ON THE NON-LINEAR THEORY OF BETATRON OSCILLATIONS

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Introduction

In considering various problems of the theory of cyclic accelerators we always have to do with the motion of particles in a magnetic field in the vicinity of a certain given curve. In recent years new types of cyclic accelerators have been suggested based on alternating-gradient focusing and other principles [1-7]. The paths of the particles in these machines, even in the ideal case, are sometimes of a rather complex shape and are complicated still more by unavoidable disturbances.

Analysis shows that despite the diversity of variations they all have much in common. A number of problems concerning the case of a plane initial curve were investigated at an earlier date by different methods, e.g., in the papers [8-11]. This paper deals in general with the motion of a particle in a magnetic field in the vicinity of an arbitrary (generally speaking, space) curve. By analogy with simpler cases we shall call this motion betatron oscillations.

More detailed consideration is given to certain disturbing resonance phenomena—both linear and non-linear—arising in magnetic periodic systems, and primarily in alternating-gradient focusing ones.

1. Natural coordinate system

Suppose we have a space curve $\mathbf{r}_0 = \mathbf{r}_0(\sigma)$ where σ is the length of its arc. The properties of the curve are characterized by two parameters: curvature $k(\sigma)$ and torsion $\varkappa(\sigma)$ [12].

$$k(\sigma) = \left| \mathbf{r}_0''(\sigma) \right|, \quad \varkappa(\sigma) = \left(\frac{[\mathbf{r}_0'(\sigma)\, \mathbf{r}_0''(\sigma)]\mathbf{r}_0'''(\sigma)}{\mathbf{r}_0''^2(\sigma)} \right) \tag{1.1}$$

where $'$ denotes differentiation with respect to σ. $\varkappa \equiv 0$ corresponds to a plane curve. In future we shall use the natural coordinate system, connected with this curve, for the unit vectors of which $\mathbf{i}_1(\sigma)$, $\mathbf{i}_2(\sigma)$, $\mathbf{i}_3(\sigma)$ we shall take

$$\mathbf{i}_1 = \mathbf{r}_0', \quad \mathbf{i}_2 = -\frac{\mathbf{r}_0''}{(\mathbf{r}'')}, \quad \mathbf{i}_3 = [\mathbf{i}_2\, \mathbf{i}_1]. \tag{1.2}$$

The vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ determining the typical directions: $\mathbf{i}_1$ – tangent, $-\mathbf{i}_2$ – main normal, $\mathbf{i}_3$ – binormal, are related by Frenet's formulae [12]

$$\mathbf{i}_1' = -k\mathbf{i}_2, \quad \mathbf{i}_2' = -\varkappa\mathbf{i}_3 + k\mathbf{i}_1, \quad \mathbf{i}_3' = \varkappa\mathbf{i}_2. \tag{1.3}$$

We designate the coordinates corresponding to $\mathbf{i}_2$, $\mathbf{i}_3$ by ρ and z, so that the radius-vector of the arbitrary point $\mathbf{r}(\sigma,\rho,z)$ equals

$$\mathbf{r} = \mathbf{r}_0(\sigma) + \rho\mathbf{i}_2 + z\mathbf{i}_3 \tag{1.4}$$

From (1.3), (1.4) we obtain the relations:

$$\begin{aligned} \partial\mathbf{r}/\partial\sigma &= \mathbf{a}_1 = (1+k\rho)\,\mathbf{i}_1 + \varkappa z\mathbf{i}_2 - \varkappa\rho\mathbf{i}_3 \,, \\ \partial\mathbf{r}/\partial\rho &= \mathbf{a}_2 = \mathbf{i}_2 \,, \quad \partial\mathbf{r}/\partial z = \mathbf{a}_3 = \mathbf{i}_3 \,, \end{aligned} \tag{1.5}$$

In contradistinction to the orthogonal system $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ the system of vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, which is not orthogonal when $\varkappa \neq 0$, possesses all the necessary proporties of curvilinear coordinate systems [13].

2. Maxwell's equations in natural coordinates

We shall make use of the auxiliary coordinate system $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. Finding the square of the length element $d\mathbf{r}^2$ according to (1.5) we obtain the expression for the metric tensor

$$g_{ij} = \begin{pmatrix} (1+k\rho)^2 + \varkappa^2(\rho^2+z^2) & \varkappa z & -\varkappa\rho \\ \varkappa z & 1 & 0 \\ -\varkappa\rho & 0 & 1 \end{pmatrix} \tag{2.1}$$

The value $g = \det(g_{ij})$ determining the volume element $dV = \sqrt{g}\, d\sigma d\rho dz$ is equal to

$$g = (1 + k\rho)^2 \tag{2.2}$$

according to (2.1).

In compliance with the general formulae for curvilinear coordinates [13] the divergence and the curl of the arbitrary vector $\mathbf{P}$ equal

$$\operatorname{div} \mathbf{P} = \frac{1}{\sqrt{g}} \sum_{i=1}^{3} \frac{\partial}{\partial u} (p^i \sqrt{g}), \tag{2.3}$$

$$\text{rot}\,\mathbf{P} = \frac{1}{\sqrt{g}}\left\{\left(\frac{\partial p_3}{\partial u^2} - \frac{\partial p_2}{\partial u^3}\right)\mathbf{a}_1 + \left(\frac{\partial p_1}{\partial u^3} - \frac{\partial p_3}{\partial u^1}\right)\mathbf{a}_2 + \left(\frac{\partial p_2}{\partial u^1} - \frac{\partial p_1}{\partial u^2}\right)\mathbf{a}_3\right\} \tag{2.4}$$

where $u^1, u^2, u^3 \rightarrow \sigma, \rho, z$; $p_i = \mathbf{P}\mathbf{a}_i$; $p^i = \mathbf{P}\mathbf{a}^i$ are respectively the contravariant and co-variant components of $\mathbf{P}$; $\mathbf{a}_i$ are mutual coordinate vectors

$$\mathbf{a}^1 = [\mathbf{a}_2\mathbf{a}_3]/V, \quad V = \mathbf{a}_1[\mathbf{a}_2\mathbf{a}_3], \tag{2.5}$$

$\mathbf{a}_2$, $\mathbf{a}_3$ are found by cyclic permutation from the formula for $\mathbf{a}^1$. Using (1.5), all the values in (2.3), (2.4) can be found. Taking the magnetic field $\mathbf{H}$ as the vector $\mathbf{P}$ we get the equation div $\mathbf{H}$ and the projections of the equation rot $\mathbf{H} = 0$ on $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, as

$$\frac{\partial H_\sigma}{\partial \sigma} + kH_\rho + (1 + k\rho)\frac{\partial H_\rho}{\partial \rho} + (1 + k\rho)\frac{\partial H_z}{\partial z} + \varkappa\left(\rho\frac{\partial H_\sigma}{\partial z} - z\frac{\partial H_\sigma}{\partial \rho}\right) = 0 \tag{2.6}$$

$$\frac{\partial H_z}{\partial \rho} - \frac{\partial H_\rho}{\partial z} = 0 \tag{2.7}$$

$$(1 + k\rho)\frac{\partial H_\sigma}{\partial z} - \frac{\partial H_z}{\partial \sigma} + \varkappa\left(H_\rho + z\frac{\partial H_\rho}{\partial z} - \rho\frac{\partial H_z}{\partial \rho}\right) = 0 \tag{2.8}$$

$$\frac{\partial H_\rho}{\partial \sigma} - kH_\sigma - (1 + k\rho)\frac{\partial H_\sigma}{\partial \rho} + \varkappa\left(H_z - z\frac{\partial H_\rho}{\partial \rho} + \rho\frac{\partial H_z}{\partial \rho}\right) = 0 \tag{2.9}$$

Differentiating (2.6) - (2.9) with respect to σ, ρ, z as many times as necessary and taking into account that $k = k(\sigma)$, $\varkappa = \varkappa(\sigma)$, we obtain the relation between the highest derivatives of the field components with respect to the coordinates. Assuming that $\rho = z = 0$, we obtain this relation for the points of the initial coordinate orbit. Thus various functions characterizing the field can be expressed by certain functions which do not depend on σ. H_z, H_σ, H_ρ, $\partial H_z/\partial\rho$, $\partial H_\rho/\partial\rho$ can be selected, for example, as such functions, for the first order; for the second order, besides those just mentioned, the functions $\partial^2 H_z/\partial\rho^2$, $\partial^2 H_\rho/\partial\rho^2$ etc. can be taken. Expansion of the magnetic field components in powers of ρ, z has the form of:

$$H_z(\rho,z) = H_{zo} + \left(\frac{\partial H_z}{\partial \rho}\right)_o \rho - \left[\left(\frac{\partial H_\sigma}{\partial \sigma}\right)_o + kH_{\rho o} + \left(\frac{\partial H_\rho}{\partial \rho}\right)_o\right] z + \ldots \tag{2.10}$$

$$H_\rho(\rho,z) = H_{\rho o} + \left(\frac{\partial H_\rho}{\partial \rho}\right)_o \rho + \left(\frac{\partial H_z}{\partial \rho}\right)_o z + \ldots \tag{2.11}$$

$$H_\sigma(\rho,z) = H_{\sigma o} + \left[\left(\frac{\partial H_\rho}{\partial \rho}\right)_o - kH_{\sigma o} + \varkappa H_{zo}\right]\rho + \left[\left(\frac{\partial H_z}{\partial \sigma}\right)_o \varkappa H_{\rho o}\right] z + \ldots \tag{2.12}$$

3. Equations of motion of a particle in natural coordinates

Let us consider a space area adjacent to the curve $\mathbf{r}_0(\sigma)$ and occupied by the magnetic field $\mathbf{H}(x, y, z)$. Let a particle be moving in this field and let its charge, momentum and speed be, respectively, e, $\vec{p}$, $\mathbf{v}$. We presume that, generally speaking, the value ρ as well as H can change slowly (adiabatically) in time. The equation of motion of the particle can be written as:

$$\frac{d\mathbf{v}^{(o)}}{dl} + \frac{1}{\rho}\frac{d\rho}{dl}\mathbf{v}^{(o)} = \frac{e}{c p}[\mathbf{v}^{(o)}\mathbf{H}] \tag{3.1}$$

where the variable l is the arc length of particle's trajectory and $\mathbf{v}^0$ the unit speed vector. In order to pass over to the main variable σ, it should be kept in mind that $\mathbf{v}^0 = \mathbf{r}'/l'$ and the expressions for l', l'', $\mathbf{r}'$, $\mathbf{r}''$ which can be obtained by means of (1.3) - (1.5) should be used. Thus, l' equals:

$$l' = [(1+k\rho)^2 + (\rho'+\varkappa z)^2 + (z'-\varkappa\rho)^2]^{1/2} \tag{3.2}$$

As a result of calculations from (3.1) we get the equations for the determination of $\rho(\sigma)$, $z(\sigma)$ solved with respect to the second derivatives ρ'', z''.

$$\rho'' + \frac{p'}{p}\frac{(\rho' + \varkappa z)}{(1 + k\rho)^2} = \frac{1}{1 + k\rho}\left\{\frac{el'}{pc}[H_\rho(\rho' + \varkappa z)(z' - \varkappa\rho) - H_z((1+k\rho)^2 + (\rho'+\varkappa z)^2) + H_\sigma(1 + k\rho)(z'-\varkappa\rho)] + k + (2k^2 + \varkappa^2)\rho - \varkappa'z - 2\varkappa'z' + k(k^2+\varkappa^2)\rho^2 + (k'\varkappa - k\varkappa')\rho z + k\varkappa^2 z^2 + k'\rho\rho' - 2k\varkappa\rho z' + 3k\varkappa\rho' z + 2k\rho'^2\right\}, \tag{3.3}$$

$$z'' + \frac{p'}{p}\frac{(z'-\varkappa\rho)l'^2}{(1+k\rho)^2} = \frac{1}{k\rho+1}\left\{\frac{el'}{pc}[H_\rho((1+k\rho)^2 + (z'-\varkappa\rho)^2) - H_z(z'-\varkappa\rho)(\rho'+\varkappa z) - H_\sigma(1+k\rho)(\rho'+\varkappa z)] + \varkappa'\rho + \varkappa^2 z + 2\varkappa\rho' + (k\varkappa'-k'\varkappa)\rho^2 + 2k\rho' z' + k\varkappa z z'\right\}. \tag{3.4}$$

The equations (3.3), (3.4) together with field component expansions of the type (2.10) - (2.12) enable us to study the motion in any order by ρ, ρ', z, z',... These equations are a generalization of the known equations of betatron oscillations for the case of motion about an arbitrary space curve.

In linear approximation (by ρ, z, ρ'...) the equations (3.3), (3.4) become

$$\rho'' + \frac{p'}{p}\rho' + \left(2k\frac{eH_z}{pc} - k^2 - \varkappa^2 + \frac{e}{pc}\frac{\partial H_z}{\partial \rho} + \varkappa\frac{eH_\sigma}{pc}\right)\rho + \left(\frac{e}{pc}\frac{\partial H_z}{\partial z} + \varkappa'\right)z - \left(\frac{eH_\sigma}{pc} - 2\varkappa\right)z' = k - \frac{eH_z}{pc}, \tag{3.5}$$

$$z'' + \frac{p'}{p}z' - \left(2k\frac{eH_\rho}{pc} + \frac{e}{pc}\frac{\partial H_\rho}{\partial \rho} + \varkappa'\right)\rho - \left[\frac{e}{pc}\left(\frac{\partial H_\rho}{\partial z} - \varkappa H_\sigma\right) + \varkappa^2\right]z + \left(\frac{eH_\sigma}{pc} - 2\varkappa\right)\rho' = \frac{eH_\sigma}{pc}, \tag{3.6}$$

which show the dependence of the frequency of the betatron oscillations, the connection between the ρ and z-oscillations and other characteristics on the curvature $\varkappa$, the torsion k, the momentum p and the shape of the field.

4. Case of a field which has a plane of symmetry

Of practical importance is the case where the magnetic field (ideal) has the plane of symmetry z = 0 in which the initial plane curve $\varkappa \equiv 0$ (4.1) lies

We express the real field **H** as the sum of an ideal field $\mathbf{H}^{id}$ and the magnitude **h** which describes the field distortions

$$\mathbf{H}(\sigma,\rho,z) = \mathbf{H}^{id}(\sigma,\rho,z) + \mathbf{h}(\rho,\sigma,z),\ |\mathbf{h}| \ll |\mathbf{H}^{id}| \tag{4.2}$$

Due to its symmetry, $\mathbf{H}^{id}$ should satisfy the following relations:

$$\left.\frac{\partial^{m+e+s} H_z^{id}(\sigma,\rho,z)}{\partial\rho^m\partial\sigma^e\partial z^s}\right|_{z=0} \equiv 0, \quad \left.\frac{\partial^{m+e} H_{\rho,\sigma}^{id}(\sigma,\rho,z)}{\partial\rho^m\partial\sigma^e}\right|_{z=0} \equiv 0 \tag{4.3}$$

where e, m are any integers, including zero and s an odd number. For the disturbing field **h** (σ, ρ, z) the derivatives indicated in (4.3) will be, in general, small values differing from zero.

We shall assume that the momentum p can differ from the fixed (equilibrium) value p_0 by a relatively small value

$$p = p_0(1 + \zeta),\ \zeta \ll 1 \tag{4.4}$$

This deviation of the momentum in accelerators is due primarily to synchronous oscillations. Considering the expansions of the field components (2.10) - (2.12) for this case and using the Maxwell equations (2.6) - (2.9), it can readily be seen that the ideal field $\mathbf{H}^{id}$ in its respective order (according to ρ and z) is fully determined by the pre-set values

$$\left.\frac{\partial^m H_z^{id}}{\partial\rho^m}\right|_{\rho = z = 0} \quad (m = 0, 1, 2, \ldots)$$

Instead of these the following dimensionless parameters, which are a generalization of the common field index n, can be employed:

$$n_m(\sigma) = (-1)^{m+1}\left.\frac{1}{k^m(\sigma)H(\sigma)}\frac{\partial^m H}{\partial\rho^m}\right|_{\rho = z = 0,} \quad H(\sigma) \equiv H_z^{id}(\sigma) \tag{4.5}$$

For the sake of brevity we introduce also the value $k_0(\sigma)$ which has the dimensions of curvature

$$k_0(\sigma) = eH(\sigma)/cp_0 \tag{4.6}$$

We denote the characteristics of the disturbing field by

$$Z(\sigma) = \frac{h_z(\sigma)}{H(\sigma)},\ R(\sigma) = \frac{h_\rho(\sigma)}{H(\sigma)},\ S(\sigma) = \frac{h_\sigma(\sigma)}{H(\sigma)},$$

$$Z(\sigma) = \frac{1}{H(\sigma)}\frac{\partial h_z}{\partial z},\ R_\sigma(\sigma) = \frac{1}{H(\sigma)}\frac{\partial h_\rho(\sigma)}{\partial\sigma}, \tag{4.7}$$

etc., where the components and their derivatives are taken for $\rho = z = 0$. We assume the values in (4.7) to be small values of the first order. Under the assumptions (4.1) - (4.7) equations for the determination of $\rho(\sigma)$, $z(\sigma)$ in any order can be obtained from the initial equations (3.3) - (3.4). Restricting ourselves to the third order of ρ, z, ρ', z',..., Z, R, Z_ρ,..., which is sufficient for the majority of practical problems, we get equations which take into account all the types of disturbances

$$\frac{\rho''}{k_0} + \frac{p'}{p}\frac{\rho'}{k_0} + \left[\frac{k}{k_0} - n + 2Z + \frac{Z_\rho}{k} + (n-2)\zeta\right]k\rho - (kR + R_\rho + S_\sigma)kz - Sz' = -1 + \frac{k}{k_0} - \zeta - Z + \left(2n - 1 + \frac{n_1}{2} - \frac{Z_{\rho\rho}}{2k^2} + \ldots\right)$$
$$\times k^2\rho^2 + (3kR_\rho + \ldots)\rho z + \tfrac{1}{2}[k^2(n_1 - n) + Z_{\rho\rho} + \ldots]z^2 + \left[2\frac{k}{k_0} - \frac{3}{2}(1 + Z)\right]\rho'^2 - \tfrac{1}{2}(1 + Z)z'^2 + \frac{k'}{k_0}\rho\rho' + R_\sigma zz'$$
$$-2\left(\frac{k}{k_0} - \frac{3n}{4}\right)k\rho\rho'^2 - \frac{k}{k_0}k'\rho^2\rho' + \frac{kn}{2}\rho z'^2 + \left[\left(\frac{3n_1}{2} - n_2 - n_1\right) + \ldots\right]\rho z^2 + \left(n - n_1 + \frac{n_2}{6}\right)k^3\rho^3 - knzp'z' + Rp'z' + \ldots \tag{4.8}$$

$$\frac{z''}{k_0} + \frac{p'}{p}\frac{z'}{k_0} + \left[n(1-\zeta) - \frac{z_\rho}{k}\right] kz - (R_\rho + 2kR)k\rho + S\rho' = R + (2kR_\rho + \ldots)\rho^2 + [k^2(n_1 - 2n) + Z_{\rho\rho} + \ldots.]\rho_1 z$$
$$+ \tfrac{1}{2}(kR_\rho + \ldots)z^2 - R_\sigma\rho\rho' - z_\sigma z\rho' + \left(2\frac{k}{k_0} - 1 - z\right)\rho' z' + \tfrac{1}{2}R_\rho'^2 + \frac{3}{2}R z'^2 + \left(2n_1 - n - \frac{n_2}{2}\right)k^3\rho^2 z$$
$$- \frac{1}{6}[k^3(n + n_1 - n_2) + \ldots.]z^3 - \frac{kn}{2}z\rho'^2 - \frac{3kn}{2}zz'^2 + \left(n - 2\frac{k}{k_0}\right)k\rho\rho' z' + \ldots, \tag{4.9}$$

where in some of the coefficients the less important terms have been omitted, particularly those containing $\partial H/\partial\sigma$, $\partial^2 H/\partial\sigma\,\partial\rho$ etc. In the equations (4.8), (4.9) all the field components and their derivatives are expressed through certain independent functions, as indicated at the end of section 2. The form in which these equations are written facilitates their comparison with the equations for the simplest case of an axially-symmetrical field. The left parts of (4.8), (4.9) contain the linear terms with respect to ρ, z and their derivatives, and the right parts the non-linear terms as well as the terms $-1 + k/k_0$, $\zeta - Z$, R determining the closed orbit around which the betatron oscillations take place.

5. Motion of a particle in magnetic periodic systems

Magnetic periodic systems consist of a certain number N of identical (in the ideal case) periodicity elements arranged along a closed curve. We substitute the arc length σ by the generalized azimuthal angle ϑ, which is equal to

$$\vartheta = 2\pi\ \sigma/\Pi = \sigma/r_{av} \tag{5.1}$$

where Π is the perimeter of the main orbit and r_{av}, a certain average radius of the orbit. The characteristics of an ideal field repeat themselves along the main orbit with a periodicity of: $\vartheta_0 = 2\pi/N$; the parameters of the disturbing field repeat, generally speaking, after each full revolution, i.e., with a periodicity of 2π.

First we consider the following system of equations of the first order from (4.8), (4.9)

$$\frac{d^2\rho}{d\vartheta^2} + \frac{1}{p}\frac{dp}{d\vartheta}\frac{d\rho}{d\vartheta} + kr_{av}^2(k - k_0 n)\rho = r_{av}^2(k - k_0) + k_0 r_{av}^2(\zeta - z) \tag{5.2}$$

$$\frac{d^2 z}{d\vartheta^2} + \frac{1}{p}\frac{dp}{d\vartheta}\frac{dz}{d\vartheta} + kk_0 r_{av}^2 nz = k_0 R \tag{5.3}$$

The solutions of the respective homogeneous equations, which are close to Hill's equations, can be written in the normal form[16])

$$\left.\begin{matrix}\rho_0(\vartheta)\\ z_0(\vartheta)\end{matrix}\right\} = \left[\frac{p(\vartheta_{init})}{p(\vartheta)}\right]^{\frac{1}{2}}\left[A_{\rho,z}\, e^{i\mu_{\rho,z}m} f_{\rho,z}(\vartheta) + \text{c.c.}\right] \tag{5.4}$$

where m is the number of the element of periodicity, c.c. a complex-conjugated value and $A_{\rho,z}$ "amplitudes" defined by the initial conditions. The factor $\left[\frac{P(\vartheta_{init})}{P(\vartheta)}\right]^{1/2}$ describes the damping of the betatron oscillations, due to adiabatic growth of the momentum of the particle. The solution (5.4) is correct with an accuracy up to the small terms of the order of $(d\rho^2/d\vartheta)^2$ and $d^2\rho/d\vartheta^2$. Approximate expressions for $f_{\rho,z}$ and $\mu_{\rho,z}$ which are, however, sufficiently accurate as a rule for practical purposes, can be easily found by assuming that within a given magnetic sector the characteristics of the ideal field do not change along an intercept of the main orbit (the intercept being a circular arc), and that there is no field at all in the intervals between the magnetic sectors (the intervals being rectilinear).

$$k \equiv k_0 \equiv \begin{cases} 1/r_0 = \text{const} & \text{(magnetic sector)} \\ 0 & \text{(rectilinear interval)} \end{cases} \tag{5.5}$$

Having the solutions of the homogeneous equations (5.4) which we shall consider to be known at (5.5) we obtain the solutions of the non-homogeneous equations:

$$\rho(\vartheta) = \rho_M(\vartheta) + \rho_0(\vartheta),\ z(\vartheta) = z_M(\vartheta) + z_0(\vartheta), \tag{5.6}$$

where $\rho_M(\vartheta)$, $z_M(\vartheta)$ are the particular periodic solutions:

$$\rho_M(\vartheta) = \rho_{MO}(\vartheta) + \rho_{M\zeta}(\vartheta) + \rho_{MZ}(\vartheta);\ \rho_{MO}(\vartheta + \vartheta_0) = \rho_{MO}(\vartheta);$$
$$\rho_{M\zeta}(\vartheta + \vartheta_0) = \rho_{M\zeta}(\vartheta);\ \rho_{MZ}(\vartheta + 2\pi) = \rho_{MZ}(\vartheta),$$
$$Z_M(\vartheta + 2\pi) = Z_M(\vartheta) \tag{5.7}$$

The function $\rho_{MO}(\vartheta)$ corresponding to the item in the right part of (5.2) describes an ideal plane closed orbit near which betatron oscillations occur. With momentum and field disturbances $Z(\vartheta)$, $R(\vartheta)$ this orbit is distorted, remaining closed

$$\left.\begin{matrix}\rho_M(\vartheta)\\ z_M(\vartheta)\end{matrix}\right\} = -\frac{ie^{i\mu_{\rho,z}m}f_{\rho z}(\vartheta)}{2(1 - e^{iM\mu_{\rho,z}})}\sum_{j=1}^{N} e^{i\mu_{\rho,z}(j-1)}\int_{\vartheta-\vartheta_0}^{\vartheta} F_{\rho,z}^{j}(\xi) f^*_{\rho,z}(\xi)\,d\xi + \text{c.c.} \tag{5.8}$$

where $F_{\rho,z}(\vartheta)$ are the right parts of the equations (5.2), (5.3) and $M = N$ (at $\vartheta_{per} = 2\pi$) or $M = 1$ (at $\vartheta_{per} = \vartheta_0$).

To explain the meaning of the function $\rho_{MO}(\vartheta)$ let us take, for example, the simplest coordinate orbit $\mathbf{r}_0(\vartheta)$ corresponding to (5.5). At the ends of the magnetic sectors facing the rectilinear sections, the characteristics of the ideal field change owing to edge effects. The rectilinear sections themselves are not, strictly speaking, free from the field either, due to the fringed field of the magnet, the windings etc. That is why the behaviour of the ideal field differs somewhat from the schematic picture: $H^{id} = \text{const}(\vartheta)$ in the sector and $H^{id} = 0$ in the rectilinear interval, and the ideal closed orbit also differs from $\mathbf{r}_0(\vartheta)$ (5.5) though these differences are as a rule very small.

To study the betatron oscillations, we select the closed orbit $\mathbf{r}_{MO}(\vartheta) = \mathbf{r}_0(\vartheta) + \rho_{MO}(\vartheta)$ as the "coordinate" curve which corresponds to:

$$k(\vartheta) = k_0(\vartheta) = \frac{eH(\vartheta)}{cp_0} \tag{5.9}$$

as can be seen from (5.2), (5.8).

6. Equations for determining the amplitudes of betatron oscillations

The equations for determining the ρ,z deviations from the closed orbit can be written as:

$$\frac{d^2\rho}{d\vartheta^2} + \frac{1}{p}\frac{dp}{d\vartheta}\frac{d\rho}{d\vartheta} + k^2(\vartheta)\, r^2_{aver}[1-n(\vartheta)]\rho = k(\vartheta) r^2_{aver} F_\rho(\vartheta,\rho,z...) \tag{6.1}$$

$$\frac{d^2z}{d\vartheta^2} + \frac{1}{p}\frac{dp}{d\vartheta}\frac{dz}{d\vartheta} + k^2(\vartheta)\, r^2_{aver}\, n(\vartheta) z = k(\vartheta) r^2_{aver} F_z(\vartheta,\rho,z,...) \tag{6.2}$$

where the expressions for F_ρ, F_z are determined from (4.8), (4.9).

We shall seek the solution of the equations (6.1), (6.2) by the method of successive approximations in a form which is a modification of the expression (5.4)

$$\left.\begin{matrix}\rho(\vartheta)\\ z(\vartheta)\end{matrix}\right\} = \lambda(\vartheta)[A_{\rho,z}(\vartheta)e^{i\mu_{\rho,z}m}f_{\rho,z}(\vartheta) + \text{c.c}], \lambda(\vartheta) = \left[\frac{p(\vartheta_{init})}{p(\vartheta)}\right]^{\frac{1}{2}} \tag{6.3}$$

Applying the requirement

$$\frac{dA_{\rho,z}}{d\vartheta} e^{i\mu_{\rho,z}m} f_{\rho,z}(\vartheta) + \text{c.c.} = 0 \tag{6.4}$$

we obtain the equations for determining $A_{\rho,z}(\vartheta)$ in the case of the disturbed field:

$$\frac{dA_{\rho,z}}{dS} \simeq - \frac{ie^{-i\mu_{\rho,z}NS}}{\lambda(\vartheta)} \sum_{j=1}^{N} e^{i\mu_{\rho,z}(j-1)} \int_0^{\vartheta_0} k(\xi) F^{\,j}_{\rho,z}(\xi,\rho(\xi),...) f^*_{\rho,z}(\xi) d\xi \tag{6.5}$$

where S is the number of the current revolution. For the ideal field it is necessary to put $N = 1$ and to substitute S by m, the number of the current periodicity element. According to (4.8) and (4.9), the right part of (6.5), in which the expressions (6.3) for ρ, z should be substituted, contains the sums of items which include the products:

$$\lambda^{(l_\rho+l_z-1)} A_\rho^{\,l_\rho} A_z^{\,l_z} e^{i(q_\rho\mu_\rho q_z\mu_z)NS} \tag{6.6}$$

where l_ρ, l_z are non negative integers, q_ρ, q_z – integers, and A_ρ, A_z can be replaced by their complex – conjugated values. As (6.5), (6.3) and (4.8), (4.9) show, owing to the relative smallness of values ρ, z allowance for non-linear terms and other disturbances lead as a rule to small corrections which have the nature of rapid oscillations and change the result (5.4), corresponding to the ideal field, very little. For approximate solution of the equations (6.5) we average them with respect to the variable S (or m) contained in explicit form. Such a method, applied for the investigation of non-linear problems by Bogoluybov and Krylov[17], was used, for example, in the papers[18,9] in considering some particular problems of the theory of weak-focusing oscillations. A considerable deviation from the oscillations corresponding to the solution of the homogeneous equations (6.1) and (6.2) may occur if any of the members in the right part of (6.5) remain approximately constant and do not become zero when averaged. This, according to (6.6), is possible in case of proximity to one of the resonance conditions:

$$q_\rho\mu_\rho + q_z\mu_z = 2k\pi \tag{6.7}$$

$$q_\rho(N\mu_\rho) + q_z(N\mu_z) = 2k\pi \tag{6.8}$$

where k is an integer.

It should be noted that in definite orders (determined by the sum l_ρ, l_z (see (6.6)) there appear terms which do not contain the argument S or m explicitly, i.e., do not depend explicitly on the "frequencies" μ_ρ and μ_z: These terms, which in (6.6) correspond to

$$q_\rho = 0 \quad q_z = 0 \tag{6.9}$$

are present in the right parts of (6.5) whether averaged or not. Let us consider, for instance, oscillations around a closed orbit (5.8). The case (6.9) occurs, for example, for the terms:

$k^2(\vartheta)\, r^2_{aver}\, \Delta n_\rho(\vartheta)\rho$ and $-\, k^2(\vartheta) r^2_{aver} \Delta n_z(\vartheta) z$

in the right parts of (1.1) and (1.2) respectively, where:

$$\Delta n_\rho(\vartheta) = -(n-2)\zeta - \frac{z_\rho}{k} - 2z + 2\left(2n-1-\frac{n_1}{2}\right) k\rho_m - \frac{4-3n}{2r^2_{aver}}\left(\frac{d\rho_M}{d\vartheta}\right)^2 - 3\left(n_1-n-\frac{n_2}{6}\right) k^2\rho_M^2 + \dots \quad (6.10a)$$

$$\Delta n_z(\vartheta) = -r\zeta - \frac{z_\rho}{k} + (n_1-2n)k\rho_M - \frac{n}{2\gamma^2_{aver}}\left(\frac{d\rho_M}{d\vartheta}\right)^2 + \left(2n_1-n-\frac{n_2}{2}\right) k^2\rho_M^2 + \dots \quad (6.10b)$$

The deviations Δn_ρ, Δn_z occur inasmuch as the oscillations take place around a disturbed closed orbit, not an ideal one, and, in accordance with (6.5), cause changes in the frequencies of the betatron oscillations μ_ρ and μ_z

$$\Delta\mu_{\rho,z} = -\frac{r^2_{aver}}{2N}\sum_{j=1}^{N}\int_0^{\vartheta_0} k^2\Delta n^j_{\rho,z}\,|f_{\rho,z}|^2 d\xi \quad (6.11)$$

The disturbances (6.11) may prove significant in alternate-gradient focusing accelerators, where the precision tolerance $\Delta\mu_{\rho,z}$ is very rigid.

7. Linear resonances

We write now the resonance conditions and the averaged equations (equations of the first approximation) for determining the amplitudes of the betatron oscillations for linear resonances (with the "detuning" δ).

A. Simple Resonance together with Parametric Resonance

$$N\mu_{\rho,z} = 2k_{\rho,z}\pi + \delta_{\rho,z};\ (k_{\rho,z}\ \text{integer}) \quad (7.1)$$

$$\frac{dA_\rho}{dS} = iN\Delta\mu_\rho A_\rho + c_\rho e^{i(\chi_\rho-\delta_\rho S)} + g_\rho e^{i(\eta_\rho-2\delta_\rho S)}A^*_\rho \quad (7.2)$$

where $\Delta\mu_\rho$ is given by the formula (1.11), S is the number of the revolution and

$$c_\rho e^{i\chi_\rho} = \frac{ir^2_{aver}\lambda}{2}\sum_{j=1}^{N} e^{-i\mu_\rho(j-1)}\int_0^{\vartheta_0} k(z^j-\zeta)f^*_\rho\, d\xi \quad (7.3)$$

$$g_\rho e^{i\eta_\rho} = \frac{ir^2_{aver}}{2}\sum_{j=1}^{N} e^{-2i\mu_\rho(j-1)}\int_0^{\vartheta_0} k(Z^j_\rho-2kZ^j)f^{*2}d\xi \quad (7.4)$$

A similar equation will result for A_z.

It should be noted that from the expression for ρ (see (7.3), (6.3)) it follows that the distortion of the closed orbit due to field disturbances does not depend on the fcator λ as might have been expected.

B. Parametric resonance

$$N\mu_{\rho,z} = k_{\rho,z}\pi + \delta_{\rho,z} \quad (7.5)$$

$$\frac{dA_\rho}{dS} = iN\Delta\mu_{\rho,z}A_\rho + g_\rho e^{i(\eta_\rho-2\delta_\rho S)}A^* \quad (7.6)$$

where the denotations are the same as in (6.11), (7.1) - (7.4). A similar equation will result for A_z.

C. Relation between the radial (ρ) and axial (z) oscillations

a) $$N(\mu_z-\mu_\rho) = 2k\pi + \delta_1 \quad (7.7)$$

$$\frac{dA_\rho}{dS} = g_\rho e^{i(\eta_\rho+\delta S)}A_z, \quad \frac{dA_z}{dS} = g_z e^{i(\eta_z-\delta S)}A_\rho, \quad (7.8)$$

$$g_\rho e^{i\eta_\rho} = -\frac{ir^2_{aver}}{2}\sum_{j=1}^{N} e^{i(\mu_z-\mu_\rho)(j-1)}\int_0^{\vartheta_0}\left\{k\,[kR^j + R_\rho^j + S_\sigma^j]\,f_z - \frac{k}{r_{aver}}S^j\frac{df_z}{d\vartheta}\right\} f_\rho^*\, d\xi, \quad (7.9)$$

$$g_z e^{i\eta_z} = -\frac{ir^2_{aver}}{2}\sum_{j=1}^{N} e^{-(\mu_z-\mu_\rho)(j-1)}\int_0^{\vartheta_0}\left\{k\left[R_\rho^j + 2kR^j\right] f_\rho + \frac{k}{r_{aver}}S^j\frac{df_\rho}{d\vartheta}\right\} f_z^*\, d\xi \quad (7.10)$$

b) $N(\mu_z + \mu_\rho) = 2k\pi + \delta,$ (7.11)

$$\frac{dA_\rho}{dS} = g_\rho e^{i(\eta_\rho - \delta S)} A_z^*, \quad \frac{dA_z}{dS} = g_z e^{i(\eta_z - \delta S)} A_\rho^* \quad (7.12)$$

where $g_\rho e^{i\eta_\rho}$ is expressed by the formula (7.9) with f_z, $df_z/d\vartheta$ changed to f_z^*, $df_z^*/d\vartheta$, and $g_z e^{i\eta_z}$ by (7.10) with f_ρ, $df_\rho/d\vartheta$ changed to f_ρ^*, $df_\rho^*/d\vartheta$.

The condition (7.11) for weak-focusing magnets (including race-tracks) can be fulfilled nowhere within the region of stability ($0 < n < 1$).

The distortions of ideal magnetic sectors through small angles $\chi(\vartheta)$ near the azimuthal direction are an example of a disturbance which gives a relation between the ρ and z oscillations of the type (7.7) or (7.11).
Here

$$R(\vartheta) \simeq -\chi(\vartheta),\ R_\rho(\vartheta) \simeq -2n(\vartheta)\chi(\vartheta),$$
$$S(\vartheta) \simeq S_\vartheta(\vartheta) \simeq 0 \quad (7.13)$$

appear.

In the case of alternating-gradient focusing ($|n| \gg 1$), according to (7.13) we need take into account only $R_\rho(\vartheta)$, so that for the coefficients in (7.8), (7.12) we get the relations:

$$g_z e^{i\eta_z} = -g_\rho e^{-i\eta_\rho} = -g e^{-i\eta}, \quad g_z e^{i\eta_z} = g_\rho e^{i\eta_\rho} = g e^{i\eta} \quad (7.14)$$

8. Non-linear resonances

It is necessary to distinguish non-linear resonances of two types, corresponding (A) to an ideal field, and (B) to a disturbed field. We write now the conditions for these resonances up to the third or fourth order, as well as averaged equations obtained with the help of (6.5) - (6.8) and determining the behaviour of the amplitudes for several cases.

(A) Ideal field (non-linear resonances)

Second order

$$3\mu_\rho = 2\pi + \delta_\rho, \quad \frac{dA_\rho}{dm} = g_\rho e^{i(\eta_\rho - \delta_\rho m)} A_\rho^{*2}, \quad (8.1)$$

$$\mu_\rho - 2\mu_z = \delta, \quad \frac{dA_\rho}{dm} = g_\rho e^{i(\eta_\rho - \delta m)} A_z^2$$

$$\frac{dA_z}{dm} = g_z e^{i(\eta_z + \delta m)} A_\rho A_z^* \quad (8.2)$$

$$\mu_\rho + 2\mu_z = 2\pi + \delta, \quad \frac{dA_\rho}{dm} = g_\rho^{(\eta_\rho - \delta m)} A_z^{*2},$$

$$\frac{dA_z}{dm} = g_z e^{i(\eta_z - \delta m)} A_\rho^* A_z^* \quad (8.3)$$

Third order

$$4\mu_{\rho,z} = 2\pi + \delta_{\rho,z}, \quad \frac{dA_{\rho,z}}{dm} = g_{\rho,z} e^{i(\eta_{\rho,z} - \delta_{\rho,z} m)} A_{\rho,z}^{*3}, \quad (8.4)$$

$$2\mu_\rho - 2\mu_z = \delta,$$

$$\frac{dA_\rho}{dm} g_\rho e^{i(\eta_\rho - \delta m)} A_\rho^* A_z^2, \quad \frac{dA_z}{dm} = g_z e^{i(\eta_z + \delta m)} A_\rho^2 A_z^* \quad (8.5)$$

$$2\mu_\rho + 2\mu_z = 2\pi + \delta, \quad \frac{dA_\rho}{dm} = g_\rho e^{i(\eta_\rho - \delta m)} A_\rho^* A_z^{*2},$$

$$\frac{dA_z}{dm} = g_z e^{i(\eta_z - \delta m)} A_\rho^{*2} A_z^* \quad (8.6)$$

In the third order, items also appear which do not depend explicitly on m (see (6.9))

$$\frac{dA_\rho}{dm} = (P_1 |A_\rho|^2 + P_2 |A_z|^2) A_\rho,$$

$$\frac{dA_z}{dm} = (Q_1 |A_\rho|^2 + Q_2 |A_z|^2) \quad (8.7)$$

Expressions for the coefficients $g_\rho e^{i\eta_\rho}$, P_i, Q_i etc., which hold good with random values of the parameters $n, n_1, n_2 \ldots$, result directly from the formulae (6.5), (4.8), (4.9).

In the case of alternate-gradient focusing, the inequalities

$$1 \ll |n| \ll |n_1| \ll |n_2|, \quad (8.8)$$

are generally satisfied[18]). Using these inequalities, the expressions for the coefficients in (8.1) - (8.7) can be considerably simplified (for the sake of simplicity we assume that the energy is at its equilibrium value, i.e., $\zeta = 0$) and be reduced to

$$g_\rho e^{i\eta_\rho} = \frac{i\lambda r_{aver}^2}{4} \int_0^{\vartheta_0} k^3 n_1 f_\rho^{*3} d\xi \quad (8.1')$$

$$g_\rho e^{i\eta_\rho} = -\frac{i\lambda r_{aver}^2}{4} \int_0^{\vartheta_0} k^3 n_1 f_z^2 f_\rho^* d\xi, \quad g_z e^{i\eta_z} = -2 g_\rho e^{-i\eta_\rho}, \quad (8.2')$$

$$g_\rho e^{i\eta_\rho} = -\frac{i\lambda r_{aver}^2}{4} \int_0^{\vartheta_0} k^3 n_1 f_z^{*2} f_\rho d\xi, \quad g_z e^{i\eta_z} = 2 g_\rho e^{i\eta_\rho}, \quad (8.3')$$

$$g_{\rho,z}e^{i\eta_{\rho,z}} = -\frac{i\lambda r^2_{aver}}{12}\int_0^{\vartheta_0} k^4 n_2 f^{*4}_{\rho,z}\,d\xi, \tag{8.4'}$$

$$g_\rho e^{i\eta_\rho} = \frac{i\lambda r^2_{aver}}{4}\int_0^{\vartheta_0} k^4 n_2 f^{*2}_\rho f^{*2}_z\,d\xi, \qquad g_z e^{i\eta_z} = -g_\rho e^{-i\eta_\rho}, \tag{8.5'}$$

$$g_\rho e^{i\eta_\rho} = \frac{i\lambda r^2_{aver}}{4}\int_0^{\vartheta_0} k^4 n_2 f^{*2}_\rho f^{*2}_z\,d\xi, \qquad g_z e^{i\eta_z} = g_\rho e^{i\eta_{\rho 1}} \tag{8.6'}$$

$$P_1,Q_2 = -\frac{i\lambda r^2_{aver}}{4}\int_0^{\vartheta_0} k^4 n_2 |f_{\rho,z}|^4\,d\xi = ip_1,\ iq_2,$$

$$P_2,Q_1 = \frac{i\lambda r^2_{aver}}{2}\int_0^{\vartheta_0} k^4 n_2 |f_\rho|^2 |f_z|^2\,d\xi = ip_2,\ iq_1 \tag{8.7'}$$

As can be seen from (8.7), (8.7′), the terms containing p_i, q_i give the non-linear effect of the dependence of the oscillation frequency on the amplitude (cf. (6.10) - (6.11)).

$$(a)\quad \Delta\mu_\rho = p_1|A_\rho|^2 + p_2|A_z|^2 \tag{8.9}$$

$$(b)\quad \Delta\mu_z = q_1|A_\rho|^2 + q_2|A_z|^2$$

In this approximation the oscillation frequency in a given (say in the ρ-th) direction depends, however, not only on the oscillation amplitude in that direction but also on the oscillation amplitude in another (z-th) direction and *vice versa*. This complicates investigations. Practically, however, the deviations (8.9a), (8.9b) can be assumed not to be related to each other. In fact, since the conditions for ρ and for z-motion must be approximately the same, the functions f_ρ and f_z obey the relations

$$f_\rho^- \simeq f_z^+,\ f_\rho^+ \simeq f_z^- \tag{8.10}$$

where the indexes – and + refer respectively to sectors with negative and positive field gradients. Furthermore, it may be assumed that the parameter n_2, in accordance with its definition (4.5), will have approximately the same absolute value but different signs in the negative and positive sectors. It follows from the above that for the coefficients (3.7) the approximative values

$$p_2 \simeq q_1 \simeq 0,\ p_1 \simeq -q_2 \simeq -\frac{\lambda r^2_{aver}}{4}\int_0^{\vartheta_0} k^4 n_2 |f|^4\,d\xi \tag{8.11}$$

are acceptable, so that the expressions (8.9a), (8.9b) are separated.

Thus the non-linear effect (3.7) in the first approximation under consideration is determined mainly by the parameter n_2 (see (4.5)). The parameter n_1 appears in the expressions (8.7) only in the second approximation which we shall not write because it is cumbersome.

B. Disturbed field (non-linear resonances)

We write out the resonance conditions for a disturbed field.

Third order

$$3N\mu_{\rho,z} = 2k_{\rho,z}\pi + \delta_{\rho,z}, \tag{8.12}$$

$$N(\mu_\rho \pm 2\mu_z) = 2k\pi + \delta, \tag{8.13}$$

$$N(\mu_z \pm 2\mu_\rho) = 2k\pi + \delta \tag{8.14}$$

Fourth order

$$4N\mu_{\rho,z} = 2k\pi + \delta_{\rho,z} \tag{8.15}$$

$$N(2\mu_\rho \pm 2\mu_z) = 2k\pi + \delta \tag{8.16}$$

The equation describing the behaviour of the amplitudes in the vicinity of a certain resonance for an ideal field in the k-th approximation has the same form for the corresponding resonance in the disturbed field in the k + 1 – m approximation if we substitute m by S and take the corresponding coefficients $g_\rho e^{i\eta_\rho}$, $g_z e^{i\eta_z}$ etc.

For example, we give the equation corresponding to (8.12)

$$\frac{dA_\rho}{dS} = g_\rho e^{i(\eta_\rho - \delta_\rho S)} A^{*2}_\rho, \tag{8.17}$$

where

$$g_\rho e^{i\eta_\rho} = \frac{i\lambda r^2_{av}}{2}\sum_{j=1}^{N} e^{-3i\mu_\rho(j-1)}\int_0^{\vartheta_0} k\left\{k^2\left(z^j + 2z^j_\rho + \tfrac{1}{2}z^j_{\rho\rho}\right) f^{*2}_\rho + \frac{3}{2r^2_{av}} z^j f'^{*2}_\rho\right\} f^*_\rho\,d\xi \tag{8.18}$$

Analogous expressions for (6.13) - (6.16) can be obtained from (4.8), (4.9) and (6.5) and, if necessary, for higher orders.

9. Influence of non-linearity on the behaviour of amplitudes near linear resonances

Let us consider the motion of particles in the vicinity of linear resonances, with allowance for the influence of the non-linear terms (8.7), (8.11). It will be shown that various possible cases can be investigated by a uniform procedure by a certain substitution of variables. For non-linearity up to the fourth order the solution for a_ρ, a_z, depending on s (or m), can be expressed by elliptic functions.

A. Parametric resonance

We shall examine the method employed in greater detail on the example of parametric resonance. According to (7.5), (7.6) and (8.7), the equation for the amplitude A_ρ (and similarly for A_z) may be written as

$$\frac{dA_\rho}{ds} = iN\Delta\mu_\rho + g_\rho e^{i(\eta_\rho - 2\delta_\rho s)} A_\rho^* + iNp_1|A_\rho|^2 A_\rho \quad (9.1)$$

where the expressions for $\Delta\mu_\rho$, $g_\rho e^{i\eta_\rho}$, p, are given by the equations (6.11), (7.4) and (8.11).

Of considerable interest, as a rule, is the behaviour of the absolute values a_ρ, a_z of the complex amplitudes A_ρ, A_z

$$A_\rho = a_\rho e^{i\alpha_\rho}, \quad A_z = a_z e^{i\alpha_z} \quad (9.2)$$

Using (9.2) instead of (9.1), we obtain a system of equations with respect to a_ρ, α_ρ

$$\frac{da_\rho}{ds} = g_\rho a \cos w_\rho, \quad \frac{d\alpha_\rho}{ds} = N\Delta\mu_\rho - g_\rho \sin w_\rho + Np_1 a_\rho^2, \quad (9.3)$$

in which we change the independent variable S to w (the index ρ is omitted).

$$\frac{da}{dw} = \frac{ga \cos w}{2(\delta_n - g \sin w + Np_1 a^2)},$$

$$\frac{d\alpha}{dw} = \frac{1}{2} - \frac{g\delta}{2(\delta_n - g \sin w + Np_1 a^2)} \quad (9.3a)$$

where

$$w = 2\alpha - \eta + 2\delta S \quad (9.4)$$

and δ_n is the complete "detuning", equal to

$$\delta_n = N\Delta\mu + \delta \quad (9.5)$$

We integrate the first equation (9.3a). We then substitute the expression for $w = w(a)$ in the first equation (9.3) and obtain its integral in the form of

$$S = \frac{2}{Np_1} \int_{a_0}^{a} \frac{a da}{[-\prod_{i=1}^{4}(a^2 - a_i^2)]^{1/2}}, \quad (9.6)$$

where the roots a_i^2 are equal to

$$a^2_{1,2,3,4} = \frac{(\pm) g - \delta_n \pm \{[g(\mp)\delta_n]^2 - 2Np_1 a_0^2 (g \sin w_0 - \delta_n - (Np_1/2) a_0^2)\}^{1/2}}{Np_1} \quad (9.7)$$

where the upper sign in brackets refers to $a_{1,2}^2$ and the lower one to $a_{3,4}^2$, and w_0, a_0 are the initial values of the respective magnitudes.

Thus S is expressed through a by means of an elliptic integral of the first kind $F\{\varphi(a^2/a_0^2), k(a_i^2/a_0^2)\}$, and a^2/a_0^2, on the other hand, is expressed by S with the help of one of the Jacobi functions $Sn\{u(s), k(a_i^2/a_0^2)\}$, the tables of which are given, for example, in [19]. The amplitude a oscillates between two extreme values (boundaries of the "potential well"): $a_l < a < a_r$ and never tends to infinity if $p_1 \neq 0$. The values a_l, a_r coincide with the two values of the roots a_i (see (9.7)), which come closest to a_0 from above and below. The period of the non-linear oscillations s_{per} equals

$$s_{per} = \frac{e}{Np_1} \int_{a_l}^{a_r} \frac{a da}{\{-\prod_{i=1}^{4}(a^2 - a_i^2)\}^{1/2}} \quad (9.8)$$

and is expressed by a complete elliptic integral of the first kind.

For the purpose of comparison we shall consider the case in which the non-linear terms are not taken into account

$$(p_1 \simeq 0).$$

Then instead of (9.6) we obtain the integral

$$S = a_0^2 \int_{a_0}^{a} \frac{a da}{\{(g^2 - \delta_n^2) a^4 - 2\delta_n (g \sin w_0 - \delta_n) a_0^2 a^2 - (g \sin w_0 - \delta_n)^2 a_0^4\}^{1/2}} \quad (9.9)$$

which, depending on the relative value of the complete "detuning" δ_n, leads to three different results:

(*a*) $\delta_n^2 > g^2$ (the "detuning" is large and the point representing it is outside the stop band):

$$a = a_0 \left(\frac{\delta_n - g\sin w_0}{\delta_n^2 - g^2}\right)^{\frac{1}{2}} \left\{\delta_n - \sin[2\sqrt{\delta_n^2 - g^2}\,S] + \arc\sin\frac{g - \delta_n \sin w_0}{\delta_n - g\sin w_0}\right\}^{\frac{1}{2}} \tag{9.10}$$

i.e. oscillation of the amplitude (modulation) takes place with the period

$$S_{per} = \pi/\sqrt{\delta_n^2 - g^2} \tag{9.11}$$

As we approach the boundary of the stop band the depth of modulation and its period increase infinitely.

(*b*) $\delta_n^2 = g^2$ (the "detuning" is small and the point representing it is on the boundary of the stop band:

$$a_{\delta_n = \pm g} = a_0 \left\{1 + 2gs\,[(1 \pm \sin w_0)\,gs + \cos w_0]\right\}^{1/2} \tag{9.12}$$

i.e., a increases infinitely with S.

(*c*) $d_n^2 < g^2$ (the "detuning" is small and the point representing it is within the stop band:

$$a \sim e^{\sqrt{g^2 - \delta_n^2}}\,S\,, \tag{9.13}$$

i.e., a increases infinitely irrespective of the initial conditions.

Thus, under condition (*a*), taking the non-linearity into consideration can lead to only small corrections to the amplitude, but under (*b*) and (*c*) the non-linearity influences the solution considerably. Let us consider the characteristic particular example when the representative point is in the middle of the instability strip $\delta_n = 0$ and the initial conditions are such that $\sin w_0 = 0$. In this case the values a_i etc. occur in the following sequence:

$$a_4^2 < a_2^2 < 0 < a_3^2 < a_0^2 < a_{min}^2 < a_1^2\,, \tag{9.14}$$

where a_{min}^2 corresponds to the minimum of the potential well and a oscillates within

$$\frac{\sqrt{g^2 + N^2 p_1{}^2 a_0^4} - g}{Np_1} = a_l < a < a_r = \frac{\sqrt{g^2 + N^2 p_1{}^2 a_0^4} + g}{Np_1} \tag{9.15}$$

with the period

$$S_{per} = \frac{2k(\gamma)}{\sqrt{g^2 + N^2 p_1{}^2 a_0{}^4}}\,,\ \sin\gamma = \frac{g}{\sqrt{g^2 + N^2 p_1^2 a_0{}^4}} \tag{9.16}$$

where $k(\gamma)$ is a complete elliptic integral of the first kind. If the maximum permissible value of the amplitude a_{max} is pre-set, then with the requirement that $a_r < a_{max}$, we obtain the restriction for the initial amplitude

$$a_0 = \left[\frac{(Np_1 a^2 max - g)^2 - g^2}{(Np_1)^2}\right]^{\frac{1}{4}} \tag{9.17}$$

With other values of δ_n, $\sin w_0$ we obtain similar results.

B. Simple resonance coupled with parametric resonance

According to (7.1) - (7.4) the equation for the amplitude may be written as

$$\frac{dA_\rho}{ds} = iN\Delta\mu_\rho A_\rho + c_\rho e^{i(\chi_\rho - \delta_\rho s)} + g e^{i(\eta_\rho - 2\delta_\rho s)} A^* + iNp_1|A_\rho|^2 A_\rho \tag{9.18}$$

and similarly for A_z.
This case corresponds to a simultaneous considerable distortion of the closed orbit (connected with the disturbance c_ρ) and an increase in the oscillation amplitude near this orbit (owing to parametric resonance). From (9.18) we obtain the system (the index ρ is omitted)

$$\frac{da}{dS} = ga\cos w + c\cos v,$$

$$\frac{d\alpha}{dS} = N\Delta\mu - g\sin w - \frac{c}{a}\sin v + Np_1 a^2\,, \tag{9.19}$$

where w is expressed by equation (9.4) and

$$v = \alpha + \delta S - \chi \tag{9.20}$$

A closed orbit with a period equal to its revolution has the solution

$$a = a_M = const \tag{9.21}$$

where a_M can be determined from the equation

$$a_M^3 + \frac{\delta_n \pm g}{Np_1} a_M \pm \frac{c}{Np_1} = 0 \tag{9.22}$$

In the absence of non-linearity ($\rho_1 \approx 0$)

$$a_M = \left|\frac{c}{\delta_n \pm g}\right| \tag{9.23}$$

The appearance of the term g in equation (9.22) and in the expression (9.23) characterizes the influence of the proximity of parametric resonance on the amplitude of the closed orbit.

The effect of parametric resonance may be considered as in section A above.

C. Connection between ρ *and z oscillations*

For a connection of the type (7.10), taking into account (7.14), we get the first integral

$$a_\rho^2 + a_z^2 = \text{const}, \tag{9.24}$$

which shows that in this case energy passes from the ρ to the z oscillations and *vice versa*, the oscillation amplitudes remaining limited.

In an other case of the relation between the ρ and z oscillations (7.11) the first integral is

$$a_\rho^2 - a_z^2 = c = \text{const} \tag{9.25}$$

from which it follows that the amplitudes of the ρ and z oscillations can increase indefinitely, and hence it may prove important to take the non-linear members into consideration. The equations (7.12), (7.14) together with (8.7), (8.11) may be re-written as:

$$\frac{dA_\rho}{dS} = ge^{i(\eta-\delta s)} A_z^* + iNp_1 |A_\rho|^2 A_\rho,$$

$$\frac{dA_z}{dS} = ge^{i(\eta-\delta s)} A_\rho^* - iNp_1 |A_z|^2 A_z, \tag{9.26}$$

from which follows the system of equations (see (9.2))

$$da_\rho/dS = ga_z \cos w, \quad d\alpha_\rho/dS = -(g\, a_z/a_\rho)\cdot \sin w + Np_1 a_\rho^2, \tag{9.27}$$

$$da_z/dS = ga_\rho \cos w, \quad d\alpha_z/dS = -(g\, a_\rho/a_z)\cdot \sin w - Np_1 a_z^2 \tag{9.28}$$

where

$$w = \alpha_\rho + \alpha_z + \delta S - \eta \tag{9.29}$$

The influence of non-linearity in this case manifests itself in the "detuning" value changing from δ to δ non-linear

$$\delta \text{ non-linear} = \delta + cNp_1 \tag{9.30}$$

where, according to (9.25), c is determined by the initial conditions. The linear and non-linear problems, therefore, are actually solved in the same manner except that δ should be changed to δ non-lin. in the final expressions.

The solution of the system (9.27) - (9.28), depending on the relative "detuning" value δ non-linear, results in three cases similar to (9.10), (9.12), (9.13):

(*a*) δ^2 non-linear $> 4g^2$ ("detuning" is high) – the amplitudes a_ρ and a_z oscillate between their extreme values, determined by the value δ non-lin., the oscillation period being equal to

$$S_{per} = \frac{2\pi}{\sqrt{(\delta + cNp_1)^2 - 4g^2}} \tag{9.31}$$

(*b*) δ^2 non-linear $= 4g^2$, (*c*) δ^2 non-linear $< 4g^2$ ("detuning" is low) – the amplitudes increase infinitely despite non-linearity.

10. Behaviour of amplitudes in the vicinity of non-linear resonances

A. Ideal field

Using the same scheme as above, the behaviour of the amplitudes in the vicinity of non-linear resonances (8.1) - (8.6) and (8.12) - (8.16) is investigated, taking into account the terms of the third order in (8.7) containing p_1 (cf. (8.7′), (8.11)).

Second order

(*a*) Let us consider in greater detail the case (8.1); omitting the index ρ

$$dA/dm = ge^{i(\eta-\delta s)} A^{*2} + ip_1|A|^2 A \tag{10.1}$$

The system for a, α is

$$da/dm = ga^2 \cos w, \quad d\alpha/dm = ga \sin w + p_1 a^2, \tag{10.2}$$

where

$$w = \eta - \delta S - 3\alpha \tag{10.3}$$

Solving the equations after the scheme (9.3) - (9.6), we obtain the result in the form of the quadrature

$$m = \int_{a_0}^{a} \frac{a\,da}{\sqrt{G(a^2)}} \tag{10.4}$$

where

$$G(a^2) = -\frac{9p_1^2}{16} a^8 + g^2 a^6 - \tfrac{1}{2}\left[\frac{\delta^2}{2} - 3p_1 a_0^2\left(\frac{3}{4}p_1 a_0^2 + \frac{\delta}{2} + ga_0 \sin w_0\right)\right] a^4 + a_0^2\,\delta\left[\frac{\delta}{2} + ga_0 \sin w_0 + \frac{3p_1 a_0^2}{4}\right] a^2$$
$$- a_0^4\left\{\left[g^2 a_0^2 \sin^2 w_0 + \delta\left(\frac{\delta}{4} + ga_0 \sin w_0\right)\right] + \frac{3p_1 a_0^2}{2}\left[\frac{\delta}{2} + \frac{3p_1 a_0^2}{8} + ga_0 \sin w_0\right]\right\} \tag{10.5}$$

Thus a is expressed by s through elliptic functions which we shall not write out. If we do not take the p_1 terms into account and if we assume that there is no "detuning" ($\delta \simeq 0$), the expressions (10.4), (10.5) become much simpler and can be presented as

$$a = a_0 (\sin w_0)^{1/3} \left[\frac{(\sqrt{3}+1)-(\sqrt{3}-1)\,\mathrm{cn}(\chi, k_0)}{1+\mathrm{cn}(\chi, k_0)}\right]^{\frac{1}{2}}, \qquad (10.6)$$

where cn $(\chi_1\, k_0)$ is an elliptic cosine, $k_0 = \sin 15°$,

$$\chi = 2\sqrt[4]{3}\; a_0 (\sin w_0)^{1/3} gm + F(\varphi_0, k),$$

$$w_0 = \eta - 3\alpha_0, \quad \cos\varphi_0 = [(\sqrt{3}+1)(\sin w_0)^{2/3} - 1]\,[(\sqrt{3}-1)(\sin w_0)^{2/3} + 1]^{-1}, \qquad (10.7)$$

and $F(\varphi_0, k)$ is an elliptic integral of the first kind. In the case (10.6) the amplitude may increase considerably in the course of a few revolutions. Thus, if the terms with p_1 are not taken into account, the representative point should not be drawn nearer to the line $3\mu_\rho = 2\pi$ i.e., a sufficiently large "detuning" should be taken

$$\delta > g a_{max} \qquad (10.8)$$

If, however, the "detuning" is small, $\delta \simeq 0$, the non-linearity of the type (8.7), (8.11) must be rather large $p_1^2 \gg g^2$ to limit the amplitude. The condition (8.8) can provide fulfilment of this inequality.

From (10.4) and (10.5) we can obtain with some precision the data on the necessary value of "detuning" and non-linearity (8.7) for various conditions. For this purpose, it is necessary to require that the root of the function $G(a^2)$ which is closest to a_0^2 should be smaller than $a^2{}_{max}$, since this root gives the limit value of a^2 attainable under conditions of oscillation.

B. Connection between non-linear ρ and z oscillations

For a connection of the type (8.2), according to (8.2′) we obtain the first integral

$$a_\rho^2 + a_z^2/2 = c^2 = \text{const}, \qquad (10.9)$$

which indicates that the amplitudes are limited. The same integral is obtained if we take into account the non-linear members of the third order (containing p_1 (8.11)). The value m is connected with $a_\rho = a$ by the same equation (10.4) where $G(a^2)$ (with $p_1 \simeq 0$) has the form of

$$G(a^2) = g^2a^6 - \left(2g^2c^2 + \frac{\delta^2}{4}\right)a^4 + \left[g^2c^4 - g\delta a_0(a_0^2-c^2)\sin w_0 + \frac{\delta^2 a_0^2}{2}\right]a^2 - a_0^2(a_0^2-c^2)^2\left[\frac{a_0\delta}{2(a_0^2-c^2)} + g\sin w_0\right]^2, \qquad (10.10)$$

$$w = -\eta + 2\alpha_z - \alpha_\rho - \delta m \qquad (10.11)$$

In the simplest case of precise resonance in the absence of "detuning" ($\delta \simeq 0$), the roots of $G(a^2)$ occur in the following sequence

$$0 < a_1^2 < a_0^2 < a_2^2 < c^2 < a_3^2 \qquad (10.12)$$

For example, with $\sin w_0 = 0$, $a_1 = 0$, $a_2^2 = c^2$; with $\sin w_0 = 1$, $a_1^2 < a_0^2$, $a_2^2 = a_0^2$, etc. In the transfer of energy from the ρ to the z oscillations (and *vice versa*) a^2 oscillates between the roots a_1^2 and a_2^2 which are the boundaries of the corresponding potential well.

The oscillation period of the value a^2 is equal

$$m_{per} = \frac{2}{g\sqrt{a_3^2-a_1^2}}\, k(\gamma), \quad \sin\gamma = \sqrt{\frac{a_2^2-a_1^2}{a_3^2-a_1^2}}, \qquad (10.13)$$

where $k(\gamma)$ is a complete elliptic integral of the first kind.

It should be noted that condition (8.2), applied to a weak-focusing continuous magnet, gives the relation

$$\sqrt{1-n} - 2\sqrt{n} = \delta \quad \text{i.e.} \quad n \simeq 0{,}2 \qquad (10.14)$$

This is the well-known coupling non-linear resonance which plays an important part in synchrocyclotrons [20]. In this case the general and not simplified expressions should be used for the coefficients (8.2′), and instead of (10.8) we get

$$a_\rho^2 + \frac{a_z^2}{4\left(1 + \frac{\sqrt{0{,}2\delta}}{n_1}\right)} = c_1^2 = \text{const} \qquad (10.15)$$

There is some difficulty in obtaining in this case expressions similar to (10.4) and (10.10) - (10.13) if we put $m \sim \vartheta$, $g \sim n_1/2\sqrt{0{,}2}$ etc.

In case of a coupling of the type (8.3) and (8.3′) we obtain the relation

$$a_\rho^2 - a_z^2/2 = C^2 = \text{const} \qquad (10.16)$$

so that the amplitudes (without considering the terms with p_1) may increase indefinitely. Again the equation (10.3) is true, in which

$$G(a^2) = g^2a^2(a^2-c^2) - a_0^2(a_0^2-c^2)^2 \left\{ g \sin w_0 - \frac{1}{2a_0(a_0^2-c^2)} \left[(\delta + 4p_1c^2)(a^2-a_0^2) - \frac{3p_1}{2}(a^4-a_0^4) \right] \right\}^2 \quad (10.17)$$

If $p_1 \simeq 0$ we obtain the same expression (10.9) but inasmuch as (10.8) is substituted by (10.15) we obtain (with $\delta \simeq 0$), instead of an exchange of energy between the ρ and z oscillations, their unrestricted excitation. Formally, this corresponds to the case where instead of (10.12) the inequality

$$a_3^2 < a_0^2 \quad (10.18)$$

is realized, a_3^2 being the largest root of (10.10). As in the case of (10.1), therefore it is not permissible, generally speaking, to approach the line $\mu_\rho + 2\mu_z = 2\pi$.

The value of the required " detuning" or non-linearity of the third order (8.7), (8.11) is determined from (10.4), (10.17).

Third order

Non-linear resonances of the third and higher orders can be examined in exactly the same way as those of the second order. The results for the cases (8.14) - (8.6) are analogous to those given above for (8.1) - (8.3) respectively.

B. Disturbed field

Where necessary, there should obviously be no great difficulty in avoiding the deleterious action of the non-linear resonances investigated above in an ideal field by selecting the working cell on the stability diagram at a sufficient distance from the respective resonance lines. In this sense, the non-linear resonances corresponding to the disturbed magnetic field may be more dangerous. The expression for the conditions of attainment of these resonances contains the value N, which stands for the number of periodicity elements of the magnet, in analogy to the case of, say, parametric resonances. In consequence, lines of non-linear resonance are found to exist on disturbances in the immediate vicinity of any representative working point. This may lead to the necessity of increasing tolerance requirements.

The behaviour of the amplitudes in the vicinity of non-linear resonances in a disturbed field is calculated in the same way as in the case of an ideal field, above. Since the corresponding equations contain functions describing disturbances, $Z_\rho(\vartheta)$, $R_\rho(\vartheta)$ etc., these functions must either be pre-set on the basis of certain additional considerations or considered as statistical values, as in estimations of the linear theory. For the resonance $3N\mu_\rho = 2k\pi + \delta_\rho$, for example, the expectation obtained is naturally similar to (10.8) in which g will then stand for the average square value

$$\bar{g}^2 \simeq \frac{\lambda N r^2_{\text{aver}}}{16} \overline{z^2_{\rho\rho}} \left| \int_0^{\vartheta_0} k f^3 d\xi \right|^2 \quad (10.19)$$

To limit the amplitude by non-linearity of the type (8.7), (8.11), the condition $p_1^2 \gg \bar{g}^2$ must be fulfilled where $\bar{g}^2$ is given by the formula (10.19).

Using the above equations for the amplitudes and employing, for example, the stationary phase method, it is possible to find the result of passing through various resonances with the slow change of parameters connected with synchrotron oscillations.

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