

Article

Noether Currents and Maxwell-Type Equations of Motion in Higher Derivative Gravity Theories

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Abstract: In general coordinate invariant gravity theories whose Lagrangians contain arbitrarily high order derivative fields, the Noether currents for the global translation and for the Nakanishi's $IOSp(8|8)$ chiral symmetry containing the BRS symmetry as its member are constructed. We generally show that for each of these Noether currents, a suitable linear combination of equations of motion can be brought into the form of a Maxwell-type field equation possessing the Noether current as its source term.

Keywords: Noether current; gravity theory; higher derivative system; Maxwell-type equation

1. Introduction

The equation of motion for the Yang-Mills field A_μ^a in the covariant gauge is given in the form

$$D^\nu F_{\mu\nu}^a + \partial_\mu B^a - ig f_{bc}^a \partial_\mu \bar{c}^b \cdot c^c = gj_\mu^a, \quad (1)$$

where $D^\nu F_{\mu\nu}^a$ is the covariant divergence of the field strength $F_{\mu\nu}^a$; j_μ^a is the color current from the matter field; and B^a , c^a and \bar{c}^a are Nakanishi-Lautrup (NL), Faddeev-Popov (FP) ghost, and anti-ghost fields, respectively. This equation was first noted by Ojima [1] to be rewritten into the form of the Maxwell-type equation of motion:

$$\partial^\nu F_{\mu\nu}^a + \{Q_B, D_\mu \bar{c}^a\} = gJ_\mu^a. \quad (2)$$

Here, Q_B is the BRS charge, and J_μ^a in the RHS is the Noether current for the global gauge transformation (=color rotation) under which all the gauge fields A_μ^a , NL, and FP ghost fields, B^a , c^a , \bar{c}^a , transform as adjoint representations, given by

$$J_\mu^a = (A^\nu \times F_{\nu\mu})^a + j_\mu^a + (A_\mu \times B)^a - i(\bar{c} \times D_\mu c)^a + i(\partial_\mu \bar{c} \times c)^a, \quad (3)$$

with $(A \times B)^a \equiv f_{bc}^a A^b B^c$. This form of YM field Equation (2) is particular, firstly, in the simple divergence form for the field strength, $\partial_\nu F^a{}^{\mu\nu}$, and secondly, in the BRS exact form for the NL and FP ghost contribution terms.

This form of YM field equation, which we call the "Maxwell-type equation of motion", played very important roles in discussing [2,3]

1. the existence of an *elementary BRS quartet* of asymptotic fields;
2. the spontaneous breaking of color symmetry and the Higgs phenomenon;
3. the unbroken color symmetry and color confinement.

From the technical viewpoint, it was also useful to simplify the computations of equal time commutation (ETC) relations for some field variables, as well as to derive Ward-Takahashi identities.

In addition, in gravity theory, there is a beautiful canonical formulation given by Nakanishi in a series of papers [4–10] based on the Einstein–Hilbert action with BRS gauge fixing in de Donder gauge. It is summarized in his textbook [11], co-authored with Ojima.



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He remarked there that the Einstein gravity field equation can also be rewritten in the form of the Maxwell-type [12]. In this formulation, he also found a beautiful theorem [11] together with Ojima in which the graviton can be identified with a *Nambu–Goldstone (NG) massless tensor particle* accompanying the spontaneous breaking of $GL(4)$ symmetry down to $SO(1, 3)$ Lorentz symmetry, thus proving the exact masslessness of the graviton in Einstein gravity theory. (Ogievetsky, independently, identified the graviton with the Nambu–Goldstone tensor in his non-linear realization theory for $GL(4)/SO(1, 3)$ [13]). This is a gravitational extension of the Ferrari–Picasso theorem [14], which proves that the photon is an *NG vector boson* accompanying the spontaneous breaking of a vector-charge Q_μ symmetry, corresponding to the gauge symmetry with a transformation parameter linear in x^μ . Nakanishi also found in his $GL(4)$ -invariant de Donder gauge that there exists an $8 + 8$ dimensional Poincaré-like $IOSp(8|8)$ supersymmetry, which he called *choral symmetry*, containing (as its member) BRS and FP ghost scale symmetries as well as the $GL(4)$ and rigid translation corresponding to the GC transformation with transformation parameter ε^μ linear in x^μ , $\varepsilon^\mu = a_\nu^\mu x^\nu + b^\mu$.

However, this work is a formal theory based on the Einstein–Hilbert action. It is perturbative non-renormalizable and may not give a well-defined theory, although there is a possibility that it may satisfy the so-called asymptotic safety [15] and give a UV complete theory. (There recently appeared an interesting paper [16] which proposes a novel perturbative approach to the Einstein–Hilbert gravity using the quadratic gravity terms as regulators which, the authors claim, can eventually be removed without harm.)

On the other hand, however, there are many investigations of higher derivative gravity theories. In particular, quadratic gravity [17–19] attracted much attention in connection with the perturbative renormalizability [20], Weyl invariant theory [21–23], and asymptotic freedom [24–26].

These higher derivative theories suffer from the massive (negative metric) ghost problem in the perturbative regime, although there have been many proposals for possible ways out (see, e.g., [19] for a review). This ghost problem is, however, outside the scope of this paper.

Even if we are much less ambitious than making gravity theory UV-complete, we still have several motivations to consider higher derivative gravity theories.

From the low energy effective field theory viewpoint, it is quite natural to consider the actions containing higher and higher order derivative fields, successively, from low to high energies. The Einstein–Hilbert action is the lowest derivative order, the quadratic gravity actions are the next derivative order, and so on.

Or, alternatively, one may simply want a *gravity theory with a UV cut-off* M valid only in the low energy region $E < M$. A simple momentum cut-off does not work here since it breaks the GC-invariance. Pauli–Villars regulators respecting the GC-invariance can be supplied by considering the covariant higher derivative terms. As noted by Stelle, the gravity field propagator behaves as $\sim 1/p^4$ in the quadratic gravity and sufficiently cuts off the UV contribution to make the theory renormalizable in 4D.

For regulators to work sufficiently enough to make all the quantities finite in 4D, however, the propagator must drop as fast as $\sim 1/p^6$. Such behavior would be supplied, for instance, by the quadratic term of covariant quantities which contain *third order derivatives of the gravity field*.

In this paper, we will consider a general gravity theory which is invariant under the general coordinate (GC) transformation and contains arbitrarily high order derivatives of gravity and matter fields, and we

1. derive a concrete form of the Noether current for the rigid translation, i.e., energy momentum tensor;
2. derive the Maxwell-type gravity equation of motion in a gauge-unfixed, i.e., classical system;
3. derive the Maxwell-type equation analogous to Equation (2) in a gauge-fixed quantum system in the de Donder–Nakanishi gauge;

4. and derive the Noether currents of the $IOSp(8|8)$ symmetry present in the de Donder-Nakanishi gauge.

The original motivation for the present author to consider this problem is to give a sound proof for the existence theorem [27] of the massless graviton, claiming that there should exist a spin 2 massless graviton in any GC invariant theory, insofar as it realizes a translational invariant vacuum with a flat Minkowski metric. This is a generalization of the Ferrari–Picasso theorem for the massless photon and the Nakanishi–Ojima theorem for the massless graviton. Those theorems were proved explicitly assuming the renormalizable QED and Einstein gravity, respectively. To prove the existence theorem generally, however, it is necessary to have the Maxwell-type gravity equation of motion in any GC invariant system assuming no particular form of action.

This paper is organized as follows. In Section 2, we present a totally general classical system containing arbitrarily high order derivative fields, which is only assumed to be GC transformation invariant. To treat such a system, we introduce a series of generalized both-side derivatives and prove some formulas they satisfy. Based on these, we derive an expression for the energy-momentum tensor for such a general system as the Noether current for the translation invariance and show that the gravity field equation of motion can be cast into the form of the Maxwell-type equation. In Section 3, these results are generalized in the gauge-fixed system by adopting the $GL(4)$ -invariant de Donder gauge à la Nakanishi. In Section 4, using the same technique, we show that each of the Noether currents of the $IOSp(8|8)$ symmetry can be written in a form of the source current of a Maxwell-type equation. Section 5 is devoted to the conclusion. Some technical points on OSp transformations are discussed in Appendices A and B. In Appendix A, the $OSp(8|8)$ transformation of the gauge-fixing plus Faddeev–Popov term is computed for the x^μ -dependent transformation parameter. In Appendix B, to obtain some familiarity with the OSp -symmetry, we briefly study the simplest model, a $OSp(2|2)$ -invariant scalar field system on flat Minkowski background; the $OSp(2|2)$ Noether current is derived, and the $IOSp(2|2)$ algebra is confirmed from the canonical (anti-)commutation relations.

2. Gravity Equation of Motion in a Generic Higher Derivative System

We consider a generic system whose action contains higher order derivative fields up to the N -th order $\partial_{\mu_1 \dots \mu_N}$:

$$S[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi, \partial_{\mu\nu} \phi, \partial_{\mu\nu\rho} \phi, \dots, \partial_{\mu_1 \dots \mu_N} \phi),$$

where ϕ stands for a collection of fields $\{\phi^i\}$ (whose index i may be suppressed when unimportant), and we use abbreviations such as

$$\partial_{\mu_1 \mu_2 \dots \mu_n} \phi^j \equiv \partial_{\mu_1} \partial_{\mu_2} \dots \partial_{\mu_n} \phi^j, \quad (4)$$

$$\mathcal{L}_j^{\mu_1 \mu_2 \dots \mu_n} \equiv \frac{\partial \mathcal{L}}{\partial (\partial_{\mu_1 \mu_2 \dots \mu_n} \phi^j)} \Big|_{\text{weight 1}}. \quad (5)$$

The suffix ‘weight 1’ in the latter means that we always keep the weight as one, irrespective of whether the n indices $\mu_1, \mu_2, \dots, \mu_n$ take the same values or not; namely, for the case $\mathcal{L} = a^{\mu\nu} \partial_{\mu\nu} \phi$, for instance, $\partial \mathcal{L} / \partial (\partial_{11} \phi) = a^{11}$ and $\partial \mathcal{L} / \partial (\partial_{12} \phi) = a^{12} + a^{21}$, but we define $\partial \mathcal{L} / \partial (\partial_{\mu\nu} \phi) \Big|_{\text{weight 1}} = (a^{\mu\nu} + a^{\nu\mu}) / 2!$ always. The functional derivative of the action S with respect to ϕ^j is given by

$$\begin{aligned} \frac{\delta S}{\delta \phi^j} &= \frac{\partial \mathcal{L}}{\partial \phi^j} - \partial_\mu \mathcal{L}_j^{\mu} + \partial_{\mu\nu} \mathcal{L}_j^{\mu\nu} - \partial_{\mu\nu\rho} \mathcal{L}_j^{\mu\nu\rho} + \dots \\ &= \sum_{n=0}^N (-)^n \partial_{\mu_1 \dots \mu_n} \mathcal{L}_j^{\mu_1 \dots \mu_n}, \end{aligned} \quad (6)$$

where N is the highest order n of the derivative fields $\partial_{\mu_1 \dots \mu_n} \phi$ contained in \mathcal{L} (so that $\mathcal{L}_j^{\mu_1 \dots \mu_n} = 0$ for $n \geq N+1$), and, for the $n=0$ case of empty set $\{\mu_1 \dots \mu_n\}$, $\partial_{\mu_1 \dots \mu_n} = 1$ and $\mathcal{L}_j^{\mu_1 \dots \mu_n} = \partial \mathcal{L} / \partial \phi^j$ are understood. The Euler–Lagrange equations are given by $\delta S / \delta \phi^j = 0$.

The Lagrangian generally changes under an infinitesimal transformation $\phi \rightarrow \phi + \delta\phi$, as

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi^j} \delta \phi^j + \mathcal{L}_j^{\mu} \partial_{\mu} \delta \phi^j + \mathcal{L}_j^{\mu\nu} \partial_{\mu\nu} \delta \phi^j + \dots = \sum_{n=0}^N \mathcal{L}_j^{\mu_1 \dots \mu_n} \partial_{\mu_1 \dots \mu_n} \delta \phi^j, \quad (7)$$

where summation over the repeated j is also implied. We consider the system which is invariant under the gauge transformation taking the form

$$\delta \phi^j(x) = G_{\rho}^j \varepsilon^{\rho}(x) + T_{\rho}^{j\mu} \partial_{\mu} \varepsilon^{\rho}(x). \quad (8)$$

For the GC transformation $x^{\rho} \rightarrow x'^{\rho} = x^{\rho} - \varepsilon^{\rho}(x)$, this field transformation reads more explicitly

$$\begin{aligned} \delta \phi^j(x) &= \varepsilon^{\rho}(x) \partial_{\rho} \phi^j + [\phi^j]_{\rho}^{\mu} \partial_{\mu} \varepsilon^{\rho}(x) \\ \text{i.e., } G_{\rho}^j &= \partial_{\rho} \phi^j, \quad T_{\rho}^{j\mu} = [\phi^j]_{\rho}^{\mu}. \end{aligned} \quad (9)$$

(Here, we are taking ε^{ρ} with an opposite sign to Nakanishi's so that the definitions of G_{ρ}^j , $T_{\rho}^{j\mu}$ and $[\phi^j]_{\rho}^{\mu}$ all have an opposite sign to Nakanishi's.) For a general tensor field $\phi^j = T_{\nu_1 \dots \nu_p}^{\sigma_1 \dots \sigma_q}$, the symbol $[\phi^j]_{\rho}^{\mu}$ is defined by [11]:

$$[T_{\nu_1 \dots \nu_p}^{\sigma_1 \dots \sigma_q}]_{\rho}^{\mu} = \sum_{i=1}^p \delta_{\nu_i}^{\mu} T_{\nu_1 \dots \nu_{i-1} \rho \nu_{i+1} \dots \nu_p}^{\sigma_1 \dots \sigma_q} - \sum_{j=1}^q \delta_{\rho}^{\sigma_j} T_{\nu_1 \dots \nu_p}^{\sigma_1 \dots \sigma_{i-1} \mu \sigma_{i+1} \dots \sigma_q}. \quad (10)$$

The GC invariance of the system implies that the Lagrangian is a scalar density so that the change of \mathcal{L} is given by a total divergence:

$$\delta \mathcal{L} = \partial_{\mu} (\mathcal{L} \varepsilon^{\mu}). \quad (11)$$

For the GC transformation $\delta \phi$ in Equation (8), we can equate expression (11) for $\delta \mathcal{L}$ with Equation (7) and obtain an identity:

$$\sum_{n=0}^N \left[\mathcal{L}_j^{\mu_1 \dots \mu_n} \partial_{\mu_1 \dots \mu_n} (G_{\rho}^j \varepsilon^{\rho}(x) + T_{\rho}^{j\mu} \partial_{\mu} \varepsilon^{\rho}(x)) \right] - \partial_{\mu} (\mathcal{L} \varepsilon^{\mu}) = 0. \quad (12)$$

This equation, if expanded in a power of derivatives ∂_{μ} on the gauge transformation parameter (function) $\varepsilon^{\rho}(x)$, yields

$$\begin{aligned} & \left(\sum_{n=0}^N \mathcal{L}_j^{\mu_1 \dots \mu_n} \partial_{\mu_1 \dots \mu_n} G_{\rho}^j - \partial_{\rho} \mathcal{L} \right) \varepsilon^{\rho} \\ & + \left(\left\{ \sum_{n=0}^{N-1} (n+1) \mathcal{L}_j^{\mu \alpha_1 \dots \alpha_n} \partial_{\alpha_1 \dots \alpha_n} G_{\rho}^j + \sum_{n=0}^N \mathcal{L}_j^{\alpha_1 \dots \alpha_n} \partial_{\alpha_1 \dots \alpha_n} T_{\rho}^{j\mu} \right\} - \mathcal{L} \delta_{\rho}^{\mu} \right) \partial_{\mu} \varepsilon^{\rho} \\ & + \sum_{k=1}^N \mathcal{K}_k^{\nu_1 \dots \nu_k \mu} \partial_{\nu_1 \dots \nu_k \mu} \varepsilon^{\rho}(x) = 0, \end{aligned} \quad (13)$$

where \mathcal{K}_k 's ($k = 0, 1, 2, \dots$) are defined by

$$\begin{aligned}\mathcal{K}_k^{\nu_1 \dots \nu_k \mu} &= \sum_{n=0}^{N-k-1} {}_{n+k+1} C_{k+1} \mathcal{L}_j^{\nu_1 \dots \nu_k \mu \alpha_1 \dots \alpha_n} \partial_{\alpha_1 \dots \alpha_n} G_\rho^j \\ &+ \sum_{n=0}^{N-k} {}_{n+k} C_k \mathcal{L}_j^{\nu_1 \dots \nu_k \alpha_1 \dots \alpha_n} \partial_{\alpha_1 \dots \alpha_n} T_\rho^{j\mu},\end{aligned}\quad (14)$$

with ${}_n C_k$ denoting the binomial coefficient $\binom{n}{k} = n! / k!(n-k)!$. Note that the first summation term $\{ \sum_{n=0}^{N-1} (\dots) + \sum_{n=0}^N (\dots) \}$ in the coefficient of $\partial_\mu \varepsilon^\rho$ in the second line in Equation (13) is only identical with the quantity $\mathcal{K}_k^{\nu_1 \dots \nu_k \mu}$ for the case $k = 0$.

Since the functions $\varepsilon^\rho, \partial_\mu \varepsilon^\rho, \dots, \partial_{\mu_1 \dots \mu_k} \varepsilon^\rho$ are mutually independent, the coefficients should vanish separately, implying the following $N+2$ identities, which we shall refer to as $\delta\mathcal{L}$ -identities below:

$$\sum_{n=0}^N \mathcal{L}_j^{\mu_1 \dots \mu_n} \partial_{\mu_1 \dots \mu_n} G_\rho^j - \partial_\rho \mathcal{L} = 0 \quad (15)$$

$$\mathcal{K}_0^\mu - \mathcal{L} \delta_\rho^\mu = 0 \quad (16)$$

$$\text{Sym}_{\{\nu_1 \dots \nu_k \mu\}} (\mathcal{K}_k^{\nu_1 \dots \nu_k \mu}) = 0 \quad \text{for } k = 1, 2, \dots, N. \quad (17)$$

Here, $\text{Sym}_{\{\nu_1 \dots \nu_k \mu\}}$ implies the totally symmetric part with respect to the $k+1$ indices ν_1, \dots, ν_k and μ . Note that only the totally symmetric part of $\mathcal{K}_k^{\nu_1 \dots \nu_k \mu}$ should vanish since it vanishes when multiplied by the totally symmetric function $\partial_{\nu_1 \dots \nu_k \mu} \varepsilon^\rho(x)$. Note also that $\mathcal{K}_k^{\nu_1 \dots \nu_k \mu}$ is manifestly symmetric with respect to the first k indices ν_1, \dots, ν_k , as is clear from the defining Equation (14).

Now, we can derive useful identities for rewriting the suitable linear combination of equations of motion (6)

$$\frac{\delta S}{\delta \phi^j} = \sum_{n=0}^N (-)^n \partial_{\mu_1 \dots \mu_n} \mathcal{L}_j^{\mu_1 \dots \mu_n}. \quad (18)$$

(We call the quantities $\delta S / \delta \phi^j$ 'equations of motion', although it is an abuse of terminology, since the equation of motion itself is the equation $\delta S / \delta \phi^j = 0$.)

First, a linear combination $-(\delta S / \delta \phi^j) G_\rho^j$ of the equations of motion is rewritten into the following form by adding the first $\delta\mathcal{L}$ -identity (15):

$$-(\delta S / \delta \phi^j) G_\rho^j = \sum_{n=0}^N \left(\mathcal{L}_j^{\mu_1 \dots \mu_n} \partial_{\mu_1 \dots \mu_n} G_\rho^j - (-)^n \partial_{\mu_1 \dots \mu_n} \mathcal{L}_j^{\mu_1 \dots \mu_n} \cdot G_\rho^j \right) - \partial_\rho \mathcal{L}. \quad (19)$$

To rewrite this more concisely, we introduce a generalized 'both-side' derivative defined for $n \geq 0$ by [28]:

$$\begin{aligned}F \overleftrightarrow{\partial}_{\mu_1 \mu_2 \dots \mu_n} G &\equiv F \partial_{\mu_1 \mu_2 \dots \mu_n} G - \partial_{\mu_1} F \cdot \partial_{\mu_2 \dots \mu_n} G \\ &+ \partial_{\mu_1 \mu_2} F \cdot \partial_{\mu_3 \dots \mu_n} G - \dots + (-)^n \partial_{\mu_1 \mu_2 \dots \mu_n} F \cdot G,\end{aligned}\quad (20)$$

for two arbitrary functions F and G , with understanding $\overleftrightarrow{\partial}_{\mu_1 \mu_2 \dots \mu_n} = 1$ when $n = 0$. This derivative is no longer symmetric under permutation of the indices but satisfies a useful formula [28]:

$$\partial_\mu [F^{\mu \alpha_1 \dots \alpha_n} \overleftrightarrow{\partial}_{\alpha_1 \dots \alpha_n} G] = F^{\mu \alpha_1 \dots \alpha_n} \partial_{\mu \alpha_1 \dots \alpha_n} G + (-)^n \partial_{\mu \alpha_1 \dots \alpha_n} F^{\mu \alpha_1 \dots \alpha_n} \cdot G, \quad (21)$$

for any totally symmetric function $F^{\mu\alpha_1\cdots\alpha_n}$ with respect to the $n+1$ indices $\{\mu, \alpha_1, \dots, \alpha_n\}$. Applying this formula, we can rewrite the identity (19) as

$$\begin{aligned} -\frac{\delta S}{\delta \phi^j} G_\rho^j &= \partial_\mu J_\rho^\mu \\ J_\rho^\mu &= \sum_{n=0}^{N-1} \mathcal{L}_j^{\mu\alpha_1\cdots\alpha_n} \overleftarrow{\partial}_{\alpha_1\cdots\alpha_n} G_\rho^j - \delta_\rho^\mu \mathcal{L}. \end{aligned} \quad (22)$$

This J_ρ^μ is the Noether current for the global GC transformation with x -independent ϵ^ρ (=translation), i.e., energy-momentum tensor, for the higher derivative system. This identity shows that it is indeed conserved when the equations of motion $\delta S/\delta\phi^j = 0$ are satisfied.

Now, in order to derive various identities from the rest of the $\delta\mathcal{L}$ -identities, (16) and (17), we need to introduce generalized both-side derivatives and some formulas for them.

We define k -th both-side derivative $\overleftarrow{\overrightarrow{\partial}}^k$ by induction both in the number k and the differential order n :

$$\overleftarrow{\overrightarrow{\partial}}_{\alpha_1\cdots\alpha_n}^k \equiv \overleftarrow{\partial}_{\alpha_1\cdots\alpha_{n-1}} \cdot \overrightarrow{\partial}_{\alpha_n} + \overleftarrow{\partial}_{\alpha_1\cdots\alpha_n}^{k-1} \quad (23)$$

with initial condition

$$\begin{cases} k = -1 & : \quad \overleftarrow{\overrightarrow{\partial}}_{\alpha_1\cdots\alpha_n}^{-1} = (-)^n \overleftarrow{\partial}_{\alpha_1\cdots\alpha_n} \text{ for } \forall n \geq 0 \\ n = 0 & : \quad \overleftarrow{\overrightarrow{\partial}}_{\alpha_1\cdots\alpha_n} \Big|_{n=0} = 1 \text{ for } \forall k \geq -1 \end{cases} \quad (24)$$

It is easy to see that the $\overleftarrow{\overrightarrow{\partial}}^0$ is just the same as the original ‘both-side’ derivative $\overleftarrow{\overrightarrow{\partial}}$ introduced above in Equation (20); indeed, it satisfies the above recursive defining relation (23) for $k = 0$ as follows:

$$\begin{aligned} \overleftarrow{\overrightarrow{\partial}}_{\alpha_1\cdots\alpha_n}^0 &= \sum_{\ell=0}^n (-)^{\ell} \overleftarrow{\partial}_{\alpha_1\cdots\alpha_\ell} \cdot \overrightarrow{\partial}_{\alpha_{\ell+1}\cdots\alpha_n} \\ &= \sum_{\ell=0}^{n-1} (-)^{\ell} \left(\overleftarrow{\partial}_{\alpha_1\cdots\alpha_\ell} \cdot \overrightarrow{\partial}_{\alpha_{\ell+1}\cdots\alpha_{n-1}} \right) \partial_{\alpha_n} + (-)^n \overleftarrow{\partial}_{\alpha_1\cdots\alpha_n} \\ &= \overleftarrow{\overrightarrow{\partial}}_{\alpha_1\cdots\alpha_{n-1}} \cdot \overrightarrow{\partial}_{\alpha_n} + \overleftarrow{\partial}_{\alpha_1\cdots\alpha_n}. \end{aligned} \quad (25)$$

Then, as a generalization of the $k = 0$ formula (21), we have the following formula, which holds for all $k \geq 0, n \geq 0$ and for any totally symmetric function $F^{\mu\alpha_1\cdots\alpha_n}$ with respect to the $n+1$ indices $\{\mu, \alpha_1, \dots, \alpha_n\}$:

$$\partial_\mu \left[F^{\mu\alpha_1\cdots\alpha_n} \overleftarrow{\overrightarrow{\partial}}_{\alpha_1\cdots\alpha_n}^k G \right] = F^{\mu\alpha_1\cdots\alpha_n} \left({}_{n+k+1} C_k \partial_{\mu\alpha_1\cdots\alpha_n} - \overleftarrow{\partial}_{\mu\alpha_1\cdots\alpha_n}^{k-1} \right) G. \quad (26)$$

The proof easily goes by induction in the number $N \equiv k+n$ in the region $k \geq 0$ and $n \geq 0$. First, note that this formula holds at $k = 0$ boundary as shown above for $\forall n \geq 0$ and also clearly holds at $n = 0$ boundary with $\forall k \geq 0$ since the relevant k -th both-side derivatives there are $\overleftarrow{\partial}_{\alpha_1\cdots\alpha_n} \Big|_{n=0} = 1$ and that of a single derivative $\overleftarrow{\partial}_\mu$ which is simply, by Equation (23),

$$\overleftarrow{\partial}_\mu^{k-1} = \overleftarrow{\partial}_\mu + \overleftarrow{\partial}_\mu^{k-2} = k \overleftarrow{\partial}_\mu + \overleftarrow{\partial}_\mu^{-1} = k \overleftarrow{\partial}_\mu - \overleftarrow{\partial}_\mu. \quad (27)$$

Thus, it is sufficient to prove the formula only for $k \geq 1$ and $n \geq 1$. Now, assume that formula (26) holds for all $k \geq 0$ and $n \geq 0$ values in the region $k+n \leq \exists N$, and let us

evaluate the LHS of the formula for any $k \geq 1$ and $n \geq 1$ with $k + n = N + 1$. If we use the defining Equation (23),

$$F^{\mu\alpha_1\cdots\alpha_n} \overset{k}{\overleftarrow{\partial}}_{\alpha_1\cdots\alpha_n} G = F^{\mu\alpha_1\cdots\alpha_n} \overset{k}{\overleftarrow{\partial}}_{\alpha_1\cdots\alpha_{n-1}} (\partial_{\alpha_n} G) + F^{\mu\alpha_1\cdots\alpha_n} \overset{k-1}{\overleftarrow{\partial}}_{\alpha_1\cdots\alpha_n} G, \quad (28)$$

the two terms on the RHS have lower values $k + n = N$ by one than the LHS, to which we can apply the formula by the induction assumption, so that

$$\begin{aligned} \partial_\mu \left[F^{\mu\alpha_1\cdots\alpha_n} \overset{k}{\overleftarrow{\partial}}_{\alpha_1\cdots\alpha_n} G \right] &= F^{\mu\alpha_1\cdots\alpha_n} \left({}_{n+k} C_k \partial_{\mu\alpha_1\cdots\alpha_{n-1}} - \overset{k-1}{\overleftarrow{\partial}}_{\mu\alpha_1\cdots\alpha_{n-1}} \right) (\partial_{\alpha_n} G) \\ &\quad + F^{\mu\alpha_1\cdots\alpha_n} \left({}_{n+k} C_{k-1} \partial_{\mu\alpha_1\cdots\alpha_n} - \overset{k-2}{\overleftarrow{\partial}}_{\mu\alpha_1\cdots\alpha_n} \right) G \\ &= F^{\mu\alpha_1\cdots\alpha_n} \left[\left({}_{n+k} C_k + {}_{n+k} C_{k-1} \right) \partial_{\mu\alpha_1\cdots\alpha_n} G \right. \\ &\quad \left. - \left(\overset{k-1}{\overleftarrow{\partial}}_{\mu\alpha_1\cdots\alpha_{n-1}} \partial_{\alpha_n} + \overset{k-2}{\overleftarrow{\partial}}_{\mu\alpha_1\cdots\alpha_n} \right) G \right]. \end{aligned} \quad (29)$$

If we note an identity (of Pascal's triangle) ${}_{n+k} C_k + {}_{n+k} C_{k-1} = {}_{n+k+1} C_k$ and again apply the defining Equation (23) with $k \rightarrow k - 1$, then we see that the last expression is simply reproducing the RHS of formula (26), finishing the proof.

Now, we are ready to derive the Maxwell-type form of gravity equation of motion. For this purpose, let us introduce the following quantity \mathcal{J}_k for $k = 0, 1, 2, \dots$:

$$\begin{aligned} \mathcal{J}_k^{\nu_1\cdots\nu_k\mu} \rho &\equiv \sum_{n=0}^{N-k-1} \mathcal{L}_j^{\nu_1\cdots\nu_k\mu\alpha_1\cdots\alpha_n} \overset{k}{\overleftarrow{\partial}}_{\alpha_1\cdots\alpha_n} G_\rho^j \\ &\quad + \sum_{n=0}^{N-k} \mathcal{L}_j^{\nu_1\cdots\nu_k\alpha_1\cdots\alpha_n} \overset{k-1}{\overleftarrow{\partial}}_{\alpha_1\cdots\alpha_n} T_\rho^{j\mu}. \end{aligned} \quad (30)$$

The first of this quantity \mathcal{J}_0^μ with $k = 0$ is a combination of the equation of motion, $(\delta S / \delta \phi^j) T_\rho^{j\mu}$, Lagrangian \mathcal{L} , and the energy-momentum tensor J_ρ^μ :

$$\mathcal{J}_0^\mu = J_\rho^\mu + \delta_\rho^\mu \mathcal{L} + \frac{\delta S}{\delta \phi^j} T_\rho^{j\mu}. \quad (31)$$

This can be seen from Equations (22) and (18) which are rewritten by using the definition of the k -th both-side derivative with $k = 0$ and -1 , respectively, into

$$\begin{aligned} \sum_{n=0}^{N-1} \mathcal{L}_j^{\mu\alpha_1\cdots\alpha_n} \overset{0}{\overleftarrow{\partial}}_{\alpha_1\cdots\alpha_n} G_\rho^j &= J_\rho^\mu + \delta_\rho^\mu \mathcal{L} \\ \sum_{n=0}^N \mathcal{L}_j^{\alpha_1\cdots\alpha_n} \overset{-1}{\overleftarrow{\partial}}_{\alpha_1\cdots\alpha_n} T_\rho^{j\mu} &= \frac{\delta S}{\delta \phi^j} T_\rho^{j\mu}. \end{aligned} \quad (32)$$

Owing to the general formula (26), the two quantities, \mathcal{J}_k introduced here (30) and \mathcal{K}_k defined previously in Equation (14), satisfy the following recurrence relation:

$$\mathcal{K}_k^{\nu_1\cdots\nu_k\mu} - \mathcal{J}_k^{\nu_1\cdots\nu_k\mu} = \partial_\tau \mathcal{J}_{k+1}^{\tau\nu_1\cdots\nu_k\mu}. \quad (33)$$

When applying the formula (26) to derive this equality, we should note that the summation over the set of $n + 1$ dummy indices $\{\tau, \alpha_1 \cdots \alpha_n\}$ contained in the RHS quantity $\partial_\tau \mathcal{J}_{k+1}^{\tau\nu_1\cdots\nu_k\mu}$ is identified with the summation over the set $\{\alpha_1 \cdots \alpha_{n+1}\}$ contained in the LHS quantities $\mathcal{K}_k^{\nu_1\cdots\nu_k\mu} - \mathcal{J}_k^{\nu_1\cdots\nu_k\mu}$ by identifying α_{n+1} as τ . This implies that the $n = 0$

terms existing in the summations $\sum_{n=0}$ in \mathcal{K}_k and \mathcal{J}_k on the LHS do not appear on the RHS. However, the $n = 0$ terms in \mathcal{K}_k and \mathcal{J}_k are seen to be the same, thus canceling themselves on the LHS.

From this relation (33), we find, suppressing the tensor indices,

$$\begin{aligned}\mathcal{J}_k &= \mathcal{K}_k - \partial \mathcal{J}_{k+1} = \mathcal{K}_k - \partial \mathcal{K}_{k+1} + \partial^2 \mathcal{J}_{k+2} \\ &= \dots = \sum_{\ell=0}^K (-\partial)^\ell \mathcal{K}_{k+\ell} + (-\partial)^{K+1} \mathcal{J}_{k+K+1}.\end{aligned}\quad (34)$$

Since $\mathcal{L}^{\nu_1 \dots \nu_k} = 0$ for $k > N$, \mathcal{J}_k and \mathcal{K}_k vanish for $k \geq N+1$. Thus, we find the following expression for \mathcal{J}_0^μ , reviving the tensor indices:

$$\mathcal{J}_0^\mu = \sum_{k=0}^N (-)^k \partial_{\nu_1 \dots \nu_k} \mathcal{K}_k^{\nu_1 \dots \nu_k \mu}.\quad (35)$$

We now insert Equation (31) into \mathcal{J}_0 on the LHS; then, noting that the term $\delta_\rho^\mu \mathcal{L}$ there cancels the $\mathcal{K}_{k=0}$ term on the RHS due to the second $\delta\mathcal{L}$ -identity (16), $\mathcal{K}_0^\mu = \delta_\rho^\mu \mathcal{L}$, we find the gravity field equation in the form

$$\frac{\delta S}{\delta \phi^j} T_{\rho}^{j\mu} = -J_{\rho}^\mu + \sum_{k=1}^N (-)^k \partial_{\nu_1 \dots \nu_k} \mathcal{K}_k^{\nu_1 \dots \nu_k \mu}.\quad (36)$$

This is still not the final form. The last summation term can be written as a divergence form of a ‘field-strength’ tensor $\mathcal{F}_{\rho}^{\nu\mu}$, but it is not yet $\nu\mu$ antisymmetric:

$$\begin{aligned}\sum_{k=1}^N (-)^k \partial_{\nu_1 \dots \nu_k} \mathcal{K}_k^{\nu_1 \dots \nu_k \mu} &\equiv -\partial_\nu \mathcal{F}_{\rho}^{\nu\mu} \\ \mathcal{F}_{\rho}^{\nu\mu} &= \sum_{k=0}^{N-1} (-)^k \partial_{\nu_1 \dots \nu_k} \mathcal{K}_{k+1}^{\nu_1 \dots \nu_k \nu \mu}.\end{aligned}\quad (37)$$

However, thanks to the remaining $\delta\mathcal{L}$ -identities (17), we can modify it into an $\nu\mu$ antisymmetric field strength $\tilde{\mathcal{F}}_{\rho}^{\nu\mu}$, satisfying

$$\partial_\nu \mathcal{F}_{\rho}^{\nu\mu} = \partial_\nu \tilde{\mathcal{F}}_{\rho}^{\nu\mu}.\quad (38)$$

As noted before, the tensor $\mathcal{K}_k^{\nu_1 \dots \nu_k \mu}$ defined in Equation (14) is manifestly totally symmetric with respect to the first k -indices $\{\nu_1, \dots, \nu_k\}$. The $\delta\mathcal{L}$ -identities (17) say that it vanishes if further symmetrized, including the last index μ ; namely, taking the cyclic permutation of the $k+1$ indices $\{\nu_1, \dots, \nu_k, \mu\}$,

$$\mathcal{K}_k^{\nu_1 \dots \nu_k \mu} + \mathcal{K}_k^{\nu_2 \dots \nu_k \mu \nu_1} + \mathcal{K}_k^{\nu_3 \dots \nu_k \mu \nu_1 \nu_2} + \dots + \mathcal{K}_k^{\mu \nu_1 \dots \nu_k} = 0.\quad (39)$$

If we apply k -ple divergence $\partial_{\nu_1 \dots \nu_k}$ to this, the ν_j indices become dummy, and, due to the manifest total symmetry among the first k indices of \mathcal{K}_k , the k terms from the second to the last yield the same quantity, and we obtain

$$\partial_{\nu_1 \dots \nu_k} \mathcal{K}_k^{\nu_1 \dots \nu_k \mu} + k \partial_{\nu_1 \dots \nu_k} \mathcal{K}_k^{\nu_2 \dots \nu_k \mu \nu_1} = 0.\quad (40)$$

Or, taking $k \rightarrow k+1$ and renaming $\nu_{k+1} \rightarrow \nu$ in the first term and $\nu_1 \rightarrow \nu$ in the second term, we have

$$\partial_\nu \left(\partial_{\nu_1 \dots \nu_k} \mathcal{K}_{k+1}^{\nu_1 \dots \nu_k \nu \mu} + (k+1) \partial_{\nu_1 \dots \nu_k} \mathcal{K}_{k+1}^{\nu_2 \dots \nu_k \mu \nu} \right) = 0..\quad (41)$$

This means that $\mathcal{K}_{k+1}^{\nu_1 \dots \nu_k \nu \mu}$ can be replaced by a $\nu \mu$ antisymmetric tensor, which we can define as

$$\tilde{\mathcal{K}}_{k+1}^{\nu_1 \dots \nu_k \nu \mu} = \frac{k+1}{k+2} \left(\mathcal{K}_{k+1}^{\nu_1 \dots \nu_k \nu \mu} - \mathcal{K}_{k+1}^{\nu_1 \dots \nu_k \mu \nu} \right). \quad (42)$$

Indeed, the difference between \mathcal{K}_{k+1} and $\tilde{\mathcal{K}}_{k+1}$ is given by

$$\mathcal{K}_{k+1}^{\nu_1 \dots \nu_k \nu \mu} - \tilde{\mathcal{K}}_{k+1}^{\nu_1 \dots \nu_k \nu \mu} = \frac{1}{k+2} \left(\mathcal{K}_{k+1}^{\nu_1 \dots \nu_k \nu \mu} + (k+1) \mathcal{K}_{k+1}^{\nu_1 \dots \nu_k \mu \nu} \right), \quad (43)$$

whose $(k+1)$ -ple divergence $\partial_\nu \partial_{\nu_1 \dots \nu_k}$ is guaranteed to vanish by Equation (41). Thus, we find that the ‘field-strength’ $\mathcal{F}_\rho^{\nu \mu}$ in Equation (37) can be replaced by the $\nu \mu$ antisymmetric one:

$$\tilde{\mathcal{F}}_\rho^{\nu \mu} = \sum_{k=0}^{N-1} (-)^k \partial_{\nu_1 \dots \nu_k} \tilde{\mathcal{K}}_{k+1}^{\nu_1 \dots \nu_k \nu \mu}. \quad (44)$$

With this antisymmetric field strength, the gravity equation of motion is finally written in the desired form of the Maxwell-type equation:

$$-\frac{\delta S}{\delta \phi^j} T_{\rho}^{j\mu} = J_{\rho}^{\mu} - \partial_{\nu} \tilde{\mathcal{F}}_{\rho}^{\mu\nu}. \quad (45)$$

This is an equation for the gauge-unfixed classical system.

Here, we note a more explicit expression for $\tilde{\mathcal{K}}_{k+1}$ in terms of the Lagrangian. Substituting expression (14) for \mathcal{K}_k into Definition (42), we note that the G_{ρ}^j -proportional part contained in $\mathcal{K}_{k+1}^{\nu_1 \dots \nu_k \nu \mu}$ is $\nu \mu$ symmetric so that only the $T_{\rho}^{j\mu}$ -proportional part contributes to $\tilde{\mathcal{K}}_{k+1}$, and obtain

$$\tilde{\mathcal{K}}_{k+1}^{\nu_1 \dots \nu_k \nu \mu} \equiv \frac{k+1}{k+2} \sum_{n=0}^{N-k-1} {}_{n+k+1} C_{k+1} \left(\mathcal{L}_j^{\nu_1 \dots \nu_k \nu \alpha_1 \dots \alpha_n} \partial_{\alpha_1 \dots \alpha_n} T_{\rho}^{j\mu} - (\nu \leftrightarrow \mu) \right) \text{ for } k = 0, 1, 2, \dots. \quad (46)$$

3. Quantum Theory with de Donder Gauge

Let us now consider the quantum system. We add the gauge-fixing and corresponding Faddeev–Popov (FP) term to the classical GC invariant Lagrangian \mathcal{L}_{cl} . (We call the Lagrangian \mathcal{L} in the previous section \mathcal{L}_{cl} hereafter.) We actually adopt Nakanishi’s simpler form of $\mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{FP}} = \mathcal{L}_{\text{GF+FP}}$ [4,5]:

$$\begin{aligned} \mathcal{L}_{\text{GF+FP}} &= h \delta_N(i\kappa^{-1} g^{\mu\nu} \partial_\mu \bar{c}_\nu) = -\kappa^{-1} \tilde{g}^{\mu\nu} \partial_\mu b_\nu - i\tilde{g}^{\mu\nu} \cdot \partial_\mu \bar{c}_\rho \cdot \partial_\nu c^\rho \\ &= \delta_B(i\kappa^{-1} \tilde{g}^{\mu\nu} \partial_\mu \bar{c}_\nu) - \partial_\mu \left(i\tilde{g}^{\lambda\nu} \partial_\lambda \bar{c}_\nu \cdot c^\mu \right), \end{aligned} \quad (47)$$

with $h \equiv \sqrt{-g}$ and $\tilde{g}^{\mu\nu} \equiv hg^{\mu\nu}$. Here, the usual BRS transformation δ_B (obtained by replacing $-\varepsilon^\mu(x) \rightarrow \kappa c^\mu(x)$ for the usual gravity/matter fields) is given by a sum of Nakanishi’s BRS δ_N and the translation $-\kappa c^\lambda \partial_\lambda$:

$$\begin{aligned} \delta_B \Phi &= \delta_N \Phi - \delta_N(x^\lambda) \partial_\lambda \Phi, \\ \delta_N(x^\lambda) &= \kappa c^\lambda, \quad \delta_N \Phi = -\kappa \partial_\mu c^\alpha \cdot [\Phi]_\alpha^\mu, \\ \delta_B \tilde{g}^{\mu\nu} &= \kappa \left(\partial_\lambda c^\mu \cdot \tilde{g}^{\lambda\nu} + \partial_\lambda c^\nu \cdot \tilde{g}^{\mu\lambda} - \partial_\lambda (c^\lambda \tilde{g}^{\mu\nu}) \right) \\ \delta_B \bar{c}_\mu &= iB_\mu, \quad \delta_N \bar{c}_\mu = i b_\mu, \quad B_\mu = b_\mu + i\kappa c^\lambda \partial_\lambda \bar{c}_\mu, \\ \delta_N c^\mu &= 0, \quad \delta_B c^\mu = -\kappa c^\lambda \partial_\lambda c^\mu. \end{aligned} \quad (48)$$

We call this gauge specified by the gauge-fixing and FP term $\mathcal{L}_{\text{GF+FP}}$ in (47) “de Donder–Nakanishi gauge”. It corresponds to the de Donder–Landau gauge possessing no $\alpha \eta^{\mu\nu} B_\mu B_\nu$ term, violating $GL(4)$ invariance by the use of $\eta^{\mu\nu}$. Since the present $\mathcal{L}_{\text{GF+FP}}$

for the de Donder–Nakanishi gauge is given in a usual BRS exact form for the de Donder–Landau gauge up to a total derivative term as shown in the last expression in Equation (47), it is also invariant under the usual BRS transformation δ_B . The use of Nakanishi’s BRS δ_N , which represents the tensorial transformation part of the usual BRS transformation δ_B , and the use of the $b_\mu = i^{-1} \delta_N \bar{c}_\mu$ field, in particular, make manifest the existence of much larger $IOSp(8|8)$ symmetry, called *choral symmetry* by Nakanishi, which contains symmetries of energy-momentum, $GL(4)$, BRS, FP-ghost scale transformation, etc., as will be discussed explicitly in the next section.

We still consider the GC transformation $x^\rho \rightarrow x'^\rho = x^\rho - \varepsilon^\rho(x)$ in this quantum theory to derive identities. The gravity/matter fields ϕ^j are transformed in the same way as before:

$$\delta\phi^j(x) = G_\rho^j \varepsilon^\rho(x) + T_{\rho}^{j\mu} \partial_\mu \varepsilon^\rho(x), \quad \text{with } G_\rho^j = \partial_\rho \phi^j, \quad T_{\rho}^{j\mu} = [\phi^j]_\rho^\mu. \quad (49)$$

We call the newly added fields b_μ , \bar{c}_μ and c^μ ghost fields collectively, and treat them all as scalar fields under GC transformation, namely denoting ghost fields by $\phi^M = (b_\mu, \bar{c}_\mu, c^\mu)$ collectively:

$$\delta\phi^M(x) = G_\rho^M \varepsilon^\rho(x) + T_{\rho}^{M\mu} \partial_\mu \varepsilon^\rho(x), \quad \text{with } G_\rho^M = \partial_\rho \phi^M, \quad T_{\rho}^{M\mu} = 0. \quad (50)$$

Of course, \mathcal{L}_{GF+FP} is not invariant under the GC transformation, but we can easily calculate the change by noting the structure of the \mathcal{L}_{GF+FP} , which is written formally as a scalar density:

$$\mathcal{L}_{GF+FP} = -\kappa^{-1} \tilde{g}^{\mu\nu} E_{\mu\nu}, \quad E_{\mu\nu} = \partial_\mu b_\nu + i\kappa \cdot \partial_\mu \bar{c}_\rho \cdot \partial_\nu c^\rho. \quad (51)$$

If the ghost part tensor $E_{\mu\nu}$ truly behaved as a $\mu\nu$ covariant tensor, \mathcal{L}_{GF+FP} would be a scalar density transforming only into the total divergence $\partial_\mu (\mathcal{L}_{GF+FP} \varepsilon^\mu)$. This is actually true for the FP ghost part $i\kappa \cdot \partial_\mu \bar{c}_\rho \cdot \partial_\nu c^\rho$ in $E_{\mu\nu}$ since c_ρ and \bar{c}^ρ are regarded as scalars, so that their simple derivatives $\partial_\mu \bar{c}_\rho$ and $\partial_\nu c^\rho$ behave as μ and ν vectors, giving the desired $\mu\nu$ tensor as a product. But the NL field part $\partial_\mu b_\nu$ transforms only as a μ vector since b_ν is regarded as a scalar, so that the ν leg rotation part of the transformation of $\tilde{g}^{\mu\nu}$, i.e., $\delta\tilde{g}^{\mu\nu} \supset -\tilde{g}^{\mu\rho} \partial_\rho \varepsilon^\nu$, is not canceled. We thus see

$$\delta\mathcal{L}_{GF+FP} = \kappa^{-1} \tilde{g}^{\mu\rho} \partial_\mu b_\nu \cdot \partial_\rho \varepsilon^\nu + \partial_\mu (\mathcal{L}_{GF+FP} \varepsilon^\mu). \quad (52)$$

Thus, the total Lagrangian in our quantum gravity theory

$$\mathcal{L} = \mathcal{L}_{cl} + \mathcal{L}_{GF+FP} \quad (53)$$

changes under the GC transformation as

$$\delta\mathcal{L} = \partial_\mu (\mathcal{L} \varepsilon^\mu) + \kappa^{-1} \tilde{g}^{\mu\rho} \partial_\mu b_\nu \cdot \partial_\rho \varepsilon^\nu = \partial_\rho \mathcal{L} \cdot \varepsilon^\rho + (\mathcal{L} \delta_\rho^\mu + \kappa^{-1} \tilde{g}^{\mu\nu} \partial_\nu b_\rho) \partial_\mu \varepsilon^\rho. \quad (54)$$

This namely differs from Equation (11) in the classical system case, only in the point that the $\kappa^{-1} \tilde{g}^{\mu\nu} \partial_\nu b_\rho$ term is added in the first order derivative term $\propto \partial_\mu \varepsilon^\rho$. Therefore, the $\delta\mathcal{L}$ -identities in the previous section almost all remain the same and only the first order $\propto \partial_\mu \varepsilon^\rho$ identity (16) is slightly changed into

$$\mathcal{K}_0^\mu_\rho - (\mathcal{L} \delta_\rho^\mu + \kappa^{-1} \tilde{g}^{\mu\nu} \partial_\nu b_\rho) = 0. \quad (55)$$

Note that we should now understand that \mathcal{L} is the total Lagrangian containing the ghost part \mathcal{L}_{GF+FP} , and the fields ϕ^j cover not only the gravity/matter fields ϕ^j but also the ghost fields $\phi^M = (b_\mu, \bar{c}_\mu, c^\mu)$. The equation of motion $\delta S/\delta\phi$, of course, takes the same form (18) as before. The zeroth order $\delta\mathcal{L}$ -identity (15), in particular, remains the same, and the global translation current (energy-momentum tensor) is given by the same form of equation as Equation (22):

$$\begin{aligned}
-\frac{\delta S}{\delta \phi^j} G_\rho^j &= \partial_\mu J_\rho^\mu \\
J_\rho^\mu &= \sum_{n=0}^{N-1} \mathcal{L}_j^{i\mu\alpha_1\cdots\alpha_n} \overleftrightarrow{\partial}_{\alpha_1\cdots\alpha_n} G_\rho^j - \delta_\rho^\mu \mathcal{L}.
\end{aligned} \tag{56}$$

Thus, Equation (31) for \mathcal{J}_0 holds unchanged. The identity (35) also holds as it stands. In going from Equation (35) to the gravity Equation (36), however, the $\mathcal{L}\delta_\rho^\mu$ term from \mathcal{J}_0 now does not totally cancel the first term $\mathcal{K}_{k=0}^\mu = \mathcal{L}\delta_\rho^\mu + \kappa^{-1}\tilde{g}^{\mu\nu}\partial_\nu b_\rho$ but leaves the $\kappa^{-1}\tilde{g}^{\mu\nu}\partial_\nu b_\rho$ term. Thus, Equation (36) is now replaced by

$$\frac{\delta S}{\delta \phi^j} T_\rho^{j\mu} = -J_\rho^\mu + \kappa^{-1}\tilde{g}^{\mu\nu}\partial_\nu b_\rho + \sum_{k=1}^N (-)^k \partial_{\nu_1\cdots\nu_k} \mathcal{K}_k^{\nu_1\cdots\nu_k\mu}. \tag{57}$$

Note, here, that the implicit summation over ϕ^j also contains the ghost fields ϕ^M which contribute only to the $n = 0$ terms since the ghost fields appear only in the first order derivatives in the de Donder–Nakanishi gauge Lagrangian (47).

The final form of the Maxwell-type gravity field equation is therefore given by

$$-\frac{\delta S}{\delta \phi^j} T_\rho^{j\mu} = J_\rho^\mu - \kappa^{-1}\tilde{g}^{\mu\nu}\partial_\nu b_\rho - \partial_\nu \tilde{\mathcal{F}}_\rho^{\mu\nu}. \tag{58}$$

in place of the previous classical one (45). The expressions in Equation (44) for the field-strength $\tilde{\mathcal{F}}_\rho^{\mu\nu}$ and Equation (46) for the quantities $\tilde{\mathcal{K}}_{k+1}^{\nu_1\cdots\nu_k\nu\mu}$ remain the same as before. Here, \mathcal{L} is understood to be the total Lagrangian, but only the classical Lagrangian part actually contributes in this case, because all the ghost fields $\phi^M = \{b_\mu, \bar{c}_\mu, c^\mu\}$ have vanishing contributions since $T_{M\rho}^\mu = 0$ for them. That is, *the field-strength is in fact the same as that in the classical theory with Lagrangian L_{cl}* .

One may wonder why the final Maxwell-type gravity equation of motion (58) is slightly different from the Yang–Mills case, since the present ghost field term $\kappa^{-1}\tilde{g}^{\mu\nu}\partial_\nu b_\rho$ is not written in a *BRS exact form* such as $\{Q_B, D_\mu \bar{c}\}$ in the latter. It is actually possible to rewrite Equation (58) into such a form. Indeed, the term $\kappa^{-1}\tilde{g}^{\mu\nu}\partial_\nu b_\rho$ is in fact BRS exact up to a divergence of an antisymmetric tensor:

$$-\kappa^{-1}\tilde{g}^{\mu\nu}\partial_\nu b_\rho = \delta_B(i\kappa^{-1}\tilde{g}^{\mu\nu}\partial_\nu \bar{c}_\rho) - \partial_\nu \left(i(c^\mu \tilde{g}^{\nu\sigma} - c^\nu \tilde{g}^{\mu\sigma}) \partial_\sigma \bar{c}_\rho \right). \tag{59}$$

The gravity field Equation (58), therefore, can be rewritten into quite a similar form as the Maxwell-type YM equation:

$$\partial_\nu \tilde{\mathcal{F}}_\rho^{\mu\nu} + \left\{ Q_B, \kappa^{-1}\tilde{g}^{\mu\nu}\partial_\nu \bar{c}_\rho \right\} = J_\rho^\mu, \tag{60}$$

where we have written $\delta_B(\cdots) = \{iQ_B, \cdots\}$ in terms of the BRS charge Q_B and defined a modified field strength $\tilde{\mathcal{F}}_\rho^{\mu\nu}$:

$$\tilde{\mathcal{F}}_\rho^{\mu\nu} = \tilde{\mathcal{F}}_\rho^{\mu\nu} + i(c^\mu \tilde{g}^{\nu\sigma} - c^\nu \tilde{g}^{\mu\sigma}) \partial_\sigma \bar{c}_\rho. \tag{61}$$

This form of the Maxwell-type equation (60) with the BRS exact term was also derived for the Einstein theory case by Nakanishi [12].

4. Noether Current for the Choral Symmetries in a Generic Higher Derivative System

BRS symmetry or, more generally, choral symmetries $IOSp(8|8)$ exist for any GC transformation invariant systems if one adopts the gauge-fixing Lagrangian (47) of the de

Donder–Nakanishi gauge [11]. This is because the currents of the choral symmetries are conserved as far as the equations of motion

$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu X^M) = 0 \quad (62)$$

hold for the 16 component ‘fields’ (=4d coordinate x^μ and three fields) [9,10].

$$X^M = (\hat{x}^\mu, b_\mu, c^\mu, \bar{c}_\mu), \quad \hat{x}^\mu \equiv x^\mu/\kappa. \quad (63)$$

Indeed, this equation of motion for the coordinate $X^M \propto x^\rho$ actually implies the de Donder condition on the gravity field:

$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu x^\rho) = \partial_\mu \tilde{g}^{\mu\rho} = 0. \quad (64)$$

In addition, the FP ghost equations of motion

$$\frac{\partial \mathcal{L}_{\text{FP}}}{\partial \bar{c}_\rho} = -i\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu c^\rho) = 0, \quad \frac{\partial \mathcal{L}_{\text{FP}}}{\partial c^\rho} = +i\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu \bar{c}_\rho) = 0 \quad (65)$$

directly follow from the gauge-fixing Lagrangian (47), implying the equations for $X^M = c^\rho$ and \bar{c}_ρ . The equation for $X^M = b_\rho$ may be a bit non-trivial, but we now already know the Maxwell-type gravity equation of motion (58), the divergence ∂_μ of which immediately leads to

$$\partial_\mu(\tilde{g}^{\mu\nu}\partial_\nu b_\rho) = 0. \quad (66)$$

These 16 components’ *d’Alembert’s equations of motion* hold if and only if the gauge-fixing Lagrangian $\mathcal{L}_{\text{GF+FP}}$ is given by that of the de Donder–Nakanishi (47), which can be written in a manifestly $OSp(8|8)$ invariant form. (The following discussion on the $OSp(8|8)$ invariance may be viewed as a mere recapitulation of Nakanishi’s paper [10,11], but we have simplified and made in particular the signs and i factors more tractable by introducing a hermitian $OSp(8|8)$ metric (68). The derivation of the $OSp(8|8)$ Noether current in the higher derivative system is, of course, new.)

$$\begin{aligned} \mathcal{L}_{\text{GF+FP}} &= -\kappa^{-1}hE = -\kappa^{-1}hg^{\mu\nu}E_{\mu\nu}, \\ E_{\mu\nu} &= \frac{1}{2}(\partial_\mu b_\nu + i\kappa\partial_\mu \bar{c}_\rho \cdot \partial_\nu c^\rho + (\mu \leftrightarrow \nu)) \\ &= \frac{1}{2}(\partial_\mu b_\rho \cdot \partial_\nu x^\rho + i\kappa\partial_\mu \bar{c}_\rho \cdot \partial_\nu c^\rho + (\mu \leftrightarrow \nu)) \\ &= \frac{\kappa}{2}\eta_{NM}\partial_\mu X^M\partial_\nu X^N. \end{aligned} \quad (67)$$

where η_{MN} is the $OSp(8|8)$ metric given by

$$\eta_{NM} = \begin{pmatrix} \delta_\mu^\nu & & \\ \hline & & \\ \delta_\nu^\mu & & i\delta_\mu^\nu \\ \hline & & -i\delta_\nu^\mu \end{pmatrix} = \eta^{NM} \quad (\text{inverse}). \quad (68)$$

Note the symmetry property of this (*c*-number) metric:

$$\eta_{MN} = (-)^{|M|\cdot|N|}\eta_{NM} = (-)^{|M|}\eta_{NM} = (-)^{|N|}\eta_{NM} \equiv \tilde{\eta}_{NM}, \quad (69)$$

where the statistics index $|M|$ is 0 or 1 when X^M is bosonic or fermionic, respectively. This property (69) is because η_{MN} is ‘diagonal’ in the sense that its off-diagonal, boso-fermi, and fermi-bose matrix elements vanish, i.e., $\eta_{MN} = 0$ when $|M| \neq |N|$, so that

$|M| = |N| = |M| \cdot |N|$ in front of η_{MN} . Note also that the $\tilde{\eta}_{NM}$ introduced here is simply the transposed metric $\eta_{NM}^T = \eta_{MN}$. Thus, we have

$$\begin{aligned}\frac{\kappa}{2} \eta_{NM} X^M X^N &= \frac{\kappa}{2} X^M \tilde{\eta}_{MN} X^N \\ &= \frac{\kappa}{2} (\hat{x}^\mu b_\mu c^\mu \bar{c}_\mu) \begin{pmatrix} \delta_\nu^\mu & \delta_\mu^\nu \\ \hline & -i\delta_\mu^\nu \\ & +i\delta_\nu^\mu \end{pmatrix} \begin{pmatrix} \hat{x}^\nu \\ b_\nu \\ c^\nu \\ \bar{c}_\nu \end{pmatrix} \\ &= \frac{\kappa}{2} (\hat{x}^\mu b_\mu + b_\mu \hat{x}^\mu - i c^\mu \bar{c}_\mu + i \bar{c}_\mu c^\mu) = b_\mu x^\mu + i \kappa \bar{c}_\mu c^\mu.\end{aligned}$$

Noting the d'Alembert's equations of motion for X^M , Nakanishi constructed the conserved currents.

$$\mathcal{M}^{MN\mu} \equiv \tilde{g}^{\mu\nu} (X^M \overset{\leftrightarrow}{\partial}_\nu X^N) \quad (70)$$

$$\mathcal{P}^{M\mu} \equiv \tilde{g}^{\mu\nu} \partial_\nu X^M = \tilde{g}^{\mu\nu} (1 \overset{\leftrightarrow}{\partial}_\nu X^M). \quad (71)$$

(Our current $\mathcal{M}^{MN\mu}$ presented here is not exactly equal to Nakanishi's original one [11], $\mathcal{M}^\mu(X^M, X^N)$, but the precise relation reads $\mathcal{M}^\mu(X^M, X^N) = i^{|M| \cdot |N|} \mathcal{M}^{MN\mu}$.)

He showed from the equal-time commutation relations (ETCR) derived in the Einstein gravity theory that their charge operators

$$\begin{aligned}M^{MN} &\equiv \int d^3x \mathcal{M}^{MN0} = (-)^{1+|M|\cdot|N|} M^{NM}, \\ P^M &\equiv \int d^3x \mathcal{P}^{M0}\end{aligned} \quad (72)$$

generate the following transformations on all the fields Φ , gravity and matter fields ϕ^j , and the $OSp(8|8)$ ghost-'fields' $X^M = (\hat{x}^\mu, b_\mu, c^\mu, \bar{c}_\mu)$:

$$[iM^{MN}, \Phi] \equiv \delta^{MN}\Phi = \delta_N^{MN}\Phi - \kappa(\delta_N^{MN}\hat{x}^\rho)\partial_\rho\Phi, \quad (73)$$

where the Nakanishi transformation δ_N^{MN} is an OSp rotation for the $OSp(8|8)$ ghost-fields X^L , given by

$$\delta_N^{MN} X^L = -\tilde{\eta}^{NL} X^M + (-)^{|M|\cdot|N|} \tilde{\eta}^{ML} X^N. \quad (74)$$

This transformation, in particular, gives for the coordinate \hat{x}^ρ

$$\delta_N^{MN} \hat{x}^\rho = -\tilde{\eta}^{N\hat{x}^\rho} X^M + \tilde{\eta}^{M\hat{x}^\rho} X^N \equiv (-\kappa)^{-1} \mathcal{E}^{MN\rho}, \quad (75)$$

which is nonvanishing only when X^M or/and X^N is b_μ . Additionally, the Nakanishi transformation of the gravity/matter fields ϕ^j is given by

$$\delta_N^{MN} \phi^j = \partial_\mu \mathcal{E}^{MN\nu} \cdot [\phi^j]_\nu^\mu. \quad (76)$$

Therefore, if either X^M or X^N equals b_μ , the δ^{MN} transformation is just the GC transformation with transformation parameter $\varepsilon^\rho(x) \rightarrow \mathcal{E}^{MN\rho} = -\kappa(\delta_N^{MN}\hat{x}^\rho)$ for the gravity/matter fields ϕ^j ,

$$\begin{aligned}\delta^{MN} \phi^j &= \mathcal{E}^{MN\rho} G_\rho^j + \partial_\mu \mathcal{E}^{MN\rho} \cdot T_\rho^{j\mu} \quad (G_\rho^j = \partial_\rho \phi^j, \quad T_\rho^{j\mu} = [\phi^j]_\rho^\mu), \\ &\text{with field dependent parameter } \mathcal{E}^{MN\rho} = -\kappa(\delta_N^{MN}\hat{x}^\rho),\end{aligned} \quad (77)$$

and, for the $OSp(8|8)$ ghost-fields X^L , the GC transformation as scalar fields *plus* an OSp rotation:

$$\delta^{MN} X^L = \mathcal{E}^{MN\rho} G_\rho^L - (\tilde{\eta}^{NL} X^M - (-)^{|M| \cdot |N|} \tilde{\eta}^{ML} X^N) \quad (G_\rho^L = \partial_\rho X^L). \quad (78)$$

Note, here, that the ‘transformation parameter’ $\mathcal{E}^{MN\rho}$ may now be fermionic when $|M| + |N| = 1$. We have therefore put the factors G_ρ^j and $T_\rho^{j\mu}$ linear in ϕ^j behind the parameter $\mathcal{E}^{MN\rho}$ in Equations (77) and (78) to avoid the sign factor $(-1)^{|\phi^j|(|M|+|N|)}$.

The choral invariance of our total Lagrangian $\mathcal{L} = \mathcal{L}_{\text{cl}} + \mathcal{L}_{\text{GF+FP}}$ is now clear; the gravity/matter fields only receive a special GC transformation with parameter $\varepsilon^\rho(x) = \mathcal{E}^{MN\rho}$ in Equation (77), and so the \mathcal{L}_{cl} part is invariant. The gauge-fixing Lagrangian $\mathcal{L}_{\text{GF+FP}}$ is also clearly invariant since in this form of the Lagrangian (67), the OSp -vector field components X^M , including the coordinate \hat{x}^ρ , are treated as scalar fields, and hence $E_{\mu\nu}$ is clearly $\mu\nu$ tensor, and $hg^{\mu\nu}E_{\mu\nu}$ is manifestly a GC scalar density. Moreover, $E_{\mu\nu}$ written in the form (67) is manifestly invariant under (global) $OSp(8|8)$ rotation. Note that this invariance is made manifest by making the mere parameter coordinate x^μ transform as if being a field both under the GC transformation and the OSp rotation; in fact, those two transformations on \hat{x}^μ cancel each other, and the coordinate x^μ remains intact under δ^{MN} as any non-field parameters should be: indeed, Equation (73) indicates for $\Phi = \hat{x}^\mu$,

$$\delta^{MN} \hat{x}^\mu = \delta_N^{MN} \hat{x}^\mu - \kappa (\delta_N^{MN} \hat{x}^\rho) \partial_\rho \hat{x}^\mu = 0, \quad (79)$$

where the first term is the OSp rotation, and the second term is the GC transformation of \hat{x}^μ regarded as a ‘scalar field’.

Let us now compute the Noether currents corresponding to these choral symmetries in our general higher derivative GC invariant system. We shall show that the Noether currents coincide with the Nakanishi’s simple form (70), aside from the divergence of an antisymmetric tensor.

To do this systematically, we devise a local version of the choral symmetry transformation (73), or (77) and (78). We multiply them by a local graded transformation parameter $\varepsilon_{NM}(x)$ *from the left*, so that it reduces to the original $OSp(8|8)$ transformation in the global limit $\varepsilon_{NM}(x) \rightarrow \varepsilon_{NM}:\text{const.}$; namely, we define the transformation,

$$\begin{aligned} \delta\phi^j &= (\varepsilon_{NM} \mathcal{E}^{MN\rho}) G_\rho^j + \partial_\mu (\varepsilon_{NM} \mathcal{E}^{MN\rho}) \cdot T_\rho^{j\mu} \quad \text{for gravity/matter fields } \phi^j, \\ \delta X^L &= (\varepsilon_{NM} \mathcal{E}^{MN\rho}) G_\rho^L \\ &\quad - \varepsilon_{NM} (\tilde{\eta}^{NL} X^M - (-)^{|M| \cdot |N|} \tilde{\eta}^{ML} X^N) \quad \text{for } OSp \text{ coordinate fields } X^L. \end{aligned} \quad (80)$$

We take our parameter ε_{NM} Grassmann even or odd according to $|N| + |M| = 0$ or 1, respectively, so that the product $(\varepsilon_{NM} \mathcal{E}^{MN\rho})$ always becomes an ordinary bosonic ‘parameter’ and hence can be moved to anywhere without worrying about sign changes. Note, however, that, in order to correctly obtain the Noether current corresponding to the OSp transformation δ^{MN} in Equation (73), or Equations (77) and (78), we have to factor out the parameter ε_{NM} from the left since it is multiplied from the left here. However, the general procedure explained in the previous sections to derive the Noether current in the higher derivative theories, which we follow now, has placed the transformation parameter at the *most right end*, and the troublesome point is that the transformation parameter for the OSp transformation is the graded one, ε_{NM} , but not the bosonic product $(\varepsilon_{NM} \mathcal{E}^{MN\rho})$. It is necessary to move these graded quantities separately and freely to apply the general procedure to this case, although the transformation parameter ε_{NM} eventually has to be factored out from the left. The best way to forget about the bothering sign factors appearing in changing the order of the graded quantities is to adopt a convention similar to the so-called ‘*implicit grading*’ [29]. We take as a *natural order* of those graded quantities, ε_{NM} first, $\mathcal{E}^{MN\rho}$ second, and other graded quantities such as X^M , G_ρ^j , and $T^{j\mu\nu}_\rho$ third. Initially,

these quantities appear in this natural order, since the product factor $(\varepsilon_{NM}\mathcal{E}^{MN\rho})$ appearing in Equation (80) is bosonic and can be placed on the most left in any case. Then, from this natural order, we freely move those factors separately anywhere without writing any sign factors. The implicit grading scheme means that the correct sign factors should be recovered when necessary; that is, in any terms containing those graded quantities, the necessary sign factor can be found by counting how many instances of changing order are necessary to bring those factors into the natural order. We hereafter adopt this implicit grading scheme.

We should note that the GC transformation part of this transformation (80) now takes exactly the same form as the GC transformation (8) with the (bosonic) transformation parameter $\varepsilon^\rho(x) \equiv \varepsilon_{NM}\mathcal{E}^{MN\rho}$:

$$\delta\Phi^j = G_\rho^j \varepsilon^\rho + T_\rho^{j\mu} \partial_\mu \varepsilon_\rho \quad (81)$$

for all the fields $\Phi^j = (\phi^j, X^M)$, and so the total action is still invariant, meaning that the total Lagrangian transforms as a scalar density: $\delta\mathcal{L} = \partial_\mu(\mathcal{L}\varepsilon^\mu) = \partial_\mu(\mathcal{L}\varepsilon_{NM}\mathcal{E}^{MN\rho})$.

As for the remaining $OSp(8|8)$ rotation part of the OSp coordinate ‘fields’ X^L ,

$$\delta^{OSp} X^L \equiv -\varepsilon_{NM}(\tilde{\eta}^{NL} X^M - (-)^{|M|\cdot|N|} \tilde{\eta}^{ML} X^N), \quad (82)$$

However, the \mathcal{L}_{GF+FP} in Equation (67) is no longer invariant under the rotation with the x -dependent parameter ε_{MN} . As shown explicitly in Appendix A, we can immediately find the change of \mathcal{L}_{GF+FP} as Equation (A6):

$$\delta\mathcal{L}_{GF+FP} = \tilde{g}^{\mu\nu} \partial_\mu \varepsilon_{NM} \cdot (X^M \overset{\leftrightarrow}{\partial}_\nu X^N) = \tilde{g}^{\mu\nu} (X^M \overset{\leftrightarrow}{\partial}_\nu X^N) \partial_\mu \varepsilon_{NM}, \quad (83)$$

where it should be noted that we have already used ‘implicit grading’ in the last equality. Following this implicit grading scheme, we can write the change of our total Lagrangian $\mathcal{L} = \mathcal{L}_{cl} + \mathcal{L}_{GF+FP}$ under our transformation (80) in the form

$$\begin{aligned} \delta\mathcal{L} &= \partial_\mu(\mathcal{L}\mathcal{E}^{MN\mu}\varepsilon_{NM}) + \tilde{g}^{\mu\nu} (X^M \overset{\leftrightarrow}{\partial}_\nu X^N) \cdot \partial_\mu \varepsilon_{NM} \\ &= \partial_\mu(\mathcal{L}\mathcal{E}^{MN\mu}) \cdot \varepsilon_{NM} + (\mathcal{L}\mathcal{E}^{MN\mu} + \tilde{g}^{\mu\nu} (X^M \overset{\leftrightarrow}{\partial}_\nu X^N)) \partial_\mu \varepsilon_{NM}. \end{aligned} \quad (84)$$

Now, we can rewrite our transformation in the same form as the general gauge transformation (8), which contains the zero-th and first order differentiation of the transformation parameter ε_{NM} ; that is, it is unifiedly given for gravity/matter and ghost fields $\Phi^I = (\phi^j, X^L)$ in the form

$$\delta\Phi^I = \mathcal{G}^{IMN}\varepsilon_{NM} + \mathcal{T}^{IMN\mu}\partial_\mu\varepsilon_{NM}, \quad (85)$$

where the coefficients \mathcal{G}^{IMN} and $\mathcal{T}^{IMN\mu}$ are given as

$$\begin{aligned} \mathcal{G}^{IMN} &= \begin{cases} G_\rho^j \mathcal{E}^{MN\rho} + T_\rho^{j\mu} \partial_\mu \mathcal{E}^{MN\rho} & \text{for } \Phi^I = \phi^j: \text{gravity/matter} \\ G_\rho^L \mathcal{E}^{MN\rho} \\ -(\tilde{\eta}^{NL} X^M - (-)^{|M|\cdot|N|} \tilde{\eta}^{ML} X^N) & \text{for } \Phi^I = X^L: \text{ghost} \end{cases} \\ \mathcal{T}^{IMN\mu} &= \begin{cases} T_\rho^{j\mu} \mathcal{E}^{MN\rho} & \text{for } \Phi^I = \phi^j: \text{gravity/matter} \\ 0 & \text{for } \Phi^I = X^L: \text{ghost} \end{cases}. \end{aligned} \quad (86)$$

Thus, we can now follow the general discussions presented in the previous two sections to derive the Noether currents in this system. We should also note that the Lagrangian change $\delta\mathcal{L}$ is now given by Equation (84) in place of Equation (11). Then, we see that the previous $\delta\mathcal{L}$ -identities following from the coefficients of n -th order derivatives $\partial_{\mu_1 \dots \mu_n} \varepsilon^\rho$ of the transformation parameter ε^ρ , Equations (15) and (17), now also hold with

the understanding that the coefficients G_ρ and T_ρ^μ of the gauge transformation (8) are now replaced by those of the present transformation: that is, by making the replacement

$$G_\rho^j \rightarrow \mathcal{G}^{IMN}, \quad T_\rho^\mu \rightarrow \mathcal{T}^{IMN\mu}, \quad (87)$$

and taking account of the form of $\delta\mathcal{L}$ given in Equation (84), we find the identities as coefficients of n -th order derivatives $\partial_{\mu_1 \dots \mu_n} \mathcal{E}_{NM}$ with $n = 0, 1$ and $n \geq 2$, respectively,

$$\sum_{n=0}^N \mathcal{L}_I^{\mu_1 \dots \mu_n} \partial_{\mu_1 \dots \mu_n} \mathcal{G}^{IMN} - \partial_\rho (\mathcal{L} \mathcal{E}^{MN\rho}) = 0 \quad (88)$$

$$\mathcal{K}_0^{MN\mu} - \mathcal{L} \mathcal{E}^{MN\mu} - \tilde{g}^{\mu\nu} (X^M \overset{\leftrightarrow}{\partial}_\nu X^N) = 0 \quad (89)$$

$$\text{Sym}_{\{\nu_1 \dots \nu_k \mu\}} (\mathcal{K}_k^{MN\nu_1 \dots \nu_k \mu}) = 0 \quad \text{for } k = 1, 2, \dots, \quad (90)$$

where $\mathcal{K}_k^{MN\nu_1 \dots \nu_k \mu}$ now reads

$$\begin{aligned} \mathcal{K}_k^{MN\nu_1 \dots \nu_k \mu} \equiv & \sum_{n=0}^{N-k-1} {}_{n+k+1} C_{k+1} \mathcal{L}_I^{\nu_1 \dots \nu_k \mu \alpha_1 \dots \alpha_n} \partial_{\alpha_1 \dots \alpha_n} \mathcal{G}^{IMN} \\ & + \sum_{n=0}^{N-k} {}_{n+k} C_k \mathcal{L}_I^{\nu_1 \dots \nu_k \alpha_1 \dots \alpha_n} \partial_{\alpha_1 \dots \alpha_n} \mathcal{T}^{IMN\mu}. \end{aligned} \quad (91)$$

Then, by combining the equation of motion (6), we first obtain a conservation equation for the $OSp(8|8)$ Noether current from the 0-th order $\delta\mathcal{L}$ -identity (88) as an analogue of Equation (22) or (56):

$$\begin{aligned} -\frac{\delta S}{\delta \Phi^I} \mathcal{G}^{IMN} &= \partial_\mu J^{MN\mu} \\ J^{MN\mu} &= \sum_{n=0}^{N-1} \mathcal{L}_I^{\mu \alpha_1 \dots \alpha_n} \overset{\longleftrightarrow}{\partial}_{\alpha_1 \dots \alpha_n} \mathcal{G}^{IMN} - \mathcal{L} \mathcal{E}^{MN\mu}. \end{aligned} \quad (92)$$

Second, as an analogue of Equation (45) or Equation (58), we find the desired equation from the first order $\delta\mathcal{L}$ -identity (89) and second or higher order $\delta\mathcal{L}$ -identity (90):

$$-\frac{\delta S}{\delta \Phi^I} \mathcal{T}^{IMN\mu} = J^{MN\mu} - \tilde{g}^{\mu\nu} (X^M \overset{\leftrightarrow}{\partial}_\nu X^N) - \partial_\nu \tilde{\mathcal{F}}^{MN\mu\nu}. \quad (93)$$

where $\tilde{\mathcal{F}}^{MN\mu\nu}$ is the antisymmetric tensor (as an ambiguity term of OSp Noether current) given by

$$\tilde{\mathcal{F}}^{MN\mu\nu} = \sum_{k=0}^{N-1} (-)^k \partial_{\nu_1 \dots \nu_k} \tilde{\mathcal{K}}_{k+1}^{MN\nu_1 \dots \nu_k \mu\nu} \quad (94)$$

with

$$\begin{aligned} \tilde{\mathcal{K}}_{k+1}^{MN\nu_1 \dots \nu_k \mu\nu} &\equiv \frac{k+1}{k+2} \sum_{n=0}^{N-k-1} {}_{n+k+1} C_{k+1} \left(\mathcal{L}_I^{\nu_1 \dots \nu_k \nu \alpha_1 \dots \alpha_n} \partial_{\alpha_1 \dots \alpha_n} \mathcal{T}^{IMN\mu} - (\nu \leftrightarrow \mu) \right) \\ &\text{for } k = 0, 1, 2, \dots. \end{aligned} \quad (95)$$

Equation (93) shows that the Noether current of $OSp(8|8)$ symmetry takes the form on-shell:

$$J^{MN\mu} = \tilde{g}^{\mu\nu} (X^M \overset{\leftrightarrow}{\partial}_\nu X^N) + \partial_\nu \tilde{\mathcal{F}}^{MN\mu\nu}, \quad (96)$$

so that the charge can be given by Nakanishi's simple form:

$$M^{MN} = \int d^3x \, hg^{0\nu} (X^M \overset{\leftrightarrow}{\partial}_\nu X^N) \quad (97)$$

as symmetry generators for any local operators. (It is, however, quite another problem whether the RHS volume integral of (97) gives a well-defined charge even when the symmetry is spontaneously unbroken. Generally, the volume integral converges only when the current contains the ambiguity term $\partial_\nu \tilde{\mathcal{F}}^{MN\mu\nu}$ with a suitable coefficient, which is, in fact, a key point when discussing the spontaneous breaking or non-breaking of the symmetry of the charge.)

Finally, recall that we are adopting implicit grading. Then, the factor $\mathcal{E}^{MN\mu}$ or its derivatives contained in \mathcal{G}^{IMN} or $\mathcal{T}^{IMN\mu}$ in the definitions of the current $J^{NM\mu}$ (92), the field strength $\tilde{\mathcal{F}}^{MN\mu\nu}$ (94), and $\tilde{\mathcal{K}}_{k+1}$ (95) should be placed at the furthest left, and if they are kept in the place as written, then the grading sign factor should be set as necessary for bringing them there from the furthest left. Fortunately, however, the sign factors actually turn out to be unnecessary here. This is because the other graded quantities to jump over when bringing the factor $\mathcal{E}^{MN\mu}$ to the furthest left are essentially only the bosonic Lagrangian \mathcal{L} as a whole. For instance, the Noether current (92) contains the terms

$$\mathcal{L}_I^{i\mu\alpha_1\cdots\alpha_n} \overset{\longleftrightarrow}{\partial}_{\alpha_1\cdots\alpha_n} \mathcal{G}^{IMN} - \mathcal{L} \mathcal{E}^{MN\mu}. \quad (98)$$

In the second term, it is the Lagrangian \mathcal{L} itself for $\mathcal{E}^{MN\mu}$ to jump over. In the first term, on the other hand, $\mathcal{L}_I^{i\mu\alpha_1\cdots\alpha_n} = \partial\mathcal{L}/\partial_{\mu\alpha_1\cdots\alpha_n}\phi^I$ is fermionic when the field ϕ^I is a fermion. However, it is immediately followed by the factor $G_\rho^I = \partial_\rho\phi^I$ in the $\mathcal{G}^{IMN} = G_\rho^I \mathcal{E}^{MN\rho}$, so that the net factor $\mathcal{L}_I^{i\mu\alpha_1\cdots\alpha_n} G_\rho^I$ in front of $\mathcal{E}^{MN\rho}$ is always bosonic, carrying the same statistics as the original \mathcal{L} .

5. Conclusions

In this paper, we considered the general GC invariant theory which contains arbitrarily high order derivative fields. We identified the explicit expression for the energy-momentum as the Noether current corresponding to the rigid case of GC transformation and showed that a linear combination of the equations of motion can be rewritten into the form of a Maxwell-type field equation, which has the total energy-momentum as its source. This was done both for the gauge-unfixed classical system and for the gauge-fixed quantum system in the de Donder–Nakanishi gauge. The Maxwell-type field equation in the latter formally has an additional term coming from the gauge-fixing compared with that in the former, which was shown to take a BRS-exact form, just as in the well-known Yang–Mills case.

By using the same technique, we derived similar expressions of Maxwell-type equations for the Noether currents for the $IOSp(8|8)$ chiral symmetries. It confirmed that Nakanishi's original result persists to any GC invariant system.

These results will be useful for proving the existence theorem [27] of the graviton (photon) in any GC (local $U(1)$) gauge invariant system as far as the translation (global $U(1)$) symmetry is not spontaneously broken. We hope we can report on this matter in the near future.

The techniques used for the higher derivative systems in this paper will also be useful for studying more general problems. For instance, basic problems such as the (non-)equivalence between canonical quantization and path-integral quantization may be discussed in a general fashion for the higher derivative systems. This is because the present technique can easily be combined with Ostrogradsky's canonical formalism for higher derivative systems [30].

[Note added in proof] We are informed by the author of Ref. [31] in which similar results to ours were obtained in arbitrary higher derivative gravity systems. We see that he essentially proved the gravity equations of motion taking the form of Maxwell-type equations in the case of gauge-unfixed classical system, although the detailed comparison of equations is not

easy since rather different notations are used. His assumption on the gauge transformation is, however, more restrictive than ours to prove the anti-symmetric property of the field strength (which he calls quasilocality).

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Appendix A. OSp Transformation of \mathcal{L}_{GF+FP}

Note that the parameter ε_{NM} and the metric $\eta_{NM} = \eta^{NM}$ have the following statistics and transposition properties:

$$\begin{aligned} |\varepsilon_{NM}| &= |M| + |N|, \quad \text{i.e., } \varepsilon_{NM} X^L = (-)^{|L|(|M|+|N|)} X^L \varepsilon_{NM} \\ \varepsilon_{NM} &= (\varepsilon^T)_{MN} = (-)^{1+|M|\cdot|N|} \varepsilon_{MN} \\ \eta_{NM} &= (\eta^T)_{MN} = (-)^{|M|} \eta_{MN} = (-)^{|N|} \eta_{MN} \equiv \tilde{\eta}_{MN}. \end{aligned} \quad (A1)$$

Using these properties and also noting that the metric η_{MN} is a usual c-number (i.e., bosonic) quantity, we can rewrite the two terms in the OSp transformation (82) of the OSp coordinates X^L as

$$\begin{aligned} -\varepsilon_{NM} \tilde{\eta}^{NL} X^M &= -\eta^{LN} \varepsilon_{NM} X^M = -(\eta^{-1} \varepsilon X)^L \\ + \varepsilon_{NM} (-)^{|M|\cdot|N|} \tilde{\eta}^{ML} X^N &= (-)^{|M|\cdot|N|} \eta^{LM} (-)^{1+|M|\cdot|N|} \varepsilon_{MN} X^N \\ &= -\eta^{LM} \varepsilon_{MN} X^N. = -(\eta^{-1} \varepsilon X)^L, \end{aligned} \quad (A2)$$

where X , η^{-1} , and ε are regarded as a column vector and matrices in the final expressions. Thus, the two terms reduce to the same expression, and so the OSp transformation (82) is rewritten concisely into

$$\delta^{OSp} X^M = -2(\eta^{-1} \varepsilon X)^M = -2\eta^{MN} \varepsilon_{NL} X^L. \quad (A3)$$

Now, it is easy to see how the gauge-fixing Lagrangian $\mathcal{L}_{GF+FP} = -\kappa^{-1} \tilde{g}^{\mu\nu} E_{\mu\nu}$ changes under the OSp transformation (82). Noting the expression of $E_{\mu\nu}$ in Equation (67), we find

$$\begin{aligned} \delta^{OSp} \mathcal{L}_{GF+FP} &= \delta^{OSp} (-\kappa^{-1} \tilde{g}^{\mu\nu} E_{\mu\nu}) \\ &= -\frac{1}{2} \tilde{g}^{\mu\nu} \eta_{NM} \delta^{OSp} (\partial_\mu X^M \partial_\nu X^N) = -\tilde{g}^{\mu\nu} \eta_{NM} (\delta^{OSp} \partial_\mu X^M) \partial_\nu X^N \\ &= -\tilde{g}^{\mu\nu} \eta_{NM} \partial_\mu (-2\eta^{-1} \varepsilon X)^M \partial_\nu X^N = 2\tilde{g}^{\mu\nu} \partial_\mu (\varepsilon_{NM} X^M) \partial_\nu X^N \\ &= 2\tilde{g}^{\mu\nu} \left(\partial_\mu \varepsilon_{NM} \cdot X^M \partial_\nu X^N + \varepsilon_{NM} \partial_\mu X^M \partial_\nu X^N \right). \end{aligned} \quad (A4)$$

Now, we note that the OSp transformation parameter ε_{NM} must be graded antisymmetric

$$\varepsilon_{NM} = -(-)^{|N|\cdot|M|} \varepsilon_{MN} \quad (A5)$$

in order for the Lagrangian to be global OSp invariant. Indeed, only the second term remains in the global transformation, and it vanishes if and only if ε_{NM} is graded antisymmetric.

metric since $\partial_\mu X^M \partial_\nu X^N$, multiplied by $\tilde{g}^{\mu\nu}$, is graded symmetric. The second term thus also vanishes here, and the remaining first term is rewritten into

$$\begin{aligned}\delta^{OSp} \mathcal{L}_{GF+FP} &= 2\tilde{g}^{\mu\nu} \partial_\mu \varepsilon_{NM} \cdot (X^M \partial_\nu X^N) \\ &= \tilde{g}^{\mu\nu} \partial_\mu \varepsilon_{NM} \cdot (X^M \partial_\nu X^N - (-)^{|M|\cdot|N|} X^N \partial_\nu X^M) \\ &= \tilde{g}^{\mu\nu} \partial_\mu \varepsilon_{NM} \cdot (X^M \partial_\nu X^N - \partial_\nu X^M \cdot X^N) \\ &= \tilde{g}^{\mu\nu} \partial_\mu \varepsilon_{NM} \cdot (X^M \overset{\leftrightarrow}{\partial}_\nu X^N),\end{aligned}\quad (A6)$$

where, in going to the second line, we have exchanged the dummy indices $N \leftrightarrow M$ and used the graded antisymmetry property of ε_{NM} , (A5).

Appendix B. $OSp(2|2)$ -Invariant Scalar Field System

To see the property of the OSp symmetry, let us consider here the OSp -invariant system on a flat Minkowski background in which scalar fields belong to an $OSp(2|2)$ vector representation.

$$\mathcal{L} = -\frac{1}{2} \eta_{NM} \partial_\mu \phi^M \partial^\mu \phi^N, \quad (A7)$$

where ϕ^M is a $2+2$ -component OSp -vector whose first 2-components are bosons and the remaining 2-components are fermions, and the $OSp(2|2)$ metric is given by

$$\eta_{NM} = \begin{pmatrix} \sigma_1 & \\ & -\sigma_2 \end{pmatrix} = \eta^{NM}. \quad (A8)$$

Note that we have added an overall minus sign to our Lagrangian (A7) in order to make it coincide with the convention of the gauge-fixing Lagrangian \mathcal{L}_{GF+FP} in Equation (67), although it is not physically important, in any case, since the OSp metric is neither positive- nor negative-definite.

The infinitesimal OSp rotation is parametrized as

$$\delta\phi^M = \varepsilon^M_N \phi^N. \quad (A9)$$

Under this rotation, the quadratic kinetic Lagrangian (A7) is transformed as

$$\begin{aligned}\delta\mathcal{L} &= -\frac{1}{2} \eta_{NM} (\varepsilon^M_L \partial_\mu \phi^L \partial^\mu \phi^N + \partial_\mu \phi^M \varepsilon^N_L \partial^\mu \phi^L) \\ &= -\frac{1}{2} (\eta_{NM} \varepsilon^M_L \partial_\mu \phi^L \partial^\mu \phi^N + \partial_\mu \phi^M \eta_{NM} \varepsilon^N_L \partial^\mu \phi^L) \\ &= -\frac{1}{2} (\varepsilon_{NL} \partial_\mu \phi^L \partial^\mu \phi^N + \partial_\mu \phi^M (-)^{|M|} \varepsilon_{ML} \partial^\mu \phi^L) \\ &= -\frac{1}{2} (\varepsilon_{NL} \partial_\mu \phi^L \partial^\mu \phi^N + \varepsilon_{ML} \partial^\mu \phi^L \partial_\mu \phi^M) = -\varepsilon_{NM} \partial_\mu \phi^M \partial^\mu \phi^N\end{aligned}\quad (A10)$$

with $\varepsilon_{NM} \equiv \eta_{NL} \varepsilon^L_M$. This can further be rewritten as

$$\begin{aligned}\varepsilon_{NM} \partial_\mu \phi^M \partial^\mu \phi^N &= \varepsilon_{MN} \partial_\mu \phi^N \partial^\mu \phi^M = (-)^{|M|\cdot|N|} \varepsilon_{MN} \partial_\mu \phi^M \partial^\mu \phi^N \\ &= \frac{1}{2} ((-)^{|M|\cdot|N|} \varepsilon_{MN} + \varepsilon_{NM}) \partial_\mu \phi^M \partial^\mu \phi^N.\end{aligned}\quad (A11)$$

Thus, if the transformation parameter ε_{NM} is graded antisymmetric, i.e.,

$$\varepsilon_{NM} = -(-)^{|M|\cdot|N|} \varepsilon_{MN}, \quad (A12)$$

then the Lagrangian (A7) is OSp -invariant.

We define the canonical conjugate variable π_M by the right-derivative

$$\pi_M \equiv \partial \mathcal{L} / \partial \dot{\phi}^M = -\eta_{MN} \dot{\phi}^N. \quad (\text{A13})$$

Then, the ETCR is given by

$$[\phi^M, \pi_N] = i\delta_N^M \rightarrow [\phi^M, \dot{\phi}^N] = -\eta^{NL} [\phi^M, \pi_L] = -\eta^{NL} i\delta_L^M = -i\eta^{NM}, \quad (\text{A14})$$

so that

$$[\phi^M, \dot{\phi}^N] = -i(-)^N \eta^{MN} \equiv -i\tilde{\eta}^{MN}, \quad [\phi^M, \phi^N] = i(-)^N \eta^{MN} = i\tilde{\eta}^{MN}. \quad (\text{A15})$$

(If the conjugate momentum π_M were defined by the left derivative, then the ETCR should be given by $[\pi_M, \phi^N] = -i\delta_M^N$.)

The Noether current $J^{MN\mu}$ for the $OSp(2|2)$ transformation is defined by

$$\begin{aligned} -\frac{1}{2} \varepsilon_{NM} J^{MN\mu} &\equiv (\partial \mathcal{L} / \partial (\partial_\mu \phi^M)) \delta \phi^M = -\eta_{MN} \partial^\mu \phi^N \varepsilon_L^M \phi^L \\ &= -(-)^N \partial^\mu \phi^N \varepsilon_{NL} \phi^L = -\varepsilon_{NL} \phi^L \partial^\mu \phi^N \\ &= -\varepsilon_{NM} \frac{1}{2} (\phi^M \partial^\mu \phi^N - (-)^{|N| \cdot |M|} \phi^N \partial^\mu \phi^M) \\ &= -\varepsilon_{NM} \frac{1}{2} (\phi^M \partial^\mu \phi^N - \partial^\mu \phi^M \cdot \phi^N) = -\frac{1}{2} \varepsilon_{NM} (\phi^M \overset{\leftrightarrow}{\partial}^\mu \phi^N), \end{aligned} \quad (\text{A16})$$

where we have used the graded-antisymmetry property of ε_{NM} in going from the second to third lines. Thus, we have

$$J^{MN\mu} = \phi^M \overset{\leftrightarrow}{\partial}^\mu \phi^N, \quad (\text{A17})$$

so that the charge is given by

$$M^{MN} = \int d^3x J^{MN0} = \int d^3x (\phi^M \dot{\phi}^N - \dot{\phi}^M \phi^N). \quad (\text{A18})$$

This actually generates the original OSp rotation by the ETCRs:

$$\begin{aligned} [iM^{MN}, \phi^L] &= \int d^3x (i\phi^M [\dot{\phi}^N, \phi^L] - i(-)^{|N| \cdot |L|} [\dot{\phi}^M, \phi^L] \phi^N) \\ &= -\phi^M \tilde{\eta}^{NL} + (-)^{|N| \cdot |L|} \tilde{\eta}^{ML} \phi^N \\ &= -\phi^M \tilde{\eta}^{NL} + (-)^{|N| \cdot |M|} \phi^N \tilde{\eta}^{ML}, \end{aligned} \quad (\text{A19})$$

$$\begin{aligned} [iM^{MN}, \dot{\phi}^L] &= \int d^3x (i(-)^{|N| \cdot |L|} [\phi^M, \phi^L] \phi^N - i\dot{\phi}^M [\phi^N, \phi^L]) \\ &= (-)^{|N| \cdot |L|} \tilde{\eta}^{ML} \phi^N - \dot{\phi}^M \tilde{\eta}^{NL} \\ &= -\dot{\phi}^M \tilde{\eta}^{NL} + (-)^{|N| \cdot |M|} \dot{\phi}^N \tilde{\eta}^{ML}. \end{aligned} \quad (\text{A20})$$

The $IOSp(2|2)$ algebra is confirmed as

$$\begin{aligned}
 [iM^{MN}, M^{RS}] &= [iM^{MN}, \int d^3x (\phi^R \dot{\phi}^S - \dot{\phi}^R \phi^S)] \\
 &= \int d^3x \left[-\phi^M \tilde{\eta}^{NR} \dot{\phi}^S + \dot{\phi}^M \tilde{\eta}^{NR} \phi^S \right. \\
 &\quad \left. + (-)^{|M|+|N|+|R|} (-\phi^R \dot{\phi}^M \tilde{\eta}^{NS} + \dot{\phi}^R \phi^M \tilde{\eta}^{NS}) - (-)^{|M|+|N|} (M \leftrightarrow N) \right] \\
 &= -M^{MS} \tilde{\eta}^{NR} + (-)^{|R|+|S|} M^{MR} \tilde{\eta}^{NS} \\
 &\quad - (-)^{|M|+|N|} (-M^{NS} \tilde{\eta}^{MR} + (-)^{|R|+|S|} M^{NR} \tilde{\eta}^{MS}) \\
 &= -M^{MS} \tilde{\eta}^{NR} + \left(3 \text{ graded antisymmetrization terms} \right. \\
 &\quad \left. \text{under } (M \leftrightarrow N) \text{ and } (R \leftrightarrow S) \right), \\
 [iM^{MN}, P^R] &= [iM^{MN}, \int d^3x (\dot{\phi}^R)] \\
 &= \int d^3x \left[-\phi^M \tilde{\eta}^{NR} + (-)^{|M|+|N|} (M \leftrightarrow N) \right] \\
 &= -P^M \tilde{\eta}^{NR} + (-)^{|M|+|N|} P^N \tilde{\eta}^{MR}. \tag{A21}
 \end{aligned}$$

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