OPEN ACCESS

PAPER

Phys. Educ. **59** (2024) 015013 (8pp)

iopscience.org/ped

Space-time is doomed! Introducing Planck scale physics in the classroom

Andrew M Low

Department of Physics, Liverpool University, Liverpool, United Kingdom

E-mail: Andrew.low@liverpool.ac.uk

Abstract

This article provides an overview of some tentative ideas relating to Planck scale physics at a level suitable for advanced high school students, beginning undergraduate students, and teachers. After discussing the ratio of the electric and gravitational forces, Heisenberg's uncertainty principle is combined with Einstein's theory of special relativity to estimate the scale at which the strength of gravity becomes comparable to the electrostatic force. It is shown that the energy required to probe a region of space smaller than the Planck length may lead to the formation of a black hole, placing a fundamental lower limit on the distances that can be meaningfully probed in Nature. The Planck length, mass and time are then obtained using a combination of dimensional analysis and the fundamental constants appearing in theories of quantum mechanics, relativity and gravity. Finally, a heuristic derivation of the generalised uncertainty principle is provided.

Keywords: Planck scale, space, time, quantum mechanics, quantum gravity, high school, undergraduates

1. Introduction

In 2010, the world renowned theoretical physicist Nima Arkani-Hamed delivered the third in a series of five Messenger Lectures[1] titled 'Space-time is doomed. What replaces it?'[1, 2] in which he argued that space and time have no operational meaning at very short distance and time scales[2]. The goal of the current article is show how some of these fundamental ideas can be explored in the classroom using only introductory knowledge of gravitation, relativity, quantum mechanics and dimensional analysis. The article is aimed at advanced high school students, teachers, and

[1] The Messenger Lectures are a famous series of talks hosted by Cornell University that have been running since 1924. Famous speakers have included Richard Feynman, Steven Weinberg, and Leonard Susskind.

[2] It is important to note that Arkani-Hamed was not the first person to suggest this. For a comprehensive overview of the history and development of ideas relating to Planck-scale physics, see [3–10].

beginning undergraduate students who wish to understand some of the tentative ideas proposed by theoretical physicists relating to the fundamental limits of space and time. The goal is to engage and inspire students by addressing some unsolved problems in fundamental physics, and hopefully to encourage independent research into Planck scale physics and theories of quantum gravity.

2. Gravity is weak

Consider two electrons, each of mass m, separated by a distance r. From a classical physics perspective, the ratio of the electrostatic force of repulsion and the gravitational force of attraction can be calculated using a combination of Newton's Universal Law of Gravitation and Coulomb's Law. The resulting equation, F_E/F_G, can be expressed as follows:

$$\frac{F_E}{F_G} = \frac{e^2/4\pi\varepsilon_0 r^2}{Gm^2/r^2} = \frac{e^2}{4\pi\varepsilon_0 Gm^2} \approx 10^{42} \tag{1}$$

where G is Newton's gravitational constant, e is the charge of the electron, and ε_0 is the permittivity of free space. This calculation shows that the electrostatic force between two electrons is approximately 10^{42} times greater than the gravitational force. In other words, gravity is very weak![3] Additionally, we see from (1) that the ratio F_E/F_G remains constant irrespective of the separation of the two electrons, suggesting that the magnitude of the electric force between two electrons is always greater than the magnitude of the gravitational force by a factor of 10^{42}.

[3] Most atoms are electrically neutral, with balanced positive and negative charges. This balance of electric charge conceals the strength of the electric force. Gravity, however, is always attractive meaning there is no cancellation of gravitational forces. This allows us to observe gravity's effects over large distances despite its weak strength.

3. Heisenberg's uncertainty principle and QFT

Consider what happens when we push two electrons closer together, causing them to occupy a smaller region of space. This phenomenon can, in part, be explained by Heisenberg's uncertainty principle, which states that there is a fundamental limit to how precisely we can know both the momentum and position of a particle. Heisenberg's uncertainty principle can be written as:

$$\Delta p \gtrsim \frac{\hbar}{\Delta x} \tag{2}$$

where Δp represents the uncertainty in momentum, Δx represents the uncertainty in position, and $\hbar = h/2\pi$ is the reduced Planck constant. As we confine an electron to a smaller region of space (reducing Δx), the inherent uncertainty in momentum (Δp) and energy (ΔE) will increase. If we assume the electron is confined to a region of size r, we can approximate Δx as r. The energy uncertainty, ΔE, can then be expressed as[4]:

$$\Delta E \gtrsim \frac{\hbar c}{r}. \tag{3}$$

We see from (3) that as r decreases, ΔE increases. In other words, confining an electron to a smaller region of space increases the energy uncertainty of the system[5]. A natural question arises: what happens if ΔE exceeds the energy required to create an electron-positron pair from the vacuum? We can estimate the value of r that

[4] From a relativistic perspective, momentum and energy are on an equivalent footing, therefore an uncertainty in momentum implies an uncertainty in energy. A quick check of units will show that this relation is dimensionally correct. The relativistic energy of an object with mass m and momentum p is given by: $E^2 = p^2c^2 + m^2c^4$. If $E \sim pc$ it follows that $\Delta p \sim \Delta E/c$.

[5] Indeed, Heisenberg's uncertainty principle plays a crucial role in the generation of electron degeneracy pressure, which ultimately supports white dwarf stars against gravitational collapse. See [11] for a basic overview.

corresponds to a value of ΔE that is comparable to the mass-energy of an electron. To do this, we simply substitute $\Delta E \sim mc^2$ into (3) x̀and find[6]:

$$r \sim \frac{\hbar}{mc}. \tag{4}$$

If we substitute the mass of an electron into (4) we find a value of $r \sim 10^{-13}m$. This signifies the distance scale where the principles of quantum mechanics and relativity intertwine, enabling the spontaneous generation of electron–positron pairs from the vacuum[7]. As we probe shorter distance scales, the concept of a fixed number of particles becomes inadequate, necessitating a theory that accommodates particle creation and annihilation. Such a theory exists and is known as quantum field theory[8].

4. Is gravity always weak?

If you add more mass to a region of space, it will increase the gravitational attraction in that region. Similarly, if you add more *energy* to a region of space, it will also contribute to the gravitational attraction in the region (since both mass and energy have a gravitational effect). As we have already seen from (3), Heisenberg's uncertainty principle tells us that as Δx gets smaller, ΔE will get larger. Consequently, as we push two electrons into a smaller region of space, the gravitational strength within that region will intensify due to the increased energy uncertainty. This leads to a decrease in the ratio of the electrostatic force to gravitational force. To estimate the length scale at which the magnitude of the gravitational force becomes comparable to the electrostatic force, we set $\frac{F_E}{F_G}$ equal to unity and rearrange for m :

$$m = \sqrt{\frac{e^2}{4\pi\varepsilon_0 G}}. \tag{5}$$

Next, we use $E = mc^2$ to convert this mass into an energy equivalent. We can then use the energy uncertainty relation (3) to estimate a value for r:

$$r \sim \sqrt{\frac{4\pi\varepsilon_0 G\hbar^2}{e^2c^2}} \sim 10^{-34}m. \tag{6}$$

This is suggesting that at a length scale of around $10^{-34}m$, the gravitational force and the electrostatic force become comparable in strength[9]. By squeezing our initial pair of electrons into a smaller region of space, we effectively enhance the mass content (and consequently the strength of gravity) within that region. So, what happens if we go to even shorter distance scales and add even more energy (and therefore mass) to the region? According to general relativity, there exists a maximum limit to the mass that can be compressed into a given volume of space. Eventually, a critical threshold will be surpassed leading to the formation of a black hole[10].

5. Black holes and the Planck length

Consider a spherical volume of space with radius r containing mass m. If the radius is reduced sufficiently, then the region of space will collapse to form a black hole. The radius at which this would occur is known as the Schwarzschild radius and is given by[11]:

$$r \sim \frac{Gm}{c^2}. \tag{7}$$

We have already seen from (4) that by combining Hesienberg's uncertainty principle with Einstein's theory of relativity it is possible to estimate the effective mass contained within our system for a given separation: $r \sim \frac{\hbar}{mc}$.

[6] This distance scale is referred to as the (reduced) Compton wavelength.

[7] This phenomenon is known as vacuum polarisation.

[8] For an introduction to QFT, see [12].

[9] For an alternative discussion of this point, see [5].

[10] It was first noted in 1936 by Matvei Bronstein that gravity does not allow arbitrarily high concentrations of mass in small regions of space, and that a black hole may form if a critical threshold is exceeded [15].

[11] A rigorous treatment of black holes requires knowledge of Einstein's General Theory of Relativity. However, a heuristic derivation of the Schwarzschild radius can be obtained by considering the escape velocity from the surface of a spherical object of mass M and radius r: $v_{esc} = \sqrt{\frac{2GM}{r}}$. If you set the escape velocity equal to the speed of light, and rearrange for r, you find $r = \frac{2GM}{c^2} \sim \frac{GM}{c^2}$. For a full treatment, see [13].

Additionally, equation (7) establishes a relationship between the Schwarzschild radius and mass. By combining equations (4) and (7) we can estimate the length scale at which a black hole will form due to the interplay of quantum mechanics, relativity and gravity:

$$r \sim \sqrt{\frac{G\hbar}{c^3}} = l_p \sim 10^{-35} m. \tag{8}$$

This equation is suggesting that the energy required to probe length scales of order $10^{-35} m$ is so immense that it will lead to the formation of a black hole[12]. This result would seem to suggest that there is no operational way of probing nature at distances smaller than this, since a black hole will inevitably form. The length scale, l_p, defined by (8), corresponds to the Planck length, which is believed to represent a fundamental minimum length scale in Nature. We will have more to say about the interpretation of this length in sections 9 and 10.

6. The Planck mass

To continue our exploration of Planck scale physics, we can equate the Schwarzschild radius with the Planck length, l_p, and then use equation (7) to estimate how much mass we would have to squeeze into a Planck-size region of space to create a black hole. To do this, we substitute (8) into (7) and rewrite in terms of m:

$$m \sim \frac{rc^2}{G} = \left(\sqrt{\frac{G\hbar}{c^3}}\right)\frac{c^2}{G} = \sqrt{\frac{\hbar c}{G}} \sim 22\mu\text{g}. \tag{9}$$

The mass $m_p = \sqrt{\frac{\hbar c}{G}}$ is known as the Planck mass and has a value which is roughly equivalent to the mass of a flea egg. It is remarkable to think that a fundamental minimum length scale is revealed by squeezing the mass equivalent of a flea-egg into a spherical region of space of order $10^{-35} m$.

[12] For this analysis we have assumed that the volume is roughly spherical. For a discussion of gravitational collapse in more general scenarios see [14].

7. The Planck time

Finally, we can calculate the Planck time, t_p, by considering how long it would take light to travel a distance equal to the Planck length, l_p:

$$t_p = \frac{l_p}{c} = \sqrt{\frac{\hbar G}{c^5}} \sim 10^{-43}\text{s}. \tag{10}$$

In the context of Big Bang Cosmology, the 'Planck era' represents the phase of the universe that occurred less than $10^{-43} s$ after the Big Bang. Most physicists agree that a theory of quantum gravity is required to understand physical interactions at the earliest stages of the universe when the energy density of the universe was unimaginably large. Currently, there is no complete theory that is able to describe such minute time scales. Many wonder about what occurred *before* the Big Bang, but the truth is that we encounter a problem even before reaching the Big Bang itself, at 10^{-43} seconds, when a quantum theory of gravity is required.

8. Dimensional analysis

In 1899 Max Planck noticed that it was possible to construct units of length, mass, and time by combining $\hbar$ and c with Newton's Gravitational constant, G, using dimensional analysis [17]. It is conventional to label the dimension of time as 'T', the dimension of length as 'L', and the dimension of mass as 'M'. We can then express the fundamental constants c, $\hbar$, and G in terms of their base dimensions as:

$$\begin{aligned}[c] &= LT^{-1} \\ [\hbar] &= ML^2T^{-1} \\ [G] &= M^{-1}L^3T^{-2}\end{aligned} \tag{11}$$

where square brackets surrounding a quantity should be read as 'dimensions of…'. Using the relations in (11), it is possible to construct a new quantity with dimension of length by writing:

$$l_p \propto G^{\alpha}\hbar^{\beta}c^{\gamma} \tag{12}$$

where α, β, and γ are constants that can be fixed using dimensional analysis. To do this, we simply demand that the dimensions of both sides of the equation should match:

$$[l_p] = [G]^{\alpha}[\hbar]^{\beta}[c]^{\gamma}. \tag{13}$$

If we then substitute (11) into (13) and simplify the exponents, we find:

$$M^0 L^1 T^0 = M^{\beta-\alpha} L^{3\alpha+2\beta+\gamma} T^{-2\alpha-\beta-\gamma}. \tag{14}$$

Next, if we match exponents on either side of (14) we find a set of simultaneous equations:

$$\begin{aligned} 0 &= \beta - \alpha \\ 1 &= 3\alpha + 2\beta + \gamma \; . \\ 0 &= -2\alpha - \beta - \gamma \end{aligned} \tag{15}$$

Solving these equations results in:

$$\begin{aligned} \alpha &= 1/2 \\ \beta &= 1/2 \\ \gamma &= -3/2. \end{aligned} \tag{16}$$

If we then substitute these values back into our initial expression for the Planck length (12) we find that:

$$l_p \propto G^{\frac{1}{2}} \hbar^{\frac{1}{2}} c^{-\frac{3}{2}} = \sqrt{\frac{G\hbar}{c^3}}. \tag{17}$$

This matches the expression (8) we arrived at earlier by considering the black hole argument. We can play the same dimensional analysis game to construct the Planck mass (9) and Planck time (10), as well as other Planckian quantities[13].

9. The GUP

We will now show that the Planck length emerges as a minimal length scale within the context of a 'generalised uncertainty principle'[14]. Imagine that you want to determine the position of an electron by scattering electromagnetic radiation off the electron and into a microscope. According to the theory of classical optics, if the wavelength of the electromagnetic wave is λ then we cannot obtain better precision in determining the location of the electron than $\Delta x \sim \lambda$. From the standpoint of quantum mechanics, an electromagnetic wave consists of photons, with each photon carrying momentum $p = \frac{h}{\lambda}$. When a photon scatters off an electron, it will transfer some of its momentum to the electron[15]. This introduces an uncertainty in the electron's momentum of order $\Delta p \sim p$ where p is the initial momentum of the photon. If we then multiply $\Delta p \sim \hbar/\lambda$ by $\Delta x \sim \lambda$ we arrive at the familiar uncertainty relation:

$$\Delta x \Delta p \sim \hbar. \tag{18}$$

Einstein's theory of general relativity asserts that both mass and energy contribute to the curvature of spacetime. Consequently, during the scattering process, a photon (which possesses energy) is expected to gravitationally interact with an electron. For the sake of this simplified argument, let us assume that a photon behaves as a classical particle with energy $E = hf = \hbar\omega$ where f is the frequency and $\omega = 2\pi f$. The 'effective mass' of the photon will then be:

$$m_{\text{eff}} = \frac{E}{c^2} = \frac{\hbar\omega}{c^2}. \tag{19}$$

As a result, the photon will exert a gravitational force on the electron, leading to an acceleration and a subsequent change in position of the electron. Assuming an effective interaction distance of r, we can estimate the acceleration as:

$$a \sim \frac{G m_{\text{eff}}}{r^2} \sim \frac{G\hbar\omega}{c^2 r^2} \tag{20}$$

where we have used (19) to rewrite the effective mass in terms of the photon frequency. Assuming the interaction takes place during an interval of time equal to $t = r/c$, we can use (20) to estimate the change in position, Δx_e, of the electron:

$$\Delta x_e \sim at^2 \sim \frac{G\hbar\omega}{c^2}\left(\frac{t}{r}\right)^2 \sim \frac{G\hbar\omega}{c^4}. \tag{21}$$

[13] For example, the Planck energy can be written as $E_p \sim \sqrt{\frac{\hbar c^5}{G}} \sim m_p c^2$.

[14] For a more thorough treatment of the GUP see [3, 5, 16].

[15] The amount of momentum transferred to the electron depends on the scattering angle.

We can then use the fact that $\hbar\omega = E = pc$ to write:

$$\Delta x_e \sim \frac{G\hbar\omega}{c^4} \sim \frac{Gp}{c^3}. \tag{22}$$

Next, if we substitute $\Delta p \sim p$ into (22) and use the expression for the Planck length given in (17), we find:

$$\Delta x_e \sim l_p^2 \left(\frac{p}{\hbar}\right) \sim l_p^2 \left(\frac{\Delta p}{\hbar}\right). \tag{23}$$

If we then combine Δx_e with the standard position-momentum uncertainty relation (2) we obtain an expression for the 'generalised uncertainty principle' (GUP):

$$\Delta x_{\text{total}} \sim \frac{\hbar}{\Delta p} + l_p^2 \left(\frac{\Delta p}{\hbar}\right). \tag{24}$$

We can determine the minimum position uncertainty, Δx_{min}, by differentiating (24) with respect to Δp and setting the result equal to zero, in which case we find:

$$\frac{d\Delta x_{\text{total}}}{d\Delta p} = -\frac{\hbar}{\Delta p^2} + \frac{l_p^2}{\hbar} = 0. \tag{25}$$

It follows that the uncertainty in the position of the particle has a minimum when $\Delta p \sim \frac{\hbar}{l_p}$. If we sub this back into (24) we find:

$$\Delta x_{\text{min}} \sim l_p. \tag{26}$$

To summarise, we see that in the process of determining an electron's position, the photon's interaction with the electron not only imparts momentum but also influences the electron's position due to gravity. This dual influence leads to a 'generalized uncertainty principle' which suggests that there exists an inherent minimum uncertainty in the electrons position which is of the order of the Planck length, around $10^{-35}m$. The implications of the GUP are significant. As one tries to probe distances close to the Planck length, the uncertainty in position increases to the point where it becomes comparable to, or even larger than, the measured length itself. Consequently, the concept of a point-like particle or a precise location becomes uncertain at these scales[16].

10. The GUP and microscopic black holes

We will end by briefly considering an alternative derivation of the GUP [18] that relates to the black hole argument from section 5. This will help tie together everything that has been covered so far. As we have already seen from equation (3), probing a region of space of size Δx requires energy of order $\Delta E \sim \frac{\hbar c}{\Delta x}$. Furthermore, we see from equation (7) that the Schwarzschild radius associated with energy ΔE is of order $r \sim \Delta x \sim \frac{G\Delta E}{c^4}$, where we have used $m = \Delta E/c^2$. We now have two expressions for Δx in terms of ΔE:

$$\Delta x \sim \frac{G\Delta E}{c^4} \tag{27}$$

and

$$\Delta x \sim \frac{\hbar c}{\Delta E}. \tag{28}$$

If we combine (27) and (28) and rewrite the resulting expression in terms of Δp using $\Delta E \sim c\Delta p$, we find:

$$\Delta x_{\text{total}} \sim \frac{\hbar c}{\Delta E} + \frac{G\Delta E}{c^4} \sim \frac{\hbar}{\Delta p} + \frac{G\Delta p}{c^3}. \tag{29}$$

Finally, if we sub in $l_p^2 = G\hbar/c^3$ we arrive at the same expression for the GUP stated in (24). Again, we can show that $\Delta x_{\text{min}} \sim l_p$ by minimising (29). To see why this represents a *minimum* length scale, consider what would happen if we attempted to probe shorter distance scales. In that case, we would need to concentrate even more energy into the region of space (than we did when the black hole originally formed). The net effect would be to increase the Schwarzschild radius of the black hole, thereby increasing the size of the region of space that is 'hidden' behind the event horizon of the black hole. In other words, by attempting to probe shorter distance scales, we actually make the situation worse by increasing the size of the black hole!

[16] The GUP has been studied in the context of loop quantum gravity, string theory, and other alternative approaches to quantum gravity. See [3] for an overview.

11. Wrapping up

In this brief article we have presented a series of simple arguments that physicists have used to suggest the existence of a fundamental length and time scale in Nature[17]. This notion challenges our understanding of space and time, as summarised by Arkani-Hamed [2]:

> *Every time we have encountered ideas in physics that cannot even in principle be observed, we have come to see such ideas as approximate notions. However, this instance is particularly disturbing because the notion that emerges as approximate is that of space-time itself.*

At this juncture, two broad perspectives can be taken. The first is to believe in the reality of space and time, acknowledging that their measurement becomes infeasible at extreme length and time scales. The alternative is to recognise that [2]:

> *Attempts to make sense of quantum mechanics and gravity at the smallest distance scales lead inexorably to the conclusion that space-time is an approximate notion that must emerge from more primitive building blocks.*

That is to say space-time loses its conventional characteristics at minute distance and time scales and we therefore require a new fundamental framework. It is in this sense that 'space-time is doomed'. One suggestion is to propose the existence of discrete 'atoms' of space-time. However, formulating a concept of discrete space-time that aligns with the laws of special relativity is immensely difficult. The truth is that our understanding of what occurs at these extreme length and time scales remains limited, and a new theory of quantum gravity is likely needed[18]. Perhaps someone reading this article will be inspired to take on the challenge.

[17] It is worth noting that there is much debate surrounding the meaning and interpretation of physics at the Planck scale, with many physicists suggesting that extrapolation to these scales is problematic without a fully-fledged quantum theory of gravity.

[18] For a summary of some of the key approaches to quantum gravity see [19].

Data availability statement

No new data were created or analysed in this study.

ORCID iD

Andrew M Low https://orcid.org/0000-0002-7004-1399

Received 10 July 2023
Accepted for publication 28 September 2023
https://doi.org/10.1088/1361-6552/acfe54

References

[1] Arkani-Hamed N 2010 Space-time is doomed. Messenger lectures, Cornell (available at: www.cornell.edu/video/nima-arkanihamed-spacetime-is-doomed)

[2] Arkani-Hamed N 2012 The future of fundamental physics *Daedalus* **141** 53–66

[3] Hossenfelder S 2013 Minimal length scale scenarios for quantum gravity *Living Rev. Relativ.* **16** 2

[4] Mead C A 1964 Possible connection between gravitation and fundamental length *Phys. Rev.* **135** B849–62

[5] Adler R J 2010 Six easy roads to the Planck scale *Am. J. Phys.* **78** 925–32

[6] Witten E 1996 Reflections on the fate of spacetime *Phys. Today* **49** 24–30

[7] Gross D 2008 Einstein and the quest for a unified theory *Einstein for the 21st Century: His Legacy in Science, Art, and Modern Culture* ed P L Galison, G Holton and S S Schweber (Princeton University Press) pp 287–97

[8] Wilczek F 2001 Scaling mount Planck I: a view from the bottom *Phys. Today* **54** 12–13
Garay L J 1994 Quantum gravity and minimum length *Int. J. Mod. Phys.* A **10** 145–66

[9] Salecker H and Wigner E P 1958 Quantum limitations of the measurement of space-time distances *Phys. Rev.* **109** 571–7

[10] Ng Y J and van Dam H 1995 Limitation to quantum measurements of space-time distances *Ann. New York Acad. Sci.* **755** 579–84

[11] Low A M 2023 The Chandrasekhar limit: a simplified approach *Phys. Educ.* **58** 045008

[12] Srednicky M 2007 *Quantum Field Theory* (Cambridge University Press)

[13] Hobson M, Efstathiou G and Lasenby A 2006 Frontmatter *General Relativity: An*

Introduction for Physicists (Cambridge University Press) pp I–VI

[14] Adler R J, Bazin M and Schiffer M 1965 *Introduction to General Relativity* (McGraw Hill) 2nd edn

[15] Gorelik G E 2005 Matvei Bronstein and quantum gravity: 70th anniversary of the unsolved problem *Phys. Usp.* **48** 1039–53

[16] Adler R J and Santiago D I 1999 On gravity and the uncertainty principle *Mod. Phys. Lett.* A **14** 1371

[17] Planck M 1900 Ueber irreversible Strahlungsvorgänge *Ann. Phys.* **1** 69

[18] Scardigli F 1999 Generalised uncertainty principle in quantum gravity from micro-black hole gedanken experiment *Phys. Lett.* B **452** 39–44

[19] Oriti D 2009 *Approaches to Quantum Gravity* (Cambridge University Press)

A M Low completed his PhD in Theoretical Physics at the Centre for Research in String Theory, Queen Mary, University of London. He is now a lecturer in Physics at the University of Liverpool. His interests lie in physics education and physics outreach, with a particular focus on Quantum Mechanics, QFT, and Relativity.