Solving two-dimensional conformal field theories using the bootstrap approach

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Solution de théories conformes à deux dimensions par la méthode du bootstrap

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ÉCOLE DOCTORALE
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Titre : Solution de théories conformes à deux dimensions par la méthode du bootstrap
Mots clés : Théorie conforme des champs, Bootstrap conforme, Le modèle de Potts, Le modè de $O(n)$

Résumé : On construit des représentations logarithmiques de l'algèbre de Virasoro à charge centrale générique en utilisant des dérivées de champs primaires (nuls ou non). Dans le cas de champs nuls, les représentations résultantes sont paramétrés par les couplages logarithmiques, qui peuvent être complètement déterminés en utilisant l'existence des champs dégénérés. On écrit également des expressions fermées pour les blocs conformes à quatre points de ces représentations logarithmiques. Comme application, des représentations logarithmiques, générées par la dérivée première de champs nuls, complètent la détermination de l'action de l'algèbre de Virasoro sur les spectres de theorie conforme décrivant les points critiques du modèle de Potts et du modèle $O(n)$ en deux dimensions, également connu sous le nom des theories de Potts et $O(n)$.

De plus, on commençe une étude systématique des fonctions génériques à quatre points des theories conformes de Potts et $O(n)$ à charge centrale générique. Les fonctions à quatre points de ces deux theories conformes sont soumises à deux contraintes : la symétrie croisement et les contraintes de symétrie globale. On résout ensuite l'équation de symétrie de croisément pour plusieurs de leurs fonctions à quatre points. Pour la theorie conforme $O(n)$, on trouve que les solutions de l'équation de la symétrie de croisement sont toujours cohérentes avec la symétrie $O(n)$. Dans le cas de la theorie conforme de Potts, il peut cependant y avoir des solutions supplémentaires, qui sont incompatibles avec la symétrie S_Q et n'ont pas encore d'interprétation claire. En particulier, pour les deux theories conformes, on a déterminé leurs nombres de solutions de symétrie de croiément, plusieurs spectres exacts, plusieurs formules analytiques de leurs constantes de structure à quatre points et quelques règles de fusion correspondantes. On discute aussi nos résultats préliminaires sur le bootstrap de la theorie conforme $O(n)$ à $n = 0$, ce qui correspond à la marche aléatoire auto-évitante critique en deux dimensions.

Ensuite, on considère les limites rationnelles des fonctions à quatre points des modèles dits minimaux généralisés. On trouve que les fonctions à quatre points résultantes peuvent avoir nos représentations logarithmiques peuvant impliquer. Ces fonctions à quatre points conduisent également à des produits de fusion non chiraux, dont la projection chirale coïncide avec certaines règles de fusion des modèles minimaux logarithmiques chiraux proposés par P. Mathieu et D. Ridout. Ceci suggère qu'il peut exister des modèles minimaux logarithmiques non-chiraux.

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ÉCOLE DOCTORALE
Physique en Île-de-France (PIF)

Title : Solving two-dimensional conformal field theories using the bootstrap approach
Keywords : Conformal field theories, Conformal bootstrap, The Q-state Potts model, The $O(n)$ model.

Abstract : This thesis consists of three main results. Firstly, we build logarithmic representations of the Virasoro algebra at generic central charge by using derivatives of primary fields (null or not). In the case of null fields, the resulting representations are parametrized by the logarithmic couplings, which can be completely determined by using the existence of the degenerate fields. We also write down closed expressions for four-point conformal blocks of these logarithmic representations. As an application, logarithmic representations, generated by the first-order derivative of null fields, complete the determination of the action of the Virasoro algebra on the spectra of CFTs describing the critical points of the Potts model and the $O(n)$ model in two dimensions, also known as the Potts and $O(n)$ CFTs.

Secondly, we initiate a systematic study of generic four-point functions of the Potts and $O(n)$ CFTs at generic central charge. Four-point functions of these two CFTs are subject to two constraints : the crossing-symmetry equation and constraints from their global symmetries. We then solve the crossing-symmetry equation for several of their four-point functions. For the $O(n)$ CFT, we find that solutions to the crossing-symmetry equation are always consistent with $O(n)$ symmetry of the $O(n)$ CFT. In the case of the Potts CFT, there however can be extra solutions, which are inconsistent with S_Q symmetry of the Potts CFT and do not yet have clear interpretations. In particular, for both CFTs, we have determined their numbers of crossing-symmetry solutions, several exact spectra, several analytic formulae of their four-point structure constants, and a few corresponding fusion rules. We also discuss our preliminary results on bootstrapping the $O(n)$ CFT at $n = 0$, which corresponds to the critical self-avoiding random walk in two dimensions.

Thirdly, we consider rational limits of four-point functions of the so-called generalized minimal models. We find that the resulting four-point functions can have our logarithmic representations propagating in their channels. These four-point functions also lead to non-chiral fusion products, whose chiral projection coincides with some fusion rules of chiral logarithmic minimal models proposed by P. Mathieu and D. Ridout. This suggests that there may exist logarithmic minimal models in the bulk.

Declaration of authorship

Publications

Main results in this thesis have previously appeared in the following publications:

1. *Logarithmic CFT at generic central charge: from Liouville theory to the Q-state Potts model*, R. Nivesvivat and S. Ribault, arXiv:2007.04190 [hep-th], SciPost Phys 10 1 21
2. *Global symmetry and conformal bootstrap in the two-dimensional $O(n)$ model*, L. Grans-Samuelsson, R. Nivesvivat, J.L. Jacobsen, S. Ribault, H. Saleur, arXiv:2111.01106 [hep-th], SciPost Phys 12 5 147
3. *Global symmetry and conformal bootstrap in the two-dimensional Q-state Potts model*, R. Nivesvivat, arXiv:2205.09349 [hep-th], submitted to SciPost Phys

Unpublished results

1. Results for rational limits of generalized minimal models in Chapter 6 are based on unpublished work with S. Ribault.
2. Results for bootstrapping the $O(n)$ CFT at $n = 0$ in Chapter 7 are based on unpublished work with L. Grans-Samuelsson and H. Saleur

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Contents

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CHAPTER 1

Introduction

1.1 Brief story of 2D CFT

Conformal field theory (CFT) is a quantum field theory with conformal symmetry. Unlike most other quantum field theories, CFTs do not need any Lagrangian for their definitions. Conformal symmetry is powerful enough to provide CFTs a simple definition, known as CFT data: a spectrum of primary fields and their operator-product expansion (OPE) coefficients which satisfy the consistency conditions, imposed by full consequences of the model's symmetries [1]. This way of solving CFT is known as *the conformal bootstrap* [2]. In practice, full consequences of conformal symmetry manifest themselves in *the crossing-symmetry equation*, a constraint which requires four-point functions of CFTs to be crossing symmetric. With only CFT data, it is expected that we can compute all physical observables of CFTs.

In two dimensions, the conformal algebra is an infinite dimensional Lie algebra, also known as the Virasoro algebra. The Virasoro algebra comes with an additional parameter, known as the central charge c, to be discussed in more details in the next Chapter. Therefore, CFT data of two-dimensional CFTs depends explicitly on the central charge, and different values of the central charge give us different CFTs. With the rich structure of conformal symmetry, two-dimensional CFTs are then more constrained and more feasible to be solvable than those in higher dimensions. Moreover, unlike in higher-dimensions where unitarity plays a crucial role [3], solving two-dimensional CFTs does not rely too much on unitarity. In some cases, with only conformal symmetry and OPE associativity, we can already completely solve two-dimensional CFTs, for instance Liouville theory and minimal models [4, 5]. With these special phenomena, two-dimensional CFT therefore deserves special attention and become a subject of its own.

Applications

Besides their rich mathematical structure, two-dimensional CFTs are well-known to provide applications in vast areas of theoretical physics: from describing the worldsheet in string theory [6] to the critical points of many two-dimensional statistical systems [4]. In this thesis, we will be particularly interested in the latter case. In the continuum limit, many lattice models in two dimensions exhibit scaling invariance and can be described by two-dimensional CFTs, for example the Ising model in two dimensions. It could also

happen that different physical systems on the lattice coincide in the scaling limit and belong to the same universality class. The concept of universality class of course does not only exist in two dimensions but also in higher dimensions, for instance the universality class of Ising model in three dimensions [3].

Apart from their applications in two dimensions, two-dimensional CFTs also have relations with higher-dimensional quantum field theories, for instance a one-to-one correspondence between correlation functions in $N = 2$ supersymmetric gauge theories in four dimensions and correlation functions in Liouville theory, known as the Alday-Gaiotto-Tachikawa (AGT) relation [7]. This relation also later led to combinatorial expression for conformal blocks in two dimensions [8].

1.2 CFTs at generic central charge

This thesis mainly concerns solving two-dimensional CFTs at generic central charge c by using the conformal bootstrap. Let us also clarify here that our intentions of solving these CFTs amount to computing all of their correlation functions or at least having all necessary ingredients such as lists of primary fields, conformal blocks and OPE coefficients. We are interested in CFTs on the Riemann sphere, and we do not discuss any theory, which involves enhanced symmetry such as the $\mathcal{W}$-algebra symmetry or supersymmetry. Furthermore, throughout this thesis, we will always parametrize the central charge c as the following:

$$c = 13 - 6\beta^{-2} - 6\beta^{2} \ , \tag{1.2.1}$$

where the parameter β^2 in general can take generic values on the complex plane. For instance, Liouville theory is a consistent CFT for $c \in \mathbb{C} - (-\infty, 1)$ [5], an extension of Liouville theory for the central charge $c < 1$ was also proposed in [9]. Generalized minimal models, to be discussed in Section 4.4, are also examples of CFTs whose central charge $c \in \mathbb{C}$. However, in this thesis, we wil always restrict the central charge to the following c-half-plane:

$$\boxed{\Re(\beta^2) > 0 \Longleftrightarrow \Re(c) < 13} \ . \tag{1.2.2}$$

The above inequality ensures the convergence of four-point functions of these particular CFTs that we consider [10, 11]. Examples of CFTs, which are only valid in the c-half-plane (1.2.2), are the Potts and $O(n)$ CFTs, which will be introduced in the next Section. Another example is the odd CFT of [10, 12], which we will not consider in this thesis.

Certainly, CFTs at rational central charge are more well-known than the case of generic central charge due to their various applications. For instance the critical self-avoiding random walk in two dimensions is described by a CFT with the central charge $c = 0$, the two-dimensional Ising model at the critical point has the central charge $c = \frac{1}{2}$, and the critical two-dimensional Ashkin–Teller model comes with the central charge $c = 1$. However, CFTs at generic central charge are simpler and more tractable. For instance, Verma modules of Virasoro algebra at generic central can have at most one null vector, whereas there can be infinitely many null vectors in the case of rational central charge. Therefore, in some cases, it is simpler to solve CFTs at rational central charge by considering their counterparts at generic central charge, whose structure is more well-understood. Then take limits of these CFTs at generic central charge such that their central charges approaching rational numbers, to arrive at the desired CFTs at rational central charge. For example, the critical self-avoiding random walk can be obtained by taking the limit $c \to 0$ of the $O(n)$ CFT at generic central charge. This type of limits is known *rational limits* and will also be discussed in Chapter 6.

Two types of the degenerate fields

Solving CFTs at generic central charge depends crucially on the existence of the degenerate fields [10], to be introduced in Chapter 4. For example, generalized minimal models are completely solved due to infinitely many degenerate fields in their spectra [13]. In practice, there are two independent types of these degenerate fields [5], and they lead to two independent constraints on correlation functions, to be discussed in details in Section 4.3. Therefore, the degenerate fields also put constraints on CFT data, in addition to the crossing-symmetry equation. Let us then classify CFTs at generic central charge according to the existence of the degenerate fields.

1. **Two independent degenerate fields:** Most of CFTs at generic central charge, which have both types of the degenerate fields, admit exact solutions, for instance generalized minimal models that we previously mentioned. Liouville theory is another example. Although, the degenerate fields do not exist in the spectrum of Liouville theory with the central charge $c \in \mathbb{C} - (-\infty, 1)$, their OPE coefficients obey constraints from both types of the degenerate fields [5].

2. **One independent degenerate field:** Examples of CFTs in this category are the Potts and $O(n)$ CFTs [14]. This type of CFT is therefore less constrained and can be much more difficult to solve. For instance, constraints from one independent degenerate field only help us to compute correlation functions of the Potts and $O(n)$ CFTs semi-analytically [15].

3. **No degenerate field:** We do not know yet how to solve CFTs at generic central charge without any constraint from the degenerate fields, neither do we know if such CFTs exist.

Logarithmic CFTs

Like the case of rational central charge [16], CFTs at generic central charge can also be logarithmic. Logarithmic CFTs are simply CFTs whose spectra contain logarithmic fields that belong to logarithmic representations of the Virasoro algebra. These representations are well-known to result in Jordan blocks of the Virasoro generators L_0 and $\bar{L}_0$ and also lead to correlation functions which depend logarithmically on the positions.

Building consistent logarithmic CFT on the Riemann sphere usually involves gluing the right- and left-moving quantities, while requiring correlation functions in the resulting theory to be single-valued, for instance [17]. Such a procedure in general is not easy. To overcome this difficulty, in Chapters 2 and 3, we will obtain logarithmic fields by simply taking derivatives of primary fields on the Riemann sphere with respect to the conformal dimensions [18] because single-valued objects remain single-valued under differentiation. This procedure also works in higher dimensions [19].

This way of constructing logarithmic representations also allows us to write down their corresponding conformal blocks explicitly by taking derivatives of some known results with respect to the conformal dimensions, as we will see in Section 3.6. Furthermore, we will argue that some of these derivatives of fields describe the logarithmic structure of the Potts and $O(n)$ CFTs at generic central charge in Chapter 5 and also play important roles in some rational limits of generalized minimal models in Chapter 6. It may look speculative and artificial to describe these CFTs, which have discrete spectra, by using our approach since taking derivatives with respect to the conformal dimensions requires having continuous values of conformal dimensions. Therefore, to validate our arguments,

we will numerically bootstrap four-point functions of these models with our logarithmic blocks at arbitrary precision.

Rational limits

Rational limits are limits of objects in CFTs at generic central charge, taken such that the central charge c is approaching some rational numbers. These objects may be correlation functions, fusion rules, fields, or even the whole CFTs themselves. For instance, rational limits of Liouville theory with $c \in \mathbb{C} - (\infty, 1)$ results in CFTs at rational central charge, known as Runkel-Watts type theories [5]. Rational limits are however in general very complicated due to the analyticity of conformal blocks and structure constants, and there are still many open questions in this area. We will now discuss a few of them.

1. Rational limits of the Potts and $O(n)$ CFTs describe very interesting statistical physic systems. For example, the limit $c \to 0$ of the Potts CFTs coincides with critical bond percolation. Recently, there has been some progress in this area in [20] where the authors demonstrated how to regularized some divergences in the limit $c \to 0$ of the four-point connectivities, to be defined in the next section. Nevertheless, these limits are not well-understood in general and are extremely complicated. Let us now mention simpler problems below.

2. Rational limits of generalized minimal models have not been completely understood and in general do not lead to minimal models. For instance, the author of [12] considered some rational limits of four-point functions of generalized minimal models, whose results appear to be logarithmic in position and do not yet have interpretations. We will visit this problem for some particular four-point functions of generalized minimal models in Chapter 6.

3. Another interesting problem is the rational limit of conformal blocks. At generic central charge, the Zamolodchikov recursion [21], to be discussed in Section 3.4.2, provides a closed expression for four-point conformal blocks. Rational limits of the Zamolodchikov recursion are therefore expected to provide closed expressions for conformal blocks at rational central charge. This recursion is however not manifestly analytic for rational values of central charge, and its rational limits have not been completely understood. While there has been a recent progress on this subject in [22], finding closed expression of conformal blocks at rational central charge remains an open problem in two-dimensional CFTs.

1.3 The Potts and $O(n)$ CFTs

The main interests of this thesis are the Potts and $O(n)$ CFTs. The Potts and $O(n)$ CFTs describe the critical points of the two-dimensional Q-state Potts model with $Q \in [0, 4]$ and the two-dimensional $O(n)$ model with $n \in [-2, 2]$ [14]. The parameters Q and n of these two models are related to the central charge c in (1.2.1) as follows:

$$Q = 4\cos(\pi\beta^2)^2 \quad \text{and} \quad n = -2\cos(\pi\beta^{-2}) \quad \text{with} \quad \frac{1}{2} \leq \beta^2 \leq 1 \ , \tag{1.3.1}$$

where Q and n are not required to be integers. In terms of statistical physics, the Potts model with non-integer Q describes the so-called Fortuin-Kesteleyn random cluster [23] while the $O(n)$ model with generic n is equivalent to the dilute phase of the loop model [4]. With these descriptions, correlation of the two lattice models exist even for $Q, n \in \mathbb{C}$

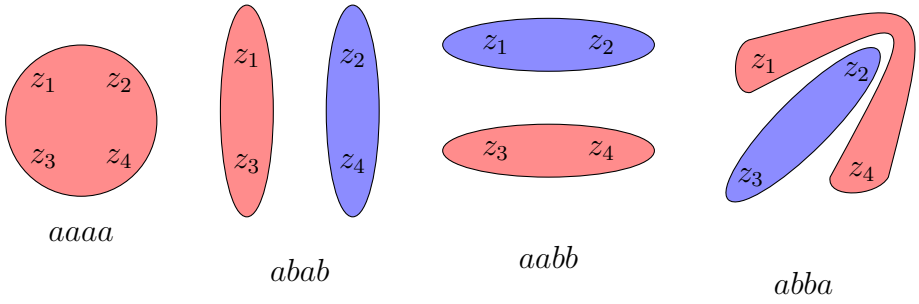
[24]. With the relation (1.3.1), it therefore makes sense to expect CFTs describing these two models to be valid for generic central charge as well. This led us to the definitions of the Potts and $O(n)$ CFT [25, 11] as the following:

The Potts and $O(n)$ CFTs are analytic continuations in the central charge c of the critical Q-state Potts model and the critical $O(n)$ model such that c is subject to the constraint (1.2.2)

Therefore, the Potts and $O(n)$ CFTs are defined as two families of CFTs, which are characterized by the parameter β^2 in (1.2.2), and each of them lives on a β^2-half-plane in (1.2.2), or equivalently on a double cover of the c-half-plane. However, we do not know yet the statistical interpretation of the both CFTs at generic central charge. Thus, these two CFTs should be considered as theories, which include the Potts and $O(n)$ models as special cases. Moreover, It also turns out that the Potts and $O(n)$ CFTs are logarithmic in general. For instance, the author of [26] found that the currents of $O(n)$ CFT belong to a logarithmic representation of the Virasoro algebra. In [27], the authors also found that Jordan blocks of L_0 and $\bar{L}_0$ only appear in the continuum limit of the Potts and $O(n)$ model at generic Q and n. In other words, these models are not logarithmic at finite scales.

In addition to conformal symmetry, the Potts and $O(n)$ CFTs also have global symmetries: S_Q and $O(n)$ respectively. Representation theories of these two symmetries can be formulated as tensor categories for generic Q and n [28]. As CFT data, solving these two CFTs amounts to finding their spectra and solving for their OPE coefficients from the consistency conditions: the crossing-symmetry equation and constraints from their global symmetries. The list of primary fields in both CFTs were first obtained in [14]. However, the complete action of the Virasoro algebra and global symmetries on their spectra were only recently determined in [29] and [30], respectively. Moreover, conformal blocks for four-point functions of primary fields in both CFTs are completely known, including the logarithmic case [29]. The next step in solving these two CFTs is then to compute their OPE coefficients. Numerically, this can be done by solving four-point structure constants from the crossing-symmetry equation with the approach of [31].

In recent years, much of the interest has been focusing on the simplest four-point function of the Potts CFT, namely the four-point connectivities. The four-point connectivities compute the probability of how the four points belong to the Fortuin-Kasteleyn clusters. There are four different configurations of these connectivities, namely P_{aaaa}, P_{abab}, P_{aabb}, and P_{abba}, which can be represented as follows:



, (1.3.2)

where different colors indicate different Fortuin-Kasteleyn clusters. The spectra of these connectivities was first proposed by [31], however it was later shown in [24], by using the transfer-matrix method on the lattice model, that there are infinitely many fields missing in the proposal of [31]. The complete spectra for the four-point connectivities of [24] have also been validated by the numerical conformal bootstrap in [15, 29]. Moreover, the

authors of [15, 32] also found several analytic ratios of four-point structure constants in the four-point connectivities, which suggest that the Potts CFT can be exactly solved.

1.4 Outline

Let us now discuss the outline for three main results of this thesis.

Logarithmic representations

- In Chapter 2, we build logarithmic representations at generic central charge by using derivatives of primary fields with respect to the conformal dimensions while also reviewing basic cases such as Verma modules and the degenerate representations. Our main results are the non-chiral logarithmic representations: $\mathcal{W}^{\kappa}_{(r,s)}$ and $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$ in Section 2.5. These representations are parametrized by their logarithmic couplings κ and also lead to second- or third-rank Jordan blocks of the dilatation generators L_0 and $\bar{L}_0$. In Section 4.5 of Chapter 4, we compute the couplings κ for both representations by using the existence of the degenerate fields.

- In Chapter 3, we discuss how to translate Verma modules and logarithmic representations, introduced in Chapter 2, into the concept of fields, and we review how to compute some correlation functions of these fields by using the Ward identities. Particularly, in Section 3.6, we argue how to write down closed expressions for four-point conformal blocks of logarithmic representations introduced in Chapter 2. In Section 4.5.2 of Chapter 4, we explain how the degenerate fields fix the undetermined constants of logarithmic blocks.

Towards solving the Potts and $O(n)$ CFTs

- Chapter 4 is mainly a review on the degenerate fields such as their basic properties: the degenerate fusion rules and the BPZ equations. We also review consequences of the existence of the degenerate fields such as the degenerate-shift equation of [10], then we give an example of CFTs whose spectra are completely made of the degenerate fields, known as generalized minimal models. At the end of this Chapter, we also discuss the interchiral blocks of [15], which will play crucial roles in bootstrapping four-point functions of the Potts and $O(n)$ CFTs.

- In Chapter 5, we review spectra of the Potts and $O(n)$ CFTs, including the action of their global symmetries on primary fields, which was recently determined in [30]. Then we discuss how to solve the crossing-symmetry equation for their four-point functions by our approach in [25] in Section 5.2 where we explain how to count the numbers of crossing-symmetry solutions and how to compare them with predictions from global symmetries. Thereafter, in Sections 5.3 and 5.4, we discuss in several examples how to numerically compute four-point functions of both CFT from [25, 33]. In particular, we conclude several exact results such as their numbers of crossing-symmetry solutions, exact spectra, vanishing three-point functions, exact ratios of structure constants, and fusion rules.

- In Subsection 7.2.3 of Chapter 7, we discuss our preliminary numerical results of bootstrapping the $O(n)$ CFT at $n = 0$ with vanishing central charge. This particular case corresponds to the critical self-avoiding random walk in two dimensions.

Numerical data for Chapter 5 can be found in Notebooks: `On4pt.ipynb` and `Potts4pt.ipynb` in [34].

Rational central charge

In Chapter 6, we discuss consequences of having rational central charge, then we briefly review the A-series minimal models. In Section 6.3, we discuss chiral logarithmic minimal models of [35]. This section is the only part of this thesis where we consider CFTs which do not live on the Riemann sphere, but the upper-half plane. Furthermore, we consider some rational limits of generalized minimal models, then we compare results from these limits with chiral logarithmic minimal models and speculate on the existence of logarithmic minimal models on the Riemann sphere.

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CHAPTER 2

Representations of the Virasoro algebra

We start with a brief review of conformal transformations in two dimensions, then we discuss various representations of the Virasoro algebra at generic central charge, from the mainstream case of Verma modules to new results on logarithmic representations.

2.1 Conformal symmetry in two dimensions

We are interested in conformal transformations on the Riemann sphere, on which we assign the local coordinates z and $\bar{z}$, called the left- and right-moving coordinates respectively. Therefore, z and $\bar{z}$ live on the complex plane $\mathbb{C}$ plus the point at ∞. In two dimensions, any holomorphic function is a conformal transformation. To write down the generators of two-dimensional conformal transformations, let us then consider the infinitesimal-local conformal transformation: $z \to z+\epsilon(z)$ where $\epsilon(z)$ is a holomorphic function. The Laurent expansion of $\epsilon(z)$ around $z = 0$ reads

$$\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n z^{n+1} \ . \tag{2.1.1}$$

For any function $f(z)$, we can then write

$$\begin{aligned} f(z+\epsilon(z)) - f(z) &= -\sum_n \epsilon_n z^{n+1} \frac{\partial}{\partial z} f(z) + \mathcal{O}(\epsilon^2) \ , \\ &= \sum_n \epsilon_n \ell_n f(z) + \mathcal{O}(\epsilon^2) \ , \end{aligned} \tag{2.1.2}$$

where the same analysis also holds for the infinitesimal transformations of $\bar{z}$, and we have introduced the generators ℓ_n, to be defined below. With (2.1.2), we find the generators of local conformal transformations as follows:

$$\ell_n = -z^{n+1}\frac{\partial}{\partial z} \quad \text{and} \quad \bar{\ell}_n = -\bar{z}^{n+1}\frac{\partial}{\partial \bar{z}} \quad \text{for} \quad n \in \mathbb{Z} \ . \tag{2.1.3}$$

The generators ℓ_n form an infinite-dimensional Lie algebra called the Witt algebra [4]. Conformal transformations on the Riemann sphere are then generated by two copies of the Witt algebra:

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m} \quad , \quad [\bar{\ell}_n, \bar{\ell}_m] = (n-m)\bar{\ell}_{n+m} \quad \text{and} \quad [\ell_n, \bar{\ell}_m] = 0 \ . \tag{2.1.4}$$

Among infinitely many generators in (2.1.3), only the generators ℓ_n and $\bar{\ell}_n$ with $n \in \{-1, 0, 1\}$ can be globally defined at any point on the Riemann sphere and form the $\mathfrak{sl}(2) \times \bar{\mathfrak{sl}}(2)$ algebra, which generates *the global conformal transformations*. The group of global conformal transformations on the Riemann sphere is equivalent to the group $PSL(2, \mathbb{C})$ or namely the group of Möbius transformations: $(z, \bar{z}) \to (w(z), \bar{w}(\bar{z}))$,

$$w(z) = \frac{az + b}{cz + d} \quad \text{for} \quad a, b, c, d \in \mathbb{C} \quad \text{with} \quad ab - cd \neq 0 \ . \tag{2.1.5}$$

The function $w(z)$ in (2.1.5) then provide us the finite form of global conformal transformations. It is also worth mentioning the following notable transformations which are special cases of (2.1.5):

1. Translation: $z \to z + b$
2. Rotation: $z \to az$ with $|a| = 1$
3. Scaling: $z \to az$ with $a \in \mathbb{R}$
4. Inversion: $z \to \frac{1}{z}$

At first glance, it may seem bizarre to have an inversion of z, which is singular at $z = 0$, as a global transformation. However, since we are on the Riemann sphere, the inversion of $z = 0$ simply takes us to the point $z = \infty$, and vice versa.

The Virasoro algebra

While two-dimensional CFTs indeed come with infinite local conformal symmetry [36], the Witt algebra (2.1.4) does not accommodate conformal symmetry of CFTs in two dimensions since describing symmetry of a quantum theory requires the use of projective representations. Lifting representations of the Witt algebra to be projective is equivalent to centrally extending the algebra itself. The resulting algebra is known as *the Virasoro algebra*, generated by the Virasoro generators L_n. Sometimes we will refer to them as the Virasoro modes. The Witt algebra has infinitely many central extensions, which however are equivalent up to a trivial redefinition of the generators $L_n \to L_n + g(n)C$ for some functions g and constant C [5]. We write the Virasoro algebra as follows:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{n+m} \ . \tag{2.1.6}$$

Since we are interested in CFTs which live on the Riemann sphere, let us also introduce the right-moving generators $\bar{L}_n$, which satisfy

$$[\bar{L}_n, \bar{L}_m] = (n - m)\bar{L}_{n+m} + \frac{c}{12} n(n^2 - 1)\delta_{n+m} \ , \tag{2.1.7a}$$

$$[L_n, \bar{L}_m] = 0 \ . \tag{2.1.7b}$$

The parameters c is called the central charge, which was defined in (1.2.1). In general, the left- and right-moving algebras are allowed to have different values of central charge but we are only interested in the case where they have the same central charge. Furthermore, we always assume that the conformal algebra of CFT on the Riemann sphere is a direct product between the left- and right-moving Virasoro algebra, which led us to (2.1.7b).

2.2 Verma modules

Verma modules of the Virasoro algebra are highest-weight representations generated by the Virasoro generators. Since the left- and right-moving generators commute, it is sufficient to study Verma modules generated by either L_n or $\bar{L}_n$. In this representation space, the generator L_0 can be completely diagonalized, therefore we define the highest-weight states, also called *the primary states* $|\Delta\rangle$ for the Verma module $\mathcal{V}_\Delta$ as follows:

$$L_0|\Delta\rangle = \Delta|\Delta\rangle \quad \text{and} \quad L_{m>0}|\Delta\rangle = 0 \ , \tag{2.2.1}$$

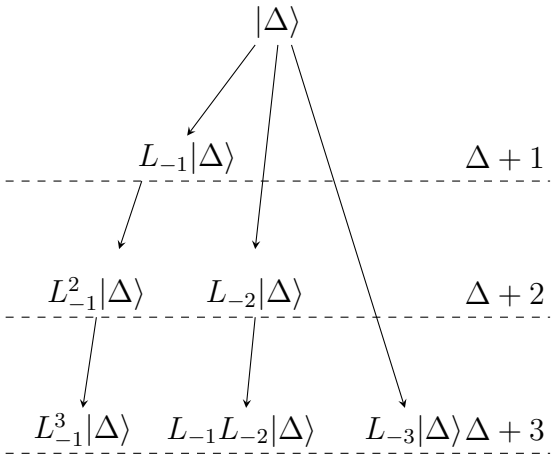
where the conformal dimension Δ is related to the central charge c in (1.2.1) as follows,

$$\boxed{\Delta(P) = \frac{c-1}{24} + P^2} \ , \tag{2.2.2}$$

where the parameter P is called *the momentum*. For generic Δ at generic central charge, the Verma module $\mathcal{V}_\Delta$ has $|\Delta\rangle$ as the only highest-weight state and is always irreducible. Acting on the primary state $|\Delta\rangle$ with the negative Virasoro modes $L_{n<0}$ then gives us a tower of infinitely many states, called *the descendants state*. These descendants can be characterized according to the degree of the Virasoro generators acting upon them. Let us then introduce the bases of the descendant at level N,

$$\prod_{i=1}^{p} L_{-n_i}|\Delta\rangle \quad \text{with} \quad 0 < n_1 \ldots \leq n_{p-1} \leq n_p \ , \tag{2.2.3}$$

where the sets of integers $\{n_p, n_{p-1}, \ldots, n_1\}$ are all possible integer partitions of N. Using the relation (2.1.7b), one can easily show that any descendant of $|\Delta\rangle$ at level N is an eigenvector of L_0 with the conformal dimension $\Delta + N$. Let us now write down examples of (2.2.3) explicitly up to level 3,



(2.2.4)

Therefore, the number of linearly-independent vectors in (2.2.3) at level N is equivalent to the number of integer partitions for N, which is given by the coefficient p_N in the generating function:

$$\sum_{N=0}^{\infty} p_N x^N = \prod_{i=1}^{\infty} (1-x^i)^{-1} = 1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots \ , \tag{2.2.5}$$

Thus, the vector space of the Verma module $\mathcal{V}_\Delta$ is infinite-dimensional.

Let us now also introduce right-moving Verma module $\bar{\mathcal{V}}_{\bar{\Delta}}$ which are generated by the right-moving generators $\bar{L}_n$ and come with the highest-weight state $|\bar{\Delta}\rangle$. Therefore, for

a given value of the central charge c, we can also build a non-chiral Verma module as the direct product $\mathcal{V}_\Delta \otimes \bar{\mathcal{V}}_{\bar{\Delta}}$ whose highest-weight state is *the non-diagonal primary states* $|\Delta, \bar{\Delta}\rangle$, given by

$$|\Delta, \bar{\Delta}\rangle = |\Delta\rangle \otimes |\bar{\Delta}\rangle \ . \tag{2.2.6}$$

In our notations, the generators L_n always act the left factor on the product $|\Delta\rangle \otimes |\bar{\Delta}\rangle$, whereas $\bar{L}_n$ acts on the right. Moreover, whenever we have the coincidence $\Delta = \bar{\Delta}$, the highest-weight state of $\mathcal{V}_\Delta \otimes \bar{\mathcal{V}}_\Delta$ is called *the diagonal primary state* $|\Delta\rangle^D$,

$$|\Delta\rangle^D = |\Delta\rangle \otimes |\Delta\rangle \ . \tag{2.2.7}$$

2.3 Degenerate representations

Although basis vectors of descendants at each level for the primary $|\Delta\rangle$ in (2.2.3) are not themselves primaries, their linear combinations at the same level can lead to another primary for some special values of Δ. In such scenarios, the Verma modules $\mathcal{V}_\Delta$ become reducible and contain non-trivial subrepresentations whose highest-weight states are called *the null vectors*. In other words, the null vectors are states which are both descendant and primary. Let us now demonstrate of how to compute these null vectors.

Consider the descendant at level 1: $L_{-1}|\Delta\rangle$. While $L_{-1}|\Delta\rangle$ is annihilated by the generators $L_{n>2}$, acting upon $L_{-1}|\Delta\rangle$ with L_1 yields

$$L_1 L_{-1}|\Delta\rangle = 2L_0|\Delta\rangle = 2\Delta|\Delta\rangle \ . \tag{2.3.1}$$

Thus, the descendant $L_{-1}|\Delta\rangle$ is a primary if and only if $\Delta = 0$. This motivates us to define the null vector at level 1 as

$$|\eta\rangle = L_{-1}|\Delta\rangle\Big|_{\Delta=0}. \tag{2.3.2}$$

At level 2, we write the null vector, $|\eta'\rangle$, as the linear combination:

$$|\eta'\rangle = aL_{-1}^2|\Delta\rangle + bL_{-2}|\Delta\rangle \ , \tag{2.3.3}$$

To solve for a and b, we recall that $|\eta'\rangle$ must be annihilated by the positive Virasoro modes $L_{n>0}$. However, the constraints from the generators $L_{n>2}$ are unnecessary since they can be obtained as commutators of L_1 and L_2. For instance, we can write $L_3 = [L_1, L_2]$. Therefore, it is sufficient to only consider the action of L_1 and L_2 on $|\eta'\rangle$. We require

$$L_1|\eta'\rangle = 0 \quad \text{and} \quad L_2|\eta'\rangle = 0 \ . \tag{2.3.4}$$

These two equations give us a linear system for the two unknowns a and b in (2.3.3),

$$\begin{pmatrix} 4\Delta + 2 & 3 \\ 6\Delta & 4\Delta + \frac{c}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \ . \tag{2.3.5}$$

Thus, the linear system (2.3.5) has non-trivial solutions if and only if the determinant of the square matrix in (2.3.5) is zero. We have

$$\det \begin{pmatrix} 4\Delta + 2 & 3 \\ 6\Delta & 4\Delta + \frac{c}{2} \end{pmatrix} = 0 \Longrightarrow \Delta = \Delta_\pm = \frac{1}{16}\left(5 - c \pm \sqrt{(c-25)(c-1)}\right) \ . \tag{2.3.6}$$

In contrast to the level-1 null vector, null vectors at level 2 come with two possible conformal dimensions: $\Delta_\pm$. One can then write down $|\eta'\rangle$ explicitly by solving (2.3.5) with Δ in (2.3.6).

$$|\eta'\rangle_\pm \sim L_{-1}^2|\Delta_\pm\rangle - \frac{12\Delta_\pm}{8\Delta_\pm + c}|\Delta_\pm\rangle \ . \tag{2.3.7}$$

For higher-level null vectors, the primary state $|\Delta_{(r,s)}\rangle$ with $r,s \in \mathbb{N}^*$ has the null vector $|\eta_{(r,s)}\rangle$ at level rs where the conformal dimension $\Delta_{(r,s)}$ is parametrized by the *the Kac indices* (r,s),

$$\boxed{\Delta_{(r,s)} = \Delta(P_{(r,s)}) \quad \text{with} \quad P_{(r,s)} = \frac{1}{2}\left(r\beta - \frac{s}{\beta}\right)} \ . \tag{2.3.8}$$

We also call $\Delta_{(r,s)}$ with $r,s \in \mathbb{N}^*$ as the degenerate conformal dimensions. The null vector $|\eta_{(r,s)}\rangle$ associated to the primary state $|\Delta_{(r,s)}\rangle$ is then given by

$$|\eta_{(r,s)}\rangle = \mathcal{L}_{(r,s)}|\Delta_{(r,s)}\rangle \quad \text{for} \quad r,s \in \mathbb{N}^* \ , \tag{2.3.9}$$

where the operator $\mathcal{L}_{(r,s)}$ denotes *the null vector operator*, a linear combination of negative Virasoro modes of degree rs which creates the null vector $|\eta_{(r,s)}\rangle$ at level rs. For instance, in the case of $(r,s) = (1,1)$, one simply has $\mathcal{L}_{(1,1)} = L_{-1}$. Furthermore, we shall be writing $\mathcal{L}_{(r,s)}$ entirely in terms of β^2 by using (2.3.8) for the rest of this thesis. Since the null vectors are primary states,

$$L_0|\eta_{(r,s)}\rangle = \Delta_{(r,-s)}|\eta_{(r,s)}\rangle \quad \text{and} \quad L_{n>0}|\eta_{(r,s)}\rangle = 0 \ . \tag{2.3.10}$$

where we have used the following identity for the dimensions of $|\eta_{(r,s)}\rangle$,

$$\Delta_{(r,s)} + rs = \Delta_{(r,-s)} \ . \tag{2.3.11}$$

To compute (2.3.9), there are formulae for $\mathcal{L}_{(1,s)}$ in [4]. However, there seems to be no closed-form expressions of $\mathcal{L}_{(r,s)}$ for general values of r and s. In practice, we have written a small program in Mathematica to calculate $\mathcal{L}_{(r,s)}$, which can easily reach the level $rs = 20$. The program is also available on request. Let us now display examples of $\mathcal{L}_{(r,s)}$ for $rs \leq 4$:

(r,s)	$\Delta_{(r,s)}$	$\mathcal{L}_{(r,s)}$
$(1,1)$	0	L_{-1}
$(2,1)$	$-\frac{1}{2} + \frac{3}{4}\beta^2$	$L_{-1}^2 - \beta^2 L_{-2}$
$(1,2)$	$-\frac{1}{2} + \frac{3}{4}\beta^{-2}$	$L_{-1}^2 - \beta^{-2} L_{-2}$
$(3,1)$	$-1 + 2\beta^2$	$L_{-1}^3 - 4\beta^2 L_{-2}L_{-1} + (4\beta^4 - 2\beta^2)\, L_{-3}$
$(1,3)$	$-1 + 2\beta^{-2}$	$L_{-1}^3 - 4\beta^{-2} L_{-2}L_{-1} + (4\beta^{-4} - 2\beta^{-2})\, L_{-3}$
$(4,1)$	$-\frac{3}{2} + \frac{15}{4}\beta^2$	$L_{-1}^4 - 10\beta^2 L_{-2}L_{-1}^2 + 9\beta^4 L_{-2}^2 + 2(12\beta^4 - 5\beta^2)L_{-3}L_{-1} - (6\beta^2 - 4\beta^4 + 6\beta^6)L_{-4}$
$(2,2)$	$-\frac{3}{2} + \frac{3}{4}(\beta^2 + \beta^{-2})$	$L_{-1}^4 - 2(\beta^2 + \beta^{-2})L_{-2}L_{-1}^2 + (\beta^4 - \beta^{-4})^2 L_{-2}^2 - 2(\beta^2 - 3 + \beta^{-2})L_{-3}L_{-1} - 3(\beta^2 - \beta^{-2})^2 L_{-4}$
$(1,4)$	$-\frac{3}{2} + \frac{15}{4}\beta^{-2}$	$L_{-1}^4 - 10\beta^{-2} L_{-2}L_{-1}^2 + 9\beta^{-4} L_{-2}^2 + 2(12\beta^{-4} - 5\beta^{-2})L_{-3}L_{-1} - (6\beta^{-2} - 4\beta^{-4} + 6\beta^{-6})L_{-4}$

(2.3.12)

Observe that the null vectors $|\eta_{(r,s)}\rangle$ are invariant under

$$r \leftrightarrow s \quad \text{simultaneously with} \quad \beta \to \beta^{-1} \ , \tag{2.3.13}$$

which naturally follows from (2.3.8) and (2.3.10). At generic central charge, there is only one null vector for each pair (r, s), in contrast to the case of rational central charge where $\mathcal{V}_{\Delta_{(r,s)}}$ can have infinitely many null vectors, which we shall see in Chapter 6.

From (2.3.9), the Verma module $\mathcal{V}_{\Delta_{(r,s)}}$ is then reducible and contains a non-trivial subrepresentation: $\mathcal{V}_{\Delta_{(r,-s)}}$, whose highest-weight state is $|\eta_{(r,s)}\rangle$. We can then build an irreducible representation by taking a quotient between $\mathcal{V}_{\Delta_{(r,s)}}$ and $\mathcal{V}_{\Delta_{(r,-s)}}$. The result is known as *the degenerate representation* $\mathcal{R}_{(r,s)}$:

$$\mathcal{R}_{(r,s)} = \mathcal{V}_{\Delta_{(r,s)}}/\mathcal{V}_{\Delta_{(r,-s)}} \ . \tag{2.3.14}$$

Thus, the resulting representation $\mathcal{R}_{(r,s)}$ is irreducible. We also stress the construction (2.3.14) is only valid at generic central charge. Let us also define the highest-weight state of $\mathcal{R}_{(r,s)}$ as $|\langle r, s\rangle\rangle$, which comes with conformal dimensions $\Delta_{(r,s)}$ and the vanishing null descendants:

$$\mathcal{L}_{(r,s)}|\langle r, s\rangle\rangle = 0 \ . \tag{2.3.15}$$

Likewise to their left-moving counter parts, we define the right-moving degenerate representation $\bar{\mathcal{R}}_{(r,s)}$ as $\bar{\mathcal{V}}_{\Delta_{(r,s)}}/\bar{\mathcal{V}}_{\Delta_{(r,-s)}}$. At generic central charge, it is not possible to build non-diagonal degenerate representations: $\mathcal{R}_{(r,s)} \otimes \bar{\mathcal{R}}_{(r',s')}$ due to the constraints $\Delta_{(r',s')} - \Delta_{(r,s)} \in \mathbb{Z}$ from the single-valuedness. However, such constraints are always satisfied trivially by diagonal representations. We then write the diagonal degenerate representations $\mathcal{R}^D$ as follows:

$$\mathcal{R}^D = \mathcal{R}_{(r,s)} \otimes \bar{\mathcal{R}}_{(r,s)} \ . \tag{2.3.16}$$

whose height-weight states are

$$|\langle r, s\rangle\rangle^D = |\langle r, s\rangle\rangle \otimes |\overline{\langle r, s\rangle}\rangle \ , \tag{2.3.17}$$

whose left- and right-null descendants vanish identically,

$$\mathcal{L}_{(r,s)}|\langle r, s\rangle\rangle^D = \bar{\mathcal{L}}_{(r,s)}|\langle r, s\rangle\rangle^D = 0 \ . \tag{2.3.18}$$

2.4 Derivatives of primary states

Let us now consider representations of the Virasoro algebra, upon which the action of L_0 and $\bar{L}_0$ cannot be diagonalizable, known as logarithmic representations. That is to say the matrices of L_0 and $\bar{L}_0$ come with Jordan blocks. This kind of representations is well-known to lead to correlation functions which are logarithmic in positions. The simplest way of introducing Jordan blocks to L_0 and $\bar{L}_0$ is to use derivatives with respect to the conformal dimensions [18]. For convenience, let us now introduce a formal notation:

$$\frac{\partial^n}{\partial \Delta^n}|\Delta\rangle^D = |\Delta^{(n)}\rangle^D \ . \tag{2.4.1}$$

For simplicity, we only consider derivatives of the diagonal primary state $|\Delta\rangle^D$ with respect to Δ in details. The non-diagonal case is however more complicated since the single-valuedness requires that the action of $L_0 - \bar{L}_0$ on any state must always be diagonalizable

[37]. Now recall that the Δ-derivatives commute with all of the Virasoro generators, therefore acting on the states $|\Delta^{(1)}\rangle^D$ and $|\Delta\rangle^D$ with L_0 yields

$$L_0|\Delta^{(1)}\rangle^D = |\Delta\rangle^D + \Delta|\Delta^{(1)}\rangle^D \quad \text{and} \quad L_0|\Delta\rangle = \Delta|\Delta\rangle^D \ . \tag{2.4.2}$$

Thus, taking the derivative of $|\Delta\rangle^D$ with respect to Δ results in a second-rank Jordan block of L_0. That is to say the logarithmic state $|\Delta^{(1)}\rangle^D$ comes with a Jordan partner $|\Delta\rangle^D$, or equivalently the primary state $|\Delta\rangle^D$ has $|\Delta^{(1)}\rangle^D$ as its logarithmic partner. Moreover, acting on $|\Delta^{(1)}\rangle$ with negative Virasoro modes simply gives us the logarithmic descendants which have the descendants of the primary state $|\Delta\rangle$ as their Jordan partners. For instance,

$$L_0 L_{n<0}|\Delta^{(1)}\rangle^D = L_{n<0}|\Delta\rangle^D + (\Delta+n)L_{n<0}|\Delta^{(1)}\rangle^D \ . \tag{2.4.3}$$

In other words, taking the Δ-derivative of the primary states $|\Delta\rangle^D$ amount to differentiating their whole Verma modules:

$$\frac{\partial}{\partial\Delta}\left(\mathcal{V}_\Delta \otimes \bar{\mathcal{V}}_\Delta\right) \ , \tag{2.4.4}$$

where the derivative in (2.4.4) means that we are taking the Δ-derivative of every state in $\mathcal{V}_\Delta \otimes \bar{\mathcal{V}}_\Delta$. Therefore, the representations in (2.4.4) are no longer holomorphically factorized. Moreover, notice that (2.4.4) is invariant under changing of the normalizations:

$$|\Delta\rangle^D \to \lambda(\Delta)|\Delta\rangle^D \Longrightarrow |\Delta^{(1)}\rangle^D \to \lambda(\Delta)|\Delta^{(1)}\rangle^D + \lambda'(\Delta)|\Delta\rangle^D \ , \tag{2.4.5}$$

where $\lambda(\Delta)$ are arbitrary functions. The transformations (2.4.5) preserve the Jordan blocks of L_0 and do not change the structure of (2.4.4). Thus, the derivatives of Verma modules in (2.4.4) have no free parameters. Let us now consider the case of higher-order derivatives briefly. We introduce the vector $\vec{\Delta}^{(n)}$,

$$\vec{\Delta}^{(n)} = \left[\frac{1}{n!}|\Delta^{(n)}\rangle^D, \ldots, \frac{1}{2!}|\Delta^{(2)}\rangle^D, |\Delta^{(1)}\rangle^D, |\Delta\rangle\right]^T \ . \tag{2.4.6}$$

Therefore, we have

$$\hat{L}_0\vec{\Delta}^{(n)} = \hat{\Delta}\vec{\Delta}^{(n)} \quad \text{and} \quad \hat{L}_{m>0}\vec{\Delta}^{(n)} = 0 \ , \tag{2.4.7}$$

where $\hat{L}_n$ is a diagonal matrix whose diagonal elements are L_n. The matrix $\hat{\Delta}$ is a $n \times n$ matrix with rank-$n+1$ Jordan blocks,

$$\hat{\Delta} = \begin{bmatrix} \Delta & 1 & 0 & \cdots & 0 \\ 0 & \Delta & 1 & \ddots & 0 \\ 0 & 0 & \Delta & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & \Delta \end{bmatrix} \ . \tag{2.4.8}$$

Thus, the n^{th}-derivatives of primary states lead to rank-n+1 Jordan blocks of L_0. Similarly to (2.4.4), the states $\vec{\Delta}^{(n)}$ transform in the representations $\frac{\partial^n}{\partial\Delta^n}\left(\mathcal{V}_\Delta \otimes \bar{\mathcal{V}}_\Delta\right)$, which are also invariant under renormalizing the primary state $|\Delta\rangle^D$.

2.5 Derivatives of null vectors

We now consider representations of Virasoro algebra which are logarithmic and also include the null vectors $|\eta_{(r,s)}\rangle^D$. To do so, we simply use the derivatives of $|\eta_{(r,s)}\rangle^D$ with respect to its conformal dimensions. Since we are working with Verma modules, we stress that

$$|\eta_{(r,s)}\rangle^D = \mathcal{L}_{(r,s)}\bar{\mathcal{L}}_{(r,s)}|\Delta_{(r,s)}\rangle^D \neq 0 \ . \tag{2.5.1}$$

Let us then write down the n^{th}-derivatives of the null vectors $|\eta_{(r,s)}\rangle^D$ as the following:

$$|\eta^{(n)}_{(r,s)}\rangle^D = \frac{\partial^n}{\partial\Delta^n}|\Delta + rs\rangle^D\Big|_{\Delta=\Delta_{(r,s)}} \ , \tag{2.5.2a}$$

$$|\mu^{(n)}_{(r,s)}\rangle^D = \mathcal{L}_{(r,s)}\bar{\mathcal{L}}_{(r,s)}\frac{\partial^n}{\partial\Delta^n}|\Delta\rangle^D\Big|_{\Delta=\Delta_{(r,s)}} \ , \tag{2.5.2b}$$

where these two states are normalized such that

$$|\mu^{(0)}_{(r,s)}\rangle^D = |\eta_{(r,s)}\rangle^D \ . \tag{2.5.3}$$

By their definitions, the action of the Virasoro algebra on the states $|\eta^{(n)}_{(r,s)}\rangle^D$ can be obtained immediately by setting $\Delta = \Delta_{(r,-s)}$ in (2.4.7) while the case of $|\mu^{(n)}_{(r,s)}\rangle^D$ is less straightforward, and we only discuss it up to $n = 2$.

2.5.1 First-order derivatives

We start with the first-order derivatives of the null vectors in (2.5.2) by introducing the following linear combination,

$$\boxed{|W^\kappa_{(r,s)}\rangle = (1-\kappa)|\eta^{(1)}_{(r,s)}\rangle^D + \kappa|\mu^{(1)}_{(r,s)}\rangle^D} \ . \tag{2.5.4}$$

With (2.5.2) and (2.3.10), the null vector $|\eta_{(r,s)}\rangle^D$ and the state $|W^\kappa_{(r,s)}\rangle$ then form second-rank Jordan blocks of L_0 and $\bar{L}_0$:

$$\bar{L}_0|W^\kappa_{(r,s)}\rangle = L_0|W^\kappa_{(r,s)}\rangle = |\eta_{(r,s)}\rangle^D + \Delta_{(r,s)}|W^\kappa_{(r,s)}\rangle \ . \tag{2.5.5}$$

Therefore, representations generated by $|W^\kappa_{(r,s)}\rangle$, which we shall denote as $\mathcal{W}^\kappa_{(r,s)}$, are in general logarithmic and also non-chiral. Moreover, the parameter κ is known as *the logarithmic coupling*, which characterizes logarithmic representations $\mathcal{W}^\kappa_{(r,s)}$. That is to say different values of κ lead to different $\mathcal{W}^\kappa_{(r,s)}$.

Let us now analyse the complete structure of primary states in the representations $\mathcal{W}^\kappa_{(r,s)}$. To do so, we must understand how the generators L_n act upon $|W^\kappa_{(r,s)}\rangle$. Let us then introduce the annihilation and creation operators $\mathcal{A}^N$ and $\mathcal{L}^N$ as combinations of negative and positive Virasoro modes of degree N, respectively. For any $\mathcal{A}^N$ and $\mathcal{L}^M$, one can write

$$\mathcal{A}^N\mathcal{L}^M|\Delta^{(1)}\rangle^D = P_{[\mathcal{A}^N,\mathcal{L}^M]}(\Delta)|\Delta^{(1)}\rangle^D + P'_{[\mathcal{A}^N,\mathcal{L}^M]}(\Delta)|\Delta\rangle^D \ , \tag{2.5.6}$$

where the prime symbol denotes the first-order derivative of $P_{[\mathcal{A}^N,\mathcal{L}^M]}(\Delta)$ with respect to

Δ. Moreover, the functions $P_{[\mathcal{A}^N,\mathcal{L}^M]}(\Delta)$ depend polynomially on Δ. For instance,

$\mathcal{A}^N$	$P_{[\mathcal{A}^N,L_{-1}^5]}(\Delta)$
L_1	$10(\Delta+2)L_{-1}^4$
L_1^2	$40(\Delta+2)(2\Delta+3)\Delta L_{-1}^3$
L_1^3	$240(\Delta+1)(\Delta+2)(2\Delta+3)L_{-1}^2$
L_1^4	$480(\Delta+1)(\Delta+2)(2\Delta+1)(2\Delta+3)L_{-1}$
L_1^5	$960\Delta(\Delta+1)(\Delta+2)(2\Delta+1)(2\Delta+3)$

(2.5.7)

More precisely, $P_{[\mathcal{A}^{N\leq M},\mathcal{L}^M]}(\Delta)$ are combinations of the generators $L_{n<0}$ whose coefficients are polynomials in Δ of degree at most N. Choosing $\mathcal{L}^M$ to be the null vector operator $\mathcal{L}_{(r,s)}$ and setting $\Delta=\Delta_{(r,s)}$ in (2.5.6) yields

$$\mathcal{A}^N\mathcal{L}_{(r,s)}|\Delta_{(r,s)}\rangle=\mathcal{A}^N|\eta_{(r,s)}\rangle^D=0\Longrightarrow P_{[\mathcal{A}^N,\mathcal{L}_{(r,s)}]}(\Delta_{(r,s)})=0\ . \tag{2.5.8}$$

At generic central charge, the function $P_{[\mathcal{A}^N,\mathcal{L}_{(r,s)}]}(\Delta)$ always has a simple zero at $\Delta=\Delta_{(r,s)}$. Using (2.5.8), let us now write

$$\begin{aligned}\mathcal{A}^N|\mu_{(r,s)}^{(1)}\rangle^D&=P'_{[\mathcal{A}^N,\mathcal{L}_{(r,s)}]}(\Delta_{(r,s)})\bar{\mathcal{L}}_{(r,s)}|\Delta_{(r,s)}\rangle^D\ ,\\&=P'_{[\mathcal{A}^N,\bar{\mathcal{L}}_{(r,s)}]}(\Delta_{(r,s)})|\Delta_{(r,s)},\Delta_{(r,-s)}\rangle\ .\end{aligned} \tag{2.5.9}$$

Taking into account the above with (2.4.7), we have

$$\mathcal{A}^{rs}|W_{(r,s)}^{\kappa}\rangle=|\Delta_{(r,s)},\Delta_{(r,-s)}\rangle\ , \tag{2.5.10a}$$

$$\bar{\mathcal{A}}^{rs}|W_{(r,s)}^{\kappa}\rangle=|\Delta_{(r,-s)},\Delta_{(r,s)}\rangle\ , \tag{2.5.10b}$$

where $\mathcal{A}^{rs}$ and $\bar{\mathcal{A}}^{rs}$ are normalized such that

$$P'_{[\mathcal{L}_{(r,s)},\mathcal{A}^{rs}]}(\Delta_{(r,s)})=P'_{[\mathcal{L}_{(r,s)},\bar{\mathcal{A}}^{rs}]}(\Delta_{(r,s)})=1\ . \tag{2.5.11}$$

Let us also stress here that the normalizations (2.5.11) are only valid at generic value of central charge whereas $P'_{[\mathcal{L}_{(r,s)},\mathcal{A}^{rs}]}(\Delta_{(r,s)})$ could vanish non-trivially at rational central charge. From (2.5.10), the representation $\mathcal{W}_{(r,s)}^{\kappa}$ then contains two non-diagonal Verma modules as its subrepresentations. We have

$$\frac{(\mathcal{V}_{\Delta_{(r,s)}}\otimes\bar{\mathcal{V}}_{\Delta_{(r,-s)}})\oplus(\mathcal{V}_{\Delta_{(r,-s)}}\otimes\bar{\mathcal{V}}_{\Delta_{(r,s)}})}{(\mathcal{V}_{\Delta_{(r,-s)}}\otimes\bar{\mathcal{V}}_{\Delta_{(r,-s)}})}\subset\mathcal{W}_{(r,s)}^{\kappa}\ , \tag{2.5.12}$$

which then implies that $\mathcal{W}_{(r,s)}^{\kappa}$ is indecomposable but reducible. We summarize the structure of $\mathcal{W}_{(r,s)}^{\kappa}$ in the figure 2.1. Now, let us climb down from the two non-diagonal states in (2.1) to the null vector $|\eta_{(r,s)}\rangle^D$ by using the null vector operators $\mathcal{L}_{(r,s)}$ or $\bar{\mathcal{L}}_{(r,s)}$. More explicitly, we have the relation:

$$\boxed{\bar{\mathcal{L}}_{(r,s)}\bar{\mathcal{A}}^{rs}|W_{(r,s)}^{\kappa}\rangle=\mathcal{L}_{(r,s)}\mathcal{A}^{rs}|W_{(r,s)}^{\kappa}\rangle=\kappa(L_0-\Delta_{(r,s)})|W_{(r,s)}^{\kappa}\rangle}\ . \tag{2.5.13}$$

The above equation then defines the logarithmic state $|W_{(r,s)}^{\kappa}\rangle$ and the logarithmic coupling κ algebraically. In other words, one can also define the representation $\mathcal{W}_{(r,s)}^{\kappa}$ by using (2.5.13) without relying on any derivative of null vector in (2.5.2). However, our derivative

formalism will show its strength when computing logarithmic conformal blocks in Chapter 3. Furthermore, with the normalizations (2.5.11), the parameter κ does not depend on the choice of $\mathcal{A}^{rs}$ and $\bar{\mathcal{A}}^{rs}$, and observe from (2.5.13) that $\mathcal{W}^{\infty}_{(r,s)}$ is not a logarithmic representation wherein $|W^{\infty}_{(r,s)}\rangle$ are eigenvectors of L_0 and $\bar{L}_0$. The case of $\mathcal{W}^{\kappa}_{(r,s)}$ with any finite value of κ is however always logarithmic. Morevover, likewise to (2.4.5), the equation (2.5.13) is invariant under renormalizing the null vector $|\eta_{(r,s)}\rangle^D$:

$$|\eta_{(r,s)}\rangle^D \to \lambda(\Delta_{(r,-s)})|\eta_{(r,s)}\rangle^D \Longrightarrow |W^{\kappa}_{(r,s)}\rangle \to \lambda(\Delta_{(r,-s)})|W^{\kappa}_{(r,s)}\rangle + \lambda'(\Delta_{(r,-s)})|\eta_{(r,s)}\rangle^D \ . \tag{2.5.14}$$

Thus, the representation $\mathcal{W}^{\kappa}_{(r,s)}$ is unchanged under the gauge transformations (2.5.14), and any logarithmic state related by (2.5.14) generates the same representation $\mathcal{W}^{\kappa}_{(r,s)}$.

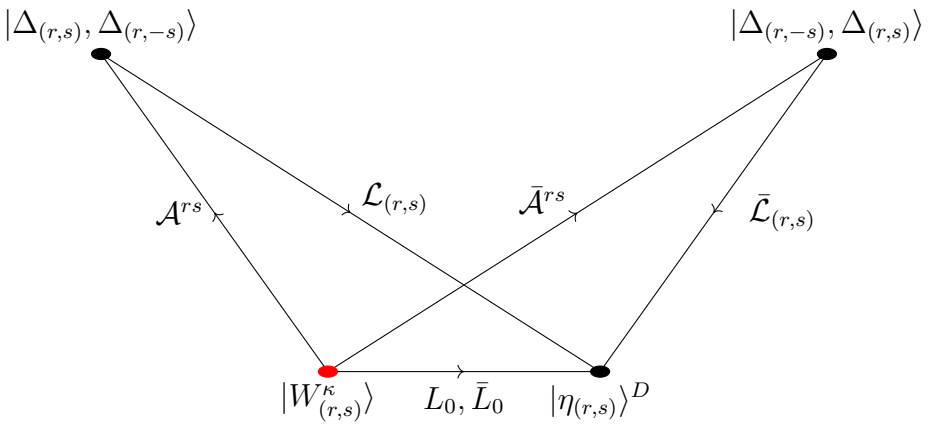


Figure 2.1: Primary and logarithmic states in $\mathcal{W}^{\kappa}_{(r,s)}$

2.5.2 Second-order derivatives

Let us now discuss a more complicated case: the second-order derivatives of the null vectors in (2.5.2). We consider the linear combination:

$$|\widetilde{W}^{\kappa}_{(r,s)}\rangle = \frac{1-\kappa}{2}|\eta^{(2)}_{(r,s)}\rangle^D + \frac{\kappa}{2}|\mu^{(2)}_{(r,s)}\rangle^D \tag{2.5.15}$$

We write $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$ for representations generated by the state $|\widetilde{W}^{\kappa}_{(r,s)}\rangle$. Acting on $|\widetilde{W}^{\kappa}_{(r,s)}\rangle$ with the generators L_0 and $\bar{L}_0$ yields

$$\bar{L}_0|\widetilde{W}^{\kappa}_{(r,s)}\rangle = L_0|\widetilde{W}^{\kappa}_{(r,s)}\rangle = |W^{\kappa}_{(r,s)}\rangle + \Delta_{(r,-s)}|\widetilde{W}^{\kappa}_{(r,s)}\rangle \ , \tag{2.5.16a}$$

$$(L_0 - \Delta_{(r,-s)})^2|\widetilde{W}^{\kappa}_{(r,s)}\rangle = (\bar{L}_0 - \Delta_{(r,-s)})^2|\widetilde{W}^{\kappa}_{(r,s)}\rangle = |\eta_{(r,s)}\rangle^D \ . \tag{2.5.16b}$$

Thus, the Jordan partner of the logarithmic state $|\widetilde{W}^{\kappa}_{(r,s)}\rangle$ is $|W^{\kappa}_{(r,s)}\rangle$, whose Jordan partner is given by the null vector $|\eta_{(r,s)}\rangle^D$. These three states then form third-rank Jordan blocks of L_0 and $\bar{L}_0$ in (2.5.16). To see the structure of the representation $\widetilde{W}^{\kappa}_{(r,s)}$, we start acting on $|\widetilde{W}^{\kappa}_{(r,s)}\rangle$ with the annihilation operators $\mathcal{A}^{rs}$ and $\bar{\mathcal{A}}^{rs}$. Using (2.5.8), we have

$$\begin{aligned}\mathcal{A}^{rs}|\widetilde{W}^{\kappa}_{(r,s)}\rangle &= \frac{\kappa}{2}\mathcal{A}^{rs}|\mu^{(2)}_{(r,s)}\rangle^D \ , \\ &= \kappa P'_{[\mathcal{A}^{rs},\mathcal{L}_{(r,s)}]}(\Delta_{(r,s)})\bar{\mathcal{L}}_{(r,s)}|\Delta^{(1)}_{(r,s)}\rangle^D + \frac{\kappa}{2}P''_{[\mathcal{A}^{rs},\mathcal{L}_{(r,s)}]}(\Delta_{(r,s)})\bar{\mathcal{L}}_{(r,s)}|\Delta_{(r,s)}\rangle^D \ .\end{aligned} \tag{2.5.17}$$

The second derivative $P''_{[\mathcal{A}^{rs},\mathcal{L}_{(r,s)}]}(\Delta_{(r,s)})$ does not vanish in general, however the term $P''_{[\mathcal{A}^{rs},\mathcal{L}_{(r,s)}]}(\Delta_{(r,s)})$ in (2.5.17) is redundant since it can always be absorbed in the gauge transformations (2.5.21), which leave the representation $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$ invariant. Using the normalizations (2.5.11), we now write

$$\mathcal{A}^{rs}|\widetilde{W}^{\kappa}_{(r,s)}\rangle = \kappa\bar{\mathcal{L}}_{(r,s)}|\Delta^{(1)}_{(r,s)}\rangle^{D} \ , \tag{2.5.18a}$$

$$\bar{\mathcal{A}}^{rs}|\widetilde{W}^{\kappa}_{(r,s)}\rangle = \kappa\mathcal{L}_{(r,s)}|\Delta^{(1)}_{(r,s)}\rangle^{D} \ , \tag{2.5.18b}$$

$$\bar{\mathcal{A}}^{rs}\mathcal{A}^{rs}|\widetilde{W}^{\kappa}_{(r,s)}\rangle = \kappa|\Delta_{(r,s)}\rangle^{D} \ . \tag{2.5.18c}$$

From the above primary states, let us now start descending by acting upon them with the null vector operators $\mathcal{L}_{(r,s)}$ and $\bar{\mathcal{L}}_{(r,s)}$,

$$\begin{aligned}\mathcal{L}_{(r,s)}\mathcal{A}^{rs}|\widetilde{W}^{\kappa}_{(r,s)}\rangle = \bar{\mathcal{L}}_{(r,s)}\bar{\mathcal{A}}^{rs}|\widetilde{W}^{\kappa}_{(r,s)}\rangle &= \kappa\mathcal{L}_{(r,s)}\bar{\mathcal{L}}_{(r,s)}|\Delta^{(1)}_{(r,s)}\rangle^{D} \ , \\ &= \kappa|\mu^{(1)}_{(r,s)}\rangle^{D} \ . \end{aligned} \tag{2.5.19}$$

Now recall (2.5.16), therefore both $|W^{\kappa}_{(r,s)}\rangle$ and $|\mu^{(1)}_{(r,s)}\rangle^{D}$ transform in the representation $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$. Since one can always take linear combinations of those two states to obtain $|\mathcal{W}^{\kappa'}_{(r,s)}\rangle$ with any value of κ', the representation $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$ then contains $\mathcal{W}^{\kappa'}_{(r,s)}$ with arbitrary κ' as non-trivial subrepresentations. To summarize, as shown in the figure 2.2, $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$ have 4 primary states and 4 logarithmic states. Acting on the diagonal primary state $|\Delta_{(r,s)}\rangle^{D}$ with both $\mathcal{L}_{(r,s)}$ and $\bar{\mathcal{L}}_{(r,s)}$ simply brings us back to the null vector $|\eta_{(r,s)}\rangle^{D}$ in (2.5.16),

$$\boxed{\mathcal{L}_{(r,s)}\bar{\mathcal{L}}_{(r,s)}\mathcal{A}^{rs}\bar{\mathcal{A}}^{rs}|\widetilde{W}^{\kappa}_{(r,s)}\rangle = \kappa(L_0 - \Delta_{(r,-s)})^2|\widetilde{W}^{\kappa}_{(r,s)}\rangle} \ , \tag{2.5.20}$$

which provides an algebraic definition for the representation $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$. Similarly to the case of $\mathcal{W}^{\infty}_{(r,s)}$, the logarithmic state $|\widetilde{W}^{\infty}_{(r,s)}\rangle$ are eigenvectors of L_0 and $\bar{L}_0$, consequently the representation $\widetilde{\mathcal{W}}^{\infty}_{(r,s)}$ is not logarithmic. Furthermore, the equation (2.5.20) is unchanged under transforming $|\eta^{D}_{(r,s)}\rangle \to \lambda(\Delta_{(r,-s)})|\eta^{D}_{(r,s)}\rangle$, or equivalently

$$\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}\rangle \to \widetilde{\mathcal{W}}^{\kappa}_{(r,s)}\rangle + \lambda''|\eta_{(r,s)}\rangle^{D} + \kappa\lambda'|\mu^{(1)}_{(r,s)}\rangle^{D} + (1-\kappa)\lambda'|\eta^{(1)}_{(r,s)}\rangle^{D} \ , \tag{2.5.21a}$$

$$\mathcal{A}^{rs}|\widetilde{W}^{\kappa}_{(r,s)}\rangle \to \mathcal{A}^{rs}|\widetilde{W}^{\kappa}_{(r,s)}\rangle + \kappa\lambda'\bar{\mathcal{L}}_{(r,s)}|\Delta_{(r,s)}\rangle^{D} \ , \tag{2.5.21b}$$

$$\bar{\mathcal{A}}^{rs}|\widetilde{W}^{\kappa}_{(r,s)}\rangle \to \bar{\mathcal{A}}^{rs}|\widetilde{W}^{\kappa}_{(r,s)}\rangle + \kappa\lambda'\mathcal{L}_{(r,s)}|\Delta_{(r,s)}\rangle^{D} \tag{2.5.21c}$$

where we have written down the last two equations in the above by acting on (2.5.21a) with $\mathcal{A}^{rs}$ and $\bar{\mathcal{A}}^{rs}$. Hence, the representation $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$ is invariant the gauge transformations (2.5.21).

Using derivatives of null vectors, we have then constructed logarithmic representations of the Virasoro algebra, which are non-chiral and also valid at generic central charge. In Chapter 4, we will fix the values of these coupling by using the existence of the degenerate fields. In particular the representations $\mathcal{W}^{\kappa}_{(r,s)}$ with such fixed values of κ play important in bootstrapping four-point functions of the Potts and $O(n)$ CFTs. While we do not know yet applications of the representations $\widetilde{\mathcal{W}}^{\kappa_0}_{(r,s)}$ at generic central charge, the authors of [20] found that our logarithmic representation $\widetilde{W}^{\kappa_0}_{(1,1)}$ with $\kappa_0 = -\frac{1}{48}$ at $c = 0$ appears as the vacuum module of the Potts and $O(n)$ CFTs at the central charge $c = 0$. This vacuum module is also an example of the non-chiral staggered module of [38].

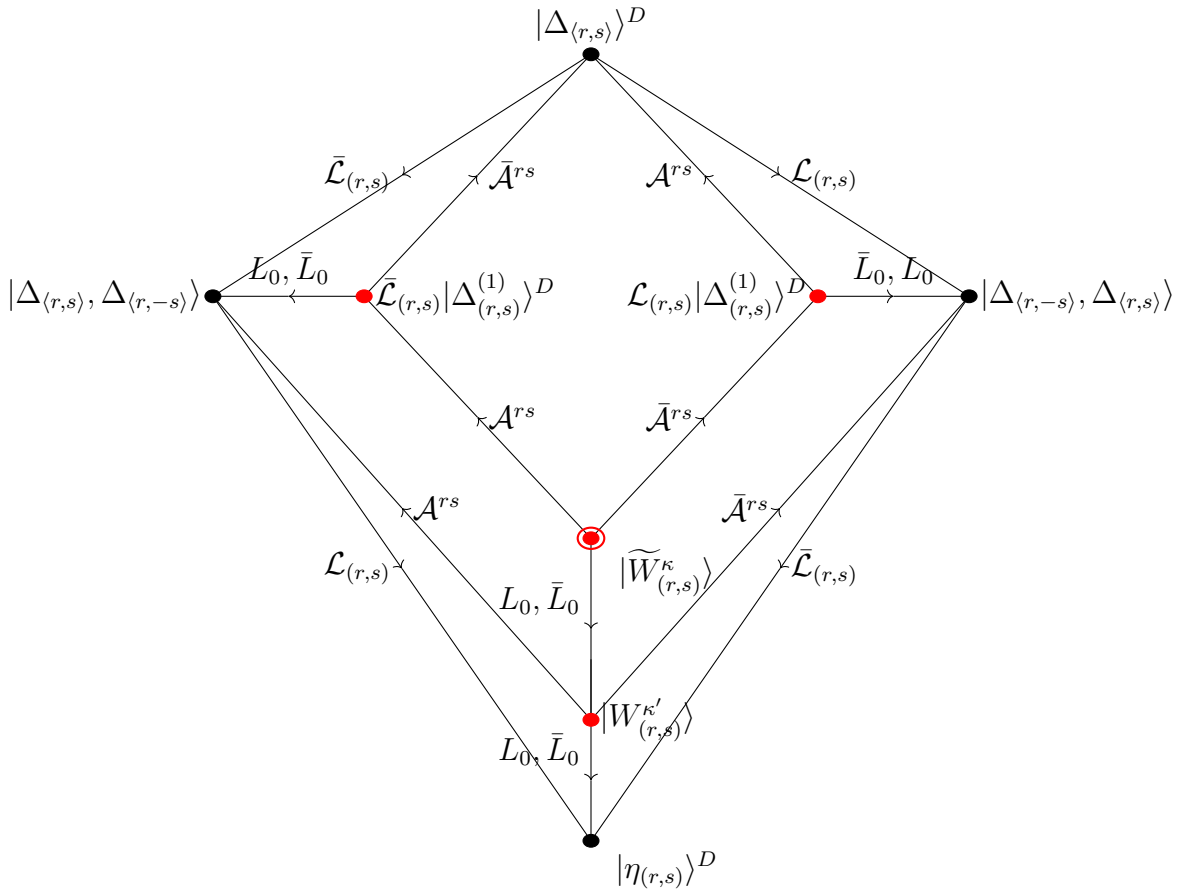


Figure 2.2: Primary and logarithmic states in $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$

CHAPTER 3

Primary fields and logarithmic fields

One of the fundamental principles of CFTs is the existence of a one-to-one map between states and operators, which is known as *the operator-state correspondence* [4]. In the case of two-dimensional CFT, one can then uniquely associate each field with each state in the representation of the Virasoro algebra. Consequently, correlation functions of fields in two-dimensional CFTs are also strongly constrained by conformal symmetry. Let us then discuss how to translate representations of the Virasoro algebra into fields, and how their correlation functions are subject to conformal symmetry. In this Chapter, we only focus on primary fields and logarithmic fields at generic central charge, while the degenerate fields will be discussed in Chapter 4.

3.1 Primary fields

The primary state $|\Delta, \bar{\Delta}\rangle$ gives rise to *the primary field* $V_{\Delta,\bar{\Delta}}(z,\bar{z})$ with the left- and right-conformal dimensions $(\Delta, \bar{\Delta})$. In other words, the primary field $V_{\Delta,\bar{\Delta}}(z,\bar{z})$ is an object that depends on the coordinates z and $\bar{z}$ and also transforms in the Verma module $\mathcal{V}_\Delta \otimes \bar{\mathcal{V}}_{\bar{\Delta}}$ as its highest-weight state. More explicitly, we have

$$L_0^{(z)} V_{\Delta,\bar{\Delta}}(z,\bar{z}) = \Delta V_{\Delta,\bar{\Delta}}(z,\bar{z}) \ , \tag{3.1.1a}$$

$$\bar{L}_0^{(\bar{z})} V_{\Delta,\bar{\Delta}}(z,\bar{z}) = \bar{\Delta} V_{\Delta,\bar{\Delta}}(z,\bar{z}) \ , \tag{3.1.1b}$$

$$\bar{L}_{n>0}^{(\bar{z})} V_{\Delta,\bar{\Delta}}(z) = L_{n>0}^{(z)} V_{\Delta,\bar{\Delta}}(z,\bar{z}) = 0 \ , \tag{3.1.1c}$$

Acting on $V_{\Delta,\bar{\Delta}}(z,\bar{z})$ with the negative Virasoro modes simply gives us *the descendant fields*, which are in correspondences with the descendant states of $|\Delta, \bar{\Delta}\rangle$,

$$\prod_{i=1}^{p} L_{n_i<0}^{(z)} V_{\Delta,\bar{\Delta}}(z,\bar{z}) \longleftrightarrow \prod_{i=1}^{p} L_{n_i<0} |\Delta\rangle \otimes |\bar{\Delta}\rangle \ . \tag{3.1.2}$$

Moreover, we refer to primary fields with the same left- and right-conformal dimensions as *diagonal primary fields*, denoted by $V_\Delta^D(z,\bar{z})$. Let us now consider how the primary field $V_{\Delta,\bar{\Delta}}(z,\bar{z})$ transforms under finite global conformal transformations $w(z)$ and $\bar{w}(\bar{z})$ in (2.1.5). From [4], we have

$$V_{\Delta,\bar{\Delta}}(w(z),\bar{w}(\bar{z})) = \left(\frac{\partial w}{\partial z}\right)^{-\Delta} \left(\frac{\partial \bar{w}}{\partial \bar{z}}\right)^{-\bar{\Delta}} V_{\Delta,\bar{\Delta}}(z,\bar{z}) \ . \tag{3.1.3}$$

We also stress that while the definition of primary fields in (3.1.1) automatically imply the global transformations (3.1.3), the transformations (3.1.3) do not define primary fields but can serve as a definition for *quasi-primary fields*. In other words, any primary field is a quasi-primary, but the converse is not always true. An example of such a phenomenon will be given at the end of next subsection. Let us now consider (3.1.3) for some special cases of $w(z)$ and $\bar{w}(\bar{z})$,

- Under the scaling: $(z, \bar{z}) \to (\lambda z, \lambda \bar{z})$, we have

$$V_{\Delta,\bar{\Delta}}(\lambda z, \bar{\lambda}\bar{z}) = \lambda^{-(\Delta+\bar{\Delta})} V_{\Delta,\bar{\Delta}}(z, \bar{z}) \ . \tag{3.1.4}$$

  Therefore, the total dimension $\Delta + \bar{\Delta}$, also known as *the scaling dimension*, dictates how primary fields scale under scaling the coordinates z and $\bar{z}$.

- Under the rotation: $(z, \bar{z}) \to (e^{i\theta} z, e^{-i\theta} \bar{z})$, we have

$$V_{\Delta,\bar{\Delta}}(e^{i\theta} z, e^{-i\theta} \bar{z}) = e^{i\theta(\Delta-\bar{\Delta})} V_{\Delta,\bar{\Delta}}(z, \bar{z}) \ . \tag{3.1.5}$$

  The quantity $\Delta - \bar{\Delta}$ is called the conformal spin of $V_{\Delta,\bar{\Delta}}(z, \bar{z})$ and is an eigenvalue of the operator $L_0 - \bar{L}_0$. Notice that conformal spins of diagonal primary fields are always zero.

- Under the inversion $(z, \bar{z}) \to (1/z, 1/\bar{z})$. The transformation (3.1.3) then gives us the behaviour of $V_{\Delta,\bar{\Delta}}(z, \bar{z})$ near the point at infinity:

$$\lim_{z,\bar{z}\to\infty} V_{\Delta,\bar{\Delta}}(z, z) \sim \frac{V_{\Delta,\bar{\Delta}}(0)}{z^{2\Delta} \bar{z}^{2\bar{\Delta}}} + \dots \ . \tag{3.1.6}$$

The limit (3.1.6) then provides us a well-behaved definition of primary fields at infinity,

$$V_{\Delta,\bar{\Delta}}(\infty) = \lim_{z,\bar{z}\to\infty} z^{2\Delta} \bar{z}^{2\bar{\Delta}} V_{\Delta,\bar{\Delta}}(z, z) \ . \tag{3.1.7}$$

More precisely, primary fields are analytic over the whole Riemann sphere, including the point at infinity, except for the points where the other fields are located. The validity of the latter argument will be clear when discussing Ward identities in Sections 3.3.

3.2 The stress-energy tensors

As functions of z and $\bar{z}$, the Virasoro generators generate infinitesimal conformal transformations. Therefore we can immediately write down the action of L_{-1} and $\bar{L}_{-1}$ as differential operators on $O(z, \bar{z})$ as follows:

$$\boxed{L_{-1}^{(z)} O(z, \bar{z}) = \frac{\partial}{\partial z} O(z, \bar{z}) \quad \text{and} \quad \bar{L}_{-1}^{(\bar{z})} O(z, \bar{z}) = \frac{\partial}{\partial \bar{z}} O(z, \bar{z})} \ . \tag{3.2.1}$$

While we have been adding the positional dependence on the Virasoro generators, the action of the Virasoro generators at different positions indeed describe the same Virasoro algebra [5]. For instance, consider $[\frac{\partial}{\partial z}, L_n^{(z)}] = (n+1) L_{n+1}^{(z)}$, which implies that L_n acting on different positions are linearly related and belong to the same vector space.

To consider the action of the other generators, we need the field description of the Virasoro algebra: *the stress-energy tensors* $T(z)$ and $\bar{T}(\bar{z})$, which can be defined as generating functions of the generators L_n and $\bar{L}_n$,

$$T(z) = \sum_{n\in\mathbb{Z}} \frac{L_n^{(y)}}{(z-y)^{n+2}} \ , \tag{3.2.2a}$$

$$\bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \frac{\bar{L}_n^{(y)}}{(\bar{z}-\bar{y})^{n+2}} \ , \tag{3.2.2b}$$

where the series in (3.2.2) converge, provided that $(z,\bar{z})$ is sufficiently closed to $(y,\bar{y})$. Now recall from [14] that $T(z)$ and $\bar{T}(\bar{z})$ are quasi-primary fields with the conformal dimensions: $(2,0)$ and $(0,2)$, respectively. Since quasi-primary fields transform under global conformal transformations as in (3.1.3), using (3.1.6), we find that $T(z)$ behaves asymptotically as follows,

$$T(z) \overset{z\to\infty}{\sim} \frac{1}{z^4} \ , \tag{3.2.3}$$

which means that $T(z)$ is smooth at $z=\infty$, as well as $\bar{T}(\bar{z})$. As a matter of facts, the stress-energy tensors are holomorphic functions on the whole Riemann sphere. More precisely, the equations in (3.2.2) suggest that $T(z)$ and $\bar{T}(\bar{z})$ satisfy the Cauchy–Riemann equations:

$$\frac{\partial}{\partial\bar{z}}T(z) = 0 \quad \text{and} \quad \frac{\partial}{\partial z}\bar{T}(\bar{z}) = 0 \ , \tag{3.2.4}$$

which then allow us to expand $T(z)$ and $\bar{T}(z)$ around any point. For compactness, from now, we only write the left-moving objects, whenever their analyses also hold for the right-moving ones. To consider the action of Virasoro algebra on fields, we insert an arbitrary field $O(z,\bar{z})$ into (3.2.2),

$$T(x)O(z,\bar{z}) = \sum_{n\in\mathbb{Z}} \frac{L_n^{(z)}O(z,\bar{z})}{(x-z)^{n+2}} \ . \tag{3.2.5}$$

The product of two fields in (3.2.5) is an example of the operator-product expansion (OPE), which shall be discuss in more details in Section 3.3.3. Applying the residue theorem to (3.2.5), we arrive at the action of $L_n^{(z)}$ on $O(z,\bar{z})$:

$$\boxed{L_n^{(z)}O(z,\bar{z}) = \oint_{\mathcal{C}_z} \frac{dx}{2\pi i}(x-z)^{n+1}T(x)O(z,\bar{z})} \ , \tag{3.2.6}$$

where $\mathcal{C}_z$ is a smooth curve, which encloses the point $x=z$. Since the formula (3.2.6) acts as a one-to-one map between the OPE in (3.2.5) and $L_n^{(z)}O(z,\bar{z})$, therefore knowing the action of Virasoro algebra on the field $O(z,\bar{z})$ also allows us to compute $T(x)O(\bar{z},z)$. For instance, using (3.1.1), we can write the OPE between $T(x)$ and the primary field $V_{\Delta,\bar{\Delta}}(z,\bar{z})$ as follows:

$$T(x)V_{\Delta,\bar{\Delta}}(z,\bar{z}) = \frac{\Delta}{(x-z)^2}V_{\Delta,\bar{\Delta}}(z,\bar{z}) + \frac{1}{x-z}\frac{\partial}{\partial z}V_{\Delta,\bar{\Delta}}(z,\bar{z}) + \mathcal{O}(1) \ . \tag{3.2.7}$$

Therefore, together with its right-moving counterpart, the OPE (3.2.7) provides us an alternative way of defining primary fields.

As fields describing the Virasoro algebra, the product of the stress-energy tensor with themselves are equivalent to the Virasoro algebra [4, 5]. For example, using the left-moving Virasoro algebra (2.1.6) with (3.2.6) gives us

$$T(x)T(z) = \frac{c}{2(x-z)^4}\mathbb{1} + \frac{2}{(x-z)^2}T(z) + \frac{1}{x-z}\frac{\partial}{\partial z}T(z) + \mathcal{O}(1) \ , \tag{3.2.8}$$

where we have introduced the identity field $\mathbb{1}$, a constant field which has vanishing conformal dimension and does not depend on the positions. While the product of the right-moving stress-energy tensors is comparable to (3.2.8), the vanishing commutators $[L_n, \bar{L}_m]$ constrain the the OPE $T(x)\bar{T}(\bar{z})$ to have no singular terms. Using (3.2.8) with (3.2.6), one can write the action of $L^{(z)}_{n\geq 0}$ on $T(z)$ as the following:

$$L^{(z)}_0 T(z) = 2 \quad , \quad L^{(z)}_1 T(z) = 0 \quad , \quad L^{(z)}_2 T(z) = \frac{c}{2}\mathbb{1} \quad \text{and} \quad L^{(z)}_{n>2} T(z) = 0 \ . \tag{3.2.9}$$

For $c \neq 0$, the stress energy tensors are examples of quasi-primary fields which are not primary fields since $L^{(z)}_2 T(z) \neq 0$.

3.3 Ward identities

Conformal Ward identities, which we simply call Ward identities, are constraints from conformal symmetry on correlation functions. In two-dimensional CFTs, Ward identities then tell us how correlation functions are subject to the Virasoro algebra. To begin, we introduce the n-point functions $\left\langle \prod_{i=1}^n O_i(z_i, \bar{z}_i) \right\rangle$ of arbitrary fields $O_i(z_i, \bar{z}_i)$. Now requiring that these n-point functions are translational invariant gives us [4]

$$\left\langle \prod_{i=1}^n O_i(z_i, \bar{z}_i) \right\rangle = h(z_{12}, z_{13}, \dots z_{n\,n+1}; \bar{z}_{12}, \bar{z}_{13}, \dots \bar{z}_{n\,n+1}) \quad \text{with} \quad z_{ij} = z_i - z_j \ , \tag{3.3.1}$$

where h are arbitrary functions. The result (3.3.1) then suggests that the n-point functions in (3.3.1) can become singular if two or more of their external fields are located at the same points, which then led us to an important axiom of correlation functions in CFTs:

$$\left\langle \prod_{i=1}^n O_i(z_i, \bar{z}_i) \right\rangle \textit{ are smooth functions for } (z_i, \bar{z}_i) \neq (z_j, \bar{z}_j) \textit{ with } i, j \in \{1, \dots n\}.$$

To consider how the rest of conformal symmetry constrains correlation functions, we insert infinitesimal conformal transformations $\epsilon(z)T(z)$ into the n-point functions $\left\langle \prod_{i=1}^n O_i(z_i, \bar{z}_i) \right\rangle$. The resulting correlation functions are then analytic provided that $z \in \mathbb{C} - \{z_1, \dots, z_n\}$. Now consider the behaviour of $\left\langle \epsilon(z)T(z) \prod_{i=1}^n O_i(z_i, \bar{z}_i) \right\rangle$ as z approached infinity. Using (3.2.3), one finds [5]

$$\boxed{\oint_{\mathcal{C}_\infty} dz \left\langle \epsilon(z)T(z) \prod_{i=1}^n O_i(z_i, \bar{z}_i) \right\rangle = 0 \quad \text{provided} \quad \epsilon(z) \overset{z\to\infty}{\sim} O(z^2)} \ . \tag{3.3.2}$$

The above equation is known as *the Ward identities*, which are characterized by the functions $\epsilon(z)$. If $\epsilon(z)$ correspond to the global conformal transformations, we refer to (3.3.2) as *global Ward identities*, while the case of local conformal transformations lead to *local Ward identities*.

3.3.1 Local Ward identities

Local Ward identities can be derived by assuming that $\epsilon(z)$ in (3.3.2) is a meromorphic function, which only has poles at the points $z \in \{z_1, \dots z_n\}$. In particular, we choose

$$\epsilon(z) = \frac{1}{(z - z_j)^{n-1}} \quad \text{for} \quad n > 1 \ . \tag{3.3.3}$$

With the above choice of $\epsilon(z)$ and using (3.2.6), we then pull the contour integral in (3.3.2) over the Riemann sphere, pick up the poles at $z \in \{z_1, \dots, z_n\}$, then arrive at

$$\left(L_{-n}^{(z_j)} + \sum_{i \neq j} \sum_{k=-1}^{\infty} \binom{n+k-1}{k+1} \frac{L_k^{(z_i)}}{z_{ij}^{n+k}} \right) \left\langle \prod_{i=1}^{n} O_i(z_i, \bar{z}_i) \right\rangle = 0 \ . \tag{3.3.4}$$

Let us also specialize (3.3.4) to the case of all fields $O_i(z_i, \bar{z}_i)$ being primary fields. Therefore, any term in (3.3.4) with positive Virasoro modes vanishes, and we simply find the well-known result [4]:

$$\left\langle L_{-n}^{(z_j)} V_{\Delta_i, \bar{\Delta}_i}(z_i, \bar{z}_i) \prod_{i \neq j} V_{\Delta_i, \bar{\Delta}_i}(z_i, \bar{z}_i) \right\rangle = - \sum_{i \neq j} \left(\frac{1}{z_{ji}^{n-1}} \frac{\partial}{\partial z_j} - \frac{n-1}{z_{ji}^{n}} \Delta_i \right) \left\langle \prod_{i=1}^{n} V_{\Delta_i, \bar{\Delta}_i}(z_i, \bar{z}_i) \right\rangle . \tag{3.3.5}$$

In particular, the equation (3.3.5) tells us how the generators $L_{-n}^{(z_j)}$ act on correlation functions of primary fields as differential operators. The local Ward identities (3.3.5) and (3.3.4) then allow us to compute correlation functions of descendant fields from correlation functions of their primary fields, while computing correlation functions of primary fields require solving global Ward identities, which we shall now discuss.

3.3.2 Global Ward identities

Global Ward identities are obtained by setting $\epsilon(z) \in \{1, z, z^2\}$ in (3.3.2), which amounts to choosing $\epsilon(z)$ which are holomorphic functions on the whole Riemann sphere. With these choices of $\epsilon(z)$, we arrive at

$$\epsilon = 1 \to \sum_{i=1}^{n} L_{-1}^{(z_i)} \left\langle \prod_{i=1}^{n} O_i(z_i, \bar{z}_i) \right\rangle = 0 \ , \tag{3.3.6a}$$

$$\epsilon = z \to \sum_{i=1}^{n} \left(L_0^{(z_i)} + z_i L_{-1}^{(z_i)} \right) \left\langle \prod_{i=1}^{n} O_i(z_i, \bar{z}_i) \right\rangle = 0 \ , \tag{3.3.6b}$$

$$\epsilon = z^2 \to \sum_{i=1}^{n} \left(L_1^{(z_i)} + 2 z_i L_0^{(z_i)} + z_i^2 L_{-1}^{(z_i)} \right) \left\langle \prod_{i=1}^{n} O_i(z_i, \bar{z}_i) \right\rangle = 0 \ . \tag{3.3.6c}$$

For the case of correlation functions with only primary fields, the terms with $L_1^{(z_i)}$ then give no contributions, and one can also express the action of L_{-1}, L_0, and L_1 in (3.3.6) as differential operators by using the OPE (3.2.7). Let us now discuss results from solving the global Ward identities for n-point functions of primary fields up to $n = 4$. Detailed calculations of these cases exist in many literatures. For example, see [4].

One-point functions

Since zero-point functions on the Riemann sphere is just a constant, which depends on the central charge c [5], we start with the case of one-point functions. Writing the first two equations in (3.3.6) for $\langle V_{\Delta,\bar{\Delta}}(z,\bar{z})\rangle$ yields

$$\langle \frac{\partial}{\partial z} V_{\Delta,\bar{\Delta}}(z,\bar{z})\rangle = 0 \ , \tag{3.3.7a}$$

$$\Delta V_{\Delta,\bar{\Delta}}(z,\bar{z}) + z\frac{\partial}{\partial z}\langle V_{\Delta,\bar{\Delta}}(z,\bar{z})\rangle = 0 \ , \tag{3.3.7b}$$

where the last equation in (3.3.6) gives a trivial constraint in this case. The equation (3.3.7a) is equivalent to $L_{-1}V_{\Delta,\bar{\Delta}}(z,\bar{z}) = 0$. Subsequently, solving the second equation in (3.3.7), as well as its right-moving counterpart, give us $\Delta = \bar{\Delta} = 0$. The only field, which satisfies both of these two requirements, is the identity field $\mathbb{1}$. Thus, the identity field is the only field with non-vanishing one point function of CFTs on the Riemann sphere,

$$\langle V_{\Delta,\bar{\Delta}}(z,\bar{z})\rangle \neq 0 \iff V_{\Delta,\bar{\Delta}}(z,\bar{z}) = \alpha \quad \text{for some constants } \alpha \ . \tag{3.3.8}$$

Two-point functions

Global Ward identities completely fix two-point functions of primary fields up to normalizations. Assuming that $(\Delta_1,\bar{\Delta}_1)$ and $(\Delta_2,\bar{\Delta}_2)$ take continuous values, we have

$$\langle V_{\Delta_1,\bar{\Delta}_1}(z_1,\bar{z}_1)V_{\Delta_2,\bar{\Delta}_2}(z_2,\bar{z}_2)\rangle = \delta(\Delta_1-\Delta_2)\delta(\bar{\Delta}_1-\bar{\Delta}_2)\frac{B(V_{\Delta_1,\bar{\Delta}_1})}{z_{12}^{\Delta_1}\bar{z}_{12}^{\bar{\Delta}_1}} \ , \tag{3.3.9}$$

where $B(V_{\Delta_1,\bar{\Delta}_1})$ is also known as the two-point structure constant. Moreover, we replace the Dirac-delta functions in (3.3.9) with the Kronecker-delta symbols $\delta_{\Delta_1,\Delta_2}\delta_{\bar{\Delta}_1,\bar{\Delta}_2}$ if the conformal dimensions in (3.3.9) take discrete values. Therefore, two-point functions of two primary fields vanish if their dimensions do not coincide, but the converse is not always correct. For instance, two-point functions of two identical null fields vanish. Moreover, $B(V_{\Delta,\bar{\Delta}})$ cannot always be normalized to one. For instance, normalizing $B_{V_{\Delta,\bar{\Delta}}}$ to one is inconsistent with the analyticity of three-point functions in Liouville theory [5].

Three-point functions

Solving (3.3.6), one finds that three-point functions of primary fields are given by

$$\Big\langle \prod_{i=1}^{3} V_{\Delta_i,\bar{\Delta}_i}(z_i,\bar{z}_i)\Big\rangle = C_{123}|\mathcal{F}^{(3)}(\Delta_i|z_i)|^2 \ , \tag{3.3.10}$$

where C_{123} are unfixed coefficients also known as the three-point structure constants. Unlike two-point functions, the coefficients C_{123} cannot be fixed by using field renormalizations. Furthermore, $\mathcal{F}^{(3)}(\Delta_i;z_i)$ are the three-point conformal blocks, defined by

$$\mathcal{F}^{(3)}(\Delta_i;z_i) = z_{12}^{\Delta_3-\Delta_1-\Delta_2}z_{13}^{\Delta_2-\Delta_1-\Delta_3}z_{23}^{\Delta_1-\Delta_2-\Delta_3} \ . \tag{3.3.11}$$

In (3.3.10), we have also introduced the modulus square of arbitrary functions $f(z;\Delta)$:

$$|f(z;\Delta)|^2 = f(z;\Delta)\bar{f}(\bar{z};\bar{\Delta}) \ . \tag{3.3.12}$$

Furthermore, the three-point structure constants C_{123} are functions of the conformal dimensions of the three primary fields in (3.3.10) and also depend explicitly on the central charge c. In the case of all three fields in (3.3.10) being bosonic, C_{123} are then full symmetric under exchanging the indices [10]. Although it is not possible to compute C_{123} by using only conformal symmetry, C_{123} can be computed by taking into account associativity of the OPE.

Four-point functions

In contrast to the previous three cases, the dependence on the positions of four-point functions of primary fields cannot be completely fixed by the global Ward identities, which only allow us to write $\left\langle \prod_{i=1}^{4} V_{\Delta_i,\bar{\Delta}_i}(z_i,\bar{z}_i) \right\rangle$ as the following

$$\left\langle \prod_{i=1}^{4} V_{\Delta_i,\bar{\Delta}_i}(z_i,\bar{z}_i) \right\rangle = |z_{13}^{-2\Delta_1} z_{23}^{\Delta_1-\Delta_2+\Delta_3+\Delta_4} z_{34}^{\Delta_1+\Delta_2-\Delta_3-\Delta_4} z_{24}^{-\Delta_1-\Delta_2+\Delta_3-\Delta_4} F(\Delta_i,c,x)|^2 \ , \quad (3.3.13)$$

where $F(\Delta_i,c,x)$ is an arbitrary function, which depends on the conformal dimensions of $V_{\Delta_i,\bar{\Delta}_i}$, the central charge c, and also the cross-ratios x and $\bar{x}$:

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}} \quad \text{and} \quad \bar{x} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}} \ . \quad (3.3.14)$$

Let us now also observe that four-point functions on the left-handside of (3.3.15) has 8 positional variables: $(z_1,\bar{z}_1),\ldots,(z_4,\bar{z}_4)$ while the dependence of the position of its right-hand can be determined by only 2 variables: the cross-ratios x and $\bar{x}$. This follows directly from having $2\times 3 = 6$ Virasoro generators for the global conformal transformations, which put constraints on positions of three fields on the left-hand side of (3.3.15). For instance, one can use global conformal transformations to fix the positions of three primary fields in (3.3.15) as the following:

$$\langle V_{\Delta_1,\bar{\Delta}_1}(z,\bar{z}) V_{\Delta_2,\bar{\Delta}_2}(0) V_{\Delta_3,\bar{\Delta}_3}(\infty) V_{\Delta_4,\bar{\Delta}_4}(1)\rangle = |F(\Delta_i,c,z)|^2 \ . \quad (3.3.15)$$

Throughout this thesis, we shall always compute four-point functions as in (3.3.15).

3.3.3 The OPE-Ward identities

The operator-product expansions (OPE) is a general idea in quantum field theory of writing a product between two fields, which are sufficiently close to each other, as a sum of other fields in the model's spectrum. We are interested in how two-dimensional conformal symmetry constrains the product of two arbitrary scaling fields $O_1(z_1,\bar{z}_1)$ and $O_2(z_2,\bar{z}_2)$. Now consider

$$O_1(z_1,\bar{z}_1)O_2(z_2,z_2) = \sum_O C^O_{O_1O_2}(z_{12},\bar{z}_{12})O(z_2,\bar{z}_2) \ . \quad (3.3.16)$$

Taking into account the scaling and translational symmetry, we can then write down the dependence on positions of $C^O_{O_1O_2}(z_{12},\bar{z}_{12})$ as follows:

$$C^O_{O_1O_2}(z_1,z_2) = C^O_{O_1O_2}|z_{12}^{\Delta_O-\Delta_{O_1}-\Delta_{O_2}}| \ . \quad (3.3.17)$$

The coefficients $C^O_{O_1O_2}$ are known as *the OPE coefficients*. The OPE coefficients $C^O_{O_1O_2}$ have no dependence on positions but depend on the fields O_1, O_2, and O. Likewise to how conformal symmetry gives rise to the Ward identities (3.3.2), the OPE (3.3.16) are constrained by the Virasoro algebra as follows [5],

$$\left(L_n^{(z_2)} + \sum_{m=-1}^{n} \binom{n+1}{m+1} z_{12}^{n-m} L_m^{(z_1)} \right) O_1O_2 = \sum_O C^O_{O_1O_2}|z_{12}|^{\Delta_O-\Delta_{O_1}-\Delta_{O_2}} L_n^{(z_2)}O \ . \quad (3.3.18)$$

The above is also known as *the OPE-Ward identities*. For simplicity, we will discuss the equation (3.3.18) for the case of CFTs with only Verma modules. Specializing (3.3.16) to this case, the OPE of two primary fields reads

$$V_{\Delta_1,\bar{\Delta}_1}(z_1,\bar{z}_1)V_{\Delta_2,\bar{\Delta}_2}(z_2,\bar{z}_2) = \sum_{\Delta}\sum_{\mathcal{L}} |C_{12}^{\Delta,\mathcal{L}} z_{12}^{\Delta_1-\Delta_2-\Delta-\sum_{\mathcal{L},\bar{\mathcal{L}}}\mathcal{L}}|\mathcal{L}\bar{\mathcal{L}}V_{\Delta,\bar{\Delta}}(z_2,\bar{z}_2) \ , \quad (3.3.19)$$

where the operators $\mathcal{L} \in \{L_{-1}, L_{-1}^2, L_{-2}, \ldots\}$, that is to say the sum in (3.3.19) is not only taken over primary fields but also all of their descendants. Inserting the right-handside of (3.3.19) into the three-point functions (3.3.10) then allows us to express the OPE coefficients of primary fields C_{12}^{Δ} through the two- and three-point structure constants:

$$C_{12}^{\Delta} = \frac{C_{12\Delta}}{B_{\Delta}} \ . \quad (3.3.20)$$

OPE coefficients of the descendant fields

Although two-dimensional CFTs come with infinite conformal symmetry, it is not possible to fix OPE coefficients of primary fields with only conformal symmetry. Nevertheless, through the OPE-Ward identities (3.3.18), conformal symmetry still allows us to fix the relative ratio between the OPE coefficients of primary fields and their descendants. We rewrite the OPE coefficients of the descendant fields $\mathcal{L}V_{\Delta}$ as the following,

$$C_{12}^{\Delta,\mathcal{L}} = C_{12}^{\Delta} f_{\Delta}^{\mathcal{L}} \ . \quad (3.3.21)$$

We call $f_{\Delta}^{\mathcal{L}}$ as *the relative OPE coefficients of* $\mathcal{L}V_{\Delta}$ *and its primary* V_{Δ}. Let us also stress here that $f_{\Delta}^{\mathcal{L}}$ also depends on Δ_1 and Δ_2, as we will see in examples below. With (3.3.21) and (3.3.19), the OPE-Ward identities (3.3.18) become [5]

$$\sum_{|\mathcal{L}|=N-n} (\Delta_j + N - n + n\Delta_1 - \Delta_2) f_{\Delta}^{\mathcal{L}} \mathcal{L}V_{\Delta} = \sum_{|\mathcal{L}|=N} f_{\Delta}^{\mathcal{L}}[L_n, \mathcal{L}]V_{\Delta} \quad \text{for} \quad n > 0 \ , \quad (3.3.22)$$

where the derivatives in z from the generator L_{-1} can be wiped out completely by considering the OPE-Ward identities (3.3.18) with $n = 0$,

$$\left(z_{12}\frac{\partial}{\partial z_1} - L_0^{(z_2)}\right) V_{\Delta_1,\bar{\Delta}_1}(z_1,\bar{z}_1)V_{\Delta_2,\bar{\Delta}_2}(z_1,\bar{z}_1) = 0 \ . \quad (3.3.23)$$

The equation (3.3.22) then provides us the system of linear equations for $f_{\Delta}^{\mathcal{L}}$. The number of unknowns $f_{\Delta}^{\mathcal{L}}$ is therefore given by the number of partitioning the integer N. Moreover, let us again remind ourselves that the equation (3.3.22) with $n > 2$ is redudant since they are just the commutators of the case $n = 1, 2$. For each value of N, it is therefore sufficient to consider the equation with $n = 1, 2$. For instance, at $N = 2$, we have

$$\begin{aligned}
&(N,n) = (1,1) \quad : 2\Delta f_{\Delta}^{L_{-1}} = \Delta + \Delta_1 - \Delta_2 \ , \quad (3.3.24)\\
&(N,n) = (2,1) \quad : 6\Delta f_{\Delta}^{L_{-1}^2} + \left(4\Delta + \tfrac{c}{2}\right) f_{\Delta}^{L_{-2}} = \Delta + 2\Delta_1 - \Delta_2 \ ,\\
&(N,n) = (2,2) \quad : (4\Delta + 2) f_{\Delta}^{L_{-1}^2} + 3 f_{\Delta}^{L_{-2}} = (\Delta + 1 + \Delta_1 - \Delta_2) f_{\Delta}^{L_{-1}} \ ,
\end{aligned}$$

where it is easy to see that both sides of the equation (3.3.22) become zero for $(N, n) = (1, 2)$. Solving the first equation in (3.3.24) simply yields

$$f_{\Delta}^{L_{-1}} = \frac{\Delta_1 - \Delta_2 - \Delta}{2\Delta} \ . \quad (3.3.25)$$

Therefore, $f_\Delta^{L_{-1}}$ has a pole at Δ equals zero, which is the degenerate dimension $\Delta_{(1,1)}$, previously defined in (2.3.8). For the last two equations in (3.3.24), we have

$$f_\Delta^{L_{-2}} = -\frac{6f_\Delta^{L_{-1}}\Delta\left(\Delta_1-\Delta_2+\Delta+1\right)-2\left(2\Delta_1-\Delta_2+\Delta\right)\left(2\Delta+1\right)}{2(\Delta-\Delta_-)(\Delta-\Delta_+)}\,, \tag{3.3.26a}$$

$$f_\Delta^{L_{-1}^2} = \frac{f_\Delta^{L_{-1}}\left(\Delta_1-\Delta_2+\Delta+1\right)\left(c+8\Delta\right)-6\left(2\Delta_1-\Delta_2+\Delta\right)}{4(\Delta-\Delta_-)(\Delta-\Delta_+)}\,, \tag{3.3.26b}$$

where $\Delta_\pm$ were defined in (2.3.6) as the degenerate conformal dimensions at level 2. Moreover, one can check that the numerators of both $f_\Delta^{L_{-1}^2}$ and $f_\Delta^{L_{-2}}$ do not have zeroes at $\Delta=\Delta_\pm$ for generic Δ_1 and Δ_2. Therefore, these two coefficients have the following poles in Δ,

$$f_\Delta^{\mathcal{L}} \sim \frac{1}{(\Delta-\Delta_+)(\Delta-\Delta_-)} \quad \text{for} \quad \mathcal{L}\in\{L_{-1}^2, L_{-2}\}\,. \tag{3.3.27}$$

The general result for $f_\Delta^{\mathcal{L}}$ comes with the following poles [21]:

$$\boxed{f_\Delta^{\mathcal{L}} \sim \prod_{\substack{rs=N\\ r,s\in\mathbb{N}^*}} (\Delta_j-\Delta_{(r,s)})^{-1} \quad \text{for} \quad |\mathcal{L}|=N}\,. \tag{3.3.28}$$

Having simple poles at each value of degenerate conformal dimension in the above is also consistent with the Zamolodchikov recursion for conformal blocks [21], to be introduced in (3.4.15).

3.4 Conformal blocks

The concept of the OPE also holds inside correlation functions [6], therefore we can reduce n-point functions to lower-point functions by performing the OPE on a string of fields inside correlation functions. This led us to the definition of conformal blocks as bases of correlation functions of CFTs, arising from using the OPE. Conformal blocks are universal objects which are solutions to Ward identities and can be completely determined by only conformal symmetry. For instance, applying the OPE to two identical primary fields in their two-point functions gives us a sum of one-point functions,

$$\langle V_{\Delta_1,\bar\Delta_1} V_{\Delta_1,\bar\Delta_1}\rangle \sim \sum_\Delta \langle V_{\Delta,\bar\Delta}\rangle \sim \langle \mathbb{1}\rangle = 1\,. \tag{3.4.1}$$

Thus, the only non-vanishing two-point conformal block corresponds to the identity field. For three-point conformal blocks, we simply have the three-point conformal block $\mathcal{F}^{(3)}$ in (3.3.10). In CFTs, we are usually interested up to only four-point conformal blocks since all dynamical data is already encoded in the four-point functions a priori [2]. For convenience, let us discuss the case of CFTs with only Verma modules. Now consider the four-point function $\left\langle \prod_{i=1}^4 V_{\Delta_i,\bar\Delta_i}(z_i,\bar z_i)\right\rangle$ in which we perform the OPEs of the following pairs of these non-diagonal primary fields: $\{V_{\Delta_1,\bar\Delta_1}, V_{\Delta_2,\bar\Delta_2}\}$ and $\{V_{\Delta_3,\bar\Delta_3}, V_{\Delta_4,\bar\Delta_4}\}$. We then define the s-channel conformal blocks $\mathcal{F}_\Delta^{(s)}(x;\Delta_1,\Delta_2,\Delta_3,\Delta_4)$ for the Verma module $\mathcal{V}$ from the s-channel decomposition:

$$\left\langle \prod_{i=1}^4 V_{\Delta_i,\bar\Delta_i}(z_i,\bar z_i)\right\rangle = \sum_\Delta D_\Delta^{(s)} |\mathcal{F}_\Delta^{(s)}(x;\Delta_1,\Delta_2,\Delta_3,\Delta_4)|^2\,, \tag{3.4.2}$$

where we have introduced the four-point structure constants,

$$D_{\Delta}^{(s)} = \frac{C_{12\Delta}C_{\Delta 34}}{B_{\Delta}} \ . \tag{3.4.3}$$

If we had chosen to perform the OPE on different pairs of fields in (3.4.2), we would have arrived at different channels of $\langle\prod_{i=1}^{4} V_{\Delta_i,\bar{\Delta}_i}(z_i,\bar{z}_i)\rangle$, namely the t- and u- channels. Conformal blocks in these two channels are related to the s-channel as follows:

$$\mathcal{F}_{\Delta}^{(t)}(x;\Delta_1,\Delta_2,\Delta_3,\Delta_4) = \mathcal{F}_{\Delta}^{(s)}(1-x;\Delta_1,\Delta_4,\Delta_3,\Delta_2) \ , \tag{3.4.4a}$$

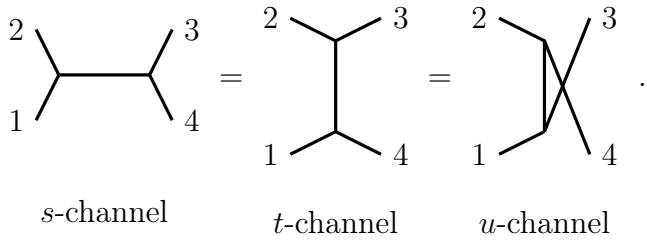
$$\mathcal{F}_{\Delta}^{(u)}(x;\Delta_1,\Delta_2,\Delta_3,\Delta_4) = x^{-\Delta_1}\mathcal{F}_{\Delta}^{(s)}\left(\frac{1}{x};\Delta_1,\Delta_3,\Delta_2,\Delta_4\right) \ . \tag{3.4.4b}$$

That is to say the t-channel is related to the s-channel via the permutation: $1 \leftrightarrow 4$ and $2 \leftrightarrow 3$, while one arrives at the u-channel by permutting $1 \leftrightarrow 3$ and $2 \leftrightarrow 4$ in the s-channel. Therefore, it also follows that four-point structure constants in the t- and u channels are given by

$$D_{\Delta}^{(t)} = \frac{C_{14\Delta}C_{\Delta 23}}{B_{\Delta}} \quad \text{and} \quad D_{\Delta}^{(s)} = \frac{C_{13\Delta}C_{\Delta 24}}{B_{\Delta}} \ . \tag{3.4.5}$$

Crossing symmetry

Since the OPE obey associativity, different channel expansions of four-point functions must coincide. Schematically, we have



(3.4.6)

The equation (3.4.6) is known as *the crossing-symmetry equation*, which constrains four-point functions of CFTs to be crossing symmetric. Furthermore, if all conformal blocks in each channel of (3.4.6) are known, the crossing-symmetry equation is then a system of quadratic equations for three-point structure constants, or equivalently a system of linear equations for four-point structure constants. It is also interesting to point here that the crossing-symmetry of two channels does not always imply the coincidence of all three channels.

3.4.1 Pedestrian computation

We now show how to compute conformal blocks in(3.4.2) by using the pedestrian computation [5]. This formalism is not only valid for conformal blocks of Verma modules but can also be applied to the case of generic representations of the Virasoro algebra. To start, let us write the s-channel decomposition in (3.4.2) more explicitly. Using (3.3.19),

we have

$$\begin{aligned}
&\langle V_{\Delta_1,\bar{\Delta}_1}(z,\bar{z})V_{\Delta_2,\bar{\Delta}_2}(0)V_{\Delta_3,\bar{\Delta}_3}(\infty)V_{\Delta_4,\bar{\Delta}_4}(1)\rangle \\
&= \sum_{\Delta} \frac{C_{12\Delta}}{B_\Delta} \sum_{\mathcal{L}} \left| f_\Delta^{\mathcal{L}} z^{-\Delta_1-\Delta_2+\Delta+|\mathcal{L}|} \right| \langle \mathcal{L}\bar{\mathcal{L}} V_{\Delta,\bar{\Delta}}(0)V_{\Delta_3,\bar{\Delta}_3}(\infty)V_{\Delta_4,\bar{\Delta}_4}(1)\rangle \ , \\
&= \sum_{\Delta} \frac{C_{12\Delta}C_{\Delta 34}}{B_\Delta} \underbrace{\sum_{\mathcal{L}} \left| f_\Delta^{\mathcal{L}} z^{-\Delta_1-\Delta_2+\Delta+|\mathcal{L}|} \right| \frac{\langle \mathcal{L}\bar{\mathcal{L}} V_{\Delta_j,\bar{\Delta}}(0)V_{\Delta_3,\bar{\Delta}_3}(\infty)V_{\Delta_4,\bar{\Delta}_4}(1)\rangle}{\langle V_{\Delta_j,\bar{\Delta}}(0)V_{\Delta_3,\bar{\Delta}_3}(\infty)V_{\Delta_4,\bar{\Delta}_4}(1)\rangle}}_{\mathcal{F}_\Delta^{(s)}\bar{\mathcal{F}}_{\bar{\Delta}}^{(s)}} \ ,
\end{aligned} \tag{3.4.7}$$

where we have recalled that four-point functions of primary fields factorize into the product of their left- and right-moving parts. From (3.4.7), we can then write $\mathcal{F}_\Delta$ as the following,

$$\mathcal{F}_\Delta^{(s)} = \sum_{\mathcal{L}} z^{\Delta-\Delta_1-\Delta_2+|\mathcal{L}|} f_\Delta^{\mathcal{L}} g_\Delta^{\mathcal{L}} \ , \tag{3.4.8}$$

where $f_\Delta^{\mathcal{L}}$ can be computed from (3.3.22). Using (3.3.28), the expansion (3.4.8) then suggests that $\mathcal{F}_\Delta^{(s)}$ then can have a pole at Δ equals to the degenerate conformal dimension $\Delta_{(r,s)}$ in (2.3.8). Furthermore, the coefficients $g_\Delta^{\mathcal{L}}$ are ratios of three-point structure constants,

$$g_\Delta^{\mathcal{L}} = \frac{\langle \mathcal{L}V_{\Delta,\bar{\Delta}}(0)V_{\Delta_1,\bar{\Delta}_1}(\infty)V_{\Delta_2,\bar{\Delta}_2}(1)\rangle}{\langle V_{\Delta,\bar{\Delta}}(0)V_{\Delta_1,\bar{\Delta}_1}(\infty)V_{\Delta_2,\bar{\Delta}_2}(1)\rangle} \tag{3.4.9}$$

For example, we write

$$g_\Delta^{L_{-1}} = \Delta - \Delta_1 + \Delta_2 \ , \tag{3.4.10a}$$

$$g_\Delta^{L_{-2}} = \Delta - \Delta_1 + 2\Delta_2 \ , \tag{3.4.10b}$$

$$g_\Delta^{L_{-1}^2} = (\Delta - \Delta_1 + \Delta_2)(\Delta - \Delta_1 + \Delta_2 + 1) \ . \tag{3.4.10c}$$

Using the above to compute (3.4.8) for the first few terms yields

$$\mathcal{F}_\Delta^{(s)}(z) = z^{\Delta-\Delta_1-\Delta_2}\left(1 + \frac{(\Delta-\Delta_1-\Delta_2)(\Delta-\Delta_3-\Delta_4)}{2\Delta} z + \dots\right) \ . \tag{3.4.11}$$

For $\Delta < \Delta_1 + \Delta_2$, the s-channel conformal blocks $\mathcal{F}_\Delta^{(s)}(z)$ then become singular at $z = 0$. Using (3.4.4), it follows that $\mathcal{F}_\Delta^{(t)}(z)$ and $\mathcal{F}_\Delta^{(u)}(z)$ can also have singularities at $z = 1$ and $z = \infty$ respectively. With the crossing-symmetry equation (3.4.6), four-point functions of CFTs on the Riemann sphere are then not well-defined at the points $z \in \{0, 1, \infty\}$. From now, we shall neglect writing the channel index of conformal blocks whenever the discussion holds for three channels. Using (3.4.8), the conformal blocks $\mathcal{F}_\Delta(z)$ are then linear in three-point functions of Verma modules $\mathcal{V}_\Delta$. This argument can also be extended to other representations of the Virasoro algebra.

3.4.2 The Zamolodchikov recursion

The conformal blocks $\mathcal{F}_\Delta(z)$ are analytic over the whole Riemann sphere, except for the points $z \in \{0, 1, \infty\}$. This can be seen by rewriting $\mathcal{F}_\Delta(z)$ as infinite power series of the elliptic nome $q(z)$, which maps the complex plane $\mathbb{C}$ to a unit disk [39]. This representation

of the conformal blocks $\mathcal{F}_\Delta(z)$ is also known as *the Zamolodchikov recursion*. Let us write $q(z)$ as follows:

$$q(z) = \exp\left\{ i\pi \frac{K(1-z)}{K(z)} \right\} \iff z = \left(\frac{\theta_2(q)}{\theta_3(q)} \right)^4 , \tag{3.4.12}$$

where $K(z)$ is the complete elliptic integral of the first kind of modulus z, while the functions $\theta_2(q)$ and $\theta_3(q)$ are the Jacobi theta functions. The nome $q(z)$ is analytic over the whole complex plane $\mathbb{C}$. For example, we have

$$q(z) \overset{z\to 0}{=} \frac{z}{16} + \frac{z^2}{32} + \dots \tag{3.4.13}$$

Let us now write the s-channel conformal blocks $\mathcal{F}_\Delta^{(s)}(z)$ by using the Zamolodchikov recursion. We have

$$\mathcal{F}_\Delta^{(s)}(z) = (16q)^{\Delta_s + \frac{c-1}{24}} z^{-\frac{c-1}{24}-\Delta_1-\Delta_2} (1-z)^{-\frac{c-1}{24}-\Delta_1-\Delta_2} \theta_3(q)^{-\frac{c-1}{8}-4(\Delta_1+\Delta_2+\Delta_3+\Delta_4)} \mathcal{H}_\Delta(\Delta_i|q) . \tag{3.4.14}$$

The functions $\mathcal{H}_{\Delta_s}(\Delta_i|q)$ satisfy the so-called Zamolodchikov recursion,

$$\mathcal{H}_\Delta(\Delta_i|q) = 1 + \sum_{m,n=1}^{\infty} \frac{(16q)^{mn}}{\Delta - \Delta_{(m,n)}} R_{(m,n)} \mathcal{H}_{\Delta_{(m,-n)}}(\Delta_i|q) , \tag{3.4.15}$$

where the residues $R_{m,n}$ are defined by

$$R_{m,n} = \frac{2P_{(0,0)}P_{(m,n)}}{\prod_{r=1-m}^{m} \prod_{s=1-n}^{n} 2P_{(r,s)}} \prod_{r\overset{2}{=}1-m}^{m-1} \prod_{s\overset{2}{=}1-n}^{n-1} (P_2 \pm P_1 + P_{(r,s)})(P_3 \pm P_4 + P_{(r,s)}) , \tag{3.4.16}$$

where the momenta $P_{(r,s)}$ were introduced in (2.3.8). From several numerical observations, we expect that the conformal blocks $\mathcal{F}_\Delta(z)$ in all three channels converge for $|q(z)| < 1$, or equivalently $z \in \mathbb{C} - \{0, 1, \infty\}$. However, proving the convergence of the Zamolodchikov recursion is still an open problem. Furthermore, the Zamolodchikov recursion shows explicitly that $\mathcal{F}_\Delta(z)$ can have a pole in the conformal dimensions at $\Delta = \Delta_{(m,n)}$, which agrees with our discussion for the pedestrian computation.

3.5 Logarithmic fields

Similarly to primary states and primary fields, logarithmic states give rise to logarithmic fields. Let us then define the derivatives of diagonal primary fields $V_\Delta^{D^{(n)}}$ as follows:

$$V_\Delta^{D^{(n)}} = \frac{1}{n!} \frac{\partial^n}{\partial \Delta^n} V_\Delta^D , \tag{3.5.1}$$

which satisfy the following conditions:

$$L_0 V_\Delta^{D^{(n)}} = V_\Delta^{D^{(n-1)}} + \Delta V_\Delta^{D^{(n)}} \quad \text{and} \quad L_{n>0} V_\Delta^{D^{(n)}} = 0 . \tag{3.5.2}$$

The order-n derivatives of primary fields in (3.5.1) then lead to a rank-$n+1$ Jordan block of L_0, as shown in (2.4.8). Assuming that the null fields $\eta_{(r,s)}^D$ do not vanish, let us now

introduce the derivatives of these null fields, which correspond to logarithmic states in (2.5.2). We have

$$\eta^{D^{(n)}}_{(r,s)} = V^{D^{(n)}}_{\Delta}\Big|_{\Delta=\Delta_{(r,-s)}} , \tag{3.5.3a}$$

$$\mu^{D^{(n)}}_{(r,s)} = \mathcal{L}_{(r,s)}\bar{\mathcal{L}}_{(r,s)}V^{D^{(n)}}_{\Delta}\Big|_{\Delta=\Delta_{(r,s)}} . \tag{3.5.3b}$$

Therefore, the representations $\mathcal{W}^{\kappa}_{(r,s)}$ and $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$ are generated by the logarithmic fields:

$$W^{\kappa}_{(r,s)} = (1-\kappa)\eta^{D^{(1)}}_{(r,s)} + \kappa\mu^{D^{(1)}}_{(r,s)} , \tag{3.5.4}$$

$$\widetilde{W}^{\kappa}_{(r,s)} = \frac{(1-\kappa)}{2}\eta^{D^{(2)}}_{(r,s)} + \frac{\kappa}{2}\mu^{D^{(2)}}_{(r,s)} . \tag{3.5.5}$$

The action of Virasoro algebra on logarithmic fields in (3.5.4) and (3.5.5) is the same as the logarithmic states in (2.5.4) and (2.5.15), respectively. For example, the logarithmic fields $\mathcal{W}^{\kappa}_{(r,s)}$ satisfy the equation:

$$\mathcal{L}_{(r,s)}\mathcal{A}^{rs}W^{\kappa}_{(r,s)} = \bar{\mathcal{L}}_{(r,s)}\bar{\mathcal{A}}^{rs}W^{\kappa}_{(r,s)} = \kappa(L_0 - \Delta_{(r,-s)})W^{\kappa}_{(r,s)} , \tag{3.5.6}$$

where the operators $\mathcal{A}^{rs}$ and $\bar{\mathcal{A}}^{rs}$ are always normalized as in (2.5.11). For $\widetilde{W}^{\kappa}_{(r,s)}$, we have

$$\mathcal{L}_{(r,s)}\bar{\mathcal{L}}_{(r,s)}\mathcal{A}^{rs}\bar{\mathcal{A}}^{rs}\widetilde{W}^{\kappa}_{(r,s)} = \kappa(L_0 - \Delta_{(r,-s)})^2\widetilde{W}^{\kappa}_{(r,s)} . \tag{3.5.7}$$

Furthermore, let us write $V_{(r,s)}$ for non-diagonal primary fields with the conformal dimensions $(\Delta_{(r,s)}, \Delta_{(r,-s)})$. For $r, s \in \mathbb{Z} - \{0\}$, the non-diagonal fields $V_{(r,s)}$ therefore correspond to the states $|\Delta_{(r,s)}, \Delta_{(r,-s)}\rangle$ in the figures 2.1 and 2.2. It then follows that $V_{(r,s)}$ have non-vanishing null descendants:

$$\mathcal{L}_{(r,s)}V_{(r,s)} = \bar{\mathcal{L}}_{(r,s)}V_{(r,-s)} = \eta^{D}_{(r,s)} \neq 0 . \tag{3.5.8}$$

3.5.1 Logarithmic correlation functions

Since we are expressing logarithmic fields as derivatives of fields, it should therefore be possible to obtain correlation functions of logarithmic fields in (3.5.1)-(3.5.5) by just taking derivative of correlation functions with respect to the conformal dimensions. For instance, using (3.3.10) gives us

$$\begin{aligned}\langle V^{D^{(1)}}_{\Delta_1}(z_1,\bar{z}_1)V^{D}_{\Delta_2}(z_2,\bar{z}_2)V^{D}_{\Delta_3}(z_3,\bar{z}_3)\rangle &= \frac{\partial}{\partial\Delta_1}\langle\prod_{i=1}^{3}V^{D}_{\Delta_i}(z_i,\bar{z}_i)\rangle , \\ &= \left(\frac{\partial}{\partial\Delta_1}\log C_{123} - 2\log\left|\frac{z_{12}z_{13}}{z_{23}}\right|\right)\langle\prod_{i=1}^{3}V^{D}_{\Delta_i}(z_i,\bar{z}_i)\rangle .\end{aligned} \tag{3.5.9}$$

While this trick seems to work for higher-point functions as well, computing two-point functions of logarithmic fields is not as straightforward as in (3.5.9). For instance, in the case of CFT with continuous spectrum, we cannot obtain the correct results for the two-point functions $\langle V^{D^{(1)}}_{\Delta_1}V^{D}_{\Delta_1}\rangle$ by directly taking the Δ_1-derivative of $\langle V^{D}_{\Delta_1}V^{D}_{\Delta_2}\rangle$, whose expression comes with the Dirac-delta function $\delta(\Delta_1 - \Delta_2)$.

Logarithmic Ward identities

While it is not always possible to take derivatives of correlation functions as in (3.5.9), we can always differentiate the Ward identities and solve the differentiated Ward identities, also known as *logarithmic Ward identities*. For example, we write the Ward identities for two-point functions, which involve $V_\Delta^{D^{(1)}}$ and V_Δ by taking the Δ-derivative of (2.1.5). For $X, Y \in \{V_\Delta^{D^{(1)}}, V_\Delta\}$, we write

$$\langle XY \rangle = \langle X(z_1, \bar{z}_1) Y(z_2, \bar{z}_2) \rangle \ . \tag{3.5.10}$$

Acting on (3.5.10) with global Ward identities in (2.1.5) then yields

$$\sum_{i=1}^{2} \frac{\partial}{\partial z_i} \langle XY \rangle = 0 \ , \tag{3.5.11a}$$

$$\sum_{i=1}^{2} (\Delta + \hat{\delta}^1 + z_i \frac{\partial}{\partial z_i}) \langle XY \rangle = 0 \ , \tag{3.5.11b}$$

$$\sum_{i=1}^{2} (2z_i(\Delta + \hat{\delta}^1) + z_i^2 \frac{\partial}{\partial z_i}) \langle XY \rangle = 0 \ , \tag{3.5.11c}$$

where we have defined the operator $\hat{\delta}$ as follows [40].

$$\hat{\delta}^1 V_\Delta^{D^{(1)}} = V_\Delta \quad \text{and} \quad \hat{\delta}^1 V_\Delta = 0 \ . \tag{3.5.12}$$

The system (3.5.11) is then a set of linear differential equations for the two-point functions: $\langle V_\Delta^D V_\Delta^D \rangle$, $\langle V_\Delta^{D^{(1)}} V_\Delta^D \rangle$, and $\langle V_\Delta^{D^{(1)}} V_\Delta^{D^{(1)}} \rangle$. These equations have non-trivial solutions, provided that the two-point function $\langle V_\Delta^D V_\Delta^D \rangle$ vanishes. Solving the set of equations (3.5.11) and its right-moving counterpart then led us to the well-known results [16]:

$$\begin{bmatrix} \langle V_\Delta^{D^{(1)}} V_\Delta^{D^{(1)}} \rangle & \langle V_\Delta^D V_\Delta^{D^{(1)}} \rangle \\ \langle V_\Delta^D V_\Delta^{D^{(1)}} \rangle & \langle V_\Delta^D V_\Delta^D \rangle \end{bmatrix} = \frac{1}{|z_{12}|^{4\Delta}} \begin{bmatrix} k_1 - k_0 \log |z_{12}|^2 & k_0 \\ k_0 & 0 \end{bmatrix} \ , \tag{3.5.13}$$

where k_1 is an unfixed coefficient, while the coefficient k_0 is defined up to the field renormalizations: $V_\Delta^D \to \lambda(\Delta) V_\Delta^D$, which led us to

$$V_\Delta^{D^{(1)}} \to \lambda V_\Delta^{D^{(1)}} + \lambda' V_\Delta^D \quad \text{for} \quad \alpha \in \mathbb{C} \ . \tag{3.5.14}$$

Similar situations also happen in the case of higher-point functions. For instance, the structure constants $\frac{\partial}{\partial \Delta_1} \log C_{123}$ in (3.5.9) is uniquely defined up to the transformation (3.5.14). Furthermore, observe from (3.5.13) and (3.5.9) that correlation functions involving $V_\Delta^{D^{(1)}}$ in general are not factorized into their left- and right-moving parts, which agrees with the structure of the representation (2.4.4).

3.5.2 Two-point functions of higher-order derivatives

Let us now compute two-point functions of the fields $V_\Delta^{D^{(n)}}, V_\Delta^{D^{(n-1)}}, \ldots, V^{D_\Delta}$. Therefore it is convenient to first define the matrix $\hat{K}_\Delta$ of their two-point functions:

$$\hat{K}_\Delta^{ij} = \langle V_\Delta^{D^{(n-i)}}(z_1, \bar{z}_1) V_\Delta^{D^{(n-j)}}(z_2, \bar{z}_2) \rangle \ . \tag{3.5.15}$$

Likewise to (3.5.11), one can write down explicitly the Ward identities for (3.5.15) by differentiating (3.3.6). The resulting Ward identities then have non-trivial solutions if and only if

$$(L_0^{(i)} - L_0^{(j)})\hat{K}_\Delta^{ij} = 0 \quad \text{or equivalently iff} \quad \hat{\Delta}\hat{K}_\Delta = \hat{K}_\Delta \hat{\Delta}^T \ , \tag{3.5.16}$$

where the matrix $\hat{\Delta}$ was defined in (2.4.8). Solving (3.5.16), we find that the matrix $\hat{K}_\Delta$ takes the form:

$$\hat{K}_\Delta = g(\hat{\Delta})\hat{K}_0 = \hat{K}_0 g(\hat{\Delta}^T) \ , \tag{3.5.17}$$

where g is not yet determined, and $\hat{K}_0$ is a triangle matrix with unfixed elements,

$$\hat{K}_0 = \begin{bmatrix} k_n & k_{n-1} & \cdots & & k_0 \\ k_{n-1} & \iddots & & k_0 & 0 \\ \vdots & & \iddots & & \vdots \\ & k_0 & & & \\ k_0 & 0 & \cdots & & 0 \end{bmatrix} \ . \tag{3.5.18}$$

Therefore, the matrix $\hat{K}_0$ leads to vanishing two-point functions:

$$i + j < n \implies \langle V_\Delta^{D^{(i)}} V_\Delta^{D^{(j)}} \rangle = 0 \ . \tag{3.5.19}$$

The function g in (3.5.17) can be determined by solving the full Ward identities for (3.5.15), which gives us

$$g(\hat{\Delta}) = \frac{1}{|z_{12}|^{2\hat{\Delta}}} \ . \tag{3.5.20}$$

Additionally, Ward identities also constrain the coefficients k_i to be independent of the positions. For example, in the case of K_{ij} with $n = 1$, we substitute $\hat{\Delta}$ as the second-rank Jordan block: $\begin{bmatrix} \Delta & 1 \\ 0 & \Delta \end{bmatrix}$ and recover the results in (3.5.13). For two-point functions with at most second derivatives, we have

$$\begin{aligned} &\begin{bmatrix} \langle \frac{1}{2} V_\Delta^{D^{(2)}} \frac{1}{2} V_\Delta^{D^{(2)}} \rangle & \langle \frac{1}{2} V_\Delta^{D^{(2)}} V_\Delta^{D^{(1)}} \rangle & \langle \frac{1}{2} V_\Delta^{D^{(2)}} V_\Delta \rangle \\ \langle V_\Delta^{D^{(1)}} \frac{1}{2} V_\Delta^{D^{(2)}} \rangle & \langle V_\Delta^{D^{(1)}} V_\Delta^{D^{(1)}} \rangle & \langle V_\Delta^{D^{(1)}} V_\Delta \rangle \\ \langle V_\Delta \frac{1}{2} V_\Delta^{D^{(2)}} \rangle & \langle V_\Delta V_\Delta^{D^{(1)}} \rangle & \langle V_\Delta V_\Delta \rangle \end{bmatrix} \\ &= \frac{1}{|z_{12}|^{4\Delta}} \begin{bmatrix} k_2 - k_1 \log|z_{12}|^4 + \frac{1}{2} k_0 (\log|z_{12}|^4)^2 & k_1 - k_0 \log|z_{12}|^4 & k_0 \\ k_1 - k_0 \log|z_{12}|^4 & k_0 & 0 \\ k_0 & 0 & 0 \end{bmatrix} \ . \end{aligned} \tag{3.5.21}$$

With this matrix formalism, one can also compute the two-point functions which involve $V_\Delta^{D^{(n)}}$ with arbitrary n. For example, see general results in [41].

3.5.3 Two-point functions of derivatives of null fields

Let us now discuss two-point functions of the representation $\mathcal{W}^\kappa_{(r,s)}$, generated by the logarithmic field $W^\kappa_{(r,s)}$ in (3.5.4). We only consider two-point functions of the logarithmic

field $W^{\kappa}_{(r,s)}$, its Jordan partner $\eta^{D}_{(r,s)}$, and the non-diagonal primary field $V_{(r,s)}$, while results which include $V_{(r,-s)}$ can be obtained in similar manners. We write

$$\begin{bmatrix} \langle W^{\kappa}_{(r,s)} W^{\kappa}_{(r,s)} \rangle & \langle W^{\kappa}_{(r,s)} \eta^{D}_{(r,s)} \rangle & \langle W^{\kappa}_{(r,s)} V_{(r,s)} \rangle \\ \langle \eta^{D}_{(r,s)} W^{\kappa}_{(r,s)} \rangle & \langle \eta^{D}_{(r,s)} \eta^{D}_{(r,s)} \rangle & \langle \eta^{D}_{(r,s)} V_{(r,s)} \rangle \\ \langle V_{(r,s)} W^{\kappa}_{(r,s)} \rangle & \langle V_{(r,s)} \eta^{D}_{(r,s)} \rangle & \langle V_{(r,s)} V_{(r,s)} \rangle \end{bmatrix}$$
$$= \frac{1}{|z_{12}|^{4\Delta_{(r,-s)}}} \begin{bmatrix} k_1 - \log|z_{12}|^4 & 1 & \frac{2\omega_{(r,s)}}{\rho_{(r,s)}} z_{12}^{rs} \\ 1 & 0 & 0 \\ \frac{2\omega_{(r,s)}}{\rho_{(r,s)}} z_{21}^{rs} & 0 & \frac{2}{\kappa\rho_{(r,s)}} z_{12}^{rs} z_{21}^{rs} \end{bmatrix} . \quad (3.5.22)$$

Recall the definition of $P_{(r,s)}$ in (2.3.8), the functions $\omega_{(r,s)}$ and $\rho_{(r,s)}$ are given by

$$\omega_{(r,s)} = \frac{2P_{(r+1,s+1)}}{2P_{(1,1)}} \frac{\prod_{i=0}^{r}\prod_{j=0}^{s} 2P_{(i,j)}}{2P_{(0,0)}2P_{(r,0)}2P_{(0,s)}2P_{(r,s)}} \prod_{i=1}^{r-1}\prod_{j=1}^{s-1} 2P_{(i,j)} , \quad (3.5.23a)$$

$$\rho_{(r,s)} = -\frac{\prod_{i=1-r}^{r}\prod_{j=1-s}^{s} 2P_{(i,j)}}{2P_{(0,0)}P_{(r,s)}} . \quad (3.5.23b)$$

The results of $\langle W^{\kappa}_{(r,s)} W^{\kappa}_{(r,s)} \rangle$, $\langle W^{\kappa}_{(r,s)} \eta^{D}_{(r,s)} \rangle$, $\langle \eta^{D}_{(r,s)} W^{\kappa}_{(r,s)} \rangle$, and $\langle \eta^{D}_{(r,s)} \eta^{D}_{(r,s)} \rangle$ can be immediately taken from (3.5.13). However, it is also interesting to point out here that the two-point function $\langle W^{\kappa}_{(r,s)} W^{\kappa}_{(r,s)} \rangle$ is not annihilated by the generator L_1 in the Ward identities (3.5.11) but satisfy the non-trivial constrain:

$$\langle L_1 W^{\kappa}_{(r,s)} W^{\kappa}_{(r,s)} \rangle + \langle W^{\kappa}_{(r,s)} L_1 W^{\kappa}_{(r,s)} \rangle = 0 . \quad (3.5.24)$$

Furthermore, the two-point functions $\langle \eta^{D}_{(r,s)} V_{(r,s)} \rangle$ and $\langle V_{(r,s)} \eta^{D}_{(r,s)} \rangle$ contain two primary fields with different dimensions, therefore they vanish. We now discuss the other cases, which come with the functions $\rho_{(r,s)}$ and $\omega_{(r,s)}$ in details.

The case of $\langle V_{(r,s)} V_{(r,s)} \rangle$

The two-point function $\langle V_{(r,s)} V_{(r,s)} \rangle$ is a two-point function of two non-diagonal primary fields, but its two-point structure constant is controlled by the equation (3.5.6). Let us now restrict ourselves to the case of generic real central charge for a moment. Therefore we can translate two-point structure constants into the norms of states [4]. For any creation operator $\mathcal{A}^{rs}$ with the normalization (2.5.11), we can write

$$\begin{aligned} &\lim_{z_2\to\infty}\lim_{z_1\to 0}\frac{1}{\kappa}\langle \eta^{D}_{(r,s)}(z_1,\bar{z}_1) W^{\kappa}_{(r,s)}(z_2,\bar{z}_2)\rangle \\ &\quad = \lim_{z_2\to\infty}\lim_{z_1\to 0}\langle \mathcal{L}_{(r,s)}\mathcal{A}^{rs} W^{\kappa}_{(r,s)}(z_1,\bar{z}_1) W^{\kappa}_{(r,s)}(z_2,\bar{z}_2)\rangle , \\ &\quad = \langle \mathcal{A}^{rs} W^{\kappa}_{(r,s)} | \mathcal{L}^{\dagger}_{(r,s)} W^{\kappa}_{(r,s)}\rangle , \\ &\quad = \langle \Delta_{(r,s)}, \Delta_{(r,-s)} | \mathcal{L}^{\dagger}_{(r,s)} \mathcal{A}^{rs\dagger} \mathcal{L}^{\dagger}_{(r,s)} \mathcal{L}_{(r,s)} | \Delta_{(r,s)}, \Delta_{(r,-s)}\rangle , \\ &\quad = P'_{\mathcal{L}^{\dagger}_{(r,s)},\mathcal{L}_{(r,s)}}(\Delta_{(r,s)}) \langle \Delta_{(r,s)}, \Delta_{(r,-s)} | \Delta_{(r,s)}, \Delta_{(r,-s)}\rangle , \\ &\quad = \lim_{z_2\to\infty}\lim_{z_1\to 0}(-1)^{rs} P'_{\mathcal{L}^{\dagger}_{(r,s)},\mathcal{L}_{(r,s)}}(\Delta_{(r,s)}) \langle V_{(r,s)}(z_1,\bar{z}_1) V_{(r,s)}(z_2,\bar{z}_2)\rangle , \end{aligned} \quad (3.5.25)$$

where the function $P'_{\mathcal{L}^{\dagger}_{(r,s)},\mathcal{L}_{(r,s)}}(\Delta_{(r,s)})$ is defined by the equation (2.5.6). Moreover, we have the factor $(-1)^{rs}$ because of the non-diagonality of the field $V_{(r,s)}$. We then arrive at

the desired result for $\langle \eta^D_{(r,s)} W^\kappa_{(r,s)} \rangle$ shown in (3.5.22),

$$\boxed{\frac{\langle V_{(r,s)} V_{(r,s)} \rangle}{\langle \eta^D_{(r,s)} W^\kappa_{(r,s)} \rangle} = \frac{(-1)^{rs}}{\kappa} z^{2rs} \rho_{(r,s)}} \ , \tag{3.5.26}$$

where we have defined

$$\rho_{(r,s)} = P'_{\mathcal{L}^\dagger_{(r,s)}, \mathcal{L}_{(r,s)}}(\Delta_{(r,s)}) \ . \tag{3.5.27}$$

The function $P'_{\mathcal{L}^\dagger_{(r,s)}, \mathcal{L}_{(r,s)}}(\Delta_{(r,s)})$ was first computed by Zamolodchikov in [21]. Indeed, we have computed the ratio (3.5.26) by assuming the central charge is real. However, the ratio (3.5.26) has also been shown to hold for generic complex central charge in [29] by solving the Ward identities for two-point functions on the left-handside of (3.5.26).

The case of $\langle V_{(r,s)} W^\kappa_{(r,s)} \rangle$

To start, it is useful to recall the follwing relations from (2.5.10),

$$\mathcal{A}^{rs} \mathcal{W}^\kappa_{(r,s)} = \kappa V_{(r,s)} \ , \tag{3.5.28a}$$
$$\bar{\mathcal{A}}^{rs} \mathcal{W}^\kappa_{(r,s)} = \kappa V_{(r,-s)} \ . \tag{3.5.28b}$$

Let us now consider the two-point functions $\langle V_{(r,s)} L_1^n W^\kappa_{(r,s)} \rangle$, which do not vanish for $n \leq rs$. The L_{-1}- and L_0-Ward identities in (3.3.6) constrain $\langle V_{(r,s)} L_1^n W^\kappa_{(r,s)} \rangle$ to take the form:

$$\langle V_{(r,s)} L_1^n W^\kappa_{(r,s)} \rangle = \frac{A^{L_1^n}}{z_{12}^{rs+n} |z_{12}|^{4\Delta_{(r,s)}}} \ , \tag{3.5.29}$$

where the coefficients $A^{L_1^n}$ depend on n and will be determined below. Now consider the L_1-Ward identity for (3.5.29). We have

$$\left(2(z_1 + z_2)\Delta_{(r,s)} + 2z_2 rs + z_1^2 \frac{\partial}{\partial z_1} + z_2^2 \frac{\partial}{\partial z_2} \right) \langle V_{(r,s)} L_1^n W^\kappa_{(r,s)} \rangle = -\langle V_{(r,s)} L_1^{n-1} W^\kappa_{(r,s)} \rangle \tag{3.5.30}$$

Substituting (3.5.29) into (3.5.30) results in the recursion for the coefficients $A^{L_1^n}$:

$$A^{L_1^{n+1}} = (rs - n) A^{L_1^n} \quad \text{with} \quad A^{L_1^{rs}} = \frac{P'_{L_1^{rs}, \mathcal{L}_{(r,s)}}(\Delta_{(r,s)})}{\rho(r,s)} \quad \text{and} \quad 0 \leq n \leq rs \ . \tag{3.5.31}$$

By using the relation (3.5.28), we find that the case of $n = rs$ in (3.5.29) is simply the two-point function $\langle V_{(r,s)} V_{(r,s)} \rangle$ with the extra renormalization factor $A^{L_1^{rs}}$. Solving the recursion (3.5.31) yields

$$\boxed{\frac{\langle V_{(r,s)} W^\kappa_{(r,s)} \rangle}{\langle V_{(r,s)} V_{(r,s)} \rangle} = z_{12}^{-rs} \frac{2\omega(r,s)}{\rho(r,s)}} \ , \tag{3.5.32}$$

where we have defined

$$\omega_{(r,s)} = \frac{P'_{L_1^{rs}, \mathcal{L}_{(r,s)}}}{(rs)!} \ . \tag{3.5.33}$$

Unlike the function $\rho_{(r,s)}$, there seems to be no derivations of $\omega_{(r,s)}$ in any literature. With the normalization $\mathcal{L}_{(r,s)} = L_{-1}^{rs} + \ldots$, let us then compute explicitly $\omega_{(r,s)}$ for $rs \leq 6$:

$$\begin{aligned}
&\omega_{(1,1)} = 2 \ , \\
&\omega_{(2,1)} = 2P_{(3,2)} \cdot 2P_{(1,0)} \ , \\
&\omega_{(3,1)} = 2P_{(4,2)} \cdot 2P_{(2,1)} \cdot 2P_{(2,0)} \cdot 2P_{(1,0)} \ , \\
&\omega_{(4,1)} = 2P_{(5,2)} \cdot 2P_{(3,1)} \cdot 2P_{(2,1)} \cdot 2P_{(3,0)} \cdot 2P_{(2,0)} \cdot 2P_{(1,0)} \ , \\
&\omega_{(2,2)} = 2P_{(3,3)} \cdot 2P_{(2,1)} \cdot 2P_{(1,2)} \cdot 2P_{(1,1)} \cdot 2P_{(0,1)} \cdot 2P_{(1,0)} \ , \\
&\omega_{(5,1)} = 2P_{(6,2)} \cdot 2P_{(4,1)} \cdot 2P_{(3,1)} \cdot 2P_{(2,1)} \cdot 2P_{(4,0)} \cdot 2P_{(3,0)} \cdot 2P_{(2,0)} \cdot 2P_{(1,0)} \ , \\
&\omega_{(2,3)} = 2P_{(3,4)} \cdot 2P_{(1,3)} \cdot 2P_{(2,2)} \cdot 2P_{(2,1)} \cdot 2P_{(0,2)} \cdot 2P_{(1,0)} \cdot 2P_{(0,1)} \cdot (2P_{(1,2)})^2 \cdot 2P_{(1,1)} \ ,
\end{aligned} \tag{3.5.34}$$

where we have taken into account the relation $\omega_{(r,s)} = \omega_{(s,r)}(\beta \to \beta^{-1})$. The above results led us to propose the analytic formula of $\omega_{(r,s)}$ in (3.5.23a). We have also further checked that (3.5.23a) always agree with explicit calculations of (3.5.33) for $rs \leq 20$.

3.6 Logarithmic conformal blocks

It is well-known in logarithmic CFTs that computing conformal blocks of logarithmic representations can be very tedious since the representations themselves can be quite complicated, as we have seen in (2.2). However, using derivatives of fields allows us to simply write conformal blocks of logarithmic representations as derivatives of the non-logarithmic blocks. For convenience, we now neglect the positional dependence of conformal blocks and our analysis here holds for conformal blocks in all three channels.

Derivatives of primary fields

We start with introducing chiral primary fields v_Δ with the conformal dimensions Δ. Now recall that conformal blocks are linear functions of primary fields. Since derivatives are also linear operators, we can therefore write conformal blocks of $\frac{\partial}{\partial\Delta}v_\Delta$ as follows:

$$\mathcal{F}\left(\frac{\partial}{\partial\Delta}v_\Delta\right) = \frac{\partial}{\partial\Delta}\mathcal{F}(v_\Delta) = \mathcal{F}'_\Delta \ , \tag{3.6.1}$$

where the prime denotes the Δ-derivative. Taking into account the right-moving fields, general results for conformal blocks of $V_\Delta^{D^{(n)}}$ are then given by

$$\frac{\partial^n}{\partial\Delta^n}(\mathcal{F}_\Delta\bar{\mathcal{F}}_\Delta) \ . \tag{3.6.2}$$

However, the above is not the full conformal block of representations generated by $V_\Delta^{D^{(n)}}$. For instance, recall that the structure of representation generated by $V_\Delta^{D^{(1)}}$ is invariant under changing the normalization: $V_\Delta^D \to \lambda(\Delta)V_\Delta^D$, which is equivalent to

$$V_\Delta^{D'} \to \lambda(\Delta)V_\Delta^{D'} + \lambda'(\Delta)V_\Delta^D \ . \tag{3.6.3}$$

Taking into account how the conformal block $(\mathcal{F}_\Delta\bar{\mathcal{F}}_\Delta)'$ changes under the above transformation, the full conformal block of representation generated by $V_\Delta^{D^{(1)}}$ is given by

$$(\mathcal{F}_\Delta\bar{\mathcal{F}}_\Delta)' + c_\alpha\mathcal{F}_\Delta\bar{\mathcal{F}}_\Delta \ , \tag{3.6.4}$$

where c_α are unfixed structure constants.

Derivatives of null fields

For the derivatives of null fields in (3.5.3a)-(3.5.3b), conformal blocks of the fields $\eta^{D^{(n)}}_{(r,s)}$ can be obtained by simply setting $\Delta = \Delta_{(r,-s)}$ in (3.6.2). For the case of $\mu^{D^{(n)}}_{(r,s)}$, we will argue heuristically how to arrive at their conformal blocks. We expand the chiral descendant field $\mathcal{L}_{(r,s)}v_\Delta$ as the following,

$$\frac{1}{\Delta-\Delta_{(r,s)}}\mathcal{L}_{(r,s)}v_\Delta \overset{\Delta\to\Delta_{(r,s)}}{\sim} \frac{1}{\Delta-\Delta_{(r,s)}}\mathcal{L}_{(r,s)}v_{\Delta_{r,s}} + \sum_{n=1}^{\infty}\frac{1}{n!}(\Delta-\Delta_{(r,s)})^{n-1}\mathcal{L}_{(r,s)}\frac{\partial^n}{\partial\Delta^n}v_\Delta\Big|_{\Delta=\Delta_{(r,s)}} . \quad (3.6.5)$$

While computing conformal blocks of $\mathcal{L}_{(r,s)}v_\Delta$ explicitly looks complicated, we know that the resulting blocks must be approaching $\mathcal{F}_{\Delta_{(r,-s)}}$ as Δ approaching the degenerate conformal dimensions $\Delta_{(r,s)}$ in (2.3.8). Now recall from the Zamolodchikov recursion (3.4.15) that the conformal block $\mathcal{F}_\Delta$ has a pole at the degenerate dimension $\Delta = \Delta_{(r,s)}$. The expansion (3.6.5) is therefore comparable to expanding $\mathcal{F}_\Delta$ as follows:

$$\frac{\mathcal{F}_\Delta}{R_{r,s}} \overset{\Delta\to\Delta_{(r,s)}}{\sim} \frac{\mathcal{F}_{\Delta_{(r,-s)}}}{\Delta-\Delta_{(r,s)}} + \sum_{n=1}^{\infty}(\Delta-\Delta_{(r,s)})^{n-1}\mathcal{F}^{\text{reg-n}}_{\Delta_{(r,s)}} . \quad (3.6.6)$$

As a consistency check, notice that the leading conformal block $\mathcal{F}_{\Delta_{(r,-s)}}$ in (3.6.6) is consistent with the leading field in (3.6.5). Moreover, the conformal blocks $\mathcal{F}^{\text{reg-n}}_{\Delta_{(r,s)}}$ are defined by

$$\mathcal{F}^{\text{reg-n}}_{\Delta_{(r,s)}} = \oint_{\mathcal{C}_{\Delta_{(r,s)}}}\frac{d\Delta}{2\pi i}(\Delta-\Delta_{(r,s)})^{-n}\frac{\mathcal{F}_\Delta}{R_{r,s}} , \quad (3.6.7)$$

where $\mathcal{C}_{\Delta_{(r,s)}}$ is a curve which lives on the Δ-plane and encloses the point $\Delta = \Delta_{(r,s)}$. The conformal blocks $\mathcal{F}^{\text{reg-n}}_{\Delta_{(r,s)}}$ are in general logarithmic For instance, in the case of $\mathcal{F}^{\text{reg-1}}_{\Delta_{(r,s)}}$, we have

$$\begin{aligned}\mathcal{F}^{\text{reg-1}}_{\Delta_{(r,s)}} &= \oint_{\mathcal{C}_{\Delta_{(r,s)}}}\frac{d\Delta}{2\pi i}(\Delta-\Delta_{(r,s)})^{-1}\frac{\mathcal{F}_\Delta}{R_{r,s}} ,\\ &\sim \oint_{\mathcal{C}_{\Delta_{(r,s)}}}\frac{d\Delta}{2\pi i}\left\{\frac{q^{\Delta+rs}}{(\Delta-\Delta_{(r,s)})^2}+\dots\right\} ,\\ &\sim \log(16q)+\dots . \end{aligned} \quad (3.6.8)$$

where $q = q(z)$ is the elliptic norm, previously introduced in (3.4.12). Using (3.4.13), the conformal block $\mathcal{F}^{\text{reg-1}}_{\Delta_{(r,s)}}$ is then linear in $\log(z)$. In general, $\mathcal{F}^{\text{reg-n}}_{\Delta_{(r,s)}}$ comes with the terms $\log^n(z)$. Hence, the word "reg" in (3.6.6) actually means that $\mathcal{F}^{\text{reg-n}}_{\Delta_{(r,s)}}$ are logarithmic regularizations of $\mathcal{F}_\Delta$ at $\Delta = \Delta_{(r,s)}$. Let us now write down the conformal blocks of $\mu^{D^{(n)}}_{(r,s)}$. Taking into account the right-moving counterpart of (3.6.6), we have

$$\mathcal{F}(\mu^{D^{(n)}}_{(r,s)}) = \oint_{\mathcal{C}_{\Delta_{(r,s)}}}\frac{d\Delta}{2\pi i}(\Delta-\Delta_{(r,s)})^{-n}\frac{\mathcal{F}_\Delta}{R_{(r,s)}}\frac{\bar{\mathcal{F}}_{\bar\Delta}}{\bar R_{r,s}} . \quad (3.6.9)$$

Using (3.6.9) and (3.6.2) with (3.5.4), we can write down the conformal blocks of the logarithmic fields $W^\kappa_{(r,s)}$ explicitly as follows:

$$\mathcal{G}^\kappa_{(r,s)} = (1-\kappa)\left(\mathcal{F}_{\Delta_{(r,-s)}}\bar{\mathcal{F}}_{\Delta_{(r,-s)}}\right)' + \frac{\kappa}{R_{r,s}}\left[\mathcal{F}_{\Delta_{(r,-s)}}\bar{\mathcal{F}}^{\text{reg-1}}_{\Delta_{(r,s)}} + \mathcal{F}^{\text{reg-1}}_{\Delta_{(r,s)}}\bar{\mathcal{F}}_{\Delta_{(r,-s)}}\right] . \quad (3.6.10)$$

In case of the logarithmic fields $\widetilde{W}^{\kappa}_{(r,s)}$ in (3.5.5), we have

$$\widetilde{\mathcal{G}}^{\kappa}_{(r,s)} = \frac{\kappa}{2}\left\{ \frac{\mathcal{F}^{\text{reg-2}}_{\Delta_{(r,s)}}\bar{\mathcal{F}}^{\text{reg-2}}_{\Delta_{(r,s)}}}{R^2_{r,s}} + \frac{1}{R_{r,s}}\left(\mathcal{F}^{\text{reg}-1}_{\Delta_{(r,s)}}\bar{\mathcal{F}}_{\Delta_{(r,-s)}} + \bar{\mathcal{F}}^{\text{reg}-1}_{\Delta_{(r,s)}}\mathcal{F}_{\Delta_{(r,-s)}}\right)\right\} + \frac{(1-\kappa)}{2}\left(\mathcal{F}_{\Delta_{(r,s)}}\bar{\mathcal{F}}_{\Delta_{(r,-s)}}\right)'' , \quad (3.6.11)$$

Nevertheless, recall that the representation $\mathcal{W}^{\kappa}_{(r,s)}$ is defined up to the gauge transformation (2.5.14). Consequently, the full conformal blocks of the representation $\mathcal{W}^{\kappa}_{(r,s)}$ is given by

$$\boxed{\mathcal{G}^{\kappa}_{(r,s)} + c_{\beta}\left(\mathcal{F}_{\Delta_{(r,-s)}}\bar{\mathcal{F}}_{\Delta_{(r,-s)}}\right)} , \quad (3.6.12)$$

where c_{β} is an unfixed structure constants. Likewise, using (2.5.21), conformal blocks of the representation $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$ read

$$\boxed{\widetilde{\mathcal{G}}^{\kappa}_{(r,s)} + c_{\delta}\left(\mathcal{F}_{\Delta_{(r,-s)}}\bar{\mathcal{F}}_{\Delta_{(r,-s)}}\right) + c_{\sigma}\mathcal{G}^{\kappa'}_{(r,s)}} , \quad (3.6.13)$$

where c_{δ} and c_{σ} are unfixed structure constants, and κ' is an unfixed coupling. We will see in the next Chapter that these unfixed parameters can be completely fixed by imposing the constraints from the degenerate fields.

Therefore, we have demonstrated that using derivatives of fields is convenient for computing conformal blocks. Moreover, we were also able to compute all two-point functions of the representations $\mathcal{W}^{\kappa}_{(r,s)}$, shown in (3.5.22). While our results in this section were obtained by some sort of heuristic arguments such as comparing (3.6.5) to (3.6.6), some of our results have also been checked to agree with the pedestrian computation. For instance, the authors of [27] used the pedestrian computation to calculate the conformal blocks of $\mathcal{W}^{\kappa}_{(r,s)}$ for the case $rs = 2$, and their results agree with our exact results in (3.6.12). These closed formulae of logarithmic blocks are crucial ingredients for bootstrapping four-point functions of the Potts and $O(n)$ CFTs at arbitrary precision in Chapter 5.

CHAPTER 4

Degenerate fields

In this Chapter, we discuss consequences of the existence of the degenerate fields in CFTs at generic central charge. We introduce *the degenerate field* $V^D_{\langle r,s\rangle}$, associated to the degenerate states $|\langle r,s\rangle\rangle^D$ in (2.3.17), as a diagonal primary field which has the conformal dimension $\Delta_{(r,s)}$ and also comes with a vanishing null vector:

$$\mathcal{L}_{(r,s)}V^D_{\langle r,s\rangle} = \bar{\mathcal{L}}_{(r,s)}V^D_{\langle r,s\rangle} = 0 \ , \tag{4.0.1}$$

where $\mathcal{L}_{(r,s)}$ is a combination of negative Virasoro modes, defined by the equation (2.3.9). That is to say $V^D_{\langle r,s\rangle}$ transforms as the highest-weight state in the degenerate representation $\mathcal{R}^D_{(r,s)}$. Notice that the degenerate field $V^D_{\langle 1,1\rangle}$ has vanishing conformal dimensions and satisfies $L_{-1}V^D_{\langle 1,1\rangle} = \bar{L}_{-1}V^D_{\langle 1,1\rangle} = 0$. Therefore, $V^D_{\langle 1,1\rangle}$ coincides with the identity field,

$$V^D_{\langle 1,1\rangle} = \mathbb{1} \ . \tag{4.0.2}$$

At generic central charge, the degenerate field $V^D_{\langle r,s\rangle}$ only has one vanishing null descendant at level rs, unlike the case of rational central charge where each degenerate field can have infinitely many null descendants. The latter case will be discussed in Chapter 6.

4.1 The degenerate fusion rules

The vanishing null descendants $\mathcal{L}_{(r,s)}V^D_{\langle r,s\rangle}$ lead to highly-nontrivial constraints that completely determine the fusion rules of the degenerate fields $V^D_{\langle r,s\rangle}$. For any field O_1 and O_2, using (4.0.1), we can write the following vanishing three-point functions,

$$\langle \mathcal{L}_{(r,s)}V^D_{\langle r,s\rangle}O_1O_2\rangle = 0 \ . \tag{4.1.1}$$

Now consider the case of $rs = 1$ in (4.1.1). Using (3.2.1), we write

$$\begin{aligned}\langle L_{-1}V^D_{\langle 1,1\rangle}O_1O_2\rangle &= \frac{\partial}{\partial z_1}\langle V^D_{\langle 1,1\rangle}O_1O_2\rangle \ , \\ &\sim (\Delta_1 - \Delta_2)\langle V^D_{\langle 1,1\rangle}O_1O_2\rangle \ .\end{aligned} \tag{4.1.2}$$

Taking into account the right-moving part of (4.1.2), we then conclude that $\langle V^D_{\langle 1,1\rangle}O_1O_2\rangle$ do not vanish if $O_1 = O_2$. In terms of the fusion rules, we have

$$V^D_{\langle 1,1\rangle} \times O_1 = O_1 \ . \tag{4.1.3}$$

For the null fields at level 2, we only consider three-point functions of diagonal fields in details. We now parametrize primary fields with their momenta in (2.2.2), instead of their conformal dimensions. Let $V^D_{P_1}$ and $V^D_{P_2}$ be diagonal primary fields with the momenta P_1 and P_2, we consider the three-point function $\langle \mathcal{L}_{(1,2)} V^D_{\langle 1,2\rangle} V^D_{P_1} V^D_{P_2}\rangle$. Using (2.3.12) and (3.3.5) with this three-point function yields

$$\begin{aligned}\langle \mathcal{L}_{(1,2)} V^D_{\langle 1,2\rangle} V^D_{P_1} V^D_{P_2}\rangle &= \left\{ \frac{\partial^2}{\partial z_1^2} - \beta^{-2} \sum_{i=2,3} \left(\frac{1}{z_{1i}} \frac{\partial}{\partial z_1} - \frac{1}{z_{1i}^2} \Delta_i \right) \right\} \langle V^D_{\langle 1,2\rangle} V^D_{P_1} V^D_{P_2}\rangle \ , \\ &\sim \prod_{\pm} \left(P_1 - P_2 \pm \frac{\beta^{-1}}{2} \right) \langle V^D_{\langle 1,2\rangle} V^D_{P_1} V^D_{P_2}\rangle \ . \end{aligned} \tag{4.1.4}$$

Using (4.1.1) with the above, we have

$$\langle V^D_{\langle 1,2\rangle} V^D_{P_1} V^D_{P_2}\rangle \neq 0 \Longrightarrow P_1 - P_2 = \pm \frac{1}{2\beta} \ . \tag{4.1.5}$$

We can then translate (4.1.5) into the fusion rule of $V^D_{\langle 1,2\rangle}$ with any diagonal primary field V^D_P as follows:

$$V^D_{\langle 1,2\rangle} \times V^D_P = V^D_{P+\frac{1}{2\beta}} + V^D_{P-\frac{1}{2\beta}} \ . \tag{4.1.6}$$

To obtain the fusion rules of $V^D_{\langle 2,1\rangle}$, we simply apply the transformation $\beta \to \beta^{-1}$ to the fusion rule (4.1.6). We have

$$V^D_{\langle 2,1\rangle} \times V^D_P = V^D_{P+\frac{\beta}{2}} + V^D_{P-\frac{\beta}{2}} \ . \tag{4.1.7}$$

Although there are infinitely many degenerate fields $V^D_{\langle r,s\rangle}$ since r and s can be any positive integer, knowing the fusion rules of $V^D_{\langle r,s\rangle}$ up to $rs = 2$ already allows us to compute all degenerate fusion rules by using associativity [5]. For example, we write $V^D_{\langle 1,3\rangle} \times V^D_P$ as follows:

$$\begin{aligned} V^D_{\langle 1,3\rangle} \times V^D_P &= \left(V^D_{\langle 1,2\rangle} \times V^D_{\langle 1,2\rangle} - V^D_{\langle 1,1\rangle} \right) \times V^D_P \ , \\ &= V^D_{P+\beta^{-1}} + V^D_{P-\beta^{-1}} + V^D_P \ . \end{aligned} \tag{4.1.8}$$

Therefore, we can repeatedly use the fusion rules (4.1.3), (4.1.6), and (4.1.7) to arrive at the following general result:

$$\boxed{V^D_{\langle r,s\rangle} \times V^D_P = \sum_{i \overset{2}{=} 1-r}^{r-1} \sum_{j \overset{2}{=} 1-s}^{s-1} V^D_{P+P_{(i,j)}}} \ . \tag{4.1.9}$$

In other words, the product between the degenerate representation $\mathcal{R}^D_{(r,s)}$ and the Verma modules $\mathcal{V}_\Delta \otimes \bar{\mathcal{V}}_\Delta$ with generic Δ can always be decomposed into a direct sum of finitely many Verma modules. We will see in Section 4.5 that the fusion rules (4.1.9) are no longer valid for some special values of Δ due to the poles in the OPE coefficients (3.3.28). For the fusion rules of the degenerate fields with themselves, the fusion rules (4.1.9) become

$$\boxed{V^D_{\langle r,s\rangle} \times V^D_{\langle r',s'\rangle} = \sum_{i \overset{2}{=} |r'-r|+1}^{r+r'-1} \sum_{j \overset{2}{=} |s'-s|+1}^{s+s'-1} V^D_{\langle i,j\rangle}} \ , \tag{4.1.10}$$

where the constraint (4.0.1) requires any field with non-positive indices to vanish in (4.1.10). Therefore, the degenerate representations are always closed under fusions with themselves. Let us also display the product between $V^D_{\langle 1,2\rangle}$ and non-diagonal primary fields,

$$V^D_{\langle 1,2\rangle} \times V_{P,\bar{P}} = V_{P+\frac{1}{2\beta},\bar{P}-\frac{1}{2\beta}} + V_{P-\frac{1}{2\beta},\bar{P}+\frac{1}{2\beta}} \ , \tag{4.1.11}$$

where $V_{P,\bar{P}}$ denotes non-diagonal primary fields denote the left- and right-momenta P and $\bar{P}$. In the case of $V^D_{\langle 2,1\rangle}$, we have

$$V^D_{\langle 2,1\rangle} \times V_{P,\bar{P}} = V_{P+\frac{\beta}{2},\bar{P}+\frac{\beta}{2}} + V_{P-\frac{\beta}{2},\bar{P}-\frac{\beta}{2}} \ . \tag{4.1.12}$$

Similarly to (4.1.9), using associativity of the OPE with the fusion rules (4.1.11), (4.1.12) and (4.1.3), one can also arrive at the general result for $V_{\langle r,s\rangle} \times V_{P,\bar{P}}$. Let us also translate these two non-diagonal fusions into the Kac indices. Recall that the non-diagonal primary fields $V_{(r,s)}$ come with the conformal dimensions $(\Delta_{(r,s)}, \Delta_{(r,-s)})$ for $rs \in \mathbb{Z}$. Using (4.1.11) and (4.1.12), we can then write

$$V^D_{\langle 1,2\rangle} \times V_{(r,s)} = V_{(r,s+1)} + V_{(r,s-1)} \ , \tag{4.1.13a}$$

$$V^D_{\langle 2,1\rangle} \times V_{(r,s)} = V_{(r+1,s)} + V_{(r-1,s)} \ . \tag{4.1.13b}$$

Notice from the definition of $P_{(r,s)}$ in (2.3.8) that opposite signs in each subscript of fields in (4.1.11) lead to shifts in the second Kac indices of (4.1.13a).

Furthermore, the degenerate fusion rules also allow us to distinguish non-diagonal fields and diagonal fields more accurately. For example, consider the non-diagonal fields $V_{P,0}$ and the diagonal fields V^D_P, both of which have the same conformal dimension and vanishing conformal spins. However, these two fields behave differently under the degenerate fusion rules. For instance, the product $V^D_{\langle 1,2\rangle} \times V_{P,0}$ yield a sum of two non-diagonal primary fields with respect to (4.1.12) but $V^D_{\langle 1,2\rangle} \times V^D_P$ is decomposed into a sum of two diagonal primary fields as in (4.1.7).

4.2 The BPZ equations

Since we can always write the action of $\mathcal{L}_{(r,s)}$ on correlation functions as linear differential operators by using the local Ward identities (3.3.4) and (3.3.5), inserting the vanishing null vector $\mathcal{L}_{(r,s)}V^D_{(r,s)}$ into correlation functions of any field O_i then leads to linear differential equations, also known as the *the Belavin–Polyakov–Zamolodchikov (BPZ) equations* of order rs:

$$\langle \mathcal{L}_{(r,s)} V^D_{\langle r,s\rangle} \prod_{i=1}^{n-1} O_i \rangle = 0 \ . \tag{4.2.1}$$

As an example, we now discuss the second order BPZ equation for the four-point function of diagonal primary fields: $\langle V^D_{\langle 1,2\rangle} \prod_{i=1}^3 V^D_{P_i}\rangle$. As solutions to the Ward identities, this four-point function can be factorized into the left- and right- moving parts as follows:

$$\langle V^D_{\langle 1,2\rangle}(z,\bar{z}) V^D_{P_1}(0) V^D_{P_2}(\infty) V^D_{P_3}(1)\rangle = \sum_{i\in s,t,u} \mathcal{V}^{(i)}(z)\bar{\mathcal{V}}^{(i)}(\bar{z}) \ . \tag{4.2.2}$$

With (3.3.5), acting on the degenerate field $V^D_{\langle 1,2\rangle}$ in the above four-point function with the operator $\mathcal{L}_{(1,2)}$ gives us

$$\left\{ -\beta^2 z(1-z)\frac{\partial^2}{\partial z^2} + (2z-1)\frac{\partial}{\partial z} + \Delta_{(1,2)} + \frac{\Delta_1}{z} - \Delta_2 + \frac{\Delta_3}{1-z} \right\} \mathcal{V}^{(i)}(z) = 0 \ . \tag{4.2.3}$$

The natural ansatz for solving the equation (4.2.3) is to write $\mathcal{V}^{(i)}(z)$ as an expansion in z around the singular points z_0. We now focus on the s-channel solution: $\mathcal{V}^{(s)}(z)$. For convenience, let us drop the channel indices whenever possible. We write

$$\mathcal{V}(z) = (z-z_0)^{\lambda(z_0)} \left(1 + \sum_{n=1}^{\infty} (z-z_0)^n\right) \ , \tag{4.2.4}$$

where $\lambda(z_0)$ is called the characteristic exponent of $\mathcal{V}(z)$, which depends on the choice of z_0. Recall our previous discussion on conformal blocks, $\mathcal{V}(z)$ then have three singular points at $z_0 \in \{0, 1, \infty\}$, as four-point functions of CFTs on the Riemann sphere. Using (4.2.3) with (4.2.4) at $z_0 = 0$, we find

$$\lambda_{\pm}(0) = \pm\beta P_1 + \frac{1}{2}\left(\beta^2 - 1\right) \ . \tag{4.2.5}$$

The expansion of $\mathcal{V}(z)$ around $z = 0$ is then comparable with the s-channel expansion in (3.4.11). Using the characteristic exponents (4.2.5), we can therefore compute the conformal dimensions of chiral primary fields associated to $\mathcal{V}(z)$ in (4.2.4),

$$\lambda_{\pm}(0) = \Delta_{(1,2)} + \Delta_1 - \Delta \Longrightarrow P_{\mp} = P_1 \pm \frac{1}{2\beta} \ , \tag{4.2.6}$$

Similarly, considering the cases $z_0 \in \{1, \infty\}$ results in the characteristic exponents for the t- and u-channels of the four-point functions $\langle V^D_{\langle 1,2\rangle} \prod_{i=1}^3 V^D_{P_i}\rangle$, respectively.

Hypergeometric conformal blocks

Let us now rewrite the equation (4.2.3) as differential equations for the hypergeometric functions ${}_2F_1(a,b;c;z)$. Recall that ${}_2F_1(a,b;c;z)$ satisfy the following equation,

$$\left\{ z(1-z)\frac{\partial^2}{\partial z^2} + [c - (a+b+1)z]\frac{\partial}{\partial z} - ab \right\} {}_2F_1(a,b;c;z) = 0 \ . \tag{4.2.7}$$

Comparing (4.2.7) to (4.2.3), we then choose the following values of a, b, and c:

$$a = \frac{1}{2} + \beta^{-1}(P_1 + P_2 + P_3) \ , \tag{4.2.8a}$$

$$b = \frac{1}{2} + \beta^{-1}(P_1 - P_2 + P_3) \ , \tag{4.2.8b}$$

$$c = 1 + 2\beta^{-1}P_1 \ . \tag{4.2.8c}$$

Therefore, we can write down explicitly the two linearly-independent s-channel solutions of (4.2.3) through the hypergeometric functions ${}_2F_1$ as the following:

$$\mathcal{V}^{(s)}_{P_-}(z) = z^{-\frac{P_1}{\beta} - \frac{1}{2}\left(1 - \frac{1}{\beta^2}\right)} (1-z)^{\frac{P_3}{\beta} - \frac{1}{2}\left(1 - \frac{1}{\beta^2}\right)} {}_2F_1(a,b;c;z) \ , \tag{4.2.9a}$$

$$\mathcal{V}^{(s)}_{P_+}(z) = z^{\frac{P_1}{\beta} - \frac{1}{2}\left(1 - \frac{1}{\beta^2}\right)} (1-z)^{\frac{P_3}{\beta} - \frac{1}{2}\left(1 - \frac{1}{\beta^2}\right)} {}_2F_1(c-a,c-b;2-c;z) \ , \tag{4.2.9b}$$

where the solutions $\mathcal{V}^{(s)}_{P_\pm}$ correspond to the characteristic exponents $\lambda_{\mp}(0)$ in (4.2.6) and are also known as *hypergeometric conformal blocks*. Permuting $1 \leftrightarrow 3$ simultaneously with $z \leftrightarrow 1 - z$ in the s-channel solutions (4.2.9) then brings us to the t-channel solutions:

$$\mathcal{V}^{(t)}_{P_+}(z) = z^{\frac{P_1}{\beta} - \frac{1}{2}\left(1 - \frac{1}{\beta^2}\right)} (1-z)^{\frac{P_3}{\beta} - \frac{1}{2}\left(1 - \frac{1}{\beta^2}\right)} {}_2F_1(a,b;a+b-c+1;1-z) \ , \tag{4.2.10a}$$

$$\mathcal{V}^{(t)}_{P_-}(z) = z^{\frac{P_1}{\beta} - \frac{1}{2}\left(1 - \frac{1}{\beta^2}\right)} (1-z)^{-\frac{P_3}{\beta} - \frac{1}{2}\left(1 - \frac{1}{\beta^2}\right)} {}_2F_1(c-a,c-b;c-a-b+1;1-z) \ . \tag{4.2.10b}$$

Indeed, one can also arrive at the u-channel from the s-channel solutions by $1 \leftrightarrow 2$ simultaneously with $z \leftrightarrow \frac{1}{z}$. Since the s- and t- channels are just different bases of the same linear space of solutions to (4.2.3), the two sets of solutions are then related by linear transformation, known as the degenerate fusing matrix:

$$\begin{pmatrix} \mathcal{V}^{(s)}_{P_-}(z) \\ \mathcal{V}^{(s)}_{P_+}(z) \end{pmatrix} = \begin{pmatrix} F_{--} & F_{-+} \\ F_{+-} & F_{++} \end{pmatrix} \begin{pmatrix} \mathcal{V}^{(t)}_{P_-}(z) \\ \mathcal{V}^{(t)}_{P_+}(z) \end{pmatrix} , \tag{4.2.11}$$

The elements of this fusing matrix are given by

$$F_{\epsilon,\bar{\epsilon}} = \frac{\Gamma(1+2\beta^{-1}\epsilon P_1)\Gamma(-2\beta^{-1}\bar{\epsilon}P_3)}{\prod_{\pm}\Gamma\left(\frac{1}{2}+\beta^{-1}(\epsilon P_1 \pm -\bar{\epsilon}P_3)\right)} \quad \text{with} \quad \det\begin{pmatrix} F_{--} & F_{-+} \\ F_{+-} & F_{++} \end{pmatrix} = -\frac{P_1}{P_3} . \tag{4.2.12}$$

Single-valued solutions

Taking into account the right-moving counterpart of (4.2.9), we write the s-channel decomposition of the four-point function in (4.2.2) as follows:

$$\langle V^D_{\langle 1,2\rangle}(z,\bar{z})V^D_{P_1}(0)V^D_{P_2}(\infty)V^D_{P_3}(1)\rangle = \sum_{\epsilon,\bar{\epsilon}=\pm} c^{(s)}(P_\epsilon, P_{\bar{\epsilon}})\mathcal{V}^{(s)}_{P_\epsilon}(z)\bar{\mathcal{V}}^{(s)}_{P_{\bar{\epsilon}}}(\bar{z}) , \tag{4.2.13}$$

where $c^{(s)}(P_\epsilon, P_{\bar{\epsilon}})$ are structure constants. Since diagonal fields always have zero spins, the four-point function in (4.2.13) must have trivial monodromies around the singular points $z \in \{0, 1, \infty\}$. Requiring the right-hand side of (4.2.13) to be single-valued around $z = 0$ gives us

$$c^{(s)}(P_-, P_+) = c^{(s)}(P_+, P_-) = 0 . \tag{4.2.14}$$

In other words, the single-valuedness only allows (4.2.13) diagonal primary fields to propagate in the s-channel of (4.2.13). This statement also holds for the t- and u- channels. Moreover, with vanishing structure constants (4.2.14), we find that the expansion in (4.2.13) is consistent with the degenerate fusion rules (4.1.6). Let us now applying the crossing-symmetry constraint to (4.2.13). We have

$$\sum_{\epsilon=\pm} c^{(s)}(P_\epsilon)\mathcal{V}^{(s)}_{\epsilon}(z)\bar{\mathcal{V}}^{(s)}_{\epsilon}(\bar{z}) = \sum_{\epsilon=\pm} c^{(t)}(P_\epsilon)\mathcal{V}^{(t)}_{\epsilon}(z)\bar{\mathcal{V}}^{(t)}_{\epsilon}(\bar{z}) \tag{4.2.15}$$

Using the fusing matrix (4.2.11) with (4.2.15), we can express $c^{(t)}$ in terms of $c^{(s)}$ and then compute the ratio of structure constants in (4.2.13) as follows:

$$\frac{c^{(s)}(P_+)}{c^{(s)}(P_-)} = \frac{\Gamma(2\beta^{-1}P_1)}{\Gamma(-2\beta^{-1}P_1)}\frac{\Gamma(1+2\beta^{-1}P_1)}{\Gamma(1-2\beta^{-1}P_1)}\frac{\prod_{\pm,\pm}\Gamma\left(\frac{1}{2}+\beta^{-1}P_1 \pm \beta^{-1}P_2 \pm \beta^{-1}P_2\right)}{\prod_{\pm,\pm}\Gamma\left(\frac{1}{2}-\beta^{-1}P_1 \pm \beta^{-1}P_2 \pm \beta^{-1}P_2\right)} . \tag{4.2.16}$$

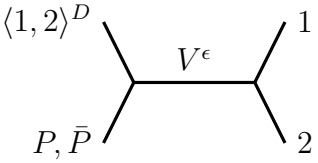
The result in (4.2.16) is an example of *the degenerate-shift equation* and can also be generalized to non-diagonal four-point functions which have at least one degenerate field as their external fields [42, 10]. Moreover, we can immediately obtain the degenerate-shift equation for $\langle V^D_{\langle 2,1\rangle}\prod_{i=1}^3 V^D_{P_i}\rangle$ by transforming (4.2.16) under $\beta \to \beta^{-1}$. In general, ratios of four-point structure constants in the four-point functions $\langle V^D_{\langle r,s\rangle}\prod_{i=1}^3 V^D_{P_i}\rangle$ can be completely determined by using only the degenerate-shift equation of $V^D_{\langle 1,2\rangle}$ and $V^D_{\langle 2,1\rangle}$ [10].

4.3 The degenerate-shift equations

In general, the existence of the degenerate fields $V^D_{\langle r,s\rangle}$ constrains four-point functions of arbitrary fields through three-point functions of the type (4.1.1). That is to say the existence of the degenerate fields $V^D_{\langle r,s\rangle}$ provides us access to the degenerate-shift equations of four-point structure constants in generic four-point function, similarly to (4.2.16).

Let us begin with the four-point function $\langle V^D_{\langle 1,2\rangle} V_{P,\bar{P}} V_1 V_2\rangle$ where V_1 and V_2 are arbitrary primary fields. The s-channel decomposition of this four-point function reads

$$\langle V^D_{\langle 1,2\rangle} V_{P,\bar{P}} V_1 V_2\rangle = \sum_{\epsilon=\pm} d^{(s)}(V^\epsilon)$$



, (4.3.1)

where $d^{(s)}(V^\epsilon)$ are four-point structure constants, and the diagram in (4.3.1) represents the product of left- and right moving conformal blocks. Moreover, we simply write the non-diagonal primary fields in the channel as follows:

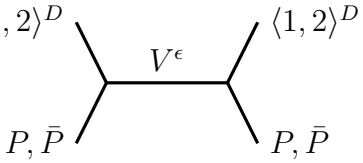
$$V^\epsilon = V_{P+\epsilon\beta^{-1},\bar{P}-\epsilon\beta^{-1}} \ . \tag{4.3.2}$$

Using (3.4.3), we express the ratio between the structure constants $d^{(s)}(V^+)$ and $d^{(s)}(V^-)$ as their two- and three-point structure constants,

$$\rho(V_{P,\bar{P}}|V_1,V_2) = \frac{B(V^-)}{B(V^+)}\frac{C(V_{\langle 1,2\rangle},V_{P,\bar{P}},V^+)}{C(V_{\langle 1,2\rangle},V_{P,\bar{P}},V^-)}\frac{C(V_1,V_2,V_{P,\bar{P}})}{C(V_1,V_2,V_{P,\bar{P}})} \ . \tag{4.3.3}$$

Since the four-point function in (4.3.1) satisfies the second-order BPZ equation, the ratio of structure constants in (4.3.3) can been determined analytically and will be displayed below [42, 10]. Now consider the s-channel of $\langle V^D_{\langle 1,2\rangle} V_{P,\bar{P}} V^D_{\langle 1,2\rangle} V_{P,\bar{P}}\rangle$:

$$\langle V^D_{\langle 1,2\rangle} V_{P,\bar{P}} V^D_{\langle 1,2\rangle} V_{P,\bar{P}}\rangle = \sum_{\epsilon=\pm} a^{(s)}(V^\epsilon)$$



. (4.3.4)

Using (4.3.3), the ratio between the four-point structure constants $a^{(s)}(V^+)$ and $a^{(s)}(V^-)$ is therefore given by

$$\rho(V_{P,\bar{P}}) = \rho(V_{P,\bar{P}}|V^D_{\langle 1,2\rangle},V_{P,\bar{P}}) \ . \tag{4.3.5}$$

Let us now introduce four-point functions of arbitrary primary fields, $\langle\prod_{i=1}^4 V_i\rangle$, which belong to CFTs where $V^D_{\langle 1,2\rangle}$ exists. Assuming that V^ϵ propagate in their s-channel, we can then write down the degenerate-shift equation for the four-point structure constants $D^{(s)}(V^+)$ and $D^{(s)}(V^-)$ in $\langle\prod_{i=1}^4 V_i\rangle$ as follows:

$$\boxed{\frac{D^{(s)}(V^+)}{D^{(s)}(V^-)} = \frac{\rho(V_{P,\bar{P}}|V_1,V_2)\rho(V_{P,\bar{P}}|V_3,V_4)}{\rho(V_{P,\bar{P}})}} \ , \tag{4.3.6}$$

where the functions $\rho(V_1|V_2,V_3)$ can computed in a similar way as their diagonal counterpart in (4.2.16) by solving the BPZ equation for the four-point function in (4.3.1) while

taking into account also crossing symmetry and constraints from the single-valuedness. Here we only display the results from [10],

$$\begin{aligned}&\rho(V_1|V_2,V_3)\\&\quad=-(-1)^{2r_2}\frac{\Gamma(-2\beta^{-1}P_1)}{\Gamma(2\beta^{-1}P_1)}\frac{\Gamma(-2\beta^{-1}\bar{P}_1)}{\Gamma(2\beta^{-1}\bar{P}_1)}\frac{\prod_{\pm,\pm}\Gamma\left(\frac{1}{2}+\beta^{-1}P_1\pm\beta^{-1}P_2\pm\beta^{-1}P_3\right)}{\prod_{\pm,\pm}\Gamma\left(\frac{1}{2}-\beta^{-1}\bar{P}_1\pm\beta^{-1}\bar{P}_2\pm\beta^{-1}\bar{P}_3\right)}\,.\end{aligned}\tag{4.3.7}$$

The index r_2 in (4.3.7) is always zero if V_2 is diagonal while r_2 is simply the first Kac index of V_2 if V_2 is non-diagonal. For instance,

$$r_2=0\quad\text{for}\quad V_2=V^D_{P_{(\frac{1}{3},0)}}\,,\tag{4.3.8a}$$

$$r_2=1\quad\text{for}\quad V_2=V_{(1,1)}\,.\tag{4.3.8b}$$

Moreover, in the case of bosonic fields, the function $\rho(V_1|V_2,V_3)$ is always symmetric under exchanging V_2 and V_3 [10]. Using (4.3.7) with (4.3.5), let us also write down the function $\rho(V)$ explicitly,

$$\rho(V_{P,\bar{P}})=\frac{\Gamma(-2\beta^{-1}P)\Gamma(-2\beta^{-1}\bar{P})}{\Gamma(2\beta^{-1}P)\Gamma(2\beta^{-1}\bar{P})}\frac{\Gamma(\beta^{-2}+2\beta^{-1}P)\Gamma(1-\beta^{-2}+2\beta^{-1}P)}{\Gamma(\beta^{-2}-2\beta^{-1}\bar{P})\Gamma(1-\beta^{-2}-2\beta^{-1}\bar{P})}\,.\tag{4.3.9}$$

Indeed, one can also arrive similar results for (4.3.6) in the t- and u-channels by simply permuting external fields in (4.3.6). Furthermore, one can check that the degenerate-shift equation (4.2.16) is the special case of (4.3.6) by specializing (4.3.7) for the four-point function in (4.2.2). While the derivation of (4.3.6) relies on the existence of the degenerate fields $V^D_{\langle 1,2\rangle}$, one can also find that $V^D_{\langle 1,3\rangle}$ provides comparable constraints, which yield exactly the same results as in (4.3.6) [25]. More generally, the existence of $V^D_{\langle 1,s\rangle}$ with $s\in\mathbb{N}^*+1$ leads to the degenerate-shift equation (4.3.6). Examples of CFTs with this type of the degenerate-shift equations are the Potts and $O(n)$ CFTs, which we will discuss in Chapter 5. In the case of CFTs in which there exist the degenerate fields $V^D_{\langle s,1\rangle}$ instead of $V^D_{\langle 1,s\rangle}$, their four-point structure constants obey another type of the degenerate-shift equation,

$$\boxed{\frac{D^{(s)}(\widetilde{V}^+)}{D^{(s)}(\widetilde{V}^-)}=\frac{\widetilde{\rho}(\widetilde{V}_{P,\bar{P}}|V_1,V_2)\widetilde{\rho}(\widetilde{V}_{P,\bar{P}}|V_3,V_4)}{\widetilde{\rho}(\widetilde{V}_{P,\bar{P}})}}\,,\tag{4.3.10}$$

where we have introduced the fields $\widetilde{V}^\epsilon$ and the functions $\widetilde{\rho}$ as follows:

$$\widetilde{V}^\epsilon=V_{P+\epsilon\beta,\bar{P}+\epsilon\beta}\,,\tag{4.3.11a}$$

$$\widetilde{\rho}(V_1|V_2,V_3)=\rho(V_1|V_2,V_3)\big|_{\beta\to\beta^{-1},r_2\to s_2}\,,\tag{4.3.11b}$$

$$\widetilde{\rho}(\widetilde{V}_{P,\bar{P}})=\widetilde{\rho}(V_{P,\bar{P}}|V^D_{\langle 1,2\rangle},V_{P,\bar{P}})\,.\tag{4.3.11c}$$

Beware of the opposite signs in (4.3.11a) and (4.3.2). An example of CFTs which have both types of the degenerate-shift equations (4.3.10) and (4.3.6) is Liouville theory. In Liouville theory, the degenerate fields $V^D_{\langle 1,2\rangle}$ and $V^D_{\langle 2,1\rangle}$ do not exist in the spectrum, however correlation functions of these two degenerate fields exist in Liouville theory [5].

Selection rules by the degenerate fields

From [10], the existence of the degenerate fields $V^D_{\langle 1,s\rangle}$ does not only constrain four-point structure constants but also impose the following selection rules on three-point function functions. For $V_i \in \{V_{P_i,\bar{P}_i}, V^D_{P_i}\}$, we have

$$\langle \prod_{i=1}^{3} V_i \rangle \neq 0 \iff \sum_{i=1}^{3} r_i \in \mathbb{Z} \quad \text{and} \quad r_i \in \frac{\mathbb{Z}}{2} \quad \text{with} \quad r_i s_i \in \mathbb{Z} \ , \tag{4.3.12}$$

where r_i is defined in the same way as in (4.3.7): $r_i = 0$ for diagonal primary fields, r_i are the first Kac indices for non-diagonal primary fields. For instance, recall that we write $V_{(r,s)}$ for non-diagonal primary fields come with the left- and right-conformal dimensions: $(\Delta_{(r,s)}, \Delta_{(r,-s)})$. With (4.3.12), for any value of P, the following three-point functions vanish.

$$\langle V^D_P V_{(\frac{1}{2},0)} V_{(0,\frac{1}{2})} \rangle = 0 \ , \tag{4.3.13a}$$

$$\langle V^D_P V_{(\frac{1}{3},0)} V_{(\frac{1}{3},0)} \rangle = 0 \ . \tag{4.3.13b}$$

In the case of CFTs with $V^D_{\langle 1,s\rangle}$, we have

$$\langle \prod_{i=1}^{3} V_i \rangle \neq 0 \iff \sum_{i=1}^{3} s_i \in \mathbb{Z} \quad \text{and} \quad s_i \in \frac{\mathbb{Z}}{2} \quad \text{with} \quad r_i s_i \in \mathbb{Z} \ , \tag{4.3.14}$$

Likewise to (4.3.12), in (4.3.14), we define $s_i = 0$ for diagonal primary fields, and we use the second Kac indices as s_i for non-diagonal primary fields. The selection rules (4.3.12) and (4.3.14) then imply that the existence of the degenerate fields do not allow the conformal dimensions of non-diagonal fields to take continuous values.

4.4 Generalized minimal models

Generalized minimal models [13] are conformal field theories at non-rational central charge whose spectra are made of diagonal degenerate representations $\mathcal{R}^D_{(r,s)}$ and can be infinite, in contrast to the more well-known minimal models, which are only valid at rational central charge and have finite spectra. We shall discuss the latter case in Chapter 6. Spectra of generalized minimal models are then given by any set of degenerate fields which are closed under the degenerate fusion rules (4.1.10). For the largest spectrum, we have

$$\mathcal{S}_{\text{GMM}} = \bigoplus_{r,s\in\mathbb{N}^*} \mathcal{R}^D_{(r,s)} \ . \tag{4.4.1}$$

Indeed, there are many more subsets of (4.4.1) which are closed under the degenerate fusion rules (4.1.10) by themselves, for example $\bigoplus_{s\in\mathbb{N}^*} \mathcal{R}^D_{(1,s)}$, $\bigoplus_{r\in\mathbb{N}^*} \mathcal{R}^D_{(r,1)}$, and $\mathcal{R}^D_{(1,1)}$. We refer to CFTs with those spectra as generalized minimal models as well. As CFT data, generalized minimal models are completely solved since their three-point structure constants can be completely determined analytically by solving the BPZ equations and the crossing-symmetry equation [13]. Unlike the minimal models, generalized minimal models are expected to be consistent only on the Riemann sphere but not on higher-genus Riemann surface [5].

Three-point structure constants

Three-point structure constants of generalized minimal models can be obtained by solving the degenerate-shift equations (4.3.6) and (4.3.10). For the case of central charge $c \in \mathbb{C} - (-\infty, 1]$, one finds [13]

$$C^{\text{GMM}}(V^D_{\langle r_1,s_1\rangle}, V^D_{\langle r_2,s_2\rangle}, V^D_{\langle r_3,s_3\rangle}) = \frac{\prod_{i=1}^3 \Upsilon'_\beta(2P_{(r_i,s_i)})}{\prod_\pm \Upsilon'_\beta\left(\frac{Q}{2} + P_{(r_1,s_1)} \pm P_{(r_2,s_2)} \pm P_{(r_3,s_3)}\right)} , \tag{4.4.2}$$

where $\Upsilon_\beta(x)$ is the Upsilon function. More details on this function exist in many literatures, for instance [9] and [5]. Since the degenerate-shift equations (4.3.6) and (4.3.10) are analytic on the whole complex plane, we should also be able to compute three-point structure constants in the region $c < 1$ as well. We can analytically continue (4.4.2) to $c < 1$ as follows [13]:

$$\beta \to i\beta \quad \text{with} \quad \Upsilon_\beta(x) \to \frac{1}{\Upsilon_{i\beta}(-ix + i\beta)} . \tag{4.4.3}$$

Three-point structure constants of generalized minimal models are then analytic at generic central charge, and generalized minimal models are consistent CFT for $c \in \mathbb{C}$. The authors of [9] have also numerically checked by showing that four-point functions of generalized minimal models are crossing symmetric for $c \in \mathbb{C}$.

4.5 Constraining logarithmic representations

We now discuss how the existence of the degenerate fields completely fixes the logarithmic couplings κ of the representations $\mathcal{W}^\kappa_{(r,s)}$ and $\widetilde{\mathcal{W}}^\kappa_{(r,s)}$, as well as the unfixed coefficients in their conformal blocks (3.6.12) and (3.6.13).

4.5.1 Constraining logarithmic couplings

To start, we introduce the non-degenerate diagonal primary fields $V^D_{P_{(r,s)}}$ with the momenta $P_{(r,s)}$ defined in (2.3.8). Moreover, whenever r and s are positive integers, the null descendants of $V^D_{P_{(r,s)}}$ do not vanish,

$$\mathcal{L}_{(r,s)} V^D_{P_{(r,s)}} \neq 0 \quad \text{and} \quad \bar{\mathcal{L}}_{(r,s)} V^D_{P_{(r,s)}} \neq 0 . \tag{4.5.1}$$

For simplicity, we consider the OPE between the degenerate fields $V^D_{\langle 1,s\rangle}$ with $s \in 2\mathbb{N}^*$ and $V^D_{P_{(r,0)}+\epsilon}$ as $\epsilon \to 0$. The case of $s \in 2\mathbb{N}^* + 1$ also leads to the same conclusion but is slightly more complicated since their OPE would contain fields with vanishing second Kac indices. Using (4.1.9), (3.3.19), and (3.3.21), we write the OPE,

$$V^D_{\langle 1,s\rangle} V^D_{P_{(r,0)}+\epsilon} = \sum_{j \overset{2}{=} -s+1}^{s-1} \sum_{\mathcal{L}} f^{\mathcal{L}}_{P_{(r,j)}+\epsilon} C_{P_{(r,j)}+\epsilon} V^D_{P_{(r,j)}+\epsilon} , \tag{4.5.2}$$

where we write C_P for OPE coefficients of primary fields V^D_P, and $f^{\mathcal{L}}_P$ denotes the relative coefficients for their descendant fields. From (3.3.28), the coefficients $f^{\mathcal{L}}_P$ with $|\mathcal{L}| = rj$ become singular at $P = P_{(r,j)}$ for r and j being positive integers. Therefore, the OPE (4.5.2) could diverge in the limit $\epsilon \to 0$. However, associativity of the OPE and the degenerate fields $V^D_{\langle 1,s\rangle}$ actually constrain the OPE coefficients (4.5.2) to cancel the divergences from

$f^{\mathcal{L}}_{P_{(r,j)}}$ through the degenerate shift equation (4.3.6). To see these cancellation, we first introduce some more compact notations:

$$C_j^+(\epsilon) = C_{P_{(r,j)}+\epsilon} \quad , \quad C_j^-(\epsilon) = C_{P_{(r,-j)}+\epsilon} \quad \text{and} \quad f_j(\epsilon) = f^{\mathcal{L}_{(r,j)}}_{P_{(r,j)}+\epsilon} \ . \tag{4.5.3}$$

Now concentrate on the terms that become singular as $\epsilon \to 0$,

$$V^D_{\langle 1,s\rangle} V^D_{P_{(r,0)}+\epsilon} \sim C_j^+(\epsilon) \left(f_j(\epsilon) \mathcal{L}_{(r,j)} V^D_{P_{(r,j)}} + f_j(\epsilon) \mathcal{L}_{(r,j)} V^D_{P_{(r,j)}} + f_j(\epsilon)^2 \bar{\mathcal{L}}_{(r,j)} \mathcal{L}_{(r,j)} V^D_{P_{(r,j)}} \right) + C_j^-(\epsilon) V^D_{P_{(r,-j)}} \dots \ . \tag{4.5.4}$$

From (3.3.28), the coefficient $f_j(\epsilon)$ has a simple pole at $\epsilon = 0$. Therefore, the first two descendant fields in the first line of (4.5.4) are subleading in this limit. For the second line of (4.5.4), we recall the relation (4.2.16), therefore the degenerate field $V^D_{\langle 1,s\rangle}$ constrains the ratio of $C_j^+(\epsilon)$ and $C_j^-(\epsilon)$ such that

$$\lim_{\epsilon \to 0} \left(\frac{C_j^-(\epsilon)}{C_j^+(\epsilon)} + f_j(\epsilon)^2 \right) = 0 \ . \tag{4.5.5}$$

Hence the double pole from the coefficient of $\bar{\mathcal{L}}_{(r,j)} \mathcal{L}_{(r,j)} V^D_{P_{(r,j)}}$ is precisely cancelled by the relation (4.5.5). Furthermore, one can check that associativity of the OPE also ensures that poles at $\epsilon = 0$ in subleading terms of (4.5.4) are always cancelled. Working out explicitly the right-hand side of (4.5.4), one is left with a combination of the first-order derivative of fields, generated by the logarithmic field:

$$W^-_{(r,s)} = \left(1 - \frac{P_{(r,s)}}{P_{(r,s)} - P_{(r,-s)}} \right) \eta^{D(1)}_{(r,s)} + \frac{P_{(r,s)}}{P_{(r,s)} - P_{(r,-s)}} \mu^{D(1)}_{(r,s)} \ , \tag{4.5.6}$$

where the derivatives of null fields $\eta^{D(1)}_{(r,s)}$ and $\mu^{D(1)}_{(r,s)}$ were introduced in (3.5.3). The non-trivial coefficients in (4.5.6) come from translating the P-derivatives to the Δ-derivatives. Comparing (4.5.6) with the definition of $W^\kappa_{(r,s)}$ in (3.5.4), we then find that (4.5.6) generates the logarithmic representations $\mathcal{W}^{\kappa^-}_{(r,s)}$ whose logarithmic coupling is given by

$$\boxed{\kappa^-_{(r,s)} = \frac{P_{(r,s)}}{P_{(r,s)} - P_{(r,-s)}} = \frac{s - r\beta^2}{2s}} \ . \tag{4.5.7}$$

If we had considered the degenerate fields $V^D_{\langle s,1\rangle}$ instead of $V^D_{\langle 1,s\rangle}$, we would have arrived at a slightly different logarithmic field:

$$W^+_{(r,s)} = \left(1 - \frac{P_{(r,s)}}{P_{(r,s)} - P_{(-r,s)}} \right) \eta^{D(1)}_{(r,s)} + \frac{P_{(r,s)}}{P_{(r,s)} - P_{(-r,s)}} \mu^{D(1)}_{(r,s)} \ , \tag{4.5.8}$$

which generate the representations $\mathcal{W}^{\kappa^+}_{(r,s)}$ with

$$\boxed{\kappa^+_{(r,s)} = \frac{P_{(r,s)}}{P_{(r,s)} - P_{(-r,s)}} = \frac{r - s\beta^{-2}}{2r}} \ . \tag{4.5.9}$$

Relating κ^+ to κ^-, we find the relation between to the representations $\mathcal{W}^{\kappa^-}_{(r,s)}$ and $\mathcal{W}^{\kappa^+}_{(r,s)}$:

$$\mathcal{W}^{\kappa^+}_{(r,s)} = \mathcal{W}^{\kappa^-}_{(s,r)}(\beta \to \beta^{-1}) \ . \tag{4.5.10}$$

Indeed, we have calculated the couplings $\kappa^\pm$ by considering OPEs which include the degenerate fields themselves. However, since the ratio of OPE coefficients in (4.5.5) only relies on the existence of the degenerate fields, the logarithmic fields $W^\pm_{(r,s)}$ could actually appear in other OPEs as well, provided that their corresponding degenerate fields exist in the model's spectrum.

Second-order derivatives

Let us now consider the OPE of the degenerate fields $V^D_{\langle r,s\rangle}$ with $r,s \in 2\mathbb{N}^*$ and the primary field $V_{P_{(0,0)}+\epsilon}$. Similarly to (4.5.4), the leading terms in this OPE are of the type:

$$V^D_{\langle r,s\rangle} V^D_{P_{(0,0)}+\epsilon} \sim C_j^+(\epsilon)\left(f_j(\epsilon)^2 \bar{\mathcal{L}}_{(r,j)}\mathcal{L}_{(r,j)} V^D_{P_{(r,j)}} + \frac{C_j^-(\epsilon)}{C_j^+(\epsilon)} V^D_{P_{(r,-j)}}\right) + C_j^+(-\epsilon)\left(f_j(-\epsilon)^2 \bar{\mathcal{L}}_{(r,j)}\mathcal{L}_{(r,j)} V^D_{P_{(r,j)}} + \frac{C_j^-(-\epsilon)}{C_j^+(-\epsilon)} V^D_{P_{(r,-j)}}\right) + \dots \quad (4.5.11)$$

Likewise to (4.5.4), cancellations from each bracket lead to a combination of the first derivatives of null fields, which then further cancel each other due to their opposite signs of ϵ. In the end, one is left with the second derivatives of null fields:

$$\widetilde{W}^0_{(r,s)} = \frac{1}{2}\left(1 - \frac{P^2_{(r,s)}}{P^2_{(r,s)} - P^2_{(-r,s)}}\right)\nu^{D(2)}_{(r,s)} + \frac{1}{2}\left(1 - \frac{P^2_{(r,s)}}{P^2_{(r,s)} - P^2_{(-r,s)}}\right)\mu^{D(2)}_{(r,s)} \ , \quad (4.5.12)$$

which generates the representation $\widetilde{W}^{\kappa^0}_{(r,s)}$ with

$$\boxed{\kappa^0_{(r,s)} = \frac{P^2_{(r,s)}}{P^2_{(r,s)} - P^2_{(-r,s)}} = \frac{1}{2} - \frac{r}{4s}\beta^2 - \frac{s}{4r}\beta^{-2}} \ . \quad (4.5.13)$$

Notice that the representation $\widetilde{\mathcal{W}}^{\kappa^0}_{(r,s)}$ is invariant under transforming $\beta \to \beta^{-1}$, simultaneously with $r \leftrightarrow s$.

4.5.2 Constraining logarithmic blocks

Let us now write down explicitly logarithmic conformal blocks for the representation $\mathcal{W}^{\kappa^\pm}_{(r,s)}$ and $\widetilde{\mathcal{W}}^{\kappa^0}_{(r,s)}$, constrained by the degenerate fields. We start with the case of $\mathcal{W}^{\kappa^-}_{(r,s)}$ by translating the leading fields on the right-handside of the OPE (4.5.4) into conformal blocks as the following:

$$\mathcal{Z}_\epsilon = \mathcal{F}_{P_{(r,s)}+\epsilon}\bar{\mathcal{F}}_{P_{(r,s)}+\epsilon} + \frac{D(V^D_{P_{(r,-s)}+\epsilon})}{D(V^D_{P_{(r,s)}+\epsilon})}\mathcal{F}_{P_{(r,-s)}+\epsilon}\bar{\mathcal{F}}_{P_{(r,-s)}+\epsilon} \ , \quad (4.5.14)$$

where we now label conformal blocks with momenta instead of conformal dimensions. Recalling that $V^D_{P_{(r,-s)}} = V^D_{P_{(r,s)}+2s\beta^{-2}}$, the ratio of four-point structure constants in (4.5.14) is then obtained by taking a product of (4.3.6). Computing the limit $\epsilon \to 0$ in (4.5.14) yields

$$\mathcal{Z}_\epsilon \underset{\epsilon\to 0}{\propto} \mathcal{G}^-_{(r,s)} = 2P_{(r,s)}\left[\mathcal{F}_{\Delta_{(r,-s)}}\frac{\bar{\mathcal{F}}^{\text{reg}}_{\Delta_{(r,s)}}}{\bar{R}_{r,s}} + \frac{\mathcal{F}^{\text{reg}}_{\Delta_{(r,s)}}}{R_{r,s}}\bar{\mathcal{F}}_{\Delta_{(r,-s)}}\right] - 2P_{(r,-s)}\left(\mathcal{F}_{\Delta_{(r,-s)}}\bar{\mathcal{F}}_{\Delta_{(r,-s)}}\right)' - \ell^{(1)-}_{(r,s)}\mathcal{F}_{\Delta_{(r,-s)}}\bar{\mathcal{F}}_{\Delta_{(r,-s)}} \ . \quad (4.5.15)$$

Comparing (4.5.15) to (3.6.12), one then recovers the value of κ^- in (4.5.9), ensuring us that the resulting blocks in (4.5.15) are indeed conformal blocks of $\mathcal{W}^{\kappa^-}$. Furthermore,

the unfixed coefficient α in (3.6.12) is now completely fixed and is given by the coefficient $\ell^{(1)-}_{(r,s)}$ in the expansion:

$$\log\left(\epsilon^2\frac{D(V_{P_{(r,-s)}+\epsilon})}{D(V_{P_{(r,s)}+\epsilon})}\right)=\sum_{n=0}^{\infty}\ell^{(n)-}_{(r,s)}\epsilon^n\ . \tag{4.5.16}$$

More explicitly, we write

$$\begin{aligned}\beta\ell^{(1)-}_{(r,s)}=&-4\sum_{j=1-s}^{s}\Big\{\psi(-2\beta^{-1}P_{(r,j)})+\psi(2\beta^{-1}P_{(r,-j)})\Big\}-4\pi\cot(\pi s\beta^{-2})\\&+\sum_{j\overset{2}{=}1-s}^{s-1}\sum_{\pm,\pm}\Big\{\psi\left(\tfrac{1}{2}-\beta^{-1}(P_{(r,j)}\pm P_1\pm P_2)\right)+\psi\left(\tfrac{1}{2}+\beta^{-1}(P_{(r,j)}\pm\bar{P}_1\pm\bar{P}_2)\right)\Big\}\\&+\sum_{j\overset{2}{=}1-s}^{s-1}\sum_{\pm,\pm}\Big\{\psi\left(\tfrac{1}{2}-\beta^{-1}(P_{(r,j)}\pm P_3\pm P_4)\right)+\psi\left(\tfrac{1}{2}+\beta^{-1}(P_{(r,j)}\pm\bar{P}_3\pm\bar{P}_4)\right)\Big\}\ ,\end{aligned} \tag{4.5.17}$$

where $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function, regularized such that $\psi(-r)=\psi(r+1)$ for $r\in\mathbb{N}$. Using (4.5.10), we can write down conformal blocks of the representations $\mathcal{W}^{\kappa^+}_{(r,s)}$ by transforming $\mathcal{G}^-_{(r,s)}$ as follows:

$$\mathcal{G}^+_{(r,s)}=\mathcal{G}^-_{(s,r)}(\beta\to\beta^{-1})\ . \tag{4.5.18}$$

Similarly to (4.5.15), the conformal blocks $\mathcal{G}^+_{(r,s)}$ are constrained by the degenerate-shift equation of $V^D_{\langle s,1\rangle}$ in (4.3.10). Let us also define the structure constants $\ell^{(1)-}_{(s,r)}(\beta\to\beta^{-1})$ as $\ell^{(1)+}_{(r,s)}$, which appears in $\mathcal{G}^+_{(r,s)}$. We write

$$\ell^{(1)+}_{(r,s)}=\ell^{(1)-}_{(s,r)}(\beta\to\beta^{-1})\ . \tag{4.5.19}$$

For the representations $\widetilde{\mathcal{W}}^{\kappa^0}_{(r,s)}$, their conformal blocks appear in the subleading order in the ϵ-expansion of (4.5.14). Therefore, we have

$$\begin{aligned}\mathcal{Z}_\epsilon+\mathcal{Z}_{-\epsilon}\underset{\epsilon\to0}{\propto}\widetilde{\mathcal{G}}^0_{(r,s)}=&\left(\mathcal{F}\bar{\mathcal{F}}\right)''_{P_{(r,-s)}}-\frac{4P^2_{(r,s)}}{R_{r,s}\bar{R}_{r,s}}\left((P-P_{(r,s)})^2\mathcal{F}\bar{\mathcal{F}}\right)''_{P_{(r,s)}}\\&+\left(\ell^{(1)-}_{(r,s)}-\ell^{(1)+}_{(r,s)}\right)\left(\mathcal{F}\bar{\mathcal{F}}\right)'_{P_{(r,-s)}}+\frac{4P^2_{(r,s)}}{R_{r,s}\bar{R}_{r,s}}\left(\ell^{(1)-}_{(r,s)}+\ell^{(1)+}_{(r,s)}\right)\left((P-P_{(r,s)})^2\mathcal{F}\bar{\mathcal{F}}\right)'_{P_{(r,s)}}\\&+\left(2\ell^{(2)}_{(r,s)}-\ell^{(1)+}_{(r,s)}\ell^{(1)-}_{(r,s)}\right)\left(\mathcal{F}\bar{\mathcal{F}}\right)_{P_{(r,-s)}}\ ,\end{aligned} \tag{4.5.20}$$

where we have used the prime derivative as the P-derivative in (4.5.20), and this notation of the prime derivative is only used in this expression. Hence, we can again recover the coupling κ^0 by comparing (4.5.20) to (3.6.13). From (4.5.20), all free parameters in (3.6.13) are now completely fixed by the degenerate fields. The structure constant $\ell^{(2)}_{(r,s)}$

is defined by the expansion (4.5.16),

$$\ell^{(2)}_{(r,s)} = \ell^{(2)+}_{(r,s)} = \ell^{(2)-}_{(r,s)} = -8\Bigg\{ \sum_{j=1-s}^{s} \sum_{i=1-r}^{r} \frac{1}{(2P_{(i,j)})^2} - \frac{1}{(2P_{(0,0)})^2} \Bigg\}$$
$$- \frac{1}{2} \sum_{j\overset{2}{=}1-s}^{s-1} \sum_{j\overset{2}{=}1-r}^{r-1} \sum_{\pm,\pm} \Bigg\{ \frac{1}{(P_1 \pm P_2 \pm P_{(i,j)})^2} + \frac{1}{(\bar{P}_1 \pm \bar{P}_2 \pm P_{(i,j)})^2}$$
$$+ \frac{1}{(P_3 \pm P_4 \pm P_{(i,j)})^2} + \frac{1}{(\bar{P}_3 \pm \bar{P}_4 \pm P_{(i,j)})^2} \Bigg\} . \quad (4.5.21)$$

From the expansion (4.5.16) It may look a bit peculiar that there are no polygamma functions in (4.5.21). However, in this case, the polygamma functions $\psi^{(1)}$ simplify to rational functions in (4.5.21) because of the identitiy:

$$\psi^{(1)}(x) + \psi^{(1)}(-x) = -\frac{d^2}{dx^2} \log(x) - \frac{d}{dx} \pi \cot \pi x. \quad (4.5.22)$$

Similarly to the representation $\widetilde{\mathcal{W}}^{\kappa^0}_{(r,s)}$, it is also easy to see that the logarithmic block $\widetilde{\mathcal{G}}^0_{(r,s)}$ is invariant under transforming $\beta \to \beta^{-1}$, simultaneously with $r \leftrightarrow s$.

4.6 Interchiral blocks

With the degenerate-shift equation (4.3.6), we can glue a tower of infinitely many conformal blocks, corresponding to four-point structure constant related by (4.3.6) or (4.3.10), into the so-called *interchiral blocks*[15]. Interchiral blocks are therefore universal objects which can be completely determined by conformal symmetry and the degenerate fields. The interchiral blocks were first found in the four-point connectivities of the critical Q-state Potts model, in which the degenerate-shift equation has the interchiral symmetry as its underlying symmetry, hence the name "interchiral blocks".

Let us then discuss interchiral blocks associated to the degenerate-shift equation of $V^D_{\langle 1,s\rangle}$ explicitly since their results will also be used in Chapter 5. The selection rule (4.3.12) implies that the existence of the degenerate fields $V^D_{\langle 1,s\rangle}$ and the single-valuedness only allow non-diagonal primary fields to have discrete conformal dimensions while conformal dimensions of diagonal fields can take continuous values. Therefore, we consider four-point functions of the form:

$$\langle \prod_{i=1}^{4} V_{(r_i,s_i)} \rangle \quad \text{for} \quad r_i \in \frac{\mathbb{Z}}{2} \quad \text{with} \quad r_i s_i \in \mathbb{Z} \ , \quad (4.6.1)$$

where we will use $V_{(r_i,s_i)}$ to parametrize diagonal primary fields $V^D_{P_{(0,s_i)}}$ by simply setting $r_i = 0$ in $V_{(r_i,s_i)}$ while allowing s_i to take generic values. This way of writing diagonal primary fields is only consistent if we consider CFTs which only have the degenerate fields of the type $V^D_{\langle 1,s\rangle}$ because fusion rules of $V^D_{P_{(0,s_i)}}$ and $V_{(0,s_i)}$ always coincide in such CFTs. For instance,

$$V_{(0,s_i)} \times V^D_{\langle 1,s\rangle} = V^D_{P_{(0,s_i)}} \times V^D_{\langle 1,s\rangle} \quad \text{whereas} \quad V_{(0,s_i)} \times V^D_{\langle s,1\rangle} \neq V^D_{P_{(0,s_i)}} \times V^D_{\langle s,1\rangle} \ . \quad (4.6.2)$$

From now, we will always write primary fields with the Kac indices, let us then also label four-point structure constants by the Kac indices. We denote $D(V^D_{\langle 1,s\rangle})$ by $D_{\langle 1,s\rangle^D}$ and

write $D_{(r,s)}$ for $D(V_{(r,s)})$. In particular, the left-handside of (4.3.6) becomes

$$\frac{D(V^D_{\langle 1,s+2\rangle})}{D(V^D_{\langle 1,s\rangle})} = \frac{D_{\langle 1,s+2\rangle^D}}{D_{\langle 1,s\rangle^D}} \quad \text{and} \quad \frac{D(V_{(r,s+2)})}{D(V_{(r,s)})} = \frac{D_{(r,s+2)}}{D_{(r,s)}} \ . \tag{4.6.3}$$

Using (4.3.6) with the above notations, let us now write infinitely many degenerate conformal blocks $|\mathcal{F}^{(x)}_{\Delta_{\langle 1,s\rangle}}|^2$ as single interchiral block,

$$\boxed{\mathcal{H}^{(x)}_{\langle 1,s_0\rangle^D} = \sum_{s\in s_0+2\mathbb{N}} \frac{D_{\langle 1,s\rangle^D}}{D_{\langle 1,s_0\rangle^D}} |\mathcal{F}^{(x)}_{\Delta_{(1,s)}}|^2 \quad \text{for} \quad s_0 \in \mathbb{N}^*} \ , \tag{4.6.4}$$

where s_0 is the smallest index that is allowed by the degenerate fusion rules:

$$V^D_{\langle 1,s\rangle} \times V_{(r,s)} = \sum_{j\overset{2}{=}s-s_0+1}^{s+s_0+1} V_{(r,j)} \ . \tag{4.6.5}$$

More explicitly, we have

$$s_0 = 1 + \min(|s_1 - s_2|, |s_3 - s_4|) \quad \text{if} \quad \begin{cases} (r_1, r_3) = (r_2, r_4) \ , \\ (s_1, s_3) \equiv (s_2, s_4) \bmod (2,2) \ . \end{cases} \tag{4.6.6}$$

For the non-diagonal fields $V_{(r,s)}$ with non-integer indices, we simply write

$$\boxed{\mathcal{H}^{(x)}_{(r,s)} = \sum_{j\in s+2\mathbb{Z}} \frac{D_{(r,j)}}{D_{(r,s)}} \mathcal{F}^{(x)}_{\Delta_{(r,s+j)}} \bar{\mathcal{F}}^{(x)}_{\Delta_{(r,-s-j)}} \quad \text{for} \quad r,s \in \mathbb{Q}-\mathbb{Z}} \ . \tag{4.6.7}$$

In the case of $V_{(r,s)}$ with r and s being integers, we use the logarithmic blocks $\mathcal{G}^-_{(r,s)}$ in (4.5.15) whenever both r and s are non-zero, whereas we simply use the conformal blocks of Verma modules with $V_{(r,0)}$ and $V_{(0,s)}$. This assumption makes sense because using the conformal blocks $\mathcal{F}_{\Delta_{(r,s)}}\bar{\mathcal{F}}_{\Delta_{(r,-s)}}$ with $V_{(r,s)}$ could result in a divergence due to the pole at degenerate conformal dimensions $\Delta_{(r,s)}$ in (3.4.15). Let us now write

$$\boxed{\mathcal{H}^{(x)}_{(r,s)} = \sum_{j\in s+2\mathbb{Z}} \frac{D_{(r,j)}}{D_{(r,s)}} \mathcal{G}^{(x)-}_{(r,s+j)} \quad \text{for} \quad r,s \in \mathbb{Z}} \ , \tag{4.6.8}$$

where we have defined

$$\mathcal{G}^{(x)-}_{(r,0)} = |\mathcal{F}^{(x)}_{\Delta_{(r,0)}}|^2 \quad \text{and} \quad \mathcal{G}^{(x)-}_{(0,s)} = |\mathcal{F}^{(x)}_{\Delta_{(0,s)}}|^2 \ . \tag{4.6.9}$$

Furthermore, in the case of $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}\rangle$, one can factorize the degenerate-shift equation (4.3.6) into two shift equations as follows:

$$\frac{D_{(r,s+2)}}{D_{(r,s)}} = \frac{D_{(r,s+2)}}{D_{(r,s+1)}} \frac{D_{(r,s+1)}}{D_{(r,s)}} \ , \tag{4.6.10}$$

where each ratio on the right-handside of (4.6.10) has been computed analytically in [15]. Therefore, for this particular four-point function, we can combine the interchiral blocks $\mathcal{H}^D_{\langle 1,s\rangle}$ and $\mathcal{H}^D_{\langle 1,s+1\rangle}$ into single interchiral block, as well as $\mathcal{H}_{(r,s+1)}$ and $\mathcal{H}_{(r,s)}$. We

can also apply the transformations $\beta \to \beta^{-1}$ to (4.6.10) and obtain similar results for $\langle V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)} \rangle$,

$$\frac{D_{(r,+2,s)}}{D_{(r+1,s)}} = \frac{D_{(r+2,s)}}{D_{(r+1,s)}} \frac{D_{(r+1,s)}}{D_{(r,s)}} \ , \tag{4.6.11}$$

where the above ratio is now controlled by the degenerate fields $V^D_{\langle s,1\rangle}$, or more precisely (4.3.10). However, to keep our formalism compatible with more general cases, we do not use the factorizations (4.6.10) and (4.6.11) with any interchiral block for any computation in this thesis. It is also still an open question if these factorizations also happen in some of other four-point functions.

In general, for CFTs with the degenerate fields of the type $V^D_{\langle s,1\rangle}$, their interchiral blocks can be obtained in a similar way by considering the degenerate-shift equation (4.3.10) and using the logarithmic blocks $\mathcal{G}^+_{(r,s)}$ in (4.5.18). In the case of CFTs in which there exist both of the degenerate fields $V^D_{\langle 1,s\rangle}$ and $V^D_{\langle s,1\rangle}$, their interchiral blocks are then obtained by taking into account both the degenerate-shift equation (4.3.6) and (4.3.10). For instance the odd CFT of [10, 43] is a CFT at generic central charge, which comes with discrete spectra and has interchiral blocks associated to both (4.3.6) and (4.3.10).

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CHAPTER 5

Solving the Potts and $O(n)$ CFTs

Since spectra of the Potts and $O(n)$ CFTs were recently completely determined in [30], the next step in solving both CFTs is to compute their four-point structure constants. In this Chapter, using the result of [30], we demonstrate in several examples of how to compute four-point functions of arbitrary primary fields in both CFTs numerically. Let us now review spectra of these two CFTs.

5.1 Spectra of the two models

The list of primary fields of the Potts and $O(n)$ CFTs can be obtained from the torus partition functions in [14],

$$Z^{\text{Potts}} = \sum_{s\in\mathbb{N}^*} \chi^D_{\langle 1,s\rangle}(\mathfrak{q}) + (Q-1)\sum_{s\in\mathbb{N}+\frac{1}{2}} \chi_{(0,s)}(\mathfrak{q}) + \sum_{r\in\mathbb{N}+2}\sum_{s\in\frac{\mathbb{Z}}{r}} \lambda_{(r,s)}(Q)\chi_{(r,s)}(\mathfrak{q}) \ , \quad (5.1.1)$$

$$Z^{O(n)} = \sum_{s\in 2\mathbb{N}^*+1} \chi^D_{\langle 1,s\rangle}(\mathfrak{q}) + \sum_{r\in\frac{1}{2}\mathbb{N}^*}\sum_{s\in\frac{\mathbb{Z}}{r}} \xi_{(r,s)}(n)\chi_{(r,s)}(\mathfrak{q}) \ , \quad (5.1.2)$$

where the parameters Q and n are related to the central charge by the relation (1.3.1), which also holds for β^2 in (1.2.2). The functions $\chi^D_{\langle r,s\rangle}(\mathfrak{q})$ are characters of the degenerate representations $\mathcal{R}^D$, while $\chi_{(r,s)}(\mathfrak{q})$ denote characters of non-diagonal Verma modules with the highest weight:$(\Delta_{(r,s)}, \Delta_{(r,-s)})$. These two types of characters have the expressions:

$$\chi^D_{\langle r,s\rangle} = \left| \frac{\mathfrak{q}^{P^2_{(r,s)}} - \mathfrak{q}^{P^2_{(r,-s)}}}{\eta(\mathfrak{q})} \right|^2 \ , \quad (5.1.3a)$$

$$\chi_{(r,s)} = \frac{\mathfrak{q}^{P^2_{(r,s)}}\bar{\mathfrak{q}}^{P^2_{(r,-s)}}}{\eta(\mathfrak{q})\eta(\bar{\mathfrak{q}})} \ , \quad (5.1.3b)$$

where $\mathfrak{q}$ is an exponential of the torus modulus: $e^{2\pi i\tau}$, and $\eta(\mathfrak{q})$ is the Dedekind eta function. That is to say what we call characters are products of characters of the left- and right-moving algebras. Moreover, the non-diagonal characters $\chi_{(r,s)}$ in (5.1.1) and (5.1.2) also come with the non-trivial multiplicities: $\lambda_{(r,s)}$ and $\xi_{(r,s)}$, which reflect the fact that the Potts and $O(n)$ CFTs have global symmetries. For instance, it was first observed in [44] that multiplicities of primary fields in the Potts CFT's partition function (5.1.1)

can always be written as a sum of the dimensions of irreducible representations of S_Q symmetry with positive integer coefficients. To write down the multiplicities $\lambda_{(r,s)}$ and $\xi_{(r,s)}$, we first define the modified Chebyshev polynomials $p_d(x)$ by the recursion:

$$xp_d(x) = p_{d-1}(x) + p_{d+1}(x) \quad \text{with} \quad p_1(x) = x \quad \text{and} \quad p_0(x) = 2 \ . \tag{5.1.4}$$

For instance, we have

$$p_1(x) = x \ , \tag{5.1.5a}$$
$$p_2(x) = x^2 - 2 \ , \tag{5.1.5b}$$
$$p_3(x) = x(x^2 - 3) \ , \tag{5.1.5c}$$
$$p_4(x) = x^4 - 4x^2 + 2 \ . \tag{5.1.5d}$$

The multiplicities of non-diagonal characters in (5.1.2) then read

$$\xi_{(r,s)}(n) = \delta_{r,1}\delta_{s\in 2\mathbb{Z}+1} + \frac{1}{2r}\sum_{r'=0}^{2r-1} e^{\pi i r' s} p_{(2r)\wedge r'}(n) \ , \tag{5.1.6}$$

where $r \wedge r'$ denotes the greatest common divisor of r and r'. For the Potts CFT, we have

$$\lambda_{(r,s)}(Q) = (Q-1)(-1)^r \delta_{s\in\mathbb{Z}+\frac{r+1}{2}} + \frac{1}{r}\sum_{r'=0}^{r-1} e^{2\pi i r' s} p_{r\wedge r'}(Q-2) \quad \text{for} \quad r > 0 \tag{5.1.7}$$

$\xi_{(r,s)}$ and $\lambda_{(r,s)}$ are symmetric under $s \to -s$ and invariant under the shifts:

$$\xi_{(r,s)} = \xi_{(r,s+2\mathbb{Z})} \quad \text{and} \quad \lambda_{(r,s)} = \lambda_{(r,s+\mathbb{Z})} \tag{5.1.8}$$

It is therefore sufficient to write down $\xi_{(r,s)}$ for $0 \le s < 2$, while we need to compute $\lambda_{(r,s)}$ for $0 \le s < 1$. For example,

(r,s)	$\xi_{(r,s)}$	$\lambda_{(r,s)}$
$(\frac{1}{2},0)$	n	–
$(1,0)$	$\frac{1}{2}(n+2)(n-1)$	–
$(1,1)$	$\frac{1}{2}n(n-1)$	–
$(\frac{3}{2},0)$	$\frac{1}{3}n(n^2-1)$	–
$(\frac{3}{2},\frac{2}{3})$	$\frac{1}{3}n(n^2-4)$	–
$(2,0)$	$\frac{1}{4}n\,(n^3-3n+2)$	$\frac{Q}{2}(Q-3)$
$(2,\frac{1}{2})$	$\frac{1}{4}\,(n^4-5n^2+4)$	$\frac{1}{2}(Q-1)(Q-2)$
$(2,1)$	$\frac{1}{4}(n-2)n(n+1)^2$	$\frac{Q}{2}(Q-3)$
$(2,\frac{3}{2})$	$\frac{1}{4}\,(n^4-5n^2+4)$	$\frac{1}{2}(Q-1)(Q-2)$
$(3,0)$	$\frac{1}{6}\,(n^6-6n^4+n^3+11n^2-n-6)$	$\frac{1}{3}(Q-1)(Q^2-5Q+3)$

(5.1.9)

Observe that examples in (5.1.9) are always polynomials in Q and n with rational coefficients for $\lambda_{(r,s)}(Q)$ and $\xi_{(r,s)}(n)$. This is not apparent due to the phase factors in the formulae (5.1.6) and (5.1.7). It was however recently shown in [30] that both $\lambda_{(r,s)}(Q)$ and $\xi_{(r,s)}(n)$ are always polynomials with rational coefficients.

5.1.1 Virasoro symmetry

At generic central charge, primary fields in the Potts and $O(n)$ transform in the representations of the Virasoro algebra as follows:

Virasoro reps	Primary fields	Logarithmic fields	Characters
$\mathcal{R}^{D}_{(1,s)}$	$V^{D}_{\langle 1,s\rangle}$	$-$	$\chi^{D}_{\langle 1,s\rangle}$
$\mathcal{W}^{\kappa^-_{(r,s)}}_{(r,s)}$ with $r,s\in\mathbb{N}^*$	$V_{(r,\pm s)}$	$W^-_{(r,s)}$	$\chi_{(r,s)}+\chi_{(r,-s)}$
$\mathcal{V}_{\Delta_{(r,s)}}\otimes\bar{\mathcal{V}}_{\Delta_{(r,-s)}}$ with $s\in\mathbb{Q}-\mathbb{N}^*$	$V_{(r,s)}$	$-$	$\chi_{(r,s)}$

(5.1.10)

where we recall that the diagonal degenerate fields $V^{D}_{\langle 1,s\rangle}$ have the left- and right-conformal dimensions $(\Delta_{(1,s)},\Delta_{(1,s)})$ and $V_{(r,s)}$ denote the non-diagonal primary fields with the conformal dimensions $(\Delta_{(r,s)},\Delta_{(r,-s)})$. Let us also give some remarks:

- The degenerate fields $V^{D}_{\langle 1,s\rangle}$ come with one vanishing null descendant at level s, which can be deduced from the missing multiplicity of their descendant fields at level s in (5.1.3a) [26]. For example, the identity field $V^{D}_{\langle 1,1\rangle}$ has $\bar{L}_{-1}L_{-1}V^{D}_{\langle 1,1\rangle}=0$ as its vanishing null descendant.

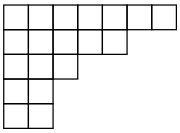
- The primary fields $V_{(1,1)}$ and $V_{(1,-1)}$ have the left- and right- conformal dimensions: $(0,1)$ and $(1,0)$, respectively. These two fields are the current of the $O(n)$ CFT [26] and belong to the non-chiral logarithmic representations $W^-_{(1,1)}$.

- The existence of the degenerate fields $V^{D}_{\langle 1,s\rangle}$ fixes the logarithmic coupling of the logarithmic field $W^-_{(r,s)}$ to be $\kappa^-_{(r,s)}$ in (4.5.9).

- Indeed, we cannot really see that the full structure of $\mathcal{W}^-_{(r,s)}$ from the torus-partition functions (5.1.1) and (5.1.2) since these torus-partition functions were obtained by taking the trace of the generators L_0 and $\bar{L}_0$ and do not carry information of any field, on which these two generators act non-diagonally. Nevertheless, the existence of $\mathcal{W}^-_{(r,s)}$ in the spectra of the Potts and $O(n)$ CFTs will be confirmed by bootstrapping several four-point functions in both CFTs with the logarithmic conformal blocks $\mathcal{G}^-_{(r,s)}$.

5.1.2 Global symmetries

At generic Q and n, irreducible representations of both S_Q and $O(n)$ can be parametrized by arbitrary size Young diagrams. We denote Young diagrams by decreasing sequences of positive integers in which each integer indicates number of boxes in each row. For example, the sequence $[7,5,3,2,2]$ represents the following diagram,

$$[\lambda_0]=[7,5,3,2,2]=[75322]=$$  $$\text{with}\quad |\lambda_0|=7+5+3+2+2=19\ ,$$

(5.1.11)

where $|\lambda_0|$ is the size of the diagram $[\lambda_0]$. Moreover, we always neglect writing commas in Young diagrams, whenever there is no ambiguity.

S_Q symmetry of the Potts CFT

Irreducible representations of symmetric group S_Q are parametrized by Young diagrams with Q boxes [45]. We then label irreducible representations of S_Q with generic Q by the Young diagrams $[\lambda]$, which are obtained by neglecting the first row of the Young diagrams $[Q-|\lambda|,\lambda]$ of symmetric group S_Q. Thus, the resulting diagrams are Q-independent. For instance, we have

S_Q reps	Integer Q	Non-integer Q
singlet	$[Q]$	$[]$
fundamental	$[Q-1,1]$	$[1]$
symmetric	$[Q-2,2]$	$[2]$
anti-symmetric	$[Q-2,1,1]$	$[11]$

(5.1.12)

Moreover, let us also point out that while we have mathematical formulation of S_Q representation theory for generic Q, it is not clear how to make sense of the S_Q group for generic Q. From [14], the degenerate fields $V^D_{\langle 1,s\rangle}$ transform as the singlet under S_Q symmetry while the non-diagonal fields $V_{(0,s)}$ belong to the fundamental representations. We write these representations as follows:

$$\Lambda_{\langle 1,s\rangle^D}=[] \quad \text{and} \quad \Lambda_{(0,s)}=[1] \tag{5.1.13}$$

From the twisted-torus partition function in [30], the action of S_Q symmetry on the other non-diagonal primary fields is given by

$$\Lambda_{(r,s)}=(-1)^r\delta_{s\in\mathbb{Z}+\frac{r+1}{2}}[1]+\frac{1}{r}\sum_{r'=0}^{r-1}e^{2\pi i r's}p_{r\wedge r'}\Big(\sum_{r''|\frac{r}{r\wedge r'}}\Lambda_{r''}-2[]\Big)\ , \tag{5.1.14}$$

where Λ_r are formal representations of S_Q defined by

$$\Lambda_r=[]+\sum_{k=0}^{r-1}(-1)^k[r-k,1^k] \quad \text{with} \quad \dim(\Lambda_1)=Q \quad \text{and} \quad \dim(\Lambda_{r\geq 2})=0\ . \tag{5.1.15}$$

To compute (5.1.14), recall the tensor product for S_Q with $Q\in\mathbb{C}$ [46]:

$$[\lambda]\otimes_{S_Q}[\mu]=\sum_\nu M_{\lambda,\mu,\nu}[\nu]\ , \tag{5.1.16}$$

where $M_{\lambda,\mu,\nu}$ are the reduced Kronecker coefficients, which are strictly positive integers [47]. Moreover, the multiplicity $M_{\lambda,\mu,\nu}$ obeys the following constraint [48]:

$$M_{\lambda,\mu,\nu}\neq 0\Longrightarrow ||\lambda|-|\mu||\leq|\nu|\leq|\lambda|+|\mu|\ . \tag{5.1.17}$$

There are simple rules of computing a product $[Q-1,1]\times[\mu]$ for symmetric group S_Q in [45], which can be rewritten for the case $[\lambda]=[1]$ in (5.1.16) as follows: the product is a sum of all possible Young diagrams obtained by removing one box from $[\mu]$, then adding at most one box to the resulting diagram where the multiplicity for each diagram is one except for the diagram $[\mu]$ itself whose coefficient is the number of different rows. In practice, we have used a program written in SageMath by [49] to compute the product

(5.1.16). Let us show a few more tensor products of S_Q with $Q \in \mathbb{C}$:

$$[1] \otimes_{S_Q} [1] = [1] + [2] + [11] + [] \ , \tag{5.1.18a}$$
$$[2] \otimes_{S_Q} [1] = [1] + [2] + [11] + [21] + [3] \ , \tag{5.1.18b}$$
$$[21] \otimes_{S_Q} [1] = 2[21] + [31] + [22] + [211] + [3] + [111] + [2] + [11] \ , \tag{5.1.18c}$$
$$[11] \otimes_{S_Q} [1] = [1] + [2] + [11] + [21] + [111] \ , \tag{5.1.18d}$$
$$[2] \otimes_{S_Q} [2] = [4] + [31] + [22] + [3] + 2[21] + [111] + 2[2] + [11] + [1] + [] \ , \tag{5.1.18e}$$

Likewise to $\lambda_{(r,s)}$, the representations $\Lambda_{(r,s)}$ are also invariant under (5.1.8). Thus, we only need to compute $\Lambda_{(r,s)}$ for $0 \leq s < 1$. Let us now display some examples of $\Lambda_{(r,s)}$:

$$\Lambda_{(2,0)} = [2] \ , \tag{5.1.19a}$$
$$\Lambda_{(2,\frac{1}{2})} = [11] \ , \tag{5.1.19b}$$
$$\Lambda_{(3,0)} = [3] + [111] \ , \tag{5.1.19c}$$
$$\Lambda_{(3,\frac{1}{3})} = [21] \ , \tag{5.1.19d}$$
$$\Lambda_{(4,0)} = [4] + [22] + [211] + [3] + [21] + 2[2] + [1] + [] \ , \tag{5.1.19e}$$
$$\Lambda_{(4,\frac{1}{4})} = [31] + [211] + [21] + [111] + [11] \ , \tag{5.1.19f}$$
$$\Lambda_{(4,\frac{1}{2})} = [31] + [22] + [1111] + [3] + [21] + [2] + [11] + [1] \ , \tag{5.1.19g}$$
$$\begin{aligned}\Lambda_{(5,0)} = {} & [5] + [32] + 2[311] + [221] + [11111] + [4] + 3[31] \\ & + 2[22] + 3[211] + [1111] + 2[3] + 4[21] + 2[111] + 2[2] + 2[11] + [1] \ .\end{aligned} \tag{5.1.19h}$$

The equation (5.1.14) then tells us how the primary fields $V_{(r,s)}$ transform under S_Q symmetry of the Potts CFT. In particular, we have

$$V_{(r,s)} \text{ in the Potts CFT} = V_{(r,s)}^{\Lambda_{(r,s)}} \ . \tag{5.1.20}$$

For instance, the representations $\Lambda_{(3,0)}$ in (5.1.19c) lead to two linearly independent primary fields

$$V_{(3,0)}^{\Lambda_{(3,0)}} \Longrightarrow V_{(3,0)}^{[3]}, V_{(3,0)}^{[111]}. \tag{5.1.21}$$

$O(n)$ symmetry of the $O(n)$ CFT

From (5.1.2), the degenerate fields $V^D_{\langle 1,s\rangle}$ in the spectrum of the $O(n)$ CFT come with multiplicity one, therefore it is natural to say that $V^D_{\langle 1,s\rangle}$ belong to the singlet representation of $O(n)$, denoted by

$$\Xi_{\langle 1,s\rangle^D} = [] \ . \tag{5.1.22}$$

The action of $O(n)$ symmetry on the other non-diagonal fields $V_{(r,s)}$ was first conjectured in [25], which was recently proven in [30]. The primary fields $V_{(r,s)}$ transform under $O(n)$ as the representations $\Xi_{(r,s)}$, defined as follows:

$$\Xi_{(r,s)} = \delta_{r,1}\delta_{s,2\mathbb{Z}+1} + \frac{1}{r}\sum_{r'=0}^{r-1} e^{\pi i r' s} p_{r\wedge r'}(\Xi_{\frac{2r}{(2r)\wedge r'}}) \ , \tag{5.1.23}$$

where Ξ_r are formal representations of $O(n)$ defined by

$$\Xi_r = \delta_{r\in 2\mathbb{Z}} + \sum_{k=0}^{r-1} (-1)^k [r-k, 1^k] \quad \text{with} \quad \dim(\Xi_r) = n \quad \text{for} \quad r \in \mathbb{N}^* \ . \tag{5.1.24}$$

To compute (5.1.23), we recall the tensor products of $O(n)$ representations at $n \in \mathbb{C}$:

$$\lambda \otimes \mu = \sum_\nu N_{\lambda,\mu,\nu} \nu \ , \tag{5.1.25}$$

where the coefficients $N_{\lambda,\mu,\nu}$ are tensor multiplicities, also known as the Newell–Littlewood numbers [50]. The sum of representations in (5.1.25) is subject to the constraint:

$$N_{\lambda,\mu,\nu} \neq 0 \Longrightarrow ||\lambda| - |\mu|| \leq \nu \leq |\lambda| + |\mu| \ . \tag{5.1.26}$$

Likewise to the tensor product of S_Q symmetry, one way of computing (5.1.25) is to use the Pieri-type rule: the product $[k] \otimes \lambda$ is a sum of all possible Young diagrams which are obtained by removing k boxes from the Young diagram λ, then adding at most k boxes to different columns of the resulting diagram. For instance, we have

$$[1] \otimes_{O(n)} [1] = [2] + [11] + [] \ . \tag{5.1.27}$$

Together with associativity, we can determine any tensor product of $O(n)$ in (5.1.25). Let us also show more examples of (5.1.25):

$$[1] \otimes_{O(n)} [2] = [21] + [3] + [1] \ , \tag{5.1.28a}$$
$$[1] \otimes_{O(n)} [11] = [111] + [21] + [1] \ , \tag{5.1.28b}$$
$$[2] \otimes_{O(n)} [2] = [4] + [31] + [22] + [2] + [11] + [] \ , \tag{5.1.28c}$$
$$[11] \otimes_{O(n)} [11] = [1111] + [211] + [22] + [2] + [11] + [] \ , \tag{5.1.28d}$$
$$[2] \otimes_{O(n)} [11] = [31] + [211] + [2] + [11] \ , \tag{5.1.28e}$$
$$[1] \otimes_{O(n)} [21] = [31] + [22] + [211] + [2] + [11] \ , \tag{5.1.28f}$$

With (5.1.25), we are now ready to compute (5.1.23). Let us again display a few examples of (5.1.23),

$$\Xi_{(\frac{1}{2},0)} = [1] \ , \tag{5.1.29a}$$
$$\Xi_{(1,0)} = [2] \ , \tag{5.1.29b}$$
$$\Xi_{(1,1)} = [11] \ , \tag{5.1.29c}$$
$$\Xi_{(\frac{3}{2},0)} = [3] + [111] \ , \tag{5.1.29d}$$
$$\Xi_{(\frac{3}{2},\frac{2}{3})} = [21] \ , \tag{5.1.29e}$$
$$\Xi_{(2,0)} = [4] + [22] + [211] + [2] + [] \ , \tag{5.1.29f}$$
$$\Xi_{(2,\frac{1}{2})} = [31] + [211] + [11] \ , \tag{5.1.29g}$$
$$\Xi_{(2,1)} = [31] + [22] + [1111] + [2] \ , \tag{5.1.29h}$$
$$\Xi_{(\frac{5}{2},0)} = [5] + [32] + 2[311] + [221] + [11111] + [3] + 2[21] + [111] + [1] \ , \tag{5.1.29i}$$
$$\Xi_{(\frac{5}{2},\frac{2}{5})} = [41] + [32] + [311] + [221] + [2111] + [3] + 2[21] + [111] + [1] \ , \tag{5.1.29j}$$
$$\begin{aligned} \Xi_{(3,0)} = {} & [6] + 2[42] + 2[411] + [33] + 2[321] + 2[3111] + 2[222] + [2211] + [21111] \\ & + 2[4] + 4[31] + 4[22] + 4[211] + 2[1111] + 4[2] + 2[11] + 2[] \ , \end{aligned} \tag{5.1.29k}$$

Likewise to the Potts CFT, each non-diagonal primary field in the $O(n)$ CFT transforms under $O(n)$ symmetry as the representations $\Xi_{(r,s)}$,

$$V_{(r,s)} \text{ in the } O(n) \text{ CFT } = V_{(r,s)}^{\Xi_{(r,s)}} \ , \tag{5.1.30}$$

which in general leads to multiple numbers of linearly independent fields with the same conformal dimensions. For instance, using (5.1.29d), one finds two linearly independent fields: $V^{[3]}_{(\frac{3}{2},0)}$ and $V^{[111]}_{(\frac{3}{2},0)}$, whose conformal dimensions coincide.

5.2 Solving the crossing-symmetry equation

We are interested in computing four-point functions of arbitrary non-diagonal primary fields in the Potts and $O(n)$ CFTs, while four-point functions, which involve at least one degenerate field, satisfy the BPZ equations and therefore can be computed analytically, as previously discussed in Chapter 4.

Since both CFTs have the degenerate fields of the type $V^D_{\langle 1,s\rangle}$ in their spectra, we decompose their four-point functions into a sum of the interchiral blocks $\mathcal{H}^{(x)}_V$, introduced in Section 4.6. The crossing-symmetry equation of the four-point functions $\langle\prod_{i=1}^4 V_{(r_i,s_i)}\rangle$ then reads

$$\sum_{V\in\mathcal{S}^{(s)}} D_V^{(s)}\mathcal{H}_V^{(s)}(z,\bar{z}) = \sum_{V\in\mathcal{S}^{(t)}} D_V^{(t)}\mathcal{H}_V^{(t)}(z,\bar{z}) = \sum_{V\in\mathcal{S}^{(u)}} D_V^{(u)}\mathcal{H}_V^{(u)}(z,\bar{z}) \ , \tag{5.2.1}$$

where $D_V^{(s)}$, $D_V^{(t)}$ and $D_V^{(u)}$ are the unknown four-point structure constants. Let us also stress here that it is necessary to solve all three channels of the crossing-symmetry equation (5.2.1) simultaneously to avoid having infinitely many solutions [15, 29]. For each model, we solve for $\langle\prod_{i=1}^4 V_{(r_i,s_i)}\rangle$ from (5.2.1) by assuming that the spectrum of each channel in (5.2.1) is the full spectrum of each model modulo the degenerate fusion rules (4.6.5). For example, in the case of the Potts CFT, the initial spectra $\mathcal{S}^{(s)}$, $\mathcal{S}^{(t)}$, and $\mathcal{S}^{(u)}$ for the four-point function $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}\rangle$ are assumed to be

$$\mathcal{S}^{\text{Potts}} = \{(r,s)\in(\mathbb{N}+2)\times(-1,1]|rs\in\mathbb{Z}\}\cup\{(0,1/2)\}\cup\{\langle 1,1\rangle^D,\langle 1,2\rangle^D\} \ , \tag{5.2.2}$$

where we always denote primary fields in spectra of four-point functions by their indices: $\langle r,s\rangle^D$ for the degenerate fields $V^D_{\langle r,s\rangle}$ and (r,s) for the non-diagonal fields $V_{(r,s)}$. The spectrum (5.2.2) is consistent with the fusion rules (4.6.5) because we have the coincidence:

$$V_{(0,\frac{1}{2})} = V_{(0,-\frac{1}{2})} \ , \tag{5.2.3}$$

which allows us to write

$$V^D_{\langle 1,1\rangle} \in V_{(0,\frac{1}{2})}\times V_{(0,\frac{1}{2})} \ , \tag{5.2.4a}$$
$$V^D_{\langle 1,2\rangle} \in V_{(0,\frac{1}{2})}\times V_{(0,-\frac{1}{2})} = V_{(0,\frac{1}{2})}\times V_{(0,\frac{1}{2})} \ . \tag{5.2.4b}$$

Moreover, with the relation (5.2.3), any primary field of the type $V_{(0,s)}$ in (5.1.1) is related to $V_{(0,\frac{1}{2})}$ by the shift: $s\to s+2\mathbb{Z}$. For instance, we have

$$V_{(0,\frac{3}{2})} = V_{(0,-\frac{1}{2}+2)} \ , \tag{5.2.5a}$$
$$V_{(0,\frac{5}{2})} = V_{(0,\frac{1}{2}+2)} \ , \tag{5.2.5b}$$
$$V_{(0,\frac{7}{2})} = V_{(0,-\frac{1}{2}+4)} \tag{5.2.5c}$$

Thus, the spectrum (5.2.2) is in fact the full spectrum of the Potts CFT modulo the shift by two in the second Kac indices. Let us also write down the full spectrum of the $O(n)$ CFT modulo the shift by two in the second Kac indices,

$$\mathcal{J}^{O(n)} = \{(r,s) \in \frac{1}{2}\mathbb{N}^* \times (-1,1] | rs \in \mathbb{Z}\} \cup \{\langle 1,1\rangle^D\} \ . \tag{5.2.6}$$

We shall always use the letter $\mathcal{J}$ to denote spectra for four-point functions of the $O(n)$ CFT and write $\mathcal{S}$ for the case of the Potts CFT. Moreover, notice from (4.6.8) that the interchiral blocks of non-diagonal primary fields with integer indices in the spectra (5.2.2) and (5.2.6) always contain the logarithmic blocks $\mathcal{G}^-_{(r,s)}$.

5.2.1 Numerical bootstrap

Because the interchiral blocks of any primary field in (5.2.2) and (5.2.6) have been completely determined [25], the crossing-symmetry equation (5.2.1) is then a linear system for infinitely many unknown four-point structure constants, which can be numerically solved by using the method of [31]. In each spectrum of (5.2.1), the tower of infinitely many fields is truncated by an upper bound on their conformal dimensions,

$$\Re(\Delta + \bar{\Delta}) \leq \Delta_{\text{max}} \ , \tag{5.2.7}$$

which led us to the truncated crossing-symmetry equation:

$$\sum_{V \in \mathcal{S}^{(s)}}^{\Re(\Delta_V+\Delta_V)\leq\Delta_{\text{max}}} D_V^{(s)} \mathcal{H}_V^{(s)}(z,\bar{z}) = \sum_{V \in \mathcal{S}^{(t)}}^{\Re(\Delta_V+\Delta_V)\leq\Delta_{\text{max}}} D_V^{(t)} \mathcal{H}_V^{(t)}(z,\bar{z}) = \sum_{V \in \mathcal{S}^{(u)}}^{\Re(\Delta_V+\Delta_V)\leq\Delta_{\text{max}}} D_V^{(u)} \mathcal{H}_V^{(u)}(z,\bar{z}) \ , \tag{5.2.8}$$

where we have written Δ_V for the total conformal dimensions of the primary field V. Computing the truncated crossing-symmetry equation (5.2.8) at random positions then gives us linear equations for the unknown structure constants. More precisely, after imposing the restriction (5.2.7) on the spectra in (5.2.1), we are left with a finite number of unknown structure constants, denoted by N_{unknowns}. In practice, we also need to normalize one structure in (5.2.8) to avoid having an indeterminate system, thus we are in fact solving (5.2.8) for $N_{\text{unknowns}} - 1$ structure consants. Since computing the truncated crossing-symmetry equation (5.2.8) at one position leads to two equations: differences between each pair of two different channels, it is therefore sufficient to compute (5.2.8) at $\lceil (N_{\text{unknowns}} - 1)/2 \rceil$ random positions, which results in at least $N_{\text{unknowns}} - 1$ linear equations for $N_{\text{unknowns}} - 1$ four-point structure constants. Let us now write the linear system for (5.2.8) explicitly. We first define the vector $\vec{d}$ with N_{unkowns} rows for four-point structure constants in (5.2.8),

$$\vec{d} = \begin{pmatrix} d^{(s)} \\ d^{(t)} \\ d^{(u)} \end{pmatrix} \quad \text{with} \quad d^{(x)} = \begin{bmatrix} D_{V_1}^{(x)} \\ D_{V_2}^{(x)} \\ D_{V_3}^{(x)} \\ \vdots \end{bmatrix} \ , \tag{5.2.9}$$

where we denote each field in the spectrum $\mathcal{S}^{(x)}$ as V_i. We then organize the interchiral blocks in each channel of (5.2.8), computed at $\lceil (N_{\text{unkowns}} - 1)/2 \rceil$ different positions, into

the matrix $b^{(x)}$:

$$b^{(x)} = \begin{bmatrix} \mathcal{H}^{(x)}_{V_1}(z_1) & \mathcal{H}^{(x)}_{V_2}(z_1) & \mathcal{H}^{(x)}_{V_3}(z_1) & \dots \\ \mathcal{H}^{(x)}_{V_1}(z_2) & \mathcal{H}^{(x)}_{V_2}(z_2) & \mathcal{H}^{(x)}_{V_3}(z_2) & \dots \\ \vdots & \vdots & \vdots & \end{bmatrix} . \tag{5.2.10}$$

Using (5.2.9) and (5.2.10), we can write the truncated-crossing symmetry equation as follows

$$B\vec{d} = 0 , \tag{5.2.11}$$

where

$$B = \begin{pmatrix} b^{(s)} & -b^{(t)} & 0 \\ 0 & b^{(t)} & -b^{(u)} \end{pmatrix} . \tag{5.2.12}$$

Hence, with (5.2.10), the linear system (5.2.11) has indeed $N_{\text{unkowns}} - 1$ equations as previously mentioned. Moreover, notice that the choice of (5.2.12) is not unique. For instance, we could have chosen

$$B = \begin{pmatrix} b^{(s)} & -b^{(t)} & 0 \\ b^{(s)} & 0 & -b^{(u)} \end{pmatrix} \quad \text{or} \quad B = \begin{pmatrix} 0 & b^{(t)} & -b^{(u)} \\ b^{(s)} & 0 & -b^{(u)} \end{pmatrix} . \tag{5.2.13}$$

However, the other choices of B in (5.2.13) are just different ways of grouping terms in (5.2.8) and, of course, yield the results as in (5.2.10).

Numerical errors

Since four-point structure constants in (5.2.1) do not depend on position, the numerical error for each four-point structure constant in (5.2.9), which we call the **deviation**, is given by the relative difference among structure constants of the same field, computed from different choices of positions. If the resulting four-point structure constants are indeed solutions to (5.2.1), their deviations should behave as follows,

$$\text{deviation} \to 0 \quad \text{as} \quad \Delta_{\text{max}} \to \infty . \tag{5.2.14}$$

The above phenomenon will be demonstrated in several examples.

5.2.2 Counting crossing-symmetry solutions

In the Potts and $O(n)$ CFTs, the crossing-symmetry equation can have non-trivial number of solutions due to their global symmetries [15, 25]. For instance, in the Potts CFT [15], there are four different four-point connectivities in (1.3.2), equivalent to four linearly-independent crossing symmetry solutions of the four-point function $\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} \rangle$. Let us then define the number of linearly-independent solutions to (5.2.1) for the four-point functions $\langle \prod_{i=1}^4 V_{(r_i,s_i)} \rangle$, according to the input of the crossing-symmetry equation for each CFT,

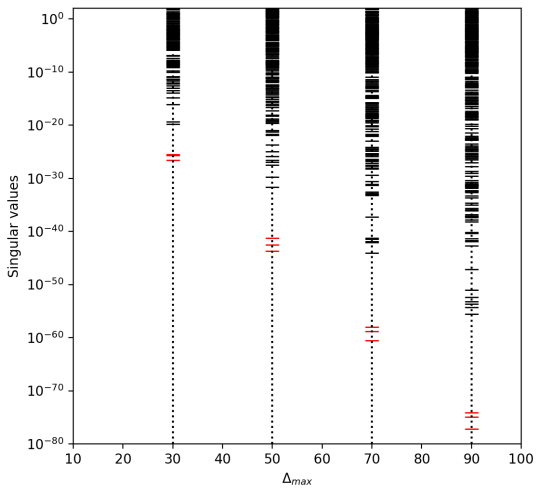
$$\begin{aligned} &\mathcal{N}_{\langle \prod_{i=1}^4 V_{(r_i,s_i)} \rangle} \\ &= \{\text{dim of solutions to (5.2.1) with the input: } Z^{\text{Potts}} \text{ modulo (4.6.5)} \} , \end{aligned} \tag{5.2.15a}$$

$$\begin{aligned} &\widetilde{\mathcal{N}}_{\langle \prod_{i=1}^4 V_{(r_i,s_i)} \rangle} \\ &= \{\text{dim of solutions to (5.2.1) with the input: } Z^{O(n)} \text{ modulo (4.6.5)} \} . \end{aligned} \tag{5.2.15b}$$

However, we stress here that each number of solution in (5.2.15) does not need to coincide with the number of crossing-symmetry solutions that belong to its corresponding CFT because crossing-symmetry solutions of the $O(n)$ and Potts CFT must also be consistent with their global symmetries. We will discuss this issue in Sections 5.3 and 5.4. Let us now discuss how to compute (5.2.15) numerically by our approach in [25].

Singular values

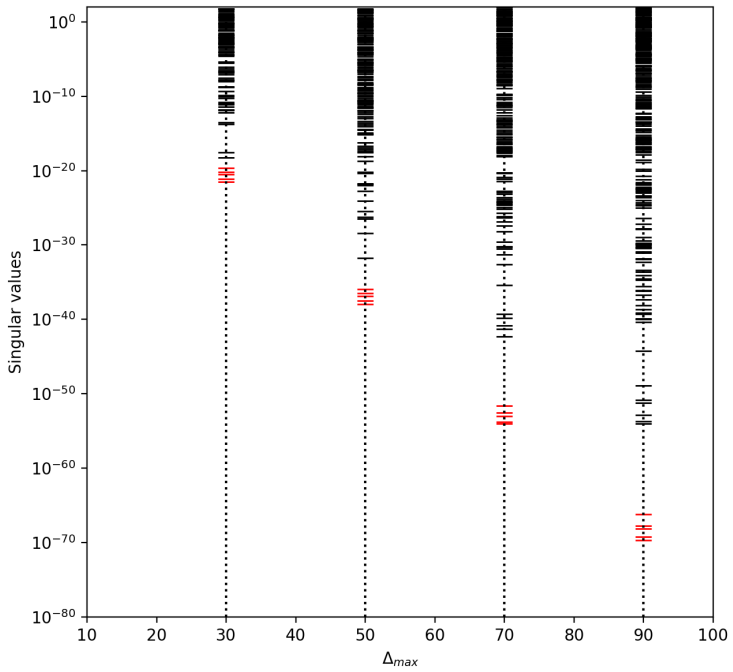
The crossing-symmetry equation (5.2.1) is a linear system, whose number of vanishing singular values can determine the number of linearly-independent crossing-symmetry solutions. We are then interested in numerically computing singular values of the crossing matrix B in (5.2.11). In practice, it is possible to distinguish vanishing singular values from the other singular values by assuming that vanishing singular values move towards zero as we increase the precision of the crossing matrix B, or equivalently increasing Δ_{max}. For example, we show the plot of some singular values of the crossing matrix B for the four-point function $\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}\rangle$ whose spectra are given by $\mathcal{S}^{(s)} = \mathcal{S}^{(t)} = \mathcal{S}^{(u)} = \mathcal{J}^{O(n)}$, computed at the central charge $\beta^{-1} = 0.8 + 0.1i$,



(5.2.16)

Therefore, from (5.2.16), there are 3 singular values, which moves away from the others as we increase Δ_{max}. These 3 singular values then vanish at infinity precision. Thus, there are 3 solutions to the crossing-symmetry equation for this four-point function. Let also display a more non-trivial example: the singular values of the crossing matrix B for the four-point function $\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{3}{2},0)}V_{(\frac{3}{2},0)}\rangle$ whose spectra are given by $\mathcal{S}^{(s)} = \mathcal{S}^{(t)} = \mathcal{S}^{(u)} = \mathcal{J}^{O(n)}$,

computed at $\beta^{-1} = 0.8 + 0.1i$,



(5.2.17)

Hence, there are 5 crossing-symmetry solutions for this case. However, separating vanishing singular values from the other singular values in general requires very high-precision computations, which consume considerable amount of time. For instance, at $\Delta_{\text{max}} = 90$, the crossing matrix B for the plot (5.2.17) has an accuracy of 70 digits, whose computation takes a standard-Desktop computer around 3 days.

Method of excluding fields

Let us discuss another method of counting crossing-symmetry solutions. The conformal bootstrap approach of [31] can easily detect if we have a unique solution to the crossing-symmetry equation. Such solution comes with deviations of structure constants, which decrease as one increases Δ_{max}. This allows us to compute the number of crossing-symmetry solutions by counting linear constraints imposed on structure constants till we find a unique solution. In practice, these constraints amount to normalizing one structure constants and setting some structure constants to be zero. More precisely, both of the number of solutions $\mathcal{N}_{\langle\prod_{i=1}^{4} V_{(r_i,s_i)}\rangle}$ and $\widetilde{\mathcal{N}}_{\langle\prod_{i=1}^{4} V_{(r_i,s_i)}\rangle}$ are given by

$$\boxed{\#(\text{linearly independent fields excluded from } \mathcal{S}^{(s)},\ \mathcal{S}^{(t)},\ \mathcal{S}^{(u)} \text{ till we find a uniqe solution }) + 1} \tag{5.2.18}$$

As an example, we again solve the crossing-symmetry equation (5.2.1) for the four-point functions $\langle V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)} \rangle$ in which all three channels have the spectrum $\mathcal{J}^{O(n)}$. From the first table in (5.2.19), we see that the deviations of all structure constants are large, except for $D^{D}_{\langle 1,1\rangle}$ which is normalized to one. However, after removing the two fields: $V_{(1,0)}$ and $V_{(1,1)}$, the deviations of the other fields become significantly small, which means that we have found a unique solution. Hence, there are $3 = 2 + 1$ solutions in this case,

which agrees with the resulting from computing the singular values in (5.2.16).

	Before		After	
(r,s)	$\Re D^{(s)}_{(r,s)}$	Deviation	$\Re D^{(s)}_{(r,s)}$	Deviation
$\langle 1,1\rangle$	1	0	1	0
$(1,0)$	1.24	0.16	–	–
$(1,1)$	-0.029	0.15	–	–
$(2,0)$	-8.9×10^{-4}	0.14	-1.5×10^{-3}	1.5×10^{-19}
$(2,\pm\frac{1}{2})$	0.3×10^{-3}	0.15	-1.1×10^{-21}	0.21
$(2,1)$	-2×10^{-3}	0.16	-2.9×10^{-22}	0.24
$(3,0)$	2.8×10^{-7}	0.15	1.3×10^{-7}	6.7×10^{-11}
$(3,\pm\frac{1}{3})$	-8.0×10^{-8}	0.15	-1.7×10^{-18}	2.6
$(3,\pm\frac{2}{3})$	2.8×10^{-7}	0.15	8.3×10^{-8}	7.4×10^{-11}

(5.2.19)

Notice that after removing two fields in (5.2.19), we are left with some vanishing structure constants, for instance $(2,\pm\frac{1}{2})$. These vanishing structure constants always have very small values of $\Re(D^{(s)}_{(r,s)})$ but come with large deviations. Indeed, from (5.2.19), we see that one does not need to have very-high precision numerical results to count solutions by the method of excluding fields, therefore we shall be using this method to count all crossing-symmetry solutions throughout this thesis. Nevertheless, it can happen that one accidentally remove fields which are linearly dependent, resulting in a miscount. In practice, to avoid such problem, we compute (5.2.18) by excluding various sets of fields and compare their outcomes.

5.3 Four-point functions of the $O(n)$ CFT

The crossing-symmetry equation only knows about conformal symmetry: Virasoro representations and their conformal blocks. Four-point functions of the $O(n)$ CFT however also transform in irreducible representations of $O(n)$. These are two independent constraints. Let us then discuss briefly how four-point functions of the $O(n)$ CFT are subject to $O(n)$ symmetry. We begin with how $O(n)$ symmetry constrains two- and three-point functions of the $O(n)$ CFT. The Schur orthogonality relations infer

$$\nu \neq \mu \Longrightarrow \langle V^\mu V^\nu\rangle = 0\ , \tag{5.3.1}$$

where V^λ is any field which transforms under $O(n)$ symmetry as the representation λ. For three-point functions, the tensor product (5.1.25) implies

$$\nu \notin \lambda \otimes_{O(n)} \mu \Longrightarrow \langle V^\lambda V^\mu V^\nu\rangle = 0\ . \tag{5.3.2}$$

Notice that reversing the statements (5.3.1) and (5.3.2) does not always leads to correct results since two- and three-point functions are also constrained by conformal symmetry and OPE associativity. For instance, two-point functions of primary fields, which transform in the same $O(n)$ representations but have different conformal dimensions, vanish. Using the OPE, vanishing three-point functions in (5.3.2) then put constraints on the spectra of four-point functions of the $O(n)$ CFT, which led us to define four-point functions of the $O(n)$ CFT as follows:

The four-point functions $\langle\prod_{i=1}^4 V_{(r_i,s_i)}\rangle$ *of the O(n) CFT are solutions to the crossing-symmetry equation* (5.2.1) *whose spectra satisfy the constraints:*

$$\mathcal{J}^{(s)} \subset \mathcal{J}^{\Xi_{(r_1,s_1)}\otimes_{O(n)}\Xi_{(r_2,s_2)}} \cap \mathcal{J}^{\Xi_{(r_3,s_3)}\otimes_{O(n)}\Xi_{(r_4,s_4)}} \ , \tag{5.3.3a}$$

$$\mathcal{J}^{(t)} \subset \mathcal{J}^{\ \Xi_{(r_1,s_1)}\otimes_{O(n)}\Xi_{(r_4,s_4)}} \cap \mathcal{J}^{\Xi_{(r_2,s_2)}\otimes_{O(n)}\Xi_{(r_3,s_3)}} \ , \tag{5.3.3b}$$

$$\mathcal{J}^{(u)} \subset \mathcal{J}^{\ \Xi_{(r_1,s_1)}\otimes_{O(n)}\Xi_{(r_3,s_3)}} \cap \mathcal{J}^{\Xi_{(r_2,s_2)}\otimes_{O(n)}\Xi_{(r_4,s_4)}} \ , \tag{5.3.3c}$$

where we have defined

$$\mathcal{J}^{\sum_i \xi_i} = \bigcup_i \mathcal{J}^{\xi_i} \quad \textit{with} \quad \mathcal{J}^{\xi} = \{\kappa \in \mathcal{J}^{O(n)} | \xi \in \Xi_\kappa\} \ . \tag{5.3.4}$$

For example, we have

$$\mathcal{J}^{[1]\times[1]} = \mathcal{J}^{[]+[11]+[2]} = \mathcal{J}^{[]} \cup \mathcal{J}^{[11]} \cup \mathcal{J}^{[2]} \ . \tag{5.3.5}$$

Using (5.1.23), let us display examples of $\mathcal{J}^\lambda$ for $|\lambda| \leq 3$. We first define

$$\mathcal{B}_0 = \{(r,s) \in \mathbb{N}^* \times (-1,1] | rs \in \mathbb{Z}\} \cup \{\langle 1,1\rangle^D\} \ , \tag{5.3.6a}$$

$$\mathcal{B}_{\frac{1}{2}} = \{(r,s) \in (\mathbb{N}+1/2) \times (-1,1] | rs \in \mathbb{Z}\} \ , \tag{5.3.6b}$$

$$\mathcal{B}_1 = \mathcal{B}_0 - \{\langle 1,1\rangle^D\} \ . \tag{5.3.6c}$$

Therefore, we have

$$\mathcal{J}^{[]} = \mathcal{B}_0 - \{(1,0),(1,1),(2,\pm 1/2),(2,1),(3,\pm 1/3),(3,1)\} \ , \tag{5.3.7a}$$

$$\mathcal{J}^{[1]} = \mathcal{B}_{\frac{1}{2}} - \{(3/2,0),(3/2,\pm 2/3)\} \ , \tag{5.3.7b}$$

$$\mathcal{J}^{[11]} = \mathcal{B}_1 - \{(1,0),(2,0),(2,1)\} \ , \tag{5.3.7c}$$

$$\mathcal{J}^{[2]} = \mathcal{B}_1 - \{(1,1),(2,\pm 1/2)\} \ , \tag{5.3.7d}$$

$$\mathcal{J}^{[3]} = \mathcal{B}_{\frac{1}{2}} - \{(1/2,0),(3/2,\pm 2/3)\} \ , \tag{5.3.7e}$$

$$\mathcal{J}^{[21]} = \mathcal{B}_{\frac{1}{2}} - \{(1/2,0),(3/2,0)\} \ , \tag{5.3.7f}$$

$$\mathcal{J}^{[111]} = \mathcal{B}_{\frac{1}{2}} - \{(1/2,0),(3/2,\pm 2/3)\} \ . \tag{5.3.7g}$$

Furthermore, we have used subsets rather than equalities in (5.3.3) because some of structure constants in these spectra could vanish non-trivially due to the crossing-symmetry equation. With (5.3.3), we now define the number of crossing-symmetry solutions that belong to $O(n)$ CFT as follows:

$$\widetilde{\mathcal{N}}^{O(n)}_{\langle\prod_{i=1}^4 V_{(r_i,s_i)}\rangle} = \dim\{\text{solutions to (5.2.1) modulo the constraints (5.3.3)}\} \ , \tag{5.3.8}$$

On the other hand, $O(n)$ symmetry can also predict the number of crossing-symmetry solutions for the four-point functions $\langle\prod_{i=1}^4 V^{\lambda_i}\rangle$. We write

$$\langle\prod_{i=1}^4 V^{\lambda_i}\rangle = \sum_i T_i^{O(n)} F_i \ , \tag{5.3.9}$$

where $T_i^{O(n)}$ are $O(n)$ invariant tensors and F_i are solutions to the crossing-symmetry equation (5.2.1). The dimension of the linear space spanned by T_i, denoted by $\mathcal{I}^{O(n)}$, then predicts the number of crossing-symmetry solutions that belong to the $O(n)$ CFT. The

invariant $\mathcal{I}^{O(n)}$ can be computed by using the tensor product (5.1.25). From [25], we have

$$\mathcal{I}^{O(n)}_{\langle\prod_{i=1}^4 V^{\lambda_i}\rangle} = \sum_{\nu} N_{\lambda_1,\lambda_2,\nu} N_{\lambda_3,\lambda_4,\nu} \ , \tag{5.3.10}$$

where $N_{\lambda,\mu,\nu}$ are multiplicities in the $O(n)$ tensor product (5.1.25). From (5.3.3), the number $\mathcal{I}^{On}_{\langle\prod_{i=1}^4 V^{\Lambda_{(r_i,s_i)}}\rangle}$ then provides an upper bound for (5.3.8),

$$\widetilde{\mathcal{N}}^{O(n)}_{\langle\prod_{i=1}^4 V_{(r_i,s_i)}\rangle} \leq \mathcal{I}^{O(n)}_{\langle\prod_{i=1}^4 V^{\Xi_{(r_i,s_i)}}\rangle} \ . \tag{5.3.11}$$

From [25], we have checked in many examples that solutions of the crossing-symmetry equation, whose input is the full spectrum of the $O(n)$ CFT, always obey (5.3.3). Let us then conjecture,

$$\boxed{\widetilde{\mathcal{N}}_{\langle\prod_{i=1}^4 V_{(r_i,s_i)}\rangle} = \widetilde{\mathcal{N}}^{O(n)}_{\langle\prod_{i=1}^4 V_{(r_i,s_i)}\rangle}} \ , \tag{5.3.12}$$

where the definition of $\widetilde{\mathcal{N}}_{\langle\prod_{i=1}^4 V_{(r_i,s_i)}\rangle}$ was given in (5.2.15). In contrast to the $O(n)$ CFT, we will see in the next section that the Potts CFT does not have similar relations as in (5.3.12).

5.3.1 Examples

We numerically solve the crossing-symmetry equation for some four-point functions, which involve the primary fields $V_{(\frac{1}{2},0)}$, $V_{(1,0)}$, and $V_{(1,1)}$. Using these numerical results, we deduce exact fusion rules among these three fields. Let us first define even- and odd-spin spectra from the spectrum $\mathcal{J}^{O(n)}$:

$$\mathcal{J}^{\text{odd}} = \{(r,s) \in \mathcal{J}^{O(n)} | rs \in 2\mathbb{Z}+1\} \ , \tag{5.3.13a}$$

$$\mathcal{J}^{\text{even}} = \{(r,s) \in \mathcal{J}^{O(n)} | rs \in 2\mathbb{Z}\} \cup \{\langle 1,1\rangle^D\} \ . \tag{5.3.13b}$$

Moreover, we only give examples of four-point functions wherein the number of solutions $\widetilde{\mathcal{N}}^{O(n)}$ coincide with the prediction from $O(n)$ symmetry: $\mathcal{I}^{O(n)}$. However, in general, we find that $\widetilde{\mathcal{N}}^{O(n)}$ is much smaller than $\mathcal{I}^{O(n)}$. For example, we find $\widetilde{\mathcal{N}}^{O(n)} = 15$ for the four-point function $\langle V_{(2,0)}V_{(2,0)}V_{(2,0)}V_{(2,0)}\rangle$ whereas $\mathcal{I}^{O(n)} \sim O(10^3)$ for this four-point function. We do not yet have clear explanations for this huge discrepancy and leave it for future work.

The four-point function $\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}\rangle$

Using (5.1.29a), the field $V_{(\frac{1}{2},0)}$ transforms as a vector under $O(n)$ symmetry, let us then write the $O(n)$ vector indices of the four-point function $\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}\rangle$ explicitly:

$$\langle V^{i_1}_{(\frac{1}{2},0)} V^{i_2}_{(\frac{1}{2},0)} V^{i_3}_{(\frac{1}{2},0)} V^{i_4}_{(\frac{1}{2},0)}\rangle = T^{O(n)}_{[]} A^{(s)}_{[]} + T^{O(n)}_{[11]} A^{(s)}_{[11]} + T^{O(n)}_{[2]} A^{(s)}_{[2]}, \tag{5.3.14}$$

where

$$T^{O(n)}_{[]} = \delta_{i_1 i_2}\delta_{i_3 i_4} \ , \tag{5.3.15a}$$

$$T^{O(n)}_{[11]} = \delta_{i_1 i_4}\delta_{i_2 i_3} - \delta_{i_1 i_3}\delta_{i_2 i_4} \ , \tag{5.3.15b}$$

$$T^{O(n)}_{[2]} = \delta_{i_1 i_3}\delta_{i_2 i_4} + \delta_{i_1 i_4}\delta_{i_2 i_3} - \frac{2}{n}\delta_{i_1 i_2}\delta_{i_3 i_4} \ . \tag{5.3.15c}$$

Therefore, $O(n)$ representation theory predicts 3 linearly-independent solutions in this case, and we indeed find 3 crossing-symmetry solutions. To write down the solution $A^{(s)}_{\lambda}$, we again impose 3 linear constraints on 3 linearly-independent structure constants in (5.3.14) such that fields, which propagate in the resulting solution, only transform under $O(n)$ as the representation λ. For example, the solution $A^{(s)}_{[]}$ can be singled out by requiring the structure constants $D^{(s)}_{(1,1)}$ and $D^{(s)}_{(1,0)}$ to vanish and fixing the normalization $D_{\langle 1,1\rangle^D} = 1$. Let us now display numerical results for $A^{(s)}_{[]}$,

$$A^{(s)}_{[]} \text{ at } \Delta_{\max} = 40 \text{ and } \beta^{-1} = 0.8 + 0.1i$$

(r,s)	$\Re D^{(s)}_{(r,s)}$	Deviation
$\langle 1,1\rangle^D$	1	0
$(2,0)$	$-1.515508647813802768 \times 10^{-3}$	2.2×10^{-19}
$(3,0)$	$1.39468476197762 \times 10^{-15}$	6.6×10^{-15}
$(3,\pm\frac{2}{3})$	$8.3751227046841 \times 10^{-8}$	1.2×10^{-14}
$(4,0)$	3.7×10^{-13}	0.87

(5.3.16)

We find that the structure constants $D^{(s)}_{(2\mathbb{N}+2,1)}$ vanish in the solutions $A^{(s)}_{[]}$, therefore we have excluded them. Therefore, we have computed the solution $\mathcal{A}^{(s)}_{[]}$ at high precision. Moreover, any structure constants with integer indices correspond to the interchiral blocks (4.6.8), which contain the logarithmic blocks $\mathcal{G}^{-}_{(r,s)}$. Therefore, we have demonstrate $\mathcal{G}^{-}_{(r,s)}$ indeed appear in a four-point function of the $O(n)$ CFT, which confirms the existence of the logarithmic representations $\mathcal{W}^{\kappa^-}_{(r,s)}$ in (5.1.10). Let us also display numerical results for the other two solutions:

$$A^{(s)}_{[11]} \text{ at } \Delta_{\max} = 40 \text{ and } \beta^{-1} = 0.8 + 0.1i$$

(r,s)	$\Re D^{(s)}_{(r,s)}$	Deviation
$(1,1)$	1	0
$(2,\pm\frac{1}{2})$	$-1.136421079784788769 \times 10^{-3}$	4.7×10^{-20}
$(3,\pm\frac{1}{3})$	$4.60859597460550 \times 10^{-7}$	4.1×10^{-15}
$(3,1)$	$2.43369637600654 \times 10^{-7}$	3.1×10^{-15}
$(4,\pm\frac{1}{4})$	5.8×10^{-13}	0.48

(5.3.17)

$$A^{(s)}_{[2]} \text{ at } \Delta_{\max} = 40 \text{ and } \beta^{-1} = 0.8 + 0.1i$$

(r,s)	$\Re D^{(s)}_{(r,s)}$	Deviation
$(1,0)$	1	0
$(2,0)$	$-6.249142617756265636 \times 10^{-3}$	2.1×10^{-19}
$(2,1)$	$-1.4658809155406988148 \times 10^{-3}$	7.9×10^{-20}
$(3,0)$	$2.06056149998946 \times 10^{-7}$	7.1×10^{-15}
$(3,\pm\frac{2}{3})$	$2.15149154275906 \times 10^{-8}$	3.9×10^{-15}
$(4,0)$	6.1×10^{-13}	0.57

(5.3.18)

Let us now summarize the spectra for each solution in (5.3.14).

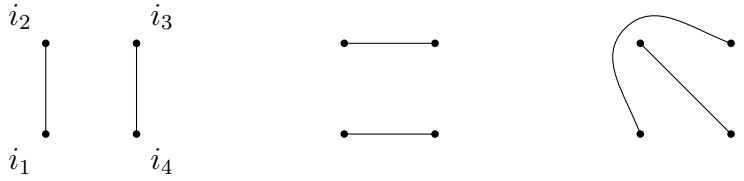
(5.3.19)

Solutions	Spectra	
	s	t, u
$A^{(s)}_{[]}$	$\mathcal{J}^{[]} \cap \mathcal{J}^{\text{even}} - (2\mathbb{N}+2, 1)$	$\mathcal{J}^{O(n)}_{r\in\mathbb{N}^*}$
$A^{(s)}_{[11]}$	$\mathcal{J}^{[11]} \cap \mathcal{J}^{\text{odd}}$	
$A^{(s)}_{[2]}$	$\mathcal{J}^{[2]} \cap \mathcal{J}^{\text{even}}$	

Indeed, one can also choose a different way of contracting the indices for the four-point function in (5.3.14). For instance,

$$\langle V^{i_1}_{(\frac{1}{2},0)} V^{i_2}_{(\frac{1}{2},0)} V^{i_3}_{(\frac{1}{2},0)} V^{i_4}_{(\frac{1}{2},0)} \rangle = \delta_{i_1 i_2}\delta_{i_3 i_4} C_1 + \delta_{i_2 i_3}\delta_{i_1 i_4} C_2 + \delta_{i_1 i_3}\delta_{i_2 i_4} C_3 \tag{5.3.20}$$

From the lattice model, the choice of bases in (5.3.20) computes the probability of how each pair of points belong on each line in the following diagrams:



$$\delta_{i_1 i_2}\delta_{i_3 i_4} C_1 \qquad \delta_{i_2 i_3}\delta_{i_1 i_4} C_2 \qquad \delta_{i_1 i_3}\delta_{i_2 i_4} C_3 \tag{5.3.21}$$

One can also relate the two sets of solutions in (5.3.14) and (5.3.20) and finds the following linear transformations:

$$C_1 = A^{(s)}_{[]} - \frac{2}{n} A^{(s)}_{[2]} \quad , \quad C_2 = A^{(s)}_{[2]} + A^{(s)}_{[11]} \quad , \quad C_3 = A^{(s)}_{[2]} - A^{(s)}_{[11]} \, . \tag{5.3.22}$$

Furthermore, while the degenerate-shift equation for the first Kac index of four-point structure constants does not exist in the $O(n)$ CFT since its spectrum does not contain the degenerate fields $V^D_{\langle s,1\rangle}$ with $s > 1$, similarly to the four-point connectivities of the Potts CFT [15], we find that the four-point function $\langle V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)} \rangle$ satisfies a renormalized version of the degenerate-shift equation (4.3.10). From [15], we write

$$\frac{D_{(r+1,s)}}{D_{(r,s)}} = \mathcal{E}_{(r,s)}(n) \left\{ 2^{-\frac{4s+2}{\beta^2}} \frac{\Gamma(\frac{1-r}{2} + \frac{s}{2\beta^2})}{\Gamma(\frac{2-r}{2} + \frac{s}{2\beta^2})} \frac{\Gamma(-\frac{r}{2} - \frac{s}{2\beta^2})}{\Gamma(\frac{1-r}{2} - \frac{s}{2\beta^2})} \frac{\Gamma(\frac{1-r}{2} + \frac{s+1}{2\beta^2})}{\Gamma(-\frac{r}{2} + \frac{s+1}{2\beta^2})} \frac{\Gamma(\frac{2-r}{2} - \frac{s+1}{2\beta^2})}{\Gamma(\frac{1-r}{2} - \frac{s+1}{2\beta^2})} \right\} . \tag{5.3.23}$$

One can check that the quantity in the curly brackets of (5.3.23) satisfies the relation (4.6.11) and is controlled by the degenerate-shift equation (4.3.10). With our numerical results, we deduce some examples of the functions $\mathcal{E}^{(s)}_{(r,s)}(n)$ in the s-channel of solutions in (5.3.22). For instance,

Solutions	(r,s)	$\mathcal{E}^{(s)}_{(r,s)}(n)$
C_1	$(1,0)$	$-\frac{n^2+3n+6}{2(n+1)}$
	$(2,0)$	$-\frac{4(n-1)\left(2n^3+4n^2-n-8\right)}{3n(n^2-2)(n^2+3n+6)}$
C_2, C_3	$(1,0)$	$-\frac{n^2}{2(n+2)}$
	$(2,0)$	$\frac{2(n-1)\left(n^4+4n^3+n^2-8n+8\right)}{3(n-2)n(n+1)(n+4)}$

(5.3.24)

It is interesting to see that rational functions in (5.3.24) do not have zeroes at rational n in general. Using the spectra in (5.3.19), let us now write the fusion rules:

$$\boxed{V_{(\frac{1}{2},0)} \times V_{(\frac{1}{2},0)} = \sum_{k\in\mathcal{J}^{[]}\cap\mathcal{J}^{\text{even}}} V_k^{[]} + \sum_{k\in\mathcal{J}^{[11]}\cap\mathcal{J}^{\text{odd}}} V_k^{[11]} + \sum_{k\in\mathcal{J}^{[2]}\cap\mathcal{J}^{\text{even}}} V_k^{[2]}} \,. \tag{5.3.25}$$

Our approach of deduce fusion rules cannot yet extract multiplicity of each field on the right-hand side of (5.3.25), as well as in other fusion rules. We hope to revisit this problem in the near future.

The four-point functions $\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(1,0)}V_{(1,0)}\rangle$ and $\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(1,1)}V_{(1,1)}\rangle$

We find 3 crossing-symmetry solutions for both cases, which agree with the prediction from $O(n)$ symmetry. Summarizingly, we have

(5.3.26)

Four-point functions	s-channel solutions	t-channel solutions
$\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(1,0)}V_{(1,0)}\rangle$	$A^{(s)}_{[]}, A^{(s)}_{[2]}, A^{(s)}_{[11]}$	$A^{(t)}_{[1]}, A^{(t)}_{[21]}, A^{(t)}_{[3]}$
$\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(1,1)}V_{(1,1)}\rangle$	$B^{(s)}_{[]}, B^{(s)}_{[2]}, B^{(s)}_{[11]}$	$B^{(t)}_{[1]}, B^{(t)}_{[21]}, B^{(t)}_{[111]}$

Solutions in different channels on the table (5.3.26) are simply related by linear transformations, that is to say they are just different choices of bases. For the s-channel solutions, each of them can be extracted by requiring two of the structure constants: $D^{(s)}_{\langle 1,1\rangle^D}$, $D^{(s)}_{(1,0)}$, and $D^{(s)}_{(1,1)}$ to vanish. For instance, we write down $A^{(s)}_{[11]}$ by imposing $D^{(s)}_{\langle 1,1\rangle^D} = D^{(s)}_{(1,0)} = 0$. Using the constraints from the single-valuedness, we find that only fields with $r \in \mathbb{N} + \frac{1}{2}$ can propagate in the t- and u-channels of these two-point functions. Therefore, spectra for the s-channel solutions are given by

(5.3.27)

Solutions	Spectra	
	s	t, u
$A^{(s)}_{[]}, B^{(s)}_{[]}$	$\mathcal{J}^{[]}\cap\mathcal{J}^{\text{even}} - (2\mathbb{N}+2,1)$	$\mathcal{J}^{O(n)}_{r\in\mathbb{N}+\frac{1}{2}}$
$A^{(s)}_{[2]}, B^{(s)}_{[2]}$	$\mathcal{J}^{[2]}\cap\mathcal{J}^{\text{even}} - (2\mathbb{N}+2,1)$	
$A^{(s)}_{[11]}$	$\mathcal{J}^{[11]}\cap\mathcal{J}^{\text{odd}} - (2\mathbb{N}+3,1)$	
$B^{(s)}_{[11]}$	$\mathcal{J}^{[11]}\cap\mathcal{J}^{\text{odd}}$	$\mathcal{J}^{O(n)}_{r\in\mathbb{N}+\frac{1}{2}} - \{(5/2,0)\}$

Likewise, for the t-channel solutions, imposing constraints on the structure constants $D^{(t)}_{(\frac{1}{2},0)}$, $D^{(t)}_{(\frac{3}{2},0)}$, $D^{(t)}_{(\frac{3}{2},\frac{2}{3})}$ allows us to write down each solution. For example, we exact the solution $A^{(t)}_{[1]}$ by imposing $D^{(t)}_{(\frac{3}{2},0)} = D^{(t)}_{(\frac{3}{2},\frac{2}{3})} = 0$ and choosing the normalization $D_{(\frac{1}{2},0)} = 1$.

Solutions	Spectra		
	s	t	u
$A^{(t)}_{[21]}, B^{(t)}_{[21]}$	$\mathcal{J}^{O(n)}_{r\in\mathbb{N}^*} - \{(\mathbb{N}+2,1)\}$	$\mathcal{J}^{[21]}$	$\mathcal{J}^{O(n)}_{r\in\mathbb{N}+\frac{1}{2}}$
$A^{(s)}_{[3]}$		$\mathcal{J}^{[3]}$	
$B^{(t)}_{[111]}$		$\mathcal{J}^{[111]}$	
$A^{(t)}_{[1]}$		$\mathcal{J}^{[1]}$	
$B^{(t)}_{[1]}$		$\mathcal{J}^{[1]} - \{(5/2,0)\}$	$\mathcal{J}^{O(n)}_{r\in\mathbb{N}+\frac{1}{2}} - \{(5/2,0)\}$

(5.3.28)

It is interesting to see that, in the four-point function $\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(1,1)}V_{(1,1)}\rangle$, the following structure constant vanishes non-trivially,

$$D^{(t,u)}_{(\frac{5}{2},0)} = 0 \ . \tag{5.3.29}$$

Using the spectra in (5.3.27) and (5.3.33), we conclude the fusion rules:

$$\boxed{V_{(\frac{1}{2},0)} \times V_{(1,0)} = \sum_{k\in\mathcal{J}^{[1]}} V_k^{[1]} + \sum_{k\in\mathcal{J}^{[21]}} V_k^{[21]} + \sum_{k\in\mathcal{J}^{[3]}} V_k^{[3]}} \tag{5.3.30}$$

and

$$\boxed{V_{(\frac{1}{2},0)} \times V_{(1,1)} = \sum_{k\in\mathcal{J}^{[1]}-\{(5/2,0)\}} V_k^{[1]} + \sum_{k\in\mathcal{J}^{[21]}} V_k^{[21]} + \sum_{k\in\mathcal{J}^{[111]}} V_k^{[111]}} \ . \tag{5.3.31}$$

The four-point function $\langle V_{(1,0)}V_{(1,0)}V_{(1,1)}V_{(1,1)}\rangle$

We find 4 crossing-symmetry solutions, which again agrees with the prediction from $O(n)$ symmetry. They can be summarized as follows:

s-channel solutions	$A^{(s)}_{[]}, A^{(s)}_{[2]}, A^{(s)}_{[11]}, A^{(s)}_{[22]}$
t-channel solutions	$A^{(t)}_{[2]}, A^{(t)}_{[11]}, A^{(t)}_{[31]}, A^{(t)}_{[211]}$

(5.3.32)

The spectra for the solutions $A^{(t)}_{[11]}$, $A^{(t)}_{[31]}$, and $A^{(t)}_{[211]}$ can be easily written down as previously explained for the other four-point functions. For the solution $A^{(t)}_{[2]}$, we need to further take into account the four-point function $\langle V_{(1,0)}V_{(1,1)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}\rangle$ wherein the field $V_{(2,0)}$ does not appear in the s-channel. Since $V_{(2,0)}$ appears in the product $V_{(\frac{1}{2},0)} \times V_{(\frac{1}{2},0)}$ and also comes with multiplicity one in the full spectrum (5.1.2), this means that $V_{(1,0)} \times V_{(1,1)}$ does not contain $V_{(2,0)}$. Therefore, we can single out $A^{(t)}_{[2]}$ by requiring $D_{(2,0)} = D_{\langle 1,1\rangle^D} = D_{(2,\frac{1}{2})} = 0$ and normalizing one structure constant. Let us now summarize the spectra for the t-channel solutions in (5.3.32):

Solutions	Spectra		
	s	t	u
$A^{(t)}_{[2]}$	$\mathcal{J}^{O(n)}_{r\in\mathbb{N}^*} - \{(2,1)\}$	$\mathcal{J}^{[2]} - \{(2,0),(3,0)\}$	$\mathcal{J}^{O(n)}_{r\in\mathbb{N}^*}$
$A^{(t)}_{[11]}$		$\mathcal{J}^{[11]}$	
$A^{(t)}_{[31]}$		$\mathcal{J}^{[31]}$	
$A^{(t)}_{[211]}$		$\mathcal{J}^{[211]}$	

(5.3.33)

where the crossing-symmetry equation constrains the field $V_{(3,0)}$ to vanish non-trivially in the solution $A^{(t)}_{[2]}$, as well as the field $V_{(2,1)}$ in the s-channel of all solutions in (5.3.32). Using (5.3.33), we now deduce the fusion rule:

$$\boxed{V_{(1,0)} \times V_{(1,1)} = \sum_{k\in\mathcal{J}^{[2]}-\{(2,0),(3,0)\}} V_k^{[2]} + \sum_{k\in\mathcal{J}^{[11]}} V_k^{[11]} + \sum_{k\in\mathcal{J}^{[31]}} V_k^{[31]} + \sum_{k\in\mathcal{J}^{[211]}} V_k^{[211]}} \ . \tag{5.3.34}$$

The four-point functions $\langle V_{(1,0)}V_{(1,0)}V_{(1,0)}V_{(1,0)}\rangle$ and $\langle V_{(1,1)}V_{(1,1)}V_{(1,1)}V_{(1,1)}\rangle$

Using the $O(n)$ tensor products (5.1.28c) and (5.1.28d), $O(n)$ representation theory predicts 6 solutions for both cases, which agrees with the results by the conformal bootstrap:

(5.3.35)

Four-point functions	s-channel solutions
$\langle V_{(1,0)}V_{(1,0)}V_{(1,0)}V_{(1,0)}\rangle$	$A^{(s)}_{[]}, A^{(s)}_{[2]}, A^{(s)}_{[11]}, A^{(s)}_{[22]}, A^{(s)}_{[31]}, A^{(s)}_{[4]}$
$\langle V_{(1,1)}V_{(1,1)}V_{(1,1)}V_{(1,1)}\rangle$	$B^{(s)}_{[]}, B^{(s)}_{[2]}, B^{(s)}_{[11]}, B^{(s)}_{[22]}, B^{(s)}_{[211]}, B^{(s)}_{[1111]}$

Therefore, to write down each solution, we need to impose 6 constraints on their structure constants, including normalizing one structure constants. With this requirement, we can only single out the solutions $A^{(s)}_{[4]}$ and $B^{(s)}_{[1111]}$ by setting

$$D^{(s)}_{\langle 1,1\rangle^D} = D^{(s)}_{(1,0)} = D^{(s)}_{(1,1)} = D^{(s)}_{(2,\frac{1}{2})} = D^{(s)}_{(2,1)} = 0 \ . \tag{5.3.36}$$

However, this does not stop us from writing down the fusion rules $V_{(1,0)} \times V_{(1,0)}$ and $V_{(1,1)} \times V_{(1,1)}$. We may not be able to numerically fix four-point structure constants of each solution in (5.3.35), but we can still see which fields appear in their spectra. For example, we write down the spectra for $A^{(s)}_{[]}$ by requiring the structure constants $D^{(s)}_{\langle 1,1\rangle^D}$, $D^{(s)}_{(1,0)}$, and $D^{(s)}_{(1,1)}$ to vanish, then we observe which structure constants survive in the resulting solution. Using this trick, we can write down the fusion rules:

$$\begin{aligned} V_{(1,0)} \times V_{(1,0)} = &\sum_{k\in\mathcal{J}^{[]}\cap\mathcal{J}^{\text{even}}} V_k^{[]} + \sum_{k\in\mathcal{J}^{[2]}\cap\mathcal{J}^{\text{even}}-\{(2,1))\}} V_k^{[2]} + \sum_{k\in\mathcal{J}^{[11]}\cap\mathcal{J}^{\text{odd}}} V_k^{[11]} \\ &+ \sum_{k\in\mathcal{J}^{[4]}\cap\mathcal{J}^{\text{even}}} V_k^{[4]} + \sum_{k\in\mathcal{J}^{[31]}\cap\mathcal{J}^{\text{odd}}} V_k^{[31]} + \sum_{k\in\mathcal{J}^{[22]}\cap\mathcal{J}^{\text{even}}} V_k^{[22]} \ , \end{aligned} \tag{5.3.37}$$

and

$$\begin{aligned} V_{(1,1)} \times V_{(1,1)} = &\sum_{k\in\mathcal{J}^{[]}\cap\mathcal{J}^{\text{even}}} V_k^{[]} + \sum_{k\in\mathcal{J}^{[2]}\cap\mathcal{J}^{\text{even}}-\{(2,1))\}} V_k^{[2]} + \sum_{k\in\mathcal{J}^{[11]}\cap\mathcal{J}^{\text{odd}}} V_k^{[11]} \\ &+ \sum_{k\in\mathcal{J}^{[1111]}\cap\mathcal{J}^{\text{even}}} V_k^{[1111]} + \sum_{k\in\mathcal{J}^{[211]}\cap\mathcal{J}^{\text{odd}}} V_k^{[211]} + \sum_{k\in\mathcal{J}^{[22]}\cap\mathcal{J}^{\text{even}}} V_k^{[22]} \ . \end{aligned} \tag{5.3.38}$$

Notice that the structure constant $D^{(s)}_{(2,1)}$ vanishes in both of the solutions $A^{(s)}_{[2]}$ and $B^{(s)}_{[2]}$, which is consistent with the s-channel solutions of the four-point functions $\langle V_{(1,0)}V_{(1,0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}\rangle$ and $\langle V_{(1,1)}V_{(1,1)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}\rangle$.

5.4 Four-point functions of the Potts CFT

Similarly to the $O(n)$ CFT, in addition to conformal symmetry and OPE associativity, S_Q symmetry also constrains correlation functions of the Potts CFT. For instance, two-point and three-point functions of the Potts CFT satisfy similar constraints as in (5.3.1) and (5.3.2). Let us now define four-point functions of the Potts CFT as follows:

The four-point functions $\langle \prod_{i=1}^4 V_{(r_i,s_i)}\rangle$ of the Potts CFT are solutions to the crossing-symmetry equation (5.2.1) *whose spectra satisfy the constraints:*

$$\mathcal{S}^{(s)} \subset \mathcal{S}^{\Lambda_{(r_1,s_1)}\otimes_{S_Q}\Lambda_{(r_2,s_2)}} \cap \mathcal{S}^{\Lambda_{(r_3,s_3)}\otimes_{S_Q}\Lambda_{(r_4,s_4)}} \ , \tag{5.4.1a}$$

$$\mathcal{S}^{(t)} \subset \mathcal{S}^{\Lambda_{(r_1,s_1)}\otimes_{S_Q}\Lambda_{(r_4,s_4)}} \cap \mathcal{S}^{\Lambda_{(r_2,s_2)}\otimes_{S_Q}\Lambda_{(r_3,s_3)}} \ , \tag{5.4.1b}$$

$$\mathcal{S}^{(u)} \subset \mathcal{S}^{\Lambda_{(r_1,s_1)}\otimes_{S_Q}\Lambda_{(r_3,s_3)}} \cap \mathcal{S}^{\Lambda_{(r_2,s_2)}\otimes_{S_Q}\Lambda_{(r_4,s_4)}} \ , \tag{5.4.1c}$$

where we have defined

$$\mathcal{S}^{\sum_i \lambda_i} = \bigcup_i \mathcal{S}^{\lambda_i} \quad \textit{with} \quad \mathcal{S}^{\lambda} = \{\kappa \in \mathcal{S}^{Potts} | \lambda \in \Lambda_\kappa\} \ . \tag{5.4.2}$$

The spectrum $\mathcal{S}^\lambda$ is a set of indices of primary fields in (5.2.2) which transform under S_Q as the irreducible representation λ. For example,

$$\mathcal{S}^{[1]\times[1]} = \mathcal{S}^{[]+[1]+[11]+[2]} = \mathcal{S}^{[]} \cup \mathcal{S}^{[1]} \cup \mathcal{S}^{[11]} \cup \mathcal{S}^{[2]} \tag{5.4.3}$$

To write down each spectrum in (5.4.3), we first define the follow set of indices:

$$\mathcal{A} = \{(r,s) \in (\mathbb{N}+5) \times [-1,1) | rs \in \mathbb{Z}\} \ . \tag{5.4.4}$$

Using (5.1.14), we have

$$\mathcal{S}^{[]} = \mathcal{A} \cup \{\langle 1,1\rangle^D, \langle 1,2\rangle^D, (4,0), (4,1)\} \ , \tag{5.4.5a}$$
$$\mathcal{S}^{[1]} = \mathcal{A} \cup \{(0,1/2), (4,0), (4,\pm 1/2), (4,1)\} \ , \tag{5.4.5b}$$
$$\mathcal{S}^{[11]} = \mathcal{A} \cup \{(2,\pm 1/2), (4,0), (4,\pm 1/4), (4,\pm 2), (4,1)\} \ , \tag{5.4.5c}$$
$$\mathcal{S}^{[2]} = \mathcal{A} \cup \{(2,0), (2,1), (4,0), (4,\pm 1/4), (4,\pm 2), (4,1)\} \ . \tag{5.4.5d}$$

With the constraints (5.4.1), we can define the number of crossing-symmetry solutions that belong to the Potts CFT as follows:

$$\mathcal{N}^{\text{Potts}}_{\langle \prod_{i=1}^4 V_{(r_i,s_i)}\rangle} = \dim\{\text{solutions to (5.2.1) modulo the constraints (5.4.1)}\} \ , \tag{5.4.6}$$

In contrast to the conjecture (5.3.12) for the $O(n)$ CFT, solving the crossing-symmetry equation (5.2.1) with the spectrum $\mathcal{S}^{\text{Potts}}$ does not always yield solutions that satisfy (5.4.1). However, we have the following inequality:

$$\mathcal{N}^{\text{Potts}}_{\langle \prod_{i=1}^4 V_{(r_i,s_i)}\rangle} \leq \mathcal{N}_{\langle \prod_{i=1}^4 V_{(r_i,s_i)}\rangle} \ , \tag{5.4.7}$$

where $\mathcal{N}_{\langle \prod_{i=1}^4 V_{(r_i,s_i)}\rangle}$ was defined in (5.2.15). We shall see in a number of examples that the inequality (5.4.7) does not always saturate. Similarly to 5.3.10 in the $O(n)$ CFT, the number of solutions $\mathcal{N}^{\text{Potts}}$ can also be deduced by S_Q symmetry. Let V^{λ_i} be a field that transforms under S_Q symmetry as the representation λ_i, we then introduce the invariant $\mathcal{I}^{S_Q}_{\langle \prod_{i=1}^4 V^{\lambda_i}\rangle}$ as the dimensions of the linear space spanned by S_Q invariant tensors of the four-point function $\langle \prod_{i=1}^4 V^{\lambda_i}\rangle$. From [25], using the tensor product (5.1.16), we can write

$$\mathcal{I}^{S_Q}_{\langle \prod_{i=1}^4 V^{\lambda_i}\rangle} = \sum_\nu M_{\lambda_1,\lambda_2,\nu} M_{\lambda_3,\lambda_4,\nu} \ , \tag{5.4.8}$$

where $M_{\lambda,\mu,\nu}$ are multiplicities in (5.1.16). From (5.4.1), the number $\mathcal{I}^{S_Q}_{\langle \prod_{i=1}^4 V^{\Lambda_{(r_i,s_i)}}\rangle}$ then provides an upper bound for (5.4.6),

$$\mathcal{N}^{\text{Potts}}_{\langle \prod_{i=1}^4 V_{(r_i,s_i)}\rangle} \leq \mathcal{I}^{S_Q}_{\langle \prod_{i=1}^4 V^{\Lambda_{(r_i,s_i)}}\rangle} \ . \tag{5.4.9}$$

5.4.1 Examples

Let us now discuss solutions to the crossing-symmetry equation (5.2.1) for some four-point functions of the Potts CFT in details. Let us start with rewriting the spectrum $\mathcal{S}^{\text{Potts}}$ in (5.2.2) with respect to their conformal spins:

$$\mathcal{S}^{\text{odd}} = \{(r,s) \in \mathcal{S}^{\text{Potts}} | rs \in 2\mathbb{Z}+1\} \ , \tag{5.4.10a}$$
$$\mathcal{S}^{\text{even}} = \{(r,s) \in \mathcal{S}^{\text{Potts}} | rs \in 2\mathbb{Z}\} \cup \mathcal{S}^{\text{deg}} \ , \tag{5.4.10b}$$

where $\mathcal{S}^{\text{deg}} = \{\langle 1,1\rangle^D, \langle 1,2\rangle^D\}$. It is also useful to display more examples of $\mathcal{S}^{\lambda}$ in (5.4.2) for $|\lambda| \leq 4$. At $|\lambda| = 3$, we have

$$\mathcal{S}^{[3]} = \mathcal{A} \cup \{(3,0),(3,1),(4,0),(4,\pm 1/2),(4,1)\} \ , \tag{5.4.11a}$$
$$\mathcal{S}^{[111]} = \mathcal{A} \cup \{(3,0),(3,1),(4,0),(4,\pm 1/4),(4,\pm 3/4),(4,1)\} \ , \tag{5.4.11b}$$
$$\mathcal{S}^{[21]} = \mathcal{A} \cup \{(3,\pm 1/3),(3,\pm 2/3),(4,0),(4,\pm 1/4),(4,\pm 1/2),(4,\pm 3/4),(4,1)\} \ . \tag{5.4.11c}$$

For $|\lambda| = 4$, we write

$$\mathcal{S}^{[4]} = \mathcal{A} \cup \{(4,0),(4,1)\} \ , \tag{5.4.12a}$$
$$\mathcal{S}^{[22]} = \mathcal{A} \cup \{(4,0),(4,\pm 1/2),(4,1)\} \ , \tag{5.4.12b}$$
$$\mathcal{S}^{[211]} = \mathcal{A} \cup \{(4,0),(4,\pm 1/4),(4,\pm 3/4),(4,1)\} \ , \tag{5.4.12c}$$
$$\mathcal{S}^{[1111]} = \mathcal{A} \cup \{(4,\pm 1/2)\} \ , \tag{5.4.12d}$$

where $\mathcal{A}$ was introduced in (5.4.4).

$\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} \rangle$: The four-point connectivities

We consider how to solve for the four-point connectivities $\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} \rangle$ in (1.3.2) from the crossing-symmetry equation and constraints of S_Q symmetry. The field $V_{(0,\frac{1}{2})}$ belongs to the fundamental representation of S_Q. Using (5.1.18a), we have the following decomposition:

$$\langle V^{[1]}_{(0,\frac{1}{2})} V^{[1]}_{(0,\frac{1}{2})} V^{[1]}_{(0,\frac{1}{2})} V^{[1]}_{(0,\frac{1}{2})} \rangle = T^{S_Q}_{[]} F^{(s)}_{[]} + T^{S_Q}_{[1]} F^{(s)}_{[1]} + T^{S_Q}_{[2]} F^{(s)}_{[2]} + T^{S_Q}_{[11]} F^{(s)}_{[11]} \ . \tag{5.4.13}$$

To solve for this four-point function, we assume that the input to the crossing-symmetry equation (5.2.1) is the spectrum $\mathcal{S}^{\text{Potts}}$ in (5.1.1). We then find four linearly independent solutions, which agree with the four-point connectivities in (1.3.2) and the decomposition (5.4.13). In this case, the crossing-symmetry equation automatically excludes any field that does not transform in irreducible representations of S_Q in (5.4.13). To write down the spectra for solutions in (5.4.13), since there are 4 solutions in (5.4.13), we single out the solution $F^{(s)}_{\lambda}$ by excluding 3 linearly-independent structure constants of any field which does not transform in the irreducible representation λ [25]. For instance, we separate the solution $F^{(s)}_{[2]}$ from the others by requiring vanishing structure constants:

$$D^{(s)}_{\langle 1,1\rangle^D} = 0 \quad , \quad D^{(s)}_{(0,\frac{1}{2})} = 0 \quad , \quad \text{and} \quad D^{(s)}_{(2,\frac{1}{2})} = 0 \ . \tag{5.4.14}$$

Applying these three constraints on (5.4.13) and normalizing one structure constant then give us a unique solution to the crossing-symmetry equation (5.2.1). Moreover, the permutation symmetry in the product $V_{(0,\frac{1}{2})} \times V_{(0,\frac{1}{2})}$ constrains $F^{(s)}_{[2]}$ to have only fields with

an even spin. Let us display the numerical results for some structure constants in the s-channel of $F^{(s)}_{[2]}$ at $\beta = 0.8 + 0.1i$:

	$\Delta_{\max} = 30$		$\Delta_{\max} = 60$	
(r,s)	$\Re D^{(s)}_{(r,s)}$	deviation	$\Re D^{(s)}_{(r,s)}$	deviation
$(2,1)$	0.09662757185	1×10^{-10}	$0.09662757185\ldots$	7.5×10^{-29}
$\langle 1,2\rangle$	-2×10^{-10}	0.50	1.0×10^{-27}	0.13
$(3,0)$	1×10^{-12}	0.38	1.0×10^{-28}	0.11
$(3,\pm\frac{2}{3})$	1×10^{-13}	0.90	1.0×10^{-30}	1.0
$(4,0)$	6.9696038×10^{-5}	7.3×10^{-8}	$6.9696038\ldots\times10^{-5}$	4.8×10^{-26}
$(4,\pm\frac{1}{2})$	3.5139509×10^{-5}	2.2×10^{-8}	$3.5139509\ldots\times10^{-5}$	1.2×10^{-26}

(5.4.15)

where we have chosen the normalization: $D^{(s)}_{(2,0)} = 1$. The structure constants $D^{(s)}_{\langle 1,2\rangle^D}$, $D^{(s)}_{(3,0)}$, and $D^{(s)}_{(3,\pm\frac{2}{3})}$ in (5.4.15) vanish since they do not transform in the representation [2]. Furthermore, all four-point structure constants in (5.4.13) with $r \in 2\mathbb{N}^* + 1$ vanish, which agree with the results of [24]. The spectra for each solution in (5.4.13) can be summarized as follows:

(5.4.16)

Solutions	Spectra	
	s	t,u
$F^{(s)}_{[]}$	$(\mathcal{S}^{[]}_{r\,\in\,2\mathbb{N}} \cup \mathcal{S}^{\text{deg}}) \cap \mathcal{S}^{\text{even}}$	$\mathcal{S}^{\text{Potts}}_{r\,\in\,2\mathbb{N}} \cup \mathcal{S}^{\text{deg}}$
$F^{(s)}_{[1]}$	$\mathcal{S}^{[1]}_{r\,\in\,2\mathbb{N}} \cap \mathcal{S}^{\text{even}}$	
$F^{(s)}_{[11]}$	$\mathcal{S}^{[11]}_{r\,\in\,2\mathbb{N}} \cap \mathcal{S}^{\text{odd}}$	
$F^{(s)}_{[2]}$	$\mathcal{S}^{[2]}_{r\,\in\,2\mathbb{N}} \cap \mathcal{S}^{\text{even}}$	

The difference between the crossing-symmetry solutions (5.4.13) and the four-point connectivities (1.3.2) is only a matter of changing bases. They are related by the linear relations:

$$F^{(s)}_{[]} = P_{aaaa} + P_{aabb} + \frac{1}{Q-1}(P_{abab} + P_{abba}) \ , \tag{5.4.17a}$$

$$F^{(s)}_{[1]} = P_{aaaa} + \frac{1}{Q-2}(P_{abab} + P_{abba}) \ , \tag{5.4.17b}$$

$$F^{(s)}_{[2]} = \frac{1}{2}(P_{abab} + P_{abba}) \ , \tag{5.4.17c}$$

$$F^{(s)}_{[11]} = \frac{1}{2}(P_{abab} - P_{abba}) \ , \tag{5.4.17d}$$

where we have normalized the four-point connectivities such that

$$D^{aabb}_{\langle 1,1\rangle^D} = 1 \quad \text{and} \quad D^{aabb}_{(0,\frac{1}{2})} = -D^{aaaa}_{(0,\frac{1}{2})}. \tag{5.4.18}$$

The linear relations in (5.4.17) can be easily computed by comparing the spectra for solutions in (5.4.13) with the spectra for the four-point connectivities in [24] and using the analytic ratios in [32]:

$$\frac{D^{aaaa}_{(0,\frac{1}{2})}}{D^{abab}_{(0,\frac{1}{2})}} = -1 \quad , \quad \frac{D^{aaaa}_{(2,0)}}{D^{abab}_{(2,0)}} = \frac{2}{2-Q} \quad , \quad \text{and} \quad \frac{D^{aaaa}_{(2,0)}}{D^{aabb}_{(2,0)}} = 1 - Q \ . \tag{5.4.19}$$

Using our numerical resulting from the conformal bootstrap, it is easy to deduce similar ratios of four-point structure constants. For example,

$$\frac{D^{aabb}_{(6,\frac{1}{3})}}{D^{aaaa}_{(6,\frac{1}{3})}} = \frac{2-Q}{2} \,, \tag{5.4.20a}$$

$$\frac{D^{abab}_{(6,\frac{1}{3})}}{D^{aaaa}_{(6,\frac{1}{3})}} = \frac{1}{4}\left(Q^5 - 9Q^4 + 27Q^3 - 28Q^2 + Q + 4\right) \,, \tag{5.4.20b}$$

$$\frac{D^{aabb}_{(6,0)}}{D^{aaaa}_{(6,0)}} = \frac{2Q^8 - 26Q^7 + 134Q^6 - 348Q^5 + 479Q^4 - 337Q^3 + 112Q^2 - 23Q + 3}{(1 - 6Q + 5Q^2 - Q^3)\,(3Q^6 - 24Q^5 + 64Q^4 - 66Q^3 + 24Q^2 - 8Q + 3)} \,, \tag{5.4.20c}$$

$$\frac{D^{abab}_{(6,0)}}{D^{aaaa}_{(6,0)}} = \frac{(2-Q)\,(Q^2 - 4Q + 1)\,(Q^6 - 9Q^5 + 30Q^4 - 40Q^3 + 13Q^2 + 4Q + 3)}{2\,(3Q^6 - 24Q^5 + 64Q^4 - 66Q^3 + 24Q^2 - 8Q + 3)} \,. \tag{5.4.20d}$$

We also announce some new ratios of structure constants, which are not in any of our articles:

$$\frac{D^{aabb}_{(8,\frac{1}{4})}}{D^{aaaa}_{(8,\frac{1}{4})}} = -\frac{\left(Q^2-5Q+5\right)\left(Q^8-14Q^7+77Q^6-209Q^5+285Q^4-169Q^3+18Q^2+12Q-4\right)}{4Q^4-32Q^3+82Q^2-76Q+20} \,, \tag{5.4.21a}$$

$$\frac{D^{abab}_{(8,\frac{1}{4})}}{D^{aaaa}_{(8,\frac{1}{4})}} = \frac{(Q-1)\left(Q^{11}-22Q^{10}+209Q^9-1119Q^8+3689Q^7-7653Q^6+9717Q^5-6834Q^4+1971Q^3-4Q^2+108Q-40\right)}{4(2Q^4-16Q^3+41Q^2-38Q+10)} \,, \tag{5.4.21b}$$

$$\frac{D^{aabb}_{(8,\frac{1}{2})}}{D^{aaaa}_{(8,\frac{1}{2})}} = \frac{(Q-2)\left(Q^9-21Q^8+182Q^7-842Q^6+2241Q^5-3437Q^4+2876Q^3-1177Q^2+232Q-32\right)}{2(2Q^8-32Q^7+208Q^6-704Q^5+1322Q^4-1360Q^3+717Q^2-184Q+32)} \,. \tag{5.4.21c}$$

All of these ratios of structure constants hold for generic Q at high precision.

The selection rules

From the numerical results, we conjecture vanishing three-point functions:

$$\boxed{\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{\lambda,a}_{(r,s)} \rangle = 0 \quad \text{for} \quad r \in 2\mathbb{N}^* + 1} \,. \tag{5.4.22}$$

The case of $r = 3$ in (5.4.22) vanishes since from (5.1.19c) and (5.1.19d), primary fields with $r = 3$ only transform in S_Q irreducible representations with three boxes, which do not appear in $[1] \otimes_{S_Q} [1]$. Furthermore, the selection rules (5.4.22) do not immediately follow from the spectra in [24] since four-point structure constants can be a sum of the product of three-point structure constants due to non-trivial multiplicities in (5.1.14). For instance, the field $V^{[2]}_{(5,0)}$ has multiplicity 2 from (5.1.19h). Thus, assuming that the two-point functions of $V^{[2],1}_{(5,0)}$ and $V^{[2],2}_{(5,0)}$ have the same normalization, we write

$$D^{[2]}_{(5,0)} \sim (C_{V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{[2],1}_{(5,0)}})^2 + (C_{V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{[2],2}_{(5,0)}})^2 \,. \tag{5.4.23}$$

Because these structure constants are complex numbers due to $Q \in \mathbb{C}$, having the four-point structure constants $D^{[2]}_{(5,0)}$ being zero does not imply that three-point structure

constants in (5.4.23) vanish independently. Nevertheless, we have checked in the simplest 22 examples for the four-point functions $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_1V_2\rangle$ that the conjecture (5.4.22) always holds for crossing-symmetry solutions of the Potts CFT. From (5.4.22), we write the fusion rule:

$$\boxed{V_{(0,\frac{1}{2})}\times V_{(0,\frac{1}{2})}=\sum_{k\in(\mathcal{S}^{[]}_{r\,\in\,2\mathbb{N}}\cup\mathcal{S}^{\rm deg})\cap\mathcal{S}^{\rm even}}V_k^{[]}+\sum_{k\in\mathcal{S}^{[1]}_{r\,\in\,2\mathbb{N}}\cap\mathcal{S}^{\rm even}}V_k^{[1]}+\sum_{k\in\mathcal{S}^{[2]}_{r\,\in\,2\mathbb{N}}\cap\mathcal{S}^{\rm even}}V_k^{[2]}+\sum_{k\in\mathcal{S}^{[11]}_{r\,\in\,2\mathbb{N}}\cap\mathcal{S}^{\rm odd}}V_k^{[11]}\,.} \tag{5.4.24}$$

$\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(2,0)}V_{(2,0)}\rangle$ and $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(2,\frac{1}{2})}V_{(2,\frac{1}{2})}\rangle$

Using (5.1.18b) and (5.1.18d), S_Q symmetry predicts 5 solutions for both cases, which agree with our findings from the conformal bootstrap. In this case, all crossing-symmetry solutions belong to the Potts CFT. They can be summarized as

Four-point functions	s-channel solutions	t-channel solutions
$\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(2,0)}V_{(2,0)}\rangle$	$F^{(s)}_{[1]},F^{(s)}_{[]},F^{(s)}_{[11]},F^{(s)}_{[2],0},F^{(s)}_{[2],1}$	$F^{(t)}_{[1]},F^{(t)}_{[2]},F^{(t)}_{[11]},F^{(t)}_{[21]},F^{(t)}_{[3]}$
$\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(2,\frac{1}{2})}V_{(2,\frac{1}{2})}\rangle$	$G^{(s)}_{[1]},G^{(s)}_{[]},G^{(s)}_{[11]},G^{(s)}_{[2],0},G^{(s)}_{[2],1}$	$G^{(t)}_{[1]},G^{(t)}_{[2]},G^{(t)}_{[11]},G^{(t)}_{[21]},G^{(t)}_{[111]}$

(5.4.25)

where the s-channel and t-channel solutions are just different bases for the same space of solutions. To singlet out each solution in (5.4.25), we again impose constraints on their structure constants. Since there are 5 solutions for both four-point functions, we can extract the solutions $F^{(t)}_{[1]}$ and $G^{(t)}_{[1]}$ from (5.4.25) by setting 4 structure constants of any primary field that does not transform in the fundamental representation to zero. For instance,

$$D^{(t)}_{(2,0)}=D^{(t)}_{(2,\frac{1}{2})}=D^{(t)}_{(3,\frac{1}{3})}=D^{(t)}_{(3,0)}=0. \tag{5.4.26}$$

Therefore, the spectra for each solution in (5.4.25) read

(5.4.27)

Solutions	Spectra		
	t	u	s
$F^{(t)}_{[3]}$	$\mathcal{S}^{[3]}$	$\mathcal{S}^{\rm Potts}-\mathcal{S}^{\rm deg}$	$\mathcal{S}^{\rm Potts}_{r\in2\mathbb{N}}\cup\{\langle1,1\rangle^D\}$
$G^{(t)}_{[111]}$	$\mathcal{S}^{[111]}$		
$F^{(t)}_{[21]},G^{(t)}_{[21]}$	$\mathcal{S}^{[21]}$		
$F^{(t)}_{[2]},G^{(t)}_{[2]}$	$\mathcal{S}^{[2]}$		
$F^{(t)}_{[11]},G^{(t)}_{[11]}$	$\mathcal{S}^{[11]}$		
$F^{(t)}_{[1]},G^{(t)}_{[1]}$	$\mathcal{S}^{[1]}$		
$F^{(s)}_{[]},G^{(s)}_{[]}$	$\mathcal{S}^{\rm Potts}-\mathcal{S}^{\rm deg}$	$\mathcal{S}^{\rm Potts}-\mathcal{S}^{\rm deg}$	$(\mathcal{S}^{[]}_{r\,\in\,2\mathbb{N}}\cup\{\langle1,1\rangle^D\})\cap\mathcal{S}^{\rm even}$
$F^{(s)}_{[1]},G^{(s)}_{[1]}$			$\mathcal{S}^{[1]}_{r\,\in\,2\mathbb{N}}\cap\mathcal{S}^{\rm even}$
$F^{(s)}_{[11]},G^{(s)}_{[11]}$			$\mathcal{S}^{[11]}_{r\,\in\,2\mathbb{N}}\cap\mathcal{S}^{\rm odd}$
$F^{(s)}_{[2],0},G^{(s)}_{[2],0}$			$\mathcal{S}^{[2]}_{r\,\in\,2\mathbb{N}}\cap\mathcal{S}^{\rm even}-\{(2,0)\}$
$F^{(s)}_{[2],1},G^{(s)}_{[2],1}$			$\mathcal{S}^{[2]}_{r\,\in\,2\mathbb{N}}\cap\mathcal{S}^{\rm even}-\{(2,1)\}$

From the fusion rules (4.6.5), only the degenerate fields with $r \in 2\mathbb{N}^* + 1$ are allowed in the s-channel while all degenerate fields are subtracted from the t, u-channels. Moreover, for each four-point function, solutions that transform in the representation [2] form a two-dimensional subspace of solutions whose bases can be chosen arbitrarily. For example, we write down each of their bases by excluding one of the fields $V_{(2,0)}$ and $V_{(2,1)}$. Let us now deduce the fusion rules of $V_{(0,\frac{1}{2})} \times V_{(2,0)}$ and $V_{(0,\frac{1}{2})} \times V_{(2,\frac{1}{2})}$,

$$\boxed{V_{(0,\frac{1}{2})} \times V_{(2,0)} = \sum_{k\in\mathcal{S}^{[3]}} V_k^{[3]} + \sum_{k\in\mathcal{S}^{[21]}} V_k^{[21]} + \sum_{k\in\mathcal{S}^{[2]}} V_k^{[2]} + \sum_{k\in\mathcal{S}^{[11]}} V_k^{[11]} + \sum_{k\in\mathcal{S}^{[1]}} V_k^{[1]}}\,, \quad (5.4.28)$$

and

$$\boxed{V_{(0,\frac{1}{2})} \times V_{(2,\frac{1}{2})} = \sum_{k\in\mathcal{S}^{[111]}} V_k^{[111]} + \sum_{k\in\mathcal{S}^{[21]}} V_k^{[21]} + \sum_{k\in\mathcal{S}^{[2]}} V_k^{[2]} + \sum_{k\in\mathcal{S}^{[11]}} V_k^{[11]} + \sum_{k\in\mathcal{S}^{[1]}} V_k^{[1]}}\,. \quad (5.4.29)$$

We have checked for several examples in Section (5.4.1) that the above fusion rules lead to consistent crossing-symmetry solutions for the four-point functions $\langle V_{(0,\frac{1}{2})} V_{(2,0)} V_1 V_2 \rangle$ and $\langle V_{(0,\frac{1}{2})} V_{(2,\frac{1}{2})} V_1 V_2 \rangle$.

More examples

We first define $L = \sum_{i=1}^4 r_i$ for $\langle \prod_{i=1}^4 V_{(r_i,s_i)} \rangle$. Let us then count the numbers of crossing-symmetry solutions, $\mathcal{N}$ and $\mathcal{N}^{\text{Potts}}$, and compute the prediction from S_Q symmetry in (5.4.8) for the simplest 28 four-point functions of the Potts CFTs with $L \leq 6$.

For convenience, these four-point functions are labelled as their indices. In 17 out of 28 cases, we find solutions that do not belong to the Potts CFT. Moreover, $\mathcal{N}^{\text{Potts}}$ aways obeys the inequality (5.4.9) and saturates the bound from S_Q symmetry in 24 out of 28 cases.

$0 \leq L \leq 2$

Four-point functions	$\mathcal{N}$	$\mathcal{N}^{\text{Potts}}$	$\mathcal{I}^{S_Q}$
$(0,\frac{1}{2})^4$	4	4	4
$(0,\frac{1}{2})^3(2,0)$	3	3	3
$(0,\frac{1}{2})^3(2,\frac{1}{2})$	3	3	3
$(0,\frac{1}{2})^3(2,1)$	3	3	3

$L = 3$

Four-point functions	$\mathcal{N}$	$\mathcal{N}^{\text{Potts}}$	$\mathcal{I}^{S_Q}$
$\left(0,\frac{1}{2}\right)^3(3,0)$	5	2	2
$\left(0,\frac{1}{2}\right)^3(3,\frac{1}{3})$	5	2	2

$L = 4$

Four-point functions	$\mathcal{N}$	$\mathcal{N}^{\text{Potts}}$	$\mathcal{I}^{S_Q}$
$(0,\frac{1}{2})^3(4,0)$	12	6	14
$(0,\frac{1}{2})^3(4,\frac{1}{2})$	12	6	13
$(0,\frac{1}{2})^3(4,\frac{1}{4})$	12	6	6
$(0,\frac{1}{2})^2(2,0)^2$	5	5	5
$(0,\frac{1}{2})^2(2,1)^2$	5	5	5
$(0,\frac{1}{2})^2(2,\frac{1}{2})^2$	5	5	5
$(0,\frac{1}{2})^2(2,\frac{1}{2})(2,-\frac{1}{2})$	5	5	5
$(0,\frac{1}{2})^2(2,0)(2,\frac{1}{2})$	4	4	4
$(0,\frac{1}{2})^2(2,1)(2,\frac{1}{2})$	4	4	4
$(0,\frac{1}{2})^2(2,0)(2,1)$	5	5	5

$L = 5$

Four-point functions	$\mathcal{N}$	$\mathcal{N}^{\text{Potts}}$	$\mathcal{I}^{S_Q}$
$\left(0,\frac{1}{2}\right)^2(2,0)(3,0)$	9	5	5
$\left(0,\frac{1}{2}\right)^2(2,\frac{1}{2})(3,0)$	9	5	5
$\left(0,\frac{1}{2}\right)^2(2,0)(3,\frac{1}{3})$	9	5	5
$\left(0,\frac{1}{2}\right)^2(2,\frac{1}{2})(3,\frac{1}{3})$	9	5	5

$L = 6$

Four-point functions	$\mathcal{N}$	$\mathcal{N}^{\text{Potts}}$	$\mathcal{I}^{S_Q}$
$(0,\frac{1}{2})(2,0)^3$	8	7	7
$(0,\frac{1}{2})(2,\frac{1}{2})^3$	8	7	7
$(0,\frac{1}{2})(2,\frac{1}{2})^2(2,0)$	8	7	7
$(0,\frac{1}{2})(2,\frac{1}{2})^2(2,-\frac{1}{2})$	8	7	7
$(0,\frac{1}{2})(2,0)^2(2,\frac{1}{2})$	8	7	7
$(0,\frac{1}{2})(2,-\frac{1}{2})(2,\frac{1}{2})(2,0)$	8	7	7
$(0,\frac{1}{2})^2(3,0)^2$	15	8	12
$(0,\frac{1}{2})^2(3,\frac{1}{3})^2$	15	8	11

Numbers of solutions and the degenerate fields

From our results for the $O(n)$ CFT in [25], observe from several examples that the numbers of solutions do not seem to depend on the second Kac indices of four-point functions, whenever their spectra do not have any degenerate field. In the Potts CFT, the same observations still hold. However, even if there are degenerate fields in some channels, the numbers of solutions for four-point functions with the same r_i but different s_i may still coincide due to the degenerate field $V^D_{\langle 1,2\rangle}$, which does not exist in the $O(n)$ CFT. For example, the numbers of solutions for $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(2,\frac{1}{2})}V_{(2,\frac{1}{2})}\rangle$ and $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(2,\frac{1}{2})}V_{(2,-\frac{1}{2})}\rangle$

match because the fusion rules (4.6.5) allow the degenerate field $V^D_{\langle 1,2\rangle}$ to propagate in the s-channel of $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(2,\frac{1}{2})}V_{(2,-\frac{1}{2})}\rangle$. Let us now propose the following conjecture:

Conjecture for both the Potts and $O(n)$ CFTs: *If there are no degenerate fields in all three channels, the numbers of crossing-symmetry solutions for the four-point functions $\langle\prod_{i=1}^4 V_{(r_i,s_i)}\rangle$ are independent of s_i.*

For the Potts CFT, the conjecture is for both $\mathcal{N}$ and $\mathcal{N}^{\text{Potts}}$. For example, observe that the four-point functions $\langle V_{(0,\frac{1}{2})}\prod_{i=1}^3 V_{(2,j_i)}\rangle$ come with $\mathcal{N}=8$ and $\mathcal{N}^{\text{Potts}}=7$, regardlessly of j_i.

5.4.2 Examples with extra solutions

We discuss some examples in Section 5.4.1 with $\mathcal{N}^{\text{Potts}}<\mathcal{N}$ and show how to pin down which solutions belong to the Potts CFT.

The four-point functions: $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(3,0)}\rangle$ and $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(3,\frac{1}{3})}\rangle$

These are the simplest cases where we have extra solutions. In both cases, we find 5 linearly-independent solutions to the crossing-symmetry equation (5.2.1), whereas S_Q representation theory predicts only two solutions.

For the case $V_{(3,0)}$, there are in fact two different fields: $V^{[3]}_{(3,0)}$ and $V^{[111]}_{(3,0)}$ from (5.1.19c). We then write

$$[3]\otimes_{S_Q}[1]=[4]+[31]+[3]+[21]+[2]\ , \tag{5.4.30a}$$
$$[111]\otimes_{S_Q}[1]=[1111]+[211]+[111]+[11]\ . \tag{5.4.30b}$$

Therefore, using (5.4.30) with (5.1.18a), the four-point functions $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V^{[3]}_{(3,0)}\rangle$ and $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V^{[111]}_{(3,0)}\rangle$ transform under S_Q symmetry as [2]+[11] in all three channels. These two four-point functions can then be built from solutions of which spectra contains only fields that can be decomposed into [2] or [11]. There are only two of these solutions, which can be obtained by requiring vanishing structure constants:

$$D^{(s)}_{(3,0)}=D^{(t)}_{(3,0)}=D^{(u)}_{(3,0)}=0\ . \tag{5.4.31}$$

In other words, there are three other solutions, in which structure constants in (5.4.31) do not vanish and will be discussed in Section 7.2.2. The two solutions that belong to the Potts CFT have the following spectra:

$\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V^{\lambda}_{(3,0)}\rangle$

λ	Spectra for s,t,u
[111]	$\mathcal{S}^{[11]}_{r\,\in\,2\mathbb{N}}\cap\mathcal{S}^{\text{odd}}$
[3]	$\mathcal{S}^{[2]}_{r\,\in\,2\mathbb{N}}\cap\mathcal{S}^{\text{even}}$

(5.4.32)

The field $V_{(3,\frac{1}{3})}$ transforms under S_Q as the irreducible representation [21]. Therefore, using (5.1.18c) and (5.1.18a), the four-point function $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V^{[21]}_{(3,\frac{1}{3})}\rangle$ can be decomposed into $[2]+[11]$ in all three channels. Similarly, we find that there are 2 out of 5

solutions which fit with such decomposition. They again satisfy (5.4.31) and come with the spectra:

$$\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{[21]}_{(3,\frac{1}{3})} \rangle$$

Solutions	Spectra	
	s	t, u
$F^{(s)}_{[11]}$	$\mathcal{S}^{[11]}_{r\,\in\,2\mathbb{N}} \cap \mathcal{S}^{\text{odd}}$	$\mathcal{S}^{[11]}_{r\,\in\,2\mathbb{N}} \cup \mathcal{S}^{[2]}_{r\,\in\,2\mathbb{N}}$
$F^{(s)}_{[2]}$	$\mathcal{S}^{[2]}_{r\,\in\,2\mathbb{N}} \cap \mathcal{S}^{\text{even}}$	

(5.4.33)

$\langle V_{(0,\frac{1}{2})} V_{(2,0)} V_{(2,0)} V_{(2,0)} \rangle$

Using (5.1.18e) and (5.1.18b), all three channels of this four-point function can be decomposed into

$$[3] + 2[21] + 2[2] + [11] + [1] \ . \tag{5.4.34}$$

That is to say S_Q symmetry predicts 7 linearly-independent solutions. However, the bootstrap approach finds 8 solutions. It turns out that only 7 out of 8 solutions have spectra which fit with the decomposition (5.4.34). The spectra of these 7 solutions are given by

(5.4.35)

Solutions	Spectra	
	s	t, u
$F^{(s)}_{[3]}$	$\mathcal{S}^{[3]} \cap \mathcal{S}^{\text{even}}$	$\mathcal{S}^{\text{Potts}} - \mathcal{S}^{\text{deg}} - \{(3,1)\}$
$F^{(s)}_{[21],\text{even}}$	$\mathcal{S}^{[21]} \cap \mathcal{S}^{\text{even}}$	
$F^{(s)}_{[2],0}$	$\mathcal{S}^{[2]} \cap \mathcal{S}^{\text{even}} - \{(2,0)\}$	
$F^{(s)}_{[2],1}$	$\mathcal{S}^{[2]} \cap \mathcal{S}^{\text{even}} - \{(2,1)\}$	
$F^{(s)}_{[1]}$	$\mathcal{S}^{[1]} \cap \mathcal{S}^{\text{even}}$	
$F^{(s)}_{[11]}$	$\mathcal{S}^{[11]} \cap \mathcal{S}^{\text{odd}}$	$\mathcal{S}^{\text{Potts}} - \mathcal{S}^{\text{deg}} - \{(3,1),(3,\pm 1/3)\}$
$F^{(s)}_{[21],\text{odd}}$	$\mathcal{S}^{[21]} \cap \mathcal{S}^{\text{odd}}$	$\mathcal{S}^{\text{Potts}} - \mathcal{S}^{\text{deg}} - \{(3,1),(2,\pm 1/2)\}$

where the degenerate fields are subtracted due to the fusion rules (4.6.5). In other words, 7 solutions in (5.4.35) can be exacted from requiring

$$D^{(s)}_{(3,1)} = D^{(t)}_{(3,1)} = D^{(u)}_{(3,1)} = 0 \ . \tag{5.4.36}$$

Vanishing structure constants in (5.4.36) do not contradict with the fusion rule (5.4.28) but are consequences of the permutation symmetry of the OPE, $V_{(2,0)} \times V_{(2,0)}$, which allows each spectrum in (5.4.35) to have either odd or even spins [25]. In this case, the crossing-symmetry equation prefers the spectrum $S^{[3]} \cap S^{\text{even}}$, over the spectrum $S^{[3]} \cap S^{\text{odd}}$. Moreover, the eighth solution comes with the spectra:

$$\mathcal{S}^{(s)} = \mathcal{S}^{\text{Potts}} \cap \mathcal{S}^{\text{odd}} - \{(2,1/2),(3,1/3)\} \ , \tag{5.4.37}$$

$$\mathcal{S}^{(t,u)} = \mathcal{S}^{\text{Potts}} - \mathcal{S}^{\text{deg}} \ . \tag{5.4.38}$$

Let us also stress here that while the fields $V_{(2,\frac{1}{2})}$ and $V_{(3,\frac{1}{3})}$ are excluded in (5.4.37), the fields $V_{(2,-\frac{1}{2})}$ and $V_{(3,-\frac{1}{3})}$ indeed appear in (5.4.37). That is to say, structure constants

in the solution with the spectra (5.4.37) and (5.4.38) do not obey the relation, $D_{(r,-s)} = D_{(r,s)}$, in contrast to the other 7 solutions in (5.4.35) where such relation always holds.

Since solutions in different channels are different choices of bases for the same space of solutions to the crossing-symmetry equation (5.2.1), one can then check that all solutions in (5.4.35) belong to the same space of solutions by numerically computing the linear relations of solutions in the s- and t- channel on the table (5.4.35),

$$F^{(t)}_{\lambda} = \sum_{\mu} \alpha_{\lambda}{}^{\mu} F^{(s)}_{\mu} \ . \tag{5.4.39}$$

We find that $\alpha_{\lambda}{}^{\mu}$ exist for any solution in (5.4.35) and do not vanish, except for $\alpha_{[11]}^{[21],\text{odd}}$ which is consistent with the t- and u- channel spectra of the solutions $F^{(s)}_{[11]}$ and $F^{(s)}_{[21],\text{odd}}$. This ensures us that the eighth solution lives in a different space of solution, and all of 7 solutions in (5.4.35) are indeed in the same space of solutions of the Potts CFT.

The four-point function $\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(4,0)} \rangle$

In this case, S_Q symmetry predicts 14 solutions, and we find 12 solutions to the crossing-symmetry equation. However, only 6 out of these 12 solutions belong to the Potts CFT.

For this example, there are 9 four-point functions from (5.1.19e). However, since the tensor products $[1] \otimes_{S_Q} [4]$, $[1] \otimes_{S_Q} [211]$, and $[1] \otimes_{S_Q} [22]$ can only be decomposed into representations with more than three boxes, which do not appear in $[1] \otimes_{S_Q} [1]$, we have

$$\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{[4]}_{(4,0)} \rangle = \langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{[211]}_{(4,0)} \rangle = \langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{[22]}_{(4,0)} \rangle = 0 \ . \tag{5.4.40}$$

The other 6 four-point functions can be decomposed into S_Q representations as follows:

$\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{\lambda}_{(4,0)} \rangle$

λ	Mul	S_Q representations in s,t,u
[3]	1	[2]
[21]	1	[2] + [11]
[2]	2	[2] + [11] + [1]
[1]	1	[2] + [11] + [1] + []
[]	1	[1]

(5.4.41)

Only 6 out of 12 solutions fit with the decompositions in (5.4.41). We extract these 6 solutions by imposing 6 constraints on 12 solutions,

$$D^{(s)}_{(3,0)} = D^{(t)}_{(3,0)} = D^{(u)}_{(3,0)} = 0 \quad \text{and} \quad D^{(s)}_{(3,\frac{1}{3})} = D^{(t)}_{(3,\frac{1}{3})} = D^{(u)}_{(3,\frac{1}{3})} = 0 \ . \tag{5.4.42}$$

The total spectra for these 6 solutions are $\mathcal{S}^{\text{Potts}}_{r\in 2\mathbb{N}}$. To single out each solution, we introduce

$$\mathcal{S}_{(r,s)} = \mathcal{S}^{\text{Potts}}_{r\in 2\mathbb{N}+6} \cup \{(r,j) | j = \pm s, \pm(s-1) \ \text{ and } \ j \in (-1,1]\} \ . \tag{5.4.43}$$

The above is equivalent to removing 5 linearly-independent structure constants with $r \leq 4$ in the spectrum $\mathcal{S}^{\text{Potts}}_{r\in 2\mathbb{N}}$. We then introduce the crossing-symmetry solutions $F^{(s)}_{(r,s)}$ which have $S_{(r,s)}$ for the s-channel spectrum and $\mathcal{S}^{\text{Potts}}_{r\in 2\mathbb{N}}$ for the t- and u-channel spectra. As

an example, let us also display the deviation of some structure constants for $F^{(s)}_{(4,\frac{1}{4})}$ at $\beta = 0.8 + 0.1i$:

		$\Delta_{\max} = 20$	$\Delta_{\max} = 40$
(r,s)	ch	deviation	deviation
$(0,\frac{1}{2})$	t	4.1×10^{-6}	2.6×10^{-14}
$(2,0)$	t	5.9×10^{-6}	6.3×10^{-14}
$(2,1)$	t	5.9×10^{-6}	6.1×10^{-14}
$(4,\pm\frac{1}{4})$	s	4.4×10^{-6}	2.3×10^{-14}
$(4,\pm\frac{3}{4})$	s	5.2×10^{-7}	7.3×10^{-14}

(5.4.44)

The 6 linearly-independent solutions in (5.4.42) are

$$F^{(s)}_{(0,\frac{1}{2})},\ F^{(s)}_{(2,0)},\ F^{(s)}_{(2,\frac{1}{2})},\ F^{(s)}_{(4,0)},\ F^{(s)}_{(4,\frac{1}{2})},\ \text{and}\ F^{(s)}_{(4,\frac{1}{4})}. \tag{5.4.45}$$

Four-point functions in (5.4.41) can then be written as linear combinations of solutions in (5.4.45) by imposing constraints on some of their structure constants:

$$\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{\lambda}_{(4,0)} \rangle$$

λ	Mul	Vanishing $D_{(r,s)}$
[3]	1	$D^{(s)}_{(0,\frac{1}{2})}, D^{(s)}_{(2,\frac{1}{2})}, D^{(s)}_{(4,\frac{1}{4})}, D^{(t,u)}_{(0,\frac{1}{2})}, D^{(t,u)}_{(2,\frac{1}{2})}$
[21]	1	$D^{(s)}_{(0,\frac{1}{2})}, D^{(t,u)}_{(0,\frac{1}{2})}$
[2]	2	–
[1]	1	–
[]	1	$D^{(s)}_{(2,0)}, D^{(s)}_{(2,\frac{1}{2})}, D^{(s)}_{(4,\frac{1}{4})}, D^{(t,u)}_{(2,0)}, D^{(t,u)}_{(2,\frac{1}{2})}$

(5.4.46)

The four-point functions $\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{[3]}_{(4,0)} \rangle$ and $\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{[]}_{(4,0)} \rangle$ can then be completely fixed up to a normalization factor because they are linear combinations of 6 solutions in (5.4.45) in which we require 5 vanishing structure constants, while the linear combination of solutions for the four-point function $\langle V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V_{(0,\frac{1}{2})} V^{[21]}_{(4,0)} \rangle$ has 3 unfixed coefficients since we only need 2 structure constants to vanish. For the cases of $\lambda = [2]$ and [1], the three linear combinations are linearly dependent and cannot be fixed with our current method.

Logarithmic blocks and precision

Using the conformal bootstrap, we have provided very strong evidence that the logarithmic representations $\mathcal{W}^{\kappa^-}_{(r,s)}$ indeed appear in spectra of the Potts and $O(n)$ CFTs. In particular, the logarithmic blocks $\mathcal{G}^{-}_{(r,s)}$ in (4.5.15) are the crucial ingredients which lead to numerical outputs at very high precision. For instance, the precision for structure constants on the tables in (5.3.16)-(5.3.18) are around 20 digits. Without the correct logarithmic blocks, such precision can never be obtained. For example, the authors of [15] bootstrapped the four-point connectivities in (1.3.2) by neglecting logarithmic blocks. Their results are only accurate up to around 5-10 digits whereas the precision of our results for the four-point

connectivities is around 30 digits, shown in (5.4.15). With our logarithmic blocks, we can actually compute four-point functions of arbitrary primary fields in both CFT at arbitrary precision by increasing the cutoff Δ_{max}.

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CHAPTER 6

Rational limits

We begin with a quick review on the structure of Verma modules at rational central charge and minimal models, then discuss the less well-known logarithmic minimal models of [35] which shall be compared with the results of taking rational limits of the generalized minimal models.

6.1 Rational central charge

Recalling (1.2.1), the central charge c with rational values can then be conveniently represented by choosing the following parametrization:

$$\boxed{\beta^2 = \frac{q}{p} \Longrightarrow c = c_{p,q} = 1 - \frac{6(p-q)^2}{pq}} \quad \text{for} \quad \frac{q}{p} \in \mathbb{Q} \ , \tag{6.1.1}$$

where we also allow $pq < 0$. Therefore, the central charge $c = c_{p,q}$ takes values of the dense line in the interval: $(-\infty, 1] \cup [25, \infty)$. Moreover, it is obvious that the central charge $c_{p,q}$ is symmetric under $p \leftrightarrow q$. For example, we have

$$c_{2,1} = c_{1,2} = -2 \ , \tag{6.1.2a}$$
$$c_{3,2} = c_{2,3} = 0 \ , \tag{6.1.2b}$$
$$c_{4,3} = c_{3,4} = \frac{1}{2} \ . \tag{6.1.2c}$$

Kac indices and Verma modules

The structure of Verma modules of the Virasoro algebra at rational central charge was studied in great details by the authors of [51]. Here, we only discuss it very briefly. Rational values of β^2 results in another reflection of the Kac indices in (2.3.8), in addition to $\Delta_{(r,s)} = \Delta_{(-r,-s)}$. More precisely, at the central charge $c = c_{p,q}$, the momenta $P_{(r,s)}$ and the conformal dimensions $\Delta_{(r,s)}$ obey the relation:

$$P_{(r,s)} = \pm P_{(p-r,q-s)} \Longleftrightarrow \Delta_{(r,s)} = \Delta_{(p-r,q-s)} \ , \tag{6.1.3}$$

which hold for any value of r and s. Consequently, for $(p-r)(q-s) \in \mathbb{N}^*$, the coincidence of the conformal dimensions in (6.1.3) led us to an extra null vector in the Verma module

$\mathcal{V}_{\Delta_{(r,s)}}$. This additional null vector has the conformal dimension:

$$\Delta_{(p-r,q-s)} + (p-r)(q-s) = \Delta_{(r,s)} + (p-r)(q-s) \ . \tag{6.1.4}$$

With (6.1.4), the Verma module $\mathcal{V}_{\Delta_{(r,s)}}$ at the central charge $c_{p,q}$ therefore has the following non-trivial subrepresentations:

$$\mathcal{V}_{\Delta_{(r,-s)}} \quad \text{and} \quad \mathcal{V}_{\Delta_{(2p-r,s)}} \ . \tag{6.1.5}$$

Using the reflection relations (6.1.3), these two subrepresentations then generate infinitely many non-trivial subrepresentations. Hence, $\mathcal{V}_{\Delta_{(r,s)}}$ at the central charge $c_{p,q}$ comes with infinitely many null vectors generated by the highest weight states of (6.1.5). To build irreducible representations from $\mathcal{V}_{\Delta_{(r,s)}}$, we require that all of these null vector vanish. Let us then introduce the degenerate representation $\mathcal{M}^{(p,q)}_{(r,s)}$ at the central charge $c_{p,q}$ as follows:

$$\mathcal{M}^{(p,q)}_{(r,s)} = \mathcal{R}_{(r,s)}\big|_{c=c_{p,q}} = \frac{\mathcal{V}_{\Delta_{(r,s)}}}{\mathcal{V}_{\Delta_{(r,-s)}} + \mathcal{V}_{\Delta_{(2p-r,s)}}} \ , \tag{6.1.6}$$

where the sum in the denominator of (6.1.6) is not a direct sum since the two representations in (6.1.5) share non-trivial subrepresentations. For more details, see [51, 4]. Let us also stress here that, in terms of representation theory, the degenerate representations $\mathcal{M}^{(p,q)}_{(r,s)}$ are well-defined for any value of r and s at the central charge $c_{p,q}$ in (6.1.1).

6.2 Minimal models

Minimal models are rational CFTs whose spectra are made of finite numbers of the degenerate fields. For simplicity, we restrict ourselves to the case of diagonal minimal models, namely *the A-series minimal models* while their non-diagonal counterparts are known as *the D-series minimal models* and *the E-series minimal models*. These three families arise as an ADE classification of minimal models. In particular, their torus-partition functions are in one-to-one correspondence with simply laced Dynkin diagrams [4].

The A-series minimal models can be characterized by their central charge: we write (p,q)-*minimal model* for a diagonal minimal model with the central charge $c_{p,q}$. The spectra of the (p,q)-minimal models are given by

$$\mathcal{S}^{\text{A-series}}_{p,q} = \frac{1}{2}\bigoplus_{i=1}^{p-1}\bigoplus_{j=1}^{q-1} \mathcal{M}^{(p,q)}_{(i,j)} \otimes \bar{\mathcal{M}}^{(p,q)}_{(i,j)} \quad \text{for} \quad p,q \in \mathbb{N}+2 \quad \text{and} \quad p \wedge q = 1 \ , \tag{6.2.1}$$

where the factor of $\frac{1}{2}$ is for avoiding overcounting due to the symmetry:

$$\mathcal{M}^{(p,q)}_{(r,s)} = \mathcal{M}^{(p,q)}_{(p-r,q-s)} \ . \tag{6.2.2}$$

Notice that the A-series minimals are invariant under exchanging p and q since the spectra (6.2.1) remain unchanged under such permutation. Moreover, the case of $|p-q| = 1$ is known as the unitary series, wherein the minimal models are unitary. However, the spectra (6.2.1) with $|p-q| \neq 1$ contain fields with negative dimensions and are therefore non-unitary. The fusion rules of the (p,q)-minimal model can be obtained by applying the identification (6.1.3) to the degenerate fusion rules (4.1.10). We consider the intersection of the following products,

$$V^D_{\langle r,s\rangle} \times V^D_{\langle r',s'\rangle} \ , \tag{6.2.3}$$

$$V^D_{\langle p-r,q-s\rangle} \times V^D_{\langle r',s'\rangle} \ , \tag{6.2.4}$$

$$V^D_{\langle p-r,q-s\rangle} \times V^D_{\langle p-r',q-s'\rangle} \ , \tag{6.2.5}$$

which brings us to

$$V^D_{\langle r,s\rangle} \times V^D_{\langle r',s'\rangle} = \sum_{i\overset{2}{=}|r-r'|+1}^{\min(2p-r-r',r+r')-1} \sum_{i\overset{2}{=}|s-s'|+1}^{\min(2q-s-s',s+s')-1} V^D_{\langle i,j\rangle} \ . \tag{6.2.6}$$

We immediately see that the degenerate fields in (6.2.1) are closed under the fusion rules (6.2.6). Likewise to the generalized minimal models, OPE coefficients of fields in (6.2.6) can be determine completely by using the BPZ equations, the crossing-symmetry equation, and the single-valuedness. However, minimal models are not only consistent on the Riemann sphere but also on higher-genus Riemann surfaces. Furthermore, while the A-series minimal models can be defined without Lagrangian by using only the spectrum (6.2.1) and the fusion rules (6.2.6), it is also possible to write down effective Lagrangians corresponding to (6.2.1) by using the Landau-Ginzburg description [14].

6.2.1 Examples

We now discuss the first few examples of (p, q)-minimal models.

$(p, q) = (3, 2)$: A trivial example

In this case, we have vanishing central charge $c_{2,3} = 0$. The spectrum of the $(3, 2)$-minimal model only contains the identity field. In particular, we have

$$V^D_{\langle 1,1\rangle} = V^D_{\langle 1,2\rangle} = \mathbb{1} \ , \tag{6.2.7}$$

The fusion rule of the above is simply

$$\mathbb{1} \times \mathbb{1} = \mathbb{1} \ , \tag{6.2.8}$$

whose OPE coefficient is one. Therefore, this case is a trivial CFT

$(p, q) = (4, 3)$: The Ising model

This minimal model is one the most well-known minimal models. The $(4, 3)$-minimal model comes with the central charge $c_{4,3} = \frac{1}{2}$ and is well-known to describe the spin representation of the critical two-dimensional Ising model.

Now consider the spectrum (6.2.1) at $(p, q) = (4, 3)$. Taking into account the factor $\frac{1}{2}$ in (6.2.1) we have 3 independent fields in this case. These 3 fields correspond to the spin operator σ, the energy operator ϵ, and the identity field $\mathbb{1}$. Using the Kac indices, we write them as the following:

$$V^D_{\langle 1,1\rangle} = V^D_{\langle 2,3\rangle} = \mathbb{1} \ , \tag{6.2.9a}$$
$$V^D_{\langle 1,2\rangle} = V^D_{\langle 3,2\rangle} = \epsilon \ , \tag{6.2.9b}$$
$$V^D_{\langle 2,1\rangle} = V^D_{\langle 2,2\rangle} = \sigma \ . \tag{6.2.9c}$$

Let us also display the conformal dimensions $\Delta_{(r,s)}$ of primary fields in (6.2.9) on the so-called Kac tables:

s \ r	1	2	3
1	0	$\frac{1}{16}$	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{1}{16}$	0

(6.2.10)

Using (6.2.6), we can write the fusion rules of fields in (6.2.9) as follows:

$$\epsilon \times \epsilon = \mathbb{1} \ , \tag{6.2.11a}$$
$$\epsilon \times \sigma = \sigma \ , \tag{6.2.11b}$$
$$\sigma \times \sigma = \mathbb{1} + \epsilon \ . \tag{6.2.11c}$$

The only OPE coefficient, which is not one, is

$$C_{\sigma\sigma\epsilon} = \frac{1}{2} \ . \tag{6.2.12}$$

Let us also stress here that the $(4,3)$-minimal model does not coincide with the limits $Q \to 2$ and $n \to 1$ of the Potts CFT and $O(n)$ CFT, which result in much more complicated CFTs describing other observables of the Ising model. For instance, while we have the coincidence $\Delta_{(2,1)} = \Delta_{(0,\frac{1}{2})}$ at $\beta^2 = \frac{3}{4}$, the four-point function $\langle\sigma\sigma\sigma\sigma\rangle$ does not coincide with the four-point connectivities at $Q = 2$ in (1.3.2). However, they satisfy a simple relation [52]:

$$\langle\sigma\sigma\sigma\sigma\rangle = (P_{aaaa} + P_{abab} + P_{aabb} + P_{abab})\Big|_{Q=2} \ , \tag{6.2.13}$$

where the authors of [52] wrote down the above relation by using arguments from the lattice model. From the view of CFTs, the relation (6.2.13) is highly non-trivial. For example, we do not yet know how to compute the limit $Q \to 2$ of each four-point connectivity in (1.3.2) explicitly but we expect them to be a sum of infinitely many interchiral blocks. From (6.2.11c), the four-point function $\langle\sigma\sigma\sigma\sigma\rangle$ is however simply a sum of two degenerate conformal blocks corresponding to $\mathbb{1}$ and ϵ. Let us also point it out that the limits $n \to 1$ or $Q \to 2$ of the partition functions (5.1.1) and (5.1.2) indeed reproduce the partition function of the $(4,3)$-minimal model [14]. However, the resulting spectrum does not have a smooth limit and therefore are not the spectrum of the $(4,3)$-minimal model.

6.3 Logarithmic minimal models

Let us now consider an extension of minimal models, known as the logarithmic minimal models proposed by Mathieu and Ridout in [35], first appearing in [53]. Their general idea is to relax the requirement on the irreducibility of Verma modules by allowing one of the two null vectors in (6.1.5) to be non-vanishing. Consequently, spectra of these logarithmic minimal models contain infinitely many fields living beyond the Kac table. There are also other extensions of minimal models, which are known as logarithmic minimal models as well, for instance [54, 55, 56]. However, we do not discuss them here.

6.3.1 Spectra of the models

Likewise to minimal models, logarithmic minimal models are characterized by their central charge. The $(2,p)$-logarithmic minimal model is a chiral CFT that has the central charge $c = c_{2,p}$ and comes with the spectrum:

$$\mathcal{S}^{(2,p)}_{\text{LMM}} = \bigoplus_{i=1}^{\infty} \mathcal{M}^{(2,p)}_{(1,pi)} \oplus \bigoplus_{i=1}^{\infty} \bigoplus_{j=1}^{p-1} \mathcal{K}^{(p)}_{(i,j)} \ . \tag{6.3.1}$$

More specifically, these logarithmic minimal models belong to a class of boundary CFTs, which are expected to live on the upper-half of the complex plane and therefore contain

only one copy of the Virasoro algebra. However, we do not know yet their corresponding bulk CFTs, whose existence will be only speculated in the next section. Furthermore, it was shown in [35] that one can also add the representations $\mathcal{M}^{(2,p)}_{(1,i)}$ for $i < p$ and still finds consistent CFTs for $p \in \{3, 5\}$. However, we shall only restrict ourselves to the spectra (6.3.1).

At the central charge $c_{2,p}$, null vectors in the degenerate representations $\mathcal{M}^{(2,p)}_{(1,pi)}$ come with the chiral degenerate fields $v_{\langle 1,pi\rangle}$, which have the conformal dimension: $\Delta_{(1,pi)}$. From (6.1.5), infinitely many null vectors in $\mathcal{M}^{(2,p)}_{(1,pi)}$ are generated by the null descendant: $\mathcal{L}_{(1,pi)} v_{\langle 1,pi\rangle} = 0$. Therefore, we can write

$$\mathcal{M}^{(2,p)}_{(1,pi)} = \frac{\mathcal{V}_{\Delta_{(1,pi)}}}{\mathcal{V}_{\Delta_{(1,-pi)}}} \ . \tag{6.3.2}$$

The logarithmic representation $\mathcal{K}^{(p)}_{(i,j)}$ is *the chiral staggered module* at the central charge $c = c_{2,p}$ and will now be discussed.

The staggered modules $\mathcal{K}^{(p)}_{(i,j)}$

Since the staggered modules $\mathcal{K}^{(p)}_{(i,j)}$ of [35] are usually introduced algebraically without relying on the use of derivatives with respect to the conformal dimensions, let us then alternatively describe $\mathcal{K}^{(p)}_{(i,j)}$ by using these derivatives. The results of two approaches are of course exactly the same. The module $\mathcal{K}^{(p)}_{(i,j)}$ is defined at the central charge $c_{2,p}$ and can be generated by the following combination of derivatives of null fields,

$$w_{(1,pi+j)} = \kappa^{-}_{(2i-1,j)} v'_{\Delta_{(1,pi+j)}} + (1 - \kappa^{-}_{(2i-1,j)}) \mathcal{L}_{(2i-1,j)} v'_{\Delta_{(1,pi-j)}} \ , \tag{6.3.3}$$

where we write $v_{\Delta_{(1,pi\pm j)}}$ for chiral primary fields with the conformal dimensions $\Delta_{(1,pi\pm j)}$, and $\kappa^{-}_{(2i-1,j)}$, to be discussed below, is related to the logarithmic coupling of $\mathcal{K}^{(p)}_{(i,j)}$. In fact, the chiral primary field $v_{\Delta_{(1,pi+j)}}$ is a null descendant of $v_{\Delta_{(1,pi-j)}}$ since we have the coincidence:

$$\Delta_{(1,pi+j)} = \Delta_{(2i-1,-j)} \quad \text{and} \quad \Delta_{(1,pi+j)} - \Delta_{(1,pi-j)} = (2i-1)j \ . \tag{6.3.4}$$

From [35], $v_{\Delta_{(1,pi-j)}}$ has the non-vanishing null descendant:

$$\mathcal{L}_{(2i-1,j)} v_{\Delta_{(1,pi-j)}} = v_{\Delta_{(1,pi+j)}} \neq 0 \ , \tag{6.3.5}$$

whereas we require

$$\mathcal{L}_{(1,pi-j)} v_{\Delta_{(1,pi-j)}} = 0 \ . \tag{6.3.6}$$

The two null fields in (6.3.5) and (6.3.6) then generate infinitely many null vectors in the staggered modules $\mathcal{K}^{(p)}_{(i,j)}$. With (6.3.5) and (6.3.3), the logarithmic field $w_{(1,pi+j)}$ then leads to a second-rank Jordan block of L_0, in which the null field $v_{\Delta_{(1,pi+j)}}$ is its Jordan partner. From (6.3.3), it is easy to see that $w_{(1,pi+j)}$ satisfies the relation:

$$\mathcal{L}^{\dagger}_{(2i-1,j)} \mathcal{L}_{(2i-1,j)} w_{(1,pi+j)} = \gamma_{(2i-1,j)} v_{\Delta(1,pi+j)} \ , \tag{6.3.7}$$

where the logarithmic coupling $\gamma_{(2i-1,j)}$ is given by $\gamma_{(2i-1,j)} = \rho_{(2i-1,j)} \kappa^{-}_{(2i-1,j)}\big|_{\beta^2 = \frac{p}{2}}$. The factor $\kappa^{-}_{(2i-1,j)}$ is given in (4.5.9), and $\rho_{(2i-1,j)}$ was defined in (3.5.23b) and arises from the

commutation relation between $\mathcal{L}^{\dagger}_{(2i-1,j)}$ and $\mathcal{L}_{(2i-1,j)}$. For $i=1$, the coupling $\gamma_{(1,j)}$ takes a simple form:

$$\gamma_{(1,j)} = \frac{4(j-1)!^2}{p^{2j}}(-1)^j \frac{p}{8} \prod_{i=-j}^{j-1}(p+2i) \ . \tag{6.3.8}$$

Let us now compare the couplings in (6.3.8) with the coupling $\hat{\beta}_{1,p+j}$ of [10], we need to beware of the following subtleties. We always normalize the operator $\mathcal{L}_{(r,s)}$ such that $\mathcal{L}_{(r,s)} = L^{rs}_{-1} + \dots$ while the authors of [35] chose the normalization: $\mathcal{L}_{(r,s)} = L_{-rs} + \dots$. Secondly, the authors of [35] label their couplings by the indices of vanishing null vectors whereas we denote them by the indices of non-vanishing null vectors. Taking into account these differences, we find

$$\hat{\beta}_{1,p+j} = \frac{p^{2j}}{4(j-1)!^2}\gamma_{(1,j)} \ , \tag{6.3.9}$$

where different normalizations of $\mathcal{L}_{(r,s)}$ lead to a non-trivial factor in front of $\gamma_{(1,j)}$ in the above. The two couplings in (6.3.9) nevertheless describe the same representation $\mathcal{K}^{(p)}_{(1,j)}$. We have also done similar comparisons in some examples for $\mathcal{K}^{(p)}_{(i,j)}$ with $i>1$. For example at $p=3$, we have

Coupling from eq. (3.9) in [35]	Normalization factor	$\gamma^{-}_{(r,s)}$
$\beta_{1,4} = -\frac{1}{2}$	1	$\gamma^{-}_{(1,1)} = -\frac{1}{2}$
$\beta_{1,5} = -\frac{5}{8}$	$\frac{4}{9}$	$\gamma^{-}_{(1,2)} = -\frac{5}{18}$
$\beta_{1,7} = -\frac{35}{3}$	36	$\gamma^{-}_{(3,1)} = -420$

(6.3.10)

From (6.3.6) and (6.3.5), the staggered module $\mathcal{K}^{(p)}_{(i,j)}$ therefore has $\frac{\mathcal{V}_{\Delta_{(1,pi+j)}}}{\mathcal{V}_{\Delta_{(1,-pi-j)}}}$ as a subrepresentation, and the quotient of $\mathcal{K}^{(p)}_{(i,j)}$ with this subrepresentation is given by $\frac{\mathcal{V}_{\Delta_{(1,pi-j)}}}{\mathcal{V}_{\Delta_{(1,-pi+j)}}}$, vice versa. We summarize this sequence in the figure 6.1. While the staggered module $\mathcal{K}^{(p)}_{(i,j)}$ is indecomposable but reducible as representation of the Virasoro algebra, $\mathcal{K}^{(p)}_{(i,j)}$ is decomposable as a vector space. That is to say we can write the vector space of $\mathcal{K}^{(p)}_{(i,j)}$ as $\frac{\mathcal{V}_{\Delta_{(1,pi-j)}}}{\mathcal{V}_{\Delta_{(1,-pi+j)}}} \oplus \frac{\mathcal{V}_{\Delta_{(1,pi+j)}}}{\mathcal{V}_{\Delta_{(1,-pi-j)}}}$.

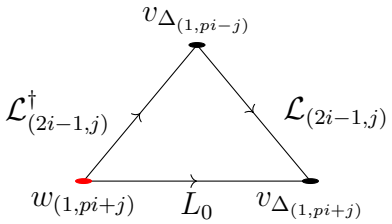

Figure 6.1: non-vanishing null vectors in the chiral stagger modules $\mathcal{K}^{(p)}_{(i,j)}$

Computing the fusion rules

The fusion rules of the representations in (6.3.1) can be computed by using the so-called Nahm-Gaberdiel-Kausch alogrithm [37], which allows us to determine how the Virasoro algebra acts on the vector space of these fusion products. The authors of [35] used this algorithm and proposed simple rules of writing the fusion rules of (6.3.1) as follows:

1. Replace the staggered modules $\mathcal{K}^{(p)}_{(i,j)}$ with the degenerate representations $\mathcal{R}_{(1,pi-j)} \oplus \mathcal{R}_{(1,pi+j)}$ at generic central charge, while also lifting $\mathcal{M}^{(2,p)}_{(1,pi)}$ to $\mathcal{R}_{(1,pi)}$. Then compute the resulting products by using the degenerate fusion rules (4.1.10). For instance, we write

$$\begin{aligned}\mathcal{M}^{(2,3)}_{(1,3)} \otimes \mathcal{K}^{(3)}_{(1,2)} &\to \mathcal{R}_{(1,3)} \otimes (\mathcal{R}_{(1,1)} \oplus \mathcal{R}_{(1,5)}) \ , \\ &= 2\mathcal{R}_{(1,3)} \oplus \mathcal{R}_{(1,5)} \oplus \mathcal{R}_{(1,7)} \ . \end{aligned} \tag{6.3.11}$$

   Let us display one more example:

$$\begin{aligned}\mathcal{K}^{(3)}_{(1,2)} \otimes \mathcal{K}^{(3)}_{(1,2)} &\to (\mathcal{R}_{(1,1)} \oplus \mathcal{R}_{(1,5)}) \otimes (\mathcal{R}_{(1,1)} \oplus \mathcal{R}_{(1,5)}) \ , \\ &= 2\mathcal{R}_{(1,1)} \oplus 3\mathcal{R}_{(1,5)} \oplus \mathcal{R}_{(1,7)} \oplus \mathcal{R}_{(1,9)} \ . \end{aligned} \tag{6.3.12}$$

2. Afterwards, replace the combination $\mathcal{R}_{(1,pi-j)} \oplus \mathcal{R}_{(1,pi-j)}$ with the staggered modules $\mathcal{K}^{(p)}_{(i,j)}$ and rewrite $\mathcal{R}_{(1,pi)}$ as $\mathcal{M}^{(2,p)}_{(1,pi)}$. There is always only one way of such replacements. For example, applying these rules to (6.3.12) and (6.3.11) gives us the fusion rules:

$$\mathcal{M}^{(2,3)}_{(1,3)} \otimes \mathcal{K}^{(3)}_{(1,2)} = 2\mathcal{M}^{(2,3)}_{(1,3)} \oplus \mathcal{K}^{(3)}_{(2,1)} \ , \tag{6.3.13a}$$

$$\mathcal{K}^{(3)}_{(1,2)} \otimes \mathcal{K}^{(3)}_{(1,2)} = 2\mathcal{K}^{(3)}_{(1,2)} \oplus \mathcal{K}^{(3)}_{(2,1)} \oplus \mathcal{M}^{(2,3)}_{(1,9)} \ . \tag{6.3.13b}$$

It has been checked in [35] extensively that these rules always always agree with direct computations from the Nahm-Gaberdiel-Kausch algorithm.

6.3.2 Example at $c = 0$

We consider the simplest case of (6.3.1): the $(2,3)$-logarithmic minimal model, which comes with vanishing central charge. This model was actually the first of its kind and is expected to underlie some observables of the continuum limit of percolation theory on the upper-half plane [53].

Let us start by displaying some conformal dimensions in the spectrum of this logarithmic minimal model,

$$\Delta_{(r,s)} = \frac{(3s-2r)^2-1}{24} \qquad \Longrightarrow \qquad \begin{array}{c|cccccccc} r \backslash s & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 1 & 0 & 0 & \frac{1}{3} & 1 & 2 & \frac{10}{3} & 5 & 7 \\ 2 & \frac{5}{8} & \frac{1}{8} & -\frac{1}{24} & \frac{1}{8} & \frac{5}{8} & \frac{35}{24} & \frac{21}{8} & \frac{33}{8} \end{array} \tag{6.3.14}$$

Therefore, we see that $\Delta_{(1,1)} = \Delta_{(1,2)} = 0$. However, the chiral primary fields $v_{\Delta_{(1,1)}}$ and $v_{\Delta_{(1,2)}}$ do not coincide since $v_{\Delta_{(1,1)}}$ belong to the staggered modules $\mathcal{K}^{(3)}_{(1,2)}$ whereas $v_{\Delta_{(1,2)}}$ belong to $\mathcal{K}^{(3)}_{(1,1)}$. In other words, the symmetry of the Kac indices in (6.1.3) no longer holds at the level of representations in logarithmic minimal models, in contrast to (6.2.2) in minimal models. To see which one of $v_{\Delta_{(1,1)}}$ and $v_{\Delta_{(1,2)}}$ is the identity field, we consider their null descendants at level one. Using (6.3.5) and (6.3.6), we find

$$L_{-1} v_{\Delta_{(1,1)}} = 0 \quad \text{but} \quad L_{-1} v_{\Delta_{(1,2)}} \neq 0 \ . \tag{6.3.15}$$

Since the identity field is required to be translational invariant, $v_{\Delta_{(1,1)}}$ is then the identity field and $\mathcal{K}^{(3)}_{(1,2)}$ is the vacuum module of the $(2,3)$-logarithmic minimal model.

Fusion rules

Let us now write down the fusion rules for the $(2,3)$-logarithmic minimal model explicitly by using the rules discussed in the previous subsection. It is convenient to first introduce the following notations:

$$\mathcal{B}_k = \mathcal{M}^{(3,2)}_{(1,3k)} \ , \tag{6.3.16a}$$

$$\mathcal{P}_k = \mathcal{K}^{(3)}_{(k,1)} \ , \tag{6.3.16b}$$

$$\mathcal{Q}_k = \mathcal{K}^{(3)}_{(k,2)} \ . \tag{6.3.16c}$$

These three families of representations have the following fusion products:

$$\mathcal{B}_{k_1} \otimes \mathcal{B}_{k_2} = \bigoplus_{k \overset{2}{=} |k_1-k_2|+1}^{k_1+k_2-1} (\mathcal{B}_k \oplus \mathcal{Q}_k) \ , \tag{6.3.17a}$$

$$\mathcal{B}_{k_1} \otimes \mathcal{P}_{k_2} = \mathcal{B}_{|k_1-k_2|} \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|+2}^{k_1+k_2-2} 2\mathcal{B}_k \oplus \mathcal{B}_{k_1+k_2} \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|+1}^{k_1+k_2-1} 2\mathcal{P}_k \ , \tag{6.3.17b}$$

$$\mathcal{B}_{k_1} \otimes \mathcal{Q}_{k_2} = \mathcal{P}_{|k_1-k_2|} \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|+2}^{k_1+k_2-2} 2\mathcal{P}_k \oplus \mathcal{P}_{k_1+k_2} \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|+1}^{k_1+k_2-1} 2\mathcal{B}_k \ , \tag{6.3.17c}$$

$$\mathcal{P}_{k_1} \otimes \mathcal{P}_{k_2} = \mathcal{P}_{|k_1-k_2|} \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|+2}^{k_1+k_2-2} 2\mathcal{P}_k \oplus \mathcal{P}_{k_1+k_2} \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|+1}^{k_1+k_2-1} (2\mathcal{Q}_k \oplus 4\mathcal{B}_k) \ , \tag{6.3.17d}$$

$$\mathcal{P}_{k_1} \otimes \mathcal{Q}_{k_2} = \mathcal{Q}_{|k_1-k_2|} \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|+2}^{k_1+k_2-2} 2\mathcal{Q}_k \oplus \mathcal{Q}_{k_1+k_2} \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|}^{k_1+k_2} 2\mathcal{B}_k \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|+1}^{k_1+k_2-1} 2\mathcal{P}_k \ . \tag{6.3.17e}$$

For sufficiently large k_1 and k_2, we can write

$$\begin{aligned} \mathcal{Q}_{k_1} \otimes \mathcal{Q}_{k_2} = \mathcal{B}_{|k_1-k_2|-1} \oplus 3\mathcal{B}_{|k_1-k_2|+1} \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|+3}^{k_1+k_2-3} 4\mathcal{B}_k \oplus 3\mathcal{B}_{k_1+k_2-1} \oplus \mathcal{B}_{k_1+k_2+1} \\ \oplus \mathcal{P}_{|k_1-k_2|} \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|+2}^{k_1+k_2-2} 2\mathcal{P}_k \oplus \mathcal{P}_{k_1+k_2} \oplus \bigoplus_{k \overset{2}{=} |k_1-k_2|+1}^{k_1+k_2-1} 2\mathcal{Q}_k \end{aligned} \tag{6.3.18}$$

Let us also write down explicitly examples of these complicated fusion rules,

$$\mathcal{B}_1 \otimes \mathcal{B}_1 = \mathcal{B}_1 \oplus \mathcal{Q}_1 \ , \tag{6.3.19a}$$

$$\mathcal{B}_1 \otimes \mathcal{P}_1 = \mathcal{B}_2 \oplus 2\mathcal{P}_1 \ , \tag{6.3.19b}$$

$$\mathcal{B}_1 \otimes \mathcal{Q}_1 = \mathcal{P}_2 \oplus 2\mathcal{B}_1 \ , \tag{6.3.19c}$$

$$\mathcal{P}_1 \otimes \mathcal{P}_1 = \mathcal{P}_2 \oplus 2\mathcal{Q}_1 \oplus 4\mathcal{B}_1 \ , \tag{6.3.19d}$$

$$\mathcal{P}_1 \otimes \mathcal{Q}_1 = 2\mathcal{B}_2 \oplus 2\mathcal{P}_1 \oplus \mathcal{Q}_1 \ , \tag{6.3.19e}$$

$$\mathcal{Q}_1 \otimes \mathcal{Q}_1 = \mathcal{B}_1 \oplus \mathcal{B}_3 \oplus \mathcal{P}_2 \oplus 2\mathcal{Q}_1 \ . \tag{6.3.19f}$$

6.4 Rational limits of generalized minimal models

It is now tempting to ask whether there exist a non-chiral extension of the $(2,p)$-logarithmic minimal models, which live on the Riemann sphere. We refer to such non-chiral extension

as *bulk-logarithmic minimal models*. We do not yet have a precise definition for bulk-logarithmic minimal models, but we expect bulk-logartihmic minimal models to be an extension of minimal models on the Riemann sphere, wherein reducible Verma modules are allowed, that is to say we allow some null vectors to be non-vanishing.

Building logarithmic CFTs in the bulk is however a very complicated problem due to constraints from the single-valuedness [37]. Therefore, we attempt to find bulk logarithmic minimal models by taking rational limits of some completely solved CFTs at generic central charge, whose spectra are characterized in a similar way as (6.3.1), since rational limits of single-valued objects remain single-valued. The natural start is then a subsector of generalized minimal models with the following spectrum:

$$\mathcal{S}_{\text{sub}}^{\text{GMM}} = \bigoplus_{i=1}^{\infty} \mathcal{R}^{D}_{(1,s)} \ . \tag{6.4.1}$$

It is easy to see that the above spectrum come with the same Kac indices as in (6.3.1). In particular, we want to know whether the the following limit leads to any consistent CFT,

$$\lim_{\beta^2 \to \frac{p}{2}} \mathcal{S}_{\text{sub}}^{\text{GMM}} = ? \tag{6.4.2}$$

More precisely, taking rational limits of this generalized minimal model amounts to knowing how the degenerate fields in (6.4.1) and their fusion rules behave under the limits. To answer this question, we consider the limit $\beta^2 \to \frac{p}{2}$ in the s-channel of the four-point functions:

$$\lim_{\beta^2 \to \frac{p}{2}} \langle V^{D}_{\langle 1,s\rangle} V^{D}_{\langle 1,s'\rangle} V^{D}_{\langle 1,s\rangle} V^{D}_{\langle 1,s'\rangle} \rangle \ , \tag{6.4.3}$$

which allows us to write down the fusion rules:

$$\lim_{\beta^2 \to \frac{p}{2}} \mathcal{R}^{D}_{(1,s)} \times \mathcal{R}^{D}_{(1,s')} \ . \tag{6.4.4}$$

As we shall see in the next subsection, it turns out that spectra of the resulting four-point functions in (6.4.3) contain the non-chiral logarithmic representations $\lim_{\beta^2 \to \frac{p}{2}} \mathcal{W}^{\kappa^-}_{(2i-1,j)}$ whose chiral projections coincide with the chiral staggered modules $\mathcal{K}^{(p)}_{(i,j)}$. In other words, we find that neglecting the right-moving quantities in the resulting representations $\lim_{\beta^2 \to \frac{p}{2}} \mathcal{W}^{\kappa^-}_{(2i-1,j)}$ gives us $\mathcal{K}^{(p)}_{(i,j)}$. This suggests that the limit $\beta^2 \to \frac{p}{2}$ of the generalized minimal model (6.4.1) could be a good candidate for study non-chiral extensions of the $(2,p)$-logarithmic minimal models. Let us now discuss results from computing (6.4.3).

6.4.1 Emergence of the logarithmic blocks

We consider the limit $\beta^2 \to \frac{p}{2}$ of the four-point function $\langle V^{D}_{\langle 1,ps\rangle} V^{D}_{\langle 1,ps'\rangle} V^{D}_{\langle 1,ps\rangle} V^{D}_{\langle 1,ps'\rangle} \rangle$ where our logarithmic blocks emerge. Using (4.1.10), the s-channel decomposition of this four-point function reads

$$\lim_{\beta^2 \to \frac{p}{2}} \langle V^{D}_{\langle 1,ps\rangle} V^{D}_{\langle 1,ps'\rangle} V^{D}_{\langle 1,ps\rangle} V^{D}_{\langle 1,ps'\rangle} \rangle = \lim_{\beta^2 \to \frac{p}{2}} \sum_{j=|s-s'|+1}^{s+s'+1} D_{(1,pj)} |\mathcal{F}_{P_{(1,pj)}}|^2 \ , \tag{6.4.5}$$

where we label conformal blocks by their momenta, and four-point structure constants in (6.4.5) can be completely determined by using the degenerate-shift equation (4.3.6) with

the normalization $D_{(1,1)} = 1$. Nevertheless, computing the limit of the right-hand side of (6.4.5) is in general very complicated due to zeroes and poles at rational values of β^2 from both the conformal blocks and the structure constants. We bypass this difficulty by studying the limit in (6.4.5) numerically for many examples, whose results strongly suggest us that the resulting four-point function in (6.4.5) is finite and could also contain the logarithmic blocks $\mathcal{G}^-_{(1,j)}$. To see how these logarithmic blocks emerge, recall the coincidence of momenta at $\beta^2 = \frac{p}{2}$,

$$P_{(1,pi\mp j)} = -P_{(2i-1,\pm j)} \ , \tag{6.4.6}$$

where i are integers and $j \in \{1, \dots, p-1\}$. Now it is easy to check that the following residues in (3.4.15) of the conformal blocks $\mathcal{F}_{P_{(1,pi-j)}}$ in (6.4.5) satisfy the following conditions:

$$R_{1,pi\pm j} = 0 \ , \tag{6.4.7a}$$
$$R_{2i-1,j} \neq 0 \ . \tag{6.4.7b}$$

Since the conformal blocks $\mathcal{F}_{P_{(1,pi-j)}}$ depend quadratically on the momenta $P_{(1,pi-j)}$, we can write

$$\begin{aligned}
\lim_{\beta^2\to\frac{p}{2}} |\mathcal{F}_{P_{(1,pi-j)}}|^2 &= \lim_{\beta^2\to\frac{p}{2}} |\mathcal{F}_{P_{(2i-1,j)}}|^2 \ , \\
&\sim \left(\frac{R_{2i-1,j}}{\beta^2 - \frac{p}{2}}\right)^2 \lim_{\beta^2\to\frac{p}{2}} |\mathcal{F}_{(2i-1,-j)}|^2 + \dots \ , \\
&\sim \left(\frac{R_{2i-1,j}}{\beta^2 - \frac{p}{2}}\right)^2 \lim_{\beta^2\to\frac{p}{2}} |\mathcal{F}_{P_{(1,pi+j)}}|^2 + \dots \ .
\end{aligned} \tag{6.4.8}$$

Nevertheless, the pole in (6.4.8) is precisely cancelled by the ratio of four-point structure constants in (6.4.5) [12],

$$\lim_{\beta^2\to\frac{p}{2}} \frac{D_{(1,pi+j)}}{D_{(1,pi-j)}} \sim -\left(\frac{R_{2i-1,j}}{\beta^2 - \frac{p}{2}}\right)^2 + \dots \ . \tag{6.4.9}$$

After studying many examples of (6.4.5) numerically, we find that the cancellation of poles in (6.4.9) and (6.4.8) leave us with the following logarithmic block:

$$\boxed{\lim_{\beta^2\to\frac{p}{2}} \left(\frac{D_{(1,pi-j)}}{D_{(1,pi+j)}} |\mathcal{F}_{P_{(1,pi-j)}}|^2 + |\mathcal{F}_{P_{(1,pi+j)}}|^2\right) \propto \lim_{\beta^2\to\frac{p}{2}} \mathcal{G}^-_{(2i-1,j)}} \ . \tag{6.4.10}$$

The relation (6.4.10) is then valid, provided that the conditions (6.4.7) hold. Thus, whenever the latter requirement is met, the conformal blocks $|\mathcal{F}_{P_{(1,pi-j)}}|^2$ and $|\mathcal{F}_{P_{(1,pi+j)}}|^2$ appearing in (6.4.5) should be replaced with the logarithmic blocks in (6.4.10). For example, we have

$$\begin{aligned}
&\lim_{\beta^2\to\frac{3}{2}} \langle V^D_{\langle 1,3\rangle} V^D_{\langle 1,3\rangle} V^D_{\langle 1,3\rangle} V^D_{\langle 1,3\rangle} \rangle \\
&= \lim_{\beta^2\to\frac{3}{2}} D_{(1,3)} |F_{P_{(1,3)}}|^2 + \lim_{\beta^2\to\frac{3}{2}} D_{(1,5)} \left(\frac{D_{(1,1)}}{D_{(1,5)}} |F_{P_{(1,1)}}|^2 + |F_{P_{(1,5)}}|^2\right) \ , \\
&= \lim_{\beta^2\to\frac{3}{2}} D_{(1,3)} |F_{P_{(1,3)}}|^2 + \lim_{\beta^2\to\frac{3}{2}} D_{(1,5)} \mathcal{G}^-_{(1,2)} \ .
\end{aligned} \tag{6.4.11}$$

We have also checked numerically that the last line of the above example is indeed a crossing symmetry four-point function at high precision. Moreover, each conformal block in the last line of (6.4.11) is finite by itself.

Null vectors

Let us now define the logarithmic representation $\widetilde{\mathcal{K}}^{(p)}_{(i,j)}$ associated to the conformal blocks in (6.4.10) as follows:

$$\widetilde{\mathcal{K}}^{(p)}_{(i,j)} = \lim_{\beta^2\to\frac{p}{2}} \mathcal{W}^{-}_{(2i-1,j)}. \tag{6.4.12}$$

To see the structure of this representation, we need to know whether the two null vectors: $\mathcal{L}_{(2i-1,j)}V_{(2i-1,j)}$ and $\mathcal{L}_{(1,pi-j)}V_{(2i-1,j)}$ vanish in this limit where we recall that the non-diagonal fields $V_{(r,s)}$ have the conformal dimensions $(\Delta_{(r,s)}, \Delta_{(r,-s)})$. This question is equivalent to figuring out whether these two null vectors decouple from all three-point functions. This can be traced from the residues of conformal blocks $R_{r,s}$ in the Zamolodchikov recursion (3.4.15). For any positive integer r and s, we have the following reasoning:

$$\forall\langle V_1V_2V_1V_2\rangle \quad R_{r,s} \quad \text{in} \quad \mathcal{F}_{P_{(r,s)}} \text{ and } R_{r,s}=0 \quad\Longrightarrow\quad \langle\mathcal{L}_{(r,s)}V_{(r,s)}V_1V_2\rangle = 0\ . \tag{6.4.13}$$

Using (6.4.7) with the above and recalling that $\Delta_{(2i-1,\pm j)} = \Delta_{(1pi\mp j)}$ for $\beta^2=\frac{p}{2}$, the limit in (6.4.12) is therefore taken such that

$$\lim_{\beta^2\to\frac{p}{2}} \mathcal{L}_{(1,pi-j)}V_{(2i-1,j)} = \lim_{\beta^2\to\frac{p}{2}} \bar{\mathcal{L}}_{(1,pi-j)}V_{(2i-1,-j)} = 0\ , \tag{6.4.14}$$

while keeping

$$\lim_{\beta^2\to\frac{p}{2}} \mathcal{L}_{(2i-1,j)}V_{(2i-1,j)} = \lim_{\beta^2\to\frac{p}{2}} \bar{\mathcal{L}}_{(2i-1,j)}V_{(2i-1,-j)} = \eta^{D}_{(2i-1,j)} \neq 0\ , \tag{6.4.15}$$

The structure of generating states in $\widetilde{\mathcal{K}}^{(p)}_{(i,j)}$ is shown in (6.2). If we normalized operators in (3.5.6) in the same way as operators in (6.3.7), the logarithmic couplings of $\widetilde{\mathcal{K}}^{(p)}_{(i,j)}$ are precisely given by $\gamma^{-}_{(i,j)}$. Taking into account (6.4.14) with (6.4.6) then comparing (6.2) with (6.1), the chiral projection of $\widetilde{\mathcal{K}}^{(p)}_{(i,j)}$ therefore equals to the chiral staggered module $\mathcal{K}^{(p)}_{(i,j)}$.

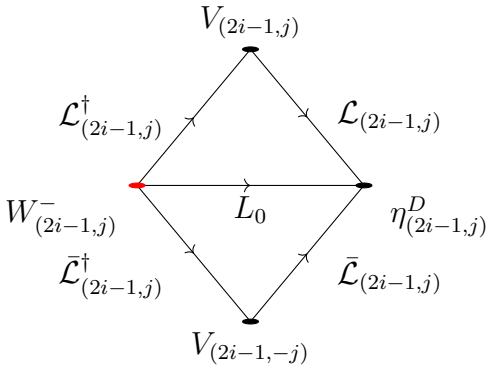

Figure 6.2: non-vanishing null vectors in $\widetilde{\mathcal{K}}^{(p)}_{(i,j)}$

However, we also stress here that $\widetilde{\mathcal{K}}^{(p)}_{(i,j)}$ does not fit with the definition of non-chiral staggered modules in [38]. For example, non-chiral staggered modules of [38] contain diagonal primary fields with no logarithmic partners whereas the only diagonal field $\eta^{D}_{(2i-1,j)}$ in $\widetilde{\mathcal{K}}^{(p)}_{(i,j)}$ is always accompanied by its logarithmic partner $W^{-}_{(2i-1,j)}$.

6.4.2 Fusion rules at $c = 0$

As an example, we consider the case $p = 3$ of four-point functions in (6.4.3), from which we deduce fusion rules of the type (6.4.4). Let us start with four-point functions involving only the degenerate fields, which live on the Kac table at $\beta^2 = \frac{3}{2}$. There are only two of them: $V^D_{\langle 1,1\rangle}$ and $V^D_{\langle 1,2\rangle}$. For example, we have

$$\lim_{\beta^2\to\frac{3}{2}} \langle V^D_{\langle 1,2\rangle} V^D_{\langle 1,2\rangle} V^D_{\langle 1,2\rangle} V^D_{\langle 1,2\rangle}\rangle = \lim_{\beta^2\to\frac{3}{2}} |F_{P_{(1,1)}}|^2 \ , \tag{6.4.16}$$

where the contribution from $|F_{P_{(1,3)}}|^2$ vanishes in this limit, and the conformal block on the right-hand side is simply the identity block. Taking into account also other four-point functions of $V^D_{\langle 1,1\rangle}$ and $V^D_{\langle 1,2\rangle}$, we find

$$\lim_{\beta^2\to\frac{3}{2}} \mathcal{R}^D_{(1,2)} \otimes \mathcal{R}^D_{(1,2)} = \lim_{\beta^2\to\frac{3}{2}} \mathcal{R}^D_{(1,1)} \otimes \mathcal{R}^D_{(1,1)} = \lim_{\beta^2\to\frac{3}{2}} \mathcal{R}^D_{(1,1)} \otimes \mathcal{R}^D_{(1,1)} = \mathbb{1} \ , \tag{6.4.17}$$

Therefore, we have recovered the fusion rules of the $(3, 2)$-minimal model. While it seems that we can recover the $(3, 2)$-minimal model by taking the rational limit of (6.4.1), let us also point it out that relations among the other minimal models in the A-series and the generalized minimal models have not yet been completely found [12].

Beyond the Kac table

For $k, j \in \mathbb{N}^*$, we consider the s-channel of the following four-point functions:

$$\lim_{\beta^2\to\frac{3}{2}} \langle V^D_{\langle 1,3k\rangle} V^D_{\langle 1,3j\rangle} V^D_{\langle 1,3k\rangle} V^D_{\langle 1,3j\rangle}\rangle \ , \tag{6.4.18}$$

whose spectra at generic central charge are given by (6.4.5). First, it is convenient to introduce the diagonal counterpart of (6.3.16a),

$$\mathcal{B}^D_i = \mathcal{B}_i \otimes \bar{\mathcal{B}}_i \ . \tag{6.4.19}$$

The degenerate representation $\mathcal{B}^D_i$ can then be obtained as rational limits as follows,

$$\mathcal{B}^D_i = \lim_{\beta^2\to\frac{3}{2}} \mathcal{R}^D_{(1,3i)} \quad \text{with} \quad \lim_{\beta^2\to\frac{3}{2}} \mathcal{L}_{(1,3i)} V^D_{\langle 1,3i\rangle} = 0 \ . \tag{6.4.20}$$

Our numerical results for the limit (6.4.18) imply the fusion rules:

$$\boxed{\lim_{\beta^2\to\frac{3}{2}} \mathcal{R}^D_{(1,3k)} \otimes \mathcal{R}^D_{(1,3j)} = \bigoplus_{i\overset{2}{=}|k-j|+1}^{k+j-1} \left(\mathcal{B}^D_i \oplus \widetilde{\mathcal{K}}^{(3)}_{(i,2)}\right)} \ . \tag{6.4.21}$$

Now it is interesting to notice that the fusion rules (6.4.21) look like non-chiral counterparts of (6.3.17a), which then suggest that it may be possible to build a bulk logarithmic minimal model whose spectrum contain the representations $\mathcal{B}^D_i$ and $\widetilde{K}^{(3)}_{(i,2)}$. The next task of reaching such goal is of course computing all fusion products among these two types of representations. We have also considered more general cases of the limit (6.4.18) but their results do not seem to be analogous to any known results and do not yet have clear interpretations.

6.4.3 Bulk-logarithmic minimal models?

Let us then assume that bulk-logarithmic minimal models are consistent CFTs on the Riemann sphere, whose spectra may contain infinitely many fields, including logarithmic fields. However, fusion products of these fields should always have finite spectra. For instance, observe that the chiral projection of the fusion rules (6.4.21) give us the fusion rules (6.3.17a) of the $(2,3)$-logarithmic minimal model. It is now tempting to build a bulk-logarithmic minimal model at $c=0$ by assuming that its spectrum contain at least the representations:

$$\mathcal{B}_i^D \quad , \quad \widetilde{\mathcal{K}}^{(3)}_{(i,1)} \quad , \quad \text{and} \quad \widetilde{\mathcal{K}}^{(3)}_{(i,2)} \quad \text{for} \quad i \in \mathbb{N}^* \ . \tag{6.4.22}$$

It is also interesting to see that the chiral projection of the above coincides with the spectrum (6.3.1) at $p=3$. Assuming that fusion products of $\mathcal{B}_i^D$ with themselves are given by the right-hand side of (6.4.21), let us now demonstrate how to compute sone of the other fusion products of representations in (6.4.22) without using the Nahm-Gaberdiel-Kausch algorithm. Recall the equations (4.5.2) and (4.5.4), we then have the following fusion products at generic central charge

$$\mathcal{R}^D_{(1,3)} \otimes \mathcal{V}^D_{P_{(1,0)}} = \mathcal{V}^D_{P_{(1,0)}} \oplus \mathcal{W}^-_{(1,2)} \ . \tag{6.4.23}$$

Now further multiplying $\mathcal{R}^D_{(1,3)}$ to both sides of (6.4.23) and using the associativity of the fusion product, we find

$$\mathcal{R}^D_{(1,3)} \otimes \mathcal{W}^-_{(1,2)} = \mathcal{V}^D_{P_{(1,0)}} \oplus \mathcal{W}^-_{(1,2)} \oplus \mathcal{W}^-_{(1,4)} \ . \tag{6.4.24}$$

Now consider the limit $\beta^2 \to \frac{3}{2}$ of the fusion product (6.4.24) with respect to (6.4.12) and (6.4.20). As OPE, such limit can then results in divergence because of the coincidence of conformal dimensions. In particular, as $\beta^2 \to \frac{3}{2}$, the representation $\mathcal{W}^-_{(1,2)}$ contains a diagonal field with dimension $\Delta_{(1,-2)} \to \Delta_{(3,1)}$, while $\mathcal{W}^-_{(1,4)}$ comes with $\Delta_{(1,-4)} \to \Delta_{(3,-1)}$. Therefore, there is a resonance between these diagonal fields in this limit. We then conjecture the following results:

$$\lim_{\beta^2 \to \frac{3}{2}} (\mathcal{W}^-_{(1,2)} \oplus \mathcal{W}^-_{(1,4)}) = \lim_{\beta^2 \to \frac{3}{2}} \mathcal{W}^-_{(3,1)} = \widetilde{\mathcal{K}}^{(3)}_{(2,1)} \ . \tag{6.4.25}$$

The above limit only makes for four-point functions in which the residues $R_{3,1}$ do not vanish, and we indeed find that $R_{3,1}$ is non-zero in four-point functions corresponding to (6.4.24). Thus, the limit of (6.4.24) reads

$$\mathcal{B}^D_1 \otimes \widetilde{\mathcal{K}}^i_{(1,1)} = \mathcal{B}^D_1 \oplus \widetilde{\mathcal{K}}^{(3)}_{(2,1)} \ , \tag{6.4.26}$$

which looks almost identical to its chiral version (6.3.19c), which has a multiplicity 2 for $\mathcal{B}_1$. In general, it is possible to predict the fusion rules of the types: $\mathcal{B}^D_i \otimes \widetilde{\mathcal{K}}^{(3)}_{(i,1)}$ and $\mathcal{B}^D_i \otimes \widetilde{\mathcal{K}}^{(3)}_{(i,2)}$ by using the rational limit of four-point functions and associativity of the fusion products. However, we do not know yet how this approach can deduce the following fusion products:

$$\widetilde{\mathcal{K}}^{(3)}_{(i,1)} \otimes \widetilde{\mathcal{K}}^{(3)}_{(j,1)} \ , \tag{6.4.27a}$$

$$\widetilde{\mathcal{K}}^{(3)}_{(i,1)} \otimes \widetilde{\mathcal{K}}^{(3)}_{(j,2)} \ , \tag{6.4.27b}$$

$$\widetilde{\mathcal{K}}^{(3)}_{(i,2)} \otimes \widetilde{\mathcal{K}}^{(3)}_{(j,2)} \ . \tag{6.4.27c}$$

Since we do not yet whether the above fusion products always return fields that belong to the assumption (6.4.22). The validity of the spectrum (6.4.22) remains unclear, as well as the existence of bulk-logarithmic minimal models.

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CHAPTER 7

Conclusion and outlook

7.1 Logarithmic representations

We build logarithmic representations of the Virasoro algebra by using derivatives of fields with respect to the conformal dimensions. Our main results are the case of derivatives of null fields, which generate non-chiral logarithmic representations that lead to second- or third-rank Jordan blocks of L_0 and $\bar{L}_0$. These representations are parametrized by logarithmic couplings which take fixed values, provided that the degenerate fields exist. The main advantage of using these derivatives is the computation of conformal blocks of logarithmic representations. This formalism allows us to simply write down closed expression for their conformal blocks as combinations of derivatives or residues of the Zamolodchikov recursion. As an application, our logarithmic representations $\mathcal{W}^{\kappa^-}_{(r,s)}$, generated by the logarithmic fields in (4.5.6), describe the logarithmic structure of the Potts and $O(n)$ CFT [29, 25] and also appear in the rational limits of some four-point functions of generalized minimal models. In particular, this completes the determination of representations of the Virasoro algebra in the Potts and $O(n)$ CFTs at generic central charge. Let us now suggest some future directions.

7.1.1 Higher-order derivatives

It should be straightforward to generalize our constructions for the representations $\mathcal{W}^{\kappa}_{(r,s)}$ and $\widetilde{\mathcal{W}}^{\kappa}_{(r,s)}$ to logarithmic representations, which include higher-order derivatives of null fields. More precisely, one should start with the combination:

$$(1-\kappa)\eta^{D^{(n)}}_{\Delta_{(r,s)}} + \kappa\mu^{D^{(n)}}_{\Delta_{(r,s)}} \ , \tag{7.1.1}$$

where $\eta^{D^{(n)}}_{\Delta_{(r,s)}}$ and $\eta^{D^{(n)}}_{\Delta_{(r,s)}}$ were defined in (3.5.3), and κ is expected to parametrize representations generated by the above field. This would result in arbitrary-rank Jordan blocks of L_0 and $\bar{L}_0$. While it is not clear if there exist higher-rank Jordan blocks of L_0 and $\bar{L}_0$ in any consistent CFT at generic central charge, it is well-known that they exist at rational central charge, for instance at $c = 0$ [57]. It may also be interesting to find if the fields (7.1.1) with $n > 2$ lead to some known representations. Moreover, using derivatives of fields will also allows us to determine the logarithmic conformal blocks for (7.1.1) analytically, similarly to the case of the logarithmic fields $W^{\kappa}_{(r,s)}$ and $\widetilde{W}^{\kappa}_{(r,s)}$.

7.1.2 Logarithmic minimal models

Our results from rational limits of generalized minimal models suggest the possiblity of consistent logarithmic minimal models in the bulk. Let us now suggest what can be done in the future.

1. Since we could not compute all fusion products of bulk representations in (6.4.22) by taking rational limits, it may be interesting to use the Nahm-Gaberdiel-Kausch alogrithm to compute fusion products of these bulk representations, which is expected to quite complicated.

2. The algebraic structure of chiral logarithmic minimal models [35] has been studied extensively, while their correlation functions seem to be left uninvestigated. Since we can describe their staggered modules by using derivatives as in (6.3.3), computing correlation functions of fields in (6.3.1) should not be too complicated, while keeping in mind that these chiral logarithmic minimal models are expected to be consistent on the upper-half plane. For instance, one may start with writing down conformal blocks for the staggered modules $\mathcal{K}^{(p)}_{(i,2)}$.

7.2 The Potts and $O(n)$ CFTs

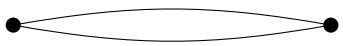
We demonstrate how to numerically solve the crossing-symmetry equation for several four-point functions of primary fields in the Potts and $O(n)$ CFTs, from which we infer some exact results such as their numbers of crossing-symmetry solutions, their spectra, and some of their fusion rules. Moreover, our results also support that both CFTs are consistent with the spectra, recently proposed by [30]. For instance, in the case of the Potts CFT, the number of crossing-symmetry solution $\mathcal{N}^{\text{Potts}}$ is always consistent with the prediction from S_Q symmetry in all of our examples wherein the inequality $\mathcal{N}^{\text{Potts}} \leq \mathcal{I}^{S_Q}$ always holds and becomes saturated in 24 out of 28 cases. Moreover, the discrepancy between $\mathcal{N}^{\text{Potts}}$ and $\mathcal{I}^{S_Q}$ suggests the Potts CFT may have a larger global symmetry than S_Q. The same signal of large global symmetry also appears in the $O(n)$ CFT [25].

We have also computed several analytic ratios of structure constants for some four-point functions of the Potts and $O(n)$ CFTs, similarly to [32, 15]. However, we have not found an analytic expression for structure constants, like the three-point connectivity in [58]. Let us now point out some plausible future directions:

7.2.1 Crossing-symmetry solutions $\leftrightarrow$ Physical Observables?

For the Potts CFT, the field $V_{(0,\frac{1}{2})}(z)$ inserts a Fortuin-Kasteleyn cluster at point z, as shown in (1.3.2) for the four-point connectivities, whereas the other non-diagonal fields $V_{(r,s)}(z)$ with $r \neq 0$ insert $2r$ lines at point z , as boundaries for clusters [24]. In the case of the $O(n)$ CFT, we only have the latter type of fields, which is expect to insert $2r$ lines at each point as well. For instance $V_{(\frac{1}{2},0)}$ inserts one line at each point in the figure 5.3.21. Let us also display the two-point function of $V_{(1,0)}$, which inserts two lines at each point,

$$\left\langle V_{(1,0)} V_{(1,0)} \right\rangle \longrightarrow \tag{7.2.1}$$



This gives us a glimpse that more general four-point functions of both CFTs should describe physical observables on the lattice with more complicated configurations. However, in general, we do not know yet precise relations between crossing-symmetry solutions and these observables of the lattice model.

A first step towards such relations is perhaps to assume that there is always a one-to-one correspondence between solutions to the crossing-symmetry equation and observables on the lattice models, similarly to the case of the four-point connectivities in (5.4.17). Then try to look for rules of drawing clusters or diagrams for four-point functions in each model such that the numbers of their configurations are always consistent with $\mathcal{N}^{\text{Potts}}$ and $\widetilde{\mathcal{N}}^{O(n)}$.

7.2.2 Extra solutions → New CFT ?

In the case of the Potts CFT, there is a significant number of extra solutions to the crossing-symmetry equation, which are inconsistent with S_Q symmetry of the Potts CFT. Let us now display the simplest examples of these extra solutions. The four-point function $\langle V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(0,\frac{1}{2})}V_{(3,0)}\rangle$ in Subsection 5.4.2 comes with three extra solutions:

Solutions	Spectra for s,t,u
$\mathcal{X}^1$	$\mathcal{S}^{\text{ex}} - \{(2,0)\}$
$\mathcal{X}^2$	$\mathcal{S}^{\text{ex}} - \{(2,1)\}$
$\mathcal{X}^3$	$\mathcal{S}^{\text{ex}} - \{(4,0)\}$

(7.2.2)

where $\mathcal{S}^{\text{ex}} = \mathcal{S}^{\text{Potts}} \cap \mathcal{S}^{\text{even}} - \mathcal{S}^{\text{deg}}$. Similarly to (7.2.2), by our set-up, the other extra solutions on the tables in Section 5.4.1 always have spectra which are subsets of the spectrum of the Potts CFT and also have vast intersections with the spectrum of the $O(n)$ CFT. Nevertheless, these extra solutions belong to neither CFTs since they do not fit with the constraints from S_Q symmetry in (5.4.1) and also come with the field $V_{(0,\frac{1}{2})}$ that does not exist in the $O(n)$ CFT. Let us suggest some plausible explanations.

Combining the Potts and $O(n)$ CFTs?

It could be that some of these extra solutions belong to a bigger CFT whose spectrum contains the spectra of the Potts and $O(n)$ CFTs as subsets. For example, we find solutions to the crossing-symmetry equation for four-point functions which mix primary fields from both the Potts and $O(n)$ CFT. For example, we find soultions from solving the crossing-symmetry equation for the four-point functions $\langle V^{\text{Potts}}V^{\text{Potts}}V^{O(n)}V^{O(n)}\rangle$. The field V^{Potts} and $V^{O(n)}$ are defined as follows:

$$V^{\text{Potts}} \in Z^{\text{Potts}} - Z^{O(n)} \,, \tag{7.2.3a}$$

$$V^{O(n)} \in Z^{O(n)} - Z^{\text{Potts}} \,. \tag{7.2.3b}$$

In other words, V^{Potts} are primary fields which appear in the spectrum of the Potts CFT but not in the spectrum of the $O(n)$ CFT, and vice versa for $V^{O(n)}$. Let us now display some numbers of crossing-symmetry solutions for the four-point functions $\langle V^{\text{Potts}}V^{\text{Potts}}V^{O(n)}V^{O(n)}\rangle$. The spectrum for each channel in these four-point functions is assumed to be $\mathcal{S}^{\text{Potts}} \cup \mathcal{J}^{O(n)}$, which were introduced in (5.2.2) and (5.2.6) respectively.

Four-point functions	Numbers of solutions
$(0,\frac{1}{2})^2(\frac{1}{2},0)^2$	2
$(0,\frac{1}{2})^3(1,0)$	3
$(0,\frac{1}{2})^2(1,0)^2$	5
$(0,\frac{1}{2})(1,0)^3$	5
$(0,\frac{1}{2})^2(1,0)(1,1)$	4
$(0,\frac{1}{2})^2(\frac{3}{2},0)^2$	8
$(0,\frac{1}{2})^2(\frac{3}{2},\frac{2}{3})(\frac{3}{2},-\frac{2}{3})$	7

Therefore, combining the spectra of both CFTs indeed gives us new crossing-symmetric solutions. From the view of the lattice model, this idea also makes sense since we know that operators from the $O(n)$ loop model also exist in the Potts model by using the loop representations of the Potts model [24]. It should therefore be interesting to investigate further to see if this kind of CFT exists.

Of course, it is also not clear whether four-point structure constants of new crossing-symmetry solutions always factorize into products of three-point structure constants since we are solving the crossing-symmetry equation for four-point structure constants. If not, they do not belong to a consistent CFT. One way of clarifying this issue is to consider the crossing-symmetry equation as a system of quadratic equations for three-point structure constants.

7.2.3 Integer Q and n

We have thus far considered only the Potts and $O(n)$ CFTs at generic central charge, that is to say generic Q and n. Integer values of Q and n are however actually more notable due to their applications in statistical physics. For instance, at the central charge $c = 0$, the Potts model with $Q = 1$ is equivalent to the bond percolation in two dimensions while the $O(n)$ model with $n = 0$ describes the problem of self-avoiding random walk. Nevertheless, some of our results at generic central charge do not hold for the case of integers Q and n. For example. the action of Virasoro algebra on the spectra of both CFTs has not been completely determined for these special cases.

At the moment, we have preliminary numerical results indicating that the Potts and $O(n)$ CFTs are consistent at integers Q and n. For instance, we have checked numerically that the limit $n \to 0$ of the four-point functions $\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}\rangle$ in the $O(n)$ CFT exists. Let us now discuss briefly our numerical results for the s-channel of the solution C_3 in the figure 5.3.21. We have also considered the other two solution in the figure 5.3.21, whose results are comparable to this case. Using (5.3.22) with (5.3.19), C_3 has the following s-channel spectrum:

$$\mathcal{S}^{(s)}_{C_3} = \{(r,s) \in \mathbb{N}^* \times (-1,1] | rs \in \mathbb{Z}\} \ . \tag{7.2.4}$$

While the interchiral block of each field in the spectrum $\mathcal{S}^{(s)}_{C_3}$ can diverge in the limit $n \to 0$, however we find that the following combinations of leading interchiral blocks are

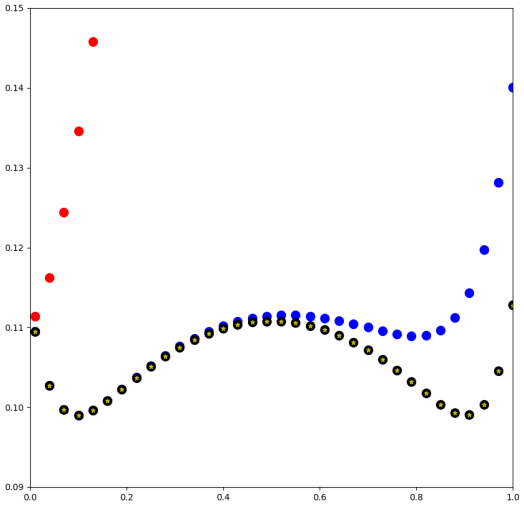


Figure 7.1: The x-axis represents the real part of the cross ratio while the y-axis shows the numerical values of C_3^{color} with $n \approx 10^{-20}$ at each position

always finite,

$$I_1 = \lim_{n\to 0} \mathcal{H}_{(1,0)} \ , \tag{7.2.5a}$$

$$I_2 = \lim_{n\to 0} \left(\mathcal{H}^{(s)}_{(1,1)} + \mathcal{H}^{(s)}_{(2,\frac{1}{2})} + \mathcal{H}^{(s)}_{(2,-\frac{1}{2})} + \mathcal{H}^{(s)}_{(3,1)} \right) \ , \tag{7.2.5b}$$

$$I_3 = \lim_{n\to 0} \left(\mathcal{H}^{(s)}_{(3,\frac{1}{3})} + \mathcal{H}^{(s)}_{(3,-\frac{1}{3})} \right) \ , \tag{7.2.5c}$$

$$I_4 = \lim_{n\to 0} \left(\mathcal{H}^{(s)}_{(3,\frac{2}{3})} + \mathcal{H}^{(s)}_{(3,-\frac{2}{3})} \right) \ , \tag{7.2.5d}$$

where $\mathcal{H}^{(s)}_{(r,s)}$ are interchiral blocks of the non-diagonal primary fields $V_{(r,s)}$, introduced in (4.6.7) and (4.6.8) . From (7.2.4), the sum of all interchiral blocks in (7.2.5) is then the leading contribution for the s-channel expansion of $\lim_{n\to 0} C_3$. Let us now consider how each interchiral block in (7.2.5) contribute to C_3. We first define

$$C_3^{\text{red}} = I_1 \ , \tag{7.2.6a}$$

$$C_3^{\text{blue}} = I_1 + I_2 \ , \tag{7.2.6b}$$

$$C_3^{\text{black}} = I_1 + I_2 + I_3 \ , \tag{7.2.6c}$$

$$C_3^{\text{yellow}} = I_1 + I_2 + I_3 + I_4 \tag{7.2.6d}$$

We then plot each C_3^{color} according to their color in the plot (7.1), which shows that the solution C_3 is finite the limit $n \to 0$. Notice that C_3^{black} and C_3^{yellow} cannot be distinguished in the plot (7.1). Similar situations also happen in the t- and u-channel of C_3, and we also have checked that $\lim_{n\to 0} C_3$ is crossing symmetric. Like (7.2.5), in general, we expect that there are always cancellations of divergences from infinitely many interchiral blocks in $\lim_{n\to 0}\langle V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)} V_{(\frac{1}{2},0)}\rangle$. These cancellations also imply that representations of the Virasoro algebra at generic central charge combine into much more complicated representations at $n = 0$, which we do not yet know. There are at least a few future directions for these results:

1. It would surely interesting to know representations of the Virasoro algebra, which underly the combinations of interchiral blocks in (7.2.5). For instance, one may

start by investigating the current modules of the primary fields $V_{(1,1)}$ and $V_{(1,-1)}$, whose conformal blocks appear in I_2 in (7.2.5).

2. At $n = 0$, the four-point function $\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}\rangle$ compute the probability of how self-avoiding random walk connects each pair of two points, and there are three configurations of these walks, as shown in the figure 5.3.21. Therefore, it would be certainly interesting to compare our numerical results of this four-point functions to experimental measurements such as the Monte Carlo method.

3. It would also be interesting to consider the limit $n \to 0$ of more general four-point functions in the $O(n)$ CFT. Right now, we find that the precision of more general four-point functions are more difficult to control in this limit.

7.2.4 Spin clusters

The authors of [59] considered another kind of physical observables in the critical Potts model, the spin-cluster connectivities. The four-point spin connectivities can be described by the four-point function $\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}\rangle$ whose channels can have a non-degenerate diagonal field with the dimensions $(\Delta_{(1,2)}, \Delta_{(1,2)})$ propagating. With our best knowledge, the only representation of Virasoro algebra at generic central charge, which has such properties, is our logarithmic representation $\widetilde{W}^{\kappa_0}_{(1,2)}$ since $\widetilde{W}^{\kappa_0}_{(1,2)}$ comes with the diagonal primary field $V^D_{\Delta_{1,2)}}$ which does not have any vanishing null descendant. To check if our guess is correct, we need to solve the crossing-symmetry equation for the four-point function $\langle V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}V_{(\frac{1}{2},0)}\rangle$ with the conformal blocks of $\widetilde{W}^{\kappa_0}_{(1,2)}$, namely $\widetilde{\mathcal{G}}^{\kappa_0}_{(1,2)}$ in (4.5.20). If we find a solution, this would also be a discovery of a higher-rank of Jordan block of L_0 at generic central charge.

CHAPTER 8

Synthèse en français

Cette thèse concerne la résolution de theorie conforme à charge centrale générique sur la sphère de Riemann en utilisant l'approche de bootstrap conforme de [31]. Nos intérêts principaux sont les theories conformes de Potts et $O(n)$ [14]. Pour les deux theories conformes, on complète la détermination des représentations conformes dans leurs spectres et calcule numériquement plusieurs exemples de leurs fonctions à quatre points en utilisant le bootstrap conforme. On commençe par une trés brève révision sur le theorie conforme. Ensuite, on discute de notre construction de la représentation logarithmique et nos résultats pour les theories conformes de Potts et $O(n)$.

8.1 Que sont les theories conformes?

La théorie conforme est une théorie des champs à symétrie conforme. Contrairement à les autres théories, les theories conformes n'ont pas besoin de lagrangien pour leurs définitions. La symétrie conforme est suffisamment puissante pour fournir aux theories conformes une définition simple, connue sous le nom de *le donnée de theorie conforme*. le donnée de theorie conforme data est une collection de champs primaires et leurs coefficients d'expansion opérateur-produit qui satisfont les conditions, imposées par toutes les conséquences des symétries du modéle [1]. En pratique, toutes les conséquences de la symétrie conforme se manifestent dans *l'équation de la symétrie de croisement*,

$$\sum_{V\in\mathcal{S}^{(s)}} D_V^{(s)} \quad \begin{matrix} 2 & & 3 \\ & V & \\ 1 & & 4 \end{matrix} \quad = \sum_{V\in\mathcal{S}^{(t)}} D_V^{(t)} \quad \begin{matrix} 2 & & 3 \\ & V & \\ 1 & & 4 \end{matrix} \quad = \sum_{V\in\mathcal{S}^{(u)}} D_V^{(u)} \quad \begin{matrix} 2 & & 3 \\ & V & \\ 1 & & 4 \end{matrix} \quad ,$$

s-channel $\qquad$ t-channel $\qquad$ u-channel (8.1.1)

óu V representent des champs primaires. De plus, $D_V^{(s)}$, $D_V^{(t)}$, et $D_V^{(u)}$ sont les coefficients a quatre points. Avec le donnée de theorie conforme seulment, on peut calculer toutes les observables physiques dans les theories conformes. Cette methode de résoudre la theorie conforme est connue sous le nom de *le bootstrap conforme* [2].

En deux dimensions, l'algébre conforme est infinie, égalment sous le nom de l'algébre de Virasoro. l'algébre de Virasoro s'accompagne d'un paramétre supplémentaire, connu sous le nom de charge centrale. Par conséquent, les données de theorie conforme des theories conformes bidimensionnels dépendent explicitement de la charge centrale, et différentes valeurs de la charge centrale nous donnent theories conformes différents. Avec la structure riche de la symétrie conforme, les theories conformes bidimensionnels sont alors plus contraints et plus réalisables pour avoir des solutions exactes que ceux de dimensions supérieures. Par exemple, la théorie de Liouville et les modéles minimaux sont des exemples de theorie conforme complétement résolus en deux dimensions [4, 5]. De plus, les theories conformes en deux dimensions ont également de nombreuses applications en physique théorique, de la description de la feuille de monde en théorie des cordes à de nombreux systèmes critiques. Avec ces phénomènes particuliers, les theories conformes bidimensionnels méritent donc une attention particulière.

8.2 Représentations logarithmiques

Les représentations de l'algèbre de Virasoro qui conduisent aux blocs de Jordan des générateurs de dilatation sont en général appelées représentations logarithmiques. Puisque les fonctions de corrélation de ces représentations logarithmiques dépendent logarithmiquement des positions.

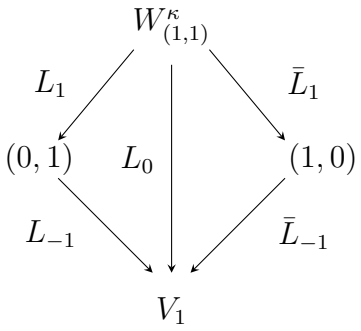
On construit des représentations logarithmiques de l'algèbre de Virasoro en utilisant les dérivées des champs primaires par rapport à leurs dimensions conformes. Dans le cas des dérivées de champs nuls, on constate que les représentations résultantes sont non chirales et sont paramétrées par des couplages logarithmiques. Par exemple, considérons la dérivée du premier ordre des champs nuls au niveau 1. On écrit leurs combinaisons linéaires comme suit :

$$W^{\kappa}_{(1,1)} = (1-\kappa)V_1' + \kappa L_{-1}\bar{L}_{-1}V_0' \tag{8.2.1}$$

où prime désigne la dérivée par rapport aux dimensions conformes et, uniquement dans cette section, le champ V_1 représente un champ nul de niveau 1 dont la dimension conforme est 1. De plus, cette construction ne donne des résultats non triviaux que si l'on suppose que le vecteur nul $V_1 = L_{-1}\bar{L}_{-1}V_0$ ne s'annule pas. Le paramètre κ est défini par l'équation:

$$\bar{L}_{-1}\bar{L}_1 W_1^{\kappa} = L_{-1}L_1 W_1^{\kappa} = \kappa(L_0 - 1)W_1^{\kappa} \ . \tag{8.2.2}$$

Le paramètre κ est connu sous le nom de couplage logarithmique, qui caractérise les représentations générées par les champs logarithmiques $W^{\kappa}_{(1,1)}$. En d'autres termes, différents κ nous donnent des représentations différentes. Dessinons la structure des représentations générées par $W^{\kappa}_{(1,1)}$,



(8.2.3)

Par conséquent, les représentations de $W^{\kappa}_{(1,1)}$ contiennent deux champs primaires non diagonaux avec des dimensions conformes à gauche et à droite : $(1,0)$ et $(0,1)$, le champ nul V_1 et le champ logarithmique $W^{\kappa}_{(1,1)}$. Dans le cas theories conformes avec champs dégénérés, le paramètre κ prend des valeurs fixes selon les types des champs dégénérés,,

$$\kappa^- = \frac{1}{2} - \frac{\beta^2}{s} \quad \text{for} \quad V^D_{\langle 1,s\rangle} \ , \tag{8.2.4}$$

$$\kappa^+ = \frac{1}{2} - \frac{1}{2\beta^2} \quad \text{for} \quad V^D_{\langle s,1\rangle} \ . \tag{8.2.5}$$

Cette construction peut être étendue à des champs nuls de niveau supérieur ainsi qu'à leurs dérivées d'ordre supérieur. En particulier, des représentations logarithmiques, générées par la dérivée première de champs nuls, complètent la détermination de l'action de l'algèbre de Virasoro sur les spectres de theorie conforme décrivant les points critiques du modèle de Potts et du modèle $O(n)$ en deux dimensions, également connu sous le nom des theories de Potts et $O(n)$.

8.3 Theories conformes de Potts and $O(n)$ en bref

Le modéle critique de Potts Q-state avec $Q \in [0,4]$ et le modéle critique $O(n)$ avec $n \in [-2,2]$ sont bien connus pour être décrits par theorie conforme dont la charge centrale c est liée à Q et n comme suit :

$$c = 13 - 6\beta^2 - 6\beta^2 \quad \text{avec} \quad Q = 4\cos(\pi\beta^2)^2 \quad \text{et} \quad n = -2\cos(\pi\beta^{-2}) \ , \tag{8.3.1}$$

où β^2 prend des valeurs dans $[\frac{1}{2}, 1]$. Bien que ces deux modèles soient remarquables en tant que généralisations du modèle d'Ising bidimensionnel dans lequel $Q \in \mathbb{N}^* + 1$ et $n \in \mathbb{N}^*$, ils ont également des descriptions qui permettent Q et n doivent être non entiers. En termes de physique statistique, le modéle de Potts avec Q non entier décrit le cluster aléatoire dit de Fortuin-Kesteleyn [23] tandis que le modèle $O(n)$ avec n générique est équivalent au dilué phase du modèle de boucle [4]. Avec ces descriptions, la corrélation des deux modèles de réseau existe même pour des valeurs complexes de Q et n [24]. Avec la relation (8.3.1), il est donc logique de s'attendre è ce que les CFT décrivant ces deux modèles soient également valables pour la charge centrale générique. Cela nous a conduit aux définitions des theories conformes de Potts et $O(n)$ [25, 11] comme suit :

theories conformes de Potts et $O(n)$ sont des prolongements analytiques dans la charge centrale c du modèle critique de Potts Q-state et du modèle critique $O(n)$ tels que

$$\boxed{\Re(c) \leq 13 \iff \Re(\beta^2) > 0} \ . \tag{8.3.2}$$

Vers la résolution des deux theories conformes

Theories conformes de Potts et $O(n)$ n'ont pas seulement une symétrie conforme mais aussi des symétries globales : respectivement S_Q et $O(n)$. Les théories de représentation de ces deux symétries peuvent être formulées sous forme de catégories tensorielles pour Q et n [28] génériques. Comme le donneé de theorie conforme, résoudre ces deux theories conformes revient à écrire leurs spectres et à résoudre leurs coefficients OPE á partir des conditions de cohérence : l'équation de la symétrie croisément et les contraintes de leurs symétries globales. Les spectres des theories conformes de Potts et $O(n)$ ont été complàtement déterminés récemment dans [30, 29]. La prochaine étape dans la résolution

des theories conformes de Potts et $O(n)$ consiste donc à résoudre leurs coefficients à quatre points en résolvant l'équation de la symétrie croisé. Cela peut être fait numériquement en utilisant notre méthode dans [25], proposée à l'origine pour la theorie conforme $O(n)$ et utilisée plus tard pour la theorie conforme de Potts.

Résultats du bootstrap conforme

Résumons les résultats du bootstrap conforme pour les deux theories conformes.

- Dans les deux theories conformes, on soutient que les théories de représentation de leur symétrie globale fournissent des limites supérieures sur le nombre de solutions de a symétrie de croisément pour toute fonction à quatre points donnée. Ces limites sont toujours respectées (saturées dans certains cas) par les résultats de la résolution de l'équation de la symétrie de croisement.

- Dans [25, 33], on résout numériquement l'équation de la symétrie croisément pour plusieurs fonctions à quatre points des deux theories conformes. De nos résultats numériques, on conclut plusieurs résultats exacts tels que leurs nombres de solutions de la symétrie de croisement, les spectres exacts, les fonctions á trois points nulles, et les régles de fusion.

- Dans le cas de la theorie conforme $O(n)$, nous constatons que les solutions de l'équation de la symétrie croisément sont toujours cohérentes avec la symétrie $O(n)$, contrairement à la theorie conforme de Potts oú nous trouvons des solutions supplémentaires, qui sont incompatibles avec la symétrie S_Q de la theorie conforme de Potts et n'ont pas encore d'interprétation claire.

La prochaine étape évidente de la résolution des theories conformes de Potts et $O(n)$ consiste alors à trouver une expression analytique pour leurs constantes de structure à quatre points. Récemment, les auteurs de [15] ont trouvé des rapports analytiques exacts pour certaines coefficients à quatre points dans la theorie conforme de Potts, et on a également trouvé des rapports similaires pour certaines fonctions à quatre points de la theorie conforme $O(n)$. Cela suggère fortement la possibilité de solutions exactes aux deux theories conformes. Cependant, on n'a pas encore trouvé d'expression analytique pour une constante de structure unique, comme la connectivité à trois points dans [58].

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