Counting initial conditions and degrees of freedom in nonlocal gravity

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Nonlocal quantum gravity is a class of fundamental theories whose classical and quantum dynamics is specified by "form factors", operators with infinitely many derivatives. After briefly reviewing this paradigm and its role in the resolution of big-bang and black-hole singularities, we count the number of nonpertubative field degrees of freedom as well as the number of initial conditions to be specified to solve the Cauchy problem. In particular, in four dimensions and for the string-related form factor, there are 8 degrees of freedom (2 graviton polarization modes, which propagate, and 6 nonpropagating rank-2 tensor modes) and 4 initial conditions. The method to obtain this result is illustrated for the case of a nonlocal scalar field.

1. Introduction

"Nonlocal quantum gravity" is an umbrella name including at least two different settings. The first group, which includes general relativity, consists in classically local gravitational theories which receive quantum corrections such that the effective one-loop action is nonlocal (i.e., it is made of operators with infinitely many derivatives). In this brief review of recent results, we will confine ourselves to the second meaning of the term, where nonlocality is fundamentally present already at the classical level and, thanks to the suppression of the graviton propagator in the ultraviolet (UV), the theory is renormalizable or finite. Nowadays, nonlocal quantum gravity has achieved a high degree of independence both from these antecedents and from other proposals, to the point where it can be considered as one of the most promising and accessible candidates for a theory where the gravitational force consistently obeys the laws of quantum mechanics. In particular, there exist several renormalization results, both at finite order and to all orders in a perturbative Feynman-diagram expansion, which showed that the good UV properties guessed at the level of power-counting indeed hold rigorously (e.g., Ref. 1).

Despite the investment of much effort in taming fundamental nonlocality, several questions remain open to date: (i) Is the Cauchy problem well-defined? (ii) If so, how many initial conditions must one specify for a solution? (iii) How many degrees of freedom are there? (iv) How to construct nontrivial solutions? (iv) Is causality violated? (v) Are singularities resolved at the classical or quantum level?

Here we will give the following answers to some of these issues: (i) Yes, for the form factors appearing in fundamental theories (not for all conceivable form factors). (ii) Two or higher for a scalar field theory and four or higher for gravity (depending on the form factor), but finite. (iii) Eight (in $D = 4$ dimensions), but only the graviton modes propagate. (iv) Via the diffusion method. Let us now examine where these cryptic responses come from. The main results can be found in Refs. 2, 3.

2. Action and form factors

The classical fundamental (not effective) action of the theory is

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-g}\, [R + G_{\mu\nu} \gamma(\Box) R^{\mu\nu}] \ , \tag{1}$$

where $G_{\mu\nu}$ is the Einstein tensor and $\gamma(\Box)$ is a weakly or quasi-polynomial nonlocal form factor. Formally, γ is an analytic function that can be expressed as an infinite series with infinite convergence radius, $\gamma(\Box) = \sum_n c_n \Box^n$, although this expansion does not actually span the whole space of solutions in general.[4] As long as we require good properties at the quantum level (in particular, locality of the counterterms), the coefficients c_n can be selected in a subclass of entire functions having special asymptotic properties.[1,5–9] There are several form factors that preserve perturbative unitarity in (1) (see Table 1). In general, we can parametrized γ as

$$\gamma(\Box) = \frac{e^{H(\Box)} - 1}{\Box} \, , \qquad H(\Box) := \alpha \int_0^{P(\Box)} d\omega \, \frac{1 - f(\omega)}{\omega} \, , \tag{2}$$

where $H(\Box)$ depends on the dimensionless combination $l^2\Box$ and l is a fixed length scale. The profile $H(\Box)$ can be defined through an integral where $\alpha > 0$ is real and $P(\Box)$ is a generic function of $l^2\Box$. The parameter α will not play any important role in what follows, but we included it to reproduce some form factors in the literature.

Table 1. Form factors in nonlocal gravity.

$H(\Box)$	$P(\Box)$	$f(\omega)$	Form factor name
$H^{\rm pol}(\Box) := \alpha\{\ln P(\Box) + \Gamma[0, P(\Box)] + \gamma_{\rm E}\}$	$-l^2\Box$ $O(\Box^n)$	$e^{-\omega}$	Kuz'min[6] Tomboulis[5,8]
$H^{\rm exp}(\Box) := \alpha P(\Box)$	$-l^2\Box$ $l^4\Box^2$	$1-\omega$	string-related[10] Krasnikov[11]

All these form factors share the common property of blowing up in the UV in momentum space. In general, this implies asymptotic freedom, i.e., interactions are subdominant at short scales.

3. The wild beast of nonlocality

Consider the scalar field theory on Minkowski spacetime

$$S_\phi = \int d^D x \left[\frac{1}{2} \phi \Box \gamma(\Box) \phi - V(\phi) \right] , \tag{3}$$

where V is an interaction potential. How could one explore its classical dynamics?

As a first attempt, one can try to truncate the nonlocal operator up to some finite order, $\gamma(\Box) \simeq \sum_{n=0}^{N} c_n \Box^n = c_0 + c_1 \Box + \cdots + c_N \Box^N$. However, the resulting finite-order dynamics is physically inequivalent to the original one and there is no smooth transition between them. The free ($V=0$) case illustrates the point well:

$$\begin{aligned}\gamma(\Box)=e^{-l^2\Box} &\Rightarrow \text{dispersion relation:} \quad -k^2e^{l^2k^2}\phi_k=0\\ &\Rightarrow \text{propagator:} \quad -\frac{e^{-l^2k^2}}{k^2} \quad\Rightarrow\quad 1\ \text{DOF}\,, \end{aligned} \tag{4}$$

$$\begin{aligned}g(\Box)\simeq 1-l^2\Box &\Rightarrow \text{dispersion relation:} \quad -k^2(1+l^2k^2)\phi_k=0\\ &\Rightarrow \text{propagator:} \quad -\frac{1}{k^2}+\frac{1}{l^{-2}+k^2} \quad\Rightarrow\quad 2\ \text{DOF, 1 ghost}\,. \end{aligned} \tag{5}$$

This example provides a good occasion to comment also on how to determine the number of field degrees of freedom (DOF). In the free case, there is only one double pole, corresponding to 1 DOF. This can also be seen by making a nonlocal field redefinition $\tilde\phi:=\sqrt{\gamma(\Box)}\phi$, so that the free Lagrangian reads $\tilde{\mathcal{L}}_\phi=(1/2)\tilde\phi\Box\tilde\phi$. This operation is safe if γ is an entire function; if γ is not entire (for instance, $\gamma=\Box^{-n}$), then the field redefinition may result in the elimination of physical modes or the introduction of spurious ones. The Lagrangian $\tilde{\mathcal{L}}_\phi$ is second-order in spacetime derivatives and features one local field $\tilde\phi$, hence there is only 1 DOF and solutions are specified by two initial conditions. However, when V is nonlinear of cubic or higher order the field redefinition does not absorb nonlocality completely and one is left with a possibly intractable problem, with extra nonperturbative degrees of freedom[12] and an infinite tower of Ostrogradsky modes.

In fact, the Cauchy problem can be naïvely stated as the assignment of an infinite number of values at some initial time t_i,

$$\phi(t_\mathrm{i}),\ \dot\phi(t_\mathrm{i}),\ \ddot\phi(t_\mathrm{i}),\ \dddot\phi(t_\mathrm{i}),\ \dots\,. \tag{6}$$

Thus, paradoxically, we can solve the dynamics only if we already know the solution:[13]

$$\phi(t)=\sum_{n=0}^{+\infty}\frac{\phi^{(n)}(t_\mathrm{i})}{n!}(t-t_\mathrm{i})^n\,. \tag{7}$$

If we do not specify all the initial conditions, the solution may be non-unique.

Finally, a word on how to rewrite nonlocalities as a convolution. It is well known that infinitely many derivatives can be traded for integrated kernel functions:

$$\gamma(\Box)\,\phi(x)=\int d^Dy\,F(y-x)\,\phi(y)\,. \tag{8}$$

However, by itself this operator bears no practical advantage. Hiding infinitely many derivatives into integrals does not help in solving the Cauchy problem, *unless* the kernel F could be found by solving some auxiliary, finite-order differential equations. This is precisely the leverage point we will focus on.

4. Diffusion method

The diffusion method was proposed some years ago[4,12,14,15] to solve nonlocal scalar field theories with exponential form factor (4), a very specific nonlocal operator that arises in string theory. By trading nonlocal operators with shifts in a fictitious extra direction r, the method allows one to count the number of DOF and of initial conditions (which are finite) and to find nonperturbative solutions. All these features can be easily illustrated by the scalar field theory

$$S_\phi = \int d^D x \left[\frac{1}{2}\phi(x)\Box e^{-l^2\Box}\phi(x) - V(\phi)\right], \tag{9}$$

where l^2 is a constant. The equation of motion is

$$\Box e^{-l^2\Box}\phi - V'(\phi) = 0\,. \tag{10}$$

Define now a *localized system*, *a priori* independent of (9), living in $D+1$ dimensions and featuring two scalars $\Phi(r,x)$ and $\chi(r,x)$:

$$\mathcal{S}[\Phi,\chi] = \int d^D x\, dr\, (\mathcal{L}_\Phi + \mathcal{L}_\chi)\,, \tag{11}$$

$$\mathcal{L}_\Phi = \frac{1}{2}\Phi(r,x)\Box\Phi(r-l^2,x) - V[\Phi(r,x)]\,, \tag{12}$$

$$\mathcal{L}_\chi = \frac{1}{2}\int_0^{l^2} dq\, \chi(r-q,x)(\partial_{r'} - \Box)\Phi(r',x), \qquad r' = r+q-l^2. \tag{13}$$

The equations of motion are

$$0 = (\partial_r - \Box)\Phi(r,x), \qquad 0 = (\partial_r - \Box)\chi(r,x), \tag{14}$$

$$0 = \frac{1}{2}[\Box\Phi(r-l^2,x) + \chi(r-l^2,x)] + \frac{1}{2}[\Box\Phi(r+l^2) - \chi(r+l^2)] - V'[\Phi(r,x)]. \tag{15}$$

The first line is telling us that the fields are diffusing along the extra direction. At this point, one assumes that there exists a constant β such that the equation of motion (15) coincides with the one of the nonlocal system (9), Equation (10), on the slice $r = \beta l^2$ (the *physical slice*). This is achieved provided the following conditions hold:

$$\Phi(\beta l^2, x) = \phi(x), \qquad \chi(\beta l^2, x) = \Box\Phi(\beta l^2, x)\,. \tag{16}$$

The conclusion is that the localized system has 4 initial conditions $\Phi(r,t_\mathrm{i},\mathbf{x})$, $\dot\Phi(r,t_\mathrm{i},\mathbf{x})$, $\chi(r,t_\mathrm{i},\mathbf{x})$, $\dot\chi(r,t_\mathrm{i},\mathbf{x})$ and 2 field DOF Φ and χ. On the physical slice, because of (16) the number of DOF reduces to 1 and the initial conditions are on the field ϕ and its first two derivatives (the initial conditions of χ are not independent):

$$\phi(t_\mathrm{i},\mathbf{x}), \quad \dot\phi(t_\mathrm{i},\mathbf{x})\,. \tag{17}$$

With traditional methods, only *perturbative* solutions of the linearized EOM or what we call “static” (in the extra direction r) solutions are available to inspection. By this name, we mean solutions where nonlocality is, in one way or another, trivialized, such as when $\Box\phi = \lambda\phi$. In contrast, the diffusion method gets access to nonperturbative solutions valid in the presence of nonlinear interactions and nontrivial nonlocality. These solutions are, in general, only approximate, and are found by searching for the value of β minimizing the equations of motion[4,15,16]. Examples are: $\phi(t) = \text{Kummer}(t)$ on a Friedmann–Robertson–Walker (FRW) background; rolling tachyon $\phi(t) = \sum_n a_n e^{nt}$ in string field theory, $V = \phi^3$; kink $\phi(x) = \text{erf}(x)$, $V = \phi^3$; $\phi(t) = \gamma(\alpha, t)$ (incomplete gamma function), $V = \phi^n$ on FRW background; instanton $\phi(x) = \text{erf}(x)$, $V = \phi^4$ (brane tension recovered at 99.8% level); kink $\phi(x) = \text{erf}(x)$, $V = (e^{\Box}\phi^2)^2$; various profiles $\phi(t)$ in bouncing and singular cosmologies.

The main reason why diffusion works is that nonlocal operators are represented as a shift in an extra direction rather than as an infinite sum of derivatives. The latter representation does not span the whole space of solutions, as one can see by a toy example.[4] Consider a $D = 4$ FRW background with Hubble expansion $H = \dot{a}/a = H_0/t$, the Laplace–Beltrami operator $\Box = -\partial_t^2 - 3H\partial_t$, and the homogeneous power-law profile $\phi(t) = t^p$. If we try to calculate the object $e^{r\Box}\phi$ as a series, the result diverges: $e^{r\Box}\phi = \sum_{n=0}^{\infty}(r\Box)^n\phi/n! = \infty$. On the other hand, with the diffusion method one interprets $\phi(t) = \Phi(t, 0)$ as the initial condition in the diffusion scale r and the profile $e^{r\Box}\Phi(t, 0) = \text{Kummer}(t, r)$ is a linear superposition of well-defined Kummer functions.

5. Initial conditions and degrees of freedom

The diffusion method has been extended to the case of gravity in Ref. 2 for the string-related and Krasnikov exponential form factors and in Ref. 3 for the asymptotically polynomial (Kuz'min and Tomboulis–Modesto) form factors. The reader can consult those papers for technical details; here we only quote the bottom line, which is that, for the string-related form factor, the localized system associated with (1) has 6 initial conditions $g_{\mu\nu}(t_\text{i}, \mathbf{x})$, $\dot{g}_{\mu\nu}(t_\text{i}, \mathbf{x})$, $\Phi_{\mu\nu}(r, t_\text{i}, \mathbf{x})$, $\dot{\Phi}_{\mu\nu}(r, t_\text{i}, \mathbf{x})$, $\chi_{\mu\nu}(r, t_\text{i}, \mathbf{x})$, $\dot{\chi}_{\mu\nu}(r, t_\text{i}, \mathbf{x})$, two for each rank-2 symmetric tensor field (the metric $g_{\mu\nu}$ and the tensors $\Phi_{\mu\nu}$ and $\chi_{\mu\nu}$). Since, on the physical slice, $\Phi_{\mu\nu}(\beta l^2, x) = G_{\mu\nu}$ and $\chi_{\mu\nu}(\beta l^2, x) = R_{\mu\nu}$, the initial conditions on these fields are not independent. However, while $\chi_{\mu\nu}$ is an auxiliary field of the localized system and depends on the dynamical degrees of freedom of the nonlocal system, $\Phi_{\mu\nu}$ is an auxiliary field already at the level of nonlocal dynamics and it encodes the two derivatives hidden in the Ricci tensor and scalar. Therefore, the solutions of the nonlocal system (1) are characterized by 4 initial conditions:

$$g_{\mu\nu}(t_\text{i}, \mathbf{x}), \quad \dot{g}_{\mu\nu}(t_\text{i}, \mathbf{x}), \quad \ddot{g}_{\mu\nu}(t_\text{i}, \mathbf{x}), \quad \dddot{g}_{\mu\nu}(t_\text{i}, \mathbf{x})\,. \tag{18}$$

Regarding the degrees of freedom, the counting for the exponential form factor is the following. (i) Graviton $g_{\mu\nu}$: symmetric $D \times D$ matrix with $D(D+1)/2$ independent entries, to which one subtracts D Bianchi identities $\nabla^\mu G_{\mu\nu} = 0$ and D diffeomorphisms (the theory is fully diffeomorphism invariant). Total: $D(D-3)/2$.

In $D = 4$, there are 2 degrees of freedom. (ii) Tensor $\phi_{\mu\nu}$: symmetric $D \times D$ matrix with $D(D+1)/2$ independent entries, to which one subtracts D transverse conditions $\nabla^\mu \phi_{\mu\nu} = 0$. Total: $D(D-1)/2$. In $D = 4$, there are 6 degrees of freedom. Similar results hold for the asymptotically polynomial Kuz'min form factor, although in that case the diffusion method requires more elaboration.[3]

The grand total is $D(D-2)$. In $D = 4$, there are 8 DOF. Two of them (the graviton) are visible already at the perturbative level, while the other 6 are of nonperturbative origin. Their role in phenomenology[3] has been determined only recently.[17] It was shown that the extra $D(D-1)/2$ tensor degrees of freedom do not propagate on Ricci-flat backgrounds, at any perturbative order.

6. Conclusions

The number of degrees of freedom and of initial conditions of fundamentally nonlocal gravitational theories with "well-behaved" form factors is finite. In the diffusion method, infinitely many initial conditions are traded for boundary conditions in an extra direction. Solving the diffusion equation and algebraic relations is way simpler than solving nonlocal equations. By making sense of the Cauchy problem in this class of theories, the doors of classical top-down phenomenology may open up.

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