EFFECT OF GRADIENT ERRORS IN THE PRESENCE OF SPACE CHARGE FORCES

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INTRODUCTION

The space charge limit of an alternating gradient synchrotron is usually estimated by computing the change in betatron frequency of individual particles within the beam due to the space charge force and then imposing the condition that the working point should stay within the original integer-half-integer diamond. The significance of such a condition is not entirely clear. For one thing, the central ray is not affected by the internal space charge forces if the system is linear, so that coherent oscillations occur at the frequency determined by external forces only. In particular, closed orbit deviations due to alignment and field errors do not depend on beam intensity except through the asymmetry of image charges and currents. Furthermore, a significant instability with respect to gradient errors does not necessarily occur just because the internal frequencies are integral or half-integral for a certain beam diameter, because the space charge force is certainly non-linear with respect to beam diameter and a tendency for the beam to increase in diameter is inhibited by the change in internal frequency. It is the purpose of this paper to obtain a quantitative estimate of this non-linear effect.

EQUATIONS OF MOTION

A very useful formulation of the problem is given in a paper by Kapchinsky and Vladimirsky [1]. Although the paper deals with space charge effects in linear accelerators, the basic equations are equally applicable to an AGS synchrotron, since the centrifugal force is negligible. The authors considered a beam which is represented in transverse phase space (x, x', y, y') as a uniform distribution on the surface of a hyper-ellipsoid. This distribution is fairly realistic; in its simplest configuration, it corresponds to a beam of circular cross section and uniform density, with transverse velocities distributed uniformly in azimuth at each point in the beam. The uniform spatial density leads to space charge forces (neglecting image effects) which are linear in displacement from the central ray whatever the axis ratio and orientation of the projections; and thus the individual particles obey linear equations of motion provided that the external forces are also linear. Under these circumstances it is possible to deduce relatively simple differential equations for the width of the beam envelope in the x and v planes. The present calculation is based on those eduations [Equations (46) and (47) of Ref. [1], with some change in notation]:

$$\frac{1}{R^2}\frac{d^2r_x}{d\theta^2}+K_x(\theta)r_x-\frac{A^2}{r_x^3}-4\frac{Ne^2}{mv^2}\frac{1}{(r_x+r_y)}=0, \quad (1)$$

$$\frac{1}{R^2}\frac{d^2r_y}{d\theta^2}+K_y(\theta)\,r_y-\frac{A^2}{r_y^3}-4\frac{Ne^2}{mv^2}\frac{1}{(r_x+r_y)}=0, \quad (2)$$

where: r_x, r_y — half-width of beam envelope in x and y planes, respectively,
K_x, K_y — external focusing functions,
A — $\frac{1}{\pi}$ times beam emittance (in displacement x angle phase space),
N — number of particles per unit length,
v — beam velocity,
R — synchrotron radius,

and the nonrelativistic form of the space charge term has been used for simplicity.

The second derivatives in (1) and (2) are each equal to the sum of three terms; namely, the external force on a boundary particle, the space charge force on a boundary particle, and a term which takes into account the finite beam quality and has the form of a centrifugal force in a two-dimensional system in which r_x or r_y is a radius in polar coordinates. The familiar laminar flow approximation is obtained by setting $A=0$.

Equations (1) and (2) are amenable to computer solution in the full generality of alternating gradient K-functions, including gradient errors, but this note shall be restricted to a simplified problem. We assume cylindrical symmetry ($r_x = r_y = r$) and set

$$K_x = K_y = \frac{\nu^2}{R^2} [1 + \varepsilon \cos n \theta], \qquad (3)$$

where $\nu = \nu_x = \nu_y =$ betatron frequency at zero intensity and ε is the amplitude of an n'th harmonic gradient error.

With a few more changes in notation, Eqs. (1) and (2) become:

$$\frac{1}{\nu^2} \frac{d^2\varrho}{d\theta^2} + \left(\varrho - \frac{1}{\varrho^3}\right) = \frac{2\Delta\nu_{sc}}{\nu} \frac{1}{\varrho} - \frac{2\Delta v_s}{\nu} \varrho \cos n\theta, \qquad (4)$$

where

$$\varrho = r \sqrt{\frac{v}{AR}},$$

$\Delta\nu_s = \frac{1}{2} \varepsilon\nu =$ total width of zero intensity stop-band

and

$$\Delta\nu_{sc} = \frac{Ne^2}{mv^2} \frac{R}{A}.$$

The radius is now measured in units of the radius of a matched, unperturbed, beam of emittance A; i. e., the solution (periodic in θ) of (4) for $\Delta\nu_{sc} = \Delta\nu_s = 0$ is $\varrho = \text{const} = 1$. $\Delta\nu_{sc}$ is the change in frequency an individual particle would see if the beam radius were constrained to its matched value; this is the shift usually used to compute the space charge limit.

In order to find the variation of ϱ due to the combined effect of space charge and gradient error, we regard $\frac{\Delta\nu_s}{\nu}$ and $\frac{\Delta\nu_{sc}}{\nu}$ as small quantities and use the variation of parameters perturbation method. The general solution of (4) with $\Delta\nu_{sc} = \Delta\nu_s = 0$ is:

$$\varrho^2 = \sqrt{1 + A^2} + A \sin(2\nu\theta + \alpha), \qquad (5)$$

where A and α are arbitrary constants. Regarding A and α as slowly varying functions of θ, related in such a way that:

$$\varrho = \frac{d\varrho}{d\theta} = \nu A \cos(2\nu\theta + \alpha), \qquad (6)$$

we deduce the following coupled first order equations from Eq. (4):

$$\frac{dA}{d\theta} = -\Delta\nu_s \sqrt{1 + A^2} \cos[(2\nu - n)\theta + \alpha], \qquad (7)$$

$$A \frac{d\alpha}{d\theta} = \Delta\nu_s \sqrt{1 + A^2} \sin[(2\nu - n)\theta + \alpha] - 2\Delta\nu_{sc} \frac{\sqrt{1 + A^2} - 1}{A}. \qquad (8)$$

All rapidly varying terms have been dropped from Eqs. (7) and (8).

Equations (7) and (8) yield a first integral:

$$\sin\varphi = -\frac{2\Delta\nu}{\Delta\nu_s} \frac{\sqrt{1 + A^2}}{A} + \frac{\Delta\nu_{sc}}{\Delta\nu_s} \frac{1}{A} \ln \frac{A^2 (\sqrt{1 + A^2} + 1}{(\sqrt{1 + A^2} - 1)} + \frac{C}{A}, \qquad (9)$$

where $\varphi = (2\nu - n)\theta + \alpha$, C is a constant of integration, and $\Delta\nu$ is the difference between the frequency at zero intensity and the half-integer at which the stop-band occurs.

Equation (9) is equivalent to the more familiar integral curves in (ϱ, ϱ') space which arise in non-linear problems after rapidly oscillating terms have been dropped. It is somewhat simpler in form than the corresponding expression in (ϱ, ϱ') and therefore preferable for the present purpose.

With the approximation, $A \ll 1$, (9) becomes:

$$\sin\varphi \sim \frac{K}{A} - \left(\frac{2\Delta v - \Delta v_{sc}}{2\Delta\nu_s}\right) A + \left(\frac{4\Delta\nu - 3\Delta\nu_{sc}}{16\Delta\nu_s}\right) A^3, \qquad (10)$$

where K is an arbitrary constant, different from C.

Since $\sin\varphi$ must lie in the range ± 1, Eq. (10) determines the range of variation of A. Of particular interest are the fixed points in (A, φ); i. e., particular values of A and φ which remain constant in time. These fixed points, which are easily seen to be stable, represent «matched» solutions in the presence of space charge and gradient errors; i. e., solutions for which the beam radius oscillates with the periodicity of the error. The remainder of this note will be concerned with determining the fixed points and calculating the maximum excursion of ϱ for the corresponding values of A, through Eq. (5).

From Eq. (7), A is constant if $\varphi = \text{const} = \pm \frac{\pi}{2}$. The condition for constant φ is then obtained from Eq. (8):

$$2A\Delta\nu - 2\Delta\nu_{sc} \frac{\sqrt{1 + A^2} - 1}{A} \pm \Delta\nu_s \sqrt{1 + A^2} = 0. \qquad (11)$$

For $A \ll 1$, Eq. (11) becomes:

$$\pm 1 = \left(\frac{2\Delta\nu - \Delta\nu_{sc}}{\Delta\nu_s}\right) A - \left(\frac{4\Delta\nu - 3\Delta\nu_{sc}}{4\Delta\nu_s}\right) A^3, \qquad (12)$$

where the appropriate sign of $\sin\varphi = \pm 1$ must be used to obtain a positive value of A.

As a simple example of the application of (12), consider the case $\Delta\nu_{sc} = 0$, $\frac{\Delta\nu_s}{\Delta\nu} \ll 1$. Then $A \sim \frac{\Delta\nu_s}{2\Delta\nu}$ and

$$\varrho^2_{\max} \sim 1 + \frac{1}{2}\left|\frac{\Delta\nu_s}{\Delta\nu}\right|. \qquad (13)$$

This is the known result for the beat factor indtroduced by the existence of stop-bands [2]. If $\left|\frac{\Delta\nu_s}{2\Delta\nu - \Delta\nu_{sc}}\right| \ll 1$, then

$$\varrho^2_{\max} \sim 1 + \frac{1}{2}\left|\frac{\Delta\nu_s}{\Delta\nu - \frac{1}{2}\Delta\nu_{sc}}\right| \qquad (14)$$

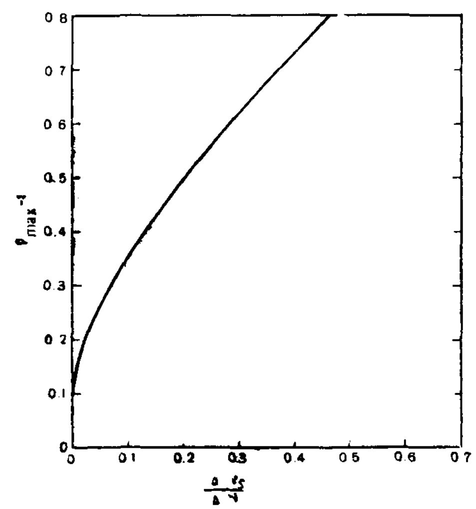

Fig. 1. $\varrho_{\max} - 1$ as a function of $\frac{\Delta\nu_s}{\Delta\nu}$ at resonance ($\Delta\nu_{sc} = 2\Delta\nu$).

It is evident from the form of Eqs. (12) and (14) that the situation leading to large excursions in radius is given by $\Delta\nu_{sc} = 2\Delta\nu$, which is twice the value usually assumed in computing the space charge limit of an accelerator. Some insight into this unconventional result can be obtained by returning to the basic Eqs. (1) and (2). If these equations are linearized for small deviations from the equilibrium radius and the gradient error set equal to zero, one finds that the system possesses a symmetric normal mode ($\delta r_x = \delta r_y$) and an anti-symmetric one ($\delta r_x = -\delta r_y$). The symmetric mode, which is driven by the symmetric perturbation (3), has a natural frequency shifted by an amount $\frac{1}{2}\Delta\nu_{sc}$ which leads to the resonance denominator of Eq. (14). The anti-symmetric mode, which would be excited by a perturbation of quadrupole character, exhibits a shift equal to $\frac{3}{4}\Delta\nu_{sc}$; the appropriate change in Eq. (14) yields a valid formula for this case. One is led to conclude that the usual criterion is too simple, for the motion of individual particles is altered by the modulations in beam diameter and it is misleading to use a frequency shift which would apply only if the modulations in beam diameter were ignored.

Unfortunately, the averaging process leading to Eqs. (7) and (8) proved to be very difficult for the anti-symmetric case, but a computational program is under way for that problem. It is anticipated that the results will be qualitatively similar to those presented below.

Returning to the symmetric case, we set $\Delta\nu_{sc} = 2\Delta\nu$ and find from Eq. (12):

$$A = \left(\frac{2\Delta\nu_s}{\Delta\nu}\right)^{1/3}. \qquad (15)$$

Therefore,

$$\varrho^2_{\max} \sim 1 + \left|\frac{2\Delta\nu_s}{\Delta\nu}\right|^{1/3}. \qquad (16)$$

As a numerical example, take $\Delta\nu = \frac{1}{4}$, $\Delta\nu_s = 0.025$. Then $\varrho^2_{\max} = 1.6$.

That is, a «matched» beam would oscillate in radius by $\pm 30\%$ about its unperturbed matched radius, and at the frequency of the perturbing gradient. Fig. 1 shows $\varrho_{\max} - 1$ at resonance ($\Delta\nu_{sc} = 2\Delta\nu$) as a function of $\frac{\Delta\nu_s}{\Delta\nu}$, as obtained numerically from the more exact expression (11). The growth is not exorbitant even for very wide stop-bands, and it would appear that a machine designed conservatively regarding aperture and injector emittance could easily handle more beam than the usual space charge limit. The treatment is, of course, not limited to the nearest stop-band; if that one can be made very narrow by suitable correction, the next ones would not be as bothersome ($\Delta\nu = 3/4, 5/4$, etc.). In a sense, space charge effects are beneficial in that they provide a nonlinear element which prevents an indefinitely increasing amplitude of oscillation.

REFEREN CES

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