SCATTERING AND PRODUCTION AMPLITUDES WITH UNSTABLE PARTICLES (*)

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In the past few years experiments have revealed the existence of a large number of resonances in elementary particle reactions. Many physicists have found attractive the idea of considering these resonances as unstable particles, decaying via strong interactions, but to be treated on an equal basis with the stable particles. This poses the challenge to the S-matrix theory of strong interactions [1] of dealing with processes in which unstable particles occur as intermediate or external particles. Forces arising from the exchange of unstable particles have been the subject of intensive investigation, i.e., the exchange of a ρ-meson ($J = I = 1$ pion-pion resonance) in pion-pion scattering. As an example, we shall be concerned here with processes such as $\pi + N \rightarrow \rho + N$, where the unstable particle occurs as an external particle, and with the effect of this process on elastic pion-nucleon scattering. One cannot insert an unstable particle state directly into an S-matrix theory, because in such a theory transitions are defined only between asymptotic states. Therefore we consider processes such as $\pi + N \rightarrow \pi + \pi + N$, using the existence of the pion-pion resonance to reduce the complexity due to the three body state [2].

In the case of an S-wave pion-pion resonance, for example, the first approximation is to assume that the amplitude T_{21} for the pion production process depends only on the invariant mass squared ω of the two final state pions in addition to the usual centre-of-mass energy squared variable s, and invariant momentum transfer variable t between the initial and final nucleon. For pion-pion resonance states of angular momentum $l > 0$ there is a further dependence on the orientation of the momentum of these pions in their centre-of-mass, which is given explicitly by the spherical harmonic of order l. We shall restrict our discussion here to the $l = 0$ case. We assume further that as a function of s, t and ω this transition amplitude is analytic except for poles and branch points associated with single particle and multiparticle intermediate states respectively. The physical values of the amplitude are obtained by approaching each of the branch cuts from the *upper* half of the corresponding complex plane. In general, the location of each cut depends on the other variables; for some range of values of these variables the cut extends below the physical threshold and into the complex plane.

We require that the discontinuity of T_{21} in s and ω satisfy a generalized form of the unitarity condition above their physical thresholds [3,4], $s = (M+\mu)^2$ and $\omega = 4\mu^2$ respectively, (M is the nucleon mass and μ is the pion mass):

$$T_{21}(s_+ t\omega) - T_{21}(s_- t\omega) = 2i \sum \{T_{21}(s_+ t'\omega) T_{11}(s_- t'') + T^c_{22}(s_+ t'\omega\omega') T_{21}(s_- t''\omega'_-)\} \quad (1)$$

$$T_{21}(st\omega_+) - T_{21}(st\omega_-) = 2i e^{i\delta(\omega)} \sin \delta(\omega) T_{21}(st\omega_-) \quad (2)$$

In Eqs. (1) and (2), T_{11} is the amplitude for pion-nucleon scattering, T^c_{22} is the part of the amplitude

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for the process $\pi+\pi+N\to\pi+\pi+N$ excluding a disconnected contribution in which the pions scatter without interacting with the nucleon, and $\delta(\omega)$ is the S-wave pion-pion phase shift. The subscript $+(-)$ indicates that the corresponding variable is taken above (below) the real axis. The primes designate intermediate variables, and the symbol Σ represents the phase space integrals over the 4-momenta of the pion and nucleon in intermediate states. An approximation has been made in keeping only the two and three particle intermediate states of lowest mass. Similar equations are also satisfied by the transition amplitudes T_{11} and T_{22}^c.

Combining these discontinuity equations together with the reality condition

$$T_{ij}^*(xy\ldots) = T_{ji}(x^*y^*\ldots) \tag{3}$$

it can be readily verified that T_{21} satisfies the physical unitarity condition. The essential point, however, is that the quantity which appears under the integrals in dispersion relations for T_{21} in the s variable, for example, is the discontinuity of T_{21} in s, which satisfies Eq. 1, while the unitarity condition gives rise to an equation for the discontinuity of T_{21} in s and ω simultaneously, and is therefore not directly useful.

The discontinuity equation for T_{21} in ω, Eq. (2), and a similar equation for T_{22}^c can be satisfied by introducing the functions

$$M_{21}(st\omega) = T_{21}(st\omega)/f(\omega)$$
$$M_{22}(st\omega\omega') = T_{22}(st\omega\omega')/[f(\omega)f(\omega')] \tag{4}$$

where $f(\omega)$ is the S-wave pion-pion scattering amplitude

$$f(\omega) = 16\pi\sqrt{\frac{\omega}{\omega-4\mu^2}}e^{i\delta(\omega)}\sin\delta(\omega) \tag{5}$$

It can be readily verified that M_{21} and M_{22} have no discontinuity in ω for $\omega\geqq 4\mu^2$. We note that Eq. (4) is the relativistic form of the final state interaction theorem of Watson [5].

Substituting Eq. (4) in Eq. (1) we then obtain an equation for the discontinuity of M_{21} in s,

$$M_{21}(s_+t\omega)-M_{21}(s_-t\omega) = 2i\sum\{M_{21}(s_+t'\omega)T_{11}(s_-t'')+M_{22}(s_+t'\omega\omega'')|f(\omega''_+)|^2M_{21}(s_-t''\omega')\} \tag{6}$$

The approximation we now make is based on the assum ption of a narrow resonance for S-wave pion-pion scattering which implies that $|f(\omega)|$ has a sharp peak at some value $\omega = m^2$, where m is defined as the mass of the unstable particle. Since M_{21} and M_{22} are analytic functions of ω, we can expand them in a Taylor series in the variable of integration ω' about m^2. Due to the factor $|f(\omega')|^2$ under the integral in Eq. (6), a good approximation is to keep only the first term in this expansion. The integral over ω' can then be carried out explicitly. Projecting out partial waves of angular momentum l we obtain

$$[M_{21}^l(s_+\omega)-M_{21}^l(s_-\omega)]/2i = M_{21}^l(s_+\omega)\rho_1(s_+)M_{11}^l(s_-)+M_{22}^l(s_+\omega m^2)\rho_2(s_+)\theta[s-(M+2\mu)^2]M_{21}^l(s_-m^2) \tag{7}$$

where

$$\rho_2(s) = \int_{4\mu^2}^{(\sqrt{s}-M)^2} d\omega'\rho_2(s\omega')|f(\omega')|^2 \tag{8}$$

and $\rho_1(s)$, $\rho_2(s,\omega)$ are the πN and $\pi\pi N$ phase space factors.

For $\omega = m^2$ Eq. (7) is identical in form to the unitarity condition for the production amplitude of a stable particle of mass m. The fact that we are dealing with an unstable particle appears in the generalized phase space integral, Eq. (8).

The procedure in establishing the integral equations satisfied by the amplitudes M_{ij} from dispersion relations is quite similar to the corresponding problem with stable particles. However, in approximating the unphysical singularities of partial wave amplitudes in the s variable, proper care has to be taken in order to insure that the analytic properties in the ω variable are satisfied. If one keeps, for example, only the contribution from the single pion intermediate state in the t-channel (see Fig. 1) corresponding to the longest part of the range of the forces in the s-channel, one finds that the amplitudes M_{21} and M_{22} have branch points at $\omega = 4\mu^2$ violating the condition that they be analytic for $\omega \geqq 4\mu^2$. The appropriate modification in this case is to include also the contribution of the single particle intermediate state indicated on Fig. 2. In this diagram the two final state pions do not interact at all, but it is essential that it be included to satisfy the unitarity condition in the present approximation.

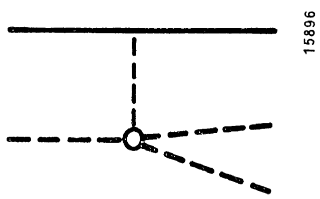

Fig. 1 1-pion exchange interaction.

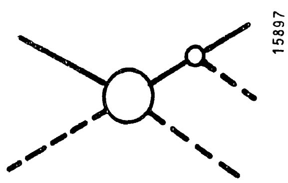

Fig. 2 Additional interaction necessary for consistent formulation of integral equations for T_{21}.

The resulting coupled non-linear and singular integral equations can be reduced by use of an extended form of the multichannel ND^{-1} method [6] to Fredholm integral equations, which can be solved by straightforward numerical methods. The first order iteration solution of the elastic pion nucleon scattering amplitude corresponds to the strip approximation [3, 7] of the Mandelstam representation [1] in which the pion-pion scattering amplitude is approximated by a resonance in a single angular momentum state [8].

We hope that the method which we have outlined here for dealing with unstable particles in an S-matrix theory of strong interactions may prove useful in the treatment of the large number of such particles which have recently been discovered experimentally.

LIST OF REFERENCES

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