Quantum Projection Filter And Feedback Stabilization

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universite
PARIS-SACLAY

NNT : 2024UPASG078

Thèse de doctorat

Quantum Projection Filter And Feedback Stabilization

Filtre De Projection Quantique Et Stabilisation Par Feedback

Thèse de doctorat de l'Université Paris-Saclay préparée à Centrale-Supélec

École doctorale n°580 Sciences et technologies de l'information et de la communication (STIC)
Spécialité de doctorat: Automatique
Graduate School: Informatique et sciences du numérique
Référent: : Centrale Supelec

Thèse préparée dans l'unité(s) de recherche : **Laboratoire des signaux et systèmes** sous la direction de **Nina Amini,** Directrice de recherche, CNRS (L2S), le co-encadrement de **Paolo Mason,** Chargé de recherche, CNRS (L2S).

Thèse présentée et soutenue à Paris Saclay, le 4 Decembre 2024, par

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Resume

Cette thèse vise à approfondir le développement de la méthode de filtrage par projection pour les systèmes quantiques ouverts soumis à des mesures imparfaites en temps continu, en mettant l'accent sur son utilisation dans la stabilisation par rétroaction. Nous commençons par discuter de la théorie du filtrage quantique, qui nous aide à comprendre l'évolution temporelle de l'opérateur densité conditionnel représentant un état quantique interagissant avec son environnement. L'évolution de l'état quantique est décrite à l'aide d'une équation différentielle stochastique à valeurs matricielles. Ensuite, nous introduisons quelques bases de la géométrie de l'information quantique pour présenter la projection quantique utilisée pour réduire la complexité quantique des systèmes quantiques ouverts. Nous nous concentrons sur l'application des filtres de projection pour les systèmes quantiques ouverts soumis à des mesures imparfaites dans le cas des mesures à non-démolition quantique (QND), et exprimons la solution exacte sur une variété de dimension inférieure. Afin de réduire encore la dimension du système étudié, nous considérons la projection sur la variété de dimension inférieure introduite initialement par Gao et al. (2019) pour le cas des mesures parfaites. Nous fournissons une analyse d'erreur de ce filtre d'approximation. La mise en œuvre en temps réel d'une rétroaction stabilisatrice reste un défi dans les expériences réelles en raison de l'échelle de temps très courte des systèmes dynamiques quantiques. Cela nous motive à faire dépendre la rétroaction du filtre de projection introduit et à démontrer son efficacité dans la stabilisation d'un système quantique ouvert dans le cas des mesures QND, tout en prenant en compte les imperfections de mesure et les imprécisions dans la spécification de l'état initial. La conception de la rétroaction repose sur une paramétrisation de la famille exponentielle utilisée pour le filtre de projection. Nous démontrons

que la rétroaction introduite garantit la convergence exponentielle de l’équation du filtre original vers un état cible prédéfini, correspondant à un état propre de l’opérateur de mesure.

Abstract

This thesis aims to further develop the projection filtering method of open quantum systems undergoing imperfect continuous-time measurements and focuses on its use in feedback stabilization. We begin by discussing quantum filtering theory, which helps us to understand the time evolution of the conditional density operator that represents a quantum state interacting with its environment. The evolution of the quantum state is described using a matrix-valued stochastic differential equation. Next, we introduce some foundations from the quantum information geometry theory to present the quantum projection that is used to reduce the quantum complexity of open quantum systems. We focus on the application of projection filters for open quantum systems undergoing imperfect measurements in the case of quantum non demolition (QND) measurements, and express the exact solution on a lower dimensional manifold. In order to further reduce the dimension of the system under study, we consider the projection on the lower dimensional manifold originally introduced in Gao et al. (2019) for the case of perfect measurements. We provide an error analysis of this approximation filter. The real time implementation of a stabilizing feedback remains a challenge in real experiments due to the short time scale of quantum dynamical systems. This motivates us to make the feedback depend on the introduced projection filter and show its efficiency in stabilizing an open quantum system in the case of QND measurements, while taking into account measurement imperfections and inaccuracies in the initial state specification. The feedback design relies on a parametrization of the exponential family utilized for the projection filter. We demonstrate that the introduced feedback guarantees exponential convergence of the original filter equation toward a predefined target state, corresponding to an eigenstate of the measurement operator.

Aperçu de la thèse

Dans cette thèse, nous étudions le problème du filtre de projection quantique sous des mesures imparfaites, puis nous procédons à l'étude de la stabilisation par retour via le filtre de projection quantique.

En ce qui concerne le premier problème, nous explorons l'utilisation de l'approche par projection pour développer des modèles approximatifs et simplifiés qui, malgré des observations défectueuses, reflètent avec succès la dynamique fondamentale de ces systèmes complexes. Notre objectif est de créer un système quantique simplifié et de dimension réduite qui capture fidèlement le système original. La question de la stabilisation par retour est abordée avec ce paradigme simplifié. La méthode générale que nous utilisons pour construire des modèles simplifiés consiste à identifier une sous-variété appropriée de l'espace entier des dynamiques possibles, vers laquelle la dynamique spécifique d'un système est généralement restreinte.

Dans le second problème, nous étudions comment réaliser une stabilisation exponentielle des systèmes quantiques ouverts couplés (le système principal et son système réduit) à l'aide d'un contrôle par retour basé sur le filtre réduit, même lorsque ces systèmes sont soumis à des mesures imparfaites en temps continu. Notre objectif est de stabiliser les systèmes vers un état cible spécifique, qui est un état pur correspondant à un vecteur propre des opérateurs de mesure.

Les bases de la théorie du filtrage quantique sont couvertes dans le Chapitre (2). Après avoir abordé les concepts mathématiques fondamentaux, ce chapitre présente les idées de base de la mécanique quantique. Ensuite, les systèmes quantiques ouverts et le filtrage quantique sont présentés respectivement. Enfin, les systèmes quantiques de spin à N niveaux sont introduits.

Le Chapitre (3) décrit la méthode de projection des équations de filtrage sur des variétés. Il commence par présenter le problème spécifique traité dans la thèse. Ensuite, il passe en revue certains outils de géométrie de l'information quantique, puis explique le concept de filtre de projection quantique. Enfin, il présente un exemple spécifique qui illustre l'idée du filtre de projection quantique.

Le Chapitre (4) est consacré à l'exploration d'un système quantique ouvert soumis à des mesures imparfaites et indirectes. Pour les mesures QND, nous démontrons que le système évolue sur une variété choisie de manière appropriée, et nous exprimons la solution exacte de l'équation du filtre quantique en utilisant une équation différentielle stochastique de dimension inférieure. Pour simplifier davantage le système, nous le projetons sur la variété de dimension réduite introduite dans [1] pour des mesures parfaites. Nous réalisons une analyse d'erreur pour évaluer la précision de ce filtre quantique approximatif, en nous concentrant spécifiquement sur les mesures QND. Par la suite, nous réappliquons la méthode de filtre de projection à une nouvelle sous-variété et obtenons ses résultats computationnels. Les simulations indiquent que le filtre de projection quantique proposé est efficace, même lorsque le contrôle stabilisant par retour, basé sur le filtre de projection, est appliqué. Ce travail est basé sur notre publication [2].

Le Chapitre (5) est consacré à l'application de l'approche de filtre de projection développée au Chapitre 4 pour la stabilisation par retour d'un système quantique de moment angulaire à N niveaux soumis à des mesures QND imparfaites, en supposant que le retour dépend du filtre de projection. Pour aborder la stabilisation par retour dans ce contexte, nous démontrons d'abord que l'analyse effectuée dans [3] peut être appliquée à ce cadre. Ensuite, afin de réduire la complexité computationnelle de l'implémentation en temps réel du retour, nous introduisons une nouvelle paramétrisation de dimension réduite de la famille exponentielle, permettant de reformuler le problème original comme un problème de stabilisation pour un système couplé décrivant l'évolution d'un couple formé par le filtre actuel et un vecteur de paramètres. Nous cherchons ensuite un retour en fonction de cette nouvelle paramétrisation. Nous fournissons des conditions suffisantes sur le contrôleur de retour garantissant la stabilisation exponentielle du système couplé vers un état cible choisi. De plus, nous donnons quelques exemples de contrôles par retour satisfaisant ces conditions. Ce travail est basé sur notre publication [4].

Le Chapitre (6) présente une conclusion sur les sujets abordés dans cette thèse et propose quelques extensions naturelles aux résultats décrits dans les chapitres mentionnés ci-dessus.

L'Annexe A fournit quelques notions et définitions de base issues de la théorie de la géométrie de l'information quantique. Elle présente également des notions fondamentales et des théorèmes issus du calcul stochastique et de la théorie du contrôle stochastique.

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Chapter 1

Introduction and Background

1.1 Motivation and context

To make a progress in quantum technologies, it is important to get hold of how to control quantum dynamics. Quantum systems are either closed or open. Closed quantum systems are isolated from their surroundings whilst open quantum systems absorb information from the surrounding world that makes them more realistic. As a consequence of this interaction, decoherence occurs and leads to information loss (see, e.g., [5]). Open quantum system dynamics can be described by Langevin equations for a quantum system driven by noise which are derived using quantum stochastic calculus [6] or input-output formalism [7].

Quantum measurements are probabilistic; they have a random back-action on the system, which is absent in classical physics. The quantum state can be described conditionally through a stochastic master equation derived from quantum filtering theory proposed by Belavkin [8, 9] and reformulated later in [10]. Also referred to as quantum trajectories within the physical and probability communities [11–13], this refers to the stochastic evolution of the filter state.

The reason for controlling open quantum systems is to suppress decoherence or make it work for us towards specific purposes [14]. Closed-loop control strategies may be more preferable than open-loop strategies because they tend to be more robust. Feedback strategies can be implemented whether or not measurements are present. Quantum feedback control is classified into two forms based on the controller: coherent feedback and measurement-based feedback. Coherent feedback involves a controller, which communicates with the quantum dynamical system via a quantum signal, such as a light beam [15]. This type of feedback has been used for several tasks, including squeezing enhancement [16,17], quantum memory, and error correction [18]. For the measurement-based feedback control, the feedback relies on partial information about the measurement outcomes when it comes to measurements. In [19,20], research indicates that certain open quantum systems can be stabilized through feedback using techniques like Lyapunov methods, geometric control and stochastic tools.

Uncertain initial states and detector inefficiency are two of the many variables that lead to defects in real-world experiments (see, e.g., [21]). Thus, it is essential to make sure that control mechanisms are robust. Numerous research, including [22–24], address this issue. Time lags that can affect the efficacy of measurement-based feedback when it is implemented in real-time provide another difficulty. The enormous dimensions of quantum filters and the comparatively lengthy processing time of classical measurements by digital electronics are the causes of these delays.

Feedback control principles for quantum systems can be designed using techniques from classical nonlinear and stochastic control. These techniques might be in some ways ideal or else created with pertinent factors (like stability) in mind. The resulting controllers are meant to be used in conjunction with some classical technologies (such as digital or analog electronics). Some experiments implementing quantum feedback controls [25–28] have led to renewed interest in the field, which is rapidly expanding [29–43]. We believe that integrating stochastic control with theoretical and experimental physics is crucial for developing quantum technologies.

In [44], the authors proposed an approximation filter for feedback design and established a feedback stabilization result for systems experiencing QND measurements. Numerical studies were used to demonstrate the effectiveness of this filter. The authors presented a formal verification of this feedback design's efficacy in [45].

For many quantum mechanical systems, describing their state effectively requires a high number of variables—often an infinite number. The goal of this thesis is to create new techniques for creating lower-dimensional reduced quantum filters and using them for open quantum system feedback control. This method reduces the complexity of quantum dynamical systems, which has a substantial impact on the engineering of quantum devices. The projection filter approach applied to a high dimensional quantum mechanical system is the main topic of this thesis.

A family of methods known as projection filters handles filtering issues in nonlinear state-space systems by utilizing information geometry and stochastic analysis [46, 47]. In essence, filtering is utilizing incomplete and noisy observations to estimate the hidden signal of a random dynamic system. Given the history of these noisy data, the objective is to ascertain the probability distribution of the signal. The computation of all pertinent statistics for the signal is made possible by this distribution. The Kushner-Stratonovich or Zakai equations are certain stochastic partial differential equations that are satisfied when this distribution has a density. It is known that the nonlinear filter density evolves in an infinite dimensional function space [48]. However, in certain special cases, the filter density evolves in a finite-dimensional space.

A finite-dimensional family of probability densities, such as Gaussian densities, Gaussian mixtures, or exponential families, can be used to mimic the infinite-dimensional filter density. Taking advantage of the geometric structure present in these selected density regions is the fundamental idea behind projection filters. A finite-dimensional stochastic differential equation (SDE) for the approximate density parameters is driven by projecting the infinite-dimensional stochastic partial differential equation (SPDE) of the optimal filter onto a finite-dimensional family [47]. To do this, the chosen finite dimensional family is equipped with a manifold structure as in information geometry.

The ideal filter for the cubic sensor problem was used to test the projection filter. Bimodal densities of the ideal filter, which would have been challenging to approximate using conventional techniques such as the extended Kalman filter, may be effectively tracked by the projection filter [46, 49].

Projection filters are well-suited for real-time estimate, because of their rapid implementation and effective operation in time, providing a finite dimensional SDE for the parameter that can be implemented efficiently [46]. These filters are also adaptable; by choosing richer approximate families, one can improve the approximation precision. The projection filtering algorithm's correction step can occasionally be made perfect using specific exponential families [47].

Certain projection filter formulations are compatible with Galerkin techniques [49] or heuristic-based assumed density filters [47]. According to certain requirements, such mean square minimization, projection filters can closely approximate the complete infinite-dimensional filter in addition to optimally approximating the SPDE coefficients [50].

Numerous domains, including navigation, ocean dynamics, quantum optics, quantum systems, fiber diameter estimation, chaotic time series estimation, and change point detection, have successfully used projection filters [51]. In this thesis, we study the projection of imperfectly measured finite high dimensional open quantum systems.

History and development (Classical case): Bernard Hanzon coined the phrase "projection filter" in 1987. During Damiano Brigo's doctoral study, the theoretical framework and numerical examples were rigorously explored and further improved in partnership with Bernard Hanzon and Francois LeGland [46,47,52]. Their work concentrated on projecting the infinite-dimensional SPDE of the optimal filter onto a selected exponential family by means of the Hellinger distance and Fisher information metric [46]. This method guarantees the accuracy of the filtering algorithm's prediction stage.

A distinct kind of projection filter based on the direct L^2 metric was presented by Armstrong and Brigo in 2016 [49]. By using this metric, filters based on Galerkin methods align with projection filters on families of mixture distributions. Later, in 2021 [50], Armstrong, Brigo, and Rossi Ferrucci derived optimal projection filters that approximate the infinite-dimensional optimum filter by satisfying certain requirements. These optimal projection filters handle the full SPDE solution, in contrast to Stratonovich-based projection filters, which optimize the approximations of the SPDE's individual coefficients on the selected manifold. Using Itô calculus directly instead of the Stratonovich version of the filter equation is what makes their method innovative. This is based on research on the geometry of Itô Stochastic differential equations on manifolds based on the jet bundle, the so-called 2-jet interpretation of Itô stochastic differential equations on manifolds [53]..

History and development (Quantum case): This expands upon the projection filter approach used for traditional filter equations. The approach, as mentioned in [46, 47, 54], is to employ differential and information geometry tools. To the best of our knowledge, Ramon Vanhandel and Mabuchi originally suggested the quantum projection filter technique in [40] in 2005 with the goal of lowering the representation complexity of quantum filters. Through unsupervised learning, the authors of [55] later in 2009 were able to determine the evolution of the system state in a lower dimensional manifold. By using local tangent space alignment, this was accomplished. In the years 2015 and 2017, respectively, an extended Kalman filter and numerical techniques were established in [56] and [57]. In 2017, a dynamical law is derived by minimizing the statistical distance in the moving basis and an equivalence with the projection filter has been shown by the authors of [58]. Recently in 2019, a quantum projection filtering approach was developed in which the dynamics is projected onto a manifold consisting of an exponential family of unnormalized density matrices [1]. Also an error analysis for a quantum projection filter based on an exponential family in the case of perfect QND measurement was performed.

Applications: Projection filters are highlighted by Jones and Soatto in [59] as possible algorithms for mapping and localization tasks, as well as for online estimation in visual-inertial navigation. Similar to this, projection filter methods were used in navigation applications by Azimi-Sadjadi and Krishnaprasad in [60]. Lermusiaux in [61] investigated the application of projection filters in ocean dynamics.

In [62], Kutschireiter, Rast, and Drugowitsch address projection filters in the context of continuous time circular filtering. Van Handel and Mabuchi introduced quantum projection filters to quantum optics for quantum systems in [63], looking at a quantum model of optical phase bistability in a tightly linked two-level atom inside an optical cavity. Further applications to quantum systems are considered in Gao, Zhang and Petersen [1].

The authors of [64] discuss projection filters in relation to hazard position estimate. To handle changepoint detection, Vellekoop and Clark in [65] extended projection filter theory. For filtering optimal point processes, the authors of [66] used projection filters in an assumed density form, with applications in neural encoding. In [67], Broecker and Parlitz examined projection filter techniques for mitigating noise in disordered time series. Additionally, the Gaussian projection filter was used by the authors of [68] in their method of determining the fiber diameters in melt-blown nonwovens.

1.2 Contribution and outline of dissertation

In this thesis, we study the quantum projection filter problem under imperfect measurements and then we proceed to study the feedback stabilization via the quantum projection filter.

Concerning the first problem, we investigate the use of the projection approach to develop approximate, simplified models that, in spite of faulty observations, successfully reflect the fundamental dynamics of these complex systems. Our goal is to create a reduced-dimensional, simplified quantum system that faithfully captures the original system. The feedback stabilization issue is handled with this simplified paradigm. Finding an appropriate submanifold of the entire space of possible dynamics, to which the specific dynamics of a system are typically restricted, is the general method we use to construct simpler models.

In the second problem, we investigate how to achieve exponential stabilization of the coupled open quantum systems (main system and its reduced one) using feedback control depending on the reduced filter, even when these systems are subject to imperfect continuous-time measurements. Our goal is to stabilize the systems towards a specific target state, which is a pure state corresponding to an eigenvector of the measurement operators.

The fundamentals of quantum filtering theory are covered in Chapter (2). After covering foundational mathematical ideas, it delves into describing the basic ideas of quantum mechanics. Next, open quantum systems and quantum filtering are presented respectively. Finally, N-level quantum spin systems are introduced. Chapter (3) outlines the method of projecting filtering equations onto manifolds. It begins with presenting the specific problem addressed in the thesis. After that, it reviews some quantum information geometry tools, then explains the concept of the quantum projection filter. Finally, it presents a specific example that illustrates the idea of the quantum projection filter.

Chapter (4) is devoted to explore an open quantum system undergoing imperfect and indirect measurements. For QND measurements, we demonstrate that the system evolves on a suitably chosen manifold, and we express the exact solution of the quantum filter equation using a lower-dimensional stochastic differential equation. To further simplify the system, we project it onto the lower-dimensional manifold introduced in [1] for perfect measurements. We conduct an error analysis to assess the accuracy of this approximate quantum filter, specifically focusing on QND measurements. After that, we repeat the previous projection filter method into a new submanifold and obtain its computational results. Simulations indicate that the proposed quantum projection filter is efficient, even when stabilizing feedback control, which relies on the projection filter, is applied. This work is based on our publication [2].

Chapter (5) is devoted to apply the projection filter approach done in Chapter 4 for feedback stabilization of a $N-$level quantum angular momentum system undergoing imperfect QND measurements, assuming that the feedback depends on the projection filter. To address feedback stabilization in this scenario, we first demonstrate that the analysis performed in [3] can be applied to this framework. Then, to reduce the computational complexity of the feedback's real-time implementation, we introduce a new lower-dimensional parametrization of the exponential family, allowing the original problem to be reformulated as a stabilization problem for a coupled system describing the evolution of a pair formed by the actual filter and a vector of parameters. We then search for a feedback as a function of this new parametrization. We provide sufficient conditions on the feedback controller ensuring the exponential stabilization of the coupled system toward a chosen target state. Furthermore, we give some examples of feedback which satisfy such conditions. This work is based on our publication [4].

Chapter (6) gives a conclusion for what we have discussed in this thesis and proposes some natural extensions to the results described in the chapters mentioned above.

Appendix A gives some basic notions and definitions from quantum information geometry theory. Also it provides some basic notions and theorems from stochastic calculus and stochastic control theory.

Chapter 2

Introduction To Feedback Control Of Open Quantum Systems

Quantum control theory is a rapidly emerging study subject, but additional efforts are needed to make it more practical for constructing quantum devices. Controlling quantum systems, such as creating and protecting desired states, is crucial for developing future quantum technology. Quantum technologies [69] are expected to outperform conventional technology. For example, quantum computing can solve certain problems quicker than conventional computing, and quantum metrology provides more exact parameter estimation than conventional techniques.

In the following, we will discuss some key postulates from quantum mechanics. Next, we will discuss open quantum systems, quantum filtering and quantum spin systems.

Roughly speaking, the stochastic master equation, derived from quantum filtering theory, describes the development of an open quantum system's state over time when interacting with an electromagnetic field under homodyne or photon counting detection.

2.1 Preliminaries

2.1.1 Hilbert Spaces, Notation, etc

Hilbert spaces are frequently used for mathematical representations in quantum physics. In this thesis we will consider finite dimensional Hilbert spaces, indicated as $\mathcal{H}$, endowed with the scalar product $\langle\cdot,\cdot\rangle$. Let's use $\mathcal{H} = \mathbf{C}^N$, where N is the dimension of complex vectors, for simplicity. The inner product between two vectors ϕ and ψ is defined as $\langle\psi,\phi\rangle = \sum_{k=1}^{n} \psi_k^* \phi_k$. Vectors are commonly represented by Dirac's kets, e.g., $\phi = |\phi\rangle \in \mathcal{H}$, where as $\psi = \langle\psi| \in \mathcal{H}^* \equiv \mathcal{H}$ is the symbol for dual vectors, often known as bras. The inner product of ϕ and ψ is expressed as

$$\langle\psi,\phi\rangle = \langle\psi\|\phi\rangle = \langle\psi \mid \phi\rangle.$$

The Banach space of bounded operators $A : \mathcal{H} \to \mathcal{H}$ is $\mathscr{B}(\mathcal{H})$. Let's examine it. The definition of $A^* \in \mathscr{B}(\mathcal{H})$ for any $A \in \mathscr{B}(\mathcal{H})$ is

$$\langle A^*\psi,\phi\rangle = \langle\psi, A\phi\rangle \text{ for all } \psi,\phi \in \mathcal{H}.$$

If $AA^* = A^*A$, then $A \in \mathscr{B}(\mathcal{H})$ is a normal operator. Two significant classes of normal operators are self adjoint operators $(A = A^*)$ and unitary operators $(A^* = A^{-1})$. The spectral theorem for a self-adjoint operator A states that:

$$A = \sum_{i|a_i \in \mathrm{spec}(A)} a_i P_{a_i}$$

where P_{a_i} is the projection onto the eigenspace corresponding to the eigenvalue a_i (diagonal representation). In the finite-dimensional case, A has a finite number of real eigenvalues a_i, $i = 1,\ldots,m \leq N$. The identity

$$\sum_{i|a_i \in \mathrm{spec}(A)} P_{a_i} = I$$

holds true.

2.1.2 Postulates of quantum mechanics

If we know the position and velocity of each point in a physical system across time, we may forecast its motion using classical mechanics. It is possible to precisely measure every physical quantity associated with the system, such as position, energy, and angular momentum, without generally affecting the system. Quantum mechanics, however, is very different from classical mechanics. If two or more quantities in quantum physics are incompatible—that is, if their associated observables do not commute—it is typically impossible to measure them exactly at the same moment. This non-commutativity gives rise to the well-known principle of Heisenberg uncertainty. The fundamental ideas of quantum mechanics are summarized here [70–72].

First postulate: A physical system's condition is represented at each given time, let's say t_0, by the state vector $|\psi(t_0)\rangle$, which is located in a separable complex Hilbert space $\mathcal{H}$. The superposition rule is suggested by this fundamental idea, which states that any linear combination of state vectors is also a state vector.

Second postulate: A Hermitian operator X acting in the Hilbert space $\mathcal{H}$ represents each measurable physical quantity $\mathscr{X}$. An observable is the name given to this operation.

Third postulate: A measurement of a physical quantity $\mathscr{X}$ must provide an eigenvalue of the associated Hermitian operator X. Any measurement of $\mathscr{X}$ will always result in a real value since X is Hermitian. The possible measurements for $\mathscr{X}$ are quantized if X only admits discrete spectrum.

Fourth postulate (finite dimensional case): The probability of obtaining the eigenvalue x_n of the corresponding observable X when measuring the physical quantity $\mathscr{X}$ in a system in the normalized state $|\psi\rangle$ is $\mathbb{P}(x_n) = \sum_{i=1}^{g_n} |\langle u_n^i \mid \psi\rangle|^2$. The multiplicity of x_n is given by g_n in this case, and the orthonormal vectors $|u_n^i\rangle$ for $i = 1, 2, \ldots, g_n$ form a basis in the eigensubspace $\mathcal{H}_n$ associated with x_n.

The relation $X|u_n^i\rangle = x_n|u_n^i\rangle$ holds for each i in the case of the eigenvalue x_n. Using this orthonormal basis, we can express the state vector $|\psi\rangle$ as follows: $|\psi\rangle = \sum_n \sum_{i=1}^{g_n} c_n^i |u_n^i\rangle$, where $c_n^i = \langle u_n^i \mid \psi\rangle$. As a result, $\mathbb{P}(x_n)$ is the probability of measuring the eigenvalue x_n.The expression $\mathbb{P}(x_n) = \sum_{i=1}^{g_n} |\langle u_n^i \mid \psi\rangle|^2 = \sum_{i=1}^{g_n} |c_n^i|^2$ corresponds to the observable X.

The projector onto $\mathcal{H}_n$ is therefore defined as follows: $P_n = \sum_{i=1}^{g_n} |u_n^i\rangle \langle u_n^i|$, satisfying $P_n^* = P_n$ and $P_n^2 = P_n$. The probability can be expressed as follows using this projector: $\mathbb{P}(x_n) = \langle\psi |P_n| \psi\rangle$. The definition of the expectation value of X is $\langle X\rangle := \sum_n x_n \mathbb{P}(x_n) = \langle\psi|X|\psi\rangle$. The observable X can be represented as $X = \sum_n x_n P_n$. An important consequence of this postulate is the following: the probabilities predicted for an arbitrary measurement are the same for two proportional state vectors, thus they represent the same physical state. Thus the states of a quantum system are rays in a Hilbert space $\mathcal{H}$.

Fifth postulate: The state of the system immediately following a measurement of a physical quantity $\mathscr{X}$ in a system initially in the state $|\psi\rangle$ changes to the normalized projection $\frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}$. The eigensubspace connected to x_n is the target of this projection. The decrease of the wave packet or collapse of the state are terms used to describe this phenomenon.

Sixth postulate: The Schrödinger equation describes the evolution of the state vector $|\psi(t)\rangle$ over time:

$$i = H(t)|\psi(t)\rangle\hbar\frac{d}{dt}|\psi(t),$$

where the system's total energy is represented by the Hamiltonian operator $H(t)$. The solution to this equation is represented by $|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$, where a unitary operator on the Hilbert space $\mathcal{H}$ is denoted by $U(t,t_0) = \exp\left(-\frac{i}{\hbar}\int_{t_0}^{t} H(\tau)d\tau\right)$. In the case of conservative systems, when H is a time-independent variable, this can be expressed as $U(t,t_0) = e^{-iH(t-t_0)/\hbar}$. It is also instructive to study the dynamics of the expectation of an observable X in a given state $|\psi(t)\rangle$,

$$\langle X\rangle(t) = \langle\psi(t)\,|X\,(t_0)|\,\psi(t)\rangle = \langle\psi\,(t_0)\,|U^*\,(t,t_0)\,X\,(t_0)\,U\,(t,t_0)|\,\psi\,(t_0)\rangle\,, \tag{2.1}$$

where we denote the observable X at t_0 as $X(t_0)$. There are two methods to deal with systems that are generally changing in time, since the time-dependent nature of the system needs to be conveyed by some combination of the state vectors and the operators:

- Schrödinger picture: While the operators don't change over time, the state vectors do. Specifically, $|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle$ holds true.

- Heisenberg picture: While the operators vary with time, the state vectors stay unchanged. In particular, $X_H(t) = U^*(t, t_0)X(t_0)U(t, t_0)$, in which the Heisenberg image is indicated by the subscript H.

With the Schrödinger picture emphasizing changing states and the Heisenberg picture emphasizing changing operators, these two methods offer distinct viewpoints on the time evolution of quantum systems.

It is evident from the equation above (2.1) that the Heisenberg and Schrödinger pictures are equivalent. The Heisenberg equation

$$i\hbar \frac{d}{dt} X_H(t) = [X_H(t), H(t)]$$

can be used to characterize the dynamics of the system in the Heisenberg picture using (2.1) and the Schrödinger equation.

Thus far, our focus has been on systems for which the states are fully known, enabling precise prediction of measurement results and tracking of their progress. In actuality, though, a system's status is frequently not entirely known. To address this, we provide the density operator, which, by accounting for imperfect information, more broadly than a state vector, describes the state of a quantum system. This scenario arises when the system is a statistical mixture of states with corresponding probabilities $p_1, p_2, \ldots$ and states $|\psi_1\rangle, |\psi_2\rangle, \ldots$. The states conditions $|\psi_1\rangle, |\psi_2\rangle, \ldots$ don't always have to be orthogonal. This statistical mixture is described by the density operator $\rho(t)$.

A system is in pure state if its state is perfectly known, that is $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$, where $|\psi(t)\rangle$ is the state vector at time t. In the case of pure states, $\mathrm{tr}(\rho^2) = 1$ and $\rho^2 = \rho$. The density operator for mixed states is $\rho = \sum_k p_k \rho_k$, where the pure state associated with $|\psi_k\rangle$ is represented by $\rho_k = |\psi_k\rangle\langle\psi_k|$. The density operator is therefore Hermitian ($\rho^* = \rho$), positive semidefinite ($\rho \geq 0$), and $\mathrm{tr}(\rho) = 1$.

The fourth, fifth, and sixth postulates are then expanded to include density operators.

Fourth postulate (finite dimensional case): The probability of finding the eigenvalue x_n of the associated observable X when measuring the physical quantity $\mathscr{X}$ in a system defined by the density operator ρ is $\mathbb{P}(x_n) = \mathrm{tr}(\rho P_n)$. This means that $\langle X \rangle = \mathrm{tr}(\rho X)$ is the expected value of X.

Fifth postulate: When a system defined by the density operator ρ is measured for a physical quantity $\mathscr{X}$ and the measurement yields an eigenvalue x_n, the state of the system immediately following the measurement is

$$\frac{P_n \rho P_n}{\mathrm{tr}(\rho P_n)}.$$

Sixth postulate: The density operator $\rho(t)$ evolves over time according to the Liouville-von Neumann equation:

$$\frac{d}{dt}\rho(t) = [H(t), \rho(t)]$$

where the system's Hamiltonian is denoted by $H(t)$.

2.2 Open system dynamics

Any quantum system interacts with external quantum systems ineluctably, such as a huge environment or a quantum heat bath is called an open quantum system.

2.2.1 Bipartite quantum systems

To create a theoretical structure for addressing these interactions, let us examine a basic scenario in which the target quantum system is comprised of two quantum subsystems, denoted as S_1 and S_2. The symbol $S_1 \otimes S_2$ represents this so-called bipartite quantum system.

Following [73], by the first postulate, the Hilbert space of the two combined systems combined is the tensor product of the individual spaces:

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_{S_1} \otimes \mathcal{H}_{S_2} \\ &= \mathrm{span}\left\{ |i\rangle_{S_1} \otimes |\mu\rangle_{S_2} \right\} \end{aligned}$$

Let us figure out the structure of a density matrix in this combined Hilbert space. We can define a pure state ensemble $\{|\Psi_a\rangle, q_a\}$ for a set of pure states $|\Psi_a\rangle \in \mathcal{H}$. Each of these states can be expanded in the basis above, i.e.,

$$|\Psi_a\rangle = \sum_{i,\mu} c_{a;i\mu}|i\rangle_{S_1} \otimes |\mu\rangle_{S_2}.$$

Thus, the associated density matrix is:

$$\rho = \sum_a q_a |\Psi_a\rangle \langle\Psi_a| = \sum_a q_a(\sum_{i,\mu} c_{a;i\mu}|i\rangle_A \otimes |\mu\rangle_B)(\sum_{j,\nu} c^*_{a;j\nu} \langle j|_A \otimes \langle\nu|_B) \ .$$

Therefore any density matrix in the combined Hilbert space can be written down as

$$\rho = \sum_{ij\mu\nu} \lambda_{ij\mu\nu}|i\rangle_{S_1}\langle j| \otimes |\mu\rangle_{S_2}\langle\nu|,$$

where $\lambda_{ij\mu\nu} = \sum_a q_a c_{a;i\mu} c^*_{a;j\nu}$. Note that if $\lambda_{ij\mu\nu} = \lambda^{S_1}_{ij}\lambda^{S_2}_{\mu\nu}$ then $\rho = \rho_{S_1} \otimes \rho_{S_2}$, where $\rho_{S_1} = \sum_{i\mu} \lambda^{S_1}_{ij}|i\rangle_{S_1}\langle j|$ and $\rho_{S_2} = \sum_{\mu\nu} \lambda^{S_2}_{\mu\nu}|\mu\rangle_{S_2}\langle\nu|$.

As a result, we may apply all six of the postulates covered in the preceding section to the situation involving bipartite quantum systems.The partial trace, which aids in averaging the complementary system, must then be introduced in order to ascertain the marginal states for bipartite quantum systems. The marginal state ρ_1 on $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be found by applying the formula

$$\rho_1 = \mathrm{tr}_{\mathcal{H}_2}(\boldsymbol{\rho}),$$

where the partial trace is defined as follows:

$$\mathrm{tr}\left(\mathrm{tr}_{\mathcal{H}_2}(\boldsymbol{\rho})X_1\right) = \mathrm{tr}\left(\boldsymbol{\rho}\left(X_1 \otimes \mathbb{1}\right)\right),$$

for each and every observable X_1 on $\mathcal{H}_1$.

2.2.2 The evolution and measurement

All bipartite quantum systems can be regarded as belonging to the family of open quantum systems. Let us describe the environment by a quantum system W on a Hilbert space $\mathcal{H}_W$, and define the quantum dynamical system of interest S on a Hilbert space $\mathcal{H}_S$. An even larger quantum system defined on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_W$ might be thought of as the system-environment $S \otimes W$. The starting states of the system and environment are denoted by $\rho_S(t_0)$ and $\rho_W(t_0)$, respectively, and are represented by the density operators $\rho_S(t)$ and $\rho_W(t)$. Then, $\boldsymbol{\rho}(t)$ gives the time evolution of system-environment represented by

$$\boldsymbol{\rho}(t) = U(t, t_0)\left(\rho_S(t_0) \otimes \rho_W(t_0)\right) U^*(t, t_0) \Rightarrow \rho_S(t) = \operatorname{tr}_{\mathcal{H}_W}(\boldsymbol{\rho}(t)).$$

The unitary operator working on $\mathcal{H}_S \otimes \mathcal{H}_W$ is denoted by $U(t, t_0)$. The observable $X_1 \otimes \mathbb{1}$ is what we measure. Let $\{x_k\}$ denotes the set of X_1 eigenvalues, and $\{P_k\}$ denotes the set of related projectors. Following the measurement, the state turns into

$$\boldsymbol{\rho}'(t) = \frac{P_n U(t, t_0)\left(\rho_S(t_0) \otimes \rho_W(t_0)\right) U^*(t, t_0) P_n}{\operatorname{tr}\left(P_n U(t, t_0)\left(\rho_S(t_0) \otimes \rho_W(t_0)\right) U^*(t, t_0) P_n\right)}$$

in which the probability of receiving x_n is given by the denominator. Consequently, the post measurement marginal state $\rho'_S(t)$ is given by

$$\rho'_S(t) = \operatorname{tr}_{\mathcal{H}_W}\left(\boldsymbol{\rho}'(t)\right) = \frac{\operatorname{tr}_{\mathcal{H}_W}\left(P_n U(t, t_0)\left(\rho_S(t_0) \otimes \rho_W(t_0)\right) U^*(t, t_0) P_n\right)}{\operatorname{tr}\left(P_n U(t, t_0)\left(\rho_S(t_0) \otimes \rho_W(t_0)\right) U^*(t, t_0) P_n\right)}.$$

This explains the open quantum system's time development while it is being measured.

2.3 Quantum filtering

Let us now imagine an open quantum system defined on $\mathcal{H}_S$ interacting with an electromagnetic field in the vacuum state defined on $\mathcal{H}_W$ undergoing continuous-time measurements. Take into account just one measurement channel for homodyne detection at a time. The electromagnetic field can be interpreted heuristically as a set of quantum harmonic oscillators, which are defined on $\mathcal{H}_W$ and do not commute with each other. These oscillators are characterized by the field operators A_t (annihilation process) and $A_t^\dagger$ (creation process). Then, the following quantum stochastic differential equation (QSDE)

$$dU_t = \left(L \otimes dA_t^\dagger - L^* \otimes dA_t - ((L^*L/2 + iH) \otimes \mathbb{1})\, dt\right) U_t, \quad U_0 = \mathbb{1},$$

describes the joint dynamics of the unitary operator U_t of the entire system, i.e., the open quantum system and the electromagnetic field, defined on the Hilbert space $\mathcal{H}_S \otimes \mathcal{H}_W$.

Then the time evolution of the observable X on $\mathcal{H}_S$ is $j_t(X) = U_t^*(X \otimes \mathbb{1})U_t$, and the observation of homodyne detection at time t is $Y_t = U_t^* \left(\mathbb{1} \otimes \left(A_t + A_t^\dagger\right)\right) U_t$. The filter estimates the observable X at time t based on the detector's output Y_t.

Using the quantum Itô calculus [6], we get

$$\begin{aligned} dj_t(X) &= \mathcal{L}\left(j_t(X)\right) dt + dA_t^\dagger \left[j_t(X), j_t(L)\right] + \left[j_t\left(L^*\right), j_t(X)\right] dA_t \\ dY_t &= \left(j_t(L) + j_t\left(L^*\right)\right) dt + dA_t + dA_t^\dagger \end{aligned}$$

where $\mathcal{L}(X) := i[H, X] + L^*XL - L^*LX/2 - XL^*L/2$ is called Lindblad generator.

Similar to classical stochastic filtering theory [74, 75], the dynamics of expectation of $j_t(X)$ conditioned on the recorded measurements Y_t up to time t, indicated by $\pi_t(X)$, are determined by the Kallianpur-Striebel formula and Itô formula. The analogues of this (non-commutative counterpart) is developed in the case of quantum filtering in [10]. A normalized filtering equation can be used to characterize the conditional expectation $\pi_t(X)$, which is the projection onto the space spanned by the measurements record $\sigma\left(Y_s, 0 \leq s \leq t\right)$. The innovation process that corresponds to the observation process is what drives this equation.

In the Schrödinger picture, we have $\pi_t(X) = \mathrm{Tr}\left(\rho_t X\right)$, where ρ_t is a density operator conditioned on the observations up to time t. Thus we can obtain a matrix-valued stochastic differential equation for the evolution of the density operator of the system under perfect continuous-time (homodyne) measurements, which is called stochastic master equation, and it is the quantum analogue of the Kushner-Stratonovich or FKK equation,

$$d\rho_t = \mathcal{L}^* (\rho_t)\, dt + (L\rho_t + \rho_t L^* - \mathrm{Tr}\,((L + L^*)\, \rho_t)\, \rho_t)\, dW_t$$
$$dY_t = dW_t + \mathrm{Tr}\,((L + L^*)\, \rho_t)\, dt$$

where W_t is a one dimensional Wiener process. Similar arguments can characterize the dynamics of the system under imperfect measurements

$$d\rho_t = \mathcal{L}^* (\rho_t)\, dt + \sqrt{\eta}\, (L\rho_t + \rho_t L^* - \mathrm{Tr}\,((L + L^*)\, \rho_t)\, \rho_t)\, dW_t. \tag{2.2}$$

The measurement efficiency is given by $\eta \in (0, 1)$. Suppose that H and L are time-invariant, then the classical expectation of the stochastic master equation (2.2) is called Lindblad master equation

$$\frac{d}{dt}\mathbb{E}\,(\rho_t) = \mathcal{L}^* (\mathbb{E}\,(\rho_t))\,.$$

Moreover by calculating the partial trace, $\mathbb{E}\,(\rho_t)$ can also be derived,

$$\mathbb{E}\,(\rho_t) = \mathrm{Tr}_{\mathcal{H}W}\,(U_t\, (\rho_0 \otimes |0\rangle\langle 0|)\, U_t^*)\,.$$

We so note that the Lindblad master equation serves as the Fokker-Plank equation for the stochastic master equation (2.2).

2.4 N-level quantum angular momentum systems

Let us investigate the N-level angular momentum system, which is a typical example in the literature of [72], where $\mathcal{H}_N$ represents the N-dimensional spin state space and $2 \leq N < \infty$. The Hermitian operators J_x, J_y, and J_z, respectively, describe the angular momentum along the x, y, and z axes. Along these three axes, the angular momenta's commutation relations are as follows:

$$[J_x, J_y] = i\hbar J_z, \quad [J_y, J_z] = i\hbar J_x, \quad [J_z, J_x] = i\hbar J_y.$$

Following, e.g., [70, 71] the eigenvalues and eigenvectors of the three angular momentum operators are as follows for all $n \in \{0, \ldots, N-1\}$: $J_x |e_n\rangle = c_n |e_n\rangle + c_{n+1} |e_{n+1}\rangle$, $J_y |e_n\rangle = -ic_n |e_{n-1}\rangle + ic_{n+1} |e_{n+1}\rangle$, $J_z |e_n\rangle = (J-n) |e_n\rangle$, where $\langle e_n \mid e_m\rangle = \delta_{m,n}$ and $J = \frac{N-1}{2}, c_n = \frac{\sqrt{(N-n)n}}{2}$. The collection of state vectors $\{|e_1\rangle, \ldots, |e_N\rangle\}$ has a basis made up in the state space $\mathcal{H}_N$. The following particular matrices can be used to describe the angular momentum operators in this basis:

$$J_x = \begin{bmatrix} 0 & c_1 & & & \\ c_1 & 0 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_2, J-1 & 0 & c_{2,J} \\ & & & c_{2,J} & 0 \end{bmatrix}, \quad J_y = \begin{bmatrix} 0 & -ic_1 & & & \\ ic_1 & 0 & -ic_2 & & \\ & \ddots & \ddots & \ddots & \\ & & ic_{2,J-1} & 0 & -ic_{2,J} \\ & & & ic_2, & 0 \end{bmatrix},$$

and

$$J_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & \ddots & & & \\ & & -J+1 & & \\ & & & & -J \end{bmatrix}.$$

In this case, the density operator ρ acting on $\mathcal{H}_N$, belongs to the state space $\mathcal{S}_N := \{\rho \in \mathbb{C}^{N\times N} \mid \rho^* = \rho, \mathrm{tr}(\rho) = 1, \rho \geq 0\}$. The corresponding N-level Bloch vector is given in [76]. The N-level angular momentum model corresponds to a special case of the general N-level quantum systems, with $2 \leq N < \infty$, whose Hamiltonian has N unique eigenvalues.

Online calculation of the quantum filter representing the N- level system is essential in many quantum technologies such as quantum state dependent feedback control. Moreover, for this N-dimensional quantum system, however, its conditional quantum state has to be obtained by calculating a system with $N^2 - 1$ SDEs, which suffers from the curse of dimension. We aim to construct a lower-dimensional approximation for this quantum filter such that the numerical procedure becomes less computationally demanding. The approximation method we follow is called quantum projection filter and it will be explained in details in Chapter (4).

In this thesis, we have considered a simplified case, in which the number of dimensions of the main system is finite, and the operators describing the interaction between the main system and the field are constant. We studied a single quantum channel in which the Hamiltonian is the sole thing affected by the control input. Based on the preliminary discussion on the quantum filtering theory of this chapter, we can adapt the results to more general cases by using more sophisticated arguments [8, 77, 78].

Chapter 3

Fundamental Tools For Quantum Projection Filtering

The dynamics of open quantum systems occur in the space of density matrices, which can be very high-dimensional space, especially when photon fields are involved. Although density matrices involving photons are technically infinite, it is common practice to use a cutoff at a high Fock state, making the density matrices finite but still large. For an N-dimensional quantum system, calculating its conditional quantum state requires dealing with a system of $N^2 - 1$ SDEs, which is computationally challenging due to the curse of dimensionality. However, since the dynamics of a specific system do not fully utilize this vast space, we aim to define a smaller, lower-dimensional space to study the system's dynamics more efficiently. We achieve this by projecting the equations of motion onto this reduced space. In this chapter, we provide an overview of how to compute the projected equations of motion, explore the projection within stochastic filtering equations, and then derive the form of the projected equations in the lower-dimensional space.

3.1 Some tools from Quantum Information Geometry

This section will introduce some foundations of quantum information geometry theory adapted to the framework of open quantum systems.

Letting $\mathcal{A}$ be the set of all Hermitian operators on $\mathbb{C}^N$, and

$$\mathcal{Q} = \{\rho \in \mathcal{A} \mid \rho \geq 0\}, \tag{3.1}$$

be the closed subset of $\mathcal{A}$ consisting of all nonnegative Hermitian operators on $\mathbb{C}^N$. We denote by $T_\rho\mathcal{Q}$, the tangent space of $\mathcal{Q}$ at the point ρ, which is identified with $\mathcal{A}$.

An m-representation of X is denoted by $X^{(m)}$, and given by a tangent vector $X \in T_\rho\mathcal{Q}$. Now if a coordinate system ϵ^i is given on $\mathcal{Q}$, each state can be parameterized by ρ_ϵ, and the m-representation of natural basis vectors in the tangent space are identified with

$$(\partial_i^{(m)}) = \partial_i, \tag{3.2}$$

where $\partial_i := \frac{\partial \rho_\epsilon}{\partial \epsilon_i}$ and $\{\partial_i\}$ are linearly independent and $T_\rho\mathcal{Q}$ =Span$\{\partial_i\}$.

As the differentiable manifold don't have by nature an inner product structure, then we define a symmetrized inner product $\{\langle\langle , \rangle\rangle_\rho, \rho \in \mathcal{Q}\}\}$ on $\mathcal{A}$ [54] by:

$$\langle\langle A, B\rangle\rangle_\rho = \frac{1}{2}\,\mathrm{tr}(\rho AB + \rho BA),\ \forall A, B \in \mathcal{A}. \tag{3.3}$$

Now we define, based on this inner product, the e-representation of a tangent vector $X \in T_\rho\mathcal{Q}$ as the self-adjoint operator $X^{(e)} \in \mathcal{A}$ satisfying

$$\langle\langle X^{(e)}, A\rangle\rangle_\rho = \mathrm{tr}(X^{(m)}A),\ \forall A \in \mathcal{A}. \tag{3.4}$$

Using the e-representation defined above, an inner product $\langle , \rangle$ on $T_\rho\mathcal{Q}$ is defined by

$$\langle X, Y\rangle_\rho = \langle\langle X^{(e)}, Y^{(e)}\rangle\rangle_\rho = \mathrm{tr}(X^{(m)}Y^{(e)}),\ \forall X, Y \in T_\rho\mathcal{Q}. \tag{3.5}$$

The quantum version of a Fisher metric is called a Riemannian metric $g = \langle , \rangle$ whose components are

$$g_{ij} = \langle \partial_i, \partial_j\rangle_\rho = \mathrm{tr}(\partial_i^{(m)}\partial_j^{(e)}). \tag{3.6}$$

More details about this information geometric tools can be found in the book [54].

3.2 Problem description

Given a N-dimensional quantum system, its state can be described by the density operator ρ, which is a positive semi-definite Hermitian matrix of trace one, i.e., ρ belongs to the set

$$\mathcal{S}_N := \left\{\rho \in \mathbb{C}^{N\times N} \mid \rho = \rho^\dagger, \rho \geq 0, \mathrm{tr}(\rho) = 1\right\}.$$

The state evolution of the quantum system undergoing continuous-time measurement can be described by the following quantum stochastic master equation (see e.g., [9])

$$\begin{aligned} d\rho_t &= -i\,[H, \rho_t]\,dt + \mathcal{F}(\rho_t)dt + \mathcal{G}(\rho_t)dW_t \\ dY_t &= dW_t + \sqrt{\eta}\,\mathrm{tr}\left(\rho_t(L + L^\dagger)\right)dt, \end{aligned} \tag{3.7}$$

where L corresponds to the measurement operator, Y represents the observation process in the case of homodyne detection, and the super-operators $\mathcal{F}$ and $\mathcal{G}$ have the following form

$$\begin{aligned} \mathcal{F}(\rho) &:= L\rho L^\dagger - \tfrac{1}{2}\left(L^\dagger L\rho + \rho L^\dagger L\right) \\ \mathcal{G}(\rho) &:= \sqrt{\eta}\left(L\rho + \rho L^\dagger - \mathrm{tr}\left[(L + L^\dagger)\rho\right]\rho\right). \end{aligned}$$

The operator H represents the Hamiltonian of the system, which is formed by a free Hamiltonian and a controlled term, i.e., $H = H_0 + uH_1$ where u is the control input. Note that, due to the dependence on the control input, the Hamiltonian is in general time-varying. We denote it simply by H for the sake of simplicity. In the following the control input will be chosen in the form of a state-feedback, i.e., $u = u(\rho)$. The efficiency of the detector is expressed by the parameter $\eta \in (0, 1]$. In equation (3.7) W_t is a classical Wiener process with respect to a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Note that, for more general observation processes, the diffusion term in the evolution equation may be driven by a complex Wiener process (for more details see, e.g., [79]).

In practice, we do not have the quantum state at our disposal. In this case an estimation of the above quantum filter is considered. A natural choice is to consider an estimate denoted by $\hat{\rho}_t$ following the same dynamics as above with an arbitrary initial state $\hat{\rho}_0$,

$$\begin{aligned} d\hat{\rho}_t &= -i\,[H,\hat{\rho}_t]\,dt + \mathcal{F}(\hat{\rho}_t)dt + \mathcal{G}(\hat{\rho}_t)(dY_t - \sqrt{\eta}\,\mathrm{tr}\left(\hat{\rho}_t(L+L^\dagger)\right)dt) \\ &= \left(-i\,[H,\hat{\rho}_t] + \mathcal{F}(\hat{\rho}_t) + \sqrt{\eta}\mathcal{G}(\hat{\rho}_t)\,\mathrm{tr}\left((\rho_t-\hat{\rho}_t)(L+L^\dagger)\right)\right)dt + \mathcal{G}(\hat{\rho}_t)dW_t. \end{aligned} \tag{3.8}$$

The stochastic master equation (3.7) (coupled or not with the equation (3.8)) has been extensively explored in various fields such as quantum state estimation and quantum feedback control [3, 19, 41, 42]. The evolution of such an Itô stochastic differential equation takes place in a N^2-1 dimensional space. For large values of N the time required to solve the equation may become extremely high. This represents an obstacle to the use of a feedback control for stabilization based on a real time evaluation of $\hat{\rho}_t$. Therefore, it is common practice to use an estimate of the system's state based on a lower dimension model of the dynamics, instead of directly computing the solution of (3.8). This motivates us to use a model reduction method called the quantum projection filter approach, that will be expressed in details in Chapter (4).

In the following we will mainly work with the Zakai equation, which is the unnormalized form of the quantum filter equation (3.7) and which is given by

$$d\check{\rho}_t = -i\,[H,\check{\rho}_t]\,dt + \left(L\check{\rho}_t L^\dagger - \frac{1}{2}\left(L^\dagger L\check{\rho}_t + \check{\rho}_t L^\dagger L\right)\right)dt + \sqrt{\eta}\left(L\check{\rho}_t + \check{\rho}_t L^\dagger\right)dY_t. \tag{3.9}$$

In particular $\rho_t = \check{\rho}_t/\mathrm{tr}(\check{\rho}_t)$. The evolution corresponding to (3.9) takes place on the space $\mathcal{Q}$.

Since the vector fields defining the dynamics are linear, hence globally Lipschitz, equation (3.9) has a unique solution [80]. Since $\rho_t = \check{\rho}_t/\mathrm{tr}(\check{\rho}_t)$, we deduce that (3.7) has a unique solution as well.

For compatibility reasons with the differential manifold structure (see e.g. [46]), we further consider the Stratonovich form of the above equation. The following Stratonovich quantum stochastic differential equation is equivalent to the Itô quantum stochastic differential equation in (3.9):

$$d\check{\rho}_t = \left(-i\left[H, \check{\rho}_t\right] - F\left(\check{\rho}_t\right)\right)dt + \sqrt{\eta}\left(L\check{\rho}_t + \check{\rho}_t L^\dagger\right) \circ dY_t \tag{3.10}$$

where

$$F\left(\check{\rho}_t\right) = (1-\eta)L\check{\rho}_t L^\dagger - \frac{\left(\eta L + L^\dagger\right)L\check{\rho}_t + \check{\rho}_t L^\dagger\left(L + \eta L^\dagger\right)}{2}.$$

3.3 Introduction to quantum projection filtering

In this section, we will give a precise overview of the procedure for projecting equations of motion from a manifold with high dimensions onto a lower-dimensional one. The explanation is coming from the idea of the methodology presented by van Handel and Mabuchi [63]. We have adapted their derivation to deal with density matrix instead of the well-known quantum quasiprobability distribution, the Q-function [81]. For this section, let us denote the space of all possible density matrices for a quantum system of interest by S_N, and the smaller subspace by K.

Consider our normalized Itô stochastic dynamical system, which can be expressed as:

$$d\rho_t = A(\rho_t)dt + \mathcal{G}(\rho_t)dW_t \tag{3.11}$$

where $A(\rho_t) = \left(-i\left[H, \rho_t\right] + \mathcal{F}(\rho_t)\right)$. In the following we will mainly work with the Zakai equation, which is the unnormalized linear form of (3.11)

$$d\check{\rho}_t = A(\check{\rho}_t)dt + \sqrt{\eta}\left(L\check{\rho}_t + \check{\rho}_t L^\dagger\right)dY_t \tag{3.12}$$

satisfying $\rho_t = \check{\rho}_t/\mathrm{tr}(\check{\rho}_t)$.

From a geometric standpoint we would like to view the right-hand side of this equation as a vector, in the tangent space to $\mathcal{Q}$ at a specific point $\theta_{\mathcal{Q}}$. When we project the equation onto K our goal is to keep the components in the tangent space to K at θ, known as $T_\theta K$. To consider the right side of equation (3.11) or (3.12) as a vector it needs to undergo a transformation which suggests an interpretation as a Stratonovich stochastic differential equation rather than an Itô equation. In situations like those discussed in the applications of projection sections one would compute the appropriate Itô correction term. The new equation is given by:

$$\mathrm{d}\check{\rho}_t = C(\check{\rho}_t)dt + \sqrt{\eta}\left(L\check{\rho}_t + \check{\rho}_t L^\dagger\right) \circ dY_t, \tag{3.13}$$

where $C(\check{\rho}_t) = -i\left[H, \check{\rho}_t\right] - F\left(\check{\rho}_t\right)$. Given a coordinate point $\theta = \left(\theta^{(1)}, \ldots, \theta^{(m)}\right)$, we assume the presence of local coordinate system on K such that we can express it as follows:

$$T_\theta K = \text{span}\left[\frac{\partial \check{\rho}_\theta}{\partial \theta^{(1)}}, \cdots, \frac{\partial \check{\rho}_\theta}{\partial \theta^{(m)}}\right]. \tag{3.14}$$

An inner product is introduced in the space of Hermitian operators :

$$\langle \rho_a, \rho_b \rangle = \mathrm{tr}\left(\rho_a \rho_b\right), \tag{3.15}$$

enabling the computation of the metric tensor in this basis:

$$\left\langle \frac{\partial \check{\rho}_\theta}{\partial \theta^{(k)}}, \frac{\partial \check{\rho}_\theta}{\partial \theta^{(j)}} \right\rangle = \mathrm{tr}\left(\frac{\partial \check{\rho}_\theta}{\partial \theta^{(k)}} \frac{\partial \check{\rho}_\theta}{\partial \theta^{(j)}}\right) = g_{kj}(\theta). \tag{3.16}$$

Leveraging the inner product and the metric, the orthogonal projection of a vector field $V[\theta]$ is defined as follows:

$$\Pi_\theta V[\theta] = \sum_{k=1}^{m} \sum_{j=1}^{m} g^{kj}(\theta) \left\langle V[\theta], \frac{\partial \check{\rho}_\theta}{\partial \theta^{(j)}} \right\rangle \frac{\partial \check{\rho}_\theta}{\partial \theta^{(k)}}, \tag{3.17}$$

Here, g^{kj} represents the (k, j) component of the inverse of the metric g_{kj} defined in equation (3.16). The objective now is to constrain the dynamics of equation (3.13) to unfold on S:

$$\mathrm{d}\check{\rho}_{\theta_t} = \Pi_{\theta_t} C\left[\check{\rho}_{\theta_t}\right] \mathrm{d}t + \Pi_{\theta_t}\left(\sqrt{\eta}(L\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t} L^\dagger)\right) \circ \mathrm{d}Y_t. \tag{3.18}$$

This equation represents a stochastic differential equation for the parameters θ_t in the Stratonovich form. Additionally, it is worthwhile mentioning that one must interpret the stochastic master equation (3.11) in the Stratonovich form as it satisfies the chain rule whereas Itô rules are incompatible with a manifold structure (see, e.g, [82]). According to the chain rule, we have

$$\mathrm{d}\check{\rho}_{\theta_t} = \sum_{k=1}^{m} \frac{\partial \check{\rho}_{\theta_t}}{\partial \theta_t^{(k)}} \circ \mathrm{d}\theta_t^{(k)}. \tag{3.19}$$

By incorporating the definition of the orthogonal projection into equation (3.13), the resulting expression is as follows:

$$
\begin{aligned}
\mathrm{d}\check{\rho}_{\theta_t} =& \sum_{k=1}^{m}\sum_{j=1}^{m} g^{kj}(\theta_t)\left\langle C\left[\check{\rho}_{\theta_t}\right], \frac{\partial \check{\rho}_{\theta_t}}{\partial \theta_t^{(j)}}\right\rangle \frac{\partial \check{\rho}_{\theta_t}}{\partial \theta_t^{(k)}}\mathrm{d}t \\
&+ \sum_{k=1}^{m}\sum_{j=1}^{m} g^{kj}(\theta_t)\left\langle \left(\sqrt{\eta}(L\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L^\dagger)\right), \frac{\partial \check{\rho}_{\theta_t}}{\partial \theta_t^{(j)}}\right\rangle \frac{\partial \check{\rho}_{\theta_t}}{\partial \theta_t^{(k)}} \circ \mathrm{d}Y_t.
\end{aligned}
$$

Comparing this expression with equation (3.19), we can isolate the equations for $\mathrm{d}\theta_t^{(k)}$:

$$
\mathrm{d}\theta_t^{(k)} = \sum_{j=1}^{m} g^{kj}(\theta_t)\left\langle C\left[\check{\rho}_{\theta_t}\right], \frac{\partial \check{\rho}_{\theta_t}}{\partial \theta_t^{(j)}}\right\rangle \mathrm{d}t + \sum_{j=1}^{m} g^{kj}(\theta_t)\left\langle \left(\sqrt{\eta}(L\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L^\dagger)\right), \frac{\partial \check{\rho}_{\theta_t}}{\partial \theta_t^{(j)}}\right\rangle \circ \mathrm{d}Y_t. \quad (3.20)
$$

It is essential to highlight that for the application of this procedure, a functional form for $\check{\rho}_\theta$ must be known. This implies the need for a mapping from the smaller space (spanned by θ) to the larger space (where ρ exists). This requirement is in addition to knowing the form of the projection from the larger space to the smaller, as facilitated by equation (3.17). Also, it is good to mention that Itô equations are required in order to carry out numerical simulations using an Itô-Euler integrator. Consequently, we need to rewrite our Stratonovich equations in Itô form. This step is also essential for the feedback stabilization problem that will be explained in details in Chapter 5.

3.4 Projecting onto a linear density matrix space

Following [40, 83, 84] (in the quantum framework), we consider the subset $\mathscr{S}$ of $\mathcal{Q}$ consisting of a linear family of unnormalized quantum density operators

$$
\mathscr{S} = \{\check{\rho}_\psi \mid \psi = (\psi_1, \ldots, \psi_l) \in \Psi\}.
$$

Here $\check{\rho}_\psi := \rho_0 + \sum_{k=1}^{l} \psi_k A_k$, ρ_0 is the initial condition for the (projected) dynamics, the operators $A_i \in \mathcal{A}$, for $i \in \{1, 2, \ldots, l\}$, are assumed to be mutually commuting and pre-designed, and

Ψ is an open subset of $\mathbb{R}^l$ containing the origin. Assume that the set $\{\frac{\partial \check{\rho}_\psi}{\partial \psi_1}, \ldots, \frac{\partial \check{\rho}_\psi}{\partial \psi_l}\}$ is linearly independent. The tangent space at some $\check{\rho}_\psi \in \mathscr{S}$ is given by $T_{\check{\rho}_\psi}\mathscr{S} = \mathrm{span}\{\check{\partial}_i, i = 1, \ldots, l\}$, where $\check{\partial}_i = \frac{\partial \check{\rho}_\psi}{\partial \psi i} = A_i$. For the linear family, we define the inner product:

$$\langle\langle A, B\rangle\rangle_{\rho_\psi} = \mathrm{tr}(AB), \qquad \forall A, B \in \mathcal{A}. \tag{3.21}$$

Using (1) and (3.21) we get $\check{\partial}_i^{(e)} = A_i$.

In analogy with (3.6) we define a Riemannian metric on $\mathscr{S}$ whose components are real-valued functions of ψ:

$$g_{ij}(\psi) = \left\langle \check{\partial}_i, \check{\partial}_j \right\rangle_{\check{\rho}_\psi} = \mathrm{tr}\left(A_i A_j\right). \tag{3.22}$$

Then, for every $\psi \in \Psi$, we can define an orthogonal projection operation Π_ψ by

$$\begin{aligned} \Pi_\psi : \mathcal{A} &\longrightarrow T_{\check{\rho}_\psi}\mathscr{S} \\ x &\longmapsto \sum_{i=1}^{m}\sum_{j=1}^{m} g^{ij}(\psi)\langle\langle x, \check{\partial}_j^{(e)}\rangle\rangle_{\check{\rho}_\psi}\check{\partial}_i, \end{aligned} \tag{3.23}$$

where the $g^{ij}(\psi)$ are the components of the inverse of the quantum information matrix $G(\psi) = g_{ij}(\psi)$.

We define the quantum projection filter on $\mathscr{S}$ by equation (3.18). Since the vector fields regulating the dynamics are everywhere tangent to $\mathscr{S}$, the solution of the previous equation is a well-defined stochastic process $\check{\rho}_{\psi_t}$ on $\mathscr{S}$, whenever $\check{\rho}_{\psi_0} = \rho_0$ belongs to $\mathscr{S}$. By using the orthogonal projection operation and the chain rule, we can easily express the dynamics of the parameter ψ_t as

$$d\psi_t = G\left(\psi_t\right)^{-1}\left\{\Xi\left(\psi_t\right)dt + \Gamma\left(\psi_t\right) \circ dY_t\right\} \tag{3.24}$$

with $\psi_i(0) = 0$, for $i = 1, \ldots, l$. Here, the j-th elements of the l-dimensional column vectors $\Xi(\psi_t)$ and $\Gamma(\psi_t)$ are

$$\Xi_j(\psi_t) = \mathrm{tr}\left(\check{\rho}_{\psi_t}\left(i\,[H, A_j] - \frac{1}{2}\big(A_j(L+L^\dagger)L + L^\dagger(L+L^\dagger)A_j\big)\right)\right)$$

and $\Gamma_j(\psi_t) = \sqrt{\eta}\,\mathrm{tr}\left(\check{\rho}_{\psi_t}\left(A_jL + L^\dagger A_j\right)\right)$. Let $\rho_{\psi_t} = \frac{\check{\rho}_{\psi_t}}{\mathrm{tr}(\check{\rho}_{\psi_t})}$ be the normalized approximate quantum information state. We note that only l SDEs need to be solved for ρ_{ψ_t} instead of $N^2 - 1$ for the original quantum filter.

For the sake of simplicity, we assume that the operator L is Hermitian. We can write $L = \sum_{i=1}^{n_0} \lambda_i P_i$, where $n_0 \leq N$ is the number of real distinct eigenvalues of L denoted by λ_i, and P_i are orthogonal projections. The following proposition can be shown.

Proposition 3.4.1. *Assume $l = n_0$ and $A_i = P_i$. Then the linear quantum projection filter equation* (3.24) *becomes*

$$d\psi_i(t) = \frac{1}{\mathrm{tr}(A_i)}\left(\mathrm{tr}\left(\check{\rho}_\psi\left(i\,[H, A_i]\right)\right) - 2\eta\lambda_i^2\,\mathrm{tr}(\check{\rho}_\psi A_i)dt + 2\sqrt{\eta}\lambda_i\,\mathrm{tr}(\check{\rho}_\psi A_i)dY_t\right), \quad i = 1, \ldots, l \tag{3.25}$$

Proof. Equation (3.25) follows directly from (3.24), with the fact that $g^{ij}(\psi) = \delta_{ij}\mathrm{tr}(A_iA_j)$. □

Chapter 4

Projection Filters For QND Imperfect Measurements

The projection filtering strategy has been developed in the classical case in [46, 47, 54], based on differential and information geometry tools. To our knowledge, the quantum projection filter scheme was first proposed in [40]. Later, in [55] the authors obtained the evolution of system state in a lower dimensional manifold by unsupervised learning. This was achieved by use of local tangent space alignment. In [58], a dynamical law is derived by minimizing the statistical distance in the moving basis and an equivalence with the projection filter has been shown. Recently, in [1], a quantum projection filtering approach was developed in which the dynamics is projected onto a manifold consisting of an exponential family of unnormalized density matrices. An extended Kalman filter and numerical approaches have been respectively established in [56] and [57].

In this chapter, we consider an open quantum system undergoing indirect measurement in presence of detection imperfections. Firstly by suitably choosing a submanifold of the state space, we show that the exact solution ρ_t of the quantum filter equation under QND measurement can be expressed in parametrized form as ρ_{ϕ_t}, where ϕ_t corresponds to the solution of a lower dimensional stochastic differential equation. Note that similar results have been derived for the particular case of qubit systems, with a different approach, in [85]. Then, in order to further reduce the complexity of the dynamics, i.e., to reduce the dimension of the parameter ϕ, we follow the projection approach introduced in [1], originally developed for perfect measurements. Specifically we adapt the computation of the approximation error in the case of imperfect measurements. We observe that under QND measurements, the asymptotic behavior of the approximate projection filter is compatible with the original filter, in the sense that both dynamics converge to the set of invariant subspaces. This motivates the application of a projection filter in a stabilizing feedback control law. This will be done in Chapter 5. Also, we did similar computational results to what discussed before, dealing with an exponential submanifold with less parameters.

4.1 System description

Let us consider a N-dimensional open quantum system undergoing indirect measurement in the case of homodyne detection. As mentioned in the previous chapter, the evolution of such a system is described by the following matrix-valued stochastic differential equation

$$d\rho_t = -i\left[H, \rho_t\right] dt + \mathcal{F}(\rho)dt + \sqrt{\eta}\left(L\rho_t + \rho_t L^\dagger - \mathrm{tr}\left[\left(L + L^\dagger\right)\rho_t\right]\rho_t\right) dW_t, \tag{4.1}$$

where $\mathcal{F}(\rho) = \left(L\rho_t L^\dagger - \frac{1}{2}(L^\dagger L\rho_t + \rho_t L^\dagger L)\right)$. In the following we will mainly work with the Zakai equation, which is the unnormalized form of the quantum filter equation (4.1) and which is given by

$$d\check{\rho}_t = \left(-i\left[H, \check{\rho}_t\right] + \mathcal{F}(\check{\rho}_t)\right) dt + \sqrt{\eta}\left(L\check{\rho}_t + \check{\rho}_t L^\dagger\right) dY_t. \tag{4.2}$$

In particular $\rho_t = \tilde{\rho}_t/\mathrm{tr}(\tilde{\rho}_t)$. Letting $\mathcal{A}$ be the set of all Hermitian operators on $\mathbb{C}^N$, the evolution corresponding to (4.2) takes place on the space

$$\mathcal{Q} = \{\rho \in \mathcal{A} \mid \rho \geq 0\}, \tag{4.3}$$

which is the closed subset of $\mathcal{A}$ consisting of all nonnegative Hermitian operators on $\mathbb{C}^N$. Since the vector fields defining the dynamics are linear, hence globally Lipschitz, Equation (4.2) has a unique solution [80]. Since $\rho_t = \tilde{\rho}_t/\mathrm{Tr}(\tilde{\rho}_t)$, we deduce that (4.1) has a unique solution as well. In the following, we also consider the corresponding Stratonovich form

$$d\check{\rho}_t = \Big(-i\,[H, \check{\rho}_t] + \hat{\mathcal{F}}\,(\check{\rho}_t)\Big)\,dt + \sqrt{\eta}\,\big(L\check{\rho}_t + \check{\rho}_t L^\dagger\big) \circ dY_t, \tag{4.4}$$

where $\hat{\mathcal{F}}\,(\rho) = (1-\eta)L\rho L^\dagger - \frac{\left(\eta L + L^\dagger\right)L\rho + \rho L^\dagger\left(L + \eta L^\dagger\right)}{2}$. Compared to the corresponding Itô form (4.2), the equation (4.4) is more compatible with the manifold structure of the state space (see, e.g, [82]).

4.2 Exact solution

This section can be seen in some sense as the quantum analogue of [86]. Under suitable assumptions, we construct a submanifold $\tilde{\mathcal{M}}$ of $\mathcal{A}$ such that the dynamics given by (4.4), with initial condition ρ_0, is confined to $\tilde{\mathcal{M}}$, and we express the dynamics in the corresponding coordinate system. In the following, we assume that L is Hermitian, that is $L = L^\dagger$, and that $[H, L] = 0$, which corresponds to QND measurements [87]. In this case, we can write $L = \sum_{k=1}^{K} \lambda_k P_k$ and $H = \sum_{j=1}^{D} \beta_j Q_j$, where the Hermitian operators P_k, Q_j are orthogonal projectors, that is $P_k P_l = \delta_{kl} P_l$ and $Q_k Q_l = \delta_{kl} Q_l$, satisfying $[P_k, Q_j] = 0$ for every k, j, and K and D are positive integers. Without loss of generality, we assume $K < N$ and $D < N$. This is justified by the fact that replacing L and H by $L - \lambda_K \mathbb{I}$ and $H - \beta_D \mathbb{I}$ respectively, does not affect the normalized dynamics given by (4.1).

Let $\theta \in \mathbb{R}^K$, $\gamma \in \mathbb{R}^D$, $\alpha = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{K-1,K}, \alpha_{KK}) \in \mathbb{R}^{\frac{K(K+1)}{2}}$, and $\phi = (\theta, \gamma, \alpha) \in \mathbb{R}^X$, with $X := K + D + \frac{K(K+1)}{2}$.

Now define $\check{\rho}_\phi := e^{\frac{1}{2}L_\theta + \frac{i}{2}H_\gamma} \rho_\alpha e^{\frac{1}{2}L_\theta - \frac{i}{2}H_\gamma}$ with $L_\theta = \sum_{k=1}^{K} \theta_k P_k$, $H_\gamma = \sum_{j=1}^{D} \gamma_j Q_j$, and $\rho_\alpha = \rho_0 + \sum_{1\le k\le j\le K} \left(P_k\rho_0 P_j + (1-\delta_{kj})P_j\rho_0 P_k\right)\left(e^{\alpha_{kj}} - 1\right)$. It can be easily verified that $\check{\rho}_\phi \in \mathcal{A}$. For the sake of simplicity, we assume that the set $\{\frac{\partial\check{\rho}_\phi}{\partial\phi_1}, \ldots, \frac{\partial\check{\rho}_\phi}{\partial\phi_X}\}$ is linearly independent. Then $\tilde{\mathcal{M}} := \{\check{\rho}_\phi \mid \phi \in$

$\mathbb{R}^X\}$ is locally a X-dimensional differential submanifold of $\mathcal{A}$, with tangent space given by

$$T_{\check{\rho}_\phi}\tilde{\mathcal{M}} = span\left\{\frac{\partial \check{\rho}_\phi}{\partial \phi_1}, \ldots, \frac{\partial \check{\rho}_\phi}{\partial \phi_X}\right\}. \tag{4.5}$$

A direct calculation yields

$$\frac{\partial \check{\rho}_\phi}{\partial \theta_k} = \frac{1}{2}(P_k \check{\rho}_\phi + \check{\rho}_\phi P_k), \tag{4.6}$$

$$\frac{\partial \check{\rho}_\phi}{\partial \gamma_j} = \frac{i}{2}(Q_j \check{\rho}_\phi - \check{\rho}_\phi Q_j), \tag{4.7}$$

$$\frac{\partial \check{\rho}_\phi}{\partial \alpha_{kj}} = e^{\frac{1}{2}L_\theta + \frac{i}{2}H_\gamma}(P_k \rho_0 P_j + (1-\delta_{kj})P_j \rho_0 P_k)e^{\alpha_{kj}} e^{\frac{1}{2}L_\theta - \frac{i}{2}H_\gamma}. \tag{4.8}$$

We have the following lemma, which follows by direct calculation and by using (4.6), (4.7) and (4.8).

Lemma 4.2.1. *The terms $i\,[H, \check{\rho}_\phi]$, $F\,(\check{\rho}_\phi)$ and $L\check{\rho}_\phi + \check{\rho}_\phi L$ appearing in* (4.4) *belong to the tangent space $T_{\check{\rho}_\phi}\tilde{\mathcal{M}}$. Furthermore,*

$$i\,[H, \check{\rho}_\phi] = 2\sum_j \beta_j \frac{\partial \check{\rho}_\phi}{\partial \gamma_j},$$

$$F\,(\check{\rho}_\phi) = (1-\eta)\sum_{k,j} \lambda_k \lambda_j \frac{\partial \check{\rho}_\phi}{\partial \alpha_{kj}} - (1+\eta)\sum_k \lambda_k^2 \frac{\partial \check{\rho}_\phi}{\partial \theta_k},$$

$$L\check{\rho}_\phi + \check{\rho}_\phi L = 2\sum_k \lambda_k \frac{\partial \check{\rho}_\phi}{\partial \theta_k}.$$

Now, we can establish the main result of this section.

Theorem 4.2.2. *The solution $\check{\rho}_t$ of the quantum filter equation (4.1) with initial condition ρ_0 coincides with $\check{\rho}_{\phi(t)}/\mathrm{tr}(\check{\rho}_{\phi(t)})$, where $\phi(t) = (\theta(t), \gamma(t), \alpha(t))$, with $\theta(t)$ satisfying the stochastic differential equation*

$$d\theta_k(t) = -(1+\eta)\lambda_k^2 dt + 2\sqrt{\eta}\lambda_k dY_t, \quad \theta_k(0) = 0,$$

and $\gamma_j(t) = -2\beta_j t$, $\alpha_{kj}(t) = (1-\eta)\lambda_k \lambda_j t$.

Proof. By the previous lemma, the solutions of (4.4) evolve (almost surely) on $\tilde{\mathcal{M}}$ and satisfy

$$d\check{\rho}_\phi = -2\sum_j \beta_j \frac{\partial\check{\rho}_\phi}{\partial\gamma_j}dt + (1-\eta)\sum_{k,j}\lambda_k\lambda_j\frac{\partial\check{\rho}_\phi}{\partial\alpha_{kj}}dt$$
$$-(1+\eta)\sum_k \lambda_k^2\frac{\partial\check{\rho}_\phi}{\partial\theta_k}dt + 2\sqrt{\eta}\sum_k \lambda_k\frac{\partial\check{\rho}_\phi}{\partial\theta_k}\circ dY_t. \tag{4.9}$$

On other hand, by the chain rule we have

$$d\check{\rho}_\phi = \sum_k \frac{\partial\check{\rho}_\phi}{\partial\theta_k}\circ d\theta_k + \sum_j \frac{\partial\check{\rho}_\phi}{\partial\gamma_j}\circ d\gamma_j + \sum_{k,j}\frac{\partial\check{\rho}_\phi}{\partial\alpha_{kj}}\circ d\alpha_{kj}. \tag{4.10}$$

To conclude, it is sufficient to identify the coefficients of the above equations with respect to the tangent space basis and solve the ordinary differential equations obtained for γ_j and α_{kj}. □

4.3 Exponential quantum projection filter and error analysis

4.3.1 The projection filter approach

The computation of the exact solution presented in Section 4.2 is valid under the assumption of QND measurements. In this section, we follow the projection filter approach explained in Chapter (3), which does not require the latter assumption and allows to further reduce the dimension of the system under study. This approach, as illustrated before, is mainly based on choosing an appropriate submanifold and suitably projecting the dynamics on it. This approach can be used to obtain an estimate of the quantum state [40].

The so-called exponential quantum projection filter represents an example of application of this idea [1,2]. The aim is to obtain an approximate solution $\check{\rho}_{\theta_t}$ of the above equation. A natural choice is to consider the following parametrized exponential family (see e.g., [1])

$$\check{\mathcal{M}} = \left\{\check{\rho}_\theta := e^{\frac{1}{2}\sum_{j=0}^{m-1}\theta_j A_j}\bar{\rho}_0 e^{\frac{1}{2}\sum_{j=0}^{m-1}\theta_j A_j} \mid \theta\in\mathbb{R}^m\right\}, \tag{4.11}$$

where $\bar{\rho}_0$ represents the initial condition of the approximated dynamics and can be chosen arbitrarily. In this chapter we suppose that $\bar{\rho}_0 = \rho_0$.

The operators A_j for $j = 0, 1, \ldots, m-1$ are assumed to be Hermitian $N \times N$ matrices. Locally, $\check{\mathcal{M}}$ is a m-dimensional submanifold of the space of positive semi-definite Hermitian operators in $\mathcal{H}$, provided that the set $\{\frac{\partial \check{\rho}_\theta}{\partial \theta_0}, \ldots, \frac{\partial \check{\rho}_\theta}{\partial \theta_{m-1}}\}$ is linearly independent.

The objective is to project the dynamics (4.4) onto $\check{\mathcal{M}}$ and to deduce the corresponding dynamics of the parameter θ. The latter can be obtained by noting that the chain rule for Stratonovich stochastic calculus yields

$$d\check{\rho}_\theta = \sum_{j=0}^{m-1} \check{\partial}_j \circ d\theta_j, \tag{4.12}$$

where $\check{\partial}_j := \frac{\partial \check{\rho}_\theta}{\partial \theta_j}$. If the operators A_j are mutually commuting, the above derivative can be explicitly computed as

$$\check{\partial}_j = \frac{1}{2}\left(A_j \check{\rho}_\theta + \check{\rho}_\theta A_j\right)$$

The choice of m and of the operators A_j should be done in a way that the computational complexity of the projected dynamics is significantly reduced compared to the original one, while keeping the dynamics of $\check{\rho}_t$ and $\check{\rho}_{\theta_t}$ as close as possible.

From now on, we assume that L is a Hermitian $N \times N$ matrix so that by spectral decomposition, we can write $L = \sum_{j=1}^{n_0} \lambda_j P_j$, where λ_j are the real distinct eigenvalues of L and P_j are the corresponding orthogonal projection operators, for $j = 1, \ldots, n_0$, with $n_0 \leq N$. In the following, we take

$$m = n_0, \quad A_j = P_{j+1}, \tag{4.13}$$

in the definition of $\check{\mathcal{M}}$. Furthermore, we assume from now on that $\bar{\rho}_0$ is chosen so that $\mathrm{tr}(\bar{\rho}_0 A_k) \neq 0$ for every $k = 0, \ldots, m-1$. This ensures that $\mathrm{tr}(\check{\rho}_\theta A_k) \neq 0$, as

$$\mathrm{tr}(\check{\rho}_\theta A_k) = \mathrm{tr}(\sum_{i,j=0}^{m-1} e^{\frac{\theta_i + \theta_j}{2}} A_i \bar{\rho}_0 A_j A_k) = e^{\theta_k} \mathrm{tr}(\bar{\rho}_0 A_k)$$

for every $\theta \in \mathbb{R}^m$.

As it is explained in Chapter (3), the projection filter is defined as follows

$$d\check{\rho}_{\theta_t} = \Pi_{\theta_t}\left(-i\left[H, \check{\rho}_{\theta_t}\right]\right)dt + \Pi_{\theta_t}\left(\hat{\mathcal{F}}\left(\check{\rho}_{\theta_t}\right)\right)dt + \Pi_{\theta_t}\left(\sqrt{\eta}(L\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L)\right) \circ dY_t. \tag{4.14}$$

From the definition, we deduce that

$$\begin{aligned} d\check{\rho}_{\theta_t} =& \sum_{j=0}^{m-1} \frac{1}{2}\left(A_j\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}A_j\right)\left(\frac{\operatorname{tr}\left(i\check{\rho}_{\theta_t}\left[H, A_j\right]\right)}{\operatorname{tr}\left(\check{\rho}_{\theta_t}A_j\right)}\right)dt - \\ & \eta(L^2\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L^2)dt + \sqrt{\eta}(L\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L) \circ dY_t \\ =& \sum_{j=0}^{m-1} \frac{1}{2}\left(A_j\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}A_j\right)\left(\frac{\operatorname{tr}\left(i\check{\rho}_{\theta_t}\left[H, A_j\right]\right)}{\operatorname{tr}\left(\check{\rho}_{\theta_t}A_j\right)}dt - 2\eta\lambda_{j+1}^2 dt + 2\sqrt{\eta}\lambda_{j+1} \circ dY_t\right). \end{aligned} \tag{4.15}$$

Now it is sufficient to use the relation (4.12) and (4.15) to get the following equation in Itô form

$$d\theta_t = G\left(\theta_t\right)^{-1} E\left(\theta_t\right)dt - 2\eta\alpha dt + 2\sqrt{\eta}\beta dY_t, \tag{4.16}$$

with

$$G\left(\theta_t\right) = \operatorname{diag}\left\{\operatorname{tr}\left(\check{\rho}_{\theta_t}A_1\right), \ldots, \operatorname{tr}\left(\check{\rho}_{\theta_t}A_m\right)\right\},$$

$$E\left(\theta_t\right) = \left(\operatorname{tr}\left(i\check{\rho}_{\theta_t}\left[H, A_1\right]\right), \ldots, \operatorname{tr}\left(i\check{\rho}_{\theta_t}\left[H, A_m\right]\right)\right)^T$$

$$\alpha = \left(\lambda_1^2, \ldots, \lambda_m^2\right)^T, \ \beta = \left(\lambda_1, \ldots, \lambda_m\right)^T.$$

Note that $G\left(\theta_t\right)$ is invertible since $\operatorname{tr}(\check{\rho}_\theta A_k) \neq 0$ for $k = 0, \ldots, m-1$, as explained above.

The unnormalized Itô stochastic differential equation takes the form

$$\begin{aligned} d\check{\rho}_{\theta_t} =& \frac{1}{2}\sum_{j=0}^{m-1}\left(A_j\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}A_j\right)\left(\frac{\operatorname{tr}\left(i\check{\rho}_{\theta_t}\left[H, A_j\right]\right)}{\operatorname{tr}\left(\check{\rho}_{\theta_t}A_j\right)}\right)dt + \\ & \eta\left(L\check{\rho}_{\theta_t}L - \frac{1}{2}(L^2\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L^2)\right)dt + \sqrt{\eta}(L\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L)dY_t. \end{aligned} \tag{4.17}$$

By applying Itô rules we can obtain the equation for the normalized solution of the projection filter

$\rho_{\theta_t} := \frac{\check{\rho}_{\theta_t}}{\mathrm{tr}\left(\check{\rho}_{\theta_t}\right)}$, which is given as follows

$$\begin{aligned} d\rho_{\theta_t} =& \frac{1}{2}\sum_{j=0}^{m-1}\left(A_j\rho_{\theta_t} + \rho_{\theta_t}A_j\right)\left(\frac{\mathrm{tr}\left(i\rho_{\theta_t}\left[H, A_j\right]\right)}{\mathrm{tr}\left(\rho_{\theta_t}A_j\right)}\right)dt + \\ & \eta\left(L\rho_{\theta_t}L - \frac{1}{2}(L^2\rho_{\theta_t} + \rho_{\theta_t}L^2)\right)dt + \mathcal{G}(\rho_{\theta_t})(dY_t - 2\sqrt{\eta}\,\mathrm{tr}\left(\rho_{\theta_t}L\right)dt). \end{aligned} \tag{4.18}$$

The evolution of the equation (4.18) starting from $\bar{\rho}_0$ takes place almost surely on the set

$$\mathcal{M} = \left\{\frac{\rho}{\mathrm{tr}(\rho)} \mid \rho \in \check{\mathcal{M}}\right\} \subset \mathcal{S}_N.$$

We note that only m SDEs need to be solved for ρ_{θ_t} instead of $N^2 - 1$ for the original quantum filter.

4.3.2 Error analysis

Following [1, 47], we define at each point $\check{\rho}_{\theta_t}$ the prediction residual as $\Omega(t) = -i\left[H, \check{\rho}_{\theta_t}\right] - \Pi_{\theta_t}\left(-i\left[H, \check{\rho}_{\theta_t}\right]\right)$ and the two correction residuals as

$$\begin{aligned} C_1(t) &= -F(\check{\rho}_{\theta_t}) - \Pi_{\theta_t}\left(-F(\check{\rho}_{\theta_t})\right), \\ C_2(t) &= \sqrt{\eta}\left(L\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L^\dagger\right) - \Pi_{\theta_t}\left(\sqrt{\eta}(L\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L^\dagger)\right), \end{aligned}$$

respectively. These residuals refer to the local approximation errors due to the projection of the vector fields $-i\left[H, \check{\rho}_{\theta_t}\right]$, $-F(\check{\rho}_{\theta_t})$ and $\sqrt{\eta}\left(L\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L^\dagger\right)$ into the tangent space $T_{\check{\rho}_\theta}\check{\mathcal{M}}$ at time t.

As mentioned in the previous section, for the sake of simplicity, we assume that the operator L is Hermitian. This assumption simplifies the analysis of the local errors.

Let us set $X_0 := -i[H, \rho_0]$. We have the following result.

Proposition 4.3.1. *Assume $m = n_0$ and $A_i = P_{i+1}$ as before. Then, the correction residuals are*

$$C_1(t) = (1-\eta)L\check{\rho}_{\theta_t}L + \frac{\eta - 1}{2}(L^2\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L^2)$$

and $C_2(t) = 0\ \forall t \geq 0$.

Moreover, if $[H,L]=0$, then the exponential quantum projection filter equation (4.16) *becomes*

$$d\theta_i(t) = -2\eta\lambda_i^2 dt + 2\sqrt{\eta}\lambda_i dY_t, \quad i=1,\ldots,m \tag{4.19}$$

and the prediction residual $\Omega(t)$ is given by

$$\Omega(t) = e^{\frac{1}{2}\sum_{i=1}^m \theta_i(t)A_i} X_0 e^{\frac{1}{2}\sum_{i=1}^m \theta_i(t)A_i}, t \geq 0. \tag{4.20}$$

Proof. By definitions of $C_1(t)$ and $C_2(t)$, one has

$$\begin{aligned}
C_1(t) &= \Pi_{\theta_t}\left(F(\check{\rho}_{\theta_t})\right) - F(\check{\rho}_{\theta_t}) \\
&= (1-\eta)L\check{\rho}_{\theta_t}L + \frac{\eta-1}{2}(L^2\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L^2), \\
C_2(t) &= \Pi_{\theta_t}\Big(\sqrt{\eta}(L\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L)\Big) - \Big(\sqrt{\eta}(L\check{\rho}_{\theta_t} + \check{\rho}_{\theta_t}L)\Big) \\
&= \sum_{k=1}^m 2\sqrt{\eta}\lambda_k \Big(\Pi_{\theta_t}(\check{\partial}_k) - \check{\partial}_k\Big) = 0.
\end{aligned}$$

Equations (4.19)-(4.20) follow from the fact that $[H,L]=0$.

□

Let $\mathbb{P}$ denote the original probability measure under which W_t is a Wiener process. By Girsanov theorem, there exists an equivalent probability measure $\mathbb{P}'$ such that Y_t in (4.1) becomes a Wiener process. Let $\mathbb{E}'$ denote the expectation with respect to the measure $\mathbb{P}'$.

To measure the gap between the filter state and its approximation, we consider the average total residual norm defined as

$$e_t := \mathbb{E}'\left(\|C_1(t)\|_F + \|C_2(t)\|_F + \|\Omega(t)\|_F\right), \tag{4.21}$$

with $e_0 = 0$. Also, set $Y_0 := (1-\eta)L\rho_0 L - \frac{(1-\eta)}{2}(L^2\rho_0 + \rho_0 L^2)$. We now state the main result of this section.

Theorem 4.3.2. *Let the assumptions of Proposition 4.3.1 hold true. If* $[H, L] = 0$, *then*

$$e_t \leq \sqrt{\mathrm{tr}(Y_0^2)} + \sqrt{\mathrm{tr}(X_0^2)}. \tag{4.22}$$

Proof. Let us firstly note that

$$e_t = \mathbb{E}' \left(\|C_1(t)\|_F + \|\Omega(t)\|_F \right) = \mathbb{E}' \|C_1(t)\|_F + \mathbb{E}' \|\Omega(t)\|_F \, .$$

Now, let $\Delta(t) := \frac{1}{2}\sum_{i=1}^{m} \theta_i(t) A_i$. We have

$$\begin{aligned} C_1(t) &= (1-\eta) L \check{\rho}_{\theta_t} L + \frac{\eta - 1}{2}(L^2 \check{\rho}_{\theta_t} + \check{\rho}_{\theta_t} L^2) \\ &= e^{\Delta(t)} Y_0 e^{\Delta(t)}. \end{aligned} \tag{4.23}$$

By using Lemma A.1 we get

$$\begin{aligned} \mathbb{E}' \|C_1(t)\|_F = \mathbb{E}' \sqrt{\mathrm{tr}(C_1(t)^2)} &= \mathbb{E}' \sqrt{\mathrm{tr}\left(e^{2\Lambda(t)} Y_0 e^{2\Lambda(t)} Y_0\right)} \\ \leq& \mathbb{E}' \sqrt{\sum_{j=0}^{m-1} s_j \left(e^{2\Lambda(t)} Y_0 e^{2\Lambda(t)} Y_0\right)} \\ \leq& \mathbb{E}' \sqrt{\sum_{j=0}^{m-1} s_j \left(e^{2\Lambda(t)} Y_0\right) s_j \left(e^{2\Lambda(t)} Y_0\right)} \\ =& \mathbb{E}' \sqrt{\sum_{j=0}^{m-1} s_j^2 \left(e^{2\Lambda(t)} Y_0\right)} \\ \leq& \mathbb{E}' \sqrt{\sum_{j=0}^{m-1} s_j^2 \left(e^{2\Lambda(t)}\right) s_j^2 \left(Y_0\right)} \\ \leq& \mathbb{E}' \sqrt{\sum_{j=0}^{m-1} s_j^2 \left(Y_0\right) s_1 \left(e^{2\Lambda(t)}\right)} = \mathbb{E}' \sqrt{\mathrm{tr}\left(Y_0^2\right)} \max_j e^{\theta_j(t)}, \end{aligned} \tag{4.24}$$

where $\max_i \mathbb{E}' e^{\theta_i(t)} = \max_i e^{\theta_i(0)} = 1$. This comes from the fact that $e^{\theta_i(t)}$ is a martingale with respect to $\mathbb{P}'$.

Similarly, we get

$$\mathbb{E}' \|\Omega(t)\|_F = \mathbb{E}' \sqrt{\operatorname{tr}(\Omega(t)^2)} \leq \sqrt{\operatorname{tr}(X_0^2)}. \tag{4.25}$$

Adding up (4.24) and (4.25), we obtain the inequality (4.22). □

Under some additional conditions, Theorem 4.3.2 leads to an equivalence between the exponential quantum projection filter equation (4.14) and the quantum filter equation (4.2).

Corollary 4.3.3. *Let the assumptions of Proposition 4.3.1 hold true and assume in addition that* $[H, L] = [H, \rho_0] = [L, \rho_0] = 0$. *Then* $\check{\rho}_t \equiv \check{\rho}_{\theta_t}$.

4.3.3 Quantum state reduction

Under the QND assumption $[H, L] = 0$, the normalized evolution of the quantum projection filter $\rho_{\theta_t} = \frac{\check{\rho}_{\theta_t}}{\operatorname{tr}(\check{\rho}_{\theta_t})}$ can be written as

$$d\rho_{\theta_t} = \eta \left(L\rho_{\theta_t} L - \frac{L^2 \rho_{\theta_t}}{2} - \frac{\rho_{\theta_t} L^2}{2} \right) dt + \sqrt{\eta} \left(L\rho_{\theta_t} + \rho_{\theta_t} L - 2 \operatorname{tr}(L\rho_{\theta_t}) \rho_{\theta_t} \right) d\hat{W}_t, \tag{4.26}$$

where $d\hat{W}_t = dY_t - 2\sqrt{\eta} \operatorname{tr}(L\rho_{\theta_t}) dt$. As in the previous section, let us write $L = \sum_{i=1}^{n_0} \lambda_i P_i$, where $n_0 \leq N$ is the number of real distinct eigenvalues of L denoted by λ_i, and P_i are orthogonal projections. The following result states that the quantum state reduction phenomenon occurs for both the evolutions given by (4.1) and by (4.26); it can be obtained by following standard stochastic LaSalle-type arguments similarly to [19], using the Lyapunov function $V(\rho) = \operatorname{tr}(L^2\rho) - \operatorname{tr}^2(L\rho)$.

Theorem 4.3.4. *For every initial condition* $\rho_0 \in S$, *the solution* ρ_{θ_t} *of* (4.26) *converge a.s. as* $t \to \infty$ *to one of the subsets* $\{\rho \in S \mid P_k \rho = \rho\}$, *for* $k = 1, \ldots, n_0$. *The same property holds true for the solution* ρ_t *of* (4.1).

Note that the previous result shows that the solutions of (4.1) and (4.26) share a similar asymptotic behavior, but it does not guarantee that such solutions converge almost surely to the same limit. The results obtained in [3, 45, 88] suggest that such limits coincide. It is then natural to expect that a feedback control depending on the quantum projection filter may be used to stabilize the system towards a chosen eigenstate of L, similarly to what was done in, e.g., [19, 20].

4.4 Exponential quantum projection filter with less parameters

In this section, we are going to present the exact solution and the projection filter for exponential manifolds with two parameters. In fact, we will show similar results as in sections (4.2) and (4.3) with less parameters (two parameters).

4.4.1 Exact solution with two parameters

In this section, we assume that L is Hermitian, that is $L = L^\dagger$, and that $[H, L] = 0$, which corresponds to QND measurements [87]. Under these assumptions, we construct a submanifold $\mathcal{N}$ of $\mathcal{A}$ such that the dynamics given by (4.4), with initial condition ρ_0, is confined to $\mathcal{N}$, and we express the dynamics in the corresponding coordinate system.
Now define $\check{\rho}_\gamma := e^{-iH\gamma_1 - \frac{\eta+1}{2}L^2\gamma_1 + \sqrt{\eta}L\gamma_2}\rho_\beta e^{+iH\gamma_1 - \frac{\eta+1}{2}L^2\gamma_1 + \sqrt{\eta}L\gamma_2}$ with $L = \sum_{k=1}^{2} \lambda_k P_k$, and $\rho_\beta = \rho_0 + \sum_{1 \le k \le j \le K} \left(P_k \rho_0 P_j + (1 - \delta_{kj}) P_j \rho_0 P_k\right) \left(e^{(1-\eta)\lambda_k \lambda_j \gamma_1} - 1\right)$, where $\gamma \in \mathbb{R}^2$. It can be easily verified that $\check{\rho}_\gamma \in \mathcal{A}$. For the sake of simplicity, we assume that the set $\{\frac{\partial \check{\rho}_\gamma}{\partial \gamma_1}, \frac{\partial \check{\rho}_\gamma}{\partial \gamma_2}\}$ is linearly independent. Then $\mathcal{N} := \{\check{\rho}_\gamma \mid \gamma \in \mathbb{R}^2\}$ is locally a 2-dimensional differential submanifold of $\mathcal{A}$, with tangent space given by

$$T_{\check{\rho}_\gamma}\mathcal{N} = span\left\{\frac{\partial \check{\rho}_\gamma}{\partial \gamma_1}, \frac{\partial \check{\rho}_\gamma}{\partial \gamma_2}\right\}. \tag{4.27}$$

We have the following lemma, which follows by direct calculation and by using (4.6), (4.7) and (4.8).

Lemma 4.4.1. *The terms* $i\,[H, \check{\rho}_\gamma]$, $F\left(\check{\rho}_\gamma\right)$ *and* $L\check{\rho}_\gamma + \check{\rho}_\gamma L$ *appearing in* (4.4) *belong to the tangent* *space $T_{\check{\rho}_\gamma}\mathcal{N}$. Furthermore,*

$$-i\left[H,\check{\rho}_\gamma\right]+\hat{\mathcal{F}}\left(\check{\rho}_\gamma\right)=\frac{\partial\check{\rho}_\gamma}{\partial\gamma_1},$$
$$\sqrt{\eta}\left(L\check{\rho}_\gamma+\check{\rho}_\gamma L\right)=\frac{\partial\check{\rho}_\gamma}{\partial\gamma_2}$$

where

$$\hat{\mathcal{F}}\left(\rho\right)=(1-\eta)L\rho L-(L^2\rho+\rho L^2).$$

Now, we can give the main result of this section.

Theorem 4.4.2. *The solution $\check{\rho}_t$ of the quantum filter equation (4.1) with initial condition ρ_0 coincides with $\check{\rho}_{\gamma(t)}/\mathrm{tr}(\check{\rho}_{\gamma(t)})$, where $\gamma(t)=(\gamma_1(t),\gamma_2(t))$, with $\gamma_1(t)=t$ and $\gamma_2(t)=Y_t$.*

Proof. By the previous lemma, the solutions of (4.4) evolve (almost surely) on $\mathcal{N}$ and satisfy

$$d\check{\rho}_\gamma=\frac{\partial\check{\rho}_\gamma}{\partial\gamma_1}dt+\frac{\partial\check{\rho}_\gamma}{\partial\gamma_2}\circ dY_t. \tag{4.28}$$

On other hand, by the chain rule we have

$$d\check{\rho}_\gamma=\frac{\partial\check{\rho}_\gamma}{\partial\gamma_1}\circ d\gamma_1+\frac{\partial\check{\rho}_\gamma}{\partial\gamma_2}\circ d\gamma_2. \tag{4.29}$$

To conclude, it is sufficient to identify the coefficients of the above equations with respect to the tangent space basis and solve the ordinary differential equations obtained for γ_1 and γ_2. □

4.4.2 Exponential projection approach

In this section, we also follow the projection filter approach illustrated in Chapter (3) that allows more dimensional reduction of the system than the previous section. For simplicity, in this section we take $\eta=1$, $L=L^\dagger$ and $H=H_0+uH_1=L+uH_1$. We consider the subset $\mathscr{E}$ of $\mathcal{Q}$ consisting of an exponential family of unnormalized quantum density operators

$$\mathscr{E} = \{\check{\rho}_\omega \mid \omega = (\omega_1, \omega_2) \in \Phi \subset \mathbb{R}^2\}.$$

Here $\check{\rho}_\omega = e^{-iH_0\omega_1 - L^2\omega_1 + L\omega_2}\rho_0 e^{+iH_0\omega_1 - L^2\omega_1 + L\omega_2}$.

In fact, if $H = H_0 = L$ (or $[H_0, L] = 0$) then the manifold $\mathcal{E}$ is invariant for the dynamics. This suggests the use of $\mathcal{E}$ for projecting.

Assuming that the set $\{\frac{\partial \check{\rho}_\omega}{\partial \omega_1}, \frac{\partial \check{\rho}_\omega}{\partial \omega_2}\}$ is linearly independent, we obtain that $\mathscr{E}$ is, locally, a 2-dimensional differential submanifold of $\mathcal{Q}$. The tangent space at some $\check{\rho}_\omega \in \mathscr{E}$ is given by $T_{\check{\rho}_\omega}\mathscr{E} = \mathrm{span}\{\check{\partial}_i, i = 1, 2\}$, where $\check{\partial}_i = \frac{\partial \check{\rho}_\omega}{\partial \omega_i}$. A direct calculation yields:

$$\frac{\partial \check{\rho}_\omega}{\partial \omega_1} = -i\,[H_0, \check{\rho}_\omega] - (L^2\check{\rho}_\omega + \check{\rho}_\omega L^2) \tag{4.30}$$

$$\frac{\partial \check{\rho}_\omega}{\partial \omega_2} = (L\check{\rho}_\omega + \check{\rho}_\omega L) \tag{4.31}$$

In analogy with (3.6) we define a Riemannian metric on $\mathscr{E}$ whose components are real-valued functions of ω:

$$g_{ij}(\omega) = \left\langle \check{\partial}_i, \check{\partial}_j \right\rangle_{\check{\rho}_\omega}. \tag{4.32}$$

Then we get,

$$G(\omega) = \begin{pmatrix} 2\,\mathrm{tr}[L^2\check{\rho}_\omega^2 - (L\check{\rho}_\omega)^2 + (L^2\check{\rho}_\omega)^2 + L^4\check{\rho}_\omega^2] & -2\,\mathrm{tr}[L\check{\rho}_\omega L^2\check{\rho}_\omega + L^3\check{\rho}_\omega^2] \\ -2\,\mathrm{tr}[L\check{\rho}_\omega L^2\check{\rho}_\omega + L^3\check{\rho}_\omega^2] & 2\,\mathrm{tr}[(L\check{\rho}_\omega)^2 + L^2\check{\rho}_\omega^2] \end{pmatrix}$$

and so

$$G^{-1}(\omega) = \alpha \begin{pmatrix} 2\,\mathrm{tr}[(L\check{\rho}_\omega)^2 + L^2\check{\rho}_\omega^2] & 2\,\mathrm{tr}[L\check{\rho}_\omega L^2\check{\rho}_\omega + L^3\check{\rho}_\omega^2] \\ 2\,\mathrm{tr}[L\check{\rho}_\omega L^2\check{\rho}_\omega + L^3\check{\rho}_\omega^2] & 2\,\mathrm{tr}[L^2\check{\rho}_\omega^2 - (L\check{\rho}_\omega)^2 + (L^2\check{\rho}_\omega)^2 + L^4\check{\rho}_\omega^2] \end{pmatrix}$$

where $\alpha = \frac{1}{detG(\omega)}$.

The matrix $G(\omega) := (g_{ij}(\omega))_{i,j=1,2}$ is a quantum Fisher information matrix. Then, for every $\omega \in \Phi$, we can define an orthogonal projection operation Π_ω by

$$\begin{aligned}\Pi_\omega : \mathcal{A} &\longrightarrow T_{\check{\rho}_\omega}\mathscr{E} \\ x &\longmapsto \sum_{i=1}^{2}\sum_{j=1}^{2} g^{ij}(\omega)\langle\langle x, \check{\partial}_j\rangle\rangle_{\check{\rho}_\omega}\check{\partial}_i,\end{aligned} \tag{4.33}$$

where the $g^{ij}(\omega)$ are the components of the inverse of the quantum information matrix $G(\omega)$. We define the quantum projection filter on $\mathscr{E}$ at time t by

$$\begin{aligned}d\check{\rho}_{\omega_t} = \Pi_{\omega_t}\left(-i\left[H, \check{\rho}_{\omega_t}\right]\right)dt + \Pi_{\omega_t}\left(-F\left(\check{\rho}_{\omega_t}\right)\right)dt \\ +\Pi_{\omega_t}\left(\left(L\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t}L\right)\right) \circ dY_t.\end{aligned} \tag{4.34}$$

By direct calculation we get,

$$\begin{aligned}\Pi(-i[H, \check{\rho}_{\omega_t}]) = 2\alpha\Bigg\{ \Bigg(&\mathrm{tr}\left[(L\check{\rho}_{\omega_t})^2 + L^2\check{\rho}_{\omega_t}^2\right]\mathrm{tr}\Big[2(L^2\check{\rho}_{\omega_t}^2 - (L\check{\rho}_{\omega_t})^2) \\ &+ (-2uH_1\check{\rho}_{\omega_t}L\check{\rho}_{\omega_t} + uH_1L\check{\rho}_{\omega_t}^2 + uLH_1\check{\rho}_{\omega_t}^2) - iu[H_1, L^2]\check{\rho}_{\omega_t}^2\Big]\Bigg) \\ &\times\left(-i\left[H_0, \check{\rho}_{\omega_t}\right] - (L^2\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t}L^2)\right) \\ &+ \Bigg(-i\,\mathrm{tr}\left[L\check{\rho}_{\omega_t}L^2\check{\rho}_{\omega_t} + L^3\check{\rho}_{\omega_t}^2\right]\mathrm{tr}\left[[L, uH_1]\check{\rho}_{\omega_t}^2\right]\Bigg) \\ &\times\left(-i\left[H_0, \check{\rho}_{\omega_t}\right] - (L^2\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t}L^2)\right) \\ &+ \Bigg(\mathrm{tr}\left[L\check{\rho}_{\omega_t}L^2\check{\rho}_{\omega_t} + L^3\check{\rho}_{\omega_t}^2\right]\mathrm{tr}\Big[2(L^2\check{\rho}_{\omega_t}^2 - (L\check{\rho}_{\omega_t})^2) \\ &+ (-2uH_1\check{\rho}_{\omega_t}L\check{\rho}_{\omega_t} + uH_1L\check{\rho}_{\omega_t}^2 + uLH_1\check{\rho}_{\omega_t}^2) - iu[H_1, L^2]\check{\rho}_{\omega_t}^2\Big]\Bigg) \\ &\times\left(L\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t}L\right) \\ &+ \Bigg(-i\,\mathrm{tr}\left[L^2\check{\rho}_{\omega_t}^2 - (L\check{\rho}_{\omega_t})^2 + (L^2\check{\rho}_{\omega_t})^2 + L^4\check{\rho}_{\omega_t}^2\right]\mathrm{tr}\left[[L, uH_1]\check{\rho}_{\omega_t}^2\right]\Bigg)\end{aligned}$$

$$
\begin{aligned}
&\times (L\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t} L)\Bigg\}, \\
\Pi\left(-(L^2\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t} L^2)\right) = 4\alpha\Bigg\{ &\Bigg(\operatorname{tr}\left[(L\check{\rho}_{\omega_t})^2 + L^2\check{\rho}^2_{\omega_t}\right] \operatorname{tr}\left[(L^2\check{\rho}_{\omega_t})^2 + L^4\check{\rho}^2_{\omega_t}\right]\Bigg) \\
&\times (-i\,[H_0, \check{\rho}_{\omega_t}] - (L^2\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t} L^2)) \\
&+ \Bigg(-\operatorname{tr}\left[L\check{\rho}_{\omega_t} L^2\check{\rho}_{\omega_t} + L^3\check{\rho}^2_{\omega_t}\right] \operatorname{tr}\left[L\check{\rho}_{\omega_t} L^2\check{\rho}_{\omega_t} + L^3\check{\rho}^2_{\omega_t}\right]\Bigg) \\
&\times (-i\,[H_0, \check{\rho}_{\omega_t}] - (L^2\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t} L^2)) \\
&+ \Bigg(\operatorname{tr}\left[L\check{\rho}_{\omega_t} L^2\check{\rho}_{\omega_t} + L^3\check{\rho}^2_{\omega_t}\right] \operatorname{tr}\left[(L^2\check{\rho}_{\omega_t})^2 + L^4\check{\rho}^2_{\omega_t}\right]\Bigg) \\
&\times (L\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t} L) \\
&+ \Bigg(-\operatorname{tr}\left[L^2\check{\rho}^2_{\omega_t} - (L\check{\rho}_{\omega_t})^2 + (L^2\check{\rho}_{\omega_t})^2 + L^4\check{\rho}^2_{\omega_t}\right] \operatorname{tr}\left[L\check{\rho}_{\omega_t} L^2\check{\rho}_{\omega_t} + L^3\check{\rho}^2_{\omega_t}\right]\Bigg) \\
&\times (L\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t} L)\Bigg\}, \\
\Pi(L\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t} L) = {} & L\check{\rho}_{\omega_t} + \check{\rho}_{\omega_t} L.
\end{aligned}
$$

Since the vector fields regulating the dynamics are everywhere tangent to $\mathscr{E}$, the solution of the previous equation is a well-defined stochastic process $\check{\rho}_{\omega_t}$ on $\mathscr{E}$, whenever $\check{\rho}_{\omega_0} = \rho_0$ belongs to $\mathscr{E}$. By using the orthogonal projection operation and the chain rule

$$
d\check{\rho}_\omega = \frac{\partial \check{\rho}_\omega}{\partial \omega_1} \circ d\omega_1 + \frac{\partial \check{\rho}_\omega}{\partial \omega_2} \circ d\omega_2, \tag{4.35}
$$

we can easily express the dynamics of the parameter $\omega_t = (\omega_1(t), \omega_2(t))'$ as

$$
\begin{aligned}
d\omega_1(t) = 2\alpha\Bigg\{ \Bigg(&\operatorname{tr}\left[(L\check{\rho}_{\omega_t})^2 + L^2\check{\rho}^2_{\omega_t}\right] \times \\
&\operatorname{tr}\left[(-2uH_1\check{\rho}_{\omega_t} L\check{\rho}_{\omega_t} + uH_1 L\check{\rho}^2_{\omega_t} + uLH_1\check{\rho}^2_{\omega_t}) - iu[H_1, L^2]\check{\rho}^2_{\omega_t}\right]\Bigg) \\
&+ \Bigg(-i\operatorname{tr}\left[L\check{\rho}_{\omega_t} L^2\check{\rho}_{\omega_t} + L^3\check{\rho}^2_{\omega_t}\right] \operatorname{tr}\left[[L, uH_1]\check{\rho}^2_{\omega_t}\right]\Bigg)\Bigg\} dt + dt
\end{aligned}
$$

$$d\omega_2(t) = 2\alpha\Bigg\{\Bigg(\operatorname{tr}\left[L\check{\rho}_{\omega_t}L^2\check{\rho}_{\omega_t} + L^3\check{\rho}^2_{\omega_t}\right]\times$$
$$\operatorname{tr}\left[(-2uH_1\check{\rho}_{\omega_t}L\check{\rho}_{\omega_t} + uH_1L\check{\rho}^2_{\omega_t} + uLH_1\check{\rho}^2_{\omega_t}) - iu[H_1, L^2]\check{\rho}^2_{\omega_t}\right]\Bigg)$$
$$+\Bigg(-i\operatorname{tr}\left[L^2\check{\rho}^2_{\omega_t} - (L\check{\rho}_{\omega_t})^2 + (L^2\check{\rho}_{\omega_t})^2 + L^4\check{\rho}_{\omega_t}\right]\operatorname{tr}\left[[L, uH_1]\check{\rho}^2_{\omega_t}\right]\Bigg)\Bigg\}dt + dY_t.$$

Let $\rho_{\omega_t} = \frac{\check{\rho}_{\omega_t}}{\operatorname{tr}(\check{\rho}_{\omega_t})}$ be the normalized approximate quantum information state. Then the normalized evolution of the quantum projection filter can be written as

$$d\rho_{\omega_t} = \Big(-2\rho_{\omega_t}\operatorname{tr}(L\rho_{\omega_t}) + (L\rho_{\omega_t} + \rho_{\omega_t}L)\Big)(d\omega_2(t) - dY_t)$$
$$+\Big(2\rho_{\omega_t}\operatorname{tr}(L^2\rho_{\omega_t}) - i[H_0, \rho_{\omega_t}] - (L^2\rho_{\omega_t} + \rho_{\omega_t}L^2)\Big)d\omega_1(t)$$
$$-2\rho_{\omega_t}\operatorname{tr}(L^2\rho_{\omega_t})dt + (L\rho_{\omega_t}L + \frac{1}{2}(L^2\rho_{\omega_t} + \rho_{\omega_t}L^2))dt$$
$$+(L\rho_{\omega_t} + \rho_{\omega_t}L - 2\rho_{\omega_t}\operatorname{tr}(L\rho_{\omega_t}))(dY_t - 2\operatorname{tr}(L\rho_{\omega_t})dt).$$

We note that only 2 SDEs need to be solved for ρ_{ω_t} instead of $N^2 - 1$ for the original quantum filter.

Remark 1: When $[H, L] = 0$, the normalized evolution of the quantum projection filter $\rho_{\omega_t} = \frac{\check{\rho}_{\omega_t}}{\operatorname{tr}(\check{\rho}_{\omega_t})}$ can be written as

$$d\rho_{\omega_t} = -i[H_0, \rho_{\omega_t}] + \left(L\rho_{\omega t}L - \frac{L^2\rho_{\omega_t}}{2} - \frac{\rho_{\omega_t}L^2}{2}\right)dt$$
$$+\left(L\rho_{\omega_t} + \rho_{\omega_t}L - 2\operatorname{tr}(L\rho_{\omega_t})\rho_{\omega_t}\right)d\bar{W}_t, \quad (4.36)$$

where $d\bar{W}_t = dY_t - 2\operatorname{tr}(L\rho_{\omega_t})dt$. This shows that when $[H, L] = 0$, we have an exact solution as explained before.

Remark 2: The goal of the exponential quantum projection filter with less parameters section is to use this approximate filter in feedback stabilization problem specifically in the case of QND measurements. This will be done in the near future work. The idea of feedback stabilization will be explained in details in the next chapter.

4.5 Numerical simulations

4.5.1 A spin-$\frac{1}{2}$ system

Here we present simulation results for the simple case of a spin-$\frac{1}{2}$ system. For a two-level quantum system, ρ can be uniquely characterized by the Bloch sphere coordinates (x, y, z) as $\rho = \frac{1}{2}\left(\begin{smallmatrix} 1+z & x-iy \\ x+iy & 1-z \end{smallmatrix}\right)$. The vector (x, y, z) belongs to the ball $B(\mathbb{R}^3) := \{(x, y, z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$. We take $H = \frac{\omega_{eg}}{2}\sigma_z$ and $L = \frac{\sqrt{M}}{2}\sigma_z$, where w_{eg} and $M > 0$ are physical parameters.

It can be verified that the dynamics in the Bloch sphere coordinates are given by

$$\begin{cases} dx_t = \left(-\frac{M}{2}x_t - \omega_{eg}y_t\right)dt - \sqrt{\eta M}x_t z_t dW(t) \\ dy_t = \left(\omega_{eg}x_t - \frac{M}{2}y_t\right)dt - \sqrt{\eta M}y_t z_t dW(t) \\ dz_t = \sqrt{\eta M}\left(1 - z_t^2\right)dW(t) \end{cases} \tag{4.37}$$

The operator L can be written as $L = \lambda_1 P_1 + \lambda_2 P_2$, where $\lambda_1 = \frac{\sqrt{M}}{2}$ and $\lambda_2 = -\frac{\sqrt{M}}{2}$, $P_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, and $P_2 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. We note that $dY(t) = dW(t) + \sqrt{\eta M}\mathrm{tr}(\sigma_z\rho_t)$ is used to drive the exponential quantum projection filter. Here, the matrices σ_x, σ_y, and σ_z correspond to the Pauli matrices. We take $t \in [0, T]$ with $T = 5$, and step size $\delta = 2^{-12}T$. Also, $w_{eg} = 1$, $\eta = 0.5$, $M = 1$, $\alpha = 7.61$, $\beta = 5$, and $\gamma = 10$. The initial state is $\rho_0 = \bar{\rho}_0 = (-1, 0, 0)$. Figure 4.1 shows the Frobenius norm of the difference between the solution ρ_t of (4.37) and the corresponding solution ρ_{θ_t} of the exponential quantum projection filter.

4.5.2 Discussion on the error in the presence of a feedback

Our goal is to study whether the approach developed in the previous sections remains effective in the presence of a controlled Hamiltonian. In particular, we wonder whether the quantum projection filter is a good candidate to replace the original filter in the stabilizing feedback law introduced in [89]. In that paper, the dynamics of a controlled spin-$\frac{1}{2}$ generalizing the dynamics

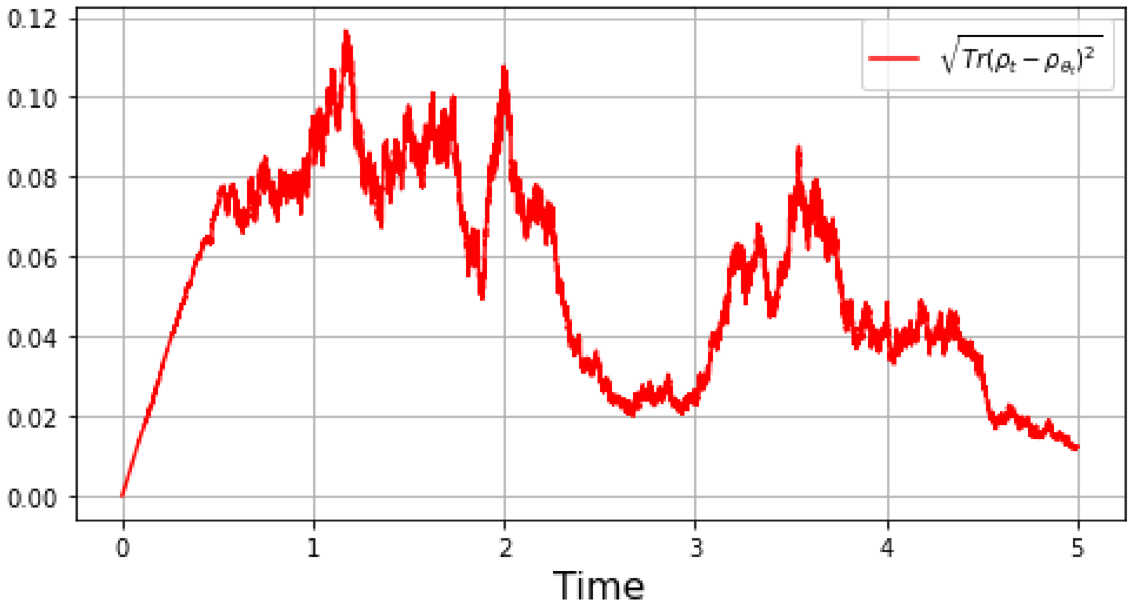

Figure 4.1: Approximation error between the quantum filter and the quantum projection filter.

(4.37) in presence of a control law u_t takes the following form

$$\begin{cases} dx_t = \left(-\frac{M}{2}x_t - \omega_{eg}y_t + u_t z_t\right)dt - \sqrt{\eta M}x_t z_t dW(t) \\ dy_t = \left(\omega_{eg}x_t - \frac{M}{2}y_t\right)dt - \sqrt{\eta M}y_t z_t dW(t) \\ dz_t = -u_t x_t dt + \sqrt{\eta M}\left(1 - z_t^2\right)dW(t) \end{cases}$$

In [89], a feedback controller $u_t = u(\rho_t)$ is applied to stabilize the above system towards the excited state ρ_e corresponding to the Bloch sphere coordinates $(0, 0, -1)$. The feedback takes the form

$$u(\rho) = \alpha[V(\rho)]^\beta - \gamma \mathrm{tr}(i[\sigma_y, \rho]\rho_e), \tag{4.38}$$

where $V(\rho) = \sqrt{1 - \mathrm{tr}(\rho\rho_e)}$, with $\alpha > 0$, $\beta \geq 0$, and $\gamma \geq 1$.

Here we assume that the feedback law (4.38) is evaluated at ρ_θ instead of ρ and we study numerically the stabilization towards the excited state. The simulation parameters are the same as before. The validity of the proposed approximation filtering scheme is checked through the Frobenius norm of the difference between ρ_t and ρ_{θ_t} in Figure 4.2. Figure 4.3 shows the convergence towards the target state. Figure 4.4 comparing the probabilities that the system is in

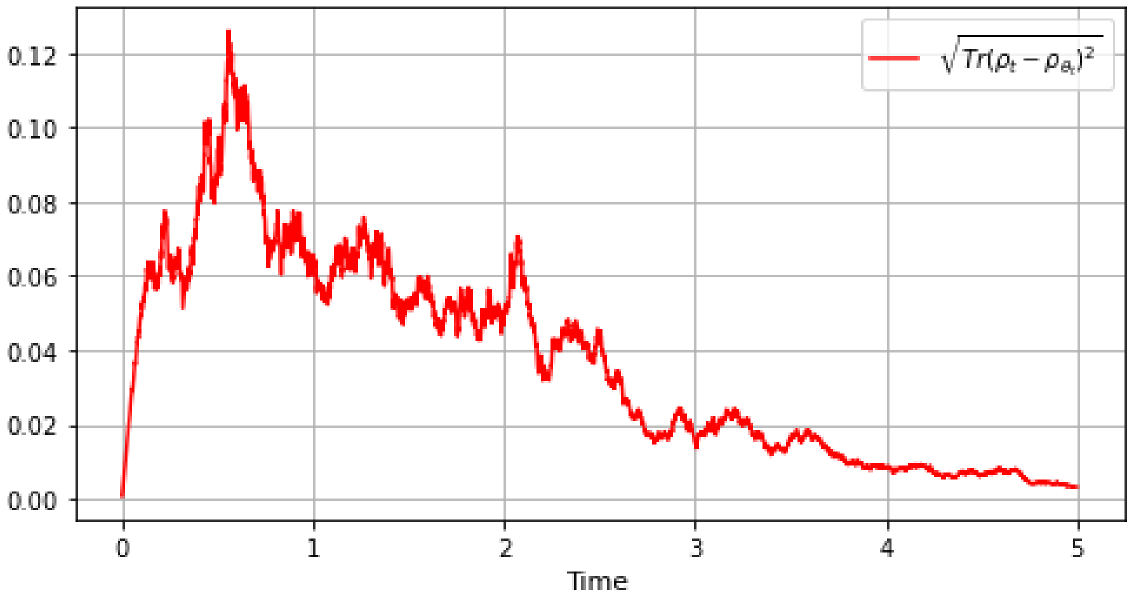

Figure 4.2: Approximation error between the quantum filter and the quantum projection filter in presence of a feedback control based on the projection filter.

the excited state, calculated using the quantum filter equation and the quantum projection filter equation, respectively.

The stabilization problem will be studied in details in the next chapter.

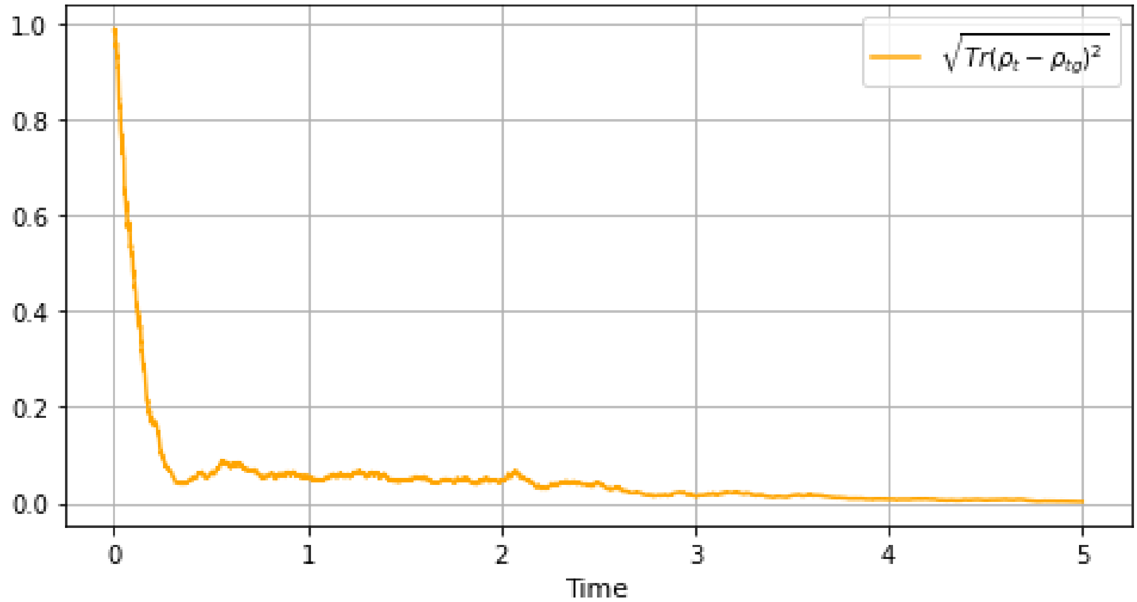

Figure 4.3: Convergence of the quantum filter to the target state ρ_e by applying a feedback control based on the projection filter.

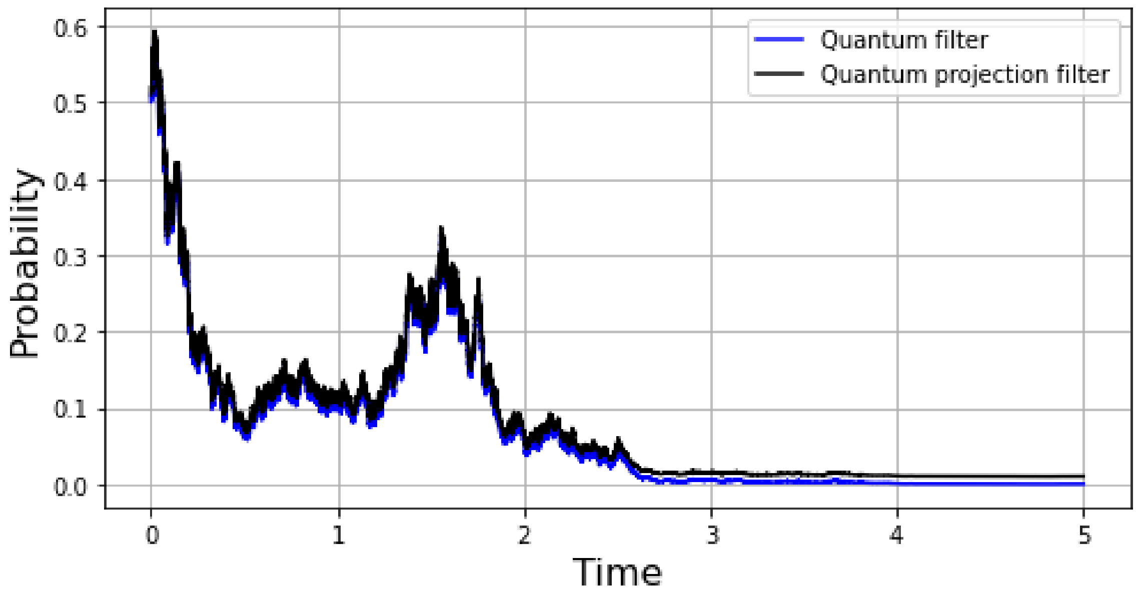

Figure 4.4: Trajectories of the quantum filter and the quantum projection filter respectively.

Chapter 5

Feedback Stabilization Via A Quantum Projection Filter

The projection filter approach may be particularly interesting in order to design feedback controls implementable in real-time, see e.g., [57] where numerical studies have been realized concerning the use of an approximate filter in feedback design. Numerous studies focused on the control of open quantum systems to achieve the preparation of pure states. These investigations are crucial for advancing quantum technologies. The pioneering work [41] addressed the problem of designing a quantum feedback controller that globally stabilizes a quantum spin-$\frac{1}{2}$ system toward an eigenstate of the measurement operator σ_z, even when imperfect measurements are involved. The controller was developed using numerical methods to find an appropriate global Lyapunov function. Subsequently, in [19], the authors employed stochastic Lyapunov techniques to analyze the stochastic flow and constructed a switching control law to stabilize N-level quantum angular momentum system toward an eigenstate of the measurement operator. The recent works [20, 89] combined local stochastic stability analysis with the support theorem to establish exponential stabilization results for spin-$\frac{1}{2}$ and spin-J systems toward stationary states of the open-loop dynamics. Unlike previous approaches based on the LaSalle method, their techniques allowed to estimate the convergence rate to the target state. This estimation is essential for practical use in quantum information processing. In [44, 90], the authors present an exponential stabilization using a noise-assisted feedback. We also refer to [91], where an exponential feedback stabilization approach in expectation toward invariant subspaces of the evolution for generic measurement has been developed.

In this chapter, we apply a projection filter approach for open quantum systems undergoing indirect measurement, in presence of detection imperfections and unknown initial state. To define a projection filter, we make use, as in [1, 2], of a parametrized exponential family. We then tackle the feedback stabilization problem for a system describing the evolution of a $N-$level quantum angular momentum system undergoing imperfect QND measurements, assuming that the feedback depends on the projection filter. In this case, we deal with a coupled stochastic equation consisting of the quantum filter and its corresponding projection filter with an arbitrary fixed initial state. To tackle the problem of feedback stabilization in this case, we first show that the analysis done in [3] can be adapted to such framework. Then, in view of reducing the computational complexity of the real-time implementation of the feedback, we introduce a new lower-dimensional parametrization of the exponential family, so that the original problem can be reformulated as a stabilization problem for a coupled system describing the evolution of a pair formed by the actual filter and a vector of parameters. We then seek for a feedback as a function of this new parametrization. We provide sufficient conditions on the feedback controller ensuring the exponential stabilization of the coupled system toward a chosen target state. Furthermore, we propose some examples of feedback which satisfy such conditions.

In the following, we will study a feedback stabilization problem based on the projection filter given in (4.18).

5.1 Quantum projection filter approach for feedback stabilization

In this section, we consider a N-level quantum angular momentum system undergoing continuous-time non-demolition measurements. This is a typical example considered in the literature, see e.g., [19]. The aim is to study feedback stabilization of such a system with a feedback which depends on the projection filter.

First let us introduce the mathematical model. The evolution of such a physical system is described by the following stochastic master equation

$$\begin{aligned} d\rho_t &= -i\left[J_z + uJ_y, \rho_t\right]dt + \mathcal{F}(\rho_t)dt + \mathcal{G}(\rho_t)dW_t \\ dY_t &= dW_t + 2\sqrt{\eta}\,\mathrm{tr}\left(J_z\rho_t\right)\right)dt \end{aligned}, \tag{5.1}$$

where

- The measurement operator is denoted by J_z which is the self-adjoint angular momentum along the z-axis. The matrix representation of J_z is

$$J_z = \begin{bmatrix} J & & & & \\ & J-1 & & & \\ & & \ddots & & \\ & & & -J+1 & \\ & & & & -J \end{bmatrix},$$

  with $J := \frac{N-1}{2}$.

- Similarly, J_y represents the self-adjoint angular momentum along the y-axis. It is defined as follows:

$$J_y = \begin{bmatrix} 0 & -ic_1 & & & \\ ic_1 & 0 & -ic_2 & & \\ & \ddots & \ddots & \ddots & \\ & & ic_{2J-1} & 0 & -ic_{2J} \\ & & & ic_{2J} & 0 \end{bmatrix},$$

  with $c_q = \frac{1}{2}\sqrt{(2J+1-q)q}$ for $q = 1, \ldots, 2J.$

- u is the feedback control law.

The following describes the time evolution of the pair $(\rho_t, \rho_{\theta_t}) \in \mathcal{S}_{\mathcal{N}} \times \mathcal{S}_{\mathcal{N}}$,

$$d\rho_t = -i\,[J_z + uJ_y, \rho_t]\,dt + \mathcal{F}(\rho_t)dt + \mathcal{G}(\rho_t)dW_t \tag{5.2}$$

$$\begin{aligned} d\rho_{\theta_t} =& \frac{1}{2}\sum_{j=0}^{2J}(A_j\rho_{\theta_t} + \rho_{\theta_t}A_j)\left(\frac{\mathrm{tr}\,(i\rho_{\theta_t}\,[uJ_y, A_j])}{\mathrm{tr}\,(\rho_{\theta_t}A_j)}\right)dt + \\ & \eta\mathcal{F}(\rho_{\theta_t})dt + 2\sqrt{\eta}\mathcal{G}(\rho_{\theta_t})\left(\mathrm{tr}(J_z\rho_t) - \mathrm{tr}(J_z\rho_{\theta_t})\right)dt + \mathcal{G}(\rho_{\theta_t})dW_t, \end{aligned} \tag{5.3}$$

where the evolution of ρ_θ given by (5.3) is obtained from (4.18).

The aim of this section is to study feedback stabilization of the original filter ρ toward one of its equilibria $A_{\bar{n}}$ with $\bar{n} \in \{0, \ldots, 2J\}$. Here we note that A_k corresponds to the rank-one projector associated with the eigenvalue $J - k$ of J_z for $k = 0, \ldots, 2J.$ It is easy to see that A_k is in $\mathrm{clos}(\mathcal{M})$ for $k = 0, \ldots, m-1,$ where $\mathrm{clos}(\cdot)$ indicates the topological closure. The key point is that the feedback u will be assumed to depend on ρ_θ. In this case the above equations are coupled, i.e., the evolution of ρ depends on ρ_θ through u and, vice versa, the evolution of ρ_θ depends on ρ through the measurement process Y.

In absence of feedback, in many papers, see e.g., [19,42,88,92], it was shown that the quantum filter (5.2) randomly selects one of the equilibria A_j. This phenomenon is called quantum state reduction. In the following theorem, we show that the same asymptotic behavior holds true for the projection filter (5.3) in absence of feedback.

Denote by $\mathbb{P}'$ the probability measure, equivalent to $\mathbb{P}$, that makes the observation process Y_t a Wiener process. The existence of such a probability measure is guarenteed by Girsanov theorem. Its corresponding expectation is defined by $\mathbb{E}'$.

Theorem 5.1.1. *For system* (5.3)*, with no external control (*$u \equiv 0$*) and initial condition* $\bar{\rho}_0 \in \mathcal{S}_N$*, the set of invariant quantum states* $\bar{E} := \{A_0, \ldots, A_{2J}\}$ *is exponentially stable both in mean and almost surely with respect to the probability measure* $\mathbb{P}'$*. The average and sample Lyapunov exponent are less than or equal to* $-\eta/2$*.*

Proof. Consider the candidate Lyapunov function

$$V(\rho_\theta) = \frac{1}{2}\sum_{\substack{\bar{n},\bar{m}=0\\ \bar{n}\neq\bar{m}}}^{2J} \sqrt{\operatorname{tr}(\rho_\theta A_{\bar{n}})\operatorname{tr}(\rho_\theta A_{\bar{m}})},$$

where $V(\rho_\theta) = 0$ if and only if $\rho_\theta \in \bar{E}$.

First, one can show that $\mathcal{L}V(\rho_\theta) \leq -\frac{\eta}{2}V(\rho_\theta)$. Then by using Itô's formula and the Grönwall inequality, one obtains an upper bound on the expected value of the Lyapunov function as a function of time, i.e, $\mathbb{E}'(V(\rho_{\theta_t})) \leq V(\bar{\rho}_0)e^{-\frac{\eta}{2}t}$.

Moreover, by comparing V with the Bures distance, one gets the following inequality:

$$C_1 d_B(\rho_\theta, \bar{E}) \leq V(\rho_\theta) \leq C_2 d_B(\rho_\theta, \bar{E}) \tag{5.4}$$

where $C_1 = 1/2$ and $C_2 = J(2J+1)$. It implies that

$$\mathbb{E}'\left(d_B\left(\rho_{\theta_t}, \bar{E}\right)\right) \leq \frac{C_2}{C_1} d_B\left(\bar{\rho}_0, \bar{E}\right) e^{-\frac{\eta}{2}t}.$$

This means that the set $\bar{E}$ is exponentially stable in mean with average Lyapunov exponent less than or equal to $-\eta/2$.

Now, we examine a stochastic process $Q(\rho_{\theta_t}, t) = e^{\frac{\eta}{2}t}V(\rho_{\theta_t}) \geq 0$, which has an infinitesimal generator of $\mathscr{L}Q(\rho_\theta, t) = e^{\frac{\eta}{2}t}(\eta/2V(\rho_\theta) + \mathscr{L}V(\rho_\theta)) \leq 0$. Using the inequality (5.4), we can obtain that the set $\bar{E}$ is a.s. exponentially stable with a sample Lyapunov exponent less than or equal to $-\eta/2$, i.e.

$$\limsup_{t\to\infty} \frac{1}{t}\log d_B\left(\rho_{\theta_t}, \bar{E}\right) \leq -\frac{\eta}{2}, \quad \text{a.s.}$$

Thus, $Q(\rho_{\theta_t}, t)$ is a positive supermartingale, and by the Doob's martingale convergence theorem [93], it converges almost surely to a finite limit as t tends to infinity. As a result, $Q(\rho_{\theta_t}, t)$ is almost surely bounded, which means that for some a.s. finite random variable A, we have $\sup_{t\geq 0} V(\rho_{\theta_t}) = Ae^{-\frac{\eta}{2}t}$ a.s. By letting t go to infinity, we can conclude that $\limsup_{t\to\infty}\frac{1}{t}\log V(\rho_{\theta_t}) \leq -\frac{\eta}{2}$ a.s. Using the inequality (5.4), we can obtain that the set $\bar{E}$ is a.s. exponentially stable with a sample Lyapunov exponent less than or equal to $-\eta/2$, i.e.

$$\limsup_{t\to\infty} \frac{1}{t} \log d_B\left(\rho_{\theta_t}, \bar{E}\right) \leq -\frac{\eta}{2}, \quad \text{a.s.}$$

□

The above theorem does not guarantee that the trajectories of the equations (5.2) and (5.3) select the same limit. The approaches developed in [88] and [94] suggest that these limits coincide (as we assumed in Chapter (4) that $\operatorname{tr}(\bar{\rho}_0 A_k) \neq 0$ for every k). Consequently, it is reasonable to guess that a feedback control mechanism, utilizing the quantum projection filter, can be employed to stabilize the system and steer it toward a desired eigenstate of L, similar to the approaches taken in previous studies like [3], [20], and [19]. In the following, we give sufficient conditions on the feedback controller ensuring stabilization.

Concretely, we will give conditions that ensure the exponential stabilization of the pair $(\rho_t, \rho_{\theta_t})$ toward the target state $(A_{\bar{n}}, A_{\bar{n}})$, with $\bar{n} \in \{0, \ldots, 2J\}$.

For this purpose, we consider the following assumptions.

A1: $u \in \mathcal{C}^1(\operatorname{clos}(\mathcal{M}), \mathbb{R})$, $u(A_{\bar{n}}) = 0$ and $u(\rho) \neq 0$ for all $\rho \in \{A_0, \ldots, A_{2J}\} \setminus A_{\bar{n}}$.

The above hypothesis ensures that the coupled system (5.2)-(5.3) contains exactly N equilibria, given by $(A_n, A_{\bar{n}})$, with $n \in \{0, \ldots, 2J\}$.

A2: $|u(\rho)| \leq c\,(1 - \operatorname{tr}(\rho A_{\bar{n}}))^p$ for $\rho \in \mathcal{M}$, with $p > 1/2$ and for some constant $c > 0$.

In the case $\bar{n} \in \{0, 2J\}$ the assumption above gives instability of $(A_n, A_{\bar{n}})$ for $n \neq \bar{n}$.

A3: $u(\rho) = 0$ for all $\rho \in B_\varepsilon(A_{\bar{n}}) \cap \mathcal{M}$, for a sufficiently small $\varepsilon > 0$.

The above hypothesis is needed in the case $\bar{n} \notin \{0, 2J\}$ for proving the instability of the target state $(A_n, A_{\bar{n}})$ as well as the reachability of an arbitrary neighborhood of $(A_{\bar{n}}, A_{\bar{n}})$. Informally speaking, reachability means that any neighborhood of the target state can be reached in finite time.

Our final assumption yields, in combination with the previous assumptions, the reachability of the target state for the coupled system.

Let us define $P_n(\rho) := J - n - \mathrm{tr}(J_z\rho)$, $\mathcal{T}(\rho,\rho_\theta) := \mathrm{tr}\,(J_z\rho) - \mathrm{tr}\,(J_z\rho_\theta)$ and $\mathbf{P}_n := \{\rho \in \mathcal{M} \mid P_n(\rho) = 0\}$ for $n \in \{0,\dots,2J\}$, and $\mathscr{V}_z(\rho) := \mathrm{tr}\,(J_z^2\rho) - \mathrm{tr}\,(J_z\rho)^2$. Set also $\Theta_n(\rho) := \mathrm{tr}(i[J_y,\rho]A_n)$.

A4: For all $\rho \in \mathbf{P}_{\bar{n}} \setminus A_{\bar{n}}$,

$$2\eta\mathscr{V}_z(\rho)\,\mathrm{tr}(\rho A_{\bar{n}}) > u(\rho)\Theta_{\bar{n}}(\rho).$$

In the following, we establish a result concerning the exponential stability of the target state $(A_{\bar{n}}, A_{\bar{n}})$ for the coupled equations (5.2)-(5.3).

Theorem 5.1.2. *Assume either that $\bar{n} \in \{0,2J\}$ and **A1**, **A2** hold true, or that $\bar{n} \notin \{0,2J\}$ and **A1**, **A3** and **A4** are verified. Assume moreover that there exists a function $V(\rho,\rho_\theta)$, which is continuous on $\mathcal{S}_N \times \mathrm{clos}(\mathcal{M})$, and twice continuously differentiable on an almost surely invariant subset Γ of $\mathcal{S}_N \times \mathrm{clos}(\mathcal{M})$, which includes $\mathrm{int}\,(\mathcal{S}_N) \times \mathcal{M}$, satisfying the following conditions*

(i) there exist positive constants C_1 and C_2 such that

$$C_1 d_B\left((\rho,\rho_\theta),(A_{\bar{n}},A_{\bar{n}})\right) \leq V(\rho,\rho_\theta) \leq C_2 d_B\left((\rho,\rho_\theta),(A_{\bar{n}},A_{\bar{n}})\right)$$

for all $(\rho,\rho_\theta) \in \mathcal{S}_N \times \mathrm{clos}(\mathcal{M})$,

(ii) there exists $C > 0$ such that

$$\limsup_{(\rho,\rho_\theta)\to(A_{\bar{n}},A_{\bar{n}})} \frac{\mathscr{L}V(\rho,\rho_\theta)}{V(\rho,\rho_\theta)} \leq -C.$$

Then, starting from Γ, the target state $(A_{\bar{n}}, A_{\bar{n}})$ is almost surely exponentially stable for the coupled system (5.2)-(5.3) *with a sample Lyapunov exponent less than or equal to $-C - \frac{K}{2}$, where $K := \liminf_{(\rho,\rho_\theta)\to(A_{\bar{n}},A_{\bar{n}})} g^2(\rho,\rho_\theta)$ and $g(\rho,\rho_\theta) := \frac{\partial V(\rho,\rho_\theta)}{\partial \rho}\frac{\mathcal{G}(\rho)}{V(\rho,\rho_\theta)} + \frac{\partial V(\rho,\rho_\theta)}{\partial \rho_\theta}\frac{\mathcal{G}(\rho_\theta)}{V(\rho,\rho_\theta)}$.*

Proof. The proof is just an adaptation of the one given in [3, Theorem 4.11] for the coupled equations of quantum filters and its corresponding projection filter. Roughly speaking, the instability of the equilibria $(A_n, A_{\bar{n}})$ for $n \neq \bar{n}$ can be shown by applying assumptions **A1** and **A2** for the case $\bar{n} \in \{0,2J\}$, and assumptions **A1** and **A3** for $\bar{n} \notin \{0,2J\}$. The reachability of the target state

$(A_{\bar{n}}, A_{\bar{n}})$ can be shown by assuming the above hypotheses and applying similar arguments as in [3]. The key point for the validity of instability of the equilibria $(A_n, A_{\bar{n}})$ for $n \neq \bar{n}$ and the reachability of the target state $(A_{\bar{n}}, A_{\bar{n}})$ is based on the fact that the diagonal element of the projection filter (5.3), i.e., $\mathrm{tr}(\rho_{\theta_t} A_n)$ and the diagonal element of the estimate filter (3.8), i.e., $\mathrm{tr}(\hat{\rho}_t A_n)$ follow the same dynamical evolution given by

$$\begin{aligned} d\,\mathrm{tr}(\rho_{\theta_t} A_n) &= -u\Theta_n(\rho_{\theta_t})dt + 4\eta P_{\bar{n}}(\rho_{\theta_t})\,\mathrm{tr}(\rho_{\theta_t} A_n)\mathcal{T}(\rho, \rho_{\theta_t})dt \\ &\quad + 4\sqrt{\eta}P_n(\rho_{\theta_t})\,\mathrm{tr}(\rho_{\theta_t} A_n)dW_t, \end{aligned} \tag{5.5}$$

where $\mathcal{T}(\rho, \rho_\theta) := \mathrm{tr}\,(J_z\rho) - \mathrm{tr}\,(J_z\rho_\theta)$. Finally, similar to the proof of [3, Theorem 4.11], one can show by using conditions (i) and (ii) the almost sure convergence toward the target state $(A_{\bar{n}}, A_{\bar{n}})$ and conclude that

$$\limsup_{t\to\infty}\frac{1}{t}\log V(\rho_t, \rho_{\theta_t}) \leq -C - \frac{K}{2}, \quad \text{a.s.}$$

□

In the following, we give the explicit rate of convergence for the coupled system (5.2)-(5.3).

Theorem 5.1.3. *Consider the coupled system* (5.2)-(5.3) *and suppose that* $\rho_0 \in \mathrm{int}\,(\mathcal{S}_N)$.

- *Assume that hypotheses* ***A1*** *and* ***A2*** *hold. Then the pair* $(A_{\bar{n}}, A_{\bar{n}})$ *for* $\bar{n} \in \{0, 2J\}$ *is almost surely exponentially stable with the sample Lyapunov exponent less than or equal to* $-\eta$.
- *Assume that hypotheses* ***A1, A3,*** *and* ***A4*** *hold. Then the target state* $(A_{\bar{n}}, A_{\bar{n}})$ *with* $\bar{n} \notin \{0, 2J\}$ *is almost surely exponentially stable with the sample Lyapunov exponent less than or equal to* $-\frac{\eta}{2}$.

Proof. For $\bar{n} \in \{0, 2J\}$, consider the Lyapunov function $V(\rho, \rho_\theta) = \sqrt{1 - \mathrm{tr}(\rho A_{\bar{n}})} + \sqrt{1 - \mathrm{tr}(\rho_\theta A_{\bar{n}})}$. It is continuous on $\mathcal{S}_N \times \mathrm{clos}(\mathcal{M})$ and twice continuously differentiable on the almost surely invariant set Γ. Moreover, $\frac{\sqrt{2}}{2} d_B\left((\rho, \rho_\theta), (A_{\bar{n}}, A_{\bar{n}})\right) \leq V(\rho, \rho_\theta) \leq d_B\left((\rho, \rho_\theta), (A_{\bar{n}}, A_{\bar{n}})\right)$ for all $(\rho, \rho_\theta) \in$

$\mathcal{S}_N \times \operatorname{clos}(\mathcal{M})$, so the condition (i) of Theorem 5.1.2 is satisfied. Also we have

$$\begin{aligned}
\mathscr{L}V(\rho,\rho_\theta) &= \frac{u\Theta_{\bar{n}}(\rho)}{2\sqrt{1-\operatorname{tr}(\rho A_{\bar{n}})}} - \frac{\eta P_{\bar{n}}^2(\rho)\operatorname{tr}(\rho A_{\bar{n}})^2}{2\left(1-\operatorname{tr}(\rho A_{\bar{n}})\right)^{3/2}} + \frac{u\Theta_{\bar{n}}(\rho_\theta)}{2\sqrt{1-\operatorname{tr}(\rho_\theta A_{\bar{n}})}} \\
&- \frac{2\eta P_{\bar{n}}(\rho_\theta)\mathcal{T}(\rho,\rho_\theta)\operatorname{tr}(\rho_\theta A_{\bar{n}})}{\sqrt{1-\operatorname{tr}(\rho_\theta A_{\bar{n}})}} - \frac{\eta P_{\bar{n}}^2(\rho_\theta)\operatorname{tr}(\rho_\theta A_{\bar{n}})^2}{2\left(1-\operatorname{tr}(\rho_\theta A_{\bar{n}})\right)^{3/2}}. \\
&\leq -\mathbf{D}_{\bar{n}}(\rho,\rho_\theta)V(\rho,\rho_\theta),
\end{aligned}$$

where $\mathbf{D}_{\bar{n}}(\rho,\rho_\theta) := \min\left\{\frac{\eta\operatorname{tr}(\rho A_{\bar{n}})^2}{2}, \frac{\eta\operatorname{tr}(\rho_\theta A_{\bar{n}})^2}{2} - 2|\mathcal{T}(\rho,\rho_\theta)|\eta\operatorname{tr}(\rho_\theta A_{\bar{n}}) + \alpha\left(1-\operatorname{tr}(\rho_\theta A_{\bar{n}})\right)^{p-\frac{1}{2}}\right\}$ with α is a positive constant that can be determined by the estimations of $\Theta_{\bar{n}}$ and $P_{\bar{n}}$. Thus, we have

$$\limsup_{(\rho,\rho_\theta)\to(A_{\bar{n}},A_{\bar{n}})} \frac{\mathscr{L}V(\rho,\rho_\theta)}{V(\rho,\rho_\theta)} \leq \limsup_{(\rho,\rho_\theta)\to(A_{\bar{n}},A_{\bar{n}})} -\mathbf{D}_{\bar{n}}(\rho,\rho_\theta) \leq -\bar{D} < 0$$

where $\bar{D} := \frac{\eta}{2} > 0$. The conditions (i) and (ii) of Theorem 5.1.2 are both satisfied, so the theorem can be applied to find the sample Lyapunov exponent. Therefore,

$$\liminf_{(\rho,\rho_\theta)\to(A_{\bar{n}},A_{\bar{n}})} \left(\frac{\partial V(\rho,\rho_\theta)}{\partial\rho}\frac{\mathcal{G}(\rho)}{V(\rho,\rho_\theta)} + \frac{\partial V(\rho,\rho_\theta)}{\partial\rho_\theta}\frac{\mathcal{G}(\rho_\theta)}{V(\rho,\rho_\theta)}\right)^2 \geq \eta.$$

The proof of the case $\bar{n} \in \{0, 2J\}$ is complete.

For $\bar{n} \notin \{0, 2J\}$, consider the candidate Lyapunov function

$$V(\rho,\rho_\theta) = V(\rho) + V(\rho_\theta) = \sum_{n\neq\bar{n}}\sqrt{\operatorname{tr}(\rho A_n)} + \sum_{n\neq\bar{n}}\sqrt{\operatorname{tr}(\rho_\theta A_n)}.$$

The function $V(\rho,\rho_\theta)$ is continuous on $\mathcal{S}_N \times \operatorname{clos}(\mathcal{M})$ and twice continuously differentiable on Γ. By applying Jensen inequality, for all $(\rho,\rho_\theta) \in \mathcal{S}_N \times \operatorname{clos}(\mathcal{M})$, we can show $\frac{\sqrt{2}}{2}d_B\left((\rho,\rho_\theta),(A_{\bar{n}},A_{\bar{n}})\right) \leq V_{\bar{n}}(\rho,\rho_\theta) \leq \sqrt{2J}d_B\left((\rho,\rho_\theta),(A_{\bar{n}},A_{\bar{n}})\right)$. Based on the estimates on $u, \Theta_{\bar{n}}$ and $P_{\bar{n}}$, and the fact that $u = 0$ for all (ρ,ρ_θ) in a sufficiently small neighbourhood of the target state (hypothesis **A3**), we have the following estimate on the infinitesimal generator of $V(\rho,\rho_\theta)$ for all $(\rho,\rho_\theta) \in B_r\left(A_{\bar{n}},A_{\bar{n}}\right)$

with $r > 0$ sufficiently small,

$$\begin{aligned}
\mathscr{L}V(\rho,\rho_\theta) \leq & -\frac{\eta}{2}\left(1-|P_{\bar{n}}(\rho)|\right)^2 V(\rho) - \frac{\eta}{2}\left(1-|P_{\bar{n}}(\rho_\theta)|\right)^2 V(\rho_\theta) \\
& + 2\eta\left(l_{\bar{n}} + |P_{\bar{n}}(\rho_\theta)|\right)|\mathcal{T}(\rho,\rho_\theta)| V_n(\rho_\theta) \\
\leq & -\mathbf{C}_{\bar{n}}(\rho,\rho_\theta)V(\rho,\rho_\theta)
\end{aligned}$$

where $l_{\bar{n}} := \max\{\bar{n}, 2J-\bar{n}\}$ and

$$\begin{aligned}
\mathbf{C}_{\bar{n}}(\rho,\rho_\theta) := \min\Big\{ & \frac{\eta}{2}\left(1-|P_{\bar{n}}(\rho)|\right)^2, \\
& \frac{\eta}{2}\left(1-|P_{\bar{n}}(\rho_\theta)|\right)^2 - 2\eta\left(l_{\bar{n}} + |P_{\bar{n}}(\rho_\theta)|\right)|\mathcal{T}(\rho,\rho_\theta)|\Big\}.
\end{aligned}$$

Thus, we have

$$\limsup_{(\rho,\rho_\theta)\to(A_{\bar{n}},A_{\bar{n}})} \frac{\mathscr{L}V(\rho,\rho_\theta)}{V(\rho,\rho_\theta)} \leq \limsup_{(\rho,\rho_\theta)\to(A_{\bar{n}},A_{\bar{n}})} -\mathbf{C}_{\bar{n}}(\rho,\rho_\theta) \leq -\frac{\eta}{2} < 0.$$

Thus we can apply Theorem 5.1.2 and the proof is complete. □

5.2 Feedback design based on the exponential family structure

By the previous sections, we know that the projection filter dynamics can be equivalently studied through the equation (4.16). In this section, based on the general results established in the previous section for N-level quantum systems, we design a stabilizing feedback based on a new parametrization of the exponential family. This makes the previous study more interesting since working directly with the new parametrization reduces the complexity of the considered filter.

Based on the fact that $\check{\rho}_\theta$ can be written as

$$\check{\rho}_\theta = \sum_{k,j=0}^{2J} e^{\frac{\theta_k+\theta_j}{2}} A_k \bar{\rho}_0 A_j,$$

we can rewrite the normalized projection filter ρ_θ in terms of θ as follows

$$\rho_\theta = \frac{\check{\rho}_\theta}{\operatorname{tr}(\check{\rho}_\theta)} = \frac{\sum_{k,j=0}^{2J} e^{\frac{\theta_k+\theta_j}{2}} A_k \bar{\rho}_0 A_j}{\sum_{k=0}^{2J} e^{\theta_k} \operatorname{tr}(A_k \bar{\rho}_0)}. \tag{5.6}$$

Note that $\rho_\theta \neq A_k$ for every $\theta \in \mathbb{R}^{2J+1}$ and $k = 0, \dots, 2J$. In particular the target state $A_{\bar{n}}$ cannot be expressed as an equilibrium of the system in the coordinates θ.

Furthermore, the parametrization $\theta \mapsto \rho_\theta$ is redundant in the sense that, for every shift $\theta' = \theta + \mu(1, \dots, 1)^T$ with $\mu \in \mathbb{R}$, one has $\rho_{\theta'} = \rho_\theta$. In order to overcome these issues, we define a new coordinate system: let us take $\xi = (\xi_0, \dots, \xi_{\bar{n}-1}, \xi_{\bar{n}+1}, \dots, \xi_{2J})^T \in \mathbb{R}^{2J}$, where $\xi_k = e^{\frac{\theta_k - \theta_{\bar{n}}}{2}}$, $k \neq \bar{n}$. Then (5.6) can be rewritten in the following form

$$\rho_\xi = \frac{A_{\bar{n}} \bar{\rho}_0 A_{\bar{n}} + \sum_{k \neq \bar{n}} \xi_k (A_k \bar{\rho}_0 A_{\bar{n}} + A_{\bar{n}} \bar{\rho}_0 A_k) + \sum_{k,j \neq \bar{n}} \xi_k \xi_j A_k \bar{\rho}_0 A_j}{\operatorname{tr}(\bar{\rho}_0 A_{\bar{n}}) + \sum_{k \neq \bar{n}} \xi_k^2 \operatorname{tr}(A_k \bar{\rho}_0)}. \tag{5.7}$$

In this representation, the target state $A_{\bar{n}}$ corresponds to the value $\xi = 0$. The other eigenstates $A_k, k \neq \bar{n}$ of the measurement operator can be obtained as the limits of ρ_ξ as ξ tends to infinity and $\frac{|\xi_k|}{\|\xi\|}$ tends to one.

The dynamics of the projection filter can now be described in terms of ξ. To obtain the dynamics of ξ_k, it is sufficient to apply Itô's formula to get [1]

$$\begin{aligned} d\xi_k(t) = &-u(\xi(t)) \sum_{p \neq k} \frac{\operatorname{tr}(iJ_y(A_p \bar{\rho}_0 A_k - A_k \bar{\rho}_0 A_p))}{2 \operatorname{tr}(\bar{\rho}_0 A_k)} \xi_p(t) dt \\ &+ u(\xi(t)) \sum_{p \neq \bar{n}} \frac{\operatorname{tr}(iJ_y(A_p \bar{\rho}_0 A_{\bar{n}} - A_{\bar{n}} \bar{\rho}_0 A_p))}{2 \operatorname{tr}(\bar{\rho}_0 A_{\bar{n}})} \xi_p(t) \xi_k(t) dt \\ &+ \eta \left(\frac{3\lambda_{\bar{n}}^2}{2} - \frac{\lambda_k^2}{2} - \lambda_k \lambda_{\bar{n}} \right) \xi_k(t) dt + 2\eta (\lambda_k - \lambda_{\bar{n}}) \operatorname{tr}(J_z \rho_t) \xi_k(t) dt \\ &+ \sqrt{\eta} (\lambda_k - \lambda_{\bar{n}}) \xi_k(t) dW_t. \end{aligned} \tag{5.8}$$

To address the stabilization problem, we shall from now on work with the coupled system (5.2)-(5.8), describing the evolution of (ρ, ξ). The assumptions on the feedback control $u(\xi)$ given in the previous section can be adapted in the current framework as follows.

[1] For the sake of simplicity, from now on we use the convention $\xi_{\bar{n}} = 1$.

A1$'$: $u \in \mathcal{C}^1\left(\mathbb{R}^{2J}, \mathbb{R}\right), u(0) = 0$ and

$\liminf_{\xi \to \infty,\, \frac{|\xi_k|}{\|\xi\|} \to 1} |u(\xi)| > 0$ for all $k \in \{0, \ldots, 2J\}$.

This ensures that $(A_k, 0) \in \mathcal{S}_N \times \mathbb{R}^{2J}$ for $k \in \{0, \ldots, 2J\}$, correspond to the equilibria for the coupled system (5.2)-(5.8), and that the pair (ρ, ρ_ξ) does not converge to (A_k, A_h) for $k \neq \bar{n}$.

A2$'$: $|u(\xi)| \leq c \max\{\|\xi\|^q, 1\}$ for $\xi \in \mathbb{R}^{2J}$ with $q > 1$ and for some constant $c > 0$.

A3$'$: $u(\xi) = 0$ if $\|\xi\| \leq \epsilon$, for a sufficiently small $\epsilon > 0$.

A4$'$: For every $\xi \neq 0$ such that $\sum_{k \neq \bar{n}} (\lambda_{\bar{n}} - \lambda_k) \xi_k^2 \operatorname{tr}(A_k \bar{\rho}_0) = 0$ it holds

$$2\eta \mathscr{V}_z(\rho_\xi) \operatorname{tr}(\rho_\xi A_{\bar{n}}) > u(\xi) \Theta_{\bar{n}}(\rho_\xi).$$

The following theorem gives an analogue of Theorem 5.1.3 in the present framework. In order to apply the stability notions in Definition A.2, we endow the space $\mathcal{S}_N \times \mathbb{R}^{2J}$ with the distance $d\left((\rho_1, \xi_1), (\rho_2, \xi_2)\right) = d_B(\rho_1, \rho_2) + \|\xi_1 - \xi_2\|$.

Theorem 5.2.1. *Consider the system* (5.2)-(5.8) *with* $\rho_0 \in \operatorname{int}(\mathcal{S}_N)$ *and* $\xi_0 = (1, \ldots, 1)^T$*,and assume that the solution of the equation* (5.8) *is well-defined on* $[0, +\infty)$*, that is, that the solution does not blow up in finite time almost surely.*

- *Assume that* $\bar{n} \in \{0, 2J\}$ *and hypotheses* ***A1***$'$ *and* ***A2***$'$ *hold. Then the pair* $(A_{\bar{n}}, 0) \in \mathcal{S}_N \times \mathbb{R}^{2J}$ *is almost surely exponentially stable with the sample Lyapunov exponent less than or equal to* $-\eta$.

- *Assume that* $\bar{n} \notin \{0, 2J\}$ *and hypotheses* ***A1***$'$*,* ***A3***$'$*, and* ***A4***$'$ *hold. Then the target state* $(A_{\bar{n}}, 0) \in \mathcal{S}_N \times \mathbb{R}^{2J}$ *is almost surely exponentially stable with the sample Lyapunov exponent less than or equal to* $-\frac{\eta}{2}$.

Proof. It is easy to verify that the conditions **A1**$'$, **A3**$'$ and **A4**$'$ imply **A1**, **A3** and **A4**, respectively.

Furthermore, we observe that

$$1 - \mathrm{tr}(\rho_\xi A_{\bar{n}}) = \frac{\sum_{k\neq\bar{n}} \xi_k^2 \mathrm{tr}(A_k\bar{\rho}_0)}{\mathrm{Tr}(A_{\bar{n}}\bar{\rho}_0) + \sum_{k\neq\bar{n}} \xi_k^2 \mathrm{tr}(A_k\bar{\rho}_0)} \geq \|\xi\|^2 \frac{\min_{k\neq\bar{n}} \mathrm{tr}(A_k\bar{\rho}_0)}{2\mathrm{tr}(A_{\bar{n}}\bar{\rho}_0)}$$

for ξ small enough. Hence **A2**′ implies **A2** for $\rho = \rho_\xi$ with ξ small enough, that is, **A2** is verified in a neighborhood of $A_{\bar{n}}$ in $\mathcal{M}$. By using $|u(\xi)| \leq c$ and the boundedness from below of $1 - \mathrm{tr}(\rho A_{\bar{n}})$ outside of that neighborhood, we can conclude the validity of **A2** for $\rho \in \mathcal{M}$. Then by Theorem 5.1.3 the system is almost surely exponentially stable. Concerning sample Lyapunov exponents, it is enough to observe that in a small enough neighborhood of the target state the following holds

$$\frac{1}{2} \min_{k\neq\bar{n}} \sqrt{\frac{\mathrm{tr}(A_k\bar{\rho}_0)}{\mathrm{tr}(A_{\bar{n}}\bar{\rho}_0)}} \|\xi\| \leq d_B(\rho_\xi, A_{\bar{n}}) \leq 2 \max_{k\neq\bar{n}} \sqrt{\frac{\mathrm{tr}(A_k\bar{\rho}_0)}{\mathrm{tr}(A_{\bar{n}}\bar{\rho}_0)}} \|\xi\|,$$

which implies that

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\xi\| = \limsup_{t\to\infty} \frac{1}{t} \log d_B(\rho_{\xi(t)}, A_{\bar{n}})$$

for every trajectory converging to the equilibrium. □

Here we give concrete examples of feedbacks which satisfy the previous assumptions and hence stabilize the coupled system (5.2)-(5.8) toward the specific target state $(A_{\bar{n}}, 0)$.

Corollary 5.2.2. *Consider the coupled system* (5.2)-(5.8) *where the initial condition* ρ_0 *belongs to* $\mathrm{int}\,(\mathcal{S}_N)$ *and* $\xi_0 = (1, \ldots, 1)^T$*. Assume* $\mathrm{tr}\,(J_y(A_p\bar{\rho}_0 A_{\bar{n}} - A_{\bar{n}}\bar{\rho}_0 A_p)) = 0$ *for* $p \neq \bar{n}$*.*

- *If* $\bar{n} \in \{0, 2J\}$*, then the feedback controller*

$$u_{\bar{n}}(\xi) = \alpha \frac{\|\xi\|^\beta}{1 + \|\xi\|^\beta} \tag{5.9}$$

  with $\alpha > 0$ *and* $\beta \geq 1$ *stabilizes the coupled system toward the state* $(A_{\bar{n}}, 0)$ *with a sample Lyapunov exponent no greater than* $-\eta$*.*

- *If* $\bar{n} \in \{1, \dots, 2J-1\}$, *then any feedback controller of the form*

$$u_{\bar{n}}(\xi) = \frac{1}{1+\|\xi\|^{2\beta}} \Big(\sum_{k \neq \bar{n}} (\lambda_{\bar{n}} - \lambda_k) \xi_k^2 \operatorname{tr}(A_k \bar{\rho}_0) \Big)^{\beta} f(\xi), \tag{5.10}$$

with $\beta \geq 1$ *and* $f \in \mathcal{C}^1(\mathbb{R}^{2J}, \mathbb{R})$ *satisfying* $f|_{B_\varepsilon(0)} \equiv 0$, $\|f\| < \infty$, *and* $\liminf_{\xi \to \infty, \frac{|\xi_k|}{\|\xi\|} \to 1} |f(\xi)| \neq 0$, *stabilizes the coupled system toward the state* $(A_{\bar{n}}, 0)$. *The sample Lyapunov exponent is no greater than* $-\eta/2$.

Note that under the condition $\operatorname{tr}(J_y(A_p \bar{\rho}_0 A_{\bar{n}} - A_{\bar{n}} \bar{\rho}_0 A_p)) = 0$ for $p \neq \bar{n}$, the solution of the equation (5.8) is well-defined on $[0, \infty)$ since the equation does not contain quadratic terms. Also, the feedback given in (5.10) vanishes on the domain of ξ considered in the assumption **A4′**, which is then satisfied. The other required assumptions given in Theorem 5.2.1 can be easily verified. An example of function f satisfying the conditions of the previous corollary is given by

$$f(\xi) = \max\left(0, \frac{(\|\xi\| - c)^3}{1 + \|\xi\|^3}\right), \tag{5.11}$$

for $c > 0$.

In the general case where $\operatorname{tr}(J_y(A_p \bar{\rho}_0 A_{\bar{n}} - A_{\bar{n}} \bar{\rho}_0 A_p)) \neq 0$ for some $p \neq \bar{n}$, ensuring that the solution of the dynamics (5.8) is well-defined in $[0, \infty)$ is not trivial, since the equation contains a quadratic term in ξ. In the following lemma, we give a condition on the feedback controller which ensures this property. Let us denote by τ_e the explosion time.

Lemma 5.2.3. *Consider the coupled system* (5.2)-(5.8) *and set*

$$u(\xi) = -\phi(\xi) \sum_{p \neq \bar{n}} \frac{\operatorname{tr}(iJ_y(A_p \bar{\rho}_0 A_{\bar{n}} - A_{\bar{n}} \bar{\rho}_0 A_p))}{2 \operatorname{tr}(\bar{\rho}_0 A_{\bar{n}})} \xi_p, \tag{5.12}$$

where $\phi(\cdot)$ *is a real bounded non-negative function such that* $|\phi(\xi)| \leq \frac{\Delta}{\|\xi\|+1}$ *with* $\Delta > 0$. *Then every solution of the equation* (5.8) *is well-defined on* $[0, +\infty)$, *i.e.,* $\mathbb{P}(\tau_e < \infty) = 0$.

Proof. We consider the dynamics of $\|\xi\|^2 = \sum_{k\neq\bar{n}} \xi_k^2$. We have

$$
\begin{aligned}
d\xi_k^2(t) &= -2\xi_k(t)u(\xi(t)) \sum_{p\neq k} \frac{\operatorname{tr}\left(iJ_y(A_p\bar{\rho}_0 A_k - A_k\bar{\rho}_0 A_p)\right)}{2\operatorname{tr}(\bar{\rho}_0 A_k)} \xi_p(t)dt \\
&+ 2\xi_k^2(t)u(\xi(t)) \sum_{p\neq\bar{n}} \frac{\operatorname{Tr}\left(iJ_y(A_p\bar{\rho}_0 A_{\bar{n}} - A_{\bar{n}}\bar{\rho}_0 A_p)\right)}{2\operatorname{tr}(\bar{\rho}_0 A_{\bar{n}})} \xi_p(t)dt \\
&+ 2\eta\left(\frac{3\lambda_{\bar{n}}^2}{2} - \frac{\lambda_k^2}{2} - \lambda_k\lambda_{\bar{n}}\right)\xi_k^2(t)dt + 4\eta\left(\lambda_k - \lambda_{\bar{n}}\right)\operatorname{tr}(J_z\rho_t)\xi_k^2(t)dt \\
&+ \eta\left(\lambda_k - \lambda_{\bar{n}}\right)^2 \xi_k^2(t)dt + 2\sqrt{\eta}\left(\lambda_k - \lambda_{\bar{n}}\right)\xi_k^2(t)dW_t.
\end{aligned} \tag{5.13}
$$

Since the feedback (5.12) is bounded, all the terms in the right hand side of (5.13) are sublinear in $\|\xi\|^2$, except for the second term which is negative. As a consequence, the infinitesimal generator of $\|\xi\|^2$ satisfies $\mathscr{L}\left(\|\xi\|^2\right) \leq c\|\xi\|^2$ with $c > 0$.

Now, define the stopping time $\tau_\varepsilon := \inf\left\{t \geq 0 \mid \|\xi(t)\|^2 \in [\frac{1}{\varepsilon}, \infty)\right\}$. By Itô's formula, for every $T > 0$ we get

$$
\mathbb{E}(e^{-c(T\wedge\tau_\varepsilon)}\|\xi(T\wedge\tau_\varepsilon)\|^2) = \|\xi(0)\|^2 + \mathbb{E}\left(\int_0^{T\wedge\tau_\varepsilon} \mathscr{L}(e^{-cs}\|\xi(s)\|^2)ds\right) \leq \|\xi(0)\|^2, \tag{5.14}
$$

the last inequality coming from the fact that $\mathscr{L}\left(e^{-ct}\|\xi\|^2\right) \leq 0$.

Now we note that by the definitions of τ_e and τ_ε, we have $\tau_\varepsilon \leq \tau_e$. We have the following bound

$$
\begin{aligned}
\mathbb{E}\left(e^{-cT}\varepsilon^{-1}\mathbb{1}_{\{\tau_e\leq T\}}\right) &\leq \mathbb{E}\left(e^{-c\tau_\varepsilon}\|\xi(\tau_\varepsilon)\|^2\mathbb{1}_{\{\tau_e\leq T\}}\right) \\
&= \mathbb{E}\left(e^{-c(T\wedge\tau_\varepsilon)}\|\xi(T\wedge\tau_\varepsilon)\|^2\mathbb{1}_{\{\tau_e\leq T\}}\right) \\
&\leq \|\xi(0)\|^2,
\end{aligned}
$$

where for the last inequality we used the inequality (5.14).

Thus, $\mathbb{E}\left(\mathbb{1}_{\{\tau_e\leq T\}}\right) = \mathbb{P}(\tau_e \leq T) \leq \varepsilon e^{cT}\|\xi(0)\|^2$. Now it is sufficient to let ε tends to zero, to conclude that $\mathbb{P}(\tau_e \leq T) = 0$. The proof is complete since T is chosen arbitrarily. □

Based on the results discussed above, below we provide further examples of feedbacks stabilizing the coupled system (5.2)-(5.8) toward the specific target state $(A_{\bar{n}}, 0)$.

Corollary 5.2.4. *Consider the coupled system* (5.2)-(5.8) *where the initial condition* ρ_0 *belongs to* $\mathrm{int}\,(\mathcal{S}_N)$ *and* $\xi_0 = (1, \dots, 1)^T$*. Assume* $\mathrm{tr}\,(J_y(A_p\bar\rho_0 A_{\bar n} - A_{\bar n}\bar\rho_0 A_p)) \neq 0$ *for every* $p \neq \bar n$*.*

- *If* $\bar n \in \{0, 2J\}$, *then the feedback controller*

$$u_{\bar n}(\xi) = -\alpha \frac{\|\xi\|^\beta}{1+\|\xi\|^{\beta+1}} \sum_{p\neq \bar n} \frac{\mathrm{tr}\,(iJ_y(A_p\bar\rho_0 A_{\bar n} - A_{\bar n}\bar\rho_0 A_p))}{2\,\mathrm{tr}(\bar\rho_0 A_{\bar n})}\,\xi_p \tag{5.15}$$

  with $\alpha > 0$ *and* $\beta \geq 1$ *stabilizes the coupled system toward the state* $(A_{\bar n}, 0)$ *with a sample Lyapunov exponent no greater than* $-\eta$*.*

- *If* $\bar n \in \{1, \dots, 2J-1\}$, *then any feedback controller of the form*

$$u_{\bar n}(\xi) = -\frac{g(\xi)}{1+\|\xi\|^{2\beta+1}} \sum_{p\neq \bar n} \frac{\mathrm{tr}\,(iJ_y(A_p\bar\rho_0 A_{\bar n} - A_{\bar n}\bar\rho_0 A_p))}{2\,\mathrm{tr}(\bar\rho_0 A_{\bar n})}\,\xi_p, \tag{5.16}$$

  where

$$g(\xi) = \Big(\sum_{k\neq \bar n}(\lambda_{\bar n} - \lambda_k)\xi_k^2\,\mathrm{tr}(A_k\bar\rho_0)\Big)^\beta f(\xi)$$

  with $\beta \geq 1$ *and* $f \in \mathcal{C}^1(\mathbb{R}^{2J}, \mathbb{R})$ *satisfying* $f|_{B_\varepsilon(0)} \equiv 0$*,* $\|f\| < \infty$*, and* $\liminf_{\xi\to\infty,\,\frac{|\xi_k|}{\|\xi\|}\to 1} |f(\xi)| \neq 0$*, stabilizes the coupled system toward the state* $(A_{\bar n}, 0)$ *. The sample Lyapunov exponent is no greater than* $-\eta/2$*.*

Here the feedbacks are in the form given in Lemma 5.2.3 and satisfy the other required assumptions given in Theorem 5.2.1. Similar as before, the function f can be chosen as in (5.11).

5.3 Simulations

In this section, we test our previous results through numerical simulations of a four-level quantum angular momentum system. Here $N = 4$ and $J = \frac{3}{2}$. A feedback control satisfying the conditions of Theorem 5.1.3 is given by

$$u(\rho_\theta) = \alpha\,(1 - \mathrm{tr}\,(\rho_\theta A_0))^\beta\,, \tag{5.17}$$

where ρ_θ is the projection filter and A_0 is the target state. By the spectral decomposition, the measurement operator can be written as $L = \sum_{j=0}^{3} \lambda_j A_j$, where $\lambda_0 = \frac{3}{2}$, $\lambda_1 = \frac{1}{2}$, $\lambda_2 = -\frac{1}{2}$ and $\lambda_3 = -\frac{3}{2}$ are the distinct eigenvalues of L, corresponding to the four projection operators given by $A_0 = \mathrm{diag}(1,0,0,0)$, $A_1 = \mathrm{diag}(0,1,0,0)$, $A_2 = \mathrm{diag}(0,0,1,0)$, and $A_3 = \mathrm{diag}(0,0,0,1)$ respectively.

We set the detector efficiency $\eta = 0.5$. Simulations have been done in the interval $[0,5]$, with step size $\delta = 5 \times 2^{-12}$. The initial states have been chosen as follows

$$\rho_0 = \begin{pmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0.3 \end{pmatrix}, \qquad \bar{\rho}_0 = \begin{pmatrix} 0.2 & 0 & 0 & 0.1i \\ 0 & 0.2 & -0.1i & 0 \\ 0 & 0.1i & 0.3 & 0 \\ -0.1i & 0 & 0 & 0.3 \end{pmatrix}.$$

Figure 5.1 shows the fidelity between the quantum filter and the quantum projection filter, the fidelity between the quantum filter and the target state A_0, and the fidelity between the quantum projection filter and the target state A_0.

Regarding the method presented in Section 5.2, we apply the feedback laws proposed in Corollary 5.2.4. If we consider the filter equation in (3.7), we should solve a stochastic differential equation in dimension 15. On the other hand, when using the presented projection method, we deal with a stochastic differential equation in dimension 3. Here we take $\rho_0, \bar{\rho}_0$ as before and we set $\xi_0 = (1,1,1)^T$.

In Figures 5.2 and 5.3, we show the convergence of the quantum filter ρ to the target states A_0 and A_1 by applying the feedback laws 5.15 and 5.16 respectively.

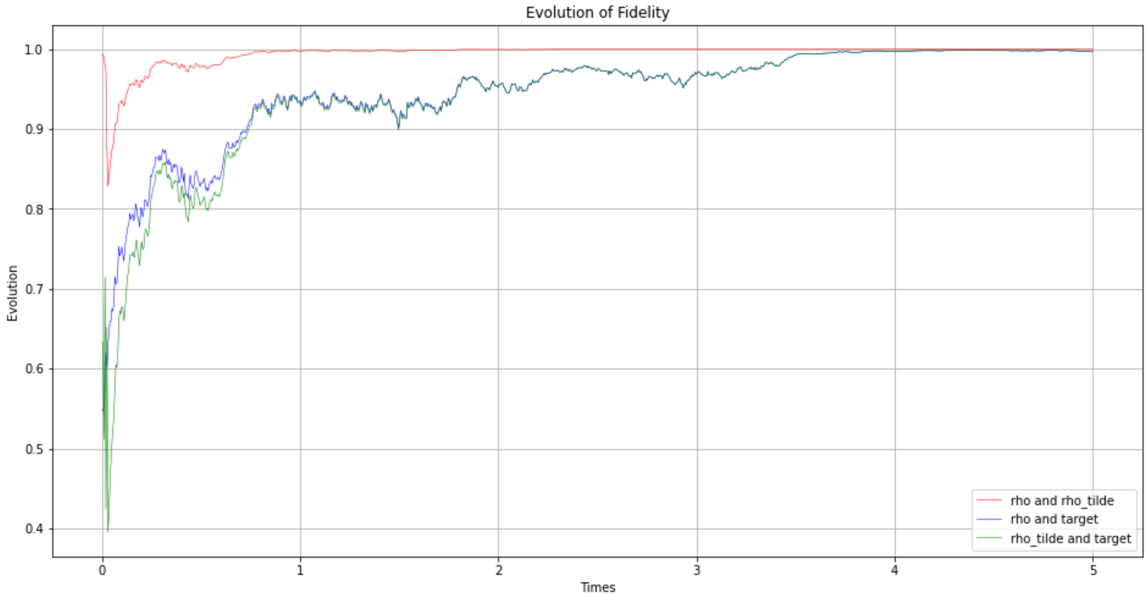

Figure 5.1: Stabilization of the coupled four-level quantum angular momentum system toward (A_0, A_0) by the feedback control (5.17) with $\alpha = 10$ and $\beta = 5$.

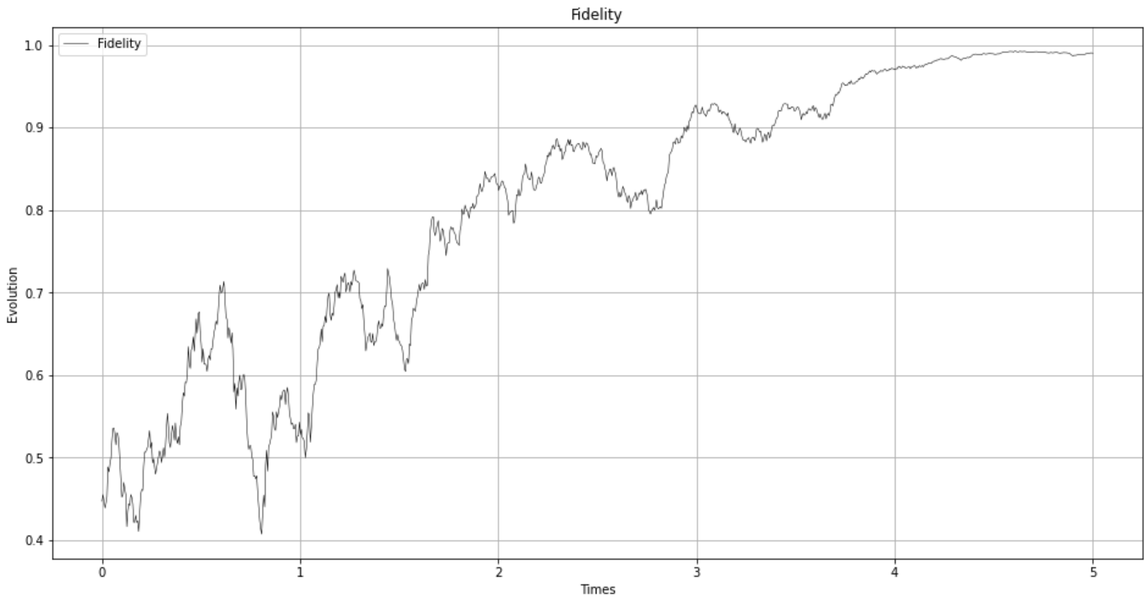

Figure 5.2: Convergence of the quantum filter (5.2) to the target state A_0, by applying the feedback control (5.15) with $\alpha = 2$ and $\beta = 2$.

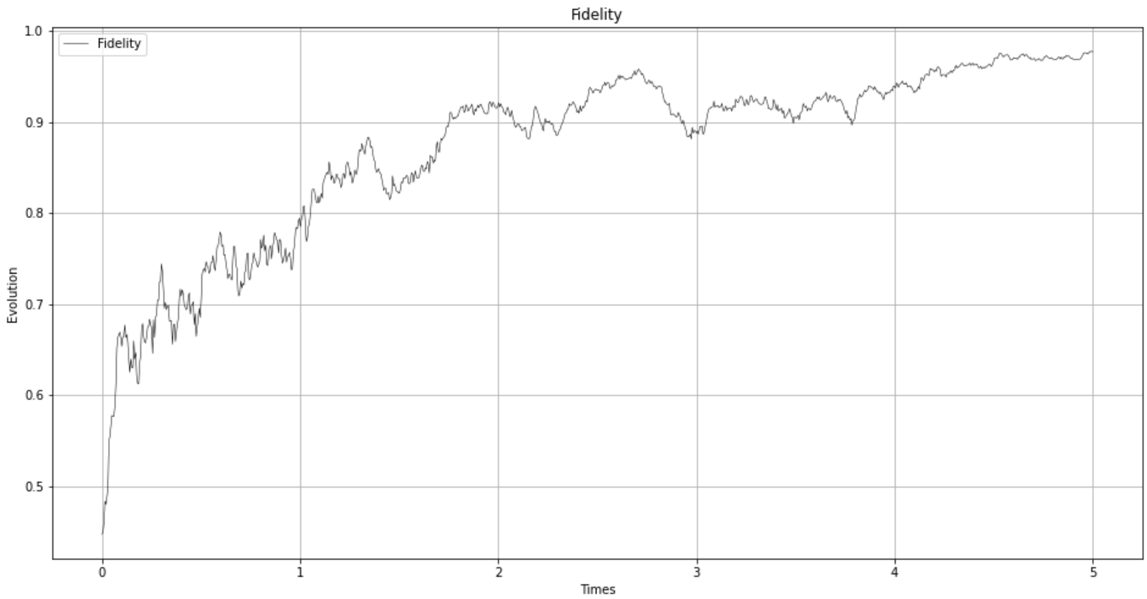

Figure 5.3: Convergence of the quantum filter (5.2) to the target state A_1, by applying the feedback control (5.16) with $\beta = 2$ and $f(\xi) = \max\left(0, \frac{(\|\xi\|-2)^3}{1+\|\xi\|^3}\right)$.

Chapter 6

Conclusion And Perspective

In this thesis we have studied the projection filter approach to open quantum systems and then we use this approximate filter in feedback stabilization problem specifically in the case of QND measurements, while taking into account measurement imperfections and inaccuracies in the initial state specification. In Chapter (2), we have presented the stochastic master equations which describe the time evolution of conditional density operators. In Chapter (3), we have introduced some tools from the quantum information geometry theory and then moved to define the quantum projection filter approach and describe the N-dimensional quantum system that this thesis is dealing with. In Chapter (4) we considered an open quantum system undergoing imperfect and indirect measurement. For QND measurement, we show that the system evolves on an appropriately chosen manifold and we express the exact solution of the quantum filter equation in terms of the solution of a lower dimensional stochastic differential equation. In order to further reduce the dimension of the system under study, we considered the projection on the lower dimensional manifold originally introduced in [1] for the case of perfect measurements. An error analysis was performed to evaluate the precision of this approximate quantum filter, focusing on the case of QND measurement. After that we repeated the projection filter approach into a new submanifold and obtained new reduced dimensional projected filters. Simulations suggested the efficiency of the proposed quantum projection filter, even in presence of a stabilizing feedback control which depends on the projection filter. In Chapter (5), we used the approximated filter illustrated in Chapter (4), in feedback stabilization problem specifically in the case of QND measurements, while taking into account measurement imperfections and inaccuracies in the initial state specification. The feedback design relies on the structure of the exponential family utilized for the projection process. We demonstrated that the introduced feedback guarantees exponential convergence of the original filter equation toward a predefined target state, corresponding to an eigenstate of the measurement operator.

Below, we resume some possible future research directions.

- One short term objective is to extend the results in Chapter (5) concerning the exponential stabilization of N-level quantum spin systems towards a predetermined state to the case of manifold with two parameters (dimension 2). The quantum state reduction is trivial as we have an exact solution when $u = 0$. The difficulty is being able to find a specific control law that stabilizes the system to the target state.
- A further short term goal is to generalize the results in Chapter (4) concerning the exact solution and the projection filter to the case of Poisson processes. The problem is more challenging as the Stratonovich from does not satisfy the chain rule in this case, and thus we are not able to project the terms in the system as they are not vector fields. This can be solved if we obtain the Marcus form instead of the Stratonovich one [95]. This part is totally new and to our knowledge it is not mentioned anywhere before.
- We also want to extend our results in Chapter (4) concerning the exact solution to the case of $[H, L] \neq 0$, i.e.,without the assumption of QND measurement. In fact, we tried different approaches to find the exact solution of the quantum filter under some conditions on the Lie algebra generated by H and L generalizing the QND assumption $[H, L] = 0$, but the problem is still challenging and we did not reach our goal.
- Further we want to explore the relation between the error analysis in Chapter (4) and the feedback stabilization in Chapter (5). In fact we want to study how the error analysis affect the feedback stabilization. Indeed, we calculated the error bounds when $[H, L] \neq 0$,

and the result was not encouraging to add them in the manuscript. The estimate of the error explodes. The error's impact on stabilization is not clear and the feedback can still stabilize the original system in our case. This leaves an open research question regarding whether there is a direct relationship between the error and the efficiency of the feedback mechanism. Another question is how to define a good error.

Appendix A

Orthogonal projection

The set $\mathcal{M}$ defined in equation (4.11) is, locally, a differential manifold of dimension m if the derivatives

$$\check{\partial}_k := \frac{\partial \check{\rho}_\theta}{\partial \theta_k}$$

form a linearly independent subset of the space $\mathcal{A}$ of Hermitian operators on $\mathbb{C}^N$. This can be easily verified if the operators A_j form a family of mutually commuting projectors. We borrow from the theory of quantum information geometry [54] some tools allowing to define a projection from $\mathcal{A}$ onto the tangent space of $\check{\mathcal{M}}$ at the point $\check{\rho}_\theta$, denoted by $T_{\check{\rho}_\theta}\check{\mathcal{M}}$. Any vector in $T_{\check{\rho}_\theta}\check{\mathcal{M}}$ can be naturally identified with an element of $\mathcal{A}$, called the mixture representation or m-representation of the tangent vector. With this identification the elements $\check{\partial}_j$ form a basis of the tangent space $T_{\check{\rho}_\theta}\check{\mathcal{M}}$. The m-representation of a vector X is indicated as $X^{(m)}$.

We will endow $\check{\mathcal{M}}$ with a Riemannian metric. For this purpose we define the inner product on $\mathcal{A}$

$$\langle\langle A, B\rangle\rangle_{\check{\rho}_\theta} = \frac{1}{2}\mathrm{tr}(\check{\rho}_\theta AB + \check{\rho}_\theta BA), \qquad \forall A, B \in \mathcal{A}.$$

In light of the aforementioned inner product, we can define an additional representation of a tangent vector $X \in T_{\check{\rho}_\theta}\check{\mathcal{M}}$ called the e-representation. This representation corresponds to a Hermitian operator denoted as $X^{(e)}$ uniquely defined by the relation

$$\langle\langle X^{(e)}, A\rangle\rangle_{\check{\rho}_\theta} = \mathrm{tr}\left(X^{(m)}A\right), \qquad \forall A \in \mathcal{A}. \tag{1}$$

This implies

$$X^{(m)} = \frac{1}{2}(\check{\rho}_\theta X^{(e)} + X^{(e)}\check{\rho}_\theta), \qquad \forall X \in T_{\check{\rho}_\theta}\check{\mathcal{M}},$$

from which it is easy to obtain $\check{\partial}_j^{(e)} = A_j$. Thanks to the e-representation, we introduce an inner product $\langle,\rangle$ on $T_{\check{\rho}_\theta}\check{\mathcal{M}}$ by

$$\begin{aligned}\langle X, Y\rangle_{\check{\rho}_\theta} &= \ll X^{(e)}, Y^{(e)} \gg_{\check{\rho}_\theta} \\ &= \mathrm{tr}\left(X^{(m)}Y^{(e)}\right), \qquad \forall X, Y \in T_{\check{\rho}_\theta}\check{\mathcal{M}},\end{aligned}$$

which is a quantum version of the Fisher metric [96]. Every element of the quantum Fisher metric is expressed as a real function of θ

$$g_{kj}(\theta) = \left\langle \check{\partial}_k, \check{\partial}_j \right\rangle_{\check{\rho}_\theta} = \mathrm{tr}\left(\check{\rho}_\theta A_k A_j\right).$$

The quantum Fisher information matrix is a real matrix of dimensions $m \times m$ and can be expressed as $G(\theta) = (g_{kj}(\theta))$. A projection operation that is orthogonal, known as Π_θ, can be defined to map vectors in $\mathcal{A}$ to vectors in $T_{\check{\rho}_\theta}\check{\mathcal{M}}$ in the following manner:

$$\begin{aligned}\Pi_\theta : \mathcal{A} &\longrightarrow T_{\check{\rho}_\theta}\check{\mathcal{M}} \\ \nu &\longmapsto \sum_{k=1}^{m}\sum_{j=1}^{m} g^{kj}(\theta)\langle \nu, \check{\partial}_j^{(e)}\rangle_{\check{\rho}_\theta}\check{\partial}_k,\end{aligned} \tag{2}$$

where the matrix $\left(g^{kj}(\theta)\right)$ refers to the inverse of the quantum information matrix $G(\theta)$.

Some standard properties of singular values [97]

Lemma A.1. Let A and B be $n \times n$ matrices. Then,

- $\sum_{i=1}^{m} s_i(AB) \le \sum_{i=1}^{m} s_i(A)s_i(B), 1 \le m \le n$;
- $s_1(AB) \le s_1(A)s_1(B)$;
- $s_i(AA^\dagger) = s_i^2(A)$;

- $\sum_i s_i(AA^\dagger) = \mathrm{tr}(AA^\dagger)$.

Some basic tools for stochastic processes and stability

Now, we revisit the key concepts and theorems from stochastic calculus and we offer a concise overview of stochastic control theory, discuss various types of stability, and outline some fundamental theorems essential for this thesis.

An aside on stochastics [84]

We will now provide a brief summary of some key concepts from stochastic calculus, which will serve as a foundation for their application in this thesis. This summary only scratches the surface of the subject; for a deeper dive, I suggest consulting Gardiner [98] and van Kampen [99]. In stochastic calculus, one frequently encounters the Wiener process (denoted as W), a continuous-time stochastic process that can be viewed as the integral of white noise. Formally, expressing stochastic equations of motion in differential form is merely a concise way to represent their integral form. When dealing with stochastic differential equations, we introduce a stochastic increment dW, analogous to an increment dt in deterministic calculus. Typically, we scale dW so that the mean value of dW^2 equals dt, or in other words, dW is approximately $\sqrt{dt}$.

In classical deterministic analysis, integration involves summing over intervals, and it doesn't matter whether we use the function's value at the start, middle, or end of these intervals when they become infinitesimally small. However, when defining the stochastic integral, this choice is crucial as it leads to different outcomes. Specifically, we must decide which point of the infinitesimal interval to use when summing. The most common conventions are those established by Itô and Stratonovich. Itô used the start of each interval, while Stratonovich chose the midpoint. Therefore, any stochastic differential equation (SDE) must indicate which convention it follows. Typically, the notation $\circ dY$ denotes a Stratonovich integral, and dW or dY denotes an Itô integral. Numerous integrators are available for stochastic systems (see [100] for an overview), but we have focused on the straightforward Itô-Euler integrator. This method essentially adds a stochastic component to the standard Euler integrator. When using this technique to numerically integrate an SDE, the equation must be in Itô form, as it implicitly uses the value at the start of each time interval. Other integrators might use the Stratonovich form or include different correction terms, such as the Milstein integrator.

Let's consider an Itô stochastic differential equation (SDE):

$$dR(x,t) = A(x,t)dt + B(x,t)dW. \tag{3}$$

To convert this into the equivalent Stratonovich SDE, we need to subtract the "Itô correction term":

$$\frac{1}{2}(\mathbf{D}B(x,t))B(x,t)dt. \tag{4}$$

Here, $\mathbf{D}$ represents the derivative. If the Itô equation (3) is linear, such as $d\mathbf{v} = A\mathbf{v}dt + B\mathbf{v}dW$, the derivative simplifies to the matrix B, and the correction term is often expressed as B^2. Conversely, if we start with a Stratonovich SDE:

$$dR(x,t) = A(x,t)dt + B(x,t) \circ dY, \tag{5}$$

we add the Itô correction term (4) to convert it to Itô form.

Another key difference between the Itô and Stratonovich forms, relevant to this thesis, is how they behave under a coordinate transformation. For a transformation $\bar{x} = \phi(x)$, the coordinates of (3) transform as follows:

$$\begin{aligned}\bar{A}(\bar{x}) &= A(x)\frac{d\phi}{dx} + \frac{1}{2}(B(x,t))^2\frac{d^2\phi}{dx^2}\\ \bar{B}(\bar{x}) &= B(x)\frac{d\phi}{dx}.\end{aligned}$$

On the other hand, in the Stratonovich form, the coordinates transform like vectors without involving second-order derivatives:

$$\begin{aligned}\bar{A}(\bar{x}) &= A(x)\frac{d\phi}{dx}\\ \bar{B}(\bar{x}) &= B(x)\frac{d\phi}{dx}.\end{aligned}$$

Projection is inherently a geometric process, which means we need to handle the stochastic increments in our equations of motion as if they were vectors. Consequently, when we project these equations of motion, they must be in the Stratonovich form [40].

We introduce in more details the stochastic tools used throughout the thesis in the next sections.

Infinitesimal generator and Itô formula

Consider a stochastic differential equation $dq_t = f(q_t)\,dt + g(q_t)\,dW_t$, where q_t takes values in $Q \subset \mathbb{R}^p$. The corresponding infinitesimal generator $\mathscr{L}$ operating on twice continuously differentiable function in space and continuously differentiable function in time such that $V : Q \times \mathbb{R}_+ \to \mathbb{R}$ is defined as follows

$$\mathscr{L}V(q,t) := \frac{\partial V(q,t)}{\partial t} + \sum_{j=1}^{p} \frac{\partial V(q,t)}{\partial q_j} f_j(q) + \frac{1}{2} \sum_{j,k=1}^{p} \frac{\partial^2 V(q,t)}{\partial q_j \partial q_k} g_j(q) g_k(q).$$

Itô formula gives [101]

$$dV(q,t) = \mathscr{L}V(q,t)dt + \sum_{j=1}^{p} \frac{\partial V(q,t)}{\partial q_j} g_j(q) dW_t,$$

which is computed according to the following Itô rules :

$$dtdt = dtdW_t = dW_t dt = 0,\ dW_t dW_t = dt.$$

Stratonovich equation

Any stochastic differential equation in Itô form in $\mathbb{R}^p$

$$dq_t = \tilde{f}(q_t)\,dt + \tilde{g}(q_t)\,dW_t, \quad q_0 = q,$$

can be written in the following Stratonovich form

$$dq_t = f\left(q_t\right)dt + g\left(q_t\right)\circ dW_t, \quad q_0 = q,$$

where $f(q) = \widetilde{f}(q) - \frac{1}{2}\sum_{l=1}^{p}\frac{\partial\tilde{g}}{\partial x_l}(x)\left(\tilde{g}\right)_l(x), \left(\widetilde{g}\right)_l$ denoting the l-th component of the vector $\widetilde{g}$, and $g(q) = \widetilde{g}(q)$.

Stochastic stability

In this section we recall some stability notions for stochastic dynamics. Consider a stochastic process q_t solution of the stochastic differential equation

$$dq_t = f\left(q_t\right)dt + g\left(q_t\right)dW_t, \quad q_0 = q,$$

where q_t takes values on a manifold $\mathcal{N}$ endowed with a metric structure d. We assume that the classical existence and uniqueness conditions for the solution are satisfied. We say that $\hat{q}$ is an equilibrium of the system if $f(\hat{q}) = g(\hat{q}) = 0$.

We have the following definition.

Definition A.2. [see [102]] Let E be a set of equilibria of the system. Then we say that

- E is locally stable in probability, if for every $\varepsilon \in (0,1)$ and for every $r > 0$, there exists $\delta = \delta(\varepsilon, r)$ such that

$$\mathbb{P}(d(q_t, E) \leq r \text{ for every } t \geq 0) \geq 1 - \varepsilon$$

  whenever $d(q_0, E) \leq \delta$,

- E is almost surely asymptotically stable in Γ, where $\Gamma \subset \mathcal{N}$ is a.s. invariant, if it is locally stable in probability and

$$\mathbb{P}(\lim_{t\to\infty} d(q_t, E) = 0) = 1$$

  whenever $q_0 \in \Gamma$,

- E is almost surely exponentially stable in Γ, where $\Gamma \subset \mathcal{N}$ is a.s. invariant, if

$$\limsup_{t\to\infty} \frac{1}{t} \log d(q_t, E) < 0$$

almost surely whenever $q_0 \in \Gamma$. The left-hand side of the inequality is referred to as the sample Lyapunov exponent of the solution,

- E is exponentially stable in mean in Γ, where $\Gamma \subset \mathcal{N}$ is a.s. invariant, if

$$\mathbb{E}(d(q_t, E)) \leq \alpha d(q_0, E) e^{-\beta t}$$

for some positive constants α and β whenever $q_0 \in \Gamma$. The smallest value $-\beta$ for which the above inequality is satisfied is called the average Lyapunov exponent.

Here we apply Definition A.2 for the notion of stochastic stability and, for the corresponding distance, we consider the Bures distance defined as follows.

Definition A.3.[Bures distance, see [103]]

- The Bures distance between two density matrices $\rho^{(1)}$ and $\rho^{(2)}$ is given by $\sqrt{2 - 2\sqrt{\mathcal{F}(\rho^{(1)}, \rho^{(2)})}}$, with the fidelity

$$\mathcal{F}(\rho^{(1)}, \rho^{(2)}) = \mathrm{tr}(\sqrt{\sqrt{\rho^{(1)}}\rho^{(2)}\sqrt{\rho^{(1)}}}).$$

- The Bures distance between ρ and a pure state σ is reduced to $d_B(\rho, \sigma) = \sqrt{2 - 2\sqrt{\mathrm{tr}(\rho\sigma)}}$.

- The Bures distance between two elements in $\mathcal{S}_N \times \mathcal{S}_N$ can be defined as

$$d_B\left((\rho^{(1)}, \tilde{\rho}^{(1)}), (\rho^{(2)}, \tilde{\rho}^{(2)})\right) = d_B\left(\rho^{(1)}, \rho^{(2)}\right) + d_B\left(\tilde{\rho}^{(1)}, \tilde{\rho}^{(2)}\right).$$

Theorem A.4. (Girsanov's theorem [104]). Assume that the probability measures $\mathbb{P}$ and $\mathbb{Q}$ are mutually absolutely continuous on $\mathcal{F}_\infty$. Let L_t be the unique continuous local martingale, $\mathcal{E}(L_t)$ be the martingale with càdlàg sample continuous paths such that, for every $t \geq 0$, $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{F}_t^W} =$

$\mathcal{E}(L_t)$. Then, if M_t is a continuous local martingale under $\mathbb{P}$, the process $\widetilde{M_t} = M_t - [M, L]_t$ is a continuous local martingale under $\mathbb{Q}$.

Theorem A.5. [Support Theorem [105]] Consider the Stratonovich equation

$$dx_t = X_0(t, x_t)\, dt + \sum_{k=1}^{n} X_k(t, x_t) \circ dW_t^k,$$

with bounded measurable function $X_0(t, x)$ that is uniformly Lipschitz continuous in x, and $X_k(t, x)$ that are twice continuously differentiable in x and continuously differentiable in t, with bounded derivatives for $k \neq 0$. Let $x_0 = x$ and $\mathbb{P}_x$ denote the probability law of the solution x_t starting at x. Also, consider a deterministic control system

$$\frac{d}{dt} x_v(t) = X_0(t, x_v(t)) + \sum_{k=1}^{n} X_k(t, x_v(t))\, v^k(t),$$

with $v^k \in \mathcal{V}$, where $\mathcal{V}$ is the set of all locally bounded measurable functions from $\mathbb{R}+$ to $\mathbb{R}$, and define $\mathcal{W}x$ as the set of all continuous paths from $\mathbb{R}+$ to $\mathbb{R}^K$ starting at x, equipped with the topology of uniform convergence on compact sets. The smallest closed subset of $\mathcal{W}_x$ such that $\mathbb{P}_x(x. \in \mathcal{I}_x) = 1$ is $\mathcal{I}_x = \overline{\{x_v(\cdot) \in \mathcal{W}_x \mid v \in \mathcal{V}^n\}} \subset \mathcal{W}_x$. This establishes a connection between the solutions of a stochastic differential equation and those of an associated deterministic one.

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ÉCOLE DOCTORALE
Sciences et technologies de l'information et de la communication (STIC)

Titre: Filtre De Projection Quantique Et Stabilisation Par Feedback

Mots clés: systemes quantiques ouverts, stabilité stochastique, filtre de projection quantique, control quantique, Feedback quantique, techniques de Lyapunov

Résumé:
Cette thèse vise à approfondir le développement de la méthode de filtrage par projection pour les systèmes quantiques ouverts soumis à des mesures imparfaites en temps continu, en mettant l'accent sur son utilisation dans la stabilisation par rétroaction. Nous commençons par discuter de la théorie du filtrage quantique, qui nous aide à comprendre l'évolution temporelle de l'opérateur densité conditionnel représentant un état quantique interagissant avec son environnement. L'évolution de l'état quantique est décrite à l'aide d'une équation différentielle stochastique à valeurs matricielles. Ensuite, nous introduisons quelques bases de la géométrie de l'information quantique pour présenter la projection quantique utilisée pour réduire la complexité quantique des systèmes quantiques ouverts. Nous nous concentrons sur l'application des filtres de projection pour les systèmes quantiques ouverts soumis à des mesures imparfaites dans le cas des mesures à non-démolition quantique (QND), et exprimons la solution exacte sur une variété de dimension inférieure. Afin de réduire encore la dimension du système étudié, nous considérons la projection sur la variété de dimension inférieure introduite initialement par Gao et al. (2019) pour le cas des mesures parfaites. Nous fournissons une analyse d'erreur de ce filtre d'approximation. La mise en œuvre en temps réel d'une rétroaction stabilisatrice reste un défi dans les expériences réelles en raison de l'échelle de temps très courte des systèmes dynamiques quantiques. Cela nous motive à faire dépendre la rétroaction du filtre de projection introduit et à démontrer son efficacité dans la stabilisation d'un système quantique ouvert dans le cas des mesures QND, tout en prenant en compte les imperfections de mesure et les imprécisions dans la spécification de l'état initial. La conception de la rétroaction repose sur une paramétrisation de la famille exponentielle utilisée pour le filtre de projection. Nous démontrons que la rétroaction introduite garantit la convergence exponentielle de l'équation du filtre original vers un état cible prédéfini, correspondant à un état propre de l'opérateur de mesure.

ÉCOLE DOCTORALE
Sciences et technologies de l'information et de la communication (STIC)

Title: Quantum Projection Filter And Feedback Stabilization

Keywords: open quantum systems, stochastic stability, quantum projection filter, quantum control, quantum feedback, Lyapunov techniques

Abstract: This thesis aims to further develop the projection filtering method of open quantum systems undergoing imperfect continuous-time measurements and focuses on its use in feedback stabilization. We begin by discussing quantum filtering theory, which helps us to understand the time evolution of the conditional density operator that represents a quantum state interacting with its environment. The evolution of the quantum state is described using a matrix-valued stochastic differential equation. Next, we introduce some foundations from the quantum information geometry theory to present the quantum projection that is used to reduce the quantum complexity of open quantum systems. We focus on the application of projection filters for open quantum systems undergoing imperfect measurements in the case of quantum non demolition (QND) measurements, and express the exact solution on a lower dimensional manifold. In order to further reduce the dimension of the system under study, we consider the projection on the lower dimensional manifold originally introduced in Gao et al. (2019) for the case of perfect measurements. We provide an error analysis of this approximation filter. The real time implementation of a stabilizing feedback remains a challenge in real experiments due to the short time scale of quantum dynamical systems. This motivates us to make the feedback depend on the introduced projection filter and show its efficiency in stabilizing an open quantum system in the case of QND measurements, while taking into account measurement imperfections and inaccuracies in the initial state specification. The feedback design relies on a parametrization of the exponential family utilized for the projection filter. We demonstrate that the introduced feedback guarantees exponential convergence of the original filter equation toward a predefined target state, corresponding to an eigenstate of the measurement operator.