity multiplet. The form of this supersymmetrization of (5.2) can be determined from the super-Weyl anomaly studied in [66–68]. For manifolds smoothly connected to the sphere, the only A-type anomaly contribution to A_0 comes from the Euler characteristic. However, the B-type anomaly gets additional contributions from the supersymmetrization of the $(\text{Weyl})^2$ term. Thus for SCFTs on smooth deformations of the sphere the coefficient of the logarithmically divergent term of the free energy is

$$A_0 = -a + \frac{c}{64\pi^2} \underbrace{\int \mathrm{d}^4x\sqrt{g}\left(C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \ldots\right)}_{\equiv I_{(\text{Weyl})^2}}. \quad (5.4)$$

In Paper III we determine the contributions to the $(\text{Weyl})^2$ term quadratic in the background fields. Functional derivatives of (5.4) can be equated to divergences of SCFT stress-tensor multiplet correlators. From the Ward identities of these correlators we get relations between the different terms in $I_{(\text{Weyl})^2}$ and thus we find that the supersymmetric completion of the $(\text{Weyl})^2$ term takes the form

$$\begin{aligned} I_{(\text{Weyl})^2} = \int \mathrm{d}^4x\sqrt{g}\Big(& C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \frac{3}{2}D^2 - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu} \\ & + 4096\nabla_\mu B^{+\mu\nu}\nabla^\sigma B^-_{\sigma\nu} + 2048 R_{\mu\nu}B^{+\mu\rho}B^{-\nu}{}_\rho \\ & + c_5 B^+_{\mu\nu}B^{+\mu\nu}B^-_{\rho\sigma}B^{-\rho\sigma}\Big). \quad (5.5) \end{aligned}$$

Here F and $\mathscr{F}$ are the field strengths of $\mathrm{U}(1)_r$ and $\mathrm{SU}(2)_R$ gauge fields respectively, B is a two-form field and D is a scalar. All of these fields are part of the supergravity multiplet. The quartic term in the third line is also allowed but it will require studying four-point correlators to determine the coefficient c_5.

5.2 The finite part of the free energy

Marginal operators $\mathscr{C}_i$ have a distinguished role in conformal field theories, since deforming a CFT in flat space by a marginal operator preserves the conformal symmetry. The space of CFTs connected through marginal deformations is called the conformal manifold. A metric on the conformal manifold is given by

$$g_{i\bar{j}} = \langle \mathscr{C}_i(0)\overline{\mathscr{C}}_j(\infty)\rangle_{\mathbb{R}^4}, \tag{5.6}$$

commonly called the Zamolodchikov metric [69]. For superconformal field theories on curved spaces, deformations by marginal operators can be made invariant under the preserved supersymmetry by including couplings to the background fields in the supergravity multiplet. The deformation term then takes the form [70]

$$\frac{1}{\pi^2}\int \mathrm{d}^4x\sqrt{g}\sum_i \tau_i \left(\mathscr{C}_i - \frac{1}{4}\mathscr{A}_i B^+_{\mu\nu}B^{+\mu\nu}\right) + h.c., \tag{5.7}$$

where $\mathscr{A}_i$ is the bottom component of the superconformal multiplet containing the marginal operator $\mathscr{C}_i$. One example of a marginal operator in an SCFT in four dimensions is the super Yang-Mills Lagrangian for the $\mathscr{N}=2$ vector multiplet.

We noted already in equation (4.11) that for $\mathscr{N}=2$ superconformal field theories on the squashed sphere, the super Yang-Mills action localizes to operators at the north and south poles of the sphere. This holds true also for marginal operators in other SCFTs without a Lagrangian description and it implies that

$$\partial_i\overline{\partial}_j \log Z_{\mathscr{M}} = (32l\widetilde{l})^2 \left\langle \mathscr{A}_i(N)\overline{\mathscr{A}}_j(S)\right\rangle_{\mathscr{M}}. \tag{5.8}$$

On the round sphere this result was found by [71, 70]. The right hand side of this equation is related to a two-point function of marginal operators through Ward identities. Hence it is proportional to the Zamolodchikov metric,

$$\partial_i\overline{\partial}_j \log Z_{\mathscr{M}} = \frac{g_{i\bar{j}}}{12}(1+\widetilde{P}(\tau_i,\overline{\tau}_i,b)), \tag{5.9}$$

where we introduced the squashing parameter by defining $l = rb$, $\widetilde{l} = r/b$. Integrating this equation we find that the finite part of the free energy takes the form

$$\log Z_{S^4_b}\Big|_{\mathrm{reg.}} = \frac{K(\tau_i,\overline{\tau}_i)}{12}(1+P(\tau_i,\overline{\tau}_i,b)) + P_h(\tau_i,b) + \overline{P}_h(\overline{\tau}_i,b), \tag{5.10}$$

where $K(\tau_i,\overline{\tau}_i)$ is the Kähler potential. $P_h(\tau_i,b)$ and $\overline{P}_h(\overline{\tau}_i,b)$ are holomorphic and anti-holomorphic functions of the marginal couplings, respectively. The term constant in b matches the round sphere result in [71].

To further fix the form of $P(\tau_i,\overline{\tau}_i,b)$ we start by looking at the part of the Weyl anomaly which involves the Kähler potential. From a general expression in [67, 68], we find in Paper III that

$$\frac{K(\tau_i,\overline{\tau}_i)}{12}P(\tau_i,\overline{\tau}_i,b) = \frac{I_{(\mathrm{Weyl})^2}}{96\pi^2}K(\tau_i,\overline{\tau}_i)\widetilde{c}(\tau_i,\overline{\tau}_i)+\gamma(\tau_i,\overline{\tau}_i,b), \tag{5.11}$$

with $\gamma(\tau_i,\overline{\tau}_i,b)$ an unambiguous, Weyl-invariant and non-local functional of the supergravity background fields. $\widetilde{c}(\tau_i,\overline{\tau}_i)$ is constrained by considering possible counterterms contributing to the finite part of the action. Kähler shifts $K\to K+F+\overline{F}$ by a holomorphic function F of the marginal couplings should be accompanied by a shift in the couplings of counterterms such that the free energy is invariant [72]. In the present case the possible supergravity counterterms are the supersymmetrized Gauss-Bonnet term and the integrated supersymmetrized Weyl squared term $I_{(\mathrm{Weyl})^2}$. This implies

$$K(\tau_i,\overline{\tau}_i)\widetilde{c}(\tau_i,\overline{\tau}_i) = \alpha K(\tau_i,\overline{\tau}_i)+\beta(\tau_i,\overline{\tau}_i), \tag{5.12}$$

for α a theory dependent constant and $\beta(\tau_i,\overline{\tau}_i)$ an unambiguous function of the moduli. Summarizing, we found in Paper III that the finite part of the free energy of an $\mathcal{N}=2$ SCFT on a curved manifold is

$$\begin{aligned}\log Z_{S^4_b}\Big|_{\mathrm{reg.}} = \frac{K(\tau_i,\overline{\tau}_i)}{12}+\frac{\alpha}{96\pi^2}K(\tau_i,\overline{\tau}_i)I_{(\mathrm{Weyl})^2}+\frac{\beta(\tau_i,\overline{\tau}_i)}{96\pi^2}I_{(\mathrm{Weyl})^2}\\ +\gamma(\tau_i,\overline{\tau}_i,b)+P_h(\tau_i,b)+\overline{P}_h(\overline{\tau}_i,b).\end{aligned} \tag{5.13}$$

5.3 Examples using localization

$\mathcal{N}=4$ super Yang-Mills is an example of a superconformal field theory where supersymmetric localization is applicable. In this section we compare the logarithm of the localized partition function of $\mathcal{N}=4$ SYM on the squashed sphere to the general form of the SCFT free energy (5.13). Keeping track of the logarithmically divergent terms in the regularization of the infinite products (4.15) we compute the integrated $(\mathrm{Weyl})^2$ term for the squashed sphere.

5.3.1 Regularizing the one-loop determinants

For $\mathcal{N} = 2^*$ SYM on the squashed four-sphere the one-loop determinant before regularization is a product of factors of the form

$$\frac{\prod\limits_{\substack{m,n\geq 0\\(m,n)\neq(0,0)}} \left(mb+nb^{-1}+\mathrm{i}x\right) \prod\limits_{m,n\geq 0} \left((m+1)b+(n+1)b^{-1}-\mathrm{i}x\right)}{\prod\limits_{m,n\geq 0} \left(\left(m+\frac{1}{2}\right)b+\left(n+\frac{1}{2}\right)+\mathrm{i}x+\mathrm{i}\mu\right)\left(\left(m+\frac{1}{2}\right)b+\left(n+\frac{1}{2}\right)-\mathrm{i}x-\mathrm{i}\mu\right)}. \tag{5.14}$$

Instead of working directly with this expression, we note that all of these factors contributing to the one-loop determinant either have $x = 0$ or they come in pairs with $\pm x$. Therefore we can look at regularizing the infinite product

$$\frac{\prod\limits_{\substack{m,n\geq 0\\(m,n)\neq(0,0)}} \left(mb+nb^{-1}+\mathrm{i}x\right) \prod\limits_{m,n\geq 0} \left((m+1)b+(n+1)b^{-1}+\mathrm{i}x\right)}{\prod\limits_{m,n\geq 0} \left(\left(m+\frac{1}{2}\right)b+\left(n+\frac{1}{2}\right)+\mathrm{i}x+\mathrm{i}\mu\right)\left(\left(m+\frac{1}{2}\right)b+\left(n+\frac{1}{2}\right)+\mathrm{i}x-\mathrm{i}\mu\right)} \tag{5.15}$$

$$= \prod_{n=1}^{\infty}\prod_{m=1}^{n}\left(1-\frac{(n-2m)^2(\gamma')^2}{(n+\mathrm{i}x')^2}\right)\left(1-\frac{(n-2m+1+\mathrm{i}\mu')^2(\gamma')^2}{(n+\mathrm{i}x')^2}\right)^{-1}, \tag{5.16}$$

where we have defined $\gamma' = \sqrt{1-\frac{4}{Q^2}}$, $\mu' = \frac{2\mu}{Q\gamma'}$ and $x' = \frac{2x}{Q}$. We can then use the Taylor series of the logarithm $\log(1-z) = -\sum_{p=1}^{\infty}\frac{z^p}{p}$ to write this in the form

$$\exp\left(-\sum_{p=1}^{\infty}\frac{(\gamma')^{2p}}{p}\sum_{n=1}^{\infty}\frac{1}{(n+\mathrm{i}x')^{2p}}\sum_{m=1}^{n}\left((n-2m)^{2p}-\left(n-2m+1+\mathrm{i}\mu'\right)^{2p}\right)\right) \tag{5.17}$$

$$= \exp\left((1+(\mu')^2)\sum_{p=1}^{\infty}(\gamma')^{2p}f_p(x',\mu')\right). \tag{5.18}$$

The functions $f_p(x',\mu')$ contain the logarithmic divergence of the one-loop determinant. Starting from the definition of the f_p we can manipulate the expression to make the divergence apparent

$$
\begin{aligned}
&f_p(x',\mu') \\
&\equiv -\sum_{n=1}^{\infty}\frac{1}{p(1+(\mu')^2)(n+\mathrm{i}x')^{2p}}\sum_{m=1}^{n}\Big((n-2m)^{2p}-\big(n-2m+1+\mathrm{i}\mu'\big)^{2p}\Big)
\end{aligned} \tag{5.19}
$$

$$
\begin{aligned}
= -\sum_{n=1}^{\infty}\frac{1}{p(1+(\mu')^2)(n+\mathrm{i}x')^{2p}}\sum_{m=1}^{n}\big(&2p(n-2m+1)^{2p-1}(-1-\mathrm{i}\mu') \\
&+p(2p-1)(n-2m+1)^{2p-2}((-1)^2+(\mathrm{i}\mu')^2) \\
&+\mathscr{O}\left((n-2m+1)^{2p-3}\right)\big)
\end{aligned} \tag{5.20}
$$

$$
= -\sum_{n=1}^{\infty}\frac{1}{(n+\mathrm{i}x')^{2p}}\left(n^{2p-1}+\mathscr{O}\left(n^{2p-2}\right)\right) \tag{5.21}
$$

$$
= -\sum_{n=1}^{\infty}\frac{1}{(n+\mathrm{i}x')}+\mathscr{O}\left((n+\mathrm{i}x')^{-2}\right). \tag{5.22}
$$

This last line makes clear the logarithmic divergence $-\log(\Lambda'_{UV})$ of the functions f_p and we see that the regularized expression is a sum of the digamma function and its derivatives[1]. The UV scale Λ'_{UV} corresponds to a cutoff for the summation but on the squashed sphere it is not obvious how to define it. We propose that the correct cutoff Λ'_{UV} equals $\frac{2\Lambda_{UV}}{Q}$ where Λ_{UV} is kept constant under squashing. This form of the cutoff is suggested by the rescaled $x'=\frac{2x}{Q}$. Summing the logarithmic divergences from the f_p functions, we see that the infinite product (5.15) diverges as

$$
\exp\left(-\log(\Lambda'_{UV})\left(\frac{Q^2}{4}-1+\mu^2\right)\right). \tag{5.24}
$$

5.3.2 The Abelian maximal super Yang-Mills

As the first example we consider the Abelian $\mathscr{N}=4$ super Yang-Mills theory. To be more precise we consider the theory of an $\mathscr{N}=2$ vector multiplet with a massless adjoint hypermultiplet. In this case the one-loop determinant is given by equation (5.14) with $x=\mu=0$ and it can be written as

$$
\log(Z_{1-\mathrm{loop}}) = -\left(\frac{Q^2}{4}-1\right)\log\Lambda'_{UV}+\log\Upsilon'(0). \tag{5.25}
$$

[1]The digamma function is defined as the series

$$
\psi(z+1) = -\gamma+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right), \tag{5.23}
$$

where γ is the Euler-Mascheroni constant.

This caputures most of the logarithmic divergence of the free energy but misses the topological piece. On the round sphere, the logarithmic divergence of the logarithm of the partition function is $-4a\log\Lambda_{UV} = -\log\Lambda_{UV}$. The U(1) instanton partition function is [44, 73–75, 58]

$$Z^{\mathrm{U}(1)}_{\mathrm{Inst.}} = \left(\prod_{i=1}^{\infty}(1-q^i)\right)^{\frac{Q^2}{4}-1}, \tag{5.26}$$

with $q = e^{2\pi \mathrm{i}\tau}$. It is holomorphic in τ and independent of the Coulomb branch parameter. The last contribution to the partition function is the Gaussian integral

$$\int d\sigma e^{-\frac{8\pi^2\sigma^2}{g^2_{YM}}} = \sqrt{\frac{g^2_{YM}}{8\pi}} = \left(\frac{1}{\mathrm{Im}(\tau-\overline{\tau})}\right)^{\frac{1}{2}}. \tag{5.27}$$

Thus the free energy of the abelian $\mathcal{N}=4$ SYM on the squashed sphere is

$$\log Z = -\frac{Q^2}{4}\log\Lambda_{UV} - \frac{1}{2}\log\mathrm{Im}(\tau-\overline{\tau}) + \log\Upsilon'(0) + \left(\frac{Q^2}{4}-1\right)\log\frac{Q}{2} + \log Z^{\mathrm{U}(1)}_{\mathrm{Inst.}}(\tau) + \log\overline{Z^{\mathrm{U}(1)}_{\mathrm{Inst.}}(\tau)}. \tag{5.28}$$

To compare with the general expressions for the free energy in equations (5.4) and (5.13) we note that for Abelian $\mathcal{N}=4$ SYM the anomaly coefficients are $a=c=\frac{1}{4}$ and the Kähler potential is $K(\tau,\overline{\tau}) = -6\log\left(\mathrm{Im}(\tau-\overline{\tau})\right)$. We thus see that

$$\begin{aligned}
I_{(\mathrm{Weyl})^2} &= -64\pi^2\left(\frac{Q^2}{4}-1\right), & \beta(\tau,\overline{\tau}) &= 0,\\
\gamma(\tau,\overline{\tau},b) &= \log\Upsilon'(0) + \left(\frac{Q^2}{4}-1\right)\log\frac{Q}{2}, & \alpha &= 0,\\
P_h(\tau,b) &= \log Z^{\mathrm{U}(1)}_{\mathrm{Inst.}}, & \overline{P}_h(\overline{\tau},b) &= \log\overline{Z^{\mathrm{U}(1)}_{\mathrm{Inst.}}}.
\end{aligned} \tag{5.29}$$

5.3.3 $\mathcal{N}=4$ super Yang-Mills at large N

As our second example we consider the large N limit of SU(N) $\mathcal{N}=2$ super Yang-Mills with a massive adjoint hypermultiplet. In this limit the instanton contributions are suppressed and the matrix integral is dominated by widely separated eigenvalues. To simplify the partition function we can then use that the functions f_p from equation (5.18) can for large x' be approximated as $f_p(x',\mu') \approx -\log\Lambda'_{UV} + \log(1+\mathrm{i}x')$. With this approximation we write the partition function as

$$Z|_{\mathrm{reg.}} \approx \int\prod_i \mathrm{d}\sigma_i \prod_{i<j}(\sigma^2_{ij})^{\frac{Q^2}{4}+\mu^2} e^{-\frac{8\pi^2}{\lambda}N\sum_i\sigma_i^2}, \tag{5.30}$$

and we dropped a prefactor which gives a subleading contribution to the free energy at large N. For a mass $\mu = \pm \mathrm{i}\frac{Q\gamma'}{2}$ this integral reduces to a Gaussian matrix model. The saddle point equation for the integral is

$$\frac{16\pi^2}{\lambda} N\sigma_i = 2\left(\frac{Q^2}{4} + \mu^2\right) \sum_{i\neq j} \frac{1}{\sigma_i - \sigma_j}. \tag{5.31}$$

This is the same saddle point equation as for a Gaussian matrix model with the coupling λ replaced by $\left(\frac{Q^2}{4} + \mu^2\right)\lambda$. Therefore the resulting free energy at large N is

$$\log Z|_{\mathrm{reg.}} \approx \frac{N^2}{2}\left(\frac{Q^2}{4} + \mu^2\right) \log\left(\left(\frac{Q^2}{4} + \mu^2\right)\lambda\right). \tag{5.32}$$

Setting the mass μ to zero and adding the divergent part we find

$$\log Z \approx N^2 \left[-\frac{Q^2}{4}\log \Lambda_{UV} + \frac{Q^2}{8}\log\lambda + \frac{Q^2}{8}\log\frac{Q^2}{4}\right]. \tag{5.33}$$

We can compare this to the general form of the $\mathscr{N} = 2$ SCFT free energy (5.13) and find

$$\begin{aligned} I_{(\mathrm{Weyl})^2} &= -64\pi^2\left(\frac{Q^2}{4} - 1\right), & \beta(\tau,\bar\tau) &= 0, \\ \gamma(\tau,\bar\tau,b) &= \frac{N^2 Q^2}{8}\log\frac{Q^2}{4}, & \alpha &= -\frac{1}{8}, \\ P_h(\tau,b) &= 0, & \overline{P}_h(\bar\tau,b) &= 0. \end{aligned} \tag{5.34}$$

We note that the value of the integrated supersymmetrized $(\mathrm{Weyl})^2$ term is the same as for the Abelian case, *cf.* equation (5.29). This was to be expected as this term only depends on the fields in the background Weyl multiplet.

Part III:
7d super Yang-Mills at negative coupling

6. Super Yang-Mills in seven dimensions

In this chapter we give a brief introduction to the seven dimensional super Yang-Mills theory. This theory has not had much attention as it is not UV-complete. However it is sensible as a low energy effective theory for its UV completion in string or M-theory. Considering a stack of D6 branes in type IIA string theory one finds that at low energies the effective theory on the worldvolume of the branes is 7d SYM with gauge group $\mathrm{U}(N)$. A Lagrangian description can then be obtained by dimensional reduction of the 10d SYM. In the following we look at the Lagrangian formulation of 7d SYM on the squashed seven-sphere and the localized partition function.

6.1 Super Yang-Mills on the seven-sphere

Super Yang-Mills theory on the seven-sphere was first constructed in [76, 77]. They started from the 10d flat space SYM Lagrangian dimensionally reduced along the time direction and two space directions. The covariantized Lagrangian was then made invariant under half of the conformal Killing spinors of the round sphere by adding correction terms to both the Lagrangian and to the supersymmetry transformations of the fermions. In this section we give a quick introduction to this Lagrangian description of the 7d SYM on S^7 and the perturbative contributions to its localized partition function.

Describing the Lagrangian as a dimensional reduction of the 10 dimensional super Yang-Mills Lagrangian, we choose all spinors in this section to be 10 dimensional Majorana-Weyl spinors and write $\Gamma^M, \widetilde{\Gamma}^N$ for the corresponding 10d Dirac matrices. To explicitly give the Killing spinors we use projective coordinates on the seven-sphere of radius r. The metric is

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2}{(1+\frac{x^2}{4r^2})^2}.$$

The Killing spinor equations and the corresponding preserved Killing spinors then take the form

$$\nabla_\mu \varepsilon = \frac{1}{2r}\widetilde{\Gamma}_\mu \Gamma^8 \widetilde{\Gamma}^9 \Gamma^0 \varepsilon, \tag{6.1}$$

$$\varepsilon = \frac{1}{\sqrt{1+\frac{x^2}{4r^2}}}\left(\varepsilon_s + \frac{1}{2r}(x\cdot\Gamma)\Gamma^8\widetilde{\Gamma}^9\Gamma^0\varepsilon_s\right), \tag{6.2}$$

parameterized by the constant spinor ε_s. We thus have a total of 16 independent Killing spinors on the seven-sphere. The 10 dimensional gauge field A_M splits up into the 7d gauge field A_μ $(\mu = 1,\ldots,7)$ and the scalars $\phi_I = A_I$ $(I = 0,8,9)$ in the vector representation of the SO$(1,2)$ R-symmetry group. The unusual choice for the indices comes from dimensionally reducing the 10d theory along the time direction $(M = 0)$ and two space directions $(M = 8,9)$. With the supersymmetry transformations

$$
\begin{aligned}
\delta_\varepsilon A_\mu &= \varepsilon \Gamma_\mu \Psi, \qquad\qquad \delta_\varepsilon \Phi_I = \varepsilon \Gamma_I \Psi, \\
\delta_\varepsilon \Psi &= \frac{1}{2}\left(\Gamma^{\mu\nu} F_{\mu\nu} + 2\Gamma^{\mu I} D_\mu \phi_I + \Gamma^{IJ}[\phi_I,\phi_J]\right)\varepsilon + \frac{8}{7}\Gamma^{\mu I}\phi_I \nabla_\mu \varepsilon,
\end{aligned}
\tag{6.3}
$$

the following 7 dimensional super Yang-Mills Lagrangian is invariant under all 16 Killing spinors

$$
\begin{aligned}
\mathscr{L} = \frac{1}{g_{YM}^2}\mathrm{Tr}\Bigg(&\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi_I D^\mu\phi^I + [\phi_I,\phi_J][\phi^I,\phi^J] - \Psi \not{D}\Psi \\
&+ \frac{3}{2r}\Psi\Gamma^8\widetilde{\Gamma}^9\Gamma^0\Psi + \frac{8}{r^2}\phi^I\phi_I - \frac{2}{r}[\phi^I,\phi^J]\phi^K\varepsilon_{IJK}\Bigg).
\end{aligned}
\tag{6.4}
$$

We use $\varepsilon_{890} = 1$ and $\not{D}\Psi = \Gamma^\mu D_\mu \Psi + \Gamma^I[\phi_I,\Psi]$. The second line of this Lagrangian are the correction terms added for the curved space.

In [8] it was noted that this construction works much more generally than just the round sphere. If a seven dimensional manifold admits a solution to the Killing spinor equation (6.1) then the Lagrangian (6.4) is invariant under the corresponding supersymmetry transformation (6.3). For supersymmetric localization a minimum of two conserved supercharges is required which means that the manifold should be Sasaki-Einstein. In the following we squash the seven-sphere to preserve only this minimum number of supercharges. This introduces four parameters ω_i restricted to satisfy $\sum_{i=1}^4 \omega_i = 4$. This deformation helps with the regularization of infinities.

6.1.1 The perturbative partition function

Given a choice of a localizing supercharge with corresponding Killing spinor ε, the localized partition function depends on the Killing vector $v^\mu = \varepsilon\Gamma^\mu\varepsilon$. Using embedding coordinates $z^i = r^i \exp(\mathrm{i}\phi^i)$ on the squashed seven-sphere the Killing vector takes the form $v = \sum_{i=1}^4 \omega_i \partial_{\phi_i}$. The full equations for the localization locus were found in [77, 8]. However the BPS locus described by these equations has not been fully determined. Only the part of the locus where the gauge field A vanishes and the BPS equations require $\phi_8 = \phi_9 = 0$ and $\phi_0 = constant$ is well understood. As in lower dimensions, this perturbative part of the partition function of super Yang-Mills on a squashed S^7 is an integral over the Cartan subalgebra for the gauge group [8]

$$Z^{pert.}_{S^7} = \int \mathrm{d}\sigma e^{-\frac{4\pi^4 r^3 \rho}{g^2_{YM}}\mathrm{Tr}(\sigma^2)} \prod_{\alpha\in\Delta} S_4(\mathrm{i}\alpha(\sigma);\omega_1,\omega_2,\omega_3,\omega_4), \tag{6.5}$$

with $\rho = (\omega_1\omega_2\omega_3\omega_4)^{-1}$ and Δ the roots of the gauge Lie algrebra. Similar to the double sine function (2.17), we define the quadruple sine function $S_4(x;\omega_1,\omega_2,\omega_3,\omega_4)$ as a the zeta funciton regularization of the infinite product [20]

$$\frac{\prod_{m,n,k,l\geq 0}(m\omega_1+n\omega_2+k\omega_3+l\omega_4+x)}{\prod_{m,n,k,l\geq 1}(m\omega_1+n\omega_2+k\omega_3+l\omega_4-x)}. \tag{6.6}$$

6.2 Contact instantons

The instanton contributions to the SYM partition function on S^7 have not been analyzed in detail in the literature. In Paper V we compute the contribution of some instantons to the partition function. The instantons we consider are the lift of point-like instantons from four dimensions and are thus three-dimensional membranes. The three-dimensional submanifold wrapped by such an instanton should be invariant under the action of the Killing vector v and thus on the squashed seven-sphere only a finite number of submanifolds can be wrapped by these membrane instantons. We restrict our attention to the instantons wrapping one of the six squashed three-spheres fixed under the action of v. Near one of these S^3 the supersymmetry approaches that of a twisted $S^3\times_{tw}\mathbb{C}^2$. In subsection 6.2.1 we present how to compute the partition function of these instantons and discuss the results in subsection 6.2.2. This provides additional factors that should be included in the S^7 partition function (6.5) similar to the non-perturbative contributions in the partition function (4.13) on the four-sphere.

Before proceeding, let us write down the seven-dimensional equivalent of the instanton equation and motivate the title of the section. First we have to define the contact one-form $\kappa = g(v,\bullet)$ from the Killing vector v. This is related to the volume element of the manifold as $\kappa\wedge\mathrm{d}\kappa^3 = -24vol_g$. It defines a contact structure on the manifold. The contact instanton equation takes the form

$$*F = \frac{1}{2}F\wedge\kappa\wedge\mathrm{d}\kappa. \tag{6.7}$$

In the appendix to Paper V we showed that a solution to this equation is the lift of a point-like instanton from 4d to 7d that has a constant wrapping around one of the fixed squashed three-spheres. All the solutions to this contact instanton equation are on the BPS locus of the seven-dimensional super Yang-Mills.

6.2.1 ADHM gauge theory

This subsection is inpsired by the discussion of D-brane bound states in [78].

To find a model in which we can compute the membrane instanton contribution to the 7d SYM partition function, we take a string theory perspective of the problem. As noted earlier, the seven dimensional super Yang-Mills theory is the low-energy effective theory on a stack of D6 branes. The D6 brane action contains the term [78]

$$\int \mathrm{Tr}(F \wedge F) \wedge C_3, \tag{6.8}$$

which is an integral over the brane worldvolume and depends on the gauge field strength F and the R-R 3-form potential C_3. The membrane instantons are non-trivial field strength configurations of co-dimension three. From (6.8) we see then that the instantons are sourced by a non-trivial 3-form potential on a three-dimensional submanifold of the D6 brane worldvolume. Such a three-form potential corresponds to D2 branes which couple to the three-form potential through $\int C_3$. Therefore we identify the membrane instantons of 7d SYM with D2 branes bound to the D6 branes.

To compute the instanton partition function we then switch our perspective. We can describe the D2-D6 brane bound state from the effective gauge theory on the worldvolume of the D2 branes. In the absence of D6 branes, the effective gauge theory on the D2 branes is 3d $\mathcal{N} = 8$ super Yang-Mills. For k D2 branes the gauge group is $\mathrm{U}(k)$. In the presence of N_c D6 branes, strings can stretch from the D2 branes to the D6 branes. The effective description of these additional strings is a hypermultiplet in the fundamental representation of both the gauge group $\mathrm{U}(k)$ and the $\mathrm{U}(N_c)$ flavor symmetry which corresponds to the gauge group of the 7d theory. Thus the effective theory is a 3d $\mathcal{N} = 4$ gauge theory with one adjoint and N_c fundamental hypermultiplets [78]. The separation of the D2 branes from the D6 branes is measured by the scalars of the vectormultiplet in the effective three dimensional theory. Therefore the instantons of the 7d theory correspond to supersymmetric configurations in the 3d theory where these vectormultiplet scalars vanish. These configurations are part of the space of vacua of the 3d $\mathcal{N} = 4$ gauge theory and their locus is called the Higgs branch. The instanton partition function is thus equal to the localized Higgs branch partition function for the effective theory on the D2 branes. However we can recover this Higgs branch partition function by evaluating the integrals in the Coulomb branch partition function reviewed in subsection 2.2.2[1].

We want to consider instantons wrapping three-spheres inside the S^7. For the computation of the partition function of instantons wrapping one of these spheres, we can consider the D6 brane worldvolume to be $S^3 \times_{tw} \mathbb{C}^2$ while the

[1] See [79] where the authors take a simple example and explicitly match a sum over residues of the Coulomb branch partition function to the Higgs branch partition function.

D2 branes wrap the three-sphere. The twist of the geometry implies that the effective 3d gauge theory lives on a squashed three-sphere and the hypermultiplets are massive with the twist parameters entering as mass parameters for some of the R-symmetry.

Guiding this whole construction are the lower dimensional results. Atiyah, Drinfeld, Hitchin and Manin noticed that they could construct all Yang-Mills instantons from a set of matrices subject to some constraint equations [80]. Nowadays we refer to these matrices as the ADHM data. Later it was noticed that the ADHM data describe the bosonic fields in the D(-1)-D3 brane bound state. In [4] Nekrasov used this ADHM data to compute the instanton partition function in the four dimensional Ω-background, where from the quantized theory he found solitons on $S^1 \times_{tw} \mathbb{C}^2$. The 2d ADHM gauge theory has also been studied on the two-sphere [81].

6.2.2 Instanton partition function

The upshot of the previous section is that the membrane instanton contribution to the S^7 partition function of super Yang-Mills is the product of the partition functions on six different squashed three-spheres

$$Z^{inst} = \prod_{1\leq i<j\leq 4} Z_{S^3_{\omega_i,\omega_j}}, \tag{6.9}$$

with the squashing parameter $\sqrt{\omega_i/\omega_j}$ for the three-sphere $S^3_{\omega_i,\omega_j}$. Each of these six contributions is the sum over the instanton number k on that squashed three-sphere, *i.e.* the rank of the ADHM gauge theory on that $S^3_{\omega_i,\omega_j}$,

$$Z_{S^3_{\omega_i,\omega_j}} = \sum_{k=0}^{\infty} e^{-\frac{4\pi^2 k}{g_{YM}^2}\frac{2\pi^2 r^3}{\omega_i\omega_j}} Z^{(k)}. \tag{6.10}$$

The weighting of each term corresponds to the contribution of the instantons to the classical 7d SYM action. For concreteness we discuss in the following the partition function on the squashed three-sphere $S^3_{\omega_1,\omega_2}$. The two parameters ω_3, ω_4 give masses to the various chiral multiplets of the ADHM theory. The partition function for k instantons wrapping $S^3_{\omega_1,\omega_2}$ is then given by the integral

$$
\begin{aligned}
Z^{(k)} = & \left(\frac{S_2\left(\frac{Q}{2}-\omega_3-\omega_4\right)}{S_2\left(\frac{Q}{4}-\omega_3\right)S_2\left(\frac{Q}{4}-\omega_4\right)} \right)^k \\
& \times \int \frac{d^k\phi\; e^{-2\pi \mathrm{i}\zeta\sum_{i=1}^k \phi_i}}{k!\,(\sqrt{\omega_1\omega_2})^k} \prod_{i\neq j} \frac{S_2(\mathrm{i}\phi_{ij})\,S_2\left(\frac{Q}{2}+\mathrm{i}\phi_{ij}-\omega_3-\omega_4\right)}{S_2\left(\frac{Q}{4}+\mathrm{i}\phi_{ij}-\omega_3\right)S_2\left(\frac{Q}{4}+\mathrm{i}\phi_{ij}-\omega_4\right)} \\
& \times \prod_{i=1}^k \prod_{A=1}^N \frac{1}{S_2\left(\frac{Q}{4}-\mathrm{i}\phi_i+\mathrm{i}\sigma_A\right)S_2\left(\frac{Q}{4}+\mathrm{i}\phi_i-\mathrm{i}\sigma_A-(\omega_3+\omega_4)\right)}, \qquad (6.11)
\end{aligned}
$$

where we use $S_2(x)\equiv S_2(x;\omega_1,\omega_2)$ and $Q=\omega_1+\omega_2$ to keep the notation more compact. The integration is on $\mathbb{R}^k$. Compared to Paper V we have adjusted two things. First, motivated by computations in lower dimensions [82], we have adjusted the R-charges of the chiral multiplets to the value suggested by $\mathcal{N}=4$ supersymmetry. Second, we have added the FI parameter ζ which is not part of the 7d SYM theory. However we use it to get a prescription for evaluating the integral. For $\zeta>0$ we can evaluate the integral by summing over all the poles in the lower half-plane. After the evaluation of the integral we then set ζ to zero.

In Paper V we used a pole prescription inspired from results in 4 and 5 dimensions [4] and evaluated the poles contributing to $Z^{(k)}$ at leading order in $e^{-2\pi\zeta\omega_1}$, $e^{-2\pi\zeta\omega_2}$. This truncated k instanton partition function takes the form

$$
\begin{aligned}
&\sum_{|Y|=k}\prod_{A,B=1}^N\prod_{s\in Y_A} S_2\left(\mathrm{i}\sigma_{BA}-(v_A(s)+1)\left(\omega_3-\frac{Q}{4}\right)+h_B(s)\left(\omega_4-\frac{Q}{4}\right)\right)^{-1} \\
&\quad\times S_2\left(\mathrm{i}\sigma_{AB}-(h_B(s)+1)\left(\omega_4-\frac{Q}{4}\right)+v_A(s)\left(\omega_3-\frac{Q}{4}\right)\right)^{-1}. \qquad (6.12)
\end{aligned}
$$

It is a sum over N-tuples of Young diagrams with in total k boxes. $v_i(s)$ is the vertical distance from box s to the boundary of the i-th tableaux and $h_j(s)$ is the horizontal distance to the boundary of the j-th tableaux. We have set the FI parameter ζ to zero. For the one instanton case this simplifies to

$$
\frac{1}{S_2\left(\frac{Q}{4}-\omega_3\right)S_2\left(\frac{Q}{4}-\omega_4\right)}\sum_{A=1}^N\prod_{B\neq A}\frac{1}{S_2(\mathrm{i}\sigma_{AB})S_2\left(\frac{Q}{2}+\mathrm{i}\sigma_{BA}-\omega_3-\omega_4\right)} \qquad (6.13)
$$

In the round sphere limit $\omega_i\to 1$ this one instanton contribution is finite and equals

$$
\frac{1}{2}\sum_{A=1}^N\prod_{B\neq A}\frac{1}{S_2(\mathrm{i}\sigma_{AB};1,1)S_2\left(\mathrm{i}\sigma_{BA}-\frac{3}{2};1,1\right)}. \qquad (6.14)
$$

For more instantons the contribution from a single fixed three-sphere is not finite in the round sphere limit. However we expect that these divergences cancel between the six fixed three-spheres and that the full instanton contribution to the round sphere partition function of 7d super Yang-Mills is finite. In five dimensions a similar cancellation of divergences is observed for instanton contributions to the sphere partition function of 5d SYM.

For the one instanton case we can explicitly write down the full partition function $Z^{(1)}$ as a sum over all the poles in the lower half-plane. After using the pseudo-periodicity of the double sine function we get the result

$$
\begin{aligned}
&Z^{(1)} = \\
&\frac{1}{S_2\left(\frac{Q}{4}-\omega_3\right)S_2\left(\frac{Q}{4}-\omega_4\right)}\sum_{A=1}^{N}\left(\prod_{B\neq A}\frac{1}{S_2(\mathrm{i}\sigma_{AB})S_2\left(\frac{Q}{2}+\mathrm{i}\sigma_{BA}-\omega_3-\omega_4\right)}\right)\\
&\times\sum_{n,m=0}^{\infty}(-2\mathrm{i})^{n+m}(-1)^{nm} \qquad (6.15)\\
&\prod_{k=1}^{n}\cos\left(\frac{\pi\left(\left(k-\frac{1}{2}\right)\omega_1-\omega_3-\omega_4\right)}{\omega_2}\right)\prod_{l=1}^{m}\cos\left(\frac{\pi\left(\left(l-\frac{1}{2}\right)\omega_2-\omega_3-\omega_4\right)}{\omega_1}\right)\\
&\prod_{B\neq A}\prod_{k=1}^{n}\frac{\cos\left(\frac{\pi\left(\left(k-\frac{1}{2}\right)\omega_1-\mathrm{i}\sigma_{AB}-\omega_3-\omega_4\right)}{\omega_2}\right)}{\sin\left(\frac{\pi(\mathrm{i}\sigma_{AB}-k\omega_1)}{\omega_2}\right)}\prod_{l=1}^{m}\frac{\cos\left(\frac{\pi\left(\left(l-\frac{1}{2}\right)\omega_2-\mathrm{i}\sigma_{AB}-\omega_3-\omega_4\right)}{\omega_1}\right)}{\sin\left(\frac{\pi(\mathrm{i}\sigma_{AB}-k\omega_2)}{\omega_1}\right)}.
\end{aligned}
$$

Looking at the products of cosines in the second to last line, we observe that in the round sphere limit the additional contributions with n or m non-zero vanish and thus $Z^{(1)}$ reduces to (6.14). We expect that this suppression of the additional poles holds for all k instanton sectors such that (6.12) captures the leading contribution close to the round sphere.

7. The negative coupling limit of 7d SYM

The usual expectation is that for a gauge theory the regime of negative coupling g_{YM}^2 is not well defined. However, in Paper V we argue that for super Yang-Mills on the seven-sphere the renormalized coupling may be negative and that at weak negative coupling the theory admits a dual description as a seven dimensional supergravity theory. We start this chapter by discussing the motivations for studying the negative coupling regime of 7d super Yang-Mills on a sphere and for expecting a supergravity theory in the small negative coupling limit. Thereafter we present the progress made in confirming this expectation using supersymmetric localization. The techniques we use are inspired from similar works in five dimensions [83].

7.1 Motivation

In the previous chapter we discussed how the 7d super Yang-Mills theory on the sphere can be studied from localization. SYM on S^7 arises as the effective theory on the spherical D6 branes of [84]. The corresponding 11d supergravity geometry is $(H^{2,2}/\mathbb{Z}_N) \times S^7$ where $H^{2,2}$ is a Wick rotation of AdS_4 with two time directions. In [10] the free energy and Wilson loop expectation value were analyzed both from supergravity and supersymmetric localization. In terms of the bare coupling g_0 and a mode number cutoff n_0, the effective coupling for the localized SYM partition function is [77]

$$\lambda_{eff}^{-1} \equiv \frac{r^3}{g_0^2 N} - \frac{n_0}{2\pi^4}. \tag{7.1}$$

The strong coupling behavior of the perturbative partition function only appears once this effective coupling is continued through the normal strong coupling point all the way to $\lambda_{eff}^{-1} \to -\infty$. To get the holographic dictionary to match, the supergravity solution then also has to be analytically continued to the point where there is an additional minus sign in the relation between the string coupling g_s and the Yang-Mills coupling g_{YM}

$$g_{YM}^2 = -2\pi g_s (2\pi l_s)^3. \tag{7.2}$$

The results of [10] show the necessity for the negative weak coupling limit of SYM on the seven-sphere in the context of holography. To understand the nature of this phase which is from the SYM perspective strongly coupled, we have to include non-perturbative contributions to the partition function.

To get some insight into what a strongly coupled phase for seven dimensional super Yang-Mills looks like, we turn to the results in [85]. Here Peet and Polchinski looked at the near-horizon geometry of a stacks of N flat Dp branes. Probing these geometries either by a Dp brane or a supergravity probe, they found that two distinct energy scales could be probed,

$$\text{supergravity}: \qquad E = \frac{U^{(5-p)/2}}{g_{YM} N^{1/2}}, \tag{7.3}$$

$$\text{Dp brane}: \qquad E = U, \tag{7.4}$$

in terms of the holographic coordinate U. Noticeably for D6 branes this implies that there exist two distinct low energy effective theories, the 7d super Yang-Mills theory in the near horizon limit at $U \to 0$ and another low energy theory at $U \to \infty$. In addition they found that both these low energy theories were weakly coupled in their respective regimes,

$$\text{supergravity}: \qquad g_{eff}^2 = \left(g_{YM}^2 N U^{p-3}\right)^{(5-p)/2}, \tag{7.5}$$

$$\text{Dp brane}: \qquad g_{eff}^2 = g_{YM}^2 N U^{p-3}. \tag{7.6}$$

This suggests that the strong coupling limit of the 7d SYM has a dual description as a weakly coupled supergravity theory.

7.2 7d SYM at weak negative coupling

In this section we present our approach for studying the weak negative coupling limit of the localized partition function of seven-dimensional super Yang-Mills.

We restrict the discussion to the gauge group SU(2). For this gauge group the perturbative partition function (6.5) simplifies to

$$Z^{pert.} = \int \mathrm{d}\sigma e^{-\frac{8\pi^4 r^3 \rho}{g_{YM}^2}\sigma^2} S_4(2\mathrm{i}\sigma;\omega_1,\omega_2,\omega_3,\omega_4) S_4(-2\mathrm{i}\sigma;\omega_1,\omega_2,\omega_3,\omega_4), \tag{7.7}$$

where we defined $\sigma \equiv \sigma_1 = -\sigma_2$. Assuming the coupling $\lambda = \frac{2g_{YM}^2}{r^3}$ is small and negative we want to determine the saddle points of this integral. We expect that in this limit of the coupling the integral is dominated by contributions from large values of σ. For large positive σ we can approximate the quadruple sines as

$$\begin{aligned} &\log\left(S_4(2\mathrm{i}\sigma;\omega_1,\omega_2,\omega_3,\omega_4) S_4(-2\mathrm{i}\sigma;\omega_1,\omega_2,\omega_3,\omega_4)\right) \\ &\quad \approx -\frac{\pi\rho}{3}\left(16\sigma^3 - 2\sigma(\omega_1\omega_2 + \omega_1\omega_3 + \omega_1\omega_4 + \omega_2\omega_3 + \omega_2\omega_4 + \omega_3\omega_4)\right), \end{aligned} \tag{7.8}$$

up to exponentially suppressed terms. The saddle point equation then takes the form

$$\sigma^2 + \frac{2\pi^3}{\lambda}\sigma - \frac{\omega_1\omega_2 + \omega_1\omega_3 + \omega_1\omega_4 + \omega_2\omega_3 + \omega_2\omega_4 + \omega_3\omega_4}{24} = \mathscr{O}\left(e^{-2\sigma}\right). \tag{7.9}$$

Solving this equation, we find the saddle point for small negative λ is at

$$\sigma = -\frac{2\pi^3}{\lambda} + \mathscr{O}(\lambda). \tag{7.10}$$

With the effective 't Hooft coupling small and negative this saddle point is at large and positive σ which confirms our expectation and warrants the approximation of the quadruple sines.

In the next step we expand both the perturbative integrand and the instanton partition function around this saddle point. We write $\sigma_{12} = 2\sigma = -\frac{2\pi^3 r^3}{g_{YM}^2} + \delta\sigma$. Then the perturbative integrand can be approximated as

$$\exp\left(\frac{2\pi^4 r^3 \rho}{g_{YM}^2}\delta\sigma^2 - \frac{\pi\rho}{3}\left(2\delta\sigma^3 - \delta\sigma\sum_{i<j}\omega_i\omega_j\right)\right). \tag{7.11}$$

The first term in this exponential is a Gaussian with the correct sign because $g_{YM}^2 < 0$ by assumption. The cubic and linear terms we expect will contribute to the one-loop determinant for the light matter in this new phase. This is motivated from the results in five dimensions, but is left to be shown by future work. From equations (6.10) and (6.12) of the previous chapter we have the leading instanton contribution

$$\sum_{k=0}^{\infty} e^{-\frac{4\pi^2|Y|}{g_{YM}^2}\frac{2\pi^2 r^3}{\omega_1\omega_2}} \sum_{|Y|=k}\prod_{A,B=1}^{2}\prod_{s\in Y_A} \frac{S_2\left(\frac{Q}{2} + \omega_3 + \omega_4 + \mathrm{i}\sigma_{AB} - (h_B(s)+1)\left(\omega_4 - \frac{Q}{4}\right) + v_A(s)\left(\omega_3 - \frac{Q}{4}\right)\right)}{S_2\left(\mathrm{i}\sigma_{AB} - (h_B(s)+1)\left(\omega_4 - \frac{Q}{4}\right) + v_A(s)\left(\omega_3 - \frac{Q}{4}\right)\right)}, \tag{7.12}$$

mutliplied by the contributions from the other five fixed three-spheres. As before the parameters of the double sine function are suppressed to shorten the expressions, *i.e.* $S_2(x) \equiv S_2(x;\omega_1,\omega_2)$. For a single pair of Young diagrams

(Y_1, Y_2) in the sum, the factors with $A = B$ take the form

$$
\begin{aligned}
&\prod_{s_1 \in Y_1} \frac{S_2\left(\frac{Q}{2} + \omega_3 + \omega_4 - (h_1(s_1)+1)\left(\omega_4 - \frac{Q}{4}\right) + v_1(s_1)\left(\omega_3 - \frac{Q}{4}\right)\right)}{S_2\left(-(h_1(s_1)+1)\left(\omega_4 - \frac{Q}{4}\right) + v_1(s_1)\left(\omega_3 - \frac{Q}{4}\right)\right)} \\
&\quad \times \prod_{s_2 \in Y_2} \frac{S_2\left(\frac{Q}{2} + \omega_3 + \omega_4 - (h_2(s_2)+1)\left(\omega_4 - \frac{Q}{4}\right) + v_2(s_2)\left(\omega_3 - \frac{Q}{4}\right)\right)}{S_2\left(-(h_2(s_2)+1)\left(\omega_4 - \frac{Q}{4}\right) + v_2(s_2)\left(\omega_3 - \frac{Q}{4}\right)\right)} \\
&= \prod_{s_1 \in Y_1} \frac{S_2\left(\frac{Q}{4} + 2 + x(s_1)\right)}{S_2\left(\frac{3Q}{4} - 2 + x(s_1)\right)} \prod_{s_2 \in Y_2} \frac{S_2\left(\frac{Q}{4} + 2 + x(s_2)\right)}{S_2\left(\frac{3Q}{4} - 2 + x(s_2)\right)}.
\end{aligned}
\tag{7.13}
$$

We have defined $x(s) = (v(s) + \frac{1}{2})(\omega_3 - \frac{Q}{4}) - (h(s) + \frac{1}{2})(\omega_4 - \frac{Q}{4})$ and used that $\sum_{i=1}^{4} \omega_i = 4$. Note that each of the factors in the last line is invariant under $\omega_{3,4} - \frac{Q}{4} \to \frac{Q}{4} - \omega_{3,4}$ or equivalently $x(s) \to -x(s)$. To expand the factors with $A \neq B$ around the saddle point we use that for $|\mathrm{Im}(z)| \gg 1$ the double sine function can be approximated as

$$
\log(S_2(z; \omega_1, \omega_2)) \approx \mathrm{sign}(\mathrm{Im}(z)) \frac{\pi \mathrm{i}\left(z^2 - (\omega_1 + \omega_2)z + \frac{\omega_1^2 + \omega_2^2 + 3\omega_1\omega_2}{6}\right)}{2\omega_1\omega_2}.
\tag{7.14}
$$

We find that the factors with $A \neq B$ can be approximated as

$$
\begin{aligned}
&\prod_{s_1 \in Y_1} \frac{S_2\left(\frac{Q}{2} + \omega_3 + \omega_4 + \mathrm{i}\sigma_{12} - (h_2(s_1)+1)\left(\omega_4 - \frac{Q}{4}\right) + v_1(s_1)\left(\omega_3 - \frac{Q}{4}\right)\right)}{S_2\left(+\mathrm{i}\sigma_{12} - (h_2(s_1)+1)\left(\omega_4 - \frac{Q}{4}\right) + v_1(s_1)\left(\omega_3 - \frac{Q}{4}\right)\right)} \\
&\times \prod_{s_2 \in Y_2} \frac{S_2\left(\frac{Q}{2} + \omega_3 + \omega_4 - \mathrm{i}\sigma_{12} - (h_1(s_2)+1)\left(\omega_4 - \frac{Q}{4}\right) + v_2(s_2)\left(\omega_3 - \frac{Q}{4}\right)\right)}{S_2\left(-\mathrm{i}\sigma_{12} - (h_1(s_2)+1)\left(\omega_4 - \frac{Q}{4}\right) + v_2(s_2)\left(\omega_3 - \frac{Q}{4}\right)\right)} \\
&\approx e^{\frac{2\pi^4 r^3}{g_{YM}^2 \omega_1 \omega_2}\left(4 - \frac{Q}{2}\right)|Y|} e^{-\frac{\pi\left(4 - \frac{Q}{2}\right)}{\omega_1\omega_2}|Y|\delta\sigma} \\
&\quad \times \prod_{s_1 \in Y_1} e^{-\frac{\pi \mathrm{i}\left(4 - \frac{Q}{2}\right)}{\omega_1\omega_2}\left((h_1(s_1) + \frac{1}{2})(\omega_4 - \frac{Q}{4}) + (v_1(s_1) + \frac{1}{2})(\omega_3 - \frac{Q}{4})\right)} \\
&\qquad \times \prod_{s_2 \in Y_2} e^{+\frac{\pi \mathrm{i}\left(4 - \frac{Q}{2}\right)}{\omega_1\omega_2}\left((h_2(s_2) + \frac{1}{2})(\omega_4 - \frac{Q}{4}) + (v_2(s_2) + \frac{1}{2})(\omega_3 - \frac{Q}{4})\right)}.
\end{aligned}
\tag{7.15}
$$

We have used equation (A.1) from Paper III to trade $h_1(s_2)$ and $h_2(s_1)$ for $h_1(s_1)$ and $h_2(s_2)$. It then follows that in the small negative coupling limit the instanton contribution from a single squashed S^3 can be factorized as

$$\mathscr{Z}(\delta\sigma;\omega_1,\omega_2;\omega_3-\frac{Q}{4},\omega_4-\frac{Q}{4})\mathscr{Z}(\delta\sigma;\omega_1,\omega_2;-\omega_3+\frac{Q}{4},-\omega_4+\frac{Q}{4}) \tag{7.16}$$

where we define

$$\mathscr{Z}(\delta\sigma;\omega_1,\omega_2;\omega_3-\frac{Q}{4},\omega_4-\frac{Q}{4})=$$
$$\sum_Y e^{-\frac{Q\pi^4 r^3}{g_{YM}^2\omega_1\omega_2}|Y|}e^{-\frac{\pi\left(4-\frac{Q}{2}\right)}{\omega_1\omega_2}\delta\sigma|Y|}\prod_{s\in Y}e^{\frac{\pi\mathrm{i}\left(4-\frac{Q}{2}\right)}{\omega_1\omega_2}\left((h(s)+\frac{1}{2})(\omega_4-\frac{Q}{4})+(v(s)+\frac{1}{2})(\omega_3-\frac{Q}{4})\right)}$$
$$\times\frac{S_2\left(\frac{Q}{4}+2+x(s)\right)}{S_2\left(\frac{3Q}{4}-2+x(s)\right)} \tag{7.17}$$

Unlike what we found in Paper V, the adjusted R-charges in the ADHM theory imply that this expression for the instantons still depends on g_{YM}^2. The saddle point should thus be shifted away from its perturbative value. We expect that at this shifted saddle three-dimensional membranes become light and can be interpreted as part of a graviton multiplet. Something similar happens for 4d $\mathscr{N}=2^*$ super Yang-Mills on the sphere [82]. In this case at weak negative coupling the naive perturbative saddle point gets shifted by instanton contributions and the resulting saddle matches with the massless monopole point of Seiberg and Witten [86].

Part IV:
The Hagedorn temperature and AdS/CFT integrability

8. The Hagedorn temperature of strings on AdS

In [87] Hagedorn studied the asymptotic form of the mass spectrum of strong interactions at high energies. Assuming that heavier and heavier hadrons appear at increasing energies, he deduced from the self-consistency of his statistical description an exponential growth for the number of states. This led him to conclude that there should be a maximum temperature. For an exponentially growing density of states $\rho(E) \propto \exp(\beta_c E)$ the thermal partition function

$$\mathrm{Tr}(e^{-E/T}) = \int_0^\infty \mathrm{d}E\,\rho(E)e^{-E/T} \propto \int_0^\infty \mathrm{d}E\,\exp((\beta_c - 1/T)E) \tag{8.1}$$

converges only if $T < \frac{1}{\beta_c}$. This limiting temperature $T_H \equiv \frac{1}{\beta_c}$ is called the Hagedorn temperature.

In this chapter we will discuss the Hagedorn temperature in the context of string theory with an emphasis on the recent developments for AdS superstrings, including in Papers VI and VII. In the next chapter, we will discuss the dual CFT perspective.

8.1 Hagedorn behavior in string theory

In bosonic string theory the exponential growth of the density of states was observed very much from the beginning. As a precursor to string theory, the Veneziano amplitude [88] and its higher point generalizations [89–91] were found from consistency requirements. Studying these amplitudes Fubini and Veneziano found a density of states growing exponentially with the energy [92], *i.e.* the Hagedorn behavior of the bosonic string theory spectrum. In flat space the bosonic string Hagedorn temperature is [93]

$$T_H^{\text{bos.}} = \frac{1}{4\pi\sqrt{\alpha'}}. \tag{8.2}$$

For superstring theory in flat space the Hagedorn temperature takes a similar form. The convergence of the free energy of superstrings in flat space was discussed in [94, 95]. For both the type I and II string theories it was found that the partition function only converges below the Hagedorn temperature

$$T_H^{\text{I/II}} = \frac{1}{2\pi\sqrt{2\alpha'}}. \tag{8.3}$$

One can reproduce the thermal partition function of a gas of strings at temperature T by putting the string theory on a Euclidean background with the time direction compactified on a circle of circumference $\frac{1}{T}$. When the thermal cycle becomes smaller than the inverse Hagedorn temperature the lightest string mode winding the thermal cycle becomes tachyonic [96–98]. In the next section we will use this winding scalar mode to compute α'-corrections to the Hagedorn temperature in AdS backgrounds.

The physical interpretation of the Hagedorn temperature is not well understood. Naively the divergence of the partition function would suggest that the Hagedorn temperature is a maximum allowed temperature. However, this does not appear to be the case in string theory. In [99] it was suggested that string theory should undergo a phase transition at the Hagedorn temperature. At this transition the light winding mode condenses and forms a self-gravitating bound state of strings, sometimes called a "string star" [100].

8.1.1 In the plane wave background

To take a first look at corrections to the Hagedorn temperature in non-trivial backgrounds, we review here the results on the plane wave background. For string theory the plane wave geometries are special as the superstring on these backgrounds can be explicitly quantized [101–103]. At special values of the fluxes these backgrounds are limits of AdS string backgrounds.

The Hagedorn temperature of the type IIB superstring in the plane wave background supported by pure RR five-form flux was studied in [104–106]. Starting from the light-cone Hamiltonian the exact free energy in this background was computed and the result is an equation for the Hagedorn temperature to all orders in α'. Tuning the five form flux to the value for the plane wave limit of the $\mathrm{AdS}_5 \times S^5$ superstring the leading orders of the equation for the Hagedorn temperature are

$$\frac{\beta^2}{4\pi\alpha'} = 2\pi - 2\beta + \frac{\beta^2 \log(2)}{\pi} + \mathscr{O}(\beta^4). \tag{8.4}$$

The same computation can be done for the type IIA superstring on the plane wave background supported by two- and four-form flux [107]. Tuning the fluxes to the value for the plane wave limit of the superstring on $\mathrm{AdS}_4 \times \mathbb{CP}^3$ the truncated equation takes the form

$$\frac{\beta^2}{4\pi\alpha'} = 2\pi - \frac{3}{2}\beta + \frac{5\beta^2 \log(2)}{8\pi} + \mathscr{O}(\beta^4). \tag{8.5}$$

In both equations (8.4) and (8.5) we get the leading contribution to the α' expansion of the Hagedorn temperature by truncating the right-hand side to the first term. This leading result matches the value in flat space $\beta_H^{(0)} = 2\pi\sqrt{2\alpha'}$. Intriguingly the linear terms in β, responsible for the first correction to the

Hagedorn temperature, give results matching with the results in the corresponding AdS backgrounds as we discuss in the next section.

8.2 The AdS Hagedorn temperature

Recently the Hagedorn temperature of superstring theory on asymptotically AdS spaces has gained much attention. In this section we discuss the results from string theory and supergravity computations and we leave the dual CFT results for the next chapter.

The extension of the string star solution [100] to asymptotically AdS space is discussed in [12] by Urbach. Here he looks at the effective equations of motion of the scalar mode winding around the thermal cycle and the dilaton in a Euclidean AdS background. He finds that the Hagedorn temperature in AdS_{d+1} is

$$T_H = \frac{1}{2\pi\sqrt{2\alpha'}} + \frac{d}{8\pi} + \mathcal{O}((\alpha')^{1/2}). \tag{8.6}$$

This result for the first correction was independently found by [13].

A different approach leading to the same result is due to [108, 109]. They study the semi-classical quantization of the string worldsheet theory around the classical solution for the string winding around the thermal cycle. In this stringy approach to the problem, the first correction to the Hagedorn temperature is due to the zero modes of the massive worldsheet scalars.

An effective supergravity model

In Papers VI and VII we propose an effective model for the string mode that winds the thermal cycle and becomes tachyonic at the Hagedorn temperature. Modeling the lowest string mode winding the cycle, we consider a scalar field χ in Euclidean AdS space only depending on the radial, or holographic, direction. At the Hagedorn temperature the string mode becomes massless and thus our scalar field does not backreact on its background. We start from an action

$$\int \mathrm{d}^{d+1}X\sqrt{g}\left(\nabla^\mu\chi\nabla_\mu\chi + m^2(R)\chi^2\right), \tag{8.7}$$

with the mass term

$$m^2(R) = (1+R^2)\left(\frac{\beta}{2\pi\alpha'}\right)^2 + C. \tag{8.8}$$

In this mass term C comes from the string zero-point energy and the rest captures the contribution from stretching the string around the thermal cycle.

From here we can write down the equations of motion for the scalar field

$$-\frac{1}{2}\frac{1}{R^{d-1}}\frac{\mathrm{d}}{\mathrm{d}R}R^{d-1}\frac{\mathrm{d}}{\mathrm{d}R}\chi(R)+\frac{1}{2}\left(\frac{\beta}{2\pi\alpha'}\right)^2 R^2\chi(R)+\Delta H\chi(R) \\ =-\frac{1}{2}\left(C+\left(\frac{\beta}{2\pi\alpha'}\right)^2\right)\chi(R). \quad (8.9)$$

This has the form of a d dimensional rotationally symmetric harmonic oscillator perturbed by

$$\Delta H=-\frac{1}{2R^{d-1}}\frac{\mathrm{d}}{\mathrm{d}R}R^{d+1}\frac{\mathrm{d}}{\mathrm{d}R}. \quad (8.10)$$

We can then use perturbation theory to compute the ground state energy and compare it to the right-hand side of (8.9). This relates the inverse Hagedorn temperature β to the string zero-point energy C. Expanding the zero-point energy around the flat space value $C=-\frac{2}{\alpha'}+\Delta C+\mathscr{O}(\alpha')$, second order perturbation theory gives us the equation

$$\frac{\beta^2}{4\pi\alpha'}=2\pi-\frac{d}{2}\beta-\pi\alpha'\Delta C-\frac{d(d+2)}{4}\pi\alpha'+\frac{d(d+2)\pi^2(\alpha')^2}{8\beta}, \quad (8.11)$$

valid up to order $(\alpha')^{3/2}$. For ΔC of order $(\alpha')^0$ this reproduces the first subleading order of the AdS Hagedorn temperature (8.6). To go to higher order we compare equation (8.11) to the pp-wave equations (8.4) and (8.5). From this comparison we conjecture that the correction to the zero-point energy takes the form

$$\Delta C=\frac{\beta^2}{2\pi^2\alpha'}\Delta c. \quad (8.12)$$

With this we find that the AdS Hagedorn temperature takes the form

$$T_H(\alpha')=\frac{1}{2\pi\sqrt{2\alpha'}}+\frac{d}{8\pi}+\frac{d(d+1)+8\Delta c}{16\sqrt{2}\pi}\sqrt{\alpha'}+\frac{d(d+2)(4d-1)}{256\pi}\alpha' \\ +\mathscr{O}\left((\alpha')^{3/2}\right). \quad (8.13)$$

In Paper VI and VII we check this equation for $d=3$ and $d=4$ by comparing with the numerical strong coupling results from the dual CFT. The numerical precision is sufficient to conjecture

$$\Delta c=-d\log(2). \quad (8.14)$$

Beyond our results

The curvature corrections to the string zero-point energy in our effective supergravity computation are very much *ad hoc*. The expression (8.12) is warranted *a posteriori* by the matching against the numerical results from field theory. Harmark suggests a more systematic approach to curvature corrections to the equations of motion of the effective scalar field [110]. Starting from the Klein-Gordon equation with the covariantized version of the mass term (8.9), he writes down the possible first order curvature corrections to both the Laplace operator and the mass term. This allows him to give a unified effective description of both the plane wave and the AdS Hagedorn temperatures. From the plane wave he fixes the coefficient of the curvature correction to the mass term and then predicts the correct value of the $\mathcal{O}(\sqrt{\alpha'})$ term for the AdS Hagedorn temperature.

To go beyond the effective description [111] uses the quadratic expansion of the worldsheet sigma-model from [108, 109]. They match the Hagedorn temperature (8.13) to order $(\alpha')^{1/2}$. This gives an explanation for the analytic result (8.14) and confirms the conjecture from CFT integrability. However, the last term in equation (8.13) cannot be computed from the current form of this worldsheet approach. To obtain the next order of the Hagedorn temperature quartic terms would need to be included in the expansion of the string sigma-model in quantum fluctuations around the winding string.

9. The CFT Hagedorn temperature and integrability

In this chapter we discuss the Hagedorn temperature in conformal field theories dual to strings on AdS, restricting to $\mathcal{N} = 4$ super Yang-Mills and ABJM theory where integrability techniques are well established. We start with a short review on perturbative results for the Hagedorn temperature in these two CFTs. We then give a practical introduction to the integrability technique used to solve the Hagedorn temperature in $\mathcal{N} = 4$ SYM and ABJM theory. We check the weak coupling solutions against the results reviewed in section 9.1, while we compare the strong coupling solution against the results of the effective AdS computation from the previous chapter. Subsection 9.2.4 contains results from Paper VI and section 9.3 is based on Paper VII.

9.1 Hagedorn temperature in CFT

Conformal field theories in $\mathbb{R}^n$ cannot have a Hagedorn temperature due to scale invariance. It is only once the CFT has been put on a compact space that Hagedorn behavior is possible. Here we choose to consider $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ and for ABJM the background is $\mathbb{R} \times S^2$. In addition the exponential growth of the density of states requires the strict large N limit. In $\mathcal{N} = 4$ SYM at finite N single-trace operators with more than N fields can be related to sums of multi-trace operators. This cuts down the number of independent states of energy larger than N and thus removes the exponential growth of the density of states. A similar statement holds for ABJM theory.

The first to calculate the free partition function of planar $\mathcal{N} = 4$ SYM was Sundborg [112] who strung beads together into necklaces to count single-trace operators. Using Pólya theory he related the bound on the convergence of the partition function to a special value of the single bead partition function. He found the free Hagedorn temperature

$$T_H(\lambda = 0) = \frac{1}{2\log(2+\sqrt{3})}, \tag{9.1}$$

where λ is the 't Hooft coupling and equals zero for the free theory. The leading correction to this Hagedorn temperature was computed by [113]. They used that the one-loop correction to the spectrum of $\mathcal{N} = 4$ SYM is related to a spin chain Hamiltonian [114, 115]. The trace of this one-loop dilatation operator is then proportional to the one-loop correction to the Hagedorn temperature. The result found by [113] is

$$T_H = \frac{1}{2\log(2+\sqrt{3})}\left(1+\frac{\lambda}{8\pi^2}+\mathscr{O}(\lambda^2)\right). \tag{9.2}$$

For ABJM theory the free Hagedorn temperature was found in [116]. Following the same route as [113], [117] computed the leading correction to the Hagedorn temperature from the trace of a spin chain Hamiltonian [118, 119]. In terms of the 't Hooft coupling λ, the ABJM Hagedorn temperature is

$$T_H = \frac{1}{2\log(3+2\sqrt{2})}\left(1+4\lambda^2(\sqrt{2}-1)+\mathscr{O}(\lambda^4)\right). \tag{9.3}$$

Note that for the $\mathscr{O}(\lambda^2)$ term we have included a subtle factor of 2 that was missing in [117] as we noted in Paper VII.

9.2 The quantum spectral curve of $\mathscr{N}=4$ SYM

For over two decades, integrability has been an important tool for the study of $\mathscr{N}=4$ SYM in the planar limit. The first decade of developments and applications were reviewed in [120]. One of the biggest successes of $\mathscr{N}=4$ integrability is solving the spectrum of operators. Techniques were developed for the exact computation of the anomalous dimensions of all the operators. These developments culminated with the quantum spectral curve (QSC) [121, 122] which gives a description of the full spectrum in terms of a finite set of functional equations. In [123] Harmark and Wilhelm set out to develop a similar integrability description of the Hagedorn temperature and started by deriving the corresponding thermodynamic Bethe ansatz (TBA) equations. In subsequent work they deduced the quantum spectral curve and solved for the Hagedorn temperature both perturbatively at weak coupling and numerically for finite coupling [11, 124].

In this section we give an introduction to the quantum spectral curve for the $\mathscr{N}=4$ Hagedorn temperature, following the results of [11, 124]. Our presentation of the Q-system follows [121, 122, 125]. This means that our notation deviates from [11, 124] and Paper VI by exchanging the **P**- and **Q**-functions. This clarifies the roles of R-symmetry and conformal symmetry and thus facilitates the deduction of the ABJM quantum spectral curve in the next section.

9.2.1 The $\mathfrak{psu}(2,2|4)$ Q-system

The global superconformal symmetry algebra of $\mathscr{N}=4$ super Yang-Mills is $\mathfrak{psu}(2,2|4)$. The corresponding Q-system consists of the 256 Q-functions

$Q_{A|I}(u)$ of the "spectral parameter" u. A and I are both anti-symmetrized multi-indices drawn from $\{1,2,3,4\}$. The indices can be raised by Hodge dualizing, *e.g.*

$$Q^{ab|ij} = \frac{1}{(6!)^2}\varepsilon^{cdab}\varepsilon^{klij}Q_{cd|kl}. \tag{9.4}$$

Several of these Q-functions play a distinguished role. First, the boundary Q-functions $Q_{\emptyset|\emptyset}$ and $Q_{1234|1234}$ are chosen to be normalized to 1,

$$Q_{\emptyset|\emptyset} = 1, \qquad Q_{1234|1234} = 1. \tag{9.5}$$

Next we define the **P**-functions

$$\mathbf{P}_a = Q_{a|\emptyset}, \qquad \mathbf{P}^a = Q^{a|\emptyset} = \frac{1}{3!}\varepsilon^{bcda}Q_{bcd|1234}. \tag{9.6}$$

These functions carry indices of the $SU(4)$ R-symmetry, with the lower one corresponding to the fundamental representation 4 and the upper one to the anti-fundamental $\overline{4}$. Similarly we have the **Q**-functions

$$\mathbf{Q}_i = Q_{\emptyset|i}, \qquad \mathbf{Q}^i = Q^{\emptyset|i}, \tag{9.7}$$

which are respectively in the fundamental and anti-fundamental representations of the global conformal symmetry $SU(2,2)$.

The Q-functions are related to each other by finite difference equations, the so-called QQ-relations. For practical calculations, the most important of these relations is

$$Q^+_{a|i} - Q^-_{a|i} = \mathbf{P}_a\mathbf{Q}_i. \tag{9.8}$$

Here we introduced the notation $f^{\pm}(u) = f(u \pm \frac{i}{2})$. From the QQ-relations many useful identities can be derived, including

$$\mathbf{Q}_i = -\mathbf{P}^a Q^+_{a|i}, \qquad \mathbf{P}_a = -\mathbf{Q}^i Q^+_{a|i}, \tag{9.9}$$

$$\mathbf{P}^a\mathbf{P}_a = 0, \qquad \mathbf{Q}^i\mathbf{Q}_i = 0. \tag{9.10}$$

Together with the normalization (9.5), the QQ-relations imply

$$Q^{a|i}Q_{a|j} = -\delta^i_j, \qquad Q^{a|i}Q_{b|i} = -\delta^a_b. \tag{9.11}$$

Using QQ-relations all of the 256 Q-functions can be expressed explicitly in terms of $\mathbf{P}_a$, $\mathbf{Q}_i$ and $Q_{a|i}$. Solving equation (9.8) for $Q_{a|i}$ as an infinite series, the Q-system can then solved in terms of a "basis" of 8 Q-functions. $\mathbf{P}_a, \mathbf{P}^a$ are often the preferred choice for the basis as they have the simplest analytic properties, *cf.* subsection 9.2.2.

In the following we work in the left-right symmetric sector of the Q-system. This assumption corresponds to a simple relation between the Q-functions in the fundamental and the anti-fundamental representation for both the R-symmetry and the conformal symmetry. As an equation this amounts to

$$\mathbf{P}^a = \chi^{ab}\mathbf{P}_b, \qquad \mathbf{Q}^i = \chi^{ij}\mathbf{Q}_j, \tag{9.12}$$

where the matrix χ has the non-zero elements $\chi^{41} = -\chi^{14} = \chi^{23} = -\chi^{32} = 1$. From the previous paragraph it is easy to see that this simplifies the Q-system significantly as the basis of the Q-functions is reduced to the four functions $\mathbf{P}_a$.

The Q-system has a large redundancy as it is invariant under the action of a $\mathrm{GL}(4)\times\mathrm{GL}(4)$ group, the so-called H-symmetry. In the following this redundancy is fixed by explicit choices for the asymptotics, parity transformations and normalizations of the **P**- and **Q**-functions.

9.2.2 Analyticity and asymptotics encode physics

To get from the Q-system to the quantum spectral curve one fixes the analytic properties of the Q-functions. The **P**-functions have the simplest analytic structure. On the first Riemann sheet they have a single square-root branch cut on the interval $u \in [-2g, 2g]$, where g is defined as

$$g = \frac{\sqrt{\lambda}}{4\pi}. \tag{9.13}$$

This single branch cut can be resolved using the Zhukovsky variable $x(u)$,

$$x(u) = \frac{1}{2g}\left(u + \sqrt{u+2g}\sqrt{u-2g}\right) \qquad u = x + \frac{1}{x}. \tag{9.14}$$

This maps the first Riemann sheet of the **P**-functions to the complement of the unit disk in the x-plane. The $\mathbf{P}_a$ functions admit a convergent Laurent series in x for $|x| > 1$. Analytically continuing $\mathbf{P}_a$ through the branch cut in the u-plane corresponds to sending x to $\widetilde{x} = \frac{1}{x}$ in this Laurent expansion. We write $\widetilde{\mathbf{P}}_a$ for this analytic continuation of $\mathbf{P}_a$.

On one Riemann sheet the **Q**-functions are upper half-plane analytic. On this same sheet they have a tower of square-root branch cuts on the intervals $u \in [-2g - \mathrm{i}n, 2g - \mathrm{i}n]$ for $n \in \mathbb{Z}_{\geq 0}$. The analytic continuation $\widetilde{\mathbf{Q}}_i$ of $\mathbf{Q}_i$ through the branch cut on the real line is lower half-plane analytic with branch cuts on $u \in [-2g + \mathrm{i}n, 2g + \mathrm{i}n]$ for $n \in \mathbb{Z}_{\geq 0}$. As the branch cuts are square-roots we find that on the real line

$$\mathbf{Q}_i + \widetilde{\mathbf{Q}}_i = \text{regular}, \qquad \frac{\mathbf{Q}_i - \widetilde{\mathbf{Q}}_i}{g(x - \frac{1}{x})} = \text{regular}. \tag{9.15}$$

Large u asymptotics

The difference between the quantum spectral curve for the spectrum [121,122] and for the Hagedorn temperature [11, 124] lies in the large u-asymptotics of the Q-functions. For the spectral problem the $\mathbf{P}_a$ and $\mathbf{Q}_i$ have powerlaw asymptotics where the leading powers encode the global charges of the state. By contrast, [11] found that the Hagedorn problem leads to exponential behavior in u for the $\mathbf{Q}_i$. Specifically, to compute the Hagedorn temperature we define[1]

$$y = e^{-\mathrm{i}\pi} \exp(1/(2T_H)), \tag{9.16}$$

and then the leading large u asymptotics of the $\mathbf{P}$- and $\mathbf{Q}$-functions are

$$\mathbf{P}_a \sim \begin{pmatrix} 1 \\ u \\ u^2 \\ u^3 \end{pmatrix}_a, \qquad \mathbf{Q}_i \sim \begin{pmatrix} y^{\mathrm{i}u} \\ u y^{\mathrm{i}u} \\ y^{-\mathrm{i}u} \\ u y^{-\mathrm{i}u} \end{pmatrix}_i. \tag{9.17}$$

This particular choice of ordering for the asymptotics of the components of $\mathbf{P}$ and $\mathbf{Q}$ fixes part of the redundancy that we mentioned at the end of the previous subsection.

While these asymptotics were first derived from the TBA we can briefly motivate their form. After conformally mapping the thermal partition function from $\mathbb{R} \times S^3$ to $S^1 \times S^3$ the inverse temperature $\frac{1}{T}$ is a fugacity for the dilatation operator. From the spectrum it is known that such a "twist" corresponds to exponential asymptotics for the Q-functions [126]. This gives the temperature dependence in the asymptotics of the $\mathbf{Q}_i$. The additional factor $e^{\mathrm{i}\pi}$ in y ensures the correct boundary conditions for the partition function on the circle S^1. Finally, the redundancies in the $\mathbf{P}_a$-functions imply that we can choose all of them to have different asymptotics and the $\mathbf{P}_a$ in (9.17) are the lowest order polynomials satisfying this constraint. Similarly, the additional powers of u in $\mathbf{Q}_2$ and $\mathbf{Q}_4$ remove the degeneracy in the asymptotics of the $\mathbf{Q}$-functions.

Gluing conditions

Solving the quantum spectral curve means determining the Q-functions that are compatible with the analytic structure and the asymptotics in (9.17). This fixes the value of y and thus gives the Hagedorn temperature.

A big step towards solving the QSC comes from understanding the analytic continuation of $\mathbf{Q}_i$ through the branch cut on the real line. As we already noted, while $\mathbf{Q}_i$ is upper half-plane analytic, the analytic continuation $\widetilde{\mathbf{Q}}_i$ is analytic on the lower half-plane. The parity transformed $\underline{\mathbf{Q}}_i \equiv \mathbf{Q}_i(-u)$ has the same analyticity properties as $\widetilde{\mathbf{Q}}_i$ and thus there should exist an analytic matrix $\mathscr{L}$ such that

$$\widetilde{\mathbf{Q}}_i = \mathscr{L}_i{}^j \underline{\mathbf{Q}}_j. \tag{9.18}$$

This gluing condition becomes constraining once the matrix $\mathscr{L}$ is determined. For the asymptotics (9.17) it was found in [11, 124] that the gluing matrix has

[1] Compared to Paper VI we have inverted y to match with the convention for ABJM in Paper VII.

the form

$$\mathscr{L} = \operatorname{diag}\left(e^{2\pi u}, -e^{2\pi u}, e^{-2\pi u}, -e^{-2\pi u}\right). \tag{9.19}$$

The zero-coupling solution

We are now ready to discuss the zero coupling solution. At zero coupling, the square-root branch cuts shrink to points and the analyticity properties outlined above forbid poles in the Q-functions. We can thus start from an *ansatz* for $\mathbf{P}_a$ and $\mathbf{Q}_i$ with the leading large u-asymptotics (9.17) and no poles at $u = 0$. This *ansatz* has a finite number of undetermined coefficients. To fix all the redundancy we choose in the *ansatz* that the $\mathbf{P}_a$- and $\mathbf{Q}_i$-functions have simple parity transformation properties and impose $\mathbf{P}_1 = 1$, $\mathbf{P}_2 = u$ and that $\mathbf{P}_3$ has no constant term. With the *ansatz* we solve equation (9.8) for the $Q_{a|i}$. The symmetric sector assumption (9.12) constrains this solution through the equation $Q^{a|i} = \chi^{ab}\chi^{ij}Q_{b|j}$. We also impose the equations (9.9) and (9.11). This fixes all the coefficients in the *ansatz* ,

$$\mathbf{P}_a = \begin{pmatrix} 1 \\ u \\ -8\mathrm{i}\cosh^4\frac{1}{4T_H}u^2 \\ -\frac{8}{3}\mathrm{i}\cosh^4\frac{1}{4T_H}u\left(u^2+3\tanh^2\frac{1}{4T_H}-2\right) \end{pmatrix}_a, \tag{9.20}$$

$$\mathbf{Q}_i = \begin{pmatrix} \coth\frac{1}{4T_H}y^{\mathrm{i}u} \\ -\mathrm{i}\coth\frac{1}{4T_H}y^{\mathrm{i}u}\left(u+\mathrm{i}\frac{1-3\tanh^2\frac{1}{4T_H}}{4\tanh\frac{1}{4T_H}}\right) \\ -\coth\frac{1}{4T_H}y^{-\mathrm{i}u} \\ \mathrm{i}\coth\frac{1}{4T_H}y^{-\mathrm{i}u}\left(u-\mathrm{i}\frac{1-3\tanh^2\frac{1}{4T_H}}{4\tanh\frac{1}{4T_H}}\right) \end{pmatrix}_i. \tag{9.21}$$

The last step in solving the quantum spectral curve at zero coupling is to check the regularity conditions (9.15) at the cut. This leads to a polynomial equation

$$y^2 + 4y + 1 = 0, \tag{9.22}$$

and thus we can solve for the Hagedorn temperature at zero coupling

$$T_H^{(0)} = \frac{1}{2\log(2+\sqrt{3})}. \tag{9.23}$$

9.2.3 The perturbative solution

The algorithm for the perturbative solution of the quantum spectral curve was first developed in [127] and adapted to the Hagedorn temperature by [11,124].

To perturbatively solve the quantum spectral curve to higher order in g^2 we start from an *ansatz* for the $\mathbf{P}$-functions. As noted earlier the $\mathbf{P}$-functions admit a convergent Laurent expansion in the Zhukovsky variable $x(u)$. Combining this with the gauge choices we made for the zero-coupling solution, the general *ansatz* is

$$\mathbf{P}_a = \begin{pmatrix} 1+\sum_{n\geq 1}^{\infty} \frac{c_{1,n} g^{2n}}{x^{2n}} \\ (xg)\left(1+\sum_{n\geq 1}^{\infty} \frac{c_{2,2n-1} g^{2(n-1)}}{x^{2n}}\right) \\ A_3 (xg)^2 \left(1+\sum_{n\geq 2}^{\infty} \frac{c_{3,2n-2} g^{2(n-2)}}{x^{2n}}\right) \\ A_4 (xg)^3 \left(1+c_{4,-1}(xg)^{-2}+\sum_{n\geq 2}^{\infty} \frac{c_{4,2n-3} g^{2(n-3)}}{x^{2n}}\right) \end{pmatrix}_a . \tag{9.24}$$

The powers of g are chosen such that all coefficients have expansions in g^2 of the form $c_{a,n} = c_{a,n}^{(0)} + c_{a,n}^{(1)} g^2 + \ldots$ and $A_a = A_a^{(0)} + A_a^{(1)} g^2 + \ldots$.

The perturbative solution of the quantum spectral curve is found iteratively order by order in g^2. We give here the algorithm to get order g^2, but higher orders can be obtained in a similar fashion. Truncating the *ansatz* for $\mathbf{P}$ to order g^2, $\mathbf{P}_a = \mathbf{P}_a^{(0)} + g^2 \mathbf{P}_a^{(1)} + \mathscr{O}(g^4)$, some coefficients are already fixed from the zero coupling solution and only a finite number of coefficients $A_a^{(1)}$ and $c_{a,n}^{(0)}$, $c_{a,n}^{(1)}$ are undetermined. We write the order g^2 term in the expansion of the $Q_{a|i}$ as $Q_{a|i}^{(1)} = Q_{a|j}^{(0)} (b^j{}_i)^+$ in terms of the zeroth order term $Q_{a|j}^{(0)}$ and some matrix $b^j{}_i(u)$. The order g^2 of the QQ-relation (9.8) can then be put into the form

$$(b^j{}_i)^{++} - b^j{}_i = -(Q^{a|j\,(0)})^+ \left(\mathbf{P}_a^{(1)} \mathbf{P}^{c\,(0)} + \mathbf{P}_a^{(0)} \mathbf{P}^{c\,(1)}\right) (Q_{c|i}^{(0)})^- . \tag{9.25}$$

This finite difference equation can be solved for $b^j{}_i$ in terms of the undetermined coefficients in the *ansatz* for $\mathbf{P}$. Some of the coefficients of $\mathbf{P}$ are then fixed by enforcing equation (9.11) to order g^2. At large u we check that $\mathbf{Q}_i = -\mathbf{P}^a Q_{a|i}^+$ truncated to $\mathscr{O}(g^2)$ satisfy

$$\frac{\mathbf{Q}_2}{\mathbf{Q}_1} = -\mathrm{i}u + \mathscr{O}(u^0), \qquad \frac{\mathbf{Q}_4}{\mathbf{Q}_3} = -\mathrm{i}u + \mathscr{O}(u^0). \tag{9.26}$$

Finally we impose that the equations (9.15) are satisfied at the branch cut. This fixes the $\mathbf{P}$-functions to order g^2.

The correction to the Hagedorn temperature can be read off from the expansion of y. Writing $T_H = T_H^{(0)} + g^2 T_H^{(1)} + \mathscr{O}(g^4)$ and $y_0 = -e^{1/(2T_H^{(0)})}$ the expansion of $y^{\mathrm{i}u}$ in powers of g takes the form

$$y^{\mathrm{i}u} = y_0^{\mathrm{i}u} - \mathrm{i}u g^2 \frac{T_H^{(1)}}{2\left(T_H^{(0)}\right)^2} y_0^{\mathrm{i}u} + \mathscr{O}(g^4). \tag{9.27}$$

This means that at order g^2 the coefficients of $uy_0^{\mathrm{i}u}$ and $y_0^{\mathrm{i}u}$ in the large u expansion of $\mathbf{Q}_1$ have the ratio $-\mathrm{i}T_H^{(1)}/(2(T_H^{(0)})^2)$. This then gives

$$T_H = \frac{1}{2\log(2+\sqrt{3})}\left(1+2g^2+\mathscr{O}(g^4)\right). \tag{9.28}$$

This matches the result in (9.2) [113].

9.2.4 The numerical solution and strong coupling

The numerical method for solving the quantum spectral curve was first developed in [128] and adapted to the computation of the Hagedorn temperature by [124].

For the perturbative solution of the quantum spectral curve we truncated the general *ansatz* (9.24) for the **P**-functions to a fixed order in g. By contrast, for the numerical solution we truncate it to a fixed order N in the inverse Zhukovsky variable $\frac{1}{x}$. The QQ-relation (9.9) implies that the functions $Q_{a|i}$ have an infinite tower of branch cuts strictly in the lower half-plane. Consequently, the subleading terms in the expansion

$$Q_{a|i} = y^{-\mathrm{i}s_i u} u^{t_{a|i}} \sum_{n=0} \frac{B_{a|i,n}}{u^n}, \tag{9.29}$$

are strongly suppressed for u large. For the numerical algorithm we truncate this expansion to a finite number of terms, K. The constants s_i and $t_{a|i}$ are determined from the zero coupling solution. We can then determine the coefficients $B_{a|i,n}$, $n \leq K$, by solving the QQ-relation

$$Q_{a|i}^{+} - Q_{a|i}^{-} = -\mathbf{P}_a \mathbf{P}^b Q_{b|i}^{+} \tag{9.30}$$

to sufficiently high order in its large u expansion.

We then choose the $|I_P|$ points $I_P = \{-2g\cos[\frac{\pi}{|I_P|}(n-\frac{1}{2})]\}_{n=1,\dots,|I_P|}$ on the cut on the real line with the aim of evaluating the **Q**-functions at $I_P + \mathrm{i}0$. To get there, we start by evaluating the truncated $Q_{a|i}$ at $I_P + \frac{U+1}{2}\mathrm{i}$ for U a large even integer such that the truncation is sensible. Then we can iterate with the QQ-relation

$$Q_{a|i}^{-} = \left(\delta_b^a + \mathbf{P}_a \mathbf{P}^b\right) Q_{b|i}^{+} \tag{9.31}$$

to get the value of $Q_{a|i}$ at $I_P + \frac{\mathrm{i}}{2}$. From these we compute the values of the **Q**-functions just above the branch cut on the real line through the relation $\mathbf{Q}_i = -\mathbf{P}^a Q_{a|i}^{+}$ where we plug in **P** evaluated at $I_P + \mathrm{i}0$. Computing also $\mathbf{Q}_i$ at $-I_P + \mathrm{i}0$ we use the gluing condition (9.18) to get the $\widetilde{\mathbf{Q}}_i$ at $I_P - \mathrm{i}0$ just below the cut.

For the exact solution of the quantum spectral curve the values of $\mathbf{Q}_i$ just above the cut on the real line should match the values of $\widetilde{\mathbf{Q}}_i$ just below the cut.

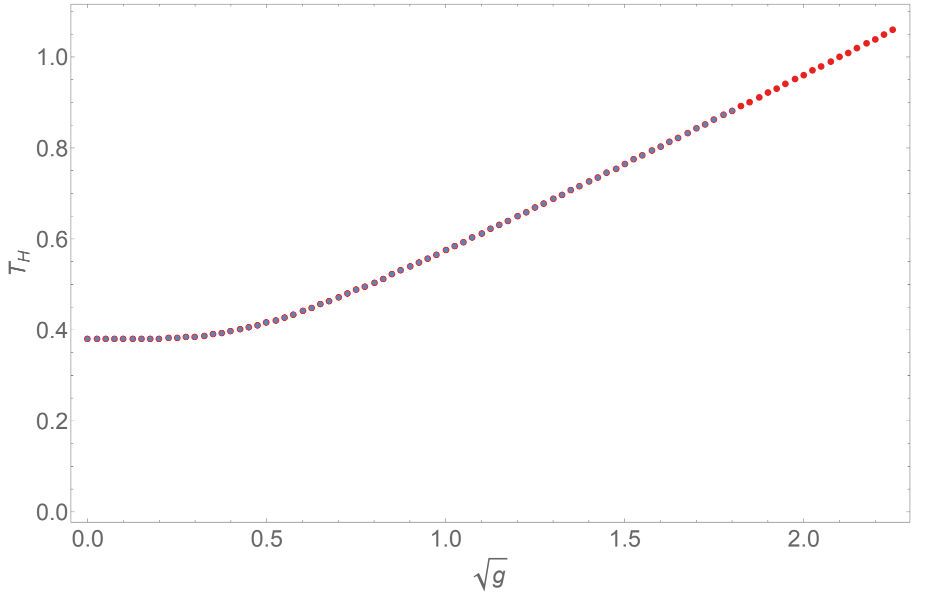


Figure 9.1. The numerical results for the Hagedorn temperature plotted against the square-root of the coupling g. In blue are the results from [124] and in red are the results from Paper VI. This figure is taken from Paper VI.

To numerically solve the quantum spectral curve we therefore minimize the function

$$F(\{c_{a,n}\},y) = \sum_{i=1,2} \sum_{p \in I_P} \left| \frac{\widetilde{\mathbf{Q}}_i(p - \mathrm{i}0)}{\mathbf{Q}_i(p + \mathrm{i}0)} - 1 \right|^2 \tag{9.32}$$

on the space of coefficients $\{c_{a,n}\}_{n \leq N}$ and y. For this numerical minimization we use the Levenberg-Marquardt method. The authors of [124] used this method to compute the Hagedorn temperature up to $\sqrt{g} = 1.8$ and in Paper VI we extended this to $\sqrt{g} = 2.25$. Figure 9.1 shows a plot of both results.

To the numerical results we can fit an expansion of the Hagedorn temperature in $\frac{1}{\sqrt{g}}$. Using that $g = \frac{1}{4\pi\alpha'}$ we can compare this fit to the result (8.13) from our effective model on AdS. In Paper VI we found that within numerical precision the integrability result matches the leading two terms and the fourth term in (8.13). We estimated that

$$\Delta c = -2.77259 \pm 1 \times 10^{-5}. \tag{9.33}$$

This estimate is equal to $-4\log(2)$ with six digits precision. These results give support to our effective supergravity model and the conjecture (8.14) for the zero-point shift.

9.3 The ABJM quantum spectral curve

In this section we discuss the quantum spectral curve for computing the Hagedorn temperature in ABJM theory developed in Paper VII. This development is based on combining the AdS_4 QSC for the spectrum from [129, 130] with insights from the $\mathcal{N}=4$ Hagedorn QSC [11, 124] that we presented in the previous section.

9.3.1 The $\mathfrak{osp}(6|4)$ Q-system

The superconformal algebra for ABJM theory in the planar limit is $\mathfrak{osp}(6|4)$ with $\mathfrak{so}(6)$ R-symmetry subalgebra and $\mathfrak{sp}(4,\mathbb{R})\simeq\mathfrak{so}(2,3)$ conformal subalgebra. Similar to the $\mathfrak{psu}(2,2|4)$ case discussed in the previous section, all the Q-functions for the $\mathfrak{osp}(6|4)$ Q-system can be written in terms of **P**-, **Q**- and $Q_{a|i}$ functions. The latter are in the spinor representations of both the SO(6) R-symmetry and the SO(2,3) conformal symmetry and they satisfy

$$\chi^{ij}Q^{a}{}_{|j}Q_{a|k}=-\delta^{i}_{j},\qquad\qquad \chi^{ij}Q^{a}{}_{|i}Q_{b|j}=\delta^{a}_{b}. \tag{9.34}$$

As the spinors of SO(2,3) do not have a chirality we use the matrix χ to lower the index $Q^{a|i}=\chi^{ij}Q^{a}{}_{|i}$. A major difference to $\mathcal{N}=4$ SYM is that the **P**- and **Q**-functions are in the vector representations of SO(6) and SO(2,3) respectively. They relate to $Q_{a|i}$ through the equations[2]

$$\mathbf{P}_A=-\frac{1}{2}Q^{+}_{a|i}\chi^{ij}\overline{\sigma}^{ab}_{A}Q^{-}_{b|j},\qquad \mathbf{Q}_I=-\frac{1}{2}(Q^{a}{}_{|i})^{+}\overline{\Sigma}^{ij}_{I}Q^{-}_{a|j}. \tag{9.35}$$

For conventions regarding the matrices $\overline{\sigma}$ and $\overline{\Sigma}$ we refer to appendix A of Paper VII and section 2 of [130].

When solving the $\mathfrak{psu}(2,2|4)$ Q-system a central piece of the perturbative and numerical algorithms is (9.8). For $\mathfrak{osp}(6|4)$ the same role is taken by the equation

$$Q^{+}_{a|i}-(\sigma^{A})_{ab}\mathbf{P}_A(Q^{b}{}_{|i})^{-}=0, \tag{9.36}$$

which comes from combining equations (9.35) and (9.34). To find the zero coupling solution we use two more QQ-relations for the $\mathfrak{osp}(6|4)$ Q-system,

$$Q_{A|IJ}=(\overline{\sigma}_A)^{ab}\left(Q^{+}_{a|i}Q^{-}_{b|j}\right)(\Sigma_{IJ})^{ij}=(\sigma_A)_{ab}\left((Q^{a}{}_{|i})^{+}(Q^{b}{}_{|j})^{-}\right)(\Sigma_{IJ})^{ij}, \tag{9.37}$$

$$Q_{AB|I}=(\sigma_{AB})_{a}{}^{b}\left((Q^{a}{}_{|i})^{+}Q^{-}_{b|j}\right)(\overline{\Sigma}_I)^{ij}=(\sigma_{AB})_{a}{}^{b}\left((Q^{a}{}_{|i})^{-}Q^{+}_{b|j}\right)(\overline{\Sigma}_I)^{ij}. \tag{9.38}$$

[2] The vector indices A and I should not be confused with the anti-symmetrized multiindices used in the previous section.

As in the previous section, the quantum spectral curve simplifies significantly by assuming that we are in the left-right symmetric sector. For the $\mathbf{P}$-functions this implies that $\mathbf{P}_5 = \mathbf{P}_6$ and it relates the functions $Q^a{}_{|i}$ to the $Q_{a|i}$ as

$$Q^a{}_{|i} = -\chi^{ab} Q_{b|j} \mathbb{K}_i^j, \tag{9.39}$$

with the matrix $\mathbb{K} = \mathrm{diag}(1,-1,-1,1)$. We plug this relation into equation (9.36) to get a linear finite difference equation for the $Q_{a|i}$ in terms of the $\mathbf{P}$-functions,

$$Q^+_{a|i} + (\sigma^A)_{ab} \mathbf{P}_A \chi^{bc} Q^-_{c|j} \mathbb{K}_i^j = 0 \tag{9.40}$$

9.3.2 Analytics, asymptotics and the Hagedorn temperature

The convention is to call h the coupling as it enters the integrability treatment of ABJM theory. The relation of this coupling to the 't Hooft coupling λ is non-trivial and the conjecture for the full relation is [131]

$$\lambda = \frac{\sinh(2\pi h)}{2\pi}\, {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\sinh^2(2\pi h)\right). \tag{9.41}$$

This can be expanded at weak and at strong coupling to give respectively

$$h(\lambda) = \lambda - \frac{\pi^2}{3}\lambda^3 + \frac{5\pi^4}{12}\lambda^5 - \frac{893\pi^6}{1260}\lambda^7 + \mathscr{O}(\lambda^9) \tag{9.42}$$

$$= \sqrt{\frac{1}{2}\left(\lambda - \frac{1}{24}\right)} - \frac{\log(2)}{2\pi} + \mathscr{O}\left(e^{-\pi\sqrt{8\lambda}}\right). \tag{9.43}$$

At strong coupling we also use the shifted 't Hooft coupling $\hat{\lambda} = \lambda - \frac{1}{24}$ which was argued to be the more natural parameter in this regime [132, 133, 40].

The first lesson we take from $\mathscr{N} = 4$ SYM is that the analytic structure for the Hagedorn QSC is the same as for the spectral problem [129, 130]. On the first Riemann sheet the $\mathbf{P}_A$ have a single square-root branch cut on the interval $u \in [-2h, 2h]$ and thus they can be written as a convergent Laurent series in the Zhukovsky variable $x(u)$ defined in (9.14). The $\mathbf{Q}_I$ have a tower of branch-cuts on the intervals $u \in [-2h - \mathrm{i}n, 2h - \mathrm{i}n]$, $n \in \mathbb{Z}_{\geq 0}$ and the $Q_{a|i}$ have a tower of branch-cuts at $u \in [-2h - \mathrm{i}\frac{2n+1}{2}, 2h - \mathrm{i}\frac{2n+1}{2}]$, $n \in \mathbb{Z}_{\geq 0}$. The two linear combinations of $\mathbf{Q}$-functions

$$\mathbf{Q}_I + \widetilde{\mathbf{Q}}_I, \qquad \frac{\mathbf{Q}_I - \widetilde{\mathbf{Q}}_I}{h\left(x - \frac{1}{x}\right)}, \tag{9.44}$$

should be regular on the branch cut $u \in [-2h, 2h]$.

The second lesson concerns the asymptotics of the Q-functions. At large u the $\mathbf{P}$ functions should have polynomial asymptotics and the $\mathbf{Q}$-functions have exponential asymptotics which encode the Hagedorn temperature. The charges under the dilation determine the powers of $y^{-\mathrm{i}u} = e^{-\pi u}\exp(-\frac{\mathrm{i}u}{2T_H})$. Consequently the asymptotics take the form

$$\mathbf{P}_A \sim \begin{pmatrix} u \\ 1 \\ u^4 \\ u^3 \\ u^2 \\ u^2 \end{pmatrix}_A, \quad \mathbf{Q}_I \sim \begin{pmatrix} y^{-\mathrm{i}u}u \\ y^{-\mathrm{i}u} \\ y^{\mathrm{i}u}u \\ y^{\mathrm{i}u} \\ 0 \end{pmatrix}_I, \quad Q_{a|i} \sim \begin{pmatrix} y^{-\mathrm{i}u} & u & 1 & y^{\mathrm{i}u} \\ y^{-\mathrm{i}u}u^2 & u^3 & u^2 & y^{\mathrm{i}u}u^2 \\ y^{-\mathrm{i}u}u & u^2 & u & y^{\mathrm{i}u}u \\ y^{-\mathrm{i}u}u^3 & u^4 & u^3 & y^{\mathrm{i}u}u^3 \end{pmatrix}_{a|i}. \tag{9.45}$$

The third and final lesson is about the gluing matrix. From $\mathcal{N}=4$ we learned that the exponential asymptotics of the $\mathbf{Q}$-functions require exponential behavior for the elements on the main diagonal of the gluing matrix. The big difference for the gluing matrix in ABJM is that it is not diagonal. This was already noted for the spectral problem [130]. Still, the ordering of the $\widetilde{\mathbf{Q}}_I$ should be the same at $u \to -\mathrm{i}\infty$ as for the $\mathbf{Q}_I$ at $u \to \mathrm{i}\infty$, *i.e.* $|\widetilde{\mathbf{Q}}_1| > |\widetilde{\mathbf{Q}}_2| > |\widetilde{\mathbf{Q}}_3| > |\widetilde{\mathbf{Q}}_4|$. This constrains the gluing matrix $\mathcal{L}$ to be upper triangular. Similarly the orderings of $\mathbf{Q}_I$ and $\widetilde{\mathbf{Q}}_I$ should be the same at $u \to -\infty$, implying $\mathcal{L}_2{}^3 = 0$. In addition, as the branch cuts are square-root the gluing matrix should satisfy $\mathcal{L}_I{}^J \underline{\mathcal{L}_J}^K = \delta_I^K$. With the assumption $\mathcal{L}_1{}^1 = -\mathcal{L}_2{}^2$ and $\mathcal{L}_3{}^3 = -\mathcal{L}_4{}^4$ we can use the H-symmetry to fix $\mathcal{L}_1{}^2 = \mathcal{L}_3{}^4 = 0$. Finally, $\mathbf{Q}_5$ should be glued to itself. We thus find that the analytic continuation of the $\mathbf{Q}$-functions through the branch cut on the real line relates to their parity transform as

$$\widetilde{\mathbf{Q}}_I = \begin{pmatrix} e^{-2\pi u} & 0 & 16\mathrm{i}\sinh^2\frac{1}{2T_H} & 0 & 0 \\ 0 & -e^{-2\pi u} & 0 & -16\mathrm{i}\sinh^2\frac{1}{2T_H} & 0 \\ 0 & 0 & -e^{2\pi u} & 0 & 0 \\ 0 & 0 & 0 & e^{2\pi u} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}_I^{\;J} \underline{\mathbf{Q}}_J. \tag{9.46}$$

Note that the above reasoning does not fix an overall constant prefactor for the exponentials on the diagonal as well as the two non-trivial off-diagonal elements. These are fixed once we solve the quantum spectral curve at tree level.

The zero-coupling solution

For $\mathcal{N}=4$ SYM we find the zero-coupling solution by starting from an *ansatz* for the $\mathbf{P}$- and $\mathbf{Q}$-functions. For ABJM we start instead with the most general *ansatz* for $Q_{a|i}$ compatible with the large u asymptotics (9.45) and that is finite at $u=0$. This *ansatz* is completely fixed up to gauge choices by the equations (9.37), (9.38) and (9.34). For the gauge choice we impose that **P**- and **Q**-functions map under parity to themselves up to a similarity transformation. We end up with the zero-coupling solution

$$\mathbf{P}_A = \begin{pmatrix} u \\ 1 \\ -\frac{4}{3}\cosh^4\left(\frac{1}{4T_H}\right)\left(u^4 - \frac{\left(2\mathrm{sech}^2\left(\frac{1}{4T_H}\right)-\frac{3}{4}\right)^2-2}{3}\right) \\ -\frac{16}{3}\cosh^4\left(\frac{1}{4T_H}\right)u\left(u^2+\frac{1}{4}\left(1-6\mathrm{sech}^2\left(\frac{1}{4T_H}\right)\right)\right) \\ 2\cosh^2\left(\frac{1}{4T_H}\right)\left(u^2+\frac{1}{6}-\mathrm{sech}^2\left(\frac{1}{4T_H}\right)\right) \\ 2\cosh^2\left(\frac{1}{4T_H}\right)\left(u^2+\frac{1}{6}-\mathrm{sech}^2\left(\frac{1}{4T_H}\right)\right) \end{pmatrix}_A \tag{9.47}$$

$$\mathbf{Q}_I = \begin{pmatrix} 8y^{-\mathrm{i}u}u\cosh^3\left(\frac{1}{4T_H}\right) \\ 8y^{-\mathrm{i}u}\cosh^3\left(\frac{1}{4T_H}\right) \\ \frac{\mathrm{i}}{8}y^{\mathrm{i}u}\mathrm{csch}^3\left(\frac{1}{4T_H}\right)\left(u-\mathrm{i}\frac{\cosh\left(\frac{1}{2T_H}\right)-3}{\sinh\left(\frac{1}{2T_H}\right)}\right) \\ \frac{\mathrm{i}}{8}y^{\mathrm{i}u}\mathrm{csch}^3\left(\frac{1}{4T_H}\right) \\ 0 \end{pmatrix}_I \tag{9.48}$$

From (9.48) together with the gluing matrix we can check the analytic properties (9.44) along the real line branch cut. This gives a polynomial equation $y^2+6y+1=0$ with the solution

$$T_H^{(0)} = \frac{1}{2\log(3+2\sqrt{2})}, \tag{9.49}$$

reproducing the leading term in (9.3),

9.3.3 The solution for non-zero coupling

The techniques to solve the ABJM quantum spectral curve perturbatively and numerically are very similar to those used for the $\mathcal{N}=4$ SYM QSC. In Paper VII we give the details of the algorithms and here we only highlight some important aspects and the results.

The perturbative solution

The perturbative solution starts from the *ansatz* for the **P** functions as a Laurent series in the Zhukovsky variable

$$\mathbf{P}_A = \begin{pmatrix} xh\left(1+\sum_{n=1}^{\infty}\frac{c_{1,2n}h^{2(n-1)}}{x^{2n}}\right) \\ 1+\sum_{n=1}^{\infty}\frac{c_{2,2n}h^{2n}}{x^{2n}} \\ A_4(xh)^4\left(1+\sum_{n=2}^{\infty}\frac{c3,2n-4h^{2(n-4)}}{x^{2n}}\right) \\ A_4(xh)^3\left(1+c_{4,-1}(xh)^{-2}+\sum_{n=2}^{\infty}\frac{c_{4,2n-3}h^{2(n-3)}}{x^{2n}}\right) \\ A_5(xh)^2\left(1+\sum_{n=1}^{\infty}\frac{c_{5,2n-2}h^{2(n-2)}}{x^{2n}}\right) \\ A_5(xh)^2\left(1+\sum_{n=1}^{\infty}\frac{c_{5,2n-2}h^{2(n-2)}}{x^{2n}}\right) \end{pmatrix}_A , \qquad (9.50)$$

where the powers of h are chosen such that all coefficients $A_A, c_{A,n}$ are $\mathcal{O}(h^0)$ at small coupling. Truncating this series expansion to finite order in h we then use the equation (9.40) to solve for $Q_{a|i}$ to the same order and fix most coefficients by satisfying equations (9.34) and requiring the correct asymptotics for the **Q**-functions constructed from (9.35). The remaining coefficients and the Hagedorn temperature are then fixed from the analytic properties of the **Q**-functions on the real line branch cut using equations (9.44) and (9.46).

We computed the perturbative Hagedron temperature to order h^8, where we find

$$T_H = \frac{1}{2\log(3+2\sqrt{2})} + \frac{\sqrt{2}-1}{\log(1+\sqrt{2})}h^2 - 2.5428h^4 + 21.778h^6 - 222.30h^8 + \mathcal{O}(h^{10}). \qquad (9.51)$$

The first two terms match the previous results [116,117] given in equation (9.3). The exact expression for the higher order terms can be found in subsection 3.2.2 and appendix C of Paper VII and can be expressed using polylogarithms. We plot the different orders of the perturbative expansion of the Hagedorn temperature in Figure 9.2.

The numerical solution

The numerical solution of the ABJM Hagedorn temperature quantum spectral curve follows the same method as for $\mathcal{N}=4$ SYM. We start from the truncation of the *ansatz* for **P**-functions to finite order in the inverse Zhukovsky variable $\frac{1}{x}$ and solve equation (9.40) for an expansion of $Q_{a|i}$ at large u truncated to finite order in u^{-1}. With the equation

$$Q^-_{a|i} = -\mathbf{P}_A(\sigma^A)_{ab}\chi^{bc}Q^+_{c|j}\mathbb{K}^j_i \qquad (9.52)$$

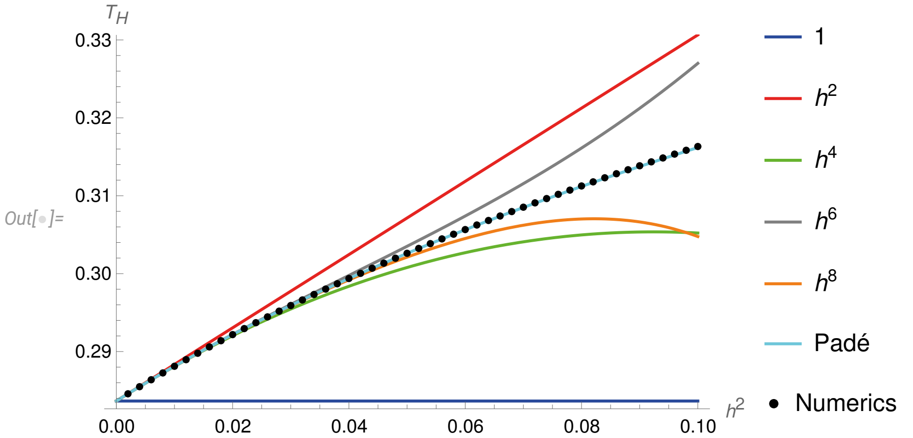


Figure 9.2. The perturbative solution of the quantum spectral curve truncated to various orders is compared to the numerical solution in the range $0 \leq h^2 \leq 0.1$ and the [4/4] Padé approximation (9.57). Figure taken from Paper VII.

we can start from values of $Q_{a|i}$ at points v with $\text{Im}(v) = \frac{U+1}{2}$ for U a large even integer and get values of $Q_{a|i}$ at points v' with $\text{Im}(v') = \frac{1}{2}$. Then we can evaluate $\mathbf{Q}_I$ just above the branch cut on the real line using

$$\mathbf{Q}_I = -\frac{1}{2}(\overline{\Sigma}_I)^{ij}\chi^{ab}Q^+_{b|k}\mathbb{K}^k_i P_A(\sigma^A)_{ac}\chi^{cd}Q^+_{d|l}\mathbb{K}^l_j \tag{9.53}$$

and then use the gluing condition (9.46) to evaluate $\widetilde{\mathbf{Q}}_I$ just below the cut. On the exact solution of the quantum spectral curve the function

$$F(\{c_{A,n}\},y) = \sum_{v\in I_P}\sum_{I=3}^{4}\left|\frac{\mathbf{Q}_I(v+\text{i}0)}{\widetilde{\mathbf{Q}}_I(v-\text{i}0)} - 1\right|^2 \tag{9.54}$$

is zero. Using the Levenberg-Marquardt algorithm we numerically minimize this function on the space of y and the coefficients $\{c_{A,n}\}_{n\leq N}$ of the truncated $\mathbf{P}_A$.

In figures 9.2 and 9.3 we show the numerical results for the Hagedorn temperature at weak and strong coupling respectively.

At strong coupling we can fit to the numerical Hagedorn temperature an expansion in $\hat{\lambda}^{-1/4}$. After checking that the leading two coefficients of the fit to the AdS result (8.6), we fix them to their exact value to get a better estimate for the subleading terms. We find that

$$T_H(\hat{\lambda}) = \frac{\hat{\lambda}^{1/4}}{2^{5/4}\sqrt{\pi}} + \frac{3}{8\pi} - (0.0308 \pm 0.0004)\hat{\lambda}^{-1/4} + (0.046 \pm 0.003)\hat{\lambda}^{-1/2} + \mathscr{O}(\hat{\lambda}^{-3/4}). \tag{9.55}$$

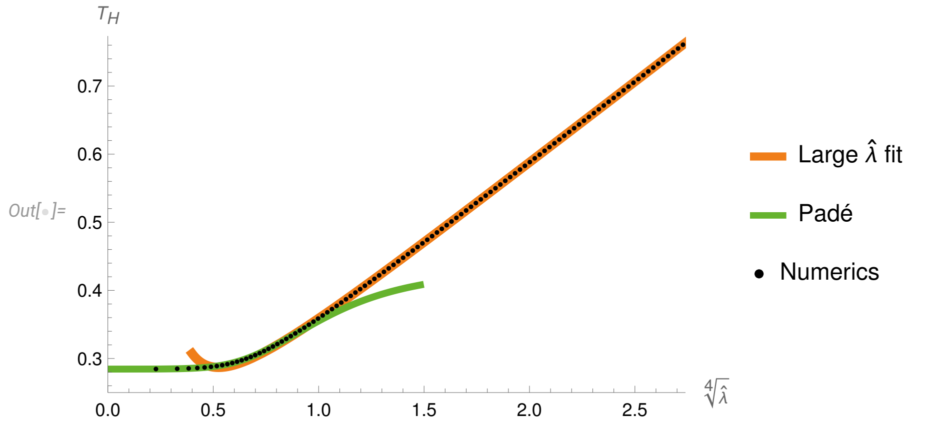


Figure 9.3. The strong coupling numerical data for $0.21 \leq \sqrt{h} \leq 2.28$ plotted against the fourth root of the shifted 't Hooft coupling $\hat{\lambda} = \lambda - \frac{1}{24}$. We also plot both the Padé approximation (9.57) from weak coupling and the strong coupling fit. Figure taken from Paper VII with an alteration to the choice of colors.

The fitted function up to order $\hat{\lambda}^{-5/4}$ is plotted in figure 9.3. The term at order $\hat{\lambda}^{-1/2}$ matches within the error estimate to the AdS result (8.13). Fixing also this coefficient to the analytic value from (8.13), the updated fit value for the $\hat{\lambda}^{-1/4}$ term gives for the string zero-point shift the estimate

$$\Delta c = -2.0782 \pm 0.0016. \tag{9.56}$$

Within the estimated error this equals $-3\log(2)$ and thus supports the conjecture (8.14).

Padé approximation

The growth of the numerical values of the subleading coefficients in the perturbative solution (9.51) indicates that this perturbative expansion for the Hagedorn temperature has a finite radius of convergence. We expect that this radius of convergence is $h^2 < \frac{1}{16}$. The same radius is expected for the perturbative solution of the spectral problem [134]. This agrees qualitatively with what we see in Figure 9.2. For the spectral problem it was suggested to use a Padé approximation to the weak coupling expansion to get a better approximation at small but finite coupling [134]. For the present case the best Padé approximation is

$$T_H^{\text{Padé}[4/4]} = \frac{0.28365 + 4.1647h^2 + 10.425h^4}{1 + 13.026h^2 + 24.137h^4}. \tag{9.57}$$

In Figure 9.2 we show this Padé approximation and the numerical results for the Hagedorn temperature for $h^2 \leq 0.1$. We observe that while the perturbative results quickly start to diverge, the Padé approximation matches well with the numerical data on the whole range shown on the plot. In Figure 9.3 we add the Padé approximation in the plot with the numerical results up to $\sqrt{h} \leq 2.28$ and the strong coupling fit. We observe that for $\hat{\lambda}^{1/4} \approx 0.85$ both the Padé approximation (9.57) and the strong coupling fit (9.55) match the numerical results within a tenth of a percent.

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Svensk Sammanfattning

Kvantfältteori utgör ryggraden i vår nuvarande förståelse av partikelfysik och framförallt är det gaugeteori som har bidragit till succén. Det mest framträdande exemplet på en kvantfältteori, Standardmodellen för partikelfysik, har varit oerhört framgångsrik i att beskriva elementarpartiklarna och deras interaktioner. Standardmodellen beskriver med stor precision processerna i dem experiment som utförs vid CERNs Large Hadron Collider. Avgörande för kvantfältteorins framgång med att beskriva acceleratorexperiment är att interaktionerna i centrum av dessa experiment är svaga. Detta gör att vi kan använda perturbativa gaugeteoritekniker för beskrivningen. Dock är explicita beräkningar i en stark kopplat gaugeteori svåra. En exempel är det så kallade massgapproblemet i Yang-Mills teorin, dvs. gaugeteorin som ligger till grund av Standardmodellen. Detta problem är på listan av de olösta Clay Millennium Prize-problemen [1].

I den här avhandlingen kommer vi att fokusera på en specifik klass av kvantfältteorier, de supersymmetriska gaugeteorierna, för att utforska icke-perturbativa beräkningar. Supersymmetri är en relation mellan bosoniska och fermioniska frihetsgrader. Supersymmetriska teorier har inte haft många succéer inom fenomenologin, men sedan deras första framträdande för cirka 50 år sedan har de utvecklats till en produktiv lekplats för att studera icke-perturbativa aspekter av kvantfältteori. Symmetrin mellan bosoner och fermioner är mycket begränsande och gör explicita beräkningar möjliga. Vi kommer att utforska tillämpningar av två av de metoder för icke-perturbativa beräkningar som har utvecklats.

I de tre första delarna av denna avhandling undersöker vi supersymmetriska gaugeteorier på tre, fyra och sju dimensionella deformerade sfärer respektive. I dessa teorier tillåter en teknik som kallas supersymmetrisk lokalisering oss att exakt beräkna partitionsfunktionen. Partitionsfunktionen innehåller mycket information om en teori.

I tre dimensioner utforskar vi först en utökning av supersymmetrin för vissa gauge teorier på den deformerade sfären vid en finjustering av en massparameter. Den ytterligare supersymmetrin leder till en förenkling av partitionsfunktionen. Därefter jämför vi partitionsfunktionerna för par av vad som kallas duala teorier. Duala teorier beskriver samma fysik. Genom att kontrollera att deras partionsfunktioner är desamma samlar vi bevis för att de två teorierna faktiskt beskriver samma fysik. Den utökade supersymmetrin möjliggör analytiska jämförelser och när symmetrin bryts faller vi tillbaka på numeriska approximation till partitionsfunktionen.

I fyra dimensioner diskuterar vi den allmänna formen för den fria energin, som är proportionell mot logaritmen av partitionsfunktionen, för så kallade superkonforma fältteorier på deformerade sfärer. Superkonforma kvantfäteorier är både supersymmetriska och skalinvarianta. Ett exempel på en sådan superkonform teori i fyra dimensioner är maximalt supersymmetrisk Yang-Mills-teori. För detta exemplet kan vi använda supersymmetrisk lokalisering och jämföra logaritmen av partitionsfunktionen med det allmänna resultat för den fria energin.

I sju dimensioner använder vi partionsfunktionen för att utforska en intressant gräns för den supersymmetriska gaugeteorin. I denna gräns förväntas teorin beskriva samma fysik som en supersymmetrisk gravitationsteori. Partitionsfunktionen av den supersymmetrisk gaugeteorin i sju dimensioner har inte utforskats mycket förut. Även om lokalisering är tillämplig så har inte alla delar av partitionsfunktionen beräknats. Vi förslår en bidrag av tredimensionella membran som kallas kontaktinstantoner. Inspirerad av beräkningar i lägre dimensioner förväntar vi oss att dessa instantoner ger ett viktigt bidrag i närheten av gränsen som vi utforskar.

I den sista delen av denna avhandling använder vi integrabilitet i vissa superkonforma gaugeteorier för att beräkna Hagedorn-temperaturen i strängteori på anti-de Sitter (AdS) geometrier. Detta är en användning av AdS/CFT dualiteten som lär oss att strängteori i AdS besrkiver samma fysik som en konform kvantfältteori i en lägre dimension. Två exemplar är dualiteten av strängteorin på $\mathrm{AdS}_5 \times S^5$ och den maximal supersymmetriska Yang-Mills teorin i fyra dimensioner ($\mathcal{N} = 4$ SYM) och dualiteten av strängteorin på $\mathrm{AdS}_4 \times \mathbb{C}\mathrm{P}^3$ och den så kallade ABJM teorin i tre dimensioner. Både $\mathcal{N} = 4$ SYM och ABJM är integrerbara teorier vilket betyder att det finns kraftfulla tekniker för exakta beräkningar. Den tekniken vid använder heter Quantum Spectral Curve (QSC) och det finns både analytiska lösningstekniker för svakt koppling och numeriska för finit koppling. Hagedorn-temperaturen i strängteori är en temperatur där en varm gas av strängar förväntas genomgå en fasövergång till en “strängstjärna”. Vi kombinerar en effektiv modell i AdS geometrin och numeriska QSC resultat i den duala konforma fältteorin för att beräkna de första tre korrigeringstermer till Hagedorn-temperaturen på grund av AdS geometrins krökning.

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