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ORIGINAL PAPER

Quantum flux effects on generalized Klein-Gordon oscillator field in a topologically charged Ellis-Bronnikov-type wormhole

F Ahmed[1], H Aounallah[2] and P Rudra[3]*

[1]Department of Physics, University of Science & Technology Meghalaya, Ri-Bhoi, Meghalaya 793101, India

[2]Department of Science and Technology, Echahid Cheikh Larbi Tebessi University, Tebessa, Algeria

[3]Department of Mathematics, Asutosh College, Kolkata 700026, India

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Abstract: The work primarily deals with the relativistic quantum dynamics of the oscillator field. It uses the generalized Klein-Gordon oscillator in an Ellis-Bronnikov-type wormhole space-time with a topological defect. The study involves deriving the radial wave equation and incorporating both Coulomb and Cornell-type potential functions. The analytical solution of the wave equation elucidates the influence of the topological defect of the geometry and the wormhole throat radius. The eigenvalue solution of the oscillator field highlights significant modifications to the results, underscoring the impact of the topological defect.

Keywords: Quantum; Wormhole; Modified theories of gravity; Relativistic wave equations; Solutions of wave equation: bound-states; Special functions

1. Introduction

The exploration of curved space-time's influence on quantum mechanical phenomena has captivated researchers since the inception of quantum mechanics. Numerous studies have investigated the quantum dynamics of both relativistic particles (via the Klein-Gordon equation, DKP equation) and non-relativistic particles (via the Schrödinger equation) in various curved space-time backgrounds, including Gödel and Gödel-type space-times, Kerr and Schwarzschild black holes, as well as curved space-times with topological defects such as cosmic strings, global monopoles, cosmic dislocations, and spinning cosmic strings. These investigations have revealed that the gravitational effects arising from curved space-time with topological defects can significantly alter the energy spectrum and shift the eigenvalue solutions of quantum particles compared to results obtained in flat space. Additionally, researchers have explored a plethora of scalar and vector potentials, including linear confining potentials, Coulomb potentials, Cornell-type potentials, Mie-type potentials, pseudoharmonic potentials, among others, in both relativistic and non-relativistic quantum systems. Furthermore, the introduction of external magnetic and quantum flux fields has been examined to analyze their effects on the eigenvalue solutions of quantum particles. Cosmic strings [1–3] and global monopoles [4–7] have been extensively studied as topological defects in quantum systems, spanning both relativistic and non-relativistic regimes. These defects are believed to have emerged in the early universe through the breaking of symmetry, giving rise to peculiar geometric structures. Although experimental observation of these objects remains elusive, theoretical investigations in various fields such as gravitation, cosmology, and solid-state physics persist. Of particular interest in recent years is the study of global monopoles in the context of quantum mechanical problems [8–16], which is the focal point of our current research endeavors.

In the realm of quantum mechanical systems, investigations into the oscillator field, whether through the Dirac oscillator, Klein-Gordon oscillator, or the DKP oscillator, continue to garner significant attention within the physics community. This oscillator field has been extensively explored in curved space-time with topological defects, amidst backgrounds of topological defects space-times featuring external magnetic and quantum flux fields. Moreover, many researchers have introduced scalar and

*Corresponding author, E-mail: prudra.math@gmail.com

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vector potentials of various forms, analyzing their effects on the eigenvalues of the oscillator field. Among these, the Klein-Gordon oscillator (KGO) stands out as one of the most prominent forms of quantum oscillators employed in relativistic quantum mechanics. This is primarily because the KG-oscillator can effectively describe the corresponding non-relativistic quantum harmonic oscillator as described by the Schrödinger equation [17]. Inspired by the Dirac oscillator [18, 19], investigations into the KG oscillator have been conducted under various theoretical frameworks. Some of these setups include explorations within the context of the Kaluza-Klein theory [20, 21], anti-de Sitter space-time [22], space-time with torsion [23], cosmic string space-time [24–26], under the influence of central potentials [27–29], and amidst Lorentz symmetry violation effects [30, 31]. Recent works on KGO in different geometry backgrounds can be found in Refs. [32–35].

Wormholes are tunnels or short-cut paths between two different regions of spacetime in the same universe or even two parallel universes [36, 37]. The wormhole spacetime is the exact solution of Einstein's Field equations violating one or more energy conditions, especially weak energy conditions (WEC) and null energy conditions (NEC). The first such attempt was undertaken by Einstein and Rosen together, which is known as the Einstein-Rosen bridge. Later on, several authors have constructed wormhole solutions in $(1+2)$-, $(1+3)$- and higher dimensions without or with the cosmological constant in the literature (see Ref. [37] for a detailed discussion). Of all the known wormhole solutions, the simplest example of a wormhole solution was given by H. G. Ellis [38] and K. Bronnikov [39] independently, which is called Ellis-Bronnikov (EB) wormhole space-time. The Raychaudhuri equation for the congruence of a radial null geodesic in the EB wormhole spacetime is given by,

$$\frac{d\Theta}{d\lambda} = -R_{\mu\nu}\kappa^{\mu}\kappa^{\nu} - \frac{\theta^2}{2} \tag{1}$$

where Θ is the congruence expansion, λ is the affine parameter connected with the geodesics, $R_{\mu\nu}$ is the Ricci tensor and κ^{μ} is a null tangent vector to the geodesic. Near the throat of the wormhole, the congruence expansion is reduced to zero, i.e., $\Theta = 0$, but $\frac{d\Theta}{d\lambda} \geq 0$, which implies from eqn. (1) $R_{\mu\nu}\kappa^{\mu}\kappa^{\nu} \leq 0$ which is a clear violation of the null convergence condition (NCC) [40]. When we take general relativity (GR) into consideration, a violation of the NCC is equivalent to a violation of the null energy condition (NEC). In GR it is known that the energy conditions specify the attractive nature of the gravitational fields, which are generated from the conventional sources of matter. For a wormhole, there is a direct violation of these conditions near its throat. This indicates that there is a requirement for some form of anti-gravitational effects to keep the throat of the wormhole open, thus sustaining it. It is theorized that these anti-gravitational effects can be generated by some form of exotic matter known as dark energy (DE). EB wormhole is a solution obtained in the background of Eddington-inspired Born-Infield (EiBI) gravity [41], which describes a static and spherically symmetric spacetime with the topological charge of GM [42, 43]. This solution was obtained by coupling the energy-momentum tensor of the exterior region of the GM core with the geometry of spacetime. The EB wormhole has been explored under various set-ups in the literature [44–46].

A static and spherically symmetric space-time describing the Ellis-Bronnikov-type wormhole with a point-like defect is represented by the following line-element ($x^0 = t, x^1 = x, x^2 = \theta, x^3 = \phi$) [44–47]

$$ds^2 = -dt^2 + \frac{dx^2}{\alpha^2} + (x^2 + a^2)\,(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2}$$

where $a = const$ is the wormhole throat radius. For $\alpha \to 1$, this space-time becomes Ellis-Bronnikov wormhole while for $a \to 0$, the space-time becomes a point-like global monopole [8–16].

Our investigation is motivated by the findings presented in Ref. [45], which explored the quantum effects of the Klein-Gordon oscillator in a topologically charged Ellis-Bronnikov-type wormhole space-time. Expanding upon this work, we delve into the quantum dynamics of the oscillator field within this wormhole space-time background using the generalized Klein-Gordon oscillator. We examine two distinct types of potential functions, namely Coulomb- and Cornell-type, and derive the corresponding eigenvalue solutions. Subsequently, we scrutinize the influences of the topological defect and the wormhole throat radius on these eigenvalue solutions. Our analysis reveals that the energy levels and wave functions of the oscillator field are indeed affected by these factors. Furthermore, we observe that the eigenvalue solution differs from the results obtained in [45] and undergoes modifications.

2. Eigenvalue solution of generalized oscillator field in wormhole space-time

In this section, we study the relativistic quantum motions of a generalized KG-oscillator in a wormhole space-time background with a point-like defect. The relativistic quantum dynamics of spin-0 scalar particles in curved space-time is described by (system of units are chosen as $c = 1 = \hbar = G$)

$$\left[-\frac{1}{\sqrt{-g}}\,\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\,\partial_\nu\right)+M^2\right]\Psi=0, \tag{3}$$

where M is the rest mass of the particles, $g_{\mu\nu}$ is the metric tensor with g being its determinant.

The oscillator field is studied by substituting $\partial_\mu \to (\partial_\mu + M\,\omega\,X_\mu)$ [33, 48–55] in the above wave equation, where ω is the oscillator frequency, and $X_\mu = (0, f(x), 0, 0)$ with $f(x)$ as an arbitrary function. If $f(x) = x$, then the quantum system is called the Klein-Gordon oscillator.

Therefore, the relativistic wave equation of the generalized oscillator field is described by [33, 48–50, 52–55]

$$\left[\frac{1}{\sqrt{-g}}(\partial_\mu + M\,\omega\,X_\mu)(\sqrt{-g}g^{\mu\nu})(\partial_\nu - M\,\omega\,X_\nu)\right]\Psi = M^2\,\Psi. \tag{4}$$

Expressing this wave Eq. (4) in the space-time background (2), we obtain the following equation

$$\begin{aligned}&\left[-\frac{\partial^2}{\partial t^2}+\frac{\alpha^2}{(x^2+a^2)}\left(\frac{\partial}{\partial x}+M\,\omega f(x)\right)\right.\\&\left\{(x^2+a^2)\left(\frac{\partial}{\partial x}-M\,\omega f(x)\right)\right\}\\&\left.\quad+\frac{1}{(x^2+a^2)}\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right\}\right]\\&\Psi=M^2\,\Psi.\end{aligned} \tag{5}$$

The wave function Ψ can be expressed in terms of different variables by the method of separation of the variables. We choose a possible function Ψ in terms of the function $\psi(x)$ given by

$$\Psi(x,\theta,\phi)=e^{-iEt}\,Y_{\ell,m}(\theta,\phi)\,\psi(x), \tag{6}$$

where $Y_{\ell,m}(\theta,\phi)$ is the spherical harmonics.

Substituting the wave function (6) in the Eq. (5) and after separating the equations for radial and angular parts, we obtain the following differential equation in terms of the wave function $\psi(x)$ as:

$$\begin{aligned}&\psi''(x)+\frac{2x}{(x^2+a^2)}\,\psi'(x)\\&+\left[b^2-\frac{2M\,\omega\,x}{(x^2+a^2)}f(x)-M\,\omega f'(x)-M^2\,\omega^2 f^2(x)-\frac{\iota^2}{(x^2+a^2)}\right]\psi=0,\end{aligned} \tag{7}$$

where different parameters are defined as

$$b^2=\frac{E^2-M^2}{\alpha^2},\quad \iota^2=\frac{\ell\,(\ell+1)}{\alpha^2}. \tag{8}$$

To solve the differential equation given in Eq.(7) using special functions, we will consider two different types of functions $f(x)$ as follows.

2.1. A coulomb-type function

In this part, we consider the function $f(x)$ to be a Coulomb-type potential form function. It is well-known that Coulomb potential ($\propto \frac{1}{x}$) is responsible for short-range interactions and has been studied in different phenomena, such as H-atom, and quark-antiquark interactions. The Coulomb-type potential function is given by [33, 48–50, 52–55]

$$f(x)=\frac{\eta}{x},\quad \eta>0. \tag{9}$$

Substituting this function (9) into the Eq. (7), we obtain the following equation

$$\begin{aligned}&\psi''(x)+\frac{2\,x}{(x^2+a^2)}\,\psi'(x)\\&+\left[b^2-\frac{(j^2-\frac{1}{4})}{x^2}-\frac{\upsilon^2}{(x^2+a^2)}\right]\psi(x)=0,\end{aligned} \tag{10}$$

where we set the parameters

$$j=\left(M\,\omega\,\eta-\frac{1}{2}\right),\quad \upsilon=\sqrt{2\,M\,\omega\,\eta+\iota^2}. \tag{11}$$

We aim to obtain solution of the quantum system under investigation. Let us consider a possible solution $\psi(x)$ as follows:

$$\psi(x)=x^{|j|+\frac{1}{2}}\,G(x), \tag{12}$$

where $G(x)$ is an unknown function.

Thereby, substituting this solution (12) into the Eq.(11), we finds

$$\begin{aligned}&G''(x)+\left[\frac{2\,x}{x^2+a^2}+\frac{(1+2\,|j|)}{x}\right]G'(x)\\&+\left[b^2+\frac{1+2\,|j|-\upsilon^2}{x^2+a^2}\right]G(x)=0.\end{aligned} \tag{13}$$

Finally, introducing a new variable via $u=-\frac{x^2}{a^2}$ in the Eq.(13), we obtain the following differential equation for G as:

$$\begin{aligned}&G''(u)+\left[\frac{1+|j|}{u}+\frac{1}{u-1}\right]G'(u)\\&+\left[\frac{-\frac{1}{4}(a^2\,b^2+1+2|j|-\upsilon^2)}{u}+\frac{\frac{1}{4}(1+2\,|j|-\upsilon^2)}{u-1}\right]\\&G(u)=0.\end{aligned} \tag{14}$$

which is the confluent Heun equation form [44–47, 57, 58], and hence, $G(u)$ is the confluent function given by

$$G(u)=H_c\left(0,|j|,0,-\frac{a^2\,b^2}{4},\frac{a^2\,b^2+1-\upsilon^2}{4};u\right). \tag{15}$$

To solve the above differential equation (14), we use the Frobenius power series solution given by [59]

$$G(u)=\sum_{i=0}^{\infty}c_i\,u^i. \tag{16}$$

Substituting this power series in the Eq.(14), we obtain the following recurrence relation

$$c_{k+2}=\frac{\left\{4\,(k+1)(k+2+|j|)+a^2\,b^2+1+2\,|j|-\upsilon^2\right\}c_{k+1}-a^2\,b^2\,c_k}{4\,(k+2)(k+2+|j|)}, \tag{17}$$

with the coefficient

$$c_1=\frac{(a^2\,b^2+1+2\,|j|-\upsilon^2)}{4\,(1+|j|)}\,c_0. \tag{18}$$

To obtain solution of the quantum system, we must truncate the power series (16) to a finite degree polynomial such that the wave function ψ is regular everywhere. Let us consider $k=(n-1)$, where the coefficient $c_{n+1}=0$ such that the wave function ψ is finite everywhere. Thereby, imposing the condition $k=(n-1)$ with the coefficient $c_{n+1}=0$, we obtain from the relation (17)

$$c_n=\frac{a^2\,b^2}{[4\,n\,(n+1+|j|)+a^2\,b^2+1+2\,|j|-\upsilon^2]}\,c_{n-1}. \tag{19}$$

Now, using relation (19), one can obtain the individual energy levels and the corresponding wave functions for the mode $n=1,2,3...$ of the oscillator field. In this work, we consider one such case defined by the mode $n=1$ which we call the ground state of the system. Thus, from (19) we obtain

$$c_1=\frac{a^2\,b^2}{[4\,(2+|j|)+a^2\,b^2+1+2\,|j|-\upsilon^2]}\,c_0. \tag{20}$$

Now, comparing Eqs.(18) and (20), we obtain the ground-state energy level as follows:

$$E_{1,\ell}=\sqrt{M^2+\frac{2\,\alpha^2}{a^2}\left\{\left(\frac{\iota^2}{2}+M\,\omega\,\eta-|j|-\frac{3}{2}\right)\pm\sqrt{(1+|j|)\,(\iota^2+2\,M\,\omega\,\eta)-(3+2\,|j|)\,|j|}\right\}}, \tag{21}$$

where $\iota^2=l\,(l+1)\alpha^2$.

The corresponding ground-state wave function will be

$$\psi_{1,\ell}=u^{\left(|j|+\frac{1}{2}\right)}\left[1+\frac{u\left\{-1\pm\sqrt{(1+|j|)(\iota^2+2\,M\,\omega\,\eta)-(3+2\,|j|)\,|j|}\right\}}{2\,(1+|j|)}\right]c_0.$$

Equation (21) denotes the lowest state energy level, and (22) represents the corresponding wave function of the oscillator field. These results are obtained by employing a Coulomb-type potential function in the background of a topologically charged Ellis-Bronnikov-type wormhole. Similarly, one can find the energy levels $E_{2,\ell},E_{3,\ell},...$ and the wave functions $\psi_{2,\ell},\psi_{3,\ell},...$ for modes $n\geq 2$. It is evident that these energy levels and wave functions are influenced by the topological defect of the geometry, characterized by the parameter α, and the constant radius a of the wormhole throat. These factors lead to modifications in the energy levels and wave functions.

2.2. A cornell-type function

Here, we consider the function $f(x)$ as a sum of two terms with the Coulomb-type potential as one of them. This is called the Cornell-type potential function. The Cornell-type potential is a special case of the quark-antiquark interactions which has one more harmonic potential term [56]. This Cornell-type potential function is given by

$$f(x)=\left(\frac{\eta}{x}+\delta\,x\right),\quad \eta>0,\delta>0. \tag{22}$$

Note that for $\eta\to 0$, this potential function becomes linear in x and the quantum system is called the KG-oscillator. For $\delta\to 0$, the function becomes a Coulomb-type which we discussed earlier. This Cornell-type potential form function has widely been investigated in the context of quantum systems by various authors in the literature [33, 48–55].

Thereby, substituting this function (22) in the Eq.(7), we obtain

$$\psi''(x)+\frac{2\,x}{(x^2+a^2)}\,\psi'(x)+\left[\epsilon^2-\frac{j^2-\frac{1}{4}}{x^2}-\frac{\varpi^2}{(x^2+a^2)}-M^2\,\omega^2\,\delta^2\,x^2\right]\psi(x)=0, \tag{23}$$

where different parameters are

$$\epsilon^2=b^2-M\,\omega\,\delta\,(3+2\,M\,\omega\,\eta),\quad \varpi=\sqrt{\upsilon^2-2\,M\,\omega\,\delta\,a^2}. \tag{24}$$

In the context of quantum mechanical systems, a fundamental requirement for the wave function is that it should be regular everywhere. This implies that the wave function, $\psi(x)$, must remain finite both as x approaches zero and as x goes to infinity. At small values of $x\to 0$, the wave function behaves as $\psi\sim x^{|j|+\frac{1}{2}}$ and at large values of $x\to\infty$, it behaves as $\psi(x)\sim e^{-\frac{1}{2}M\,\omega\,\delta\,x^2}$. Let's explore a suitable wave function, $\psi(x)$, as follows:

$$\psi(x)=x^{|j|+\frac{1}{2}}\,e^{-\frac{1}{2}M\,\omega\,\delta\,x^2}\,G(x). \tag{25}$$

Substituting this solution (25) in the Eq.(23), we obtain the following differential equation

$$G''(x)+\left[\frac{(1+2\,|j|)}{x}-2\,M\,\omega\,\delta\,x+\frac{2\,x}{(x^2+a^2)}\right]G'(x)+\left[\Pi+\frac{\Xi}{(x^2+a^2)}\right]G(x)=0, \tag{26}$$

where

$$\Pi=b^2-M\,\omega\,\delta\,(7+2\,M\,\omega\,\eta+2|j|),\quad \Xi=1+2\,|j|-\varpi^2+2\,M\,\omega\,\delta\,a^2. \tag{27}$$

By changing to a new variable via $u=-\frac{x^2}{a^2}$ in the Eq. (26), we obtain

$$G''(u)+\left[M\,\omega\,\delta\,a^2+\frac{1+|j|}{u}+\frac{1}{u-1}\right]G'(u)+\left[\frac{-\frac{1}{4}\,(\Pi\,a^2+\Xi)}{u}+\frac{(\Xi/4)}{u-1}\right]G(u)=0, \tag{28}$$

which is the confluent Heun equation form [44–47, 57, 58] and the Heun function is given by

$$G(u)=H_c\left(M\,\omega\,\delta\,a^2,|j|,0,-\frac{a^2\,\epsilon^2}{4},\frac{a^2\,\epsilon^2+1-\varpi^2}{4};u\right). \tag{29}$$

To solve the differential Eq. (28), we use a power series solution $G=\sum_{i=0}^{\infty}c_i\,x^i$ [59] into the Eq. (28), we obtain the following recurrence relation

$$c_{k+2}=\frac{\left\{4\,(k+1)(k+2+|j|-M\,\omega\,\delta\,a^2)+\Pi\,a^2+\Xi\right\}c_{k+1}+(4\,M\,\omega\,\delta\,a^2\,k-\Pi\,a^2)\,c_k}{4(k+2)(k+2+|j|)}, \tag{30}$$

with the coefficient

$$c_1=\frac{\Pi\,a^2+\Xi}{4\,(1+|j|)}\,c_0. \tag{31}$$

Similar to the previous analysis done in this paper, let us consider $k=(n-1)$ where the coefficient $c_{n+1}=0$. We finds

$$c_n=-\frac{4\,M\,\omega\,\delta\,a^2\,(n-1)-\Pi\,a^2}{\left[4\,n\,(n+1+|j|-M\,\omega\,\delta\,a^2)+\Pi\,a^2+\Xi\right]}\,c_{n-1}. \tag{32}$$

The ground state of the quantum system is defined by $n=1$. Thus, we obtain using (32) the following coefficient

$$c_1=\frac{\Pi\,a^2}{\left[4\,(2+|j|-M\,\omega\,\delta\,a^2)+\Pi\,a^2+\Xi\right]}\,c_0. \tag{33}$$

Comparing Eqs. (31) with (33) and after simplification, we obtain the energy level $E_{1,\ell}$ given by

$$E_{1,\ell}=\pm\sqrt{M^2+\frac{\alpha^2}{a^2}\left\{\Theta+M\,\omega\,\delta\,a^2\,(7+2\,M\,\omega\,\eta)\right\}}, \tag{34}$$

where we have set the parameter

$$\Theta=2\,M\,\omega\,\delta\,a^2\,|j|-2\,|j|+\varpi^2-3\pm 2\sqrt{-3\,|j|-2j^2+\varpi^2(1+|j|)-2\,M\,\omega\,\delta\,a^2\,(|j|+2)+M^2\,\omega^2\,\delta^2\,a^4},\quad \varpi=\sqrt{\imath^2+2\,M\,\omega\,(\eta-\delta\,a^2)}, \tag{35}$$

where $\imath$ is given in Eq.(8).

The ground state wave function will be

$$\psi_{1,\ell}=u^{\left(|j|+\frac{1}{2}\right)}\,exp\left(-\frac{1}{2}\,M\,\omega\,\delta\,u^2\right)\left[1+\frac{u}{2}\frac{(M\,\omega\,\delta\,a^2-1)}{(1+|j|)}\pm\frac{u}{2\,(1+|j|)}\times\left\{-3\,|j|-2\,j^2+\varpi^2(1+|j|)-2\,M\,\omega\,\delta\,a^2(|j|+2)+M^2\,\omega^2\,\delta^2\,a^4\right\}^{1/2}\right]. \tag{36}$$

Equations (34)–(35) represent the ground state energy level, while Equation (36) denotes the corresponding wave function of the oscillator field. These results are obtained by selecting a Cornell-type potential form within the backdrop of a topologically charged Ellis-Bronnikov-type wormhole. Similarly, one can derive additional energy levels $E_{2,\ell},E_{3,\ell},\ldots$ and wave functions $\psi_{2,\ell},\psi_{3,\ell},\ldots$ for modes $n \geq 2$. It is evident that the eigenvalue solutions depend on the topological defect characterized by the parameter α and the constant radius a of the wormhole throat, leading to modifications in the energy levels and wave functions.

Now, we compare our result with the one obtained in Ref. [45]. Let $\eta \to 0$, and hence, the function $f(x) \propto x$, that is, a linear function. In this case, the quantum system is called the KG oscillator. Therefore, for $\eta \to 0$, the energy eigenvalue will become

$$E_{1,\ell} = \pm\sqrt{M^2 + \frac{\alpha^2}{a^2}(\tilde{\Theta} + 7\,M\,\omega\,\delta)}, \tag{37}$$

where

$$\tilde{\Theta} = \iota^2 - M\,\omega\,\delta\,a^2 - 4$$
$$\pm 2\sqrt{-2 + \frac{3}{2}\,\iota^2 - 8\,M\,\omega\,\delta\,a^2 + M^2\,\omega^2\,\delta^2\,a^4}, \tag{38}$$

with ι given in the Eq. (8).

The corresponding ground state wave function will be

$$\psi_{1,\ell} = u\,exp\left(-\frac{1}{2}\,M\,\omega\,\delta\,u^2\right)\left[1 + u\,\frac{(M\,\omega\,\delta\,a^2 - 1)}{3}\right.$$
$$\left.\pm \frac{u}{3}\sqrt{-2 + \frac{3}{2}\,\iota^2 - 8\,M\,\omega\,\delta\,a^2 + M^2\,\omega^2\,\delta^2\,a^4}\right]. \tag{39}$$

Equations (37)–(38) represent the ground state energy level, and (39) denotes the corresponding wave function of the KG-oscillator field in the background of a topologically charged Ellis-Bronnikov-type wormhole. It is noteworthy that the results presented in this section diverge from those obtained previously, as there was an error in the earlier analysis (Ref. [45]).

3. Conclusions

Exploring quantum mechanical phenomena within the context of topological defects in space-time is a fascinating and highly consequential area of research within the scientific community. Topological defects can fundamentally change the geometric characteristics of the space-time they inhabit, thereby influencing the behavior of quantum systems within that space-time. The presence of these defects can lead to alterations in the energy eigenvalues and wave functions of quantum particles, departing from the outcomes observed in flat space and introducing a breakdown of degeneracy among energy levels.

In this investigation, we derived the radial equation for the generalized Klein-Gordon oscillator field within the framework of Ellis-Bronnikov-type wormhole space-time featuring a point-like defect. We then selected a Coulomb-type potential function, leading to the derivation of the radial wave equation. We then solved this radial equation through the special function of the confluent Heun function and presented the ground state energy level $E_{1,\ell}$ in Eq. (21) and the corresponding wave function in (22) as a particular case, and others are in the same way. Additionally, we explored a Cornell-type potential function (a linear plus Coulomb function) and converted it into the formulation of the confluent Heun equation after a few mathematical steps. By employing a power series expansion of the confluent Heun function and conducting subsequent mathematical calculations, we also obtained the ground state energy level $E_{1,\ell}$, as described by the expressions (34)–(35), and the corresponding wave function $\psi_{1,\ell}$, as given by equation (36), for the oscillator field.

It is important to note that incorporating a Cornell-type potential function in this quantum system led to modifications in the energy levels and wave functions of the oscillator field compared to the findings in Ref. [45]. In both scenarios examined, we demonstrated that the topological defect parameter α and the wormhole throat radius a (held constant) exerted influences on the eigenvalue solutions of the oscillator field, introducing adjustments compared to the case without the topological defect ($\alpha \to 1$). Furthermore, the presence of the topological defect not only broke the degeneracy of energy levels but also induced a more pronounced shift in the energy spectrum.

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Declarations

Conflict of interest There are no conflict of interest.

References

[1] A Vilenkin *Phys. Lett. B* **133** 177 (1983)
[2] W A Hiscock *Phys. Rev. D* **31** 3288 (1985)
[3] B Linet *Gen. Relativ. Gravit.* **17** 1109 (1985)
[4] M Barriola and A Vilenkin *Phys. Rev. Lett.* **63** 341 (1989)
[5] R A Battye, S J Cotterill and D G Viatic *Phys. Lett. B* **844** 138091 (2023)
[6] N Chatzifotis, N E Mavromatos and D P Theodosopoulos *Phys. Rev. D* **107** 085014 (2023)
[7] A Vilenkin and E P S Shellard *Strings and Other Topological Defects* (Cambridge: Cambridge University Press) (1994)
[8] E R Bezerra de Mello and F C Carvalho *Class. Quantum Grav.* **18** 5455 (2001)
[9] E R Bezerra de Mello and F C Carvalho *Class. Quantum Grav.* **18** 1637 (2001)

[10] E R Bezerra de Mello *Class. Quantum Grav.* **19** 5141 (2002)
[11] E R Bezerra de Mello, J Spinelly and U Freitas *Phys. Rev. D* **66** 024018 (2002)
[12] E R Bezerra de Mello and A A Saharian *Class. Quantum Grav.* **29** 135007 (2012)
[13] T R P Carames, J C Fabris, E R Bezerra de Mello and H Belich *Eur. Phys. J. C* **77** 496 (2017)
[14] E R Bezerra de Mello *Braz. J. Phys.* **31** 211 (2001)
[15] H Aounallah and A Boumali *Phys. Part. Nucl. Lett.* **16** 195 (2019)
[16] A Boumali and H Aounallah *Rev. Mex. Fis.* **66** 2 (2021)
[17] N A Rao and B A Kagali *Phys. Scr.* **77** 015003 (2008)
[18] A Benkrane and H Benzair *Eur. Phys. J. Plus.* **138** 295 (2023)
[19] M Moshinsky and A Szczepaniak *J. Phys. A: Math. Gen.* **22** L817 (1989)
[20] J. Carvalho, A. M. de M. Carvalho, E. Cavalcante and C. Furtado, Eur. Phys. J. C **76**, 365 (2016)
[21] E V B Leite, H Belich and R L L Vitoria *Braz. J. Phys.* **50** 744 (2020)
[22] B Hamil and M Merad *Eur. Phys. J. Plus* **133** 174 (2018)
[23] R L L Vitoria and K Bakke *Eur. Phys. J. Plus* **133** 490 (2018)
[24] A Boumali and N Messai *Can. J. Phys.* **92** 1490 (2014)
[25] M Hosseini, H Hassanabadi, S Hassanabadi and P Sedaghatnia *Int. J. Geom. Meths. Mod. Phys.* **16** 1950054 (2019)
[26] L C N Santos, C E Mota and C C Barros *Adv. High Energy Phys.* **2019** 2729352 (2019)
[27] K. Bakke and C. Furtado, Ann. Phys. (N. Y.) **355**, 48 (2015)
[28] R L L Vitoria and K Bakke *Eur. Phys. J. Plus* **131** 36 (2016)
[29] B-Q Wang, Z-W Long, C-Y Long and S-R Wu *Mod. Phys. Lett. A* **33** 1850025 (2018)
[30] F Ahmed *EPL* **136** 61001 (2022)
[31] F Ahmed *EPL* **138** 20001 (2022)
[32] A Saidi and M B Sedra *Eur. Phys. J. Plus* **137** 102 (2022)
[33] Marc de Montigny, H. Hassanabadi, J. Pinfold and S. Zare, Eur. Phys. J. Plus **136**, 788 (2021)
[34] L Zhong, H Chen, H Hassanabadi, Z W Long and C Yun *Int. J. Mod. Phys. A* **36** 2150129 (2021)
[35] Marc de Montigny, J. Pinfold, S. Zare and H. Hassanabadi, Eur. Phys. J. Plus **137**, 54 (2022)
[36] M S Morris and K S Thorne *Am. J. Phys.* **56** 395 (1988)
[37] M Visser *Lorentzian Wormholes: from Einstein to Hawking* (New York: American Institute of Physics) (1995)
[38] H G Ellis *J. Math. Phys.* **14** 104 (1973)
[39] K A Bronnikov *Acta Phys. Pol. B* **4** 251 (1973)
[40] S W Hawking and R Penrose *The large scale structure of space-time* (Cambridge: Cambridge University Press) (1973)
[41] M Banados and P G Ferreira *Phys. Rev. Lett.* **105** 011101 (2010)
[42] J R Nascimento, G J Olmo, AYu Petrov, P J Porfirio and A R Soares *Phys. Rev. D* **99** 064053 (2019)
[43] R D Lambaga and H S Ramadhan *Eur. Phys. J. C* **78** 436 (2018)
[44] H Aounallah, A R Soares and R L L Vitoria *Eur. Phys. J. C* **80** 447 (2020)
[45] A R Soares, R L L Vitoria and H Aounallah *Eur. Phys. J. Plus* **136** 966 (2021)
[46] A Moussa, H Aounallah, P Rudra and F Ahmed *Int. J. Geom. Meths. Mod.* **20** 2350102 (2023)
[47] F Ahmed *EPL* **141** 54001 (2023)
[48] F Ahmed *Eur. Phys. J. C* **80** 211 (2020)
[49] F Ahmed *Sci. Rep.* **11** 1742 (2021)
[50] F Ahmed *Sci. Rep.* **12** 8794 (2022)
[51] L F Deng, C Y Long, Z W Long and T Xu *Adv. High Energy Phys.* **2018** 2741694 (2018)
[52] L Zhong, H Chen, Z-W Long, C-Y Long and H Hassanabadi *Int. J. Mod. Phys. A* **36** 2150129 (2021)
[53] F Ahmed *EPL* **130** 40003 (2020)
[54] F Ahmed *Can. J. Phys.* **99** 496 (2021)
[55] F Ahmed *Can. J. Phys.* **101** 169 (2023)
[56] M K Bahar and F Yasuk *Adv. High Energy Phys.* **2013** 814985 (2013)
[57] A Ronveaux *Heun's Differential Equations* (Oxford: Oxford University Press) (1995)
[58] M Abramowitz and I A Stegun *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables* (New York: Dover) (1972)
[59] G B Arfken and H J Weber *Mathematical Methods for Physicists* (New York: Elsevier Academic Press) (2005)