THE MICROCANONICAL DENSITY OF STATES AND CAUSAL
DYNAMICAL TRIANGULATIONS

by

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Abstract

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Brown and York's gravitational microcanonical density of states is extended to general spacetime dimension and shown to be dependent upon features of the 4 dimensional gravitational action for its interpretation. Black hole entropy is calculated from the density of states path integral in general spacetime dimension, and the interpretation is shown to be likewise dependent upon the dimension of spacetime. The entropy of de Sitter and Rindler horizons are calculated using the black hole density of states and the notion of local horizon entropy density is shown to be supported. The applicability of the microcanonical ensemble to black hole mechanics is discussed at a fundamental level focussing on the absence of angular velocity as an external parameter in the gravitational Hamiltonian. The rotational ensemble and a new ensemble - the angular momentum ensemble - are introduced following Jaynes' information theory approach to statistical mechanics and proposed as more compelling candidates to calculate black hole entropy as a function of state.

A program to calculate the density of states path integral non-perturbatively using causal dynamical triangulations is initiated. Regge calculus expressions for extrinsic curvature are extended to the case of Lorentzian hypersurfaces and used to derive Regge calculus expressions for quasilocal energy-momentum. The Regge version of the black hole density of states action is derived and specialised to the $3d$ and $4d$ spacetime constructions of causal dynamical triangulations.

Finally, the recent suggestion that entropy is observer dependent is shown to be incompatible with the Tolman law for the equilibrium temperature in a gravitational field.

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Contents

Chapter 1

Introduction

Over the century since their development, quantum mechanics and Einstein's general theory of relativity have eluded all attempts to unify them into a quantum theory of gravity that is as well supported as the two theories separately. The majority of current effort in quantum gravity research falls under the umbrella of string theory, whilst a smaller research community works on loop quantum gravity and other approaches are pursued by small groups and individuals. The gravitational path integral is an approach that falls into the latter category. Work in this direction was started over 50 years ago [Misner, 1957], and advanced in the 1970s when it was applied to calculate black hole partition functions [Gibbons and Hawking, 1977a]. Notable progress into the non-perturbative realm has been achieved recently in causal dynamical triangulations [Ambjorn et al., 2005].

The basis of the path integral [Feynman and Hibbs, 1965] is the method in quantum mechanics of obtaining the probability amplitude for an event to occur by summing the amplitudes of the different ways in which it can occur. In the two slit experiment, for example, the amplitude for a particle to arrive at a point on the screen is the sum of the amplitudes to arrive by passing through the individual slits. In general, the probability amplitude for a particle in an initial state $|x_{in}\rangle$ to be found, a time T later, in a final state

$|x_{fin}\rangle$ is given by a sum of amplitudes $\phi[x(t)]$, over all possible paths $x(t)$ of duration T, starting at x_{in} and ending at x_{fin}

$$\langle x_{fin}|e^{-\frac{i\hat{H}T}{\hbar}}|x_{in}\rangle = \sum_{All\ Paths} \phi[x(t)], \qquad (1.1)$$

where $\hat{H}$ is the Hamiltonian operator of the system. Feynman proposed

$$\phi[x(t)] = const\ e^{\frac{iS[x(t)]}{\hbar}}, \qquad (1.2)$$

for the amplitude based on the following intuition: In the classical limit the action functional $S[x(t)]$ is very much greater than Planck's constant $\hbar$, and the sum is over a quickly oscillating phase factor. In general, the positive contribution to the sum of one path will be cancelled by the negative contribution of a close-by path. But the action is extremised by the path that solves the classical equations of motion and does not change greatly for nearby paths. Paths in the region of the classical path therefore make the most important contribution to the path integral. The path integral is denoted formally by

$$\langle x_{fin}|e^{-i\hat{H}T/\hbar}|x_{in}\rangle = \int_{x(0)=x_{in}}^{x(T)=x_{fin}} \mathcal{D}H\ e^{iS/\hbar}, \qquad (1.3)$$

where $\mathcal{D}H$ is a measure for the space of possible paths or "histories".

To calculate the one-particle path integral, the time duration T is first split into small intervals. Smooth paths through space are approximated by straight line segments during each time interval that change direction discontinuously between intervals. The position of the particle at the end of each interval is integrated over and the limit of the number of time intervals being sent to infinity is taken.

The path integral approach to quantum gravity is to generalise equation (1.3) to the gravitational field. The initial and final particle positions are replaced by initial and final *spatial* metrics h_{in} and h_{fin}, defined respectively on surfaces Σ_{in} and Σ_{fin}.

Particle "histories" become spacetime metrics g with initial and final boundary metrics h_{in} and h_{fin}. The action is the Einstein-Hilbert gravitational action, which is extremised by solutions to Einstein's field equations of general relativity. The gravitational path integral is expressed formally as

$$\langle h_{fin}, \Sigma_{fin} | h_{in}, \Sigma_{in} \rangle = \int_{h_{in}}^{h_{fin}} \mathcal{D}g \, e^{iS[g]}. \tag{1.4}$$

On a perturbative level quantum general relativity is non-renormalisable ['t Hooft and Veltman, 1974]. This may mean that it only makes sense as an effective field theory [Burgess, 2004]. Or, from the perspective of Wilson's renormalisation group [Wilson and Kogut, 1974], the theory could yet prove to be non-perturbatively renormalisable. This would require the existence of an ultraviolet non-Gaussian fixed point of the renormalisation group flow to make the theory assymptotically safe [Weinberg, 1979]. The Gross-Neveu model is an example of a perturbatively non-renormalisable theory that does in fact turn out to be renormalisable [Karkkainen, 1993]. The current status of the assymptotic safety scenario in quantum gravity is reviewed in [Niedermaier and Reuter, 2006].

One approach to calculating the path integral non-perturbatively is to take the smooth spacetime manifolds of general relativity and perform something analogous to the discretisation of the histories in the one-particle path integral. Regge considered lattice-like triangulations of spacetime and constructed a discrete version of the Einstein-Hilbert action from the curvature information contained in the triangulations [Regge, 1961].

Using Regge's discretised action for general relativity there are two possible routes to implement an integration over spacetimes. The first is to fix a triangulation and integrate over the lengths of the edges in the triangulation. This approach, known as quantum Regge calculus [Rocek and Williams, 1981], is technically difficult because of the large number of constraints on the edge lengths. For example, each edge belongs to at least one triangle; so its length must be restricted in the integration to values less than the sum of the triangle's other two edges. Such a constraint exists for each triangle to which

the edge belongs, and similar constraints arise from higher dimensional objects in the triangulation. Another difficultly with this approach is the implementation of an effective continuum limit; as sending the edge lengths to zero would be at odds with integrating over them.

An alternative route, in d spacetime dimensions, is to take identical d-simplices and integrate over spacetimes by summing over all possible ways of connecting varying numbers of these d-simplices [David, 1995, Agishtein and Migdal, 1992b]. This approach - dynamical triangulations - avoids the difficulties of constraints on edge lengths and provides an elegant way to take the continuum limit - let the simplicial edge lengths tend to zero. In two spacetime dimensions dynamical triangulation path integrals can be calculated analytically [Ambjorn et al., 1999]. In three and four dimensions progress has been made by Monte Carlo numerical simulations. These numerical computations were initially performed as integrations over Euclidean triangulations and soon ran into difficulty: although the simplicial building blocks used to construct the spacetime histories were 3-simplices and 4-simplices respectively, the resulting quantum spacetimes were not three and four dimensional [Boulatov and Krzywicki, 1991, Ambjorn and Varsted, 1992, Agishtein and Migdal, 1991, Ambjorn and Jurkiewicz, 1992, Agishtein and Migdal, 1992a].

A proposal in response to this dimensionality problem, [Ambjorn and Loll, 1998] influenced by earlier work [Teitelboim, 1983a, Teitelboim, 1983b], is to restrict the sum over histories to Lorentzian spacetimes that do not exhibit spatial topology change. In other words, each spacetime contributing to the path integral must be expressible as a time-series of spatial hypersurfaces having the same topology. Any spacetimes that have parts splitting off and rejoining later, for example, are ruled out. In such spacetimes there is a point - the point of splitting - at which it is not clear which of the forking spaces an arriving light ray should go into. As the causal structure of spacetime is defined by the lightcones this results in problems with causality. Dynamical triangulations with this restriction on topology are aptly named causal dynamical triangulations.

Computer calculations using Monte Carlo techniques and spacetime histories constructed from 4-simplices have shown that, surprisingly, the resulting quantum spacetimes are four dimensional at large scales but two dimensional at short distances [Ambjorn et al., 2005].

As quantum gravity has notoriously weak links to experiment it is difficult to verify or falsify many aspects of theoretical work on the subject. One area where real insight has been gained into the consequences of quantum gravity is the realm of black holes and their synthesis with thermodynamics. In particular, the ability to reproduce the relationship between a black hole's entropy and its event horizon area is possibly the most stringent test we currently have of any candidate theory of quantum gravity.

A black hole is a region of spacetime in which gravity is so strong that nothing can escape. The boundary of this region - the event horizon - acts as a one-way surface. Objects can cross the event horizon as they fall into the black hole, but, in classical physics, nothing can cross the event horizon travelling in the other direction.

A non-electrically charged black hole formed from collapsing matter eventually settles down to a stationary state with an external gravitational field determined *uniquely* by the total mass M and total angular momentum J of the black hole. This is the uniqueness theorem [Carter, 1971, Hawking, 1972] or, more colloquially, the "black holes have no hair" theorem. It is the first link between black holes and thermodynamic systems which are also described by a small number of stationary parameters in equilibrium.

The solutions to Einstein's field equations describing rotating black holes are the Kerr solutions, parametrised by M and J [Kerr, 1963]. In terms of M and J the event horizon area $A_{\mathcal{H}}$ is

$$A_{\mathcal{H}} = 8\pi \left(M^2 + \sqrt{M^4 - J^2} \right).$$
(1.5)

It is useful to introduce two other properties of Kerr black holes; their surface gravity $\kappa_{\mathcal{H}}$, so called because it is the acceleration of a particle located on, and rotating with, the event horizon viewed from an infinite distance away; and their angular velocity Ω.

These properties are related to M and J in the following way

$$\Omega = \frac{J}{2M(M^2 + (M^4 - J^2)^{1/2})},\tag{1.6}$$

$$\kappa_{\mathcal{H}} = \frac{(M^4 - J^2)^{1/2}}{2M(M^2 + (M^4 - J^2)^{1/2})}.\tag{1.7}$$

The link between black holes and thermodynamics was further strengthened when Bardeen et al. proved four laws of black hole mechanics that are strikingly similar to the laws of thermodynamics [Bardeen et al., 1973]. The zeroth law states that the surface gravity $\kappa_{\mathcal{H}}$ of a stationary black hole is constant over the event horizon. The first law is the relationship between black hole solutions varying by mass δM, event horizon area $\delta A_{\mathcal{H}}$ and angular momentum δJ

$$\delta M = \frac{\kappa_{\mathcal{H}}}{8\pi}\delta A_{\mathcal{H}} + \Omega\delta J.\tag{1.8}$$

It is clear that the surface gravity enters these two laws in the same way that temperature enters the laws of thermodynamics. At the time they were discovered, however, it was thought that this could be nothing more than an analogy because an object with a non-zero physical temperature must radiate a thermal spectrum; something a black hole could not do as nothing can escape from it. If $\kappa_{\mathcal{H}}$ really were to correspond to the temperature of the black hole then it is also clear from the first law (1.8) that the event horizon area $A_{\mathcal{H}}$ would correspond to entropy. Bekenstein, taking the idea of black hole entropy seriously, considered the efficiency of a heat engine in which one of the sinks was a non-rotating black hole. By comparing the efficiency of this heat engine with that of a Carnot cycle he provided further evidence the surface gravity $\kappa_{\mathcal{H}}$ really does correspond to a temperature [Bekenstein, 1973], but the widespread acceptance of the idea would have to wait for the work of Hawking.

Hawking showed [Hawking, 1975], in the framework of quantum field theory on a fixed

background spacetime, that black holes do in fact appear to static observers an infinite spatial distance away to be radiating like a black body with temperature

$$T = \frac{\kappa_{\mathcal{H}}}{2\pi}. \tag{1.9}$$

There is a simplified version of why this is the case; a fluctuation takes place in which a virtual pair of particles is formed just outside the horizon. One of the particles, which is in a negative-energy state and therefore would classically be repelled by the black hole, quantum-mechanically tunnels through the horizon, thereby reducing the black hole's mass. The other particle has positive energy and escapes to infinity. The overall effect is that the black hole loses mass which is carried to infinity by the flux of radiation. The most remarkable part of the result is that the flux of radiation is the same as that of a black body with temperature given by equation (1.9). With Hawking's result the relation between black hole horizon area and entropy $\mathcal{S}_{BH}$ suggested by equation (1.8) is shown to be more than an analogy, and furthermore the constant of proportionality can be determined:

$$\mathcal{S}_{BH} = \frac{A_{\mathcal{H}}}{4}, \tag{1.10}$$

in units such that $G = k_B = c = \hbar = 1$.

It is then natural to ask whether this entropy is special to black hole horizons or should there be an entropy ascribed to all horizons, such as the cosmological horizons of de Sitter spacetime or the acceleration horizons of Rindler spacetime?

In de Sitter spacetime Hawking and Gibbons showed [Gibbons and Hawking, 1977b] that an observer moving on a timelike geodesic observes a thermal bath of radiation with temperature

$$T = \frac{1}{2\pi\ell}, \tag{1.11}$$

where ℓ is the de Sitter radius. The relationship between this temperature and an entropy is less obvious than the black hole case because there is no equivalent version of the first law, equation (1.8), that is applicable to de Sitter spacetimes. As the mass M in the first law of black hole mechanics is the ADM mass [Arnowitt et al.,] it is only defined for asymptotically flat spacetimes, something de Sitter spacetime is not. Therefore, in order to relate this temperature to an entropy using the standard relation

$$\frac{\partial \mathcal{S}}{\partial E} = \frac{1}{T},$$ (1.12)

some sense would have to be made of what should correspond to E. Gibbons and Hawking circumvented the problem by considering the entropy of a Schwarzschild-de Sitter solution, in which they could take E to be the mass of the black hole. They found the same relation between the entropy and area of the cosmological event horizon as the black hole case, given in equation (1.10).

An investigation of the high frequency particles near the horizon of a black hole led Unruh to discover that an accelerating Rindler observer in Minkowski spacetime sees a thermal bath of particles with temperature

$$T = \frac{a}{2\pi},$$ (1.13)

where a is the observer's proper acceleration [Unruh, 1976]. In comparison to the black hole and de Sitter horizons, there is much less agreement as to whether an entropy should be ascribed to the Rindler horizon. Competing viewpoints can be found in [Hawking and Horowitz, 1996] and [Jacobson and Parentani, 2003].

Now that black hole entropy has been described thermodynamically it is natural to seek to reproduce it using statistical mechanics. The first attempt to do so used a path integral to calculate the canonical partition function of a black hole and de Sitter space [Gibbons and Hawking, 1977a]. The canonical partition function is defined by

$$Z = tr \ e^{-\beta H},$$ (1.14)

where β is the inverse temperature and H is the Hamiltonian of the system. If the system were a single quantum particle the trace could be taken over a position basis

$$Z = \int dx \langle x|e^{-\beta H}|x \rangle.$$ (1.15)

By comparison with equation (1.3) we see that if we set $T = -i\beta$ then

$$Z = \int \mathcal{D}H \ e^{iS/\hbar},$$ (1.16)

where the path integral is taken over paths that are periodic in imaginary time with period β. Gibbons and Hawking took this as the definition of the canonical path integral for gravity. By expanding the action as a Taylor series about the classical solution, and approximating the path integral by the first term in the expansion, they calculated the canonical partition function and from it found an entropy for both black hole and de Sitter horizons of

$$\mathcal{S} = \frac{A_{\mathcal{H}}}{4},$$ (1.17)

as expected. The calculation was applied to the Rindler horizon in [Laflamme, 1987]. Hawking and Gibbons noted that a black hole has a negative specific heat capacity and therefore a canonical ensemble analysis is not fully justifiable. As an alternative they suggested that a microcanonical ensemble be used. This direction of enquiry was picked up by Brown, York and collaborators, eventually culminating in a definition of a gravitational microcanonical density of states and a perturbative calculation of the entropy [Brown and York, 1993b].

1.1 Thesis Outline

The objective of this thesis is to establish a framework in which the gravitational path integrals of the Brown and York program can be calculated non-perturbatively using causal dynamical triangulations (CDTs). There are a number of reasons why this might be fruitful. The microcanonical density of states path integral and the path integrals in CDTs both require the spacetime histories integrated over to be finite. For the density of states this is to stabilise the ensemble [York, 1986]; in CDTs it is so that the calculations can be performed numerically without requiring an infinite number of input data. Both programs also emphasise the important role of Lorentzian spacetimes in the path integral.

CDT numerical calculations become significantly more complex in higher spacetime dimensions; so it would be beneficial to start in three spacetime dimensions. The gravitational microcanonical density of states is therefore extended to general spacetime dimension in section (2.3). Brown and York's calculation of black hole entropy involves a version of the microcanonical density of states that is modified to deal with a black hole's spacetime topology - this calculation is extended to general spacetime dimension in section (2.5).

CDTs require a positive cosmological constant, which rules out a calculation of black hole entropy in $3d$ because there are no black hole solutions to Einstein's field equations with positive cosmological constant. This motivates the application of the microcanonical density of states to de Sitter spacetime in section (2.6). The simpler topology should also afford an opportunity to evaluate the merits of the microcanonical density of states on its own, without the modifications required to calculate black hole entropy. However, it is shown that it is impossible to find appropriate boundary conditions to allow the horizon entropy to be calculated in this way. Instead, the de Sitter horizon entropy is calculated in section (2.7) by an analysis that closely follows the black hole case but sheds new light on the interpretation. The same analysis is then applied to Rindler horizons in section (2.8).

The applicability of the microcanonical ensemble to black hole mechanics is discussed at a fundamental level in section (2.10), where the discussion focuses on the role of black hole entropy as a function of state. To emphasise this role, black hole entropy is derived as a function of state in section (2.9). The rotational ensemble is proposed as a more compelling candidate to describe black hole entropy in section (2.12). As the microcanonical ensemble is central to many justifications of statistical mechanics, the rotational ensemble is introduced following Jaynes' information theory approach to statistical mechanics [Jaynes, 1957]. Jaynes' methods allow a simple derivation of a new ensemble - referred to as the angular momentum ensemble - in section (2.14).

The application of CDTs to the Brown and York density of states is undertaken in chapter (3). The required Regge calculus expressions for extrinsic curvature are extended to the case of Lorentzian hypersurfaces in section (3.3). In section (3.4) they are used to derive Regge calculus expressions for the Brown and York quasilocal energy-momentum quantities. The foliation and boundary structure of the simplicial spacetimes to be used in the CDT path integral are given in section (3.5). The Regge version of the black hole density of states action is derived in section (3.7) and specialised to the $3d$ and $4d$ cases in sections (3.8) and (3.9) respectively.

In separate work to the main thesis objective, chapter (4) is an investigation of the consequences for the equilibrium temperature in a gravitational field of the recent suggestion that entropy is observer dependent [Marolf et al., 2004].

Chapter 2

The Microcanonical Density of States

2.1 Introduction

The first step in progressing beyond the canonical ensemble formalism of Hawking and Gibbons is to deal with the problem of black holes being thermodynamically unstable. The negative heat capacity of a Schwarzschild black hole is evident upon substituting the expression for surface gravity $\kappa_{\mathcal{H}}$ (1.7) into the equation for black hole temperature (1.9) and setting $J = 0$:

$$T = \frac{1}{8\pi M}.$$ (2.1)

As the mass and temperature are inversely related, a Schwarzschild black hole that absorbs thermal energy from its surroundings increases in mass, whilst decreasing in temperature. A self-perpetuating cycle is thereby started as the cooler black hole absorbs more thermal energy and becomes even cooler.

York showed that to thermodynamically stabilise a black hole it should be placed in a spatially finite "box"; or equivalently, boundary conditions should be imposed on a

spatial surface surrounding it [York, 1986]. This is the first element of Brown and York's program: the microcanonical density of states should be a function of boundary data on a finite spatial boundary.

The next step is to stipulate what this boundary data is and how the density of states should be constructed as a function of it. Brown and York motivate their definition [Brown and York, 1993b] as a generalisation of the path integral form of the density of states $\nu(E)$ in one particle non-relativistic quantum mechanics, defined by

$$\nu(E) = tr\ \delta(E - \hat{H}), \tag{2.2}$$

where $\hat{H}$ is the Hamiltonian operator for the system. If the trace is taken over a set of position eigenstates, rather than the more standard energy eigenstates, and the delta function is expressed in its integrated exponent form, the density of states becomes:

$$\nu(E) = \int dx\ \langle x|\delta(E - \hat{H})|x\rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dT \int dx\ e^{iET/\hbar}\langle x|e^{-i\hat{H}T/\hbar}|x\rangle. \tag{2.3}$$

The matrix element in this expression can be replaced by its path integral form using equation (1.3), suitably modified so that the starting and end points are the same:

$$\langle x|e^{-i\hat{H}T/\hbar}|x\rangle = \int_{x(0)=x}^{x(T)=x} DH\ e^{iS/\hbar}. \tag{2.4}$$

Inserting this into equation (2.3) leads to

$$\nu(E) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dT \int dx \int_{x(0)=x}^{x(T)=x} DH\ e^{i(S+ET)/\hbar}. \tag{2.5}$$

Brown and York show that this form of the density of states is equivalent to

$$\nu(E) = \int DH_p\ e^{iS_E/\hbar}, \tag{2.6}$$

where S_E is Jacobi's version of the action in which the energy E is fixed, rather than the more standard time T, and the subscript p denotes periodic histories. In words: the

density of states at energy E is a path integral over histories periodic in real time, with each history weighted by a phase involving Jacobi's action of fixed energy E. Equation (2.6) is the key result that is generalised by Brown and York to the case of self-gravitating systems.

2.2 Quasilocal Energy Momentum

Before the density of states, $\nu(E)$, can be generalised to a gravitational density of states, an appropriate gravitational notion must be found to correspond to the energy E of the particle. The obvious candidate would be the ADM mass [Arnowitt et al.,], which is a state variable in black hole thermodynamics and appears in the first law of black hole mechanics, equation (1.8), as energy appears in the first law of thermodynamics. The ADM mass is only defined, however, in asymptotically flat spacetimes and will not always be compatible with boundary data imposed on a finite spatial boundary. A notion of energy-momentum that is more local than the ADM mass is therefore required.

It is not possible to find a truly local definition of energy-momentum in general relativity because a coordinate system can always be found in which the Christoffel symbols vanish at any given point. And if the gravitational field can be made to vanish at any point there can be no notion of its local energy-momentum at that point. In contrast, the Christoffel symbols can only be made to vanish over an extended region if the region has Minkowski geometry, and therefore a meaningful notion of energy-momentum can be attributed to a spatial region. Such a notion of energy-momentum is referred to as quasilocal energy-momentum. In searching for boundary data for the density of states, Brown and York were led to a definition of quasilocal energy-momentum [Brown and York, 1993a] that is more general than the ADM mass and reduces to the ADM mass when restricted to asymptotically flat spacetimes.

Searching for the quasilocal energy-momentum of a region motivates consideration of
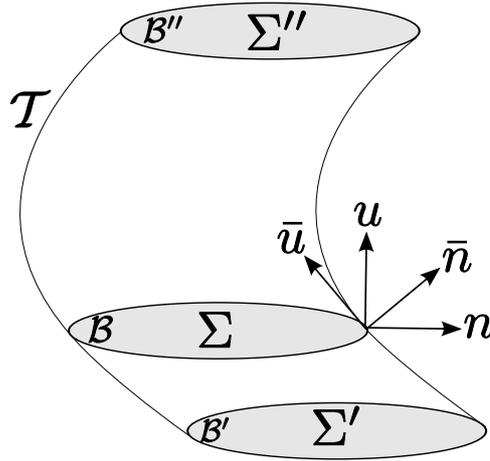
Figure 2.1: The manifold structure shown for the case of a $3d$ spacetime. $\mathcal{M}$ - the whole cylinder - has a $2d$ timelike boundary $\mathcal{T}$ - the surface of the cylinder. $\mathcal{M}$ is foliated into $2d$ spatial hypersurfaces Σ, of which one example - the shaded circle - is shown. The intersection of Σ and $\mathcal{T}$ foliates $\mathcal{T}$ in to 1d spatial hypersurfaces $\mathcal{B}$ (the circle).

a finite, d dimensional spacetime region $\mathcal{M}$; with a $(d{-}1)$ dimensional timelike boundary $\mathcal{T}$ arising from the propagation through time of a $(d{-}2)$ dimensional spatial boundary $\mathcal{B}$ that encloses a spatial region Σ (see figure(2.1)). An initial surface Σ' and final surface Σ'' also form part of the boundary of $\mathcal{M}$. Their boundaries $\mathcal{B}'$ and $\mathcal{B}''$ shall be referred to as the "corners".

In Brown and York's original analysis the condition that the spatial foliation Σ be orthogonal to the timelike boundary $\mathcal{T}$ was imposed. In this case, u_μ, the timelike unit normal of Σ, is tangent to $\mathcal{T}$ and n_μ, the spacelike unit normal of $\mathcal{B}$ lying in Σ, is also the normal vector of $\mathcal{T}$. This condition was later removed [Lau, 1996, Booth and Mann, 1999]. We will stick to the general case because a boundary on the horizon will be introduced for the black hole entropy calculation and the orthogonality condition cannot apply to a null surface. Let $\bar{n}_\mu$ be the outward pointing spacelike unit normal of $\mathcal{T}$, $\bar{u}_\mu$ be the timelike unit normal of $\mathcal{B}$ lying in $\mathcal{T}$ and define a velocity parameter $\bar{\theta}$ by $\sinh\bar{\theta} = -u^\mu\bar{n}_\mu$. The original orthogonality condition on the foliation corresponds to $u^\mu\bar{n}_\mu = 0$ and therefore

$\bar{\theta} = 0$.

With this manifold topology the gravitational action is

$$S[g] = \frac{1}{2\kappa} \int_{\mathcal{M}} d^d x \sqrt{-g} (\Re - 2\Lambda) + \frac{1}{\kappa} \int_{\Sigma'}^{\Sigma''} d^{d-1} x \sqrt{h} K$$
$$- \frac{1}{\kappa} \int_{\mathcal{T}} d^{d-1} x \sqrt{-\gamma} \,\Theta - \frac{1}{\kappa} \int_{\mathcal{B}'}^{\mathcal{B}''} d^{d-2} x \sqrt{\sigma} \bar{\theta}, \quad (2.7)$$

in which $g_{\mu\nu}$ and $\Re_{\mu\nu\alpha\beta}$ are, respectively, the metric and curvature tensor of $\mathcal{M}$; h_{ij} and K_{ij} are the metric and extrinsic curvature of Σ; γ_{ij} and Θ_{ij} are the Lorentzian metric and extrinsic curvature of $\mathcal{T}$; and σ_{ab} is the metric of $\mathcal{B}$. The notation $\int_{\Sigma'}^{\Sigma''}$ refers to $\int_{\Sigma''} - \int_{\Sigma'}$ and the constants are $\kappa = 8\pi$ and the cosmological constant Λ.

Under variation of g the first order variation of the action is

$$\delta S[g] = \frac{1}{2\kappa} \int_{\mathcal{M}} d^d x \sqrt{-g} \, G_{\mu\nu} \delta g^{\mu\nu} + \frac{2}{d-2} \int_{\Sigma'}^{\Sigma''} d^{d-1} x \, P^{ij} \delta h_{ij}$$
$$+ \frac{2}{d-2} \int_{\mathcal{T}} d^{d-1} x \, \Pi^{ij} \delta\gamma_{ij} - \frac{1}{\kappa} \int_{\mathcal{B}'}^{\mathcal{B}''} d^{d-2} x \, \bar{\theta} \delta \sqrt{\sigma}, \quad (2.8)$$

where $G_{\mu\nu}$ is the Einstein tensor

$$G_{\mu\nu} = \Re_{\mu\nu} - \frac{\Re g_{\mu\nu}}{2} - g_{\mu\nu}\Lambda, \tag{2.9}$$

P^{ij} is the canonical momentum of the surface Σ

$$P^{ij} = \frac{\sqrt{h}}{2\kappa} \left(K h^{ij} - K^{ij} \right), \tag{2.10}$$

and Π^{ij} is the momentum canonically conjugate to the metric γ_{ij}

$$\Pi^{ij} = -\frac{\sqrt{-\gamma}}{2\kappa} \left(\Theta \gamma^{ij} - \Theta^{ij} \right). \tag{2.11}$$

To obtain the equations of motion, $G_{\mu\nu} = 0$, on setting $\delta S = 0$ in equation (2.8), the variation of g must be restricted so that the boundary metrics h_{ij} and γ_{ij} are held fixed

and therefore $\delta h_{ij} = \delta \gamma_{ij} = 0$. This means an arbitrary function of the boundary metric $S_0[\gamma_{ij}]$ can be added to the action (2.7) with no change to the equations of motion.

In classical mechanics the relationship between the Hamiltonian and action of a system is given by the Hamilton-Jacobi equation

$$H = -\frac{\partial S}{\partial T}, \tag{2.12}$$

where T is the time between the initial and final configurations. In general relativity the notion of time between an initial and final configuration is encoded in the Lorentzian metric of the boundary $\mathcal{T}$. Brown and York therefore generalise the notion of energy to a quasilocal stress tensor defined by

$$\tau^{ij} = \frac{(d-2)}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma_{ij}}. \tag{2.13}$$

As τ^{ij} is a tensor defined on the boundary $\mathcal{T}$ it is actually a stress tensor surface density and must be integrated over the boundary $\mathcal{B}$ to find the total stress. From the action (2.7) and its variation (2.8) τ^{ij} is

$$\tau^{ij} = \frac{2}{\sqrt{-\gamma}} \Pi^{ij}. \tag{2.14}$$

The quasilocal energy surface density ε is defined as the normal-normal projection of the quasilocal stress tensor (2.13) onto the surface $\mathcal{B}$, the quasilocal momentum surface density j_a as the normal-tangential projection and the quasilocal spatial stress surface density s^{ab} as the tangential-tangential projection of the quasilocal stress tensor:

$$\varepsilon = \bar{u}_i \bar{u}_j \tau^{ij}, \tag{2.15}$$

$$j_a = -\sigma_{ai} \bar{u}_j \tau^{ij},$$

$$s^{ab} = \sigma^a_i \sigma^b_j \tau^{ij}.$$

Under the canonical foliation of $\mathcal{M}$ into spatial leaves Σ, the extrinsic curvature of $\mathcal{T}$ can be decomposed into components parallel and perpendicular to Σ, and a mixed component as [Brown et al., 2000]

$$\Theta_{\mu\nu} = \cosh\bar{\theta}k_{\mu\nu} + \sinh\bar{\theta}\ell_{\mu\nu} + 2\sigma^{\alpha}_{(\mu}\bar{u}_{\nu)}\left(n^{\beta}K_{\alpha\beta} - \partial_{\alpha}\bar{\theta}\right) + \bar{u}_{\mu}\bar{u}_{\nu}\bar{n}_{\beta}\bar{a}^{\beta}, \tag{2.16}$$

where $k_{\mu\nu} = -\sigma^{\alpha}_{\mu}D_{\alpha}n_{\nu}$ is the extrinsic curvature of $\mathcal{B}$ embedded in Σ; D_{α} is the covariant derivative with respect to the geometry of Σ; $\ell_{ab} = -\sigma^{c}_{a}\sigma^{d}_{b}\nabla_{c}u_{d}$ is the extrinsic curvature of $\mathcal{B}$ embedded in a surface with normal vector n_{μ}; and the acceleration of $\bar{u}_{\mu}$ is $\bar{a}_{\mu} = \bar{u}^{\nu}\nabla_{\nu}\bar{u}_{\mu}$. It follows immediately upon taking the trace of this expression that

$$\Theta = \cosh\bar{\theta}k + \sinh\bar{\theta}\ell - \bar{n}_{\beta}\bar{a}^{\beta}. \tag{2.17}$$

Substituting these canonical decompositions of $\Theta_{\mu\nu}$ and Θ into the quasilocal stress tensor components in equations (2.15) leads to

$$\kappa\varepsilon = (\cosh\bar{\theta})k + (\sinh\bar{\theta})\ell \tag{2.18}$$

$$\kappa j_{a} = \sigma^{i}_{a}K_{ij}n^{j} - \partial_{a}\bar{\theta} \tag{2.19}$$

$$\kappa s^{ab} = (\cosh\bar{\theta})(k^{ab} - k\sigma^{ab}) + \bar{n}\cdot\bar{a}\,\sigma^{ab} + (\sinh\bar{\theta})(\ell^{ab} - \ell\sigma^{ab}) \tag{2.20}$$

To obtain the total energy E of the local region Σ the quasilocal energy surface density is integrated over the spatial boundary $\mathcal{B}$

$$E = \int_{\mathcal{B}} d^{(d-2)}x\sqrt{\sigma}\varepsilon. \tag{2.21}$$

In the case of an asymptotically flat spacetime, with $\mathcal{B}$ taken out to infinity, this expression agrees with the ADM mass [Brown and York, 1993a]. If the boundary $\mathcal{B}$ has

rotational symmetry and therefore contains the orbits of a Killing vector field ϕ^i; then it is possible to define a total angular momentum J

$$J = \int_{\mathcal{B}} d^{(d-2)}x \sqrt{\sigma} j_i \phi^i. \tag{2.22}$$

Again, if the spacetime is asymptotically flat, this expression reduces to the ADM expression for angular momentum [Brown and York, 1993a].

2.3 The Microcanonical Density of States in General Spacetime Dimension

The microcanonical density of states [Brown and York, 1993b] will now be presented and extended to the case of general spacetime dimension. In preparation for this, the d-dependent factor was included in the variation of the standard action, equation (2.8). This seemingly innocuous term leads to severe difficulties in extending the microcanonical density of states beyond 4 spacetime dimensions.

The one-particle density of states $\nu(E)$ is a function of energy. In non gravitational physics energy is the value of the Hamiltonian that generates unit time translations. In general relativity time has a "many-fingered" character - there are many ways to slice the spacetime into spatial hypersurfaces. The gravitational Hamiltonian is found by expressing the action in canonical form. Given the foliation of the spacetime manifold $\mathcal{M}$ into spatial hypersurfaces Σ, the ADM form of the spacetime metric g, with lapse function N and shift vector V^i, is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N^2 dt^2 + h_{ij}(dx^i + V^i dt)(dx^j + V^j dt). \tag{2.23}$$

The ADM form of the $\mathcal{T}$ metric γ_{ij}, using the same shift vector but allowing for a different lapse function $\bar{N}$, is

$$\gamma_{ij}dx^i dx^j = -\bar{N}^2 dt^2 + \sigma_{ab}(dx^a + V^a dt)(dx^b + V^b dt). \qquad (2.24)$$

The action (2.8) in canonical form is

$$S[g] = \int_{\mathcal{M}} d^d x \sqrt{-g} \left[P^{ij}\dot{h}_{ij} - N\mathcal{H} - V^i \mathcal{H}_i \right] - \int_{\mathcal{T}} d^{d-1}x \sqrt{\sigma} \left[\bar{N}\varepsilon - V^i j_i + \bar{N}\bar{\jmath}_\vdash \theta \right], \quad (2.25)$$

where $\bar{\jmath}_\vdash = -\ell/\kappa$; $\dot{h}_{ij}$ is the Lie derivative of h_{ij} along the timelike direction;

$$\mathcal{H} = \frac{2\kappa}{\sqrt{h}} \left(P_{ij}P^{ij} - \frac{P^2}{d-2} \right) - \frac{\sqrt{h}R}{2\kappa} \qquad (2.26)$$

is the Hamiltonian constraint and

$$\mathcal{H}_i = -2D_j P_j^i \qquad (2.27)$$

is the momentum constraint. The Hamiltonian is therefore:

$$H = \int_{\mathcal{M}} d^d x \sqrt{-g} \left[N\mathcal{H} + V^i \mathcal{H}_i \right] + \int_{\mathcal{T}} d^{d-1}x \sqrt{\sigma} \left[\bar{N}\varepsilon - V^i j_i + \bar{N}\bar{\jmath}_\vdash \theta \right]. \qquad (2.28)$$

The energy surface density ε is the value (per unit boundary area) of the Hamiltonian that generates unit magnitude time translations of the boundary $\mathcal{B}$ and the momentum surface density j_a is the value (per unit boundary area) of the Hamiltonian that generates spatial diffeomorphisms on the boundary. Consequently, it is proposed in [Brown and York, 1993b] that the gravitational microcanonical density of states should be a functional of *both* ε and j_a. The validity of this will be discussed in section (2.10); but accepting it for the time being, the goal is to find the gravitational equivalent of Jacobi's action. Such an action will require that ε and j_a are held fixed on the boundary in order to obtain the equations of motion.

To find this action, it is useful to begin by making a canonical decomposition of the $\mathcal{T}$ boundary term in the variation of the standard gravitational action (2.8). The ADM decomposition of the variation of γ_{ij} is [Brown et al., 2000],

$$\delta\gamma_{ij} = \sigma^a_{(i}\sigma^b_{j)}\delta\sigma_{ab} - \frac{2}{\bar{N}}\sigma_{a(i}\bar{u}_{j)}\delta V^a - 2\bar{u}_{(i}\bar{u}_{j)}\frac{\delta\bar{N}}{\bar{N}}, \tag{2.29}$$

which can be inserted into the variation of the standard action, equation (2.8), resulting in

$$\delta S[g] = \frac{1}{2\kappa}\int_{\mathcal{M}} d^dx\sqrt{-g}\,G_{\mu\nu}\delta g^{\mu\nu} + \frac{2}{d-2}\int_{\Sigma'}^{\Sigma''} d^{d-1}x\,P^{ij}\delta h_{ij}$$
$$- \frac{2}{d-2}\int_{\mathcal{T}} d^{d-1}x\,\sqrt{\sigma}\left(\varepsilon\delta\bar{N} - j_k\delta V^k - \frac{\bar{N}}{2}s^{ab}\delta\sigma_{ab}\right) - \frac{1}{\kappa}\int_{\mathcal{B}'}^{\mathcal{B}''} d^{d-2}x\,\bar{\theta}\delta\sqrt{\sigma}. \tag{2.30}$$

To obtain the equations of motion from the standard action, the appropriate quantities to hold fixed on $\mathcal{T}$ are therefore the lapse $\bar{N}$, shift V^i and σ_{ab}, the two metric of $\mathcal{B}$.

The microcanonical action S_M is defined by the addition of a $\mathcal{T}$-surface-term to the standard action

$$S_M[g] = S[g] + \frac{2}{d-2}\int_{\mathcal{T}} d^{d-1}x\sqrt{\sigma}\left[\bar{N}\varepsilon - V^i j_i\right] \tag{2.31}$$

and has the following $\mathcal{T}$ boundary variation:

$$(\delta S_M[g])_{\mathcal{T}} = \frac{1}{2\kappa}\int_{\mathcal{M}} d^dx\sqrt{-g}\,G_{\mu\nu}\delta g^{\mu\nu} + \int_{\mathcal{T}} d^{d-1}x\left[\bar{N}\delta(\sqrt{\sigma}\varepsilon) - V^i\delta(\sqrt{\sigma}j_i) + \frac{\bar{N}}{d-2}\sqrt{\sigma}s^{ab}\delta\sigma_{ab}\right]. \tag{2.32}$$

The effect of the additional boundary term is to modify the requirement that $\bar{N}$ and V^i be held fixed on $\mathcal{T}$ to the requirement that ε and j_a be held fixed. The definition of the microcanonical action given in equation (2.31), includes a d-dependent factor, $\frac{2}{d-2}$, in front of the $\mathcal{T}$ boundary term. This new factor is necessary so that substituting δS in equation (2.30) into δS_M will result in the complete cancellation of the $\delta\bar{N}$ and δV^i terms. This term is unity in $d = 4$ and is not included in the definition of Brown and York.

As the microcanonical action describes the system at fixed ε, j_i and σ_{ab}, it is the required gravitational equivalent of the Jacobi action. Brown and York define the gravitational density of states by [Brown and York, 1993b]

$$D[\varepsilon, j, \sigma] = \sum_{\mathcal{M}} \int \mathcal{D}H_p \exp(iS_M). \tag{2.33}$$

The path integral in this expression is taken over all periodic Lorentzian spacetime histories with the appropriate "microcanonical" boundary conditions ε, j and σ_{ab}. These boundary data can be obtained from a surface $\mathcal{B}$ embedded into a classical solution to Einstein's equations. As the metric is integrated over, the boundary data is held fixed on each spatial $\mathcal{B}$ of the timelike boundary $\mathcal{T}$. In this sense the boundary data are "constant in time". The sum over $\mathcal{M}$ denotes a sum over spacetime topologies with the appropriate boundary topology.

At the zero-loop level, the density of states path integral is approximated by the exponential of the action evaluated on the classical solution

$$D[\varepsilon, j, \sigma] \approx \exp\left(iS_M|_{Classical}\right). \tag{2.34}$$

Once the density of states is known it leads to the entropy via the relationship

$$\mathcal{S}[\varepsilon, j, \sigma] = \log\left(D[\varepsilon, j, \sigma]\right), \tag{2.35}$$

in units such that Boltzmann's constant is unity. The entropy is then given approximately as:

$$\mathcal{S}[\varepsilon, j, \sigma] \approx iS_M|_{Classical}. \tag{2.36}$$

2.4 The Entropy of Minkowski Spacetime in General Spacetime Dimension

In inertial coordinates, the metric of d dimensional Minkowski spacetime is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_{d-2}^2, \tag{2.37}$$

where $d\Omega_{d-2}^2$ is the metric of a $(d-2)$-sphere. To calculate the entropy of a region of Minkowski spacetime from the microcanonical density of states, the most natural canonical foliation to employ is surfaces of constant t as spatial leaves Σ. The lapse N in inertial coordinates is then unity and the shift vector vanishes. An appropriate spatial boundary $\mathcal{B}$ from which to take boundary data is a sphere of fixed radius, with the boundary $\mathcal{T}$ given by the time history of these spheres. This is an orthogonal boundary so $\bar{N} = N$ and $\bar{\theta} = 0$. Microcanonical boundary data taken from such a $\mathcal{B}$ are independent of time.

To evaluate the microcanonical action on the classical solution it helps if it is first converted to canonical form. This can be obtained from the canonical form of the standard action, equation (2.25), and the definition of $S_M[g]$, equation (2.31):

$$S_M[g] = \int_{\mathcal{M}} d^d x \sqrt{-g} \left[P^{ij} \dot{h}_{ij} - N\mathcal{H} - V^i \mathcal{H}_i \right] + \frac{4-d}{d-2} \int_{\mathcal{T}} d^{d-1} x \sqrt{\sigma} \left[\bar{N}\varepsilon - V^i j_i \right]. \tag{2.38}$$

Evaluated on the Minkowski metric, the terms integrated over $\mathcal{M}$ vanish in any spacetime dimension because the $(d-1)$-dimensional spatial metric h_{ij} is time independent and the constraints $\mathcal{H} = 0$ and $\mathcal{H}_i = 0$ are satisfied by any classical solution. As the shift vector is zero, only one term survives and, from equation (2.36), the resulting entropy is

$$\mathcal{S}[\varepsilon, j, \sigma] \approx i \frac{4-d}{d-2} \int_{\mathcal{T}} d^{d-1} x \sqrt{\sigma}\varepsilon. \tag{2.39}$$

In 4 dimensions this entropy vanishes, which is to be expected because there are no horizons present. However, the entropy is non-vanishing in dimensions other than 4, which is clearly a problem for the Brown and York program.

The calculated entropy of Minkowski spacetime would vanish in any dimension if the d-dependent factor, $\frac{2}{d-2}$, was omitted from the definition of S_M in equation (2.31). The canonical form of S_M would then contain only the $\mathcal{M}$ term in equation (2.38). However, in dimensions other than 4, there would be additional $\delta \bar{N}$ and δV terms in δS_M, equation (2.32), and therefore the interpretation of the microcanonical action in terms of fixed microcanonical boundary conditions would be lost. It is impossible to choose a d-dependent factor in the definition of the microcanonical action such that the interpretation is valid in all d and the entropy of Minkowski spacetime vanishes in all d.

2.5 Black Hole Entropy in General Spacetime Dimension

Brown and York's calculation of black hole entropy [Brown and York, 1993b] is repeated in this section for general spacetime dimension. It will be shown that the calculation of the entropy can be made to work in all dimensions, but the interpretation of the result is lost in dimensions other than 4.

The topology of a $4d$ Schwarzschild black hole poses an immediate obstacle to an application of the microcanonical path integral - a $2d$ boundary $\mathcal{B}$, placed in the region exterior to the horizon, does not enclose a $3d$ submanifold because of the Einstein-Rosen bridge. Brown and York deal with this by introducing a second 2-boundary $\mathcal{B}_\mathcal{H}$, with time history $\mathcal{T}_\mathcal{H}$, inside $\mathcal{B}$ and $\mathcal{T}$. A tilde is used to distinguish geometrical properties of $\mathcal{T}_\mathcal{H}$ from the corresponding properties of the outer timelike boundary $\mathcal{T}$. For example, $\tilde{\gamma}_{ij}$ is the metric of $\mathcal{T}_\mathcal{H}$ and $\tilde{n}_\mu$ is its outward-pointing unit normal. The standard action (2.7) is easily modified to account for the presence of the $\mathcal{T}_\mathcal{H}$ boundary by adding tilded

versions of the $\mathcal{T}$ boundary terms:

$$S[g] = \frac{1}{2\kappa} \int_{\mathcal{M}} d^d x \sqrt{-g}(\Re - 2\Lambda) - \frac{1}{\kappa} \int_{\mathcal{T}} d^{d-1}x\sqrt{-\gamma}\,\Theta - \frac{1}{\kappa} \int_{\mathcal{B}'}^{\mathcal{B}''} d^{d-2}x\sqrt{\sigma}\bar\theta$$

$$- \frac{1}{\kappa} \int_{\mathcal{T}_{\mathcal{H}}} d^{d-1}x\sqrt{-\tilde\gamma}\,\tilde\Theta - \frac{1}{\kappa} \int_{\mathcal{B}_{\mathcal{H}}'}^{\mathcal{B}_{\mathcal{H}}''} d^{d-2}x\sqrt{\tilde\sigma}\tilde\theta. \quad (2.40)$$

Similarly to the $\mathcal{T}$ boundary, the requirement that $\mathcal{T}_{\mathcal{H}}$ be orthogonal to the spatial surfaces Σ is imposed, expressed formally as $u^\mu \tilde n_\mu = 0$. The ADM decomposition of the metric $\tilde\gamma_{ij}$ is:

$$\tilde\gamma_{ij}dx^i dx^j = -\tilde N^2 dt^2 + \tilde\sigma_{ab}(dx^a + V^a dt)(dx^b + V^b dt). \quad (2.41)$$

The standard action in canonical form can be obtained by adding tilded versions of the $\mathcal{T}$ boundary terms to the canonical form of the standard action for a single boundary in equation (2.25)

$$S[g] = \int_{\mathcal{M}} d^d x \sqrt{-g} \left[P^{ij}\dot h_{ij} - N\mathcal{H} - V^i \mathcal{H}_i \right] - \int_{\mathcal{T}} d^{d-1}x\sqrt{\sigma} \left[\bar N\varepsilon - V^i j_i + \bar N \bar j_{\vdash}\bar\theta \right]$$

$$- \int_{\mathcal{T}_{\mathcal{H}}} d^{d-1}x\sqrt{\tilde\sigma} \left[\tilde N\tilde\varepsilon - V^i \tilde j_i + \tilde N \tilde j_{\vdash}\tilde\theta \right]. \quad (2.42)$$

The variation of this action, $\delta S[g]$, can similarly be obtained, from equation (2.8), by addition of tilded terms to match those on $\mathcal{T}$.

Instead of using the microcanonical density of states to calculate black hole entropy, Brown and York actually consider a "black hole density of states". This is a path integral of the same form as the microcanonical density of states (2.33), but weighted by the following action:

$$S_{BH}[g] = S[g] + \int_{\mathcal{T}} d^{d-1}x\sqrt{\sigma} \left[\bar N\varepsilon - V^i j_i \right] - \frac{2}{d-2} \int_{\mathcal{T}_{\mathcal{H}}} d^{d-1}x\sqrt{\tilde\sigma}\frac{\tilde N\tilde s_a^a}{2}. \quad (2.43)$$

This "black hole action" differs from the microcanonical action by the use of dissimilar terms on the boundaries $\mathcal{T}$ and $\mathcal{T}_{\mathcal{H}}$. The d-dependent factors, in front of the $\mathcal{T}$ and $\mathcal{T}_{\mathcal{H}}$ boundary terms in S_{BH}, have been chosen here so that the black hole density of states leads to the expected entropy, in any dimension. Their effect on the interpretation of the result will be given after the calculation of entropy. In $4d$, S_{BH} in equation (2.43) reduces to the black hole action of Brown and York. To obtain boundary conditions that represent a black hole a closed $(d-2)$ submanifold $\mathcal{B}$ is placed in the exterior region of a stationary, axisymmetric black hole solution of Einstein's equations and a closed $(d-2)$ submanifold $\mathcal{B}_{\mathcal{H}}$ on is placed on the horizon.

The canonical form of the action S_{BH} is

$$S_{BH}[g] = \int_{\mathcal{M}} d^d x \sqrt{-g} \left[P^{ij} \dot{h}_{ij} - N\mathcal{H} - V^i \mathcal{H}_i \right] - \int_{\mathcal{T}} d^{d-1} x \sqrt{\sigma} \bar{N} \bar{\jmath}_{\vdash} \bar{\theta}$$

$$- \int_{\mathcal{T}_{\mathcal{H}}} d^{d-1} x \sqrt{\tilde{\sigma}} \left[\tilde{N}\tilde{\varepsilon} - V^i \tilde{\jmath}_i + \tilde{N} \tilde{\jmath}_{\vdash} \tilde{\theta} + \frac{\tilde{N} \tilde{s}^a_a}{d-2} \right]. \quad (2.44)$$

The trace of the spatial stress $\tilde{s}^a_a$ can be obtained from equation (2.20), suitably modified by the addition of tildes

$$\tilde{s}^a_a = (3-d)\tilde{\varepsilon} + (d-2)\frac{\tilde{n} \cdot \tilde{a}}{\kappa}. \quad (2.45)$$

Inserting this, along with the identity [York, 1979]

$$\tilde{n} \cdot \tilde{a} = \frac{\tilde{n}^i \partial_i \tilde{N}}{\tilde{N}}, \quad (2.46)$$

into S_{BH} leads to

$$S_{BH}[g] = \int_{\mathcal{M}} d^d x \sqrt{-g} \left[P^{ij} \dot{h}_{ij} - N\mathcal{H} - V^i \mathcal{H}_i \right] - \int_{\mathcal{T}} d^{d-1} x \sqrt{\sigma} \bar{N} \bar{\jmath}_{\vdash} \bar{\theta}$$

$$- \int_{\mathcal{T}_{\mathcal{H}}} d^{d-1} x \sqrt{\tilde{\sigma}} \left[\frac{\tilde{N}\tilde{\varepsilon}}{d-2} + \tilde{N} \tilde{\jmath}_{\vdash} \tilde{\theta} - V^i \tilde{\jmath}_i + \frac{1}{\kappa} \tilde{n}^i \partial_i \tilde{N} \right] \quad (2.47)$$

When S_{BH} is evaluated on the classical solution from which the boundary conditions are taken, the terms integrated over $\mathcal{M}$ vanish because the metric is stationary and the constraints are satisfied. The $\mathcal{T}$ boundary term vanishes if an orthogonal foliation is chosen at the outer boundary. The $\mathcal{T}_{\mathcal{H}}$ boundary coincides with the horizon, where the lapse function vanishes, and in static coordinates the shift vector is zero everywhere. This leaves

$$S_{BH}|_{classical} = -\int_{\mathcal{T}_{\mathcal{H}}} d^{d-1}x\sqrt{\tilde{\sigma}}\frac{1}{\kappa}(\tilde{n}^i\partial_i\tilde{N}) = -\int dt\int_{\mathcal{B}_{\mathcal{H}}} d^{d-2}x\sqrt{\tilde{\sigma}}\frac{1}{\kappa}(\tilde{n}^i\partial_i\tilde{N}). \qquad (2.48)$$

Because $\tilde{n}^i$, the normal vector of $\mathcal{B}_{\mathcal{H}}$, is time independent, it follows that

$$i\int dt\,\tilde{n}^i\partial_i\tilde{N} = \tilde{n}^i\partial_i\beta, \qquad (2.49)$$

where $\beta = i\int dt\,\tilde{N}$ is the inverse temperature of the black hole [Gibbons and Hawking, 1977a], and therefore:

$$iS_{BH}|_{classical} = -\frac{1}{\kappa}\int_{\mathcal{B}_{\mathcal{H}}} d^{d-2}x\sqrt{\tilde{\sigma}}(\tilde{n}^i\partial_i\beta). \qquad (2.50)$$

The surface $\mathcal{B}_{\mathcal{H}}$ is a bifurcate Killing horizons and must therefore satisfy [Wald, 1994]:

$$-\tilde{n}^i\partial_i\beta = 2\pi. \qquad (2.51)$$

This leaves

$$iS_{BH}|_{classical} = \frac{2\pi}{\kappa}\int_{\mathcal{B}_{\mathcal{H}}} d^{d-2}x\sqrt{\tilde{\sigma}} = \frac{A_{\mathcal{H}}}{4}, \qquad (2.52)$$

where $A_{\mathcal{H}} = \int_{\mathcal{B}_{\mathcal{H}}} d^{d-2}x\sqrt{\tilde{\sigma}}$ is the area of the horizon. The relationship between the entropy and the black hole action is the same as that of the microcanonical case, equation (2.36), so the entropy in any spacetime dimension is

$$\mathcal{S} \approx \frac{A_{\mathcal{H}}}{4}, \qquad (2.53)$$

which is the expected result.

We now return to consider the interpretation of this result, central to which is the variation of S_{BH}

$$
\delta S_{BH}[g] = \text{(equations of motion term)}
$$
$$
-\frac{1}{\kappa}\int_{\mathcal{T}} d^{d-1}x \left[\left(\frac{2}{d-2}-1\right)\sqrt{\sigma}\varepsilon\delta\bar{N} - \bar{N}\delta(\sqrt{\sigma}\varepsilon) - \left(\frac{2}{d-2}-1\right)\sqrt{\sigma}j_i\delta V^i \right.
$$
$$
\left. +V^i\delta(\sqrt{\sigma}j_i) - \frac{\bar{N}}{d-2}\sqrt{\sigma}s^{ab}\delta\sigma_{ab}\right] - \frac{1}{\kappa}\int_{\mathcal{B}'}^{\mathcal{B}''} d^{d-2}x\,\bar{\theta}\delta\sqrt{\sigma}
$$
$$
-\frac{1}{\kappa}\int_{\mathcal{T}_\mathcal{H}} d^{d-1}x \left[\frac{2}{d-2}\sqrt{\tilde{\sigma}}\tilde{\varepsilon}\delta\tilde{N} - \frac{2}{d-2}\sqrt{\tilde{\sigma}}\tilde{j}_i\delta V^i + \frac{\tilde{\sigma}_{ab}}{d-2}\delta\left(\sqrt{\tilde{\sigma}}\tilde{N}\tilde{s}^{ab}\right)\right] - \frac{1}{\kappa}\int_{\mathcal{B}_\mathcal{H}'}^{\mathcal{B}_\mathcal{H}''} d^{d-2}x\,\tilde{\theta}\delta\sqrt{\tilde{\sigma}}.
$$

$$(2.54)$$

On the $\mathcal{T}_\mathcal{H}$ boundary the quantities $\tilde{N}$, V^i and $(\sqrt{\tilde{\sigma}}\tilde{N}\tilde{s}^{ab})$ must be held fixed to obtain the equations of motion, in all spacetime dimensions. On the $\mathcal{T}$ boundary, the $\delta\tilde{N}$ and δV^i terms vanish in $4d$, leading to microcanonical boundary conditions - ε, j_i and σ_{ab} are held fixed. This is only true, however, in the case of $4d$. In other spacetime dimensions the $\mathcal{T}$ boundary conditions are more general than microcanonical.

Considering only $4d$ spacetimes, Brown and York offered two different interpretations of their result [Brown and York, 1998], see also the appendix. In the first the $\mathcal{T}_\mathcal{H}$ boundary conditions are "thermodynamically completely open", in the sense that the conserved quantities of energy and momentum are allowed to fluctuate in the path integral. The black hole entropy therefore arises because the microcanonical density of states is calculated with "missing" thermodynamic data on one boundary. The second interpretation is that the boundary $\mathcal{B}_\mathcal{H}$ is an interior boundary, and when black hole boundary data are imposed, the condition $\tilde{N} = 0$ on $\mathcal{B}_\mathcal{H}$ seals the manifold, changing its topology. In this interpretation, the boundary conditions on $\mathcal{B}_\mathcal{H}$ are "no boundary" conditions, and the black hole entropy calculation is justified as an application of the microcanonical density of states. After considering the application of the microcanonical density of states to de

Sitter spacetimes, we will return to discuss the relative merits of the two interpretations.

Importantly, the entropy calculated does not arise from a straightforward application of the microcanonical density of states in the case of a disjoint boundary. This is highlighted by the analysis of Martinez who showed that placing the second boundary component in region 3 of the Kruskal diagram and imposing microcanonical boundary conditions on both boundaries results in zero entropy [Martinez, 1995].

2.6 de Sitter Horizons and the Microcanonical Density of States

Cosmological horizons in de Sitter spacetime are a topologically simpler scenario than the black hole case and, in particular, do not require the introduction of a disjoint 2-boundary to enclose a $3d$ submanifold. In theory, microcanonical boundary data could be taken from a boundary enclosing an observer's horizon and the microcanonical density of states calculated. If this program were to work there would be a conceptual point to overcome: if the boundary $\mathcal{B}$ encloses more than one observer's horizon then what entropy would we expect the density of states to correspond to? However, as we point out in this section, a boundary $\mathcal{T}$ that encloses a horizon and has the required constant values of ε and σ_{ab}, on each leaf $\mathcal{B}$, cannot be found, and therefore the microcanonical density of states cannot be calculated. We will deal with orthogonal foliations so that $\bar{\theta} = 0$, $\bar{u}_\mu = u_\mu$ and $\bar{n}_\mu = n_\mu$.

In static coordinates, with units chosen so that the horizon is at $r = 1$, the metric of $4d$ de Sitter spacetime is

$$ds^2 = -(1 - r^2)dt^2 + \frac{dr^2}{(1 - r^2)} + r^2 d\Omega^2. \tag{2.55}$$

Consider the simple case of a surface $\mathcal{B}$ that does not enclose an observer's horizon. Spacelike surfaces Σ can be specified by constant t, within each of which the boundary

surface $\mathcal{B}$ can be specified by $r = r_{\mathcal{B}}$, where $r_{\mathcal{B}}$ is a constant less than one. Along $\mathcal{T}$, the two metric of the spatial hypersurfaces $\mathcal{B}$ is

$$\sigma_{ij}dx^i dx^j = r_{\mathcal{B}}^2 d\Omega^2, \tag{2.56}$$

which is constant in time. In fact, each leaf in the Σ foliation has the same geometry and each leaf in the foliation $\mathcal{B}$ of $\mathcal{T}$ has the same geometry. In particular, the energy-surface-density and momentum-surface-density of $\mathcal{B}$ are constant along $\mathcal{T}$. Boundary data can therefore be taken from these $\mathcal{B}$ spheres and used in the zero-loop approximation to the microcanonical density of states. As in the case of $4d$ Minkowski spacetime, the microcanonical action, equation (2.38), vanishes because the metric is stationary and the constraints are satisfied by a classical solution. As expected, the entropy of a region not containing a horizon is found to be zero. The difficulty arises in trying to take boundary data from a surface that does enclose a horizon. The boundary $\mathcal{B}$ cannot be taken outside of the horizon by specifying $r_{\mathcal{B}} > 1$ because t would no longer be a timelike coordinate.

To find a suitable surface requires a coordinate system that goes beyond a static region. In global coordinates the de-Sitter metric is

$$ds^2 = -d\tau^2 + \cosh^2 \tau d\chi^2 + \cosh^2 \tau \sin^2 \chi d\Omega^2. \tag{2.57}$$

The relationship of the static coordinate r to the global coordinates is

$$r = \cosh \tau \sin \chi. \tag{2.58}$$

The manifold can be foliated into spatial leaves Σ using constant τ hypersurfaces. If $\mathcal{B}$ were to be defined in each Σ by $\chi = constant = \chi_{\mathcal{B}}$; then the two-metric of $\mathcal{B}$ will be

$$\sigma_{ij}dx^i dx^j = \cosh^2 \tau \sin^2 \chi_{\mathcal{B}} d\Omega^2, \tag{2.59}$$

which is clearly not independent of time τ. This can be fixed by promoting $\chi_{\mathcal{B}}$ to a

function of τ specifically chosen so that σ_{ij} remains constant in time. From equation (2.59), the appropriate function $\chi_\mathcal{B}(\tau)$ must satisfy

$$\sin \chi_\mathcal{B}(\tau) = \frac{constant}{\cosh \tau}. \tag{2.60}$$

However, the energy-surface density of the resulting $\mathcal{B}$ surfaces are no longer independent of time, as we now show. The extrinsic curvature of $\mathcal{B}$ is

$$k_{ab} = -\sigma_a^c D_c n_b, \tag{2.61}$$

where D_a is the covariant derivative with respect to the geometry of Σ, which is the projection onto Σ of the covariant derivative with respect to the geometry of $\mathcal{M}$:

$$k_{ab} = -\sigma_a^c h_c^i h_d^j \nabla_i n_j. \tag{2.62}$$

The submanifold Σ has been specified by constant τ coordinate so the only non-zero components of its metric $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ are the spatial components h_{ij} and the projection tensor onto Σ is just the delta symbol: $h_j^i = \delta_j^i$. Similarly, by using the χ coordinate to specify $\mathcal{B}$ we have $\sigma_a^c = \delta_a^c$. Inserting this into the extrinsic curvature

$$k_{ab} = -\nabla_a n_b, \tag{2.63}$$

the normal vector n_j of $\mathcal{B}$ in Σ has a non-zero χ component only: $n_\chi = -\cosh \tau$, $n_\theta = n_\phi = 0$. The only non-zero components of k_{ab} are therefore

$$k_{\theta\theta} = -(\partial_\theta n_\theta - \Gamma_{\theta\theta}^\mu n_\mu) = \Gamma_{\theta\theta}^\chi n_\chi = \sin \chi \cos \chi \cosh \tau, \tag{2.64}$$

and

$$k_{\phi\phi} = \Gamma_{\phi\phi}^\chi n_\chi = \sin^2 \theta \sin \chi \cos \chi \cosh \tau. \tag{2.65}$$

Thus, the energy density on $\mathcal{B}$ is

$$\varepsilon = \frac{k}{\kappa} = \frac{1}{\kappa} \frac{2\cos\chi_{\mathcal{B}}}{\cosh\tau \sin\chi_{\mathcal{B}}}. \tag{2.66}$$

Upon substituting (2.60):

$$\varepsilon(\tau) = \frac{1}{\kappa} \frac{2\cos\chi_{\mathcal{B}}(\tau)}{constant}. \tag{2.67}$$

It is clear from this expression that ε is not independent of time τ; therefore de Sitter spacetime, with $\mathcal{B}$ defined in this way, cannot be used in the zero-loop approximation of the microcanonical density of states.

Although this analysis does not prove it is impossible to find a $\mathcal{B}$ surface in de Sitter spacetime that encloses a horizon and has both constant metric and constant energy density, there is an intuitive argument that this must be the case. As emphasised by Schrodinger [Schrodinger, 1950], a foliation of de Sitter spacetime into identical spatial leaves is restricted to a single static coordinate region. Outside this region the geometry of the spatial leaves must become time-dependent. As the spatial leaves Σ are spherically symmetric, any time-dependence is restricted to an expansion or contraction. If the metric of a 2-surface $\mathcal{B}$ embedded in Σ is fixed then its extrinsic curvature, and therefore ε, must change. Conversely, if the surface expands and contracts with Σ, its metric must change. In a time dependent-spherically symmetric $3d$ space, it is impossible to find a 2 surface that has both constant metric and constant extrinsic curvature.

This causes trouble for the definition of the microcanonical density of states, because in the space of the variables σ_{ab} and ε there are regions for which no classical solution can be found with a surface $\mathcal{T}$ that has constant σ_{ab} and ε on spatial foliations $\mathcal{B}$. If there is no classical solution then there is no stationary point in the action, and if there is no stationary point in the action then the path integral is not defined.

2.7   de Sitter Horizon Entropy

It was shown in the previous section that a single boundary, microcanonical density of states analysis cannot be applied to a de Sitter horizon. Although the motivation for introducing a disjoint boundary in the black hole entropy calculation - so that a $2d$ boundary will enclose a $3d$ submanifold - is not applicable to de Sitter spacetime, it is shown in this section that doing so anyway leads, via an analysis that parallels the black hole case, to an entropy of $A_{\mathcal{H}}/4$.

Consider a path integral weighted by the action

$$S_{dS}[g] = S[g] + \int_{\mathcal{T}} d^{d-1}x\sqrt{\sigma}\left[\bar{N}\varepsilon - V^i j_i\right] - \frac{2}{d-2}\int_{\mathcal{T}_{\mathcal{H}}} d^{d-1}x\sqrt{\tilde{\sigma}}\frac{\tilde{N}\tilde{s}_a^a}{2}, \tag{2.68}$$

which is identical to the black hole action and therefore has the same variation, equation (2.54). To obtain appropriate boundary data place a $(d-2)$-sphere $\mathcal{B}$, of radius $r_{\mathcal{B}} < 1$, into a constant t time slice of the d-dimensional de Sitter metric

$$ds^2 = -(1 - r^2)dt^2 + \frac{dr^2}{(1 - r^2)} + r^2 d\Omega_{(d-2)}^2, \tag{2.69}$$

where $d\Omega_{(d-2)}^2$ is the metric of a $(d-2)$-sphere and place a second $(d-2)$-sphere, $\mathcal{B}_{\mathcal{H}}$, of radius $r_{\mathcal{B}_{\mathcal{H}}} = 1$ on the horizon in the same time slice. The canonical form of the action S_{dS} is the same as the black hole case (2.44)

$$S_{dS}[g] = \int_{\mathcal{M}} d^dx\sqrt{-g}\left[P^{ij}\dot{h}_{ij} - N\mathcal{H} - V^i\mathcal{H}_i\right] - \int_{\mathcal{T}} d^{d-1}x\sqrt{\sigma}\bar{N}_{\bar{\jmath}\vdash}\bar{\theta}$$

$$- \int_{\mathcal{T}_{\mathcal{H}}} d^{d-1}x\sqrt{\tilde{\sigma}}\left[\tilde{N}\tilde{\varepsilon} - V^i\tilde{\jmath}_i + \tilde{N}_{\tilde{\jmath}\vdash}\tilde{\theta} + \frac{\tilde{N}\tilde{s}_a^a}{d-2}\right]. \tag{2.70}$$

The terms integrated over $\mathcal{M}$ vanish when S_{dS} is evaluated on the classical solution because the metric is stationary and the constraints are satisfied. The $\mathcal{T}$ boundary term vanishes for an orthogonal foliation. The $\mathcal{T}_{\mathcal{H}}$ boundary coincides with the horizon so the

lapse function vanishes there and in static coordinates the shift vector is zero everywhere. This leaves

$$S_{dS}|_{classical} = -\int dt \int_{\mathcal{B}_{\mathcal{H}}} d^{d-2}x\sqrt{\tilde{\sigma}}\frac{1}{\kappa}(\tilde{n}^i\partial_i\tilde{N}). \qquad (2.71)$$

The normal $\tilde{n}^i$ to $\mathcal{B}_{\mathcal{H}}$ is time independent, from which it follows that $i\int dt\tilde{n}^i\partial_i\tilde{N} = \tilde{n}^i\partial_i\beta$ and therefore

$$iS_{dS}|_{classical} = -\frac{1}{\kappa}\int_{\mathcal{B}_{\mathcal{H}}} d^{d-2}x\sqrt{\tilde{\sigma}}(\tilde{n}^i\partial_i\beta). \qquad (2.72)$$

As with the black hole case the factor appearing in the integral obeys $-\tilde{n}^i\partial_i\beta = 2\pi$ because this result applies to all bifurcate Killing horizons, including the de Sitter horizon. We therefore have

$$iS_{dS}|_{classical} = \frac{2\pi}{\kappa}\int_{\mathcal{B}_{\mathcal{H}}} d^{d-2}x\sqrt{\tilde{\sigma}} = \frac{A_{\mathcal{H}}}{4}, \qquad (2.73)$$

where $A_{\mathcal{H}} = \int_{\mathcal{B}_{\mathcal{H}}} d^{d-2}x\sqrt{\tilde{\sigma}}$ is the area of the horizon. From the relationship between the entropy and the action (2.36) the entropy in any spacetime dimension is therefore

$$\mathcal{S} \approx \frac{A_{\mathcal{H}}}{4}. \qquad (2.74)$$

The entropy of a de Sitter horizon can therefore be calculated from Brown and York's density of states in exactly the same way as the black hole entropy. The de Sitter and Black hole actions have the same variation and therefore the discussion of the dimensional dependence of boundary conditions in the black hole case applies here too. In 4d, where the $\mathcal{B}$ boundary conditions are microcanonical, the de Sitter entropy calculation is useful in assessing the merits of the two interpretations of the origin of entropy given by Brown and York. In the black hole calculation $\mathcal{B}$ was an outer boundary and $\mathcal{B}_{\mathcal{H}}$ an inner boundary. In the de Sitter case this is reversed; $\mathcal{B}$ is the inner boundary and $\mathcal{B}_{\mathcal{H}}$ is the outer boundary. It is then difficult to see how the $\tilde{N} = 0$ condition on the boundary

$\mathcal{B}_{\mathcal{H}}$ signifies the boundary is sealed up and the manifold topology changed. The "no boundary" interpretation is therefore not applicable to the de Sitter entropy calculation, but the interpretation in terms of missing thermodynamic data is still possible.

2.8 Rindler Horizon Entropy

In this section it is shown that the program of calculating the entropy as a path integral, with missing thermodynamic data on the horizon, can be extended to the Rindler horizon. In Rindler coordinates the metric of the Rindler wedge of $4d$ Minkowski spacetime is

$$ds^2 = -a^2 x^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{2.75}$$

where a is the proper acceleration of the Rindler observer. The Rindler observer's horizon corresponds to $x = 0$ and in a given constant t spatial slice is an infinite plane.

Consider a path integral weighted by the action

$$S_{Rind}[g] = S[g] + \int_{\mathcal{T}} d^{d-1}x \sqrt{\sigma} \left[\bar{N}\varepsilon - V^i j_i \right] - \frac{2}{d-2} \int_{\mathcal{T}_{\mathcal{H}}} d^{d-1}x \sqrt{\tilde{\sigma}} \frac{\tilde{N}\tilde{s}_a^a}{2}, \tag{2.76}$$

which is identical to the black hole and de Sitter density of states actions. The variation of the action and the appropriate boundary data to hold fixed in the path integral is the same as the black hole and de Sitter cases.

There are two topologically distinct boundary configurations from which to obtain boundary data to represent Rindler spacetime. The first configuration is a boundary $\mathcal{B}$ corresponding to a plane defined by $x = constant > 0$, and a boundary $\mathcal{B}_{\mathcal{H}}$ corresponding to the plane of the observer's horizon at $x = 0$. Taking boundary data from these surfaces leads, via an analysis that exactly parallels the black hole and de Sitter case, to an entropy of

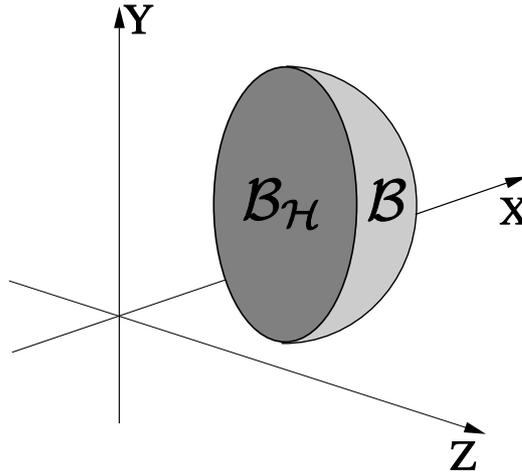$$\mathcal{S} \approx \frac{A_{\mathcal{H}}}{4}. \tag{2.77}$$

Figure 2.2: The surface used to obtain boundary conditions for the Rindler horizon in $4d$ spacetime. $\mathcal{B}_{\mathcal{H}}$ lies entirely within the plane of the horizon at $x = 0$ and $\mathcal{B}$ is a semi-spherical surface extending in the positive x-direction.

However, these boundaries do not enclose a $3d$ submanifold because the y and z directions extend to infinity. As a result, the area of the horizon is infinite and the action is not properly defined. Periodically identifying the y and z directions would serve to make the action well defined and render the horizon area finite [Hawking and Horowitz, 1996].

A second option is to consider a single, connected $(d-2)$ dimensional boundary B and impose different boundary conditions on different regions of it. For example, take B to be a hemisphere and define the circular part as $\mathcal{B}_{\mathcal{H}}$ and the rest of the hemisphere's surface as $\mathcal{B}$, so that $B = \mathcal{B} \cup \mathcal{B}_{\mathcal{H}}$ (see figure 2.2). To obtain boundary conditions locate B in Rindler spacetime so that $\mathcal{B}_{\mathcal{H}}$ lies entirely within the plane of the horizon at $x = 0$ and $\mathcal{B}$ extends in the positive x-direction.

In $4d$, microcanonical boundary data are taken from $\mathcal{B}$ and thermodynamically completely open boundary data from $\mathcal{B}_{\mathcal{H}}$. The analysis will again follow the black hole case and lead to entropy $\mathcal{S} \approx \frac{A_{\mathcal{H}}}{4}$. However, in this case the area $A_{\mathcal{H}}$ is the area of the finite spatial boundary section $\mathcal{B}_{\mathcal{H}}$.

It could be argued that obtaining different boundary conditions from different regions

of a single boundary is flawed because the boundary data cannot be integrated to obtain a constant of motion. The energy of the region enclosed by $\mathcal{B} \cup \mathcal{B}_{\mathcal{H}}$ is

$$E = \int_{\mathcal{B}_{\mathcal{H}}} d^{d-2}x \sqrt{\tilde{\sigma}} \tilde{\varepsilon} + \int_{\mathcal{B}} d^{d-2}x \sqrt{\sigma} \varepsilon, \tag{2.78}$$

and if the energy surface density $\tilde{\varepsilon}$ is not fixed on $\mathcal{B}_{\mathcal{H}}$ then this is not defined. However, this argument also applies to Brown and York's original derivation of the black hole entropy. In that case the system under study lies between the inner (horizon) boundary and the outer boundary. With microcanonical boundary conditions the energy surface density is fixed on the outer boundary but with thermodynamically completely open boundary conditions on the inner boundary $\tilde{\varepsilon}$ fluctuates in the path integral and consequently the energy of the system is not fixed.

One of the important issues in extending the notion of black hole entropy to all causal horizons is the validity of a local horizon entropy-density [Jacobson and Parentani, 2003]. The Brown and York program is particularly suited to addressing this question because it originated from the need to treat finite gravitational systems thermodynamically. This calculation of Rindler horizon entropy shows the notion of a local horizon entropy-density is supported, because $\mathcal{B}_{\mathcal{H}}$ does not cover the entire horizon and indeed could be taken arbitrarily small. This single boundary analysis for the Rindler horizon could also be applied to the black hole and de Sitter horizons. The hemisphere B could be placed into a black hole or de Sitter spacetime such that the circle $\mathcal{B}_{\mathcal{H}}$ intersects the horizon and the spherical part $\mathcal{B}$ protrudes out into the exterior region in the black hole case and into the static region in the de Sitter case. In either case taking thermodynamically open boundary conditions on $\mathcal{B}_{\mathcal{H}}$ and microcanonical boundary conditions on $\mathcal{B}$ will lead, following the exact same steps as the Rindler case, to an entropy of $A_{\mathcal{H}}/4$, where $A_{\mathcal{H}}$ is the area of the horizon intersecting $\mathcal{B}_{\mathcal{H}}$.

2.9   Black Hole Entropy as a Function of State

In Brown and York's calculation of black hole entropy the result is $\mathcal{S} = A_{\mathcal{H}}/4$, where $A_{\mathcal{H}}$ is the *physical* horizon area of the *specific solution* that was selected from the Kerr family of solutions to obtain boundary conditions. In contrast, entropy is a function of state variables, which is very different to the physical horizon area of a single solution. In the following section it will be argued that the microcanonical ensemble cannot succeed in deriving the entropy as a function of state. Before doing this it will be useful to emphasise the nature of black hole entropy as a thermodynamical function of state by deriving it as such, without the horizon area entering the calculation as an intermediate step.

At the level of thermodynamics the state of a non-electrically charged black hole is completely specified by two state variables. The mass of the black hole, for example, can be expressed as a function of state by solving the Smarr relation (1.6) for M

$$
\begin{aligned}
M(J, \Omega) = & \frac{1}{6 \cdot 3^{1/6} \Omega^{1/3} J^{1/3} (9\sqrt{16 J^2 \Omega^4 + 27} + 8\sqrt{3} J^2 \Omega^4 + 27\sqrt{3})^{1/3}} \\
& \times \left(J^{2/3} (9\sqrt{16 J^2 \Omega^4 + 27} + 8\sqrt{3} J62\Omega^4 + 27\sqrt{3}) + 4 \cdot 3^{1/3} \Omega^{8/3} J^2 \right. \\
& \left. + 2 \cdot 3^{1/6} \cdot \Omega^{4/3} J^{4/3} (9\sqrt{16 J^2 \Omega^4 + 27} + 8\sqrt{3} J^2 \Omega^4 + 27\sqrt{3})^{1/3} \right).
\end{aligned}
\tag{2.79}
$$

As a function of state, M has an exact differential dM. In contrast, the heat added $đQ$ and the work done $đW$ on a black hole during a change of state are inexact differentials because they depend on the path taken. Between two equilibrium states the first law

$$
dM = đQ + đW,
\tag{2.80}
$$

applies independently of the path. If the change is adiabatic - meaning it passes through a series of equilibrium states - then the work done is related to an exact differential; $đW = \Omega dJ$. Similarly, we look to relate the heat $đQ$ to an exact differential dS defined by

$$\eth Q = T(M, J)d\mathcal{S}, \tag{2.81}$$

where $T(M, J)$ is an integrating factor. In general, different choices of integrating factor would relate $\eth Q$ to different exact differentials. The entropy $d\mathcal{S}$ is determined uniquely by the requirement that $T(M, J)$ be the physical temperature, which is why the discovery of the Hawking temperature is crucial to deriving the entropy.

After replacing $\eth Q$ and $\eth W$ in equation (2.80) by the exact differentials the task is to find the function of state $\mathcal{S}(M, J)$ such that:

$$d\mathcal{S} = \frac{dM}{T} - \frac{\Omega}{T}dJ. \tag{2.82}$$

This can be done in the same way that the entropy of an ideal gas was first calculated - by considering the change in entropy after an initial state (M_1, J_1) is transformed adiabatically into a final state (M_2, J_2)

$$\mathcal{S}_2 - \mathcal{S}_1 = \int_1^2 d\mathcal{S} = \int_1^2 \frac{1}{T}dM - \int_1^2 \frac{\Omega}{T}dJ. \tag{2.83}$$

As $d\mathcal{S}$ is an exact differential the path between the initial and final state can be freely chosen. The easiest path, in terms of evaluating the integral, is to first hold J constant and change M_1 to M_2, thereby arriving at an intermediate state (M_2, J_1), and then hold M constant whilst J_1 is changed to J_2, arriving at the final state. Labelling the intermediate state as 3, we have

$$\mathcal{S}_2 - \mathcal{S}_1 = \int_1^3 \frac{1}{T}dM - \int_3^2 \frac{\Omega}{T}dJ. \tag{2.84}$$

Ω and T are now substituted as functions of M and J. Ω is given in the Smarr relation (1.6) and the Hawking temperature (1.9) is found by substitution of equation (1.7) for the surface gravity $\kappa_{\mathcal{H}}$

$$T = \frac{(M^4 - J^2)^{1/2}}{4\pi M (M^2 + (M^4 - J^2)^{1/2})}. \tag{2.85}$$

This gives

$$S_2 - S_1 = 2\pi \int_{M_1}^{M_2} \frac{2M(M^2 + \sqrt{M^4 - J_1^2})}{\sqrt{M^4 - J_1^2}} dM - 2\pi \int_{J_1}^{J_2} \frac{J}{\sqrt{M_2^4 - J^2}} dJ. \tag{2.86}$$

Performing the integral and rearranging the result leads to

$$S_2 - S_1 = 2\pi \left(M_2^2 + \sqrt{M_2^4 - J_2^2} - M_1^2 - \sqrt{M_1^4 - J_1^2} \right). \tag{2.87}$$

It is clear from this expression that the entropy must be

$$\mathcal{S}(M, J) = 2\pi \left(M^2 + \sqrt{M^4 - J^2} \right) + \ constant. \tag{2.88}$$

This is confirmed to be the same result as $\mathcal{S} = A_{\mathcal{H}}/4$ by substitution of equation (1.5) for $A_{\mathcal{H}}$ in terms of state variables. The entropy of a black hole can therefore be derived without recourse to the area of the horizon, and this analysis highlights that there is no logical necessity for black hole entropy to be physically related to the horizon area.

2.10 The Applicability of the Microcanonical Ensemble

External fields play an important role in statistical mechanics. The particles of a classical ideal gas, for example, are confined to a container of volume V by an external potential field $U(q_i)$ satisfying

$$U(q_i) = \begin{cases} 0 & q_i \in V \\ \infty & q_i \notin V \end{cases}, \tag{2.89}$$

where q_i is the position of the $i'th$ particle. In addition to being dependent on the phase space variables q_i and momentum p_i of each particle i, the Hamiltonian of the system is dependent upon V as an external parameter:

$$H = \sum_{i=1} \left(\frac{p_i^2}{2m} + U(q_i) \right). \tag{2.90}$$

Using the shorthand notation (q, p) for all phase space coordinates this dependency can be expressed as $H = H(q, p; V)$. It is through the Hamiltonian that the V dependency of the microcanonical ensemble phase space density arises

$$\rho_M(q, p; E, V) = \delta(E - H(q, p; V)). \tag{2.91}$$

When the entropy is calculated in the standard way,

$$S(E, V) = -k_B \int dp dq \rho(p, q; E, V) \log \rho(p, q; E, V) \tag{2.92}$$

it leads to a function of two state variables (E, V) which is appropriate in the case of fixed particle number.

Alternatively, the canonical ensemble

$$\rho_C(q, p; T, V) = \frac{e^{\frac{-H(q,p;V)}{k_B T}}}{Z(T, V)}, \tag{2.93}$$

could be used, where $Z(T, V)$ is the canonical partition function

$$Z(T, V) = \int dq dp \; e^{\frac{-H(q,p;V)}{k_B T}}. \tag{2.94}$$

Inserting the canonical phase density into the entropy (2.92) results in entropy as a function of (T, V). In both cases the entropy is dependent on V because it is an external parameter in the Hamiltonian. If this were not the case the entropy would be a function of a single state variable.

The situation is different in the case of the gravitational field where the Hamiltonian is

$$H = \int_{\mathcal{M}} d^d x \sqrt{-g} \left[N\mathcal{H} + V^i \mathcal{H}_i \right] + \int_{\mathcal{T}} d^{d-1} x \sqrt{\sigma} \left[N\varepsilon - V^i j_i \right]. \tag{2.95}$$

The angular velocity Ω is entirely absent and although the angular momentum density j_i does appear its role is different from V in the ideal gas case. The angular momentum density j_i is a function of phase space variables - it is not an external parameter. Denoting the physical degrees of freedom of the gravitational field as γ and their canonical momenta as π, the microcanonical phase space density is

$$\rho_M(\gamma, \pi) = \delta(M - H(\gamma, \pi)). \tag{2.96}$$

Using a functional integral to integrate over phase space, the entropy would then be calculated from

$$\mathcal{S} = -k_B \int D\gamma D\pi \rho_M(\gamma, \pi) \log \rho_M(\gamma, \pi). \tag{2.97}$$

The problem is that all values of j_i are integrated over and there is consequently no way of determining the entropy as a function of two state variables using the microcanonical ensemble.

Although they did not relate it to a missing external parameter, Brown and York did encounter this problem on route to their definition of the microcanonical density of states. They were forced to introduce an angular momentum dependence by fixing j_i as a boundary condition in the path integral representation of the density of states. As discussed in section (2.3), they motivated this by comparing the nature of time translations in non-gravitational physics and general relativity. In non-gravitational mechanics, energy is the value of the Hamiltonian that generates unit time translations. In gravitational physics time is "many fingered" and therefore both ε and j_i play a role in generating

unit translations of the boundary. The logic was that because ε and j_i play an analogous role to energy in non-gravitational physics they both must be fixed in the microcanonical density of states. The real reason j_i needs to be fixed in the microcanonical ensemble, however, is because it is not an external parameter.

By fixing j_i - and therefore its surface integral J - in this way it is unclear how the gravitational microcanonical ensemble should be related to a standard microcanonical phase space density in the form of equation (2.96). Is the integral over phase space in equation (2.97) to be restricted to phase points with the specified value of J? If so, how is a small change in J to be implemented in order to recover the first law?

To see how important the role of external parameters is in recovering the first law, the process will now be reviewed for the canonical ensemble. Consider a system with an energy that is dependent on an external parameter α. The phase space density of the canonical ensemble is

$$\rho_C(q,p) = \frac{e^{-\beta E(q,p;\alpha)}}{Z_C(\beta,\alpha)}, \tag{2.98}$$

where $\beta = \frac{1}{k_B T}$ and the canonical partition function is defined by

$$Z_C(\beta,\alpha) = \int dq dp \, e^{-\beta E(q,p;\alpha)}, \tag{2.99}$$

so that $\rho_C(q,p)$ is normalised. The entropy is calculated from

$$\mathcal{S} = -k_B \int dq dp \rho_C(q,p) \log \rho_C(q,p)$$

$$= k_B \beta \langle E \rangle + k_B \log Z_C, \tag{2.100}$$

where the second line is obtained by substitution of $\rho_C(q,p)$ from equation (2.98) and the notation $\langle E \rangle$ denotes the phase space average of E:

$$\langle E \rangle = \int dq dp \, E(q,p) \rho_C(q,p). \tag{2.101}$$

As a function of α and β the entropy's differential is

$$dS = k_B d\beta \langle E \rangle + k_B \beta d \langle E \rangle + k_B \frac{\partial}{\partial \alpha} \log Z_C d\alpha + k_B \frac{\partial}{\partial \beta} \log Z_C d\beta. \qquad (2.102)$$

The derivatives of the partition function can be obtained from its definition (2.99):

$$\frac{\partial}{\partial \alpha} \log Z_C(\beta, \alpha) = -\beta \left\langle \frac{\partial E}{\partial \alpha} \right\rangle, \qquad (2.103)$$

$$\frac{\partial}{\partial \beta} \log Z_C(\beta, \alpha) = - \langle E \rangle, \qquad (2.104)$$

and therefore dS is

$$dS = k_B \beta d \langle E \rangle - k_B \beta \left\langle \frac{\partial E}{\partial \alpha} \right\rangle d\alpha. \qquad (2.105)$$

By rearranging this into the form

$$d \langle E \rangle = T dS + \left\langle \frac{\partial E}{\partial \alpha} \right\rangle d\alpha, \qquad (2.106)$$

the first law of thermodynamics is recovered. If the system is an ideal gas, for example, with $\alpha = V$ then

$$d \langle E \rangle = T dS + \langle P \rangle dV, \qquad (2.107)$$

because $P = \frac{\partial E}{\partial V}$ is the definition of pressure. The key point is that the first law has only been recovered from the canonical ensemble because one of the thermodynamic state variables was an external parameter in the Hamiltonian.

2.11 The Microcanonical Ensemble and the Foundations of Statistical Mechanics

The foremost concern in the foundations of statistical mechanics is to explain why thermodynamical state variables can be related to ensemble averages so successfully. Consider a classical system, with phase space coordinates (q, p), upon which a measurement will be made of a thermodynamical property χ that is uniquely determinable from the actual microstate of the system. Experimentally, such a measurement will typically be a short time average. When the measurement begins at time t_0 the system is in state $(q(t_0), p(t_0))$. During the measurement the system follows a path through phase space, as determined by the equations of motion, and finishes in state $(q(t_0 + t_{obs}), p(t_0 + t_{obs}))$ as the measurement concludes after an elapsed time t_{obs}. In principle, the result of the measurement is

$$\chi_{obs} = \frac{1}{t_{obs}} \int_{t_0}^{t_0+t_{obs}} \chi\left(q(t), p(t)\right) dt. \tag{2.108}$$

In reality, thermal systems have so many degrees of freedom that it is impossible either to determine the initial state or to solve the equations of motion. Instead, statistical mechanics relates the result to a phase space average

$$\langle \chi \rangle = \int \chi(q, p)\rho(q, p)dqdp \ , \tag{2.109}$$

for an appropriately chosen ensemble $\rho(q, p)$. The key issue is to explain the relationship between χ_{obs} and $\langle \chi \rangle$. The ergodic hypothesis and equal a priori probabilities are the two most well known approaches [Tolman, 1938].

The ergodic hypothesis asserts that the phase space point of an isolated system passes in succession through each point of phase space compatible with the system's energy before eventually returning to its starting point. Accepting this assertion as true, it can be shown that $\chi_{obs} = \langle \chi \rangle$ as a matter of physical fact, and therefore the equality of

time and ensemble averages is a result of standard dynamics. The ergodic hypothesis fails as a rationalisation of statistical mechanics because the time-period over which a system is judged to be ergodic is usually infinity. But for χ_{obs} to equal $\langle \chi \rangle$ the system must be ergodic over a time of order t_{obs}, and if this were true of all isolated systems it would have been impossible to experimentally distinguish equilibrium states from non-equilibrium states.

To represent an isolated system of energy E, the principle of equal a priori probabilities asserts that all microstates of energy E should be given equal probability in the ensemble. It therefore leads directly to the microcanonical ensemble. The canonical ensemble is then justified by applying the microcanonical ensemble to a system consisting of subsystems exchanging heat and showing the subsystems form a canonical ensemble [Hill, 1956]. Interestingly, the subsystems are required to be weakly interacting over short distances only, so gravity would be difficult to include in this framework.

The principle of equal a priori probabilities places the microcanonical ensemble at the centre of the justification of statistical mechanics. In the previous section I claimed that the microcanonical and canonical ensemble are inadequate for black hole statistical mechanics and in the next sections will present the rotational and angular momentum ensembles as more compelling candidates. Their use, without recourse to the microcanonical ensemble, is justified through Jaynes' information theory approach to the rationalisation of statistical mechanics [Jaynes, 1957]. This can be seen as an extension of the principle of a priori equal probabilities to situations where more than one piece of information is available. Jaynes' criterion states an ensemble should be the assignment that agrees with the known information whilst being unbiased in all other respects. Claude Shannon, in the context of information theory, shows that maximising entropy, subject to the known information constraints, leads to this assignment [Shannon, 1949].

2.12   The Rotational Ensemble

An ensemble that can recover the first law of black hole mechanics, with neither J nor Ω being an external parameter in the Hamiltonian, is the rotational ensemble. It was first defined by Gibbs [Gibbs, 1902] and mentioned by Gibbons and Hawking [Gibbons and Hawking, 1977a] but referred to as the grand canonical ensemble by them.

Label the possible states of a black hole by x_i. This set includes states with all possible values of J and M. Allow the known information about the black hole to be its mass and its angular momentum. A statistical ensemble $p(x_i)$ is an assignment of a probability to each state x_i. For the ensemble to be representative, the ensemble averages $\langle M \rangle$ and $\langle J \rangle$ must equal the known values. The as yet undetermined probability assignment $p(x_i)$ must therefore satisfy the constraints:

$$\sum_{x_i} p(x_i) M(x_i) = \langle M \rangle, \tag{2.110}$$

$$\sum_{x_i} p(x_i) J(x_i) = \langle J \rangle, \tag{2.111}$$

where $M(x_i)$ and $J(x_i)$ are the mass and angular momentum of the state x_i. These constraints are in addition to the standard rule of probability:

$$\sum_{x_i} p(x_i) = 1. \tag{2.112}$$

Out of the many possible assignments $p(x_i)$ that satisfy these three constraints Jaynes' criterion [Jaynes, 1957] selects the assignment that is unbiased in all other respects by maximising the entropy

$$\mathcal{S} = -k_B \sum p(x_i) \log p(x_i), \tag{2.113}$$

subject to the constraints. Boltzmann's constant has been included here so that the units match the those of the thermodynamic entropy. The entropy $\mathcal{S}$ is maximised subject to

the constraints (2.110), (2.111) and (2.112) by the introduction of Lagrange multipliers λ, β and μ and solving

$$\frac{\partial \mathcal{S}}{\partial p(x_j)} + \beta \frac{\partial}{\partial p(x_j)} \left(\langle M \rangle - \sum_{x_i} p(x_i) M(x_i) \right)$$
$$+ \lambda \frac{\partial}{\partial p(x_j)} \left(\langle J \rangle - \sum_{x_i} p(x_i) J(x_i) \right) + \mu \frac{\partial}{\partial p(x_j)} \left(1 - \sum_{x_i} p(x_i) \right) = 0. \quad (2.114)$$

Substituting $\mathcal{S}$ from (2.113) leads to

$$p(x_i) = e^{-\mu} e^{-\beta M(x_i) - \lambda J(x_i)}, \quad (2.115)$$

where a factor of e has been absorbed into $e^{-\mu}$. The Lagrange multiplier μ can be determined from the normalisation constraint (2.112)

$$\mu = \log \left(\sum_{x_i} e^{-\beta M(x_i) - \lambda J(x_i)} \right) \equiv \log Z(\beta, \lambda), \quad (2.116)$$

where the second step is the definition of the partition function $Z(\beta, \lambda)$

$$Z(\beta, \lambda) = \sum_{x_i} e^{-\beta M(x_i) - \lambda J(x_i)}. \quad (2.117)$$

The ensemble that satisfies the constraints and is unbiased in all other respects is therefore

$$p(x_i) = \frac{e^{-\beta M(x_i) - \lambda J(x_i)}}{Z(\beta, \lambda)}. \quad (2.118)$$

The identification of β and λ in terms of T and Ω is made by comparison to the first law of black hole mechanics. Substituting the probability assignment (2.118) into the entropy (2.113) gives

$$\mathcal{S} = k_B \beta \langle M \rangle + k_B \lambda \langle J \rangle + k_B \log Z(\beta, \lambda). \quad (2.119)$$

The Lagrange multipliers λ and β are not independent variables; they are dependent on $\langle M \rangle$ and $\langle J \rangle$. This can be seen by substitution of $p(x_i)$ into the constraints (2.110) and (2.111):

$$\sum_{x_i} \frac{e^{-\beta M(x_i) - \lambda J(x_i)}}{Z(\beta, \lambda)} M(x_i) = \langle M \rangle , \tag{2.120}$$

$$\sum_{x_i} \frac{e^{-\beta M(x_i) - \lambda J(x_i)}}{Z(\beta, \lambda)} J(x_i) = \langle J \rangle . \tag{2.121}$$

Finding $d\mathcal{S}$ is helped by temporarily reversing the logical dependency and considering $\langle M \rangle$ and $\langle J \rangle$ to be functions of λ and β. This leads to the differential relationship

$$d\mathcal{S} = k_B \beta d \langle M \rangle + k_B \lambda d \langle J \rangle , \tag{2.122}$$

where use has been made of

$$\langle M \rangle = -\frac{\partial \log Z(\beta, \lambda)}{\partial \beta}, \tag{2.123}$$

and

$$\langle J \rangle = -\frac{\partial \log Z(\beta, \lambda)}{\partial \lambda} \tag{2.124}$$

By comparison of (2.122) with the first law of black hole mechanics

$$d\mathcal{S} = \frac{dM}{T} - \frac{\Omega}{T} dJ, \tag{2.125}$$

the Lagrange multipliers should be related to T and Ω as follows

$$\beta = \frac{1}{k_B T}, \tag{2.126}$$

$$\lambda = \frac{-\Omega}{k_B T}. \tag{2.127}$$

The rotational ensemble will then reproduce the first law. The final expression for the rotational ensemble is

$$p(x_i) = \frac{exp\left(-\frac{M(x_i)}{k_B T} + \frac{\Omega J(x_i)}{k_B T}\right)}{Z(T, \Omega)}. \tag{2.128}$$

The entropy calculated using this ensemble is a function of two state variables: T and Ω. The dependency on Ω arises not as an external field in the Hamiltonian but as a parameter in the definition of the ensemble.

The phase space over which $p(x_i)$ is defined includes solutions with all values of M and J. To calculate the entropy of the Schwarzschild solution, one would specify $\langle J \rangle = 0$, and then calculate the entropy from equation (2.113) with the ensemble probability (2.128). Even at the classical level there will be more than just the Schwarzschild solution contributing to the sum over states. This process is very different from looking for an ensemble to "represent" the Schwarzschild solution.

The grand canonical ensemble for an ideal gas, which is a function of the chemical potential μ, is a Legendre transformation of the microcanonical ensemble, which is a function of the number of particles N. In contrast, the rotational ensemble is not a Legendre transformation of a microcanonical ensemble because the microcanonical ensemble is neither a function of J nor Ω.

2.13   Landau's Canonical Rotational Ensemble

The difficulty of using the microcanonical and canonical ensemble in the absence of an external field in the Hamiltonian is not specific to black hole statistical mechanics; it is equally relevant to the case of a rotating system in non-gravitational classical mechanics. Landau's application [Landau and Lifshitz, 1958] of the canonical ensemble to a classical body in uniform rotation about an axis is therefore considered in this section and shown to be equivalent to the rotational ensemble.

The relationship between the energy $E_{rot}(p, q)$ of a body seen from a co-rotating frame of reference and the energy $E(p, q)$ in an inertial frame of reference is

$$E_{rot}(p, q) = E(p, q) - \Omega J(p, q). \tag{2.129}$$

The Angular velocity Ω is therefore an external parameter of $E_{rot}(p, q; \Omega)$ and

$$\frac{\partial E_{rot}(p, q; \Omega)}{\partial \Omega} = -J(p, q). \tag{2.130}$$

A canonical ensemble based on $E_{rot}(p, q; \Omega)$ and the partition function

$$Z(T, \Omega) = \int dq dp \, e^{-\frac{E_{rot}(p,q;\Omega)}{k_B T}}, \tag{2.131}$$

will lead, using equation (2.106), to a form of the first law for the rotating observer

$$d \langle E_{rot} \rangle = T d\mathcal{S} - \langle J \rangle \, d\Omega. \tag{2.132}$$

This must now be converted to the first law relevent to the fixed coordinate system. From equation (2.129), $\langle E_{rot} \rangle = \langle E \rangle - \Omega \langle J \rangle$ and therefore

$$d \langle E_{rot} \rangle = d \langle E \rangle - d\Omega \langle J \rangle - \Omega d \langle J \rangle . \tag{2.133}$$

Inserting this into (2.132) leads to

$$d \langle E \rangle = T d\mathcal{S} + \Omega d \langle J \rangle , \tag{2.134}$$

which is the same form of the first law as in black hole mechanics.

It therefore appears that, for non-gravitational rotating systems, Ω can be turned into a parameter in the energy and a canonical ensemble analysis used. However, if $E_{rot}(p, q; \Omega)$ is substituted from (2.129) into the canonical ensemble (2.131), it is seen to be the partition function of the rotational ensemble

$$Z(T,\Omega) = \int dqdp \; e^{-\frac{E(p,q)}{k_B T} + \frac{\Omega J(p,q)}{k_B T}} . \tag{2.135}$$

This method does not therefore offer a true method of applying the canonical ensemble to rotating systems and recovering the first law. In any case, the relationship (2.129) between energies in rotating and inertial coordinates is not applicable to the mass of black holes; a rotating Kerr black hole viewed from a co-rotating coordinate system is not a Schwarzschild black hole.

2.14 The Angular Momentum Ensemble

In the rotational ensemble the prior information about the thermal state of the system is the mass M (or the energy E) and angular momentum J. In this section we derive an ensemble that is applicable when the angular velocity Ω is given as prior information in place of J.

Because Ω is an external parameter and not a physical property of the states x_i it must be added as a coordinate to the phase space in order for the microstates of the system to be fully specified. In other words, microstates of the system are specified by the pair (x_i, Ω). In addition to Ω another piece of information is required. It cannot be the mass $M(x_i)$ as it is independent of Ω, and we require a function on phase space. A natural choice is to define a rotational equivalent of enthalpy:

$$\mathbf{H}(x_i, \Omega) = M(x_i) - \Omega J(x_i). \tag{2.136}$$

The given information about the system is then the expectation value of the enthalpy

$$\int_0^\infty d\Omega \sum_{x_i} p(x_i, \Omega)\mathbf{H}(x_i, \Omega) = \langle \mathbf{H} \rangle , \tag{2.137}$$

and the ensemble expectation value of the angular velocity,

$$\int_0^\infty d\Omega \sum_{x_i} p(x_i, \Omega)\Omega(x_i, \Omega) = \langle \Omega \rangle \,, \tag{2.138}$$

where $\Omega(x_i, \Omega) \equiv \Omega$.

Following Jaynes' procedure the entropy

$$\mathcal{S} = -k_B \int_0^\infty d\Omega \sum_{x_i} p(x_i, \Omega) \log p(x_i, \Omega), \tag{2.139}$$

is maximised, subject to the two constraints and the normalization condition, by the introduction of two Lagrange multipliers β and λ. This leads, following a calculation that parallels the example given in the rotational ensemble section, to the probability assignment

$$p(x_i, \Omega) = \frac{e^{-\beta \mathbf{H}(x_i) - \lambda\Omega}}{Z(\beta, \lambda)}, \tag{2.140}$$

where the angular momentum partition function is defined by

$$Z(\beta, \lambda) = \int_0^\infty d\Omega \sum_{x_i} e^{-\beta H(x_i, \Omega) - \lambda\Omega}. \tag{2.141}$$

Further information on the Lagrange multipliers can be obtained by substituting the ensemble probability into the entropy

$$\mathcal{S} = k_B\beta \langle \mathbf{H} \rangle + k_B\lambda \langle \Omega \rangle + k_B \log Z(\beta, \lambda). \tag{2.142}$$

The derivative is

$$d\mathcal{S} = k_B\beta d \langle \mathbf{H} \rangle + k_B\lambda d \langle \Omega \rangle \,, \tag{2.143}$$

where use has been made of

$$\langle \mathbf{H} \rangle = -\frac{\partial \log Z(\beta, \lambda)}{\partial \beta}, \tag{2.144}$$

$$\langle \Omega \rangle = -\frac{\partial \log Z(\beta, \lambda)}{\partial \lambda}. \tag{2.145}$$

The first law of black hole mechanics with the mass replaced by enthalpy is

$$d\mathbf{H} = Td\mathcal{S} - Jd\Omega. \tag{2.146}$$

By comparison with (2.143) the Lagrange multipliers are seen to be

$$\beta = \frac{1}{k_B T}, \tag{2.147}$$

$$\lambda = \frac{J}{k_B T}. \tag{2.148}$$

and the angular momentum ensemble is

$$p(x_i, \Omega) = \frac{exp\left(-\frac{\mathbf{H}(x_i)}{k_B T} - \frac{J\Omega}{k_B T}\right)}{Z(T, J)}. \tag{2.149}$$

In this case J is a parameter determined by the constraints, not a function of phase space as in the case of the rotational ensemble. The ensemble is equally relevant to rotating systems in non-gravitational classical statistical mechanics.

The angular momentum ensemble is mathematically identical to the pressure ensemble defined by Møller [unpublished] (see [Ter Haar and Wergeland, 1971]). Møller defined the pressure ensemble by considering an ideal gas in a canister having a moveable piston with a weight on top. In this situation the pressure is a fixed external parameter (although not one that appears in the Hamiltonian) and the volume must be given, along with the phase space point, to specify the state of the system. The pressure ensemble was re-derived by Jaynes as a demonstration of the information theory approach to statistical mechanics [Jaynes, 1957]. The Jaynesian approach is crucial in the derivation of the angular momentum ensemble where recourse can't be made to a simple physical model such as the ideal gas in the container.

2.15   Discussion

The extension of Brown and York's microcanonical density of states to general spacetime dimension results in zero entropy for Minkowski spacetime only in $4d$. The microcanonical density of states therefore appears to rely on features of the gravitational action specific to 4 dimensions. This was confirmed by the application of the black hole density of states to the calculation of black hole entropy in general spacetime dimension. The expected result can be produced in any dimension; however, away from $4d$, this comes at the expense of the interpretation, because the boundary conditions on the outer boundary are not microcanonical.

An attempt to apply the microcanonical density of states to the simpler topology of de Sitter spacetime was made, but it was shown to be impossible to find appropriate time-independent boundary conditions. Instead, the de Sitter horizon entropy was calculated in the same way as the black hole entropy. This calculation suffers from the same dimensional dependence of its interpretation but offers insight on the interpretation in the original $4d$ case. The interpretation of the horizon boundary conditions as being "no boundary" conditions - due to the vanishing lapse - is difficult to justify when the horizon is actually the outer boundary. If this program is to succeed, a return to Brown and York's interpretation in terms of missing thermodynamic data is more promising. The same calculation of horizon entropy was applied to a Rindler horizon with a modified boundary topology and it was shown that the density of states supports the notion of a local horizon entropy density.

When stripped of the details, the black hole density of states calculation depends on the following steps: The path integral is invoked to relate the entropy to the gravitational action (of whatever form) evaluated on a classical solution (2.36). As the bulk-term of the gravitational action is a sum of constraints, it will always result in zero when evaluated on the classical solution; so the entropy is dependent solely on the boundary terms of the action. As noted after equation (2.8), however, an arbitrary function of the boundary

metric can be added to the action with no change to the equations of motion. The key step is then to introduce a boundary on the horizon itself. A boundary term in the action can then be freely chosen specifically to result in an entropy of $\frac{A_{\mathcal{H}}}{4}$. The only other requirement is to limit the integral in the time direction, which is done by condition (2.51).

The program could be argued to genuinely produce the entropy if it could be shown that the choice of boundary term is not free, but determined by a relationship to a statistical mechanical ensemble. This is what Brown and York attempted to do in justifying the boundary conditions in the 4d microcanonical density of states. However, the justification for the horizon boundary conditions is much less compelling.

Boundary terms in the action play an important role in calculations of energy-momentum based on the Brown and York quasilocal stress tensor, equation (2.13), with boundaries at spatial infinity. As the spatial boundary is extended to infinity the quasilocal stress tensor typically diverges. To regulate this divergence Brown and York proposed a subtraction term in the action based on embedding the boundary in a reference spacetime and likened this to the freedom of choosing the zero-point of energy [Brown and York, 1993a]. Finding a reference spacetime into which all possible boundary geometries can be embedded, however, is impeded by Weyl's embedding theorem. This enforces the use of different reference spacetimes for different boundary geometries; which in turn affects not only the zero-point but relative energy-momentum.

In asymptotically anti-de Sitter spacetimes the AdS/conformal field theory correspondence [Maldacena, 1998] guided the way to an improved boundary term [Balasubramanian and Kraus, 1999]. In the dual CFT on the AdS boundary the divergences of the stress tensor are the standard ultraviolet divergences of quantum field theory and can be removed by adding counter terms to the action. On the gravity side of the duality the counterterms depend only on the intrinsic geometry of the boundary and therefore no reference spacetime is required. In this method the boundary terms are fixed

uniquely by requiring finiteness of the stress tensor. The "holographic renormalisation" method is not dependent upon the AdS/CFT correspondance and has been extended to asymptotically de Sitter spacetimes [Balasubramanian et al., 2002].

In asymptotically flat spacetimes Regge and Teitelboim showed that the ADM boundary term results from requiring the Hamiltonian to be stationary under all asymptotically at variations [Regge and Teitelboim, 1974]. Applying this criterion to the Einstein-Hilbert action, Marolf and Mann presented a counterterm suitable for asymptotically flat spacetimes that renders the quasilocal energy momentum finite [Mann and Marolf, 2006]. Work is ongoing to understand why different counter terms are required for asymptotically AdS and asymptotically flat spacetimes [Mann and McNees, 2010].

The role of these counterterms in black hole statistical mechanics has been considered through the canonical ensemble [Bergamin et al., 2008], which leaves the two key points of Brown and York's program unaddressed: the need for finite spacetimes and the microcanonical ensemble due to the negative heat capacity of the black hole. Nonetheless, it would be important to understand the relationship between these boundary terms and those in the microcanonical and black hole actions of Brown and York. Despite the importance of finite spatial boundaries, it is intersting that the exterior boundary of the black hole could be taken to spatial infinity whilst the black hole action, equation (2.44), remains finite.

In light of the discussion of entropy as a function of state, my own view is that the microcanonical and canonical ensembles are incapable of producing the black hole entropy and recovering the first law of black hole mechanics. The rotational ensemble, as discussed in section (2.12), and the angular momentum ensemble developed in section (2.14), are more appropriate starting points. A direction for future research would be to relate these ensembles to a path integral, in a similar way to Brown and York's effort for the microcanonical ensemble described in section (2.1).

Chapter 3

CDTs and the Density of States

3.1 Introduction

All calculations of the microcanonical and black hole density of states path integrals have been restricted to the zero-loop order of a perturbative expansion, as in equation (2.34). The goal of this section is to apply the framework of causal dynamical triangulations to calculating the black hole density of states non-perturbatively.

Causal dynamical triangulations is a non-perturbative method to calculate the path integral

$$\langle h_{fin}, \Sigma_{fin} | h_{in}, \Sigma_{in} \rangle = \int_{h_{in}}^{h_{fin}} \mathcal{D}g \, e^{iS[g]}, \tag{3.1}$$

where h_{in} and h_{fin} are initial and final spatial metrics, defined on spatial hypersurfaces Σ_{in} and Σ_{fin}. The action $S[g]$ is the standard gravitational action that requires the hypersurface metrics of any boundaries to be fixed in order to obtain the equations of motion.

In dynamical triangulations the sum over smooth spacetime geometries in the path integral (3.1) is approximated as a sum over triangulated spacetimes constructed from identical simplices

$$\langle h_{fin}, S_{fin} | h_{in}, S_{in} \rangle = \sum_{Triangulations} e^{iS_{DT}[T]}, \tag{3.2}$$

where S_{DT} is the Regge action appropriate for a discretised spacetime (Regge calculus will be described in the following section). In d dimensions the sum over triangulations is implemented as a sum over all ways in which the identical d-simplices can be connected. The simplices have spatial edges of length a and timelike edges of length $\sqrt{\alpha}a$.

The action $S[g]$ in the path integral (3.1) is real for Lorentzian metrics g. As a result, the oscillating path integral does not converge. In an effort to circumvent this problem, an imaginary time coordinate τ can be introduced such that $t = -i\tau$ and Lorentzian signatured metrics $(- + ++)$ are changed to Euclidean ones $(+ + ++)$. This adds a factor of $-i$ to the integration measure in the action. The new action $S_{Euc}[g]$ is related to the old one $S[g]$ via $S[g] = iS_{Euc}[g]$. The path integral in equation (3.1) becomes a sum over Euclidean metrics weighted by phase $-S_{Euc}$

$$\langle h_{fin}, \Sigma_{fin} | h_{in}, \Sigma_{in} \rangle = \int_{h_{in}}^{h_{fin}} \mathcal{D}g e^{-S_{Euc}[g]}. \tag{3.3}$$

The problem with this formalism is that the Euclidean gravitational action is not positive-definite and geometries can be found with arbitrarily negative S_{Euc}, thereby destroying the seemingly good convergence properties of (3.3).

Initially the problems with Euclidean path integrals were ignored, and dynamical triangulations were used to calculate a sum over Euclidean simplicial spacetimes. However, a major stumbling block - independent of the arbitrarily negative action - was soon hit. Quantum spacetimes constructed as sums over $3d$ and $4d$ simplicial spaces turned out not to have dimensions 3 and 4 respectively [Boulatov and Krzywicki, 1991, Ambjorn and Varsted, 1992, Agishtein and Migdal, 1991, Ambjorn and Jurkiewicz, 1992, Agishtein and Migdal, 1992a]. In the full models the spacetime dimension is determined by studying scattering processes in the quantum spacetime. However, because it is not at all intuitive that a sum over individual d-dimensional histories can result in anything

other than a d-dimensional quantum spacetime, I will describe a simple example from 2 dimensions.

Consider $2d$ spaces constructed from equal sized equilateral triangles meeting at a single vertex. Unless the total number of triangles is six the space will be curved. As the number of triangles increases the volume of the space increases, but the radius remains constant. From the relationship

$$Volume \sim (Radius)^{Dimension}, \qquad (3.4)$$

the dimension of the space must go to infinity as the volume does. It turns out that spaces of this form dominate the path integral and therefore the dimension of the quantum spacetime is not 2.

In causal dynamical triangulations the sum over histories is restricted to Lorentzian spacetimes that do not exhibit spatial topology change. Computer calculations using Monte Carlo techniques show that the resulting quantum spacetimes have the same dimension as their simplicial building blocks at large distances but, in the $4d$ case, are 2-dimensional at short distances [Ambjorn et al., 2005].

3.2   Regge Calculus

The normal approach to discretising a field theory is to place a lattice into spacetime and restrict attention to values of the field over the vertices of the lattice. An infinite set of values - the field at every spacetime point - is thereby reduced to a finite set; the field over the finite number of vertices. General Relativity is different because it involves the geometry of spacetime itself. Regge calculus is a discretisation of general relativity in which there are no fields, just a triangulation of spacetime [Regge, 1961]. All the information required by the theory is contained within the spacetime triangulation itself.

The Curvature of Triangulated Spaces

Consider a $2d$ smooth manifold. To probe the curvature of the surface a triangle with geodesic edges may be placed in a given region and its intrinsic geometry studied. If the geometry inside the triangle is non-Euclidean the sum of the triangle's internal angles will not be equal to π. The Gaussian integral curvature ϵ_t of a triangle with internal angles α, β and γ is defined by

$$\epsilon_t = \alpha + \beta + \gamma - \pi. \tag{3.5}$$

The local Gaussian curvature $K(P)$ at point P is defined in the limit that the triangle's area A_t shrinks to zero

$$K(P) = \lim_{A_t \to 0} \frac{\epsilon_t}{A_t}. \tag{3.6}$$

The Gaussian integral curvature is then the surface integral of the local curvature

$$\epsilon_t = \int_t K(P) dA. \tag{3.7}$$

Now imagine replacing the smooth $2d$ manifold with a triangulated $2d$ surface, consisting of flat triangles connected at their edges and use a test triangle to probe its geometry. Each face of the polyhedron is flat - so a test triangle placed entirely within a face will have zero integral curvature. Placing the triangle so that it crosses an edge but does not contain a vertex is analogous to taking a flat triangle and folding it - no curvature is introduced. We are led to the conclusion that if the polyhedral is to have curvature it must be concentrated at the vertices. In other words, $K(P) = 0$ if P is not a vertex. Any two test triangles that contain the same single vertex V must have the same integral curvature; therefore, ϵ_V is a characteristic of the vertex V and is called the deficiency of V. If there is more than one vertex inside a triangle t, then ϵ_t is the sum of the deficiencies of those vertices.
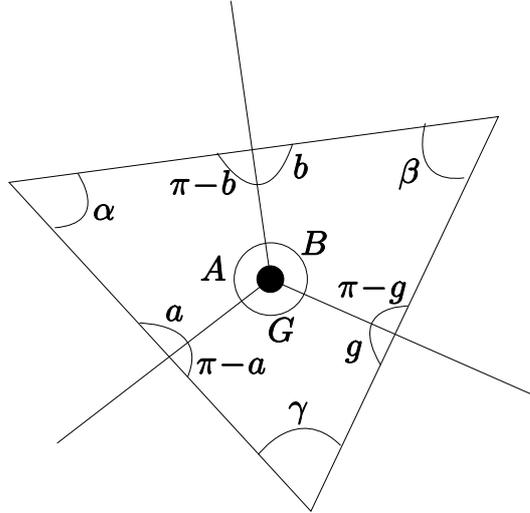
Figure 3.1: A test triangle placed over a vertex.

The restriction of the curvature to the vertices means that $K(P)$ behaves as a Dirac type distribution. Any integral of the form

$$\int_{\mathcal{M}} f(P)K(P)dA, \tag{3.8}$$

where $f(P)$ is a continuous function is to be calculated by

$$\sum_n f(V_n)\epsilon_n, \tag{3.9}$$

where ϵ_n is the deficiency of the vertex V_n and the summation is taken over all vertices of $\mathcal{M}$.

The deficiency of a vertex ϵ_V can be calculated without reference to any test triangle because the sum of the internal angles of faces that meet at the vertex is $2\pi - \epsilon_V$. To show this for the case of a vertex where three edges meet consider the test triangle in figure (3.1). The triangle has Gaussian curvature $\epsilon_V = \alpha + \beta + \gamma - \pi$. The sum of the internal angles of the faces meeting at V is

$$
\begin{aligned}
A + B + G &= [2\pi - a - (\pi - b) - \alpha] + [2\pi - b - (\pi - a) - \beta] + [2\pi - g - (\pi - \alpha) - \gamma] \\
&= 3\pi - (\alpha + \beta + \gamma) \\
&= 3\pi - (\epsilon_V + \pi) \\
&= 2\pi - \epsilon_V.
\end{aligned}
$$

A similar analysis holds if there are more edges meeting at the vertex and if there is more than one vertex within the triangle [Regge, 1961]. This result is important because it means the curvature information obtained from the geodesic test triangle is obtainable from the triangulation itself.

In higher dimensional spaces it is useful to replace the geodesic triangle as a test of curvature with the notion of parallel transporting a vector around a closed loop. Parallel transport of a vector along a path from point p to point q is a path-dependent mapping from T_pM, the tangent space at p, to T_qM, the tangent space at q. If the path is a loop that begins and ends at the same point p, then parallel transport around the loop is a mapping from T_pM into itself. The angle $\sigma(a)$ between a given vector and the same vector after it has been parallel transported around a loop a is given by the Gaussian curvature integrated over the interior of the loop:

$$
\sigma(a) = \int_a K \, dA. \tag{3.10}
$$

In the case of a $2d$ triangulated space, if the loop does not contain a vertex, the vector will be unchanged after parallel transport around it. If the loop does contain a vertex then the right-hand side of equation (3.10) is equal to the deficit angle of the vertex. Parallel transport of a vector around a vertex therefore results in a rotation by the deficit angle of the vertex.

In $3d$ a simplectic spacetime is composed of 3-simplices connected via their flat triangular faces. Because the metric in the interior of each simplex is flat, parallel transporting

a vector around a loop confined to a single simplex leaves the vector unchanged. Two 3-simplices are connected by a shared flat triangular face; so parallel transport round a loop that passes into a neighbouring 3-simplex and returns via the same triangular face - without enclosing an edge - also leaves the vector unchanged. The curvature manifests itself at the 1-simplices of the triangulation. Each 1-simplex is shared by a number of 2-simplices. If the sum of the dihedral angles (angles between planes) between these 2-simplices is 2π there is no curvature at the 1-simplex. If there is curvature then the sum of the dihedral angles is $2\pi - \epsilon$, where ϵ is the deficiency of the edge. A vector parallel transported around a loop enclosing a 1-simplex will be rotated by the deficit angle of the 1-simplex.

In general simplicial spacetimes are constructed from individually flat d-simplices connected by flat $(d-1)$-sub-simplices. The curvature is therefore localised to the $(d-2)$-dimensional sub-simplices and manifests itself as a deficiency angle. Following the terminology of Regge, we shall call these boundary simplices 'bones'; although they only correspond to lines in 3-dimensions. The sum of the angles between the $(d-1)$-simplices that meet at the bone is $2\pi - \epsilon$, where ϵ is the deficiency of the bone.

Deficit Angle and The Riemann Curvature Tensor

The curvature information contained in a simplicial spacetime is encapsulated in the deficit angles of the bones. A dictionary can be found between this curvature information and the Riemann curvature tensor of a differential manifold by considering a simplicial space with a high density of bones. This will be useful in obtaining the Regge action in the next section.

As each bone is a $(d-2)$-sub-simplex there exists a $2d$ plane perpendicular to it. Consider a bundle of parallel bones with a high density of bones per unit area ρ and, for the time being, specialise to the case of $3d$. Let each bone have the same deficit angle ϵ and assume the curvature induced by the bones is small, so that the simplicial spacetime
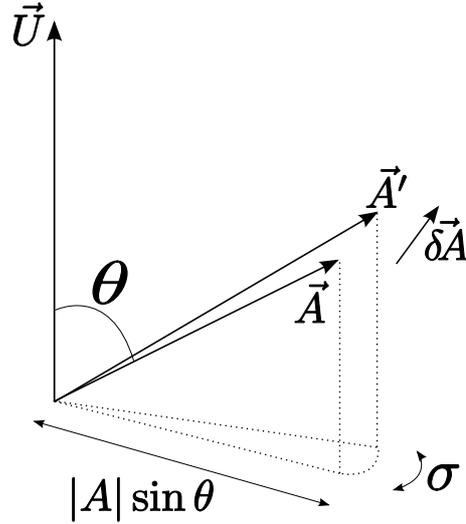
Figure 3.2: The vector $\vec{A}$ is rotated about $\vec{U}$ by angle σ.

is approximately Euclidean. Let $\vec{U}$ be a unit vector parallel to the bones. A vector $\vec{A}$ is parallel transported around a small loop of area Σ with unit normal $\vec{n}$ and corresponding vector area $\vec{\Sigma} = \vec{n}\Sigma$. The vector $\vec{A}$ can be decomposed into a component $\vec{A}^{\shortparallel}$ parallel to the bones (and therefore $\vec{U}$), and a component $\vec{A}^{\perp}$ lying in the plane perpendicular to $\vec{U}$. Under parallel transport around Σ the $\vec{A}^{\shortparallel}$ component is unchanged, whereas the component $\vec{A}^{\perp}$ is rotated by an angle $\sigma = N\epsilon$, where N is the number of bones piercing the loop. In other words, $\vec{A}$ is rotated about $\vec{U}$ by an angle σ.

From figure (3.2), where θ is the angle from $\vec{U}$ to $\vec{A}$, $|\delta A| = \sigma|A|\sin\theta = \sigma|U||A|\sin\theta$ (because $\vec{U}$ is a unit vector), and therefore:

$$\vec{\delta A} = \sigma(\vec{U} \times \vec{A}). \tag{3.11}$$

The number of bones crossing Σ is $N = \rho\vec{U} \cdot \vec{\Sigma}$; so substituting for σ in $\vec{\delta A}$ gives

$$\vec{\delta A} = \rho\epsilon \left(\vec{U} \cdot \vec{\Sigma} \right) \vec{U} \times \vec{A}. \tag{3.12}$$

In component form equation (3.12) is

$$\delta A_\mu = \rho\epsilon\epsilon_{\mu\nu\sigma}U^\nu A^\sigma U^\gamma \Sigma_\gamma, \tag{3.13}$$

where $\epsilon_{\mu\nu\sigma}$ is the anti-symmetric tensor. The correspondence between deficit angle and Riemann curvature tensor is found by comparing this expression for $\delta\vec{A}$ to that obtained from standard Riemannian geometry. When a vector $\vec{A}$ is parallel transported around an infinitesimal parallelogram with sides $\vec{dx}$ and $\vec{dy}$ then $\delta\vec{A}$ is given by [Synge and Schild, 1962]

$$\delta A_\mu = R^\sigma_{\mu\alpha\beta}A_\sigma dx^\alpha dy^\beta = R^\sigma_{\mu\alpha\beta}A_\sigma\frac{1}{2}\left(dx^\alpha dy^\beta - dx^\beta dy^\alpha\right), \tag{3.14}$$

where the symmetry property of the Riemann tensor is used in the second step. The loop area $\vec{\Sigma}$ of the parallelogram is given by $\vec{\Sigma} = (1/2)\vec{dx} \times \vec{dy}$, or in components

$$\Sigma_\mu = (1/2)\epsilon_{\mu\nu\rho}dx^\nu dy^\rho. \tag{3.15}$$

Using the identity $\epsilon^{\alpha\beta\mu}\epsilon_{\mu\nu\rho} = \delta^\alpha_\nu\delta^\beta_\rho - \delta^\beta_\nu\delta^\alpha_\rho$ we have

$$\epsilon^{\alpha\beta\mu}\Sigma_\mu = \frac{1}{2}\left(dx^\alpha dy^\beta - dx^\beta dy^\alpha\right). \tag{3.16}$$

Substituting this into equation (3.14) leads to

$$\delta A_\mu = R^\sigma_{\mu\alpha\beta}\epsilon^{\alpha\beta\gamma}\Sigma_\gamma A_\sigma. \tag{3.17}$$

Equating the two expressions (3.17) and (3.13) gives

$$R_{\sigma\mu\alpha\beta}\epsilon^{\alpha\beta\gamma}\Sigma_\gamma A^\sigma = \rho\epsilon\epsilon_{\mu\nu\sigma}U^\nu A^\sigma U^\gamma \Sigma_\gamma, \tag{3.18}$$

$$R_{\sigma\mu\alpha\beta}\epsilon^{\alpha\beta\gamma} = \rho\epsilon\epsilon_{\mu\nu\sigma}U^\nu U^\gamma. \tag{3.19}$$

Multiplying both sides by $\epsilon_{\gamma\rho\eta}$ and using the identity $\epsilon^{\alpha\beta\gamma}\epsilon_{\gamma\rho\eta} = \delta^\alpha_\rho\delta^\beta_\eta - \delta^\alpha_\eta\delta^\beta_\rho$ gives

$$R_{\sigma\mu\rho\eta} - R_{\sigma\mu\eta\rho} = \rho\epsilon(-U_{\mu\sigma})U_{\rho\eta}, \tag{3.20}$$

where the notation $U_{\rho\eta} = \epsilon_{\rho\eta\gamma}U^{\gamma}$ has been introduced. The key expression linking the deficit angle and the Riemann tensor is

$$R_{\sigma\mu\rho\eta} = \frac{1}{2}\rho\epsilon U_{\sigma\mu}U_{\rho\eta}. \tag{3.21}$$

In a triangulated spacetime of general dimension the orientation of the bones is determined by a skew symmetric tensor $U_{\mu\sigma}$ satisfying $U_{\mu\sigma}U^{\mu\sigma} = 2$ and $U_{\mu\sigma}U_{\alpha\beta} + U_{\mu\alpha}U_{\beta\sigma} + U_{\mu\beta}U_{\sigma\alpha} = 0$. If coordinates x_1 and x_2 are chosen to be the ones perpendicular to the bone then $U_{12} = -U_{21} = 1$ and all other components vanish. In this case equation (3.21) will still hold if the indices are allowed to take all values from 1 to d.

The Regge Action

The Einstein-Hilbert action of standard general relativity is

$$S_{EH} = \frac{1}{16\pi}\int d^d x \sqrt{-g}R. \tag{3.22}$$

The curvature of a simplicial spacetime is confined to the bones; so the Regge calculus equivalent of scalar curvature vanishes everywhere except on the bones b_n. The Regge action must be of the form

$$S_{Regge} = \sum_n F(b_n). \tag{3.23}$$

The sum is taken over all n bones and F is a function that must be related to curvature and be the same for all bones. As the bones are homogeneous $F(b_n)$ must be proportional to the bone's volume V_n. As the only variable related to curvature is the deficit angle $F(b_n)$ must be of the form; $F(b_n) = V_n f(\epsilon_n)$, where ϵ_n is the deficit angle of the nth bone and f is some function that is determined with the following trick. Any bone b_n can be

considered as the superposition of two bones, $b_{n'}$ and $b_{n''}$, of the same shape and area such that $\epsilon_n = \epsilon_n' + \epsilon_n''$. This requires the function f to satisfy $f(\epsilon_n) = f(\epsilon_n') + f(\epsilon_n'')$, which implies $f(\epsilon) = C\epsilon$ for some constant C. The Regge action is therefore:

$$S_{Regge} = C \sum_n \epsilon_n V_n. \tag{3.24}$$

The constant C can be determined by comparison with the Einstein-Hilbert action when the bone density is high. In this case the relationship between the deficit angle and the Ricci scalar curvature R can be obtained from equation (3.21) using the identity $U_{\mu\sigma}U^{\mu\sigma} = 2$. This gives $R = \rho\epsilon$ and by comparison with the Einstein-Hilbert action the constant is found to be $C = 1/16\pi$, leading to the action

$$S_{Regge} = \frac{1}{16\pi} \sum_n \epsilon_n V_n. \tag{3.25}$$

3.3 Extrinsic Curvature in the Regge Calculus Formalism

The curvature information contained in the deficit angles is related to the intrinsic geometry of the simplicial spacetime. The actions and quasilocal quantities of the Brown and York path integrals involve the extrinsic curvature of hypersurfaces. In this section the derivation of the Regge calculus expressions for the extrinsic curvatures of a $(d{-}1)$ dimensional spatial hypersurface Σ and a $(d{-}2)$ dimensional spatial-surface $\mathcal{B}$, embedded within Σ, are reviewed following [Dittrich and Loll, 2006]. The analysis is then extended to the extrinsic curvature of $(d{-}1)$ dimensional hyper-surfaces $\mathcal{T}$ with an induced Lorentzian metric. The dimension of spacetime shall be kept arbitrary, but to aid visualisation, relevant diagrams will be drawn for $3d$ spacetimes.

Consider two d dimensional simplices σ_1^d and σ_2^d, connected by a common $(d{-}1)$-dimensional sub-simplex $\sigma_{1\cap2}^{d-1}$ and intersected by a $(d{-}1)$-dimensional constant time-
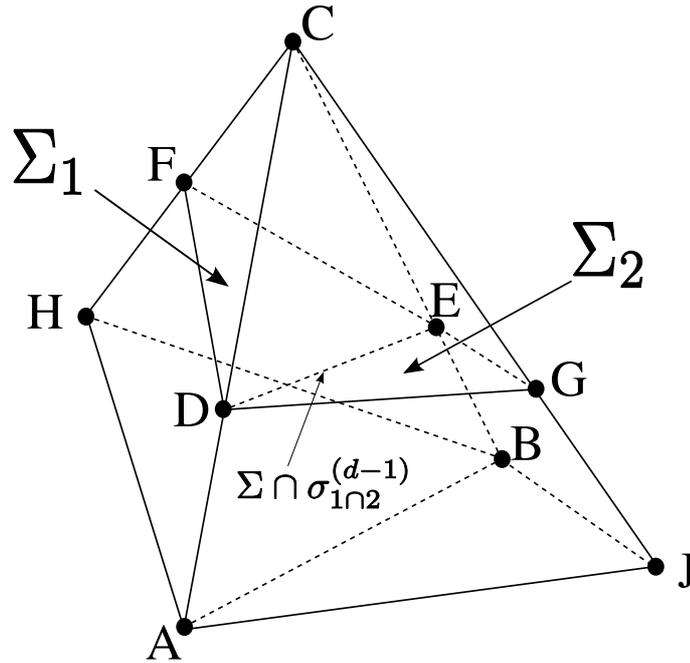
Figure 3.3: Two 3-simplices, σ_1^3 (AHBC) and σ_2^3 (AJBC) meeting at a common 2-simplex $\sigma_{1\cap2}^2$ (ABC). The intersection of Σ with the simplex σ_1^3 defines Σ_1 (DEF) and the intersection of Σ with the simplex σ_2^3 defines Σ_2 (DEG). The kink $\Sigma \cap \sigma_{1\cap2}^{d-1}$ is the intersection of Σ_1 and Σ_2, which in this case is the 1-simplex (DE).

surface Σ. As the spacetime in which it is embedded is piecewise linear, it is a natural choice to make Σ itself piecewise linear and completely flat in the interior of each d-simplex with which it intersects. In this case Σ consists of two flat sections - Σ_1 in σ_1^d and Σ_2 in σ_2^d - meeting at a "kink" in $\sigma_{1\cap2}^{d-1}$. The kink, being the intersection of Σ and $\sigma_{1\cap2}^{d-1}$, is denoted $\Sigma \cap \sigma_{1\cap2}^{d-1}$. The surface Σ is flat everywhere except at the kink; therefore the extrinsic curvature of Σ is a distribution restricted to the kink. Figure (3.3) shows the $3d$ case.

As the kink is a $(d-2)$ dimensional subspace, there exists a $2d$ plane perpendicular to it. Orthonormal basis vectors in this plane are defined as follows: $\vec{e}_0$ is the unique future oriented, timelike unit vector parallel to $\sigma_{1\cap2}^{d-1}$ and normal to the kink; $\vec{e}_1$ is the unique spacelike unit vector that is perpendicular to $\vec{e}_0$ and points from σ_1^d to σ_2^d. These basis
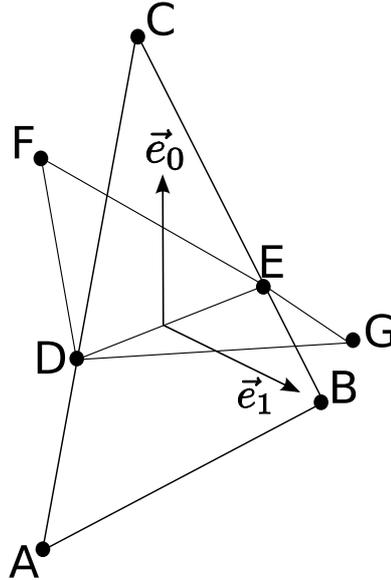
Figure 3.4: The same kink (DE) as figure (3.3) showing the basis vectors $\vec{e}_0$ and $\vec{e}_1$. Vector $\vec{e}_0$ is tangent to $\sigma^{d-1}_{1\cap2}$ (ABC) and normal to the kink $\Sigma \cap \sigma^{d-1}_{1\cap2}$ (DE). Vector $\vec{e}_1$ is normal to $\vec{e}_0$ and the kink, and points from σ^d_1 to σ^d_2

vectors are shown, for a $3d$ spacetime, in figure (3.4).

Let $\vec{u}_1$ and $\vec{u}_2$ denote the future oriented, unit normal vectors of Σ_1 and Σ_2 respectively. It is convenient to introduce an index $i = 1, 2$ and refer to $\vec{u}_i$ as the normal vector of Σ_i. The kink $\Sigma \cap \sigma^{d-1}_{1\cap2}$ is a subsimplex of both Σ_1 and Σ_2; therefore both vectors $\vec{u}_i$ are normal to the kink and can be split into components using $\vec{e}_0$ and $\vec{e}_1$ as basis vectors. Given the timelike nature of the vector $\vec{e}_0$ the decomposition is

$$\vec{u}_i = \cosh \rho_i \; \vec{e}_0 + \sinh \rho_i \; \vec{e}_1, \tag{3.26}$$

where ρ_i denotes the angle from $\vec{e}_0$ to $\vec{u}_i$, shown in figure (3.5).

Unit vectors $\vec{s}_i$ are defined to be tangent to Σ_i and perpendicular to the kink $\Sigma \cap \sigma^{d-1}_{1\cap2}$. To define their orientation - $\vec{s}_1$ points towards the kink and $\vec{s}_2$ points away from the kink. As normal vectors to the kink, $\vec{s}_i$ are also tangent to the plane spanned by $\vec{e}_0$ and $\vec{e}_1$ and can be decomposed as
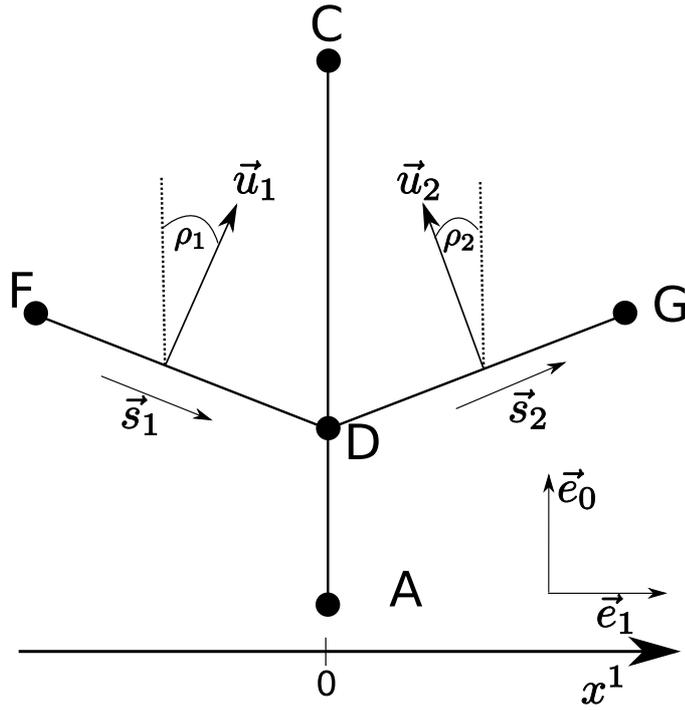
Figure 3.5: A side view of figure (3.4) showing the normal and tangent vectors $\vec{u}_i$ and $\vec{s}_i$. The edge (FD) belongs to Σ_1 and (DG) to Σ_2. Edge (AC) is a subsimplex of $\sigma_{1\cap 2}^{d-1}$ and vertex (D) belongs to the kink $\Sigma \cap \sigma_{1\cap 2}^{d-1}$

$$\vec{s}_i = \sinh \rho_i \; \vec{e}_0 + \cosh \rho_i \; \vec{e}_1. \tag{3.27}$$

A coordinate x^1, shown in figure (3.5), is introduced that measures proper distance along the integral curves of $\vec{e}_1$ and has its origin x^1 situated at $\sigma^{d-1}_{1\cap2}$. It allows the angles ρ_i to be combined into a step function of x^1

$$\rho(x^1) = \rho_1 + \Delta\rho \int_{-\epsilon}^{x^1} \delta_\epsilon(x')dx', \tag{3.28}$$

where $\Delta\rho = \rho_1 - \rho_2$. The function $\rho(x^1)$ is equal to ρ_1 if $x^1 < 0$ and ρ_2 if $x^1 > 0$. Using this we can write:

$$\vec{u}(x^1) = \cosh \rho(x^1) \; \vec{e}_0 + \sinh \rho(x^1) \; \vec{e}_1,$$

$$\vec{s}(x^1) = \sinh \rho(x^1) \; \vec{e}_0 + \cosh \rho(x^1) \; \vec{e}_1. \tag{3.29}$$

The relevant vectors have now been defined to calculate the extrinsic curvature of Σ which, by definition, is

$$K^{ab}(x^1) = -h^{ac}(x^1)\nabla_c u^b(x^1), \tag{3.30}$$

where h_{ab} is the induced metric of Σ. As the interior of each d-simplex is flat Minkowski spacetime geometry with Minkowski metric η^{ac}, the induced metric h_{ab} can be expressed as

$$h^{ac}(x^1) = \eta^{ac} + u^a(x^1)u^c(x^1). \tag{3.31}$$

In Minkowski coordinates, covariant derivatives reduce to partial derivatives. Substitution of $\vec{u}(x^1)$ from (3.29) into (3.30) and a reasonable amount of algebra leads to the required expression for the extrinsic curvature of a spatial hypersurface, as given by [Dittrich and Loll, 2006]:
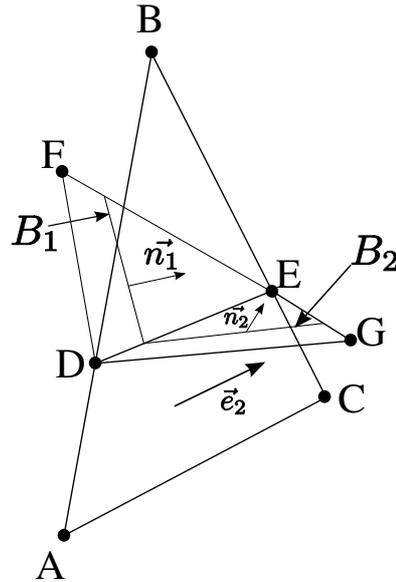
Figure 3.6: The same geometry as figure (3.3) now showing the line elements $\mathcal{B}_1$ and $\mathcal{B}_2$, in Σ_1 and Σ_2 respectively; their normal vectors $\vec{n}_1$ and $\vec{n}_2$; and the base vector parallel to the kink $\vec{e}_2$.

$$K^{ab}(x^1) = -\delta_\epsilon(x^1)\Delta\rho \cosh \rho(x^1) s^a(x^1) s^b(x^1). \tag{3.32}$$

Now consider a $(d-2)$ dimensional spatial-surface $\mathcal{B}$, embedded within Σ. It is sensible to choose this surface (or line element in a $3d$ spacetime) to be piecewise smooth and flat in the interior of each $(d-1)$-simplex of Σ with which it intersects. Let $\mathcal{B}_i$ denote the section of $\mathcal{B}$ lying in Σ_i, and define $\vec{n}_i$ to be the unit vectors normal to $\mathcal{B}_i$ embedded in Σ_i as shown in figure (3.6). With this choice of $\mathcal{B}$ its extrinsic curvature is confined to its intersection with the kink. Let $\vec{v}_i$ to be unit vectors parallel to $\mathcal{B}_i$ with their directions defined by $\vec{v}_1$ pointing towards the kink and $\vec{v}_2$ away from it. A third unit-basis vector $\vec{e}_2$ is introduced which, being perpendicular to $\vec{e}_0$ and $\vec{e}_1$, must be parallel to the kink $\Sigma \cap \sigma^{d-1}_{1\cap 2}$. In $3d$ the three $\vec{e}$ vectors span the entire spacetime. In $4d$ the kink is a $2d$ surface so another basis vector parallel to the kink is introduced and labelled $\vec{e}_3$. To fully specify the orientation of the two vectors on the kink, $\vec{e}_2$ is defined to be perpendicular

Figure 3.7: Looking down on the surfaces Σ_1 (DEF) and Σ_2 (DEG) shown in figure (3.6). $\mathcal{B}_1$ (HI) and $\mathcal{B}_2$ (IJ) are the intersections of $\mathcal{B}$ with Σ_1 and Σ_2 respectively. The extrinsic curvature of $\mathcal{B}$ embedded in Σ is restricted to its intersection with the kink, which is the vertex (I) in this $3d$ example.

to $\mathcal{B} \cap \sigma_{1\cap2}^{d-1}$ and $\vec{e}_3$ to be parallel to $\mathcal{B} \cap \sigma_{1\cap2}^{d-1}$. In $4d$, where $\mathcal{B}_i$ is a plane, $\vec{v}_i$ is specified further by the condition it is perpendicular to $\vec{e}_3$. A plan view of the kink is given in figure (3.7). The angle between $\vec{n}_i$ and $\vec{e}_2$ is defined to be ψ_i and we define

$$\psi(x^1) = \psi_1 + \Delta\psi \int_{-\epsilon}^{x^1} \delta_\epsilon(x')dx', \tag{3.33}$$

where $\Delta\psi = \psi_1 - \psi_2$.

It is useful to project the vectors $\vec{n}_i$ and $\vec{v}_i$ into components parallel and normal to the kink. Vectors $\vec{s}_i$ were defined above as unit vectors in Σ_i perpendicular to the kink and can therefore serve as one of the basis vectors in this projection. In $3d$ the only basis vector parallel to the kink is $\vec{e}_2$. In $4d$ $\vec{e}_3$ is also parallel to the kink but both $\vec{n}_i$ and $\vec{v}_i$ are, by definition, perpendicular to $\vec{e}_3$. The vectors $\vec{s}_i$ and $\vec{e}_2$ therefore serve as basis vectors in either $3d$ or $4d$ and the projections are:

$$\vec{n}(x^1) = \cos \psi(x^1)\; \vec{e}_2 + \sin \psi(x^1)\; \vec{s}(x^1);\tag{3.34}$$

$$\vec{v}(x^1) = -\sin \psi(x^1)\; \vec{e}_2 + \cos \psi(x^1)\; \vec{s}(x^1).\tag{3.35}$$

To determine the extrinsic curvature of $\mathcal{B}$ in Σ, we first need the covariant derivative of its normal $\vec{n}$ - with respect to the geometry of Σ. This is obtained by projecting the $4d$ covariant derivative of $\vec{n}$ onto Σ. Letting D_a denote the induced covariant derivative on Σ, then

$$
\begin{aligned}
D_a n^b &= h_a^c(x^1) h_d^b(x^1)(\nabla_c n^d)(x^1)\\
&= \delta_\epsilon(x^1)\Delta\psi \cosh\rho(x^1) s_a(x^1) v^b(x^1).
\end{aligned}\tag{3.36}
$$

The extrinsic curvature of $\mathcal{B}$ in Σ is this covariant derivative projected onto $\mathcal{B}$. The two metric of $\mathcal{B}$ is σ_{ab} so

$$
\begin{aligned}
k^{ab} &= -\sigma^{ac} D_c n^b\\
&= -\delta_\epsilon(x^1)\Delta\psi \cosh\rho(x^1)\cos\psi(x^1) v^a(x^1) v^b(x^1).
\end{aligned}\tag{3.37}
$$

The Brown and York microcanonical and black hole actions involve the submanifold $\mathcal{T}$ with an induced Lorentzian metric and a spacelike normal vector. The Regge calculus expression of the extrinsic curvature $\Theta_{\mu\nu}$ of a piecewise smooth $\mathcal{T}$ will now be derived. Consider two $(d-1)$ dimensional subsimplices $\mathcal{T}_1$ and $\mathcal{T}_2$ intersecting at a $(d-2)$ dimensional kink. In contrast to Σ, it will not in general be the case that $\mathcal{T}_1$ and $\mathcal{T}_2$ are part of different d-simplices σ_1^d and σ_2^d, connected by a common $(d-1)$ dimensional simplex $\sigma_{1\cap 2}^{d-1}$. For example, $\mathcal{T}_1$ and T_2 could be subsimplices of the *same* d-simplex (see figure (3.8) for a $3d$ example). The notation $\mathcal{T}_1 \cap \mathcal{T}_2$ is therefore more appropriate for the kink than $\mathcal{T} \cap \sigma_{1\cap 2}^{d-1}$.
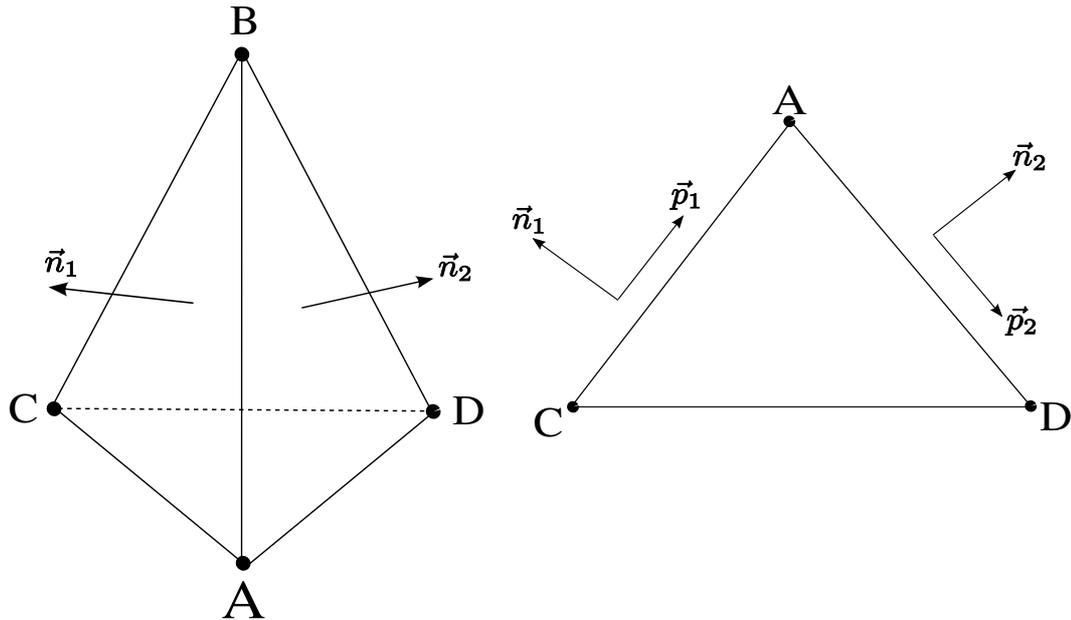
Figure 3.8: A 3-simplex with 3 spatial edges (AC), (AD), and (CD); and 3 timelike edges (AB), (CB) and (DB). $\mathcal{T}_1$ is the subsimplex (ABC) and $\mathcal{T}_2$ is the subsimplex (ABD). The kink is the edge (AB), which in this case is timelike. On the right is the same 3-simplex looking directly from (A) to (B), showing the normal vectors $\vec{n}_1$ and $\vec{n}_2$; and vectors $\vec{p}_1$ and $\vec{p}_2$, which are tangent to $\mathcal{T}_1$ and $\mathcal{T}_2$ and normal to the kink.

Define vectors $\vec{n}_i$ to be unit normals to $\mathcal{T}_i$. Although the same notation is used for the normal of $\mathcal{B}$ this will not lead to confusion because the future choice of foliation will mean they are the same vectors. Let $\vec{p}_i$ be unit vectors tangent to $\mathcal{T}_i$ and perpendicular to the kink $\mathcal{T}_1 \cap \mathcal{T}_2$. The orientation of the vectors $\vec{p}_i$ is specified such that $\vec{p}_1$ points towards the kink and $\vec{p}_2$ points away from the kink; see figure (3.8). The possible absence of $\sigma_{1 \cap 2}^{d-1}$ means that basis vectors $\vec{e}_i$ cannot be introduced as they were in the calculation of Σ's extrinsic curvature, because it is no longer possible to define $\vec{e}_0$ as perpendicular to the kink and parallel to $\sigma_{1 \cap 2}^{d-1}$. The simplest approach is to use the vectors $\vec{n}_1$ and $\vec{p}_1$ themselves as basis vectors to span the plane perpendicular to the kink. Let ϕ_i denote the angle from $\vec{n}_i$ to $\vec{n}_1$. In this case $\phi_1 = 0$, but the definition is useful as the resulting analysis closely follows that of the Σ example.

Because $\mathcal{T}$ has an induced Lorentzian metric one of the basis vectors tangent to $\mathcal{T}$ must be timelike. There are two cases to be distinguished; either the timelike basis vector is parallel to the kink or perpendicular to it. A kink with a timelike tangent vector shall be referred to as a "timelike kink" and a kink with only spacelike tangent vectors as a "spacelike kink".

In the case of a timelike kink, $\vec{n}_1$ and $\vec{p}_1$ are both spacelike and their decompositions are

$$\vec{n}_i = \cos \phi_i \ \vec{n}_1 + \sin \phi_i \ \vec{p}_1,$$
$$\vec{p}_i = -\sin \phi_i \ \vec{n}_1 + \cos \phi_i \ \vec{p}_1. \tag{3.38}$$

By defining the step function

$$\phi(x^1) = \phi_1 + \Delta\phi \int_{-\epsilon}^{x^1} \delta_\epsilon(x')dx', \tag{3.39}$$

where $\Delta\phi = \phi_1 - \phi_2$, and x^1 is a coordinate that measures proper distance along the integral curves of $\vec{p}_1$ with its origin at the kink $\mathcal{T}_1 \cap \mathcal{T}_2$, the vectors (3.38) can be written as step functions

$$\vec{n}(x^1) = \cos \phi(x^1) \ \vec{n}_1 + \sin \phi(x^1) \ \vec{p}_1,$$
$$\vec{p}(x^1) = -\sin \phi(x^1) \ \vec{n}_1 + \cos \phi(x^1) \ \vec{p}_1. \tag{3.40}$$

The extrinsic curvature is by definition

$$\Theta^{ab}(x^1) = -\gamma^{ac}(x^1)\nabla_c n^b(x^1). \tag{3.41}$$

The interior of each flat section of $\mathcal{T}$ is a section of Minkowski spacetime in which covariant derivatives reduce to partials derivatives. Using this and substituting $\vec{n}(x^1)$ from equation (3.40) into equation (3.41) gives

$$\Theta^{ab}(x^1) = -\gamma^{ac}(x^1)p^b(x^1)\partial_c\phi(x^1). \tag{3.42}$$

The derivative of $\phi(x^1)$ is seen from (3.39) to be

$$\partial_c\phi(x^1) = (\vec{p}_1)_c \; \delta(x^1) \; \Delta\phi. \tag{3.43}$$

Substituting this and the decomposition of the metric $\gamma^{ac}(x^1) = \eta^{ac} - n^a(x^1)n^c(x^1)$ into (3.42) leads to

$$\Theta^{ab}(x^1) = -\delta_\epsilon(x^1)\Delta\phi\cos\phi(x^1)p^a(x^1)p^b(x^1). \tag{3.44}$$

This is the extrinsic curvature of $\mathcal{T}$ at a timelike kink. For a spacelike kink the $\vec{p}_i$ vectors are timelike. The decompositions of $\vec{n}$ and $\vec{p}$ in this case are

$$\vec{n}(x^1) = \cosh\phi(x^1) \; \vec{n}_1 + \sinh\phi(x^1) \; \vec{p}_1, \tag{3.45}$$

$$\vec{p}(x^1) = \sinh\phi(x^1) \; \vec{n}_1 + \cosh\phi(x^1) \; \vec{p}_1. \tag{3.46}$$

This leads to the extrinsic curvature

$$\Theta^{ab}(x^1) = -\delta_\epsilon(x^1)\Delta\phi\cosh\phi(x^1)p^a(x^1)p^b(x^1), \tag{3.47}$$

which completes the extension of Dittrich and Loll's analysis to the case of a Lorentzian hypersurface.

3.4 Quasilocal Energy, Momentum and Stress Surface Densities

The relevant quantities have now been given to find Regge calculus expressions for the quasilocal energy, momentum and stress surface densities. The expression for the quasilo-

cal energy surface density given in equation (2.18) is

$$\epsilon = \frac{k}{\kappa}.$$
(3.48)

Taking the trace of k_{ab} in equation (3.37), noting that $\vec{v}_a$ is a unit vector, leads to the Regge calculus version:

$$\epsilon = -\frac{1}{\kappa}\delta_\epsilon(x^1)\Delta\psi\cosh\rho(x^1)\cos\psi(x^1).$$
(3.49)

To find the Regge calculus expression for quasilocal momentum surface density j_i we first require the expression for P^{jk}, the momentum of Σ embedded in M, repeated here from equation (2.10):

$$P^{jk} = \frac{\sqrt{h}}{2\kappa}\left(Kh^{jk} - K^{jk}\right).$$
(3.50)

Substituting the Regge calculus version of K^{ab} from equation (3.32) and the fact that $\vec{s}(x^1)$ is a unit vector gives

$$P^{jk} = \frac{\sqrt{h}}{2\kappa}\delta_\epsilon(x^1)\Delta\rho\cosh\rho(x^1)\left[s^j(x^1)s^k(x^1) - h^{jk}(x^1)\right].$$
(3.51)

The quasilocal momentum surface density was given in equation (2.19):

$$j_i = -\frac{2}{\sqrt{h}}\sigma_{ij}n_k P^{jk},$$
(3.52)

where $\sigma_{ij} = h_{ij} - n_i n_j$ is the metric of $\mathcal{B}$. The interior of each simplex is flat so h^{ij} is just the Euclidean metric. Using this we arrive at

$$j_i = -\frac{1}{\kappa}\delta_\epsilon(x^1)\Delta\rho\cosh\rho(x^1)\cos\psi(x^1)\sin\psi(x^1)u_i(x^1).$$
(3.53)

3.5   The Canonical Spacetime Foliation

A product triangulation M is topologically a Cartesian product $M = B \times F$ of a base space B and a fibre F [Dittrich and Loll, 2006]. The simplicial spacetimes used in CDTs are examples of triangulations of product type. The base space corresponds to the time direction and the fibres are the spatial hypersurfaces. The triangulation of the timelike 1-dimensional base space consists of vertices, which can be given an integer labelling, connected by timelike edges. In the case of 2 spacetime dimensions, an example of which is shown in figure (3.9), the spatial fibres are also 1 dimensional. Above each base space vertex is a spatial fibre consisting of vertices connected by spatial edges. All the vertices in a given spatial fibre inherit the integer time label of the vertex in the base space. The geometry of the triangulation is determined by the way vertices in neighbouring spatial hypersurfaces are connected. Each triangle can be described by the number of its vertices at time t and the number at time $t + 1$. Using the notation: [vertices at t, vertices at $t + 1$], the triangles are classed as type [2,1] or [1,2]. The notion of the spatial fibres can be extended to non-integer times [Dittrich and Loll, 2006]. Each triangle is foliated into $1d$ spatial leaves Σ that are parallel to the triangle's spacelike edge. At integer times the edges in the spatial fibres all have length a. At non-integer times the spatial fibres still consist of $1d$ spatial edges but not of length a. An example fibre is shown as the dotted line in figure (3.9).

Extending this notation to $3d$ spacetimes the 3-simplices fall into one of three categories: [3,1], [2,2] or [1,3]. The spatial fibres at integer times consist of 2-simplices. Simplices of type [3,1] or [1,3] are foliated by Σ into a series of triangles that are parallel to their $2d$ spatial sub-simplex. The intersection of Σ with a 3-simplex of type [2,2] is a rectangle, as shown in figure (3.10). At non-integer times, therefore, the spatial hypersurfaces are piecewise flat but do not necessarily consist solely of triangles.

The Brown and York program was simplified by foliating spacetime $\mathcal{M}$ into spatial leaves Σ that are perpendicular to the timelike boundary $\mathcal{T}$. One consequence of this
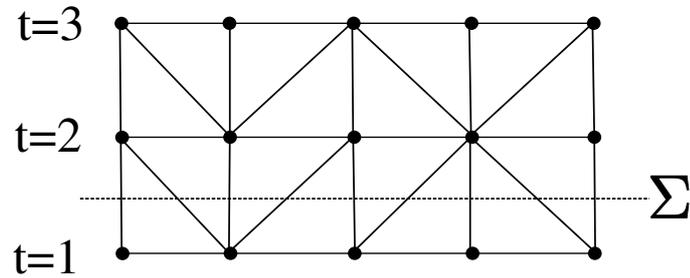
Figure 3.9: A $2d$ example of a CDT product triangulation. All the timelike edges have the same length; this is a representation of a curved spacetime on a flat page.
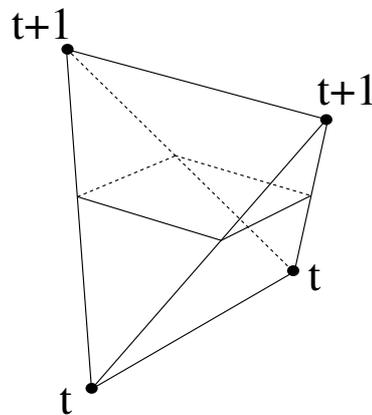

Figure 3.10: In 3 spacetime dimensions a [2,2] simplex has 2 vertices at t and 2 vertices at $t+1$. It is foliated by Σ into a series of parallel rectangles, of which one is shown.
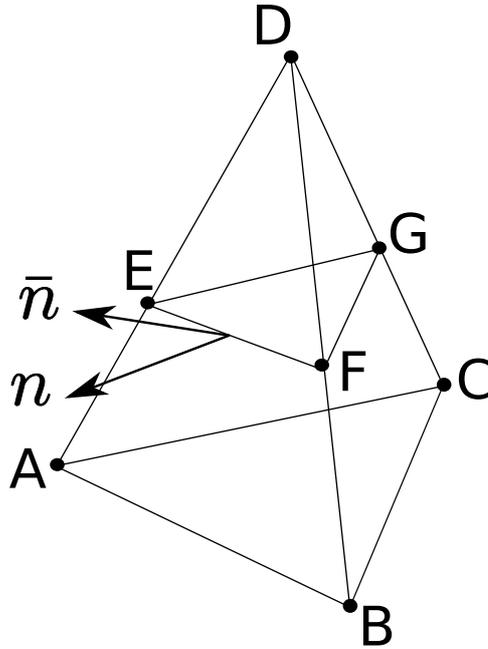
Figure 3.11: A [3,1] simplex with 3 spatial edges (AB), (BC), (CA) and 3 timelike edges (AD), (BD) and (CD). The 3-simplex is foliated by Σ into a series of parallel triangular spatial leaves, of which two are shown: the sub-simplex (ABC), and (EFG). If $\mathcal{T}$ is chosen to be the boundary 2-simplex (ABD), then it is foliated by Σ into spatial leaves $\mathcal{B}$ that are a series of 1-simplices parallel to the edge (AB). The 1-simplex (EF) is one such leaf. In this case $\bar{n}^{\mu}$, the normal of $\mathcal{B}$ embedded in Σ, does not coincide with n^{μ}, the normal vector of $\mathcal{T}$.
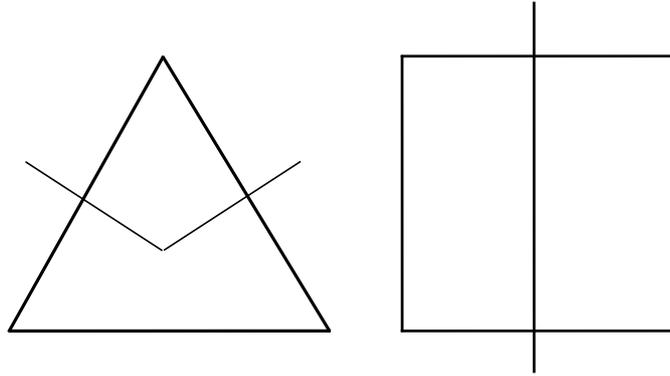
Figure 3.12: In 3 spacetime dimensions Σ consists of triangles and square sections. $\mathcal{B}$ is chosen to be dual to the edges of Σ, meaning it crosses them perpendicularly. In the case of the square $\mathcal{B}$ has no extrinsic curvature. In the case of the triangle the extrinsic curvature of $\mathcal{B}$ is restricted to the centre.

is that $\bar{n}^\mu$, the normal of $\mathcal{B}$ embedded in Σ, coincides with n^μ, the normal vector of $\mathcal{T}$. Implementing this requirement in the CDT framework requires a careful choice of boundary $\mathcal{T}$ and foliation Σ. For example, a [3,1] simplex is foliated into spatial leaves Σ so that the triangular base of the 3-simplex is a leaf of Σ and all the leaves of the foliation are parallel to this one. If $\mathcal{T}$ is chosen to be one of the timelike $2d$ sub-simplices then it will be foliated by Σ into $1d$ leaves $\mathcal{B}$ that are parallel to the spatial edge in the triangular base (see figure (3.11)). In this case n^μ and $\bar{n}^\mu$ are not the same.

This problem can be solved with a more appropriate choice of boundaries $\mathcal{B}$ and $\mathcal{T}$. An applicable type of boundary was considered by Dittrich and Loll in their work on trapped surfaces [Dittrich and Loll, 2006]. Let Σ_i be the foliation of a given simplex i and let $\mathcal{B}_i$ be the section of $\mathcal{B}$ passing through Σ_i. Instead of choosing $\mathcal{B}_i$ to be a section of the boundary of Σ_i, choose $\mathcal{B}_i$ to be "dual" to the boundary of Σ_i; in the sense that $\mathcal{B}_i$ intersects the boundary of Σ_i perpendicularly. For example, in 3 spacetime dimensions Σ_i is either a 2-simplex or a square; the corresponding choices of $\mathcal{B}_i$ are shown in figure (3.12). As these $\mathcal{B}_i$ sweep through time they lead to the boundaries $\mathcal{T}_i$ shown in figure (3.13). This choice has the further consequence of moving the extrinsic curvature of $\mathcal{B}$,
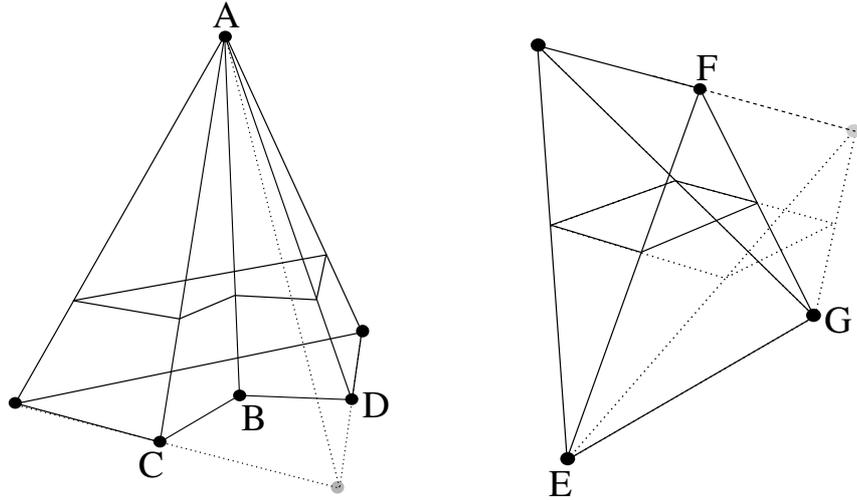
Figure 3.13: As the $\mathcal{B}_i$ shown in figure (3.12) sweep through time they lead to the boundaries $\mathcal{T}_i$ shown here. In the [3,1] simplex on the left $\mathcal{T}_i$ is the surface (ABCD). Its extrinsic curvature is restricted to the edge (AB). In the [2,2] simplex on the right $\mathcal{T}_i$ is the triangular surface (EFG); it has no extrinsic curvature.

and therefore $\mathcal{T}$, from the joints between 3-simplices to the interior of each 3-simplex. The $4d$ case will be considered in a later section.

3.6 Boundary Terms in the Action

Our strategy for obtaining Regge calculus versions of the relevent actions will simply be to substitute the Regge calculus expressions derived in the previous section for their differential geometric counterparts. In order to justify this method it is first used to reproduce the result in [Hartle and Sorkin, 1981], where the Regge action appropriate for fixed metric on a single spacelike boundary was obtained. Their result is:

$$-\frac{1}{\kappa} \int_\Sigma d^{(d-1)} x \sqrt{h} K = \frac{1}{\kappa} \sum_{b \in \Sigma} A(b) \Psi(b), \tag{3.54}$$

where b are the "bones" in the $(d-1)$ simplicial boundary Σ, $A(b)$ is the area of the bone b and $\Psi(b)$ is the angle between the normal vectors of the $(d-1)$ dimensional simplices

of Σ that meet at the bone b. The bones are equivalent to the kinks. We now re-derive this result starting from the expression for the extrinsic curvature K^{ab}:

$$K^{ab}(x^1) = -\delta_\epsilon(x^1)\Delta\rho\cosh\rho(x^1)s^a(x^1)s^b(x^1), \tag{3.55}$$

which was derived in equation (3.32). From the fact that s^a is a unit vector, the trace is:

$$K(x^1) = -\delta_\epsilon(x^1)\Delta\rho\cosh\rho(x^1). \tag{3.56}$$

Switching from x^1 to a coordinate p that measures proper distance along the integral curves of $\vec{s}_i$ (see figure (3.5)), then, remembering that x^1 is a coordinate corresponding to proper distance along integral curves of $\vec{e}_1$, it follows from equation (3.29) that $dp = \cosh\rho dx^1$ and therefore:

$$K(x^1)dx^1 = -\delta_\epsilon(p)\Delta\rho dp. \tag{3.57}$$

The three vectors that span a simplex in Σ are $\vec{p}$, $\vec{e}_2$ and $\vec{e}_3$. By introducing coordinates x^2 and x^3 to measure proper distance along the integral curves of $\vec{e}_2$ and $\vec{e}_3$ we can write

$$\frac{1}{\kappa}\int_\Sigma dpdx^2dx^3\sqrt{h}K(p) = -\frac{1}{\kappa}\int_\Sigma dpdx^2dx^3\delta_\epsilon(p)\Delta\rho, \tag{3.58}$$

noting the metric h_{ij} is flat in each simplex and therefore $\sqrt{h} = 1$. The delta function basically restricts the integral over Σ to an integral over the kink, or in other words the bone b, therefore

$$-\frac{1}{\kappa}\int_\Sigma d^3x\sqrt{h}K = \sum_{b\in\Sigma} A(b)\Psi(b). \tag{3.59}$$

I will therefore take this as a legitimate method of deriving the boundary terms in the actions.

3.7 Regge Action for the Black Hole Density of States

The black hole density of states of Brown and York uses the action given in equation (2.43)

$$S_{BH}[g] = \frac{1}{2\kappa} \int_{\mathcal{M}} d^d x \sqrt{-g}(\Re - 2\Lambda) - \frac{1}{\kappa} \int_{\mathcal{T}} d^{d-1} x \sqrt{-\gamma} \,\Theta - \frac{1}{\kappa} \int_{\mathcal{T}_{\mathcal{H}}} d^{d-1} x \sqrt{-\tilde{\gamma}} \,\tilde{\Theta}$$
$$+ \int_{\mathcal{T}} d^{d-1} x \sqrt{\sigma} \left[N\epsilon - V^i j_i \right] - \frac{2}{d-2} \int_{\mathcal{T}_{\mathcal{H}}} d^{d-1} x \sqrt{\tilde{\sigma}} \frac{N\tilde{s}_a^a}{2}. \quad (3.60)$$

There is nothing physical in the choice of coordinates used to calculate the integrals. The conversion of this action to a Regge calculus version is simplified by choosing coordinates such that $N = 1$ and $V = 0$. Although it may not be possible to find a global coordinate system satisfying this criteria, it is always possible to employ a patchwork of overlapping coordinate systems that satisfy the criteria locally. In such coordinates

$$S_{BH}[g] = \frac{1}{2\kappa} \int_{\mathcal{M}} d^d x \sqrt{-g}(\Re - 2\Lambda) + \frac{1}{\kappa} \int_{\mathcal{T}} d^{d-1} x \sqrt{-\gamma} \,(k - \Theta) - \int_{\mathcal{T}_{\mathcal{H}}} d^{d-1} x \sqrt{-\tilde{\gamma}} \left(\frac{\tilde{\Theta}}{\kappa} + \frac{\tilde{s}_a^a}{d-2} \right). \quad (3.61)$$

Inserting the expression given in equation (2.17):

$$\Theta = k - n_\beta a^\beta \quad (3.62)$$

into the $\mathcal{T}$ boundary term and the tilded version of this into the $\mathcal{T}_{\mathcal{H}}$ boundary term, along with the trace of the spatial stress (2.45)

$$\tilde{s}_a^a = \frac{1}{\kappa} \left[\tilde{k}(3-d) + (\tilde{n}\cdot\tilde{a})(d-2) \right], \quad (3.63)$$

leads to

$$S_{BH}[g] = \frac{1}{2\kappa} \int_{\mathcal{M}} d^d x \sqrt{-g}(\Re - 2\Lambda) + \frac{1}{\kappa} \int_{\mathcal{T}} d^{d-1} x \sqrt{-\gamma} n_\beta a^\beta - \frac{1}{\kappa} \int_{\mathcal{T}_{\mathcal{H}}} d^{d-1} x \sqrt{-\tilde{\gamma}} \frac{\tilde{k}}{(d-2)}. \quad (3.64)$$

As each simplex is foliated by Σ into a series of parallel hypersurfaces; $a^\beta = u^\mu \nabla_\mu u^\beta = 0$, and therefore, from equation (3.62), $\Theta = k$. This can be seen intuitively from the fact that $\mathcal{T}$ has no extrinsic curvature in the direction perpendicular to Σ; so its extrinsic curvature should be equal to that of $\mathcal{B}$. Similarly, $\tilde{a}^\beta = 0$ and therefore $\tilde{\Theta} = \tilde{k}$. In this case the black hole action is

$$S_{BH}[g] = \frac{1}{2\kappa} \int_\mathcal{M} d^d x \sqrt{-g}(\Re - 2\Lambda) - \frac{1}{\kappa} \int_{\mathcal{T}_\mathcal{H}} d^{d-1}x \sqrt{-\tilde{\gamma}}\, \frac{\tilde{\Theta}}{(d-2)}. \tag{3.65}$$

Substituting S_{Regge} for the $\mathcal{M}$ term and the trace of the Regge expression for $\tilde{\Theta}_{ab}$, derived in equation (3.47), into the $\mathcal{T}_\mathcal{H}$ boundary term gives

$$S_{BH} = S_{Regge} + \frac{1}{(d-2)\kappa} \int_{\mathcal{T}_\mathcal{H}} d^{d-1}x \sqrt{-\tilde{\gamma}}\delta_\epsilon(x^1)\Delta\phi \cos\phi(x^1). \tag{3.66}$$

Here the coordinate x^1 is a coordinate local to each kink, defined as in section (3.3), and it is implicit that the full integral is to be calculated by applying a patchwork of such coordinate systems valid in the vicinity of the individual kinks. It is useful to switch from the coordinate x^1 that measures proper time along $\vec{p}_1$ to a coordinate p that measures proper distance along $\vec{p}(x^1)$. From (3.40) we have $dp = \cos\phi(x^1)dx^1$ and the boundary term in the action becomes

$$S_{BH} = S_{Regge} + \frac{1}{(d-2)\kappa} \int_{\mathcal{T}_\mathcal{H}} dp d^{d-2}x \sqrt{-\tilde{\gamma}}\delta_\epsilon(p)\Delta\phi. \tag{3.67}$$

The delta function $\delta(p)$ restricts the integral to the kinks. With the choice of boundary $\mathcal{T}_i$ shown in figure (3.13), the angle between the normal vectors $\Delta\phi$ is constant on the kink and therefore

$$S_{BH} = S_{Regge} + \frac{1}{(d-2)\kappa} \sum_{k \in \mathcal{T}_\mathcal{H}} Vol(k)\Delta\phi, \tag{3.68}$$

where the sum is taken over all kinks k in $\mathcal{T}_\mathcal{H}$.

3.8   CDT Black Hole Action in $3d$

In 3 spacetime dimensions the extrinsic curvature of $\mathcal{T_H}$ is restricted to the simplices of type [3,1] and [1,3] that it intersects. All the kinks in $\mathcal{T_H}$ have the same volume, given by the length of the 1-simplices (AB) in figure (3.13). Although it is a fairly simple task to calculate this length using trigonometry, affine coordinates will now be introduced and used. These coordinates are overly complex in $3d$ but will prove invaluable in calculating the volume of the kinks in $4d$ spacetimes in the next section.

To introduce affine coordinates consider an arbitrary interior point P of a triangle embedded in $2d$ Euclidean space. Let the vectors $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ point from P to the three vertices of the triangle. Three appropriately chosen masses m^1, m^2 and m^3 can be placed at the vertices so that point P coincides with their centre of mass

$$P = m^1\vec{v_1} + m^2\vec{v_2} + m^3\vec{v_3}. \tag{3.69}$$

It is useful to normalise the masses, so that there is a one-to-one correspondence between points within the triangle and the three masses

$$m^1 + m^2 + m^3 = 1. \tag{3.70}$$

An affine vector is defined by the difference of two points P_1 and P_2:

$$\vec{P_1P_2} = (m_2^1 - m_1^1)\vec{v_1} + (m_2^2 - m_1^2)\vec{v_2} + (m_2^3 - m_1^3)\vec{v_3}, \tag{3.71}$$

where the notation m_2^1 refers to the mass at vertex one used to define point P_2. Introducing the following notation for the components of the affine vector:

$$y^i = m_2^i - m_1^i \tag{3.72}$$

allows it to be written as

$$\vec{P_1 P_2} = y^1 \vec{v_1} + y^2 \vec{v_2} + y^3 \vec{v_3}. \tag{3.73}$$

It follows from the normalisation condition (3.70) that the sum of the components of an arbitrary affine vector vanish

$$y^1 + y^2 + y^3 = 0. \tag{3.74}$$

Using this condition it is fairly simple to show that the vectors $\vec{v_1}$, $\vec{v_2}$ and $\vec{v_3}$ can be interchanged for the following three vectors

$$\vec{a_1} = \vec{v_1} - \frac{1}{3}(\vec{v_1} + \vec{v_2} + \vec{v_3}), \tag{3.75}$$

$$\vec{a_2} = \vec{v_2} - \frac{1}{3}(\vec{v_1} + \vec{v_2} + \vec{v_3}), \tag{3.76}$$

$$\vec{a_3} = \vec{v_3} - \frac{1}{3}(\vec{v_1} + \vec{v_2} + \vec{v_3}), \tag{3.77}$$

in other words:

$$\vec{P_1 P_2} = y^1 \vec{a_1} + y^2 \vec{a_2} + y^3 \vec{a_3} \tag{3.78}$$

The vectors $\vec{a_i}$ must be over-complete because they are 3 vectors in a $2d$ space. In fact

$$\vec{a_1} + \vec{a_2} + \vec{a_3} = 0. \tag{3.79}$$

It can be shown [Sorkin, 1975] that the Euclidean metric $\eta_{ij} a^i a^j$, in the over complete $\vec{a_i}$ basis, has components

$$\eta_{ij} = -\frac{1}{2} l_{ij}^2, \tag{3.80}$$

where l_{ij}^2 is the length-squared of the triangle's edge between vertices i and j, (note that $\eta_{ii} = 0$).

To check this formula for the metric and to get a feel for affine coordinates consider the vector from vertex 1 to vertex 2 of a triangle with sides of length a. The coordinate of vertex one is $(1,0,0)$ and the coordinate of vertex two is $(0,1,0)$. The affine vector from vertex 1 to vertex 2 therefore has components

$$\vec{P_1P_2} = (-1,1,0), \tag{3.81}$$

and the magnitude-squared of this vector is:

$$\eta_{ij}(\vec{P_1P_2})^i(\vec{P_1P_2})^j = 2\eta_{12}(\vec{P_1P_2})^1(\vec{P_1P_2})^2 = -\frac{2l_{12}^2}{2}(-1)1 = a^2, \tag{3.82}$$

as required.

The concept of affine coordinates generalises to higher dimensions and is especially useful for calculating lengths in general d-simplices. To calculate the length of the kink (AB) in figure (3.13) choose the vertices in the base triangle to correspond to the first three components of the affine coordinates and the vertex at the top to the last component. The coordinate of point B is then $(1/3,1/3,1/3,0)$ and that of point A is $(0,0,0,1)$. The vector $\vec{BA}$ therefore has components $(-1/3,-1/3,-1/3,1)$. Using the fact that the spatial edges of the 3-simplex have length a and the timelike edges have length $\sqrt{\alpha}a$ we have

$$\eta_{ij}(\vec{BA})^i(\vec{BA})^j = -a^2\left(\frac{1}{3} + \alpha\right), \tag{3.83}$$

and therefore the required volume of the kink is
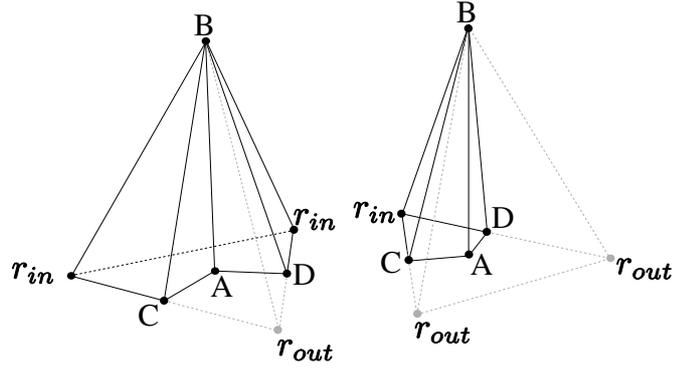
$$|\vec{BA}| = a\sqrt{\frac{1}{3} + \alpha}. \tag{3.84}$$

Figure 3.14: A simplex of type $[3,1]_{[2,1]}$ on the left and $[3,1]_{[2,1]}$ on the right. In both cases $\mathcal{T}_{\mathcal{H}}$ consists of two flat elements (ABC) and (ABD) meeting at the kink (AB). The outward pointing normal vectors of (ABC) and (ABD) in the $[3,1]_{[2,1]}$ simplex point towards each other and have angle $\pi/3$ between them. In the $[3,1]_{[1,2]}$ simplex they point away from each other by angle $-\pi/3$.

Simplices of type [3,1] or [1,3] can be further specified by the radial positions of the 3 vertices in the same spatial hypersurface. Similarly to the integer labelling of vertices with respect to time, all vertices can be give an integer label with respect to a radial direction. There are then two possibilities for the configuration of the 3 vertices in the same spatial hypersurface: two vertices at r_{in} and one at r_{out}; or one vertex at r_{in} and two at r_{out} as shown in figure (3.14). The notation $[3,1]_{[2,1]}$ will be used to refer to a [3,1] simplex with two base vertices at r_{in} and one at r_{out}. The radial positions of the vertices affects the angle ϕ between the normal vectors of the flat sections of $\mathcal{T}_{\mathcal{H}}$ that meet at the kink. For $[3,1]_{[2,1]}$ and $[1,3]_{[2,1]}$ $\Delta\phi$ is $\pi/3$ and for $[3,1]_{[1,2]}$ and $[1,3]_{[1,2]}$ $\Delta\phi$ is $-\pi/3$.

Using the notation $N_{[3,1]_{[2,1]}}$ for the number of $[3,1]_{[2,1]}$ simplices in the boundary $\mathcal{T}_{\mathcal{H}}$, and similarly for the other types of simplices, the Regge black hole action in 3 spacetime dimensions is

$$S_{BH} = S_{Regge} + \frac{1}{\kappa}\frac{\pi}{3}a\sqrt{\frac{1}{3}+\alpha}\left(N_{[3,1]_{[2,1]}} + N_{[1,3]_{[2,1]}} - N_{[3,1]_{[1,2]}} - N_{[1,3]_{[1,2]}}\right). \qquad (3.85)$$

3.9 CDT Black Hole Action in $4d$

In 4 spacetime dimensions there are four types of 4-simplices with respect to the integer time labelling of spatial hypersurfaces: [4,1]; [3,2]; [2,3]; and [1,4]. The spatial foliation Σ of the [4,1] and [1,4] simplices takes the form of a series of parallel 3-simplices. The four vertices of these spatial 3-simplices can be given an integer radial label. The [4,1] and [1,4] simplices can then be further specified by the number of vertices at r_{in} and r_{out} in a given spatial slice. Using the same style of notation as the $3d$ case, so that $[4,1]_{[3,1]}$ means a $[4,1]$ simplex with 3 vertices of its spatial foliation at r_{in} and one at r_{out} there are three classes: $[4,1]_{[3,1]}$, $[4,1]_{[1,3]}$, and $[4,1]_{[2,2]}$; and similarly for the [1,4] type. An example 3-simplex from the spatial foliation of each type, including the corresponding choice for the dual surface $\mathcal{B}_{\mathcal{H}}$ (or $\mathcal{B}$), is shown in figure (3.15). The extrinsic curvature of these elements of $\mathcal{B}_{\mathcal{H}}$ is restricted to the 1-simplex kinks that run from the triangular faces of the 3-simplices to their centre.

In the $t = 0$ spatial slice, at the base of a [4,1] simplex, the 3-simplex has sides of length a. From basic trigonometry it can be shown that, in this base 3-simplex, each of the kinks of $\mathcal{B}_{\mathcal{H}}$ have length

$$\ell = \frac{a}{2\sqrt{6}}.$$

(3.86)

With the passage of time, the 3-simplices in the foliation of the [4,1] simplex shrink to a point at the vertex in the $t = 1$ spatial fibre. Likewise, the 1-simplex kinks of $\mathcal{B}_{\mathcal{H}}$ shrink to points at $t = 1$, sweeping out the $2d$ kinks of $\mathcal{T}_{\mathcal{H}}$ as they do so. To calculate the volume of these $\mathcal{T}_{\mathcal{H}}$ kinks, we need to know the "height" of the 4-simplex. That is, we need to know the distance from the centre of the 3-simplex at the base of the 4-simplex to the vertex at the "tip" of the 4-simplex. If this vertex at the tip is assigned as the 5th vertex of an affine coordinate system then its affine coordinate is $(0,0,0,0,1)$, and the coordinate of the centre of the 3-simplex at the base of the [4,1] simplex is $(1/4,1/4,1/4,1/4,0)$. The
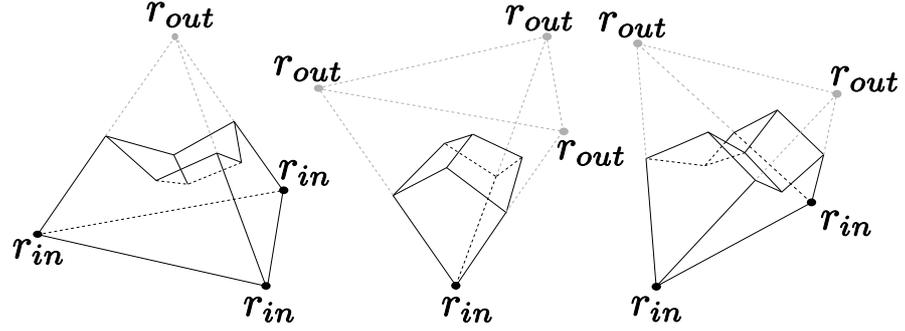
Figure 3.15: From left to right: example 3-simplices from the spatial foliation of a $[4,1]_{[3,1]}$, $[4,1]_{[1,3]}$, and $[4,1]_{[2,2]}$ simplex, showing the dual surfaces $\mathcal{B}_\mathcal{H}$ (or $\mathcal{B}$). The kinks are the 1-simplices that run from the centre of the triangular faces to the centre of the 3-simplices. There are 3 of these kinks in the $[4,1]_{[3,1]}$ and $[4,1]_{[1,3]}$ type simplices and 4 in the $[4,1]_{[2,2]}$ type.

vector $\vec{C5}$ from the centre of the base simplex to the tip of the 4-simplex therefore has components

$$\vec{C5} = (-1/4, -1/4, -1/4, -1/4, 1). \tag{3.87}$$

Using the fact that the spatial edges of the 4-simplex have length a and the timelike edges have length $\sqrt{\alpha}a$ we find

$$(\vec{C5})^2 = \eta_{ij}(\vec{C5})^i(\vec{C5})^j = -\alpha a^2 - \frac{3a^2}{8}, \tag{3.88}$$

and therefore:

$$|\vec{C5}| = \sqrt{\alpha a^2 + \frac{3a^2}{8}}. \tag{3.89}$$

The $\mathcal{T}_\mathcal{H}$ kinks of the $[4,1]$ and $[1,4]$ simplices are triangles with base length (3.86) and height (3.89) and therefore their area is simply

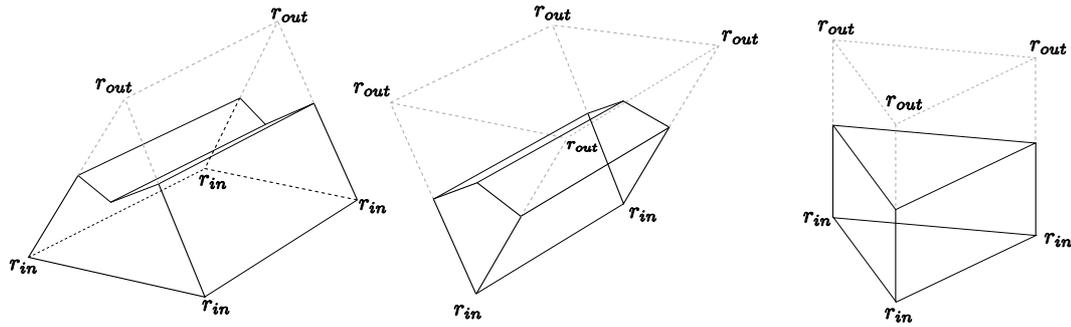$$Vol(k) = \frac{a^2\sqrt{\alpha + \frac{3}{8}}}{4\sqrt{6}}. \tag{3.90}$$

Figure 3.16: From left to right: example rectangular based pyramids from the spatial foliation of a $[3,2]_{[4,2]}$, $[3,2]_{[2,4]}$, and $[3,2]_{[3,3]}$ simplex, showing the dual surfaces $\mathcal{B}_{\mathcal{H}}$ (or $\mathcal{B}$).

In 4-simplices of type $[4,1]_{[3,1]}$ and $[1,4]_{[3,1]}$ there are three $\mathcal{T}_{\mathcal{H}}$ kinks with this area, as can be seen from the three kinks of $\mathcal{B}_{\mathcal{H}}$ in figure (3.15). The outward pointing normal vectors of $\mathcal{B}_{\mathcal{H}}$ on either side of the three kinks point towards each other by an angle of $\pi/3$. Likewise, in 4-simplices of type $[4,1]_{[1,3]}$ and $[1,4]_{[1,3]}$ there are three kinks, but the normal vectors of $\mathcal{B}_{\mathcal{H}}$ point away from each other by angle $-\pi/3$. Simplices of type $[4,1]_{[2,2]}$ and $[1,4]_{[2,2]}$ have two kinks at which the angle between the normal vectors is $\pi/3$ and two kinks at which the angle is $-\pi/3$.

The 4-simplices of type [3,2] and [2,3] are foliated into a parallel series of rectangular based pyramids. There are three possibilities with respect to the radial direction labelling of the vertices in these rectangular pyramids: $[3,2]_{[4,2]}$, $[3,2]_{[3,3]}$, and $[3,2]_{[2,4]}$, and similarly for [2,3]. Figure (3.16) shows the corresponding elements of the dual boundary $\mathcal{B}_{\mathcal{H}}$. As can be seen in the figure, there is no kink and therefore no extrinsic curvature of $\mathcal{B}_{\mathcal{H}}$ in the interior of a simplex of type $[3,2]_{[3,3]}$. When the subscript-radial labelling is either [4,2] or [2,4] the extrinsic curvature of $\mathcal{B}_{\mathcal{H}}$ is restricted to the 1-simplex kinks running through the centre of the rectangular based pyramids. In 4-simplices of type [3,2] these 1-simplex kinks have zero length in the base-triangle at time $t = 0$ (see figure (3.17)). With the passage of time they grown to length a at time $t = 1$, forming the triangular kinks of $\mathcal{T}_{\mathcal{H}}$ as they do so. In simplices of type [2,3] this is reversed; the 1-simplex kinks

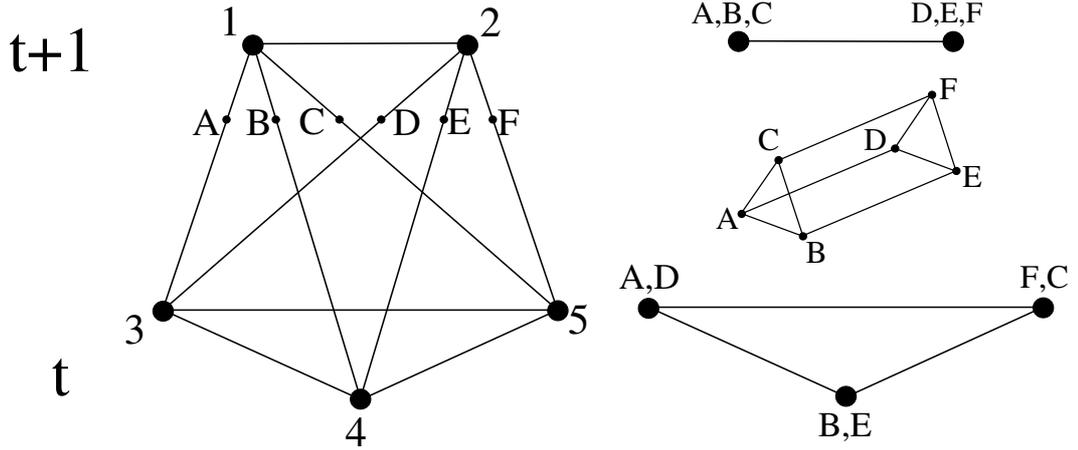Figure 3.17: A representation of a [3,2] type 4-simplex. The $1d$ sub-simplices in an intermediate spatial slice are labelled A to F. If two of these edges converge to a vertex at either t or $t+1$ they are connected, resulting in the rectangular based pyramid of the intermediate spatial foliations. The spatial foliation begins as two equilateral triangles on top of each other at t. As time passes the triangles move apart whilst simultaneously shrinking. At $t = 1$ the two triangles have shrank to points, separated by distance a.

begin with length a at $t = 0$ and shrink to zero to length at time $t = 1$, but the area of the $\mathcal{T}_{\mathcal{H}}$ kinks is the same.

To calculate the area of this triangle in the [3,2] case we need to know the distance from the centre of the base triangle at $t = 0$ to the centre of the top 1-simplex at time $t = 1$. The affine coordinate of the centre of the bottom triangle is $(1/3,1/3,1/3,0,0)$ and the centre of the top 1-simplex is $(0,0,0,1/2,1/2)$. Using this we can calculate the distance between them:

$$\sqrt{\frac{-a^2}{3} + \frac{3\alpha a^2}{4}}. \tag{3.91}$$

The area of the triangular kinks is therefore:

$$Vol(k) = \frac{1}{2}a\sqrt{\frac{-a^2}{3} + \frac{3\alpha a^2}{4}}. \tag{3.92}$$

For simplices of type $[3,2]_{[4,2]}$ and $[2,3]_{[4,2]}$ the angle $\Delta\phi$ between the normal vectors of $\mathcal{B}_{\mathcal{H}}$ that meet at the kink is $\pi/3$. For type $[3,2]_{[2,4]}$ and $[2,3]_{[2,4]}$ simplices it is $-\pi/3$.

Putting all the results of this section together, the CDT black hole action in $4d$ is

$$
\begin{aligned}
S_{BH} = S_{Regge} &+ \frac{\pi a^2}{6\kappa} \frac{\sqrt{\alpha + \frac{3}{8}}}{\sqrt{6}} \left(3N_{[4,1]_{[3,1]}} + 3N_{[1,4]_{[3,1]}} - 3N_{[4,1]_{[1,3]}} - 3N_{[1,4]_{[1,3]}} \right) \\
&\frac{\pi a^2}{12\kappa} \sqrt{\frac{-1}{3} + \frac{3\alpha}{4}} \left(N_{[3,2]_{[4,2]}} + N_{[2,3]_{[4,2]}} - N_{[3,2]_{[2,4]}} - N_{[2,3]_{[2,4]}} \right) \quad (3.93)
\end{aligned}
$$

3.10 Discussion

The first steps towards a CDT calculation of the Brown and York black hole density of states have been taken. A foliation and boundary structure of CDT simplicial spacetimes were introduced to implement an "orthogonal" spacetime foliation of the type used by Brown and York. The Regge calculus equivalent of extrinsic curvature of a timelike hypersurface was derived and used to implement Regge versions of the Brown and York quasilocal quantities. After the method was tested by deriving the Sorkin-Hartle boundary term for fixed metrics, a Regge version of the black hole action was constructed by substitution of Regge calculus expressions for their differential geometric counterparts. The black hole action was then specialised to the simplicial spacetimes used in CDTs, culminating in the actions (3.85) and (3.93), in $3d$ and $4d$ respectively.

More work is required to implement a full CDT calculation of the black hole density of states. The task is complicated by the questionable validity of the black hole density of states discussed in chapter (2). The issue of boundary conditions is the most difficult problem to overcome. Even in $4d$, where it is possible to interpret the boundary conditions as microcanonical on an outer boundary and thermodynamically completely open on the horizon, it is not clear how to implement the boundary conditions in Regge calculus. These boundary conditions involve fixed lapse and shift on the horizon, but Regge calculus

involves no notion of coordinates and therefore no notion of lapse or shift.

Moves are a set of rules by which simplices in the CDT triangulation may be changed during Monte Carlo simulations. If relevant Regge boundary conditions could be found, the next step would be to invent moves that leave these boundary conditions intact. Boundary conditions could then be taken from the de Sitter-Schwarzschild triangulations described in [Dittrich and Loll, 2006], and the CDT path integral implemented with these boundary conditions fixed.

Despite these difficulties with the microcanonical density of states, boundary terms in gravitational actions are of broad relevance. The method of deriving Regge calculus boundary terms developed in this section, and its demonstrated ability to reproduce the Hartle-Sorkin boundary term, may therefore prove to be of use outside of its application to the Brown and York program.

Chapter 4

Tolman Temperature and Observer Independent Entropy

4.1   Introduction

Bekenstein's generalized second law of thermodynamics [Bekenstein, 1973, Bekenstein, 1974], asserts that after any change of state, the sum of black hole entropy and the entropy of systems in the exterior region never decreases. The law appears to be violated if an object falling into a black hole of temperature T has an entropy $S_{obj} > E_{obj}/T$, where E_{obj} is the object's energy. In this case the black hole's entropy would increase by $dS_{BH} = E_{obj}/T$, but the exterior entropy would decrease by a greater amount: $dS_{Exterior} = -S_{obj} < -E_{obj}/T$. Marolf, Minic and Ross [Marolf et al., 2004] conjecture that the generalized second law remains valid in this scenario because the entropy attributed to a localized matter system is not an observer invariant quantity. In particular, a Rindler observer and an inertial observer in Minkowski spacetime assign different entropies to a system at rest with respect to the inertial observer. The entropy of an object that falls into a black hole, viewed by an accelerated observer who remains in the exterior, is lower than the normal entropy assigned by an inertial observer, and con-

sequently the generalized second law remains valid to observers exterior to black holes. The authors note that observer dependent entropy will have important consequences for thermodynamic discussions. In this chapter I point out one such implication of observer dependent entropy: the Tolman law [Tolman, 1934] for the local temperature of an equilibrium system in a gravitational field would no longer hold. The observer independence of entropy plays a key role in the derivation of the Tolman law and if the law is to apply to a system in a uniform gravitational field it must, by the equivalence principle, apply to an accelerating system. It is then noted that the Tolman law can be derived without recourse to the notion of entropy and therefore appears to seriously limit the useful introduction of an observer-dependent entropy.

4.2 The Tolman Law

The derivation of the Tolman law in which the required observer independence of entropy is most clearly visible is that of [Landau and Lifshitz, 1951]. An elevator containing a matter system in thermal equilibrium is held in a fixed position in a stationary gravitational field. A stationary observer with respect to the elevator describes the system by a spatially uniform temperature T such that an infinitesimal volume element of the system with entropy S and energy E satisfies

$$\frac{\partial S}{\partial E} = \frac{1}{T}. \tag{4.1}$$

Here E is the conserved Killing energy of the element of volume, associated with time translations of the elevator observer's coordinate system. An observer that is momentarily in free-fall at the location of the infinitesimal volume element measures its energy to be E_0, which is related to E by

$$E = E_0\sqrt{g_{00}}. \tag{4.2}$$

The free-fall observer will relate the temperature T_0 they measure to changes in entropy with respect to the energy E_0 via

$$\frac{\partial S_0}{\partial E_0} = \frac{1}{T_0}. \tag{4.3}$$

According to Tolman, entropy is observer independent and therefore the free-fall observer and the observer to whom the elevator is stationary both attribute the *same* entropy to the element:

$$S_0(E_0) = S(E). \tag{4.4}$$

We therefore have:

$$\frac{\partial S}{\partial E} = \frac{\partial S_0}{\partial E_0}\frac{\partial E_0}{\partial E} = \frac{1}{T_0}\frac{1}{\sqrt{g_{00}}}. \tag{4.5}$$

Comparison with equation (4.1) leads to the Tolman law for the variation of local temperature of a system in thermal equilibrium in a gravitational field:

$$T = T_0\sqrt{g_{00}}. \tag{4.6}$$

In particular, this must apply for an elevator in a uniform gravitational field described by the metric

$$ds^2 = (1 + gx)dt^2 - dx^2 - dy^2 - dz^2. \tag{4.7}$$

By the equivalence principle an elevator held fixed in a uniform gravitational field is equivalent to an elevator uniformly accelerated in flat spacetime and the above derivation of the Tolman law must therefore apply equally well to an accelerating elevator. Indeed, the metric (4.7) is the metric of Minkowski spacetime in Rindler coordinates. In this case T is the temperature measured by a Rindler observer accelerating with the elevator and T_0 is the temperature measured by a locally, momentarily co-moving Minkowski observer.

If, as proposed in [Marolf et al., 2004], entropy is observer-dependent then condition (4.4) would not hold and the Tolman law would require modification. The Tolman law can be derived, however, from purely mechanical considerations without recourse to the notion of entropy, as Tolman himself originally did [Tolman, 1930], by considering a spherically symmetric, static distribution of blackbody radiation. Given the spherical symmetry the metric of the system must take the form

$$ds^2 = e^{\nu(r)}dt^2 - e^{\mu(r)}(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2). \tag{4.8}$$

The components of the stress-energy tensor can be found in terms of the metric components through Einstein's equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu}. \tag{4.9}$$

Alternatively, the non-zero components of a spherically symmetric perfect fluid stress-energy tensor are $T_0^0 = \rho_{00}$ and $T_1^1 = T_2^2 = T_3^3 = -p_0$, where ρ_{00} is the proper energy density of the fluid and p_0 is its proper pressure. Combining the two expressions for the stress tensor results in the following expression for the change of pressure with radius:

$$\frac{\partial p_0}{\partial r} = -\frac{\rho_{00} + p_0}{2}\frac{\partial \nu}{\partial r}. \tag{4.10}$$

In the case of black body radiation the local temperature is a simple function of the energy density and pressure density:

$$\rho_{00} = aT_0^4, \tag{4.11}$$

and

$$p_0 = \frac{1}{3}aT_0^4, \tag{4.12}$$

where T_0 is the proper temperature as measured by a local free-fall observer. Substituting these into equation (4.10) leads to

$$\frac{d \ \log T_0}{dr} = -\frac{1}{2}\frac{d\nu}{dr}, \tag{4.13}$$

and upon integration

$$T = T_0 e^{\nu/2}, \tag{4.14}$$

where T is a constant of integration. With the metric (4.8) this agrees with Tolman's law.

4.3   Discussion

The key role of observer independent entropy in the derivation of Tolman's law has been highlighted and it has been shown, via the equivalence principle, that entropy should be independent of an observer's acceleration in Minkowski spacetime.

Chapter 5

Conclusions

A full calculation of the black hole density of states using CDTs is difficult to implement because of the problems inherent with the path integral approach to gravitational statistical mechanics as currently formulated.

The extension of the Brown and York microcanonical density of states to general spacetime dimension shows that it relies for its interpretation on features of the gravitational action specific to $4d$. Only in $4d$ does the entropy of Minkowski spacetime vanish with boundary conditions interpretable as microcanonical. The black hole density of states was extended to general spacetime dimension and shown to produce the expected result, in terms of horizon entropy, in any dimension; however, the boundary conditions that play an important role in the interpretation are also specific to $4d$. The calculation of de Sitter horizon entropy suffers the same dimensional dependence of its interpretation but, in four spacetime dimensions, suggests that the missing thermodynamic data interpretation is more promising than the no boundary interpretation. With the change of boundary topology motivated by the infinite area of the Rindler Horizon it was noted that the density of states of Brown and York supports the notion of a local horizon entropy density.

These difficulties led to a discussion of the microcanonical and canonical ensembles'

applicability to black hole mechanics. It was noted that the absence of angular momentum or velocity as an external parameter in the Hamiltonian is a severe difficulty in recovering the first law of black hole mechanics from these ensembles. The rotational ensemble and the angular momentum ensemble - a new ensemble developed for the first time in this thesis - were presented as more compelling candidates. A promising future route for the path integral approach would be to pursue calculations based on these ensembles.

Towards the objective of calculating the gravitational density of states non-perturbatively using causal dynamical triangulations, appropriate simplicial spacetime foliations and boundaries have been identified; Regge calculus versions of quasilocal energy-momentum have been developed; and the Regge black hole action has been derived and specialised to CDT simplicial spacetimes in $3d$ and $4d$. Before further work in this direction is pursued the problems identified with the current path integral approach to gravitational statistical mechanics should be resolved.

It has been shown that the recently proposed observer dependence of entropy would result, via the equivalence principle, in a violation of Tolman's law for the temperature variation at equilibrium in a gravitational field.

Chapter 6

Appendix

Relevant quotes on the interpretation of the black hole entropy calculation from [Brown and York, 1998].

"let ε, j_a, σ_{ab} be the stress-energy-momentum for a topologically spherical two-surface $\mathcal{B}$ within a time slice of a stationary black hole solution .... The path integral can be evaluated semiclassically by searching for metrics that extremise the action and satisfy the conditions at both $\mathcal{T}$ and $\mathcal{T}_{\mathcal{H}}$. One such metric will be the complex metric that is obtained by substituting $t \rightarrow -it$ in the Lorentzian black hole solution.......The complex metric will also satisfy the conditions at $\mathcal{T}_{\mathcal{H}}$, where $\mathcal{B}$ coincides with the intersection of the stationary time slices and the black hole horizon for the Lorentzian metric. This follows from the observation that the lapse function corresponding to the natural stationary time-slicing vanishes on the horizon of a stationary black hole."

"these conditions are "no boundary" conditions in the sense that they are precisely the conditions needed to seal the opening $\mathcal{T}_{\mathcal{H}}$ in the manifold $\mathcal{M} = \Sigma \times S^1$ and convert the manifold topology to $R^2 \times S^2$. Thus with these conditions there is no inner boundary in the spacetime four-geometry. [The conditions] are more properly called regularity conditions, rather than boundary conditions. Accordingly the path integral is *not* to be viewed as a functional of any inner boundary data."

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