Reconstruction of some cosmological models from the deceleration parameter

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Abstract. Since the discovery of the late-time acceleration of the universe, researchers are still trying to find an explanation for it. This is regarded as the most important unsolved problem in cosmology today. The most favoured explanation is dark energy, an unknown or exotic form of matter with negative pressure. One may argue that particle physics may provide the answer in time. Currently, the LambdaCDM model is regarded as the best model. Although this model is reasonably successful and widely accepted, there is growing interest in looking at alternatives. Some of the reasons for this are the fine-tuning, coincidence, inflationary paradigm and cosmological constant problems, and whether general relativity is valid on large scales. One focus in trying to understand dark energy is to assume some form of the scale, Hubble or deceleration parameter (or some other reasonable assumption), and then to see how well the model fits in with current observations. This approach is broadly called reconstruction. In this talk, we focus on the deceleration parameter. We provide a brief review of the various forms of the deceleration parameter that have been employed in the past in cosmology, and then focus on some particular forms of interest which have drawn some attention. We note that it is most worthwhile to study alternative dark energy and dark gravity models in order to fully understand the entire space of possibilities.

1. Introduction
Einstein's field equations with cosmological constant are:

$$R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} = T_{ab}. \tag{1}$$

For the Friedmann-Lemaitre-Robertson-Walker metric, which represents a homogeneous and isotropic universe,

$$ds^2 = -dt^2 + a^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + sin^2\phi d\phi^2)\right\}, \tag{2}$$

where a is the scale factor, we get the following Friedmann and Raychaudhuri-type equations:

$$3\frac{\dot{a}^2}{a^2} + \frac{3k}{a^2} - \Lambda = \rho, \tag{3}$$

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \Lambda = -p. \tag{4}$$

Now, the Hubble parameter H determines the expansion rate of the universe:

$$H = \frac{\dot{a}}{a}, \tag{5}$$

whereas the deceleration parameter q implies a small correction as the matter slows down the expansion

$$q = -\frac{\ddot{a}a}{\dot{a}^2}. \tag{6}$$

The importance of q is that it relates to the non-linear part of the time dependence of the scale factor. In terms of H and q, equations (3) and (4) can be written as:

$$3H^2 + \frac{3k}{a^2} - \Lambda = \rho, \tag{7}$$

$$H^2(1 - 2q) + \frac{k}{a^2} - \Lambda = -p. \tag{8}$$

Two other parameters that have proven useful in the analysis of dark energy models are the jerk and snap parameters. The jerk parameter j is effectively a derivative of q:

$$j = \frac{\dddot{a}}{a} \left(\frac{\dot{a}}{a}\right)^{-3}, \tag{9}$$

and the snap parameter s, which is effectively a derivative of j, is defined by:

$$s = \frac{\ddddot{a}}{a} \left(\frac{\dot{a}}{a}\right)^{-4}. \tag{10}$$

2. History of cosmology

If we take a look briefly at the history of cosmology, we notice that there have been three major revolutions. The first was the discovery of the expansion of the universe, largely due to Hubble [1]. What do the solutions to equations (3) and (4) look like for $\Lambda = 0$, $p = 0$?

Figure 1 [2] illustrates these solutions for the cases $k = -1, 0, +1$ corresponding to open, flat and closed spatial sections, respectively. Observations at that time indicated that the universe was expanding ($H > 0$), but that the expansion was decreasing, i.e., a decelerating universe ($q > 0$).

In the early 1970's, Sandage defined cosmology as simply the search for H_o & q_o: "All of observational cosmology is the search for two numbers: H_o and q_o" [3]. We notice from equation (6) that a negative sign appears on the right side. The reason for this was that there was such confidence in the expansion of the universe slowing down that a negative sign was assigned to make the value of q positive. For historical reasons, this has remained even today.

The second revolution in cosmology came around 1979 with the advent of the idea of inflation [4]. Inflation occurs during the period $10^{-36} - 10^{-34}/10^{-33}s$ after the big-bang - see figure 2 [2].

We note that inflation solves the horizon, flatness and monopole problems, but does not get rid of the big-bang singularity. Around that time, the big bang model plus inflation was believed to describe the universe to a first approximation.

In 1998, we saw the birth of the third revolution, i.e., the discovery of the accelerated expansion of the universe. One should note that this is usually interpreted as dark enery, but that there are some alternative explanations. The accelerated expansion began around 6
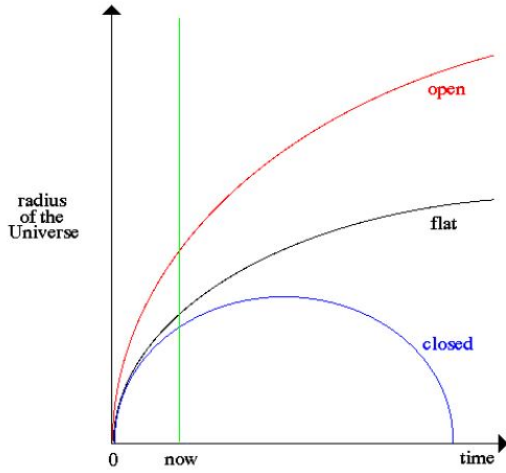
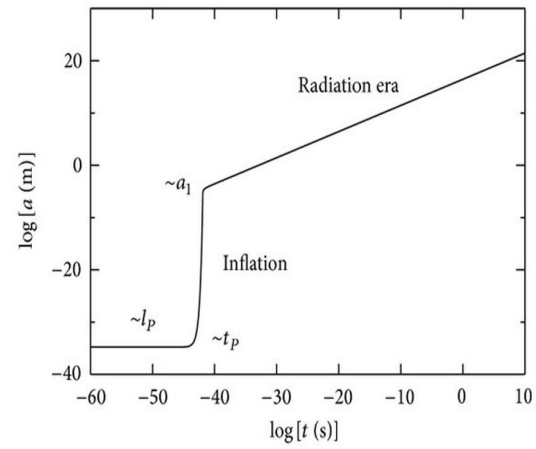**Figure 1.** a against t before DE and inflation.

Figure 2. a against t showing early inflation.

billion yrs ago. From the Friedmann (3) and Raychaudhuri-type (4) equations, we can derive a conservation type eq:

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{11}$$

We assume a barotropic eq of state:

$$p = \omega\rho, \quad \omega = const. \tag{12}$$

For $k = \Lambda = 0$, we can solve equations (3), (4) and (11) to get:

$$a \propto t^{2/(3(1+\omega))}, \quad \omega \neq -1. \tag{13}$$

For pressure-free matter ($\omega = 0$), $a \propto t^{2/3}$, and for for radiation, ($\omega = 1/3$), $a \propto t^{1/2}$. For $k = 0, \Lambda \neq 0$ or $\rho = const$, $\omega = -1$, we get:

$$a \propto e^{Ht}, \quad \rho = const, \quad \omega = -1. \tag{14}$$

This exponential type of expansion occurs during early inflation and the late-time acceleration eras.

From the definition of the deceleration parameter (6), we find that it can be written in a unified way for all three cases as:

$$q = \frac{1}{2}(1 + 3\omega). \tag{15}$$

The pressure can be written in terms of q as:

$$p = H^2(2q - 1), \tag{16}$$

and the age of the universe as:

$$t_o = \frac{1}{H_o(1 + q_o)}. \tag{17}$$

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3. Classification in terms of q

Comsological models may be classified according to q in various ways. Firstly, in terms of H as:

- $H > 0, \ q > 0$;
- $H > 0, \ q < 0$;
- $H > 0, \ q = 0$.

For $p = 0$, the relation between q and the spatial geometry can be written as:

- $q > 1/2, \ k = +1$: closed spherical surface
- $q > 1/2, \ k = 0$: open flat space
- $q > 1/2, \ k = -1$: open hyperbolic space

The age of the universe can be written in terms of q and the Einstein-de Sitter age, $t^* = (2/3)H^{-1}$, as:

- $q > 1/2, \ \text{age} < t^*$;
- $q = 1/2, \ \text{age} = t^*$;
- $q < 1/2, \ \text{age} > t^*$.

The energy conditions can be expressed in terms of q as:

- NEC $(\rho + p \geq 0), \ q \geq 1$;
- SEC $(\rho + p \geq 0, \rho + 3p \geq 0), \ q \geq 0$;
- DEC $(\rho \geq 0, \rho \geq |p|), \ q \leq 2$.

A similar analysis may be carried out for multi-fluids also. Now, radiation and matter can only produce deceleration. For $k \neq 0$, the DP (15) for all 3 cases can be written in a unified way as:

$$q = \frac{1}{2}\left(1 + \frac{k}{a^2 H^2}\right)(1 + 3\omega). \tag{18}$$

Another way of solving equations (3) and (4) without assuming an equation of state (12) is to make some assumption on q, e.g.,

$$q = const. \tag{19}$$

This yields the solutions (13) and (14) as special cases. Many authors have assumed condition (19) to find solutions in general relativity as well as alternative theories, especially before the discovery of dark energy.

4. Cosmography

Cosmography is concerned with the kinematics of cosmology, i.e., relations independent of the theory. The Hubble parameter H and the deceleration parameter q are the simplest cosmograhic parameters. The procedure in cosmography is to expand the scale factor $a(t)$ (or some other parameter) in Taylor series about the present time t_o:

$$a(t) = a(t_o) + \dot{a}(t_o)(t - t_o) + \frac{1}{2}\ddot{a}(t_o)(t - t_o)^2 + \cdots \tag{20}$$

which can also be written as:

$$\frac{a(t)}{a(t_o)} = 1 + H_o(t - t_o) + \frac{q_o}{2}H_o{}^2(t - t_o)^2 + \cdots \tag{21}$$

from which it can be seen that H and q arise naturally in the expansion.

4.1. Constant deceleration parameter

We assume the cosmological principle, i.e., homogeneity and isotropy. Then expand some measurable quantity in a Taylor series around the present time. Let the deceleration parameter q be expanded about the present time (or redshift, scale factor, Hubble parameter, etc). The simplest is, of course, $q = const$. This was essentially done by Berman [5] who proposed a law of variation of the Hubble parameter:

$$H = Da^{-n}, \quad D, \ n = consts. \tag{22}$$

This yields

$$a = (nDt + c)^{1/n}. \tag{23}$$

The deceleration parameter q in this case is

$$q = n - 1, \tag{24}$$

which is constant. So, we may equivalently regard equation (24) as a law of constant deceleration parameter. We observe that $n > 1 \implies$ deceleration, whereas $n < 1 \implies$ acceleration. The idea was to choose values of q consistent with observations. This analysis was carried out before the discovery of dark energy, and we note that a constant q cannot yield a transition from deceleration to acceleration (no "signature flip" - not to be confused with a signature change in the metric). Hence, a variable q was considered which yields the required transition.

4.2. Linearly varying deceleration parameter

After a constant q, the next simplest is a linearly varying q:

$$q(x) = q_o + q_1 x. \tag{25}$$

Some parametrisations that have been considered are:

$$q = q_o + q_1 z; \tag{26}$$

$$q = q_o + q_1 a; \tag{27}$$

$$q = q_o + q_1 t. \tag{28}$$

Akarsu and Dereli [6], and Akarsu et al [7] have given a very discussion of these particular cases and shown that they are consistent with observations. We note that linearity in one parameter $\nRightarrow$ linearity in another.

4.3. Nonlinear deceleration parameter

In this subsection, we focus on some interesting parametrisations of q. Let us firstly consider [8]:

$$H(a) = \alpha(1 + a^{-n}), \tag{29}$$

where $\alpha > 0$ and $n > 1$. For q, this yields:

$$q(z) = \frac{(n-1)(1+z)^n - 1}{1 + (1+z)^n}. \tag{30}$$

or

$$q(t) = -1 + \frac{n}{e^{n\alpha t}}, \ n, \alpha = const. \tag{31}$$

This is plotted in figure 3 for some suitable values of the parameter n. From the figure, we
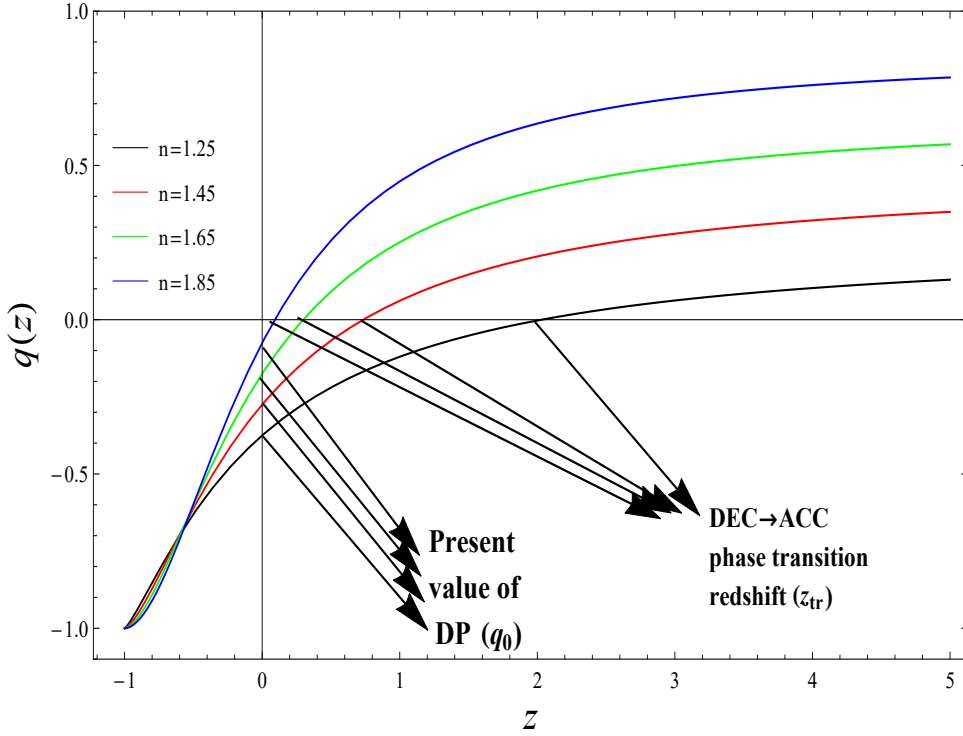
Figure 3. Plot of deceleration parameter *vs* redshift z for different n

see that the deceleration parameter q is positive during early times, implying deceleration, and negative at late times, implying acceleration. There is a transition at a redshift of around $z = 1$. One can get model which fits in with observations in $f(R, T)$ theory.

Secondly, let us consider the deceleration parameter q with a double flip [9]:

$$q = -1 + \frac{\alpha t}{1 + t^2}. \tag{32}$$

The matter source in this case is a self interacting scalar field. This allows for a double signature flip, viz., to account for early inflation, and then from deceleration to late-time acceleration. See figure 4. In this case, we notice that the graphs of q above the blue curve cut the t axis at two points. The first point corresponds to the transition form early inflation to the radiation-dominated era, and the second to the transition from the matter era to the currently accelerated era.

4.4. Other forms of the deceleration parameter
Let us illustrate some other interesting forms of q that have been considered. In general, one can write a polynomial in q as a function of the time t:

$$q = q_o + q_1 t + q_2 t^2 + ..., \tag{33}$$

We studied a quadratic form [10], and a cubic form is under preparation. One may also expand q in a series in the redshift z as [11, 12]:

$$q = q_o + (-q_o - 2q_o^2 + j_o)z + \frac{1}{2}(2q_o + 8q_o^2 + 8q_o^3 - 7q_o j_o - 4j_o - s_o)z^2 + O(z^3). \tag{34}$$
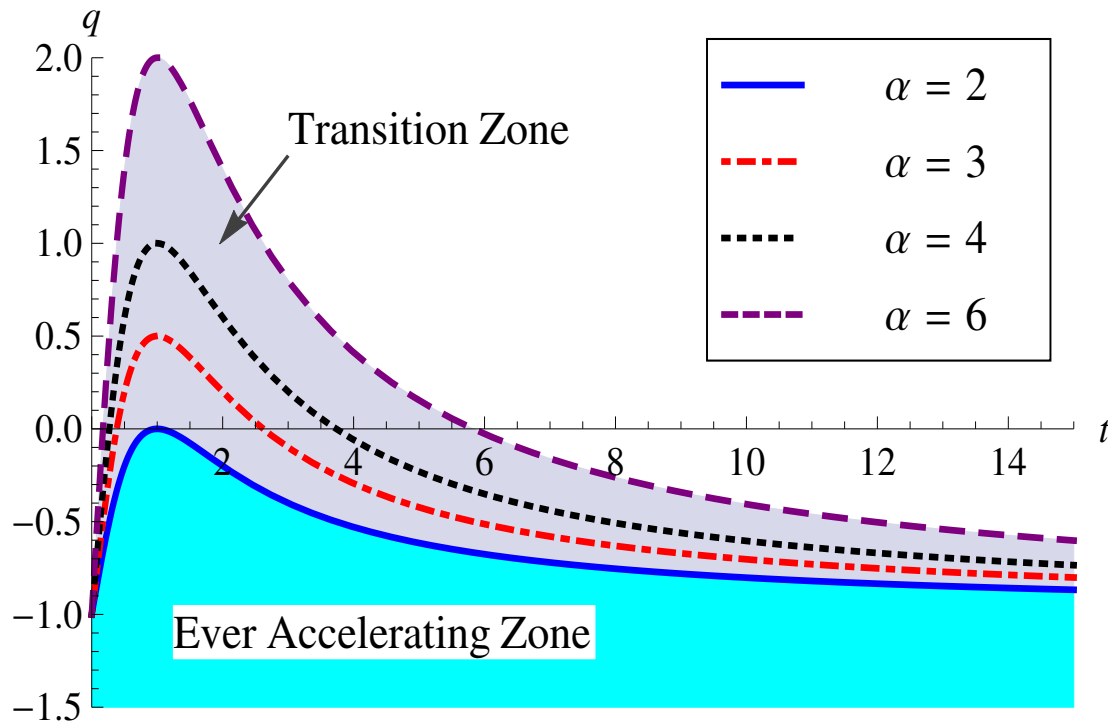
Figure 4. Plot of q against t

Finally, there has been some interest in bouncing solutions recently, i.e., cosmological solutions that do not have a big-bang, but rather a bounce with an infinite series of expansions and contractions. An example of such a model is given by the following periodic form of q:

$$q = m \, cos(nt) - 1. \tag{35}$$

This implies a bouncing solution [13].

5. Conclusion

The basic characteristics of cosmological evolution are expressed by the Hubble parameter H and the deceleration parameter q; H describes the linear part and q relates to the nonlinear part. The standard ΛCDM model is most widely accepted and fits the observations, but it has some problems. Hence, it is worthwhile to examine other possibilities. Many reconstructions have been done with q. Choosing suitable forms of q may explain all stages of evolution of the universe. In this work, we have examined this approach, and looked at the various possibilities for q. It is possible to account for the transition from deceleration to acceleration. Also, nonlinear forms have the potential to explain the initial inflation as well as late-time acceleration. One can construct models that are consistent with observations. Even choices of the scale factor a and Hubble parameter H can be re-expressed as functions of q. We are not sure what lies in store for us in cosmology in the future.

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References

[1] Hubble E 1929 *Proc. Natl. Acad. Sci. U. S. A.* **15** 168
[2] Gilson, David. The Ever Expanding Universe In Modern Cosmology
    http://www.laidback.org/ daveg/academic/expandinguniverse/index2.html
[3] Sandage A R 1970 Phys. Today **23** 34
[4] Guth A H 1981 *Phys. Rev. D* **23** 347–356
[5] Berman M S 1983 *Il Nuovo Cim.* **74** 182-186
[6] Akarsu O and Dereli T *Int. J. Theor. Phys.* **51** 612
[7] Akarsu 0, Dereli T, Kumar S and Xu L. *Eur. Phys. J Plus* **129** 22
[8] Nagpal R, Pacif S K J, Singh J K, Bamba K and Beesham A 2018 *Eur. Phys. J. C* **78** 946
[9] Singh V and Beesham A *Int. J. Geom. Meth. Mod. Phys.* **15** 1850145
[10] Tiwari R K, Beesham A and Shukla B K 2021 *Afr. Math.* **32** 983
[11] Bolotin Yu L, Cherkaskiy V A, Lemets O A, Yerokhin D A, and Zazunov L G 2015 Cosmology in terms of the deceleration parameter. Part 1 *gr-qc/1502.00811v1*
[12] Bolotin Yu L, Cherkaskiy V A, Lemets O A, Yerokhin D A, and Zazunov L G 2015 Cosmology in terms of the deceleration parameter. Part 2 *Preprint gr-qc/1506.08918v1*
[13] Hulke N, Singh G P, Bishi B K, and Beesham, A Bianchi-I Dark energy Cosmological Models with generalised Chaplygin gas and time periodic varying deceleration parameter (*Preprint*)