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PAPER

Entropy production and the generalised second law of black hole thermodynamics

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Abstract

The generalised second law of black hole thermodynamics states that the sum of a black hole's entropy and the entropy of all matter outside the black hole cannot decrease with time. The violation of the generalised second law via the process in which a distant observer extracts work by lowering a box arbitrarily close to the event horizon of a black hole has two profound ramifications: (1) that the entropy of the Universe can be decreased arbitrarily via this process; and (2) that it is not appropriate to apply the laws of thermodynamics to systems containing black holes. In this paper, we argue that for the generalised second law to not be violated, entropy must be produced during the lowering process. To demonstrate this, we begin by deriving an equation for the locally measured temperature of the vacuum state of an observer that is a finite distance from the event horizon of a Schwarzschild black hole. Then, using this locally measured temperature and the Unruh effect, we derive an equation for the force required to hold this observer in a stationary position relative to a Schwarzschild black hole. These equations form a framework for calculating the change in black hole entropy as a result of the lowering process both in the case where the process is isentropic and in the case where entropy is produced during the lowering process. In the latter case, two requirements: (1) that the resultant change in black hole entropy is finite; and (2) that the resultant change in common entropy is finite, are used to identify two conditions that the form of an entropy production function must satisfy. These, in turn, are used to identify a set of possible functions describing the production of entropy. Using this set of functions, we demonstrate that the production of entropy limits the amount of work that the distant observer can extract from the lowering process. We find that this allows for the generalised second law to be preserved, provided that a coefficient in this set of functions satisfies a given bound. To conclude, we discuss two natural choices of this coefficient that allow for the generalised second law to be preserved in this lowering process. In addition to providing a resolution to this violation of the generalised second law, the framework presented in this paper can be applied to inform theories of gravity and quantum gravity on the form of their entropy relations, such that they do not violate the generalised second law.

1. Introduction

The generalised second law of black hole thermodynamics underpins the interpretation that the governing laws of black hole mechanics are the ordinary laws of thermodynamics applied to a system containing a black hole [1–3]. Its apparent violation via the process in which a distant observer extracts work by lowering a box arbitrarily close to the event horizon of a black hole has two profound ramifications: (1) it is not appropriate to apply the ordinary laws of thermodynamics to systems containing black holes [1, 2], and (2) that the entropy of the Universe can be decreased arbitrarily via this process [4, 5]. In this paper, we argue that for the generalised second law to be preserved, entropy must be produced during the lowering process. By postulating a set of possible functions describing the production of entropy, we demonstrate that the production of entropy limits the amount of work that the distant observer can extract via the lowering process, allowing for the generalised second law to be preserved.

The argument that entropy must be produced during the lowering process is in contrast to previous attempts to resolve this violation by Unruh, Wald [2] and Bekenstein [6], where the lowering process is assumed to be isentropic. Unruh and Wald postulated the existence of a buoyancy force opposing the acceleration of the box which, by requiring work to be done to overcome, limits the amount of work that can be extracted by the distant observer [2]. This resolution, however, suffers from two major limitations: it is only applicable to reflective boxes and the buoyancy force is generally not sufficient enough to preserve the generalised second law in macroscopic systems [7, 8]. On the other hand, Bekenstein's resolution to this violation hinges on the existence of a universal entropy bound [6]. Although there are examples of systems for which such a bound does hold [7, 9–16], there are also examples of systems, particularly those with strong gravitational fields, where this bound is known to fail [3, 16–25]. Our notion that the lowering process must generate entropy for the generalised second law to remain preserved does not require the use of Unruh–Wald buoyancy or a universal entropy bound.

More recently, concepts from black hole thermodynamics have found renewed interest in the fields of quantum gravity [26–30], entropic gravity [31–36] and entropic cosmology [37–44]. The bedrock of these fields is the holographic principle [45, 46]; the conjecture that the information of the matter contained within some volume is encoded on the surface that bounds that volume [47]. A defining feature of systems that obey the holographic principle is that their entropy is proportional to one-fourth of their surface area [25, 47]. As we will see, this is a feature of black holes, whose entropy satisfies this area law. The implication of this is clear; black holes obey the holographic principle and therefore act as a case study to understand it and, the theories that invoke it, in greater detail. To physically understand why black holes obey the holographic principle, consider an observer external to the event horizon of a black hole who is watching matter accrete to the black hole. To this external observer, due to gravitational red-shifting, it would take an infinite amount of time for the matter to reach the black hole's event horizon [48, 49]. Given a sufficient amount of time, to this observer, the matter (and crucially all of its information) that has been accreted to the black hole will be contained within an arbitrarily thin volume asymptotically close to the event horizon. In this sense, one can interpret the information of the matter accreted to the black hole as being, at least approximately, stored on its event horizon. This interpretation, that the information of the contents of a black hole is stored on its event horizon, will be used regularly throughout this paper.

The outline of this paper is as follows: in section 2, the necessary fundamentals of black hole thermodynamics are introduced. This includes the four laws of black hole mechanics [50], black hole entropy, the Hawking effect [51], the Fulling–Davies–Unruh effect [49, 52, 53], the generalised second law [4, 5] and the process by which it can be violated. In section 3, a brief overview of Schwarzschild black holes and their properties is given. Following this, the core assumptions made in this paper are discussed and, in turn, used to derive an equation for the locally measured temperature of the vacuum state of an observer that is a finite distance from the event horizon of a Schwarzschild black hole. Using this locally measured temperature and the Fulling–Davies–Unruh effect, an equation for the force required to hold this observer in a stationary position relative to the black hole is derived. In section 4, using this force and by assuming that the lowering process is isentropic, the change in black hole entropy resulting from a box, with negligible volume, being lowered arbitrarily close to the black hole's event horizon is calculated. This change in black hole entropy is found to be arbitrarily small, allowing for the violation of the generalised second law [1–5]. We then argue that, for the generalised second law to be preserved in this process, entropy must be produced in the space-time surrounding the box (that is, the lowering process is not isentropic). This production of entropy will result in a loss of work to the surrounding space-time, limiting the amount of work that can be extracted by the distant observer and preventing the change in black hole entropy from being made arbitrarily small. In this case, to calculate the resultant change in black hole entropy and ensure that the generalised second law is not violated, a function describing the production of entropy, which is dependent on the box's distance from the black hole, is required. In the absence of such a function in the literature, two requirements: (1) that the resultant change in black hole entropy is finite; and (2) that the resultant change in common entropy is finite, are used to identify two conditions that such a function must satisfy. Using these conditions, a set of such functions is identified and shown to result in a change in generalised entropy that is non-negative provided that a coefficient in this set of functions satisfies a given bound. To conclude, we discuss two natural choices of this coefficient which allow for the generalised second law to be preserved in this lowering process.

2. Black hole thermodynamics

In this section, we introduce the fundamentals of black hole thermodynamics. This will include a review of the four laws of black hole mechanics, black hole entropy, the Hawking effect, the Fulling–Davies–Unruh effect and the generalised second law. At the end of this section, the process by which the generalised second law can be violated is introduced and two proposed resolutions to this violation are discussed.

2.1. Classical black hole mechanics

A black hole is a region of space-time, enclosed by a causal horizon, out of which not even light can escape [1]. This causal horizon, better known as an *event horizon*, prevents the communication of information about the region of space-time it encloses across it [36, 54]. Information about the internal structure and contents of the black hole is unavailable to an external observer. Consequently, the external observer is left to characterise the black hole using only its externally observable properties. In the case of a black hole that has reached a steady, final state –*a stationary black hole*– these parameters are the black hole's mass M, charge Q and angular momentum J [1, 54, 55]. Using these three parameters, additional parameters that are of use (such as the surface area of the black hole's event horizon) can be defined, but for any stationary black hole, the black hole's mass, charge and angular momentum are sufficient to characterise it uniquely [1, 54, 55]. As in thermodynamics, where there exists a set of laws defining the characteristic parameters of a thermodynamic system, there also exists a set of laws defining the characteristic parameters of a system containing a stationary black hole. This set of laws, which are known as *the four laws of black hole mechanics* [50], are as follows:

The Zeroth Law The surface gravity κ of a stationary black hole is uniform across the black hole's event horizon [1, 50, 55].

The First Law Following an infinitesimally small change in mass ΔM, the change in surface area ΔA, angular momentum ΔJ and charge ΔQ of a stationary black hole is given by [50, 55]

$$\kappa\Delta A = 8\pi G(\Delta M - \Omega\Delta J - \Phi\,\Delta Q)\,, \tag{1}$$

where Ω is the angular velocity of the black hole's event horizon, Φ is the electrostatic potential of the black hole and G is Newton's gravitational constant.

The Second Law The surface area A of a black hole's event horizon cannot decrease with time. Additionally, if two black holes merge, the surface area of the final black hole's event horizon must be greater than the sum of the initial black holes' event horizons [1, 50, 55–58].

The Third Law For a stationary black hole, a surface gravity κ equal to zero cannot be reached within a finite number of processes [1, 50, 55, 59].

As noted by Bardeen, Carter, and Hawking, these laws share remarkable parallels with the four laws of thermodynamics [50]. The zeroth, second, and third laws of black hole mechanics are identical to their thermodynamic counterparts if one identifies that the surface gravity and event horizon's surface area are analogous to the black hole's temperature and entropy, respectively. The first law can be made identical to its thermodynamic counterpart if one identifies that the mass, angular velocity, and angular momentum are analogous to the black hole's energy, pressure, and volume, respectively [1, 50, 55]. At this point, however, with the exception of the black hole's mass and energy $E = Mc^2$, the equivalences between the black hole's properties and general thermodynamic properties are purely mathematical [1, 50].

2.2. Black hole entropy and temperature

Motivated by entropy's interpretation as a measure of the lack of information about a system's internal configuration, and given that an external observer has no information about the internal configuration of a black hole, Bekenstein argued that it was natural to associate a physical entropy to a black hole [4]. To derive an explicit expression for a black hole's entropy, Bekenstein associated a minimum entropy increase (equivalent to one bit of information) with a minimum increase in the black hole's radius (equivalent to one Compton wavelength) for a process in which one particle is absorbed by the black hole [4]. Using this argument, Bekenstein concluded that the entropy of a black hole must be proportional to the ratio of its event horizon's surface area to the square of the Plank length [4]. Shortly after this, Hawking identified that the proportionality constant in this relationship was equal to one-fourth of the Boltzmann constant [48]. That is to say, the entropy S_{bh} of a black hole with an event horizon of area A_{bh} is given by [1, 4, 55]

$$S_{\mathrm{bh}} = \frac{k_{\mathrm{b}}c^3A_{\mathrm{bh}}}{4G\hbar}, \tag{2}$$

where k_b is the Boltzmann constant, c is the speed of light and $\hbar$ is the reduced Plank's constant. The second law of black hole mechanics can now be physically interpreted in a natural form; the entropy of a stationary black hole cannot decrease with time [1, 50, 55].

To further solidify this physical interpretation, consider the process where matter with an entropy S is accreted to a black hole. Assuming that no entropy is produced in this process, once the matter crosses the black hole's event horizon, the entropy of all matter outside of the black hole S_m (henceforth referred to as the '*common entropy*') will decrease by S [4]. Prior to crossing the event horizon, an observer can perform a measurement to obtain information about the matter's internal configuration. Doing so will decrease the matter's entropy and, by extension, the common entropy. However, once the matter has passed the event horizon of the black hole, measuring its internal configuration is no longer possible and any associated information becomes inaccessible to the observer [4]. Given that this process results in a loss of accessible information, we can equally conclude that it should increase the Universe's entropy, as opposed to decreasing it [4]. This apparent contradiction can only be solved by accepting that the source of this entropy increase is an increase in the black hole's entropy and that the Universe's entropy must consist of both the black hole's entropy and the common entropy. This way, whilst the common entropy decreases in this process, the entropy of the Universe still increases due to the black hole gaining the matter's entropy [4]. This concept can be formulated more concisely into the generalised second law [4, 5]:

The Generalised Second Law The generalised entropy S', which is the sum of the black hole's entropy S_{bh} and the common entropy S_m, cannot decrease with time. In a process that results in a change in black hole entropy ΔS_{bh} or common entropy ΔS_m, the change in generalised entropy $\Delta S'$ must be greater than, or equal to, zero [1, 4, 5];

$$\Delta S' = \Delta S_{\mathrm{bh}} + \Delta S_m \geqslant 0. \tag{3}$$

The formulation of black hole entropy and the generalised second law was fundamental steps toward reconciling the four laws of black hole mechanics with the four laws of thermodynamics. However, for this reconciliation to be complete, a physical interpretation of black hole temperature was required. This was achieved when Hawking [51] demonstrated that the vacuum state of the space-time surrounding a black hole is a thermal state with a well-defined temperature [1, 51, 60]

$$T_{\mathrm{H}} = \frac{\hbar\kappa}{2\pi k_b c}. \tag{4}$$

That is, a black hole emits thermal radiation into its surrounding space-time analogous to how a black body with a well-defined temperature emits thermal radiation [1, 48, 51]. This is known as the *Hawking effect* and it allows the remaining three laws of black hole mechanics to be physically interpreted as their thermodynamic counterparts applied to a system containing a black hole [1].

A closely related effect, known as the Fulling–Davies–Unruh effect [49, 52, 53] (henceforth referred to as the 'Unruh effect'), occurs in Minkowski and curved space-times. The Unruh effect predicts that the vacuum state of a uniformly accelerating observer will be a thermal state, populated by thermal radiation, with a well-defined temperature [1]. In Minkowski space-time, this temperature T_{U} is proportional to the observer's acceleration a and is given by [2, 53, 55]

$$T_{\mathrm{U}} = \frac{\hbar a}{2\pi k_b c}. \tag{5}$$

In a curved space-time that admits an event horizon, such as the space-time surrounding a black hole, this temperature T is given by [1, 2]

$$T = \frac{T_{\mathrm{H}}}{\chi}, \tag{6}$$

where χ is a gravitational red-shift factor which, for a given space-time metric tensor $g_{\mu\nu}$, is given by $\chi = \sqrt{g_{tt}}$ [2]. χ is dependent on the observer's distance from the event horizon and is normalised such that it is zero at the event horizon and one infinitely far from it [1, 2].

An important consequence of the Hawking effect is its ramifications on the second law of black hole mechanics. The Hawking effect provides a process by which black holes can radiate their energy into the surrounding space-time and, by doing so, reduce their mass, surface area, and entropy [1, 51]. This immediately violates the second law of black hole mechanics by allowing a black hole's entropy to decrease with time [1, 51]. The generalised second law, however, endures. This is because whilst the emission of thermal radiation decreases the black hole's entropy, it also increases the common entropy. Consequently, the change in generalised entropy remains equal to, or greater than, zero [1, 48, 51].

2.3. The violation of the generalised second law

Naturally, the resilience of the generalised second law is of considerable interest within the field of black hole thermodynamics. Its violation might suggest that it is not appropriate to apply thermodynamics to a system containing a black hole [1, 2]. One suggested violation of the generalised second law is the following process: An observer who is an infinite distance from the event horizon of a black hole –*a distant observer*– extracts work by lowering a box of negligible volume with mass m, energy mc^2 and entropy S_{box} towards the event horizon of a Schwarzschild black hole using a rope. At some distance from the event horizon, the observer allows the box to fall freely into the black hole [1–5]. Assuming that no entropy is produced during the lowering process, the change in the common entropy when the box crosses the black hole's event horizon will be $\Delta S_m = -S_{\text{box}}$ [1–5]. The black hole's entropy change will be determined by the change in its event horizon's surface area and therefore by its change in energy. However, as the box approaches the black hole's event horizon, its energy decreases due to gravitational red-shifting. As a result of this red-shifting, the energy change of the black hole ΔE_{bh} once the box crosses the event horizon will be $\Delta E_{\text{bh}} = \chi mc^2$ [1, 2]. Consequently, if the observer lowers and releases the box at an arbitrarily small distance from the event horizon, the energy of the box and therefore the black hole's energy change can be made arbitrarily small (recall that χ is zero at the event horizon of the black hole) [1–5]. Such a process would result in no entropy change for the black hole and a negative entropy change for all matter outside the black hole, violating the generalised second law [1–5].

Several resolutions to the above violation of the generalised second law have been proposed. Bekenstein argued that the violation only arises if one takes the box to have a negligible volume [4, 5]. If the box's dimensions are not treated as negligible, its centre of mass cannot be lowered arbitrarily close to the event horizon before the box crosses it. Therefore, the black hole's energy change cannot be made arbitrarily small [3–5]. By postulating the existence of a universal upper bound on the entropy of an object with a given circumscribing radius and energy –*the Bekenstein bound*– [6], Bekenstein demonstrated that the change in generalised entropy for this process was generally non-negative [2, 3, 6]. The existence of the Bekenstein bound is crucial to this resolution, and whilst there are systems for which this bound holds [7, 9–16], there are also systems, particularly those with strong gravitational fields, where this bound is known to fail [3, 16–25].

Uneasy with a universal bound on entropy, Unruh and Wald attempted to resolve the violation of the generalised second law by postulating the existence of a buoyancy force that opposes the acceleration of the box [2]. Their resolution is as follows: if a distant observer were to extract work by lowering a reflective box with a non-negligible volume towards the event horizon of a black hole, by virtue of the Unruh effect, the box's vacuum state would be a thermal state, populated by thermal radiation, with a locally measured temperature $T = T_{\text{H}}/\chi$ [1, 2]. As the box approaches the event horizon (which corresponds to the limit $\chi \to 0$), the local temperature and the density of the thermal radiation of the box's vacuum state increase. Due to the box's reflectivity, this increase in thermal radiation density will result in a buoyancy force that acts against the box's acceleration (away from the event horizon) [1, 2]. At the point where the energy densities of the box and thermal state are equal, this buoyancy force is sufficient to hold the box in a stationary position relative to the black hole [1, 2]. Consequently, if the distant observer wishes to lower the box past this point and to the event horizon, they must do work to overcome the buoyancy force. This limits the amount of work the distant observer can extract by lowering the box to the event horizon, increasing the black hole's energy gain [1, 2]. Considering this buoyancy force, it was shown by Unruh and Wald that the black hole's energy change is minimised by releasing the box at the point of equal energy densities [2]. The associated minimum change in generalised entropy at this point was shown to be non-negative, allowing for the preservation of the generalised second law [1, 2]. Whilst this resolution does not require the use of the Bekenstein bound, it has its limitations. The most evident of which is that it is only applicable to a reflective box, which severely limits its applicability as a general resolution. In addition to this, analysis by Bekenstein of this buoyancy force for the lowering of macroscopic systems has shown that it is insufficient to preserve the generalised second law [7, 8]. This is due to the point of equal energy densities being located very close to the black hole's event horizon [7, 8]. This further limits its applicability as a resolution to this violation of the generalised second law.

3. Locally measured temperature

In this section, we begin by providing a brief overview of Schwarzschild black holes and some of their properties. Following this, we will present and discuss the initial assumptions made in this paper. These, along with the concepts introduced in section 1, will then be used to derive an equation for the locally measured temperature of the vacuum state of an observer that is a finite distance from a Schwarzschild black hole. Finally, using this equation and the Unruh effect, an equation for the force required to hold this observer in a stationary position relative to the black hole will be derived. Both of these equations will be necessary for section 4, where they will be used to calculate the change in generalised entropy following the process in which a distant observer extracts work by lowering a box arbitrarily close to the event horizon of a black hole.

3.1. Schwarzschild black holes

From here on out, we will focus our analysis on Schwarzschild black holes, which have a mass M but no charge Q and no angular momentum J [1]. The curvature of space-time for an observer external to a Schwarzschild black hole is described by the Schwarzschild metric tensor. Taking the mostly minus metric signature $(+ - - -)$, the Schwarzschild metric tensor $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = \mathrm{diag}\left(\left(1-\frac{R_\mathrm{s}}{r}\right), -\left(1-\frac{R_\mathrm{s}}{r}\right)^{-1}, -r^2, -r^2\sin^2\theta\right), \tag{7}$$

where $\mu,\nu = (t,r,\theta,\phi)$ denote space-time indices, t is the time coordinate, r is the observer's distance from the centre of the black hole, θ and ϕ are the observer's polar and azimuthal angles, respectively, and R_s is the Schwarzschild radius of the black hole, which itself is given by [61]

$$R_\mathrm{s} = \frac{2GM}{c^2}. \tag{8}$$

The gravitational red-shift factor for an observer in a generic space-time is given by $\chi = \sqrt{g_{tt}}$ [2]. Using equation (7) for the Schwarzschild metric tensor, we can identify the gravitational red-shift factor for an observer in the space-time surrounding a Schwarzschild black hole as

$$\chi = \left(1-\frac{R_\mathrm{s}}{r}\right)^{\frac{1}{2}}. \tag{9}$$

An important property of χ is that it is normalised such that $\chi \to 0$ as $r \to R_\mathrm{s}$ and $\chi \to 1$ as $r \to \infty$ [1, 2].

The surface gravity κ of a Schwarzschild black hole is given by [2]

$$\kappa = \frac{c^4}{4MG}. \tag{10}$$

Hence, equation (4) for the Hawking temperature T_H of a black hole, as defined by a distant observer, can be expressed as [1, 51]

$$T_\mathrm{H} = \frac{\hbar c^3}{8\pi GMk_\mathrm{b}} \tag{11}$$

for a Schwarzschild black hole. Using this expression for the Hawking temperature, equation (2) for the entropy of a black hole and equation (10) for the surface gravity of a Schwarzschild black hole, the first law of black hole mechanics for a Schwarzschild black hole can be expressed in the following form:

$$\Delta E_\mathrm{bh} = T_\mathrm{H}\Delta S_\mathrm{bh}, \tag{12}$$

where ΔE_bh and ΔS_bh are the black hole's energy and entropy changes, respectively, following a process that alters the black hole's mass [1]. To distinguish between this form of the first law of black hole mechanics and that given in section 2, the above equation will henceforth be referred to as 'the first law of Schwarzschild black hole thermodynamics'.

3.2. The equipartition rule

Analogous to [4, 31], we assume that the maximum amount of information that can be stored in a unit area is one bit per Plank length squared $l_p{}^2$. Thus, the maximum amount of information, measured in the number of bits N, that can be stored on the event horizon of a spherically symmetric black hole with an area $A_{\rm bh} = 4\pi R_{\rm s}{}^2$ is [31]

$$N = \frac{A_{\rm bh} c^3}{G\hbar}. \tag{13}$$

Next, we associate an average thermal energy of $\frac{1}{2}k_{\rm b}T_{\rm H}$ with each bit of information stored on the black hole's event horizon [31]. This can be justified using either an equipartition theory or an information theory argument. The equipartition argument is as follows: consider a stationary black hole with N bits of information stored on its event horizon. Under the zeroth law of black hole mechanics, the temperature of the black hole must be uniform across the black hole's event horizon [1, 50, 55]. Thus, each bit of information must have the same temperature $T_{\rm H}$. By invoking the equipartition theorem, we can associate an average thermal energy of $\frac{n}{2}k_{\rm b}T_{\rm H}$ to each bit of information, where n is the number of degrees of freedom of each bit [31]. As each bit of information has one degree of freedom [4], we can associate an average thermal energy of $\frac{1}{2}k_{\rm b}T_{\rm H}$ to each bit on the event horizon. Here, our invocation of the equipartition theorem requires the black hole to be in a thermal equilibrium. This, however, is strictly not true. Due to the Hawking effect, the black hole continuously emits thermal radiation, preventing it from reaching thermal equilibrium [48]. The information theory argument does not suffer from this hamartia. This argument is as follows: consider a process in which a collection of particles with N bits of information crosses the event horizon of a black hole, thus becoming inaccessible to any external observers. To these external observers, this process corresponds to the erasure of N bits of information [36]. Landauer's principle then allows us to associate an energy of $\ln(2)k_{\rm b}T_{\rm H} \approx \frac{1}{2}k_{\rm b}T_{\rm H}$ with each bit of information that the black hole has erased [36, 62].

Having associated an average thermal energy of $\frac{1}{2}k_{\rm b}T_{\rm H}$ with each bit of information, we can express the total energy E of the information stored on the event horizon as [31]

$$E = \frac{1}{2}Nk_{\rm b}T_{\rm H}, \tag{14}$$

where N is the number of bits stored on the black hole's event horizon. Equation (11) for the Hawking temperature and equation (13) for the maximum amount of information stored on the event horizon can be substituted into equation (14) for the total energy of the information stored on the event horizon to give

$$E = \frac{A_{\rm bh}c^6}{16\pi G^2 M}. \tag{15}$$

By substituting the surface area of the event horizon $A_{\rm bh} = 4\pi R_{\rm s}{}^2$ into the above equation we obtain

$$E = \frac{R_{\rm s}{}^2 c^6}{4G^2 M}, \tag{16}$$

which, by substituting equation (8) for the Schwarzschild radius into the above result, allows us to reach

$$E = Mc^2. \tag{17}$$

That is, the total energy of the information stored on the event horizon of the black hole is equal to the total energy of the black hole [31, 63];

$$Mc^2 = \frac{1}{2}Nk_{\rm b}T_{\rm H}. \tag{18}$$

Equation (18) will henceforth be referred to as the 'equipartition rule' [31].

3.3. Derivation of locally measured temperature

We now wish to derive an equation analogous to the equipartition rule for an observer that is a finite distance from the black hole's event horizon. This will enable us to derive an equation for the locally measured temperature of such an observer's vacuum state which, in turn, will be used to derive an equation for the force required to hold such an observer in a stationary position relative to the black hole.

Consider an observer at a distance $R \geqslant 0$ from the event horizon of a black hole. This observer defines a spherically symmetric surface $\mathcal{S}$ with radius $r = R_{\rm s} + R$ that encloses the black hole and with it, the information contained on its event horizon. We assume that the maximum information $N' \geqslant N$ enclosed by $\mathcal{S}$ will be given by

$$N' = \frac{4\pi r^2 c^3}{G\hbar}. \tag{19}$$

The information contained within $\mathcal{S}$ was erased at the event horizon of the black hole, thus we can once again associate the average thermal energy of $\frac{1}{2}k_b T_H$ with each bit of information. The total energy of the information contained within $\mathcal{S}$ is therefore related to the energy of the black hole via the following equation:

$$Mc^2 = \frac{1}{2}N' k_b T_H. \tag{20}$$

As the observer is a finite distance from the black hole's event horizon, by virtue of the Unruh effect in curved space-time, the observer's vacuum state will be a thermal state with a locally measured temperature $T = T_H/\chi$ [1, 2]. This naturally allows us to account for the gravitational red-shifting, which occurs because of the observer's finite distance from the black hole's event horizon, and to relate the energy of the black hole with the red-shifted energy of the information contained within $\mathcal{S}$ via the following equation:

$$\frac{Mc^2}{\chi} = \frac{1}{2}N' k_b T. \tag{21}$$

This equation will henceforth be referred to as the 'red-shifted equipartition rule'. Substitution of equation (19) for the maximum number of bits N' contained within $\mathcal{S}$ into the red-shifted equipartition rule allows the following equation for the locally measured temperature T of the observer's vacuum state to be obtained:

$$T = \frac{\hbar GM}{2\pi k_b c r^2 \chi}, \tag{22}$$

where r is the observer's distance from the centre of a black hole. The acceleration a required to hold the observer in a stationary position relative to the black hole can be found using the Unruh effect. Equating the above equation with equation (5) for the Unruh temperature yields

$$a = \frac{GM}{r^2 \chi}. \tag{23}$$

Lastly, Newton's second law of motion can be used to equate this acceleration to the force F required to hold an observer in a stationary position relative to the black hole. Doing so yields

$$F = \frac{GMm}{r^2 \chi}, \tag{24}$$

where m is the observer's mass. This is the force as measured locally by the stationary observer. As shown in appendix A, the same expressions for both the force required to hold an observer in a stationary position and the locally measured temperature of the observer's vacuum state can be derived using general relativity by calculating the four-acceleration required to hold an observer in a stationary position relative to a Schwarzschild black hole. Our reason for displaying the derivation method presented in this section as the primary derivation method is that, by making direct use of the holographic principle and treating the information content of a black hole as erased to any external observers, it has the potential to shed light on both the holographic principle and the black hole information paradox.

4. The production of entropy

In this section, the force required to hold an observer in a stationary position relative to a black hole will be used to calculate the change in generalised entropy resulting from the process in which a box, with negligible volume, is lowered by a distant observer to the event horizon of a black hole. As will be shown, the resultant change in generalised entropy can be made negative by lowering the box arbitrarily close to the black hole's event horizon, reproducing the violation of the generalised second law discussed in [1–5]. This will motivate our conjecture that in order to preserve the generalised second law, entropy must be produced during the lowering process.

4.1. The isentropic lowering process

Consider a distant observer that extracts work by slowly lowering a box of negligible volume, mass m, energy mc^2 and entropy S_{box} to a distance $r_{\mathrm{f}} \geqslant R_{\mathrm{s}}$ from the centre of a Schwarzschild black hole of mass M and Schwarzschild radius R_{s}. Once the box reaches r_{f}, it is allowed by the distant observer to fall freely towards the black hole [1–5]. To investigate whether the generalised second law is violated in this process, the resultant change in common entropy ΔS_m and change in black hole entropy ΔS_{bh} must be calculated. If one assumes that the lowering process is isentropic, that is, during the lowering process no entropy is produced and the box's entropy does not change, the change in the common entropy once the box crosses the event horizon will be $\Delta S_m = -S_{\mathrm{box}}$ [1–5]. The black hole's change in entropy can be determined using the first law of Schwarzschild black hole thermodynamics and the black hole's corresponding change in energy ΔE_{bh}, which itself can be found by calculating the amount of work extracted by the distant observer W_∞ in lowering the box to r_{f} [1, 2, 5]. The amount of work that is extracted by lowering the box to a distance r_{f} is given by [1, 2]

$$W_\infty = \int_{r_{\mathrm{f}}}^{\infty} F \mathrm{d}r, \tag{25}$$

where F is the force exerted on the box by the distant observer as given by equation (24). Thus, the maximum amount of work the distant observer can extract by lowering the box to a distance r_{f} is given by

$$W_\infty = GMm \int_{r_{\mathrm{f}}}^{\infty} \frac{1}{r^2} \left(1 - \frac{R_{\mathrm{s}}}{r}\right)^{-\frac{1}{2}} \mathrm{d}r. \tag{26}$$

By making the change of variables $u = 1 - \frac{R_{\mathrm{s}}}{r}$, the integral in the above equation can be solved to yield

$$W_\infty = \frac{GMm}{R_{\mathrm{s}}} \int_{r=r_{\mathrm{f}}}^{r\to\infty} u^{-\frac{1}{2}} \mathrm{d}u, \tag{27}$$

$$W_\infty = \frac{2GMm}{R_{\mathrm{s}}} \left[\sqrt{1 - \frac{R_{\mathrm{s}}}{r}}\right]_{r_{\mathrm{f}}}^{\infty}, \tag{28}$$

$$W_\infty = mc^2 \left[\lim_{r\to\infty} \left(\sqrt{1 - \frac{R_{\mathrm{s}}}{r}}\right) - \sqrt{1 - \frac{R_{\mathrm{s}}}{r_{\mathrm{f}}}}\right], \tag{29}$$

$$W_\infty = mc^2 \left(1 - \sqrt{1 - \frac{R_{\mathrm{s}}}{r_{\mathrm{f}}}}\right). \tag{30}$$

Upon the box crossing the event horizon, the black hole's change in energy will be given by the energy of the box subtracted by the amount of work the distant observer has extracted [1, 2]. That is,

$$\Delta E_{\mathrm{bh}} = mc^2 - W_\infty, \tag{31}$$

$$\Delta E_{\mathrm{bh}} = mc^2 \left(1 - \frac{R_{\mathrm{s}}}{r_{\mathrm{f}}}\right)^{\frac{1}{2}}. \tag{32}$$

Thus, by lowering the box arbitrarily close to the event horizon, which in equation (32) corresponds to taking the limit $r_{\mathrm{f}} \to R_{\mathrm{s}}$, the black hole's change in energy can be made to be zero. Using the first law of Schwarzschild black hole thermodynamics, the corresponding change in the black hole's entropy will be zero ($\Delta S_{\mathrm{bh}} = 0$). Consequently, the change in generalised entropy will be

$$\Delta S' = \Delta S_{\mathrm{bh}} + \Delta S_m, \tag{33}$$

$$\Delta S' = -S_{\mathrm{box}}, \tag{34}$$

violating the generalised second law. Here we have exactly reproduced the violation of the generalised second law discussed in section 2 and [1–5].

Recalling that to an external observer it would take an infinite amount of time for the box to reach the event horizon, the perceptive reader might note that, for the distant observer, it would take an infinite amount of time to lower the box such that the black hole's entropy change is zero. Whilst this is true, to violate the generalised second law, the distant observer need only lower the box to a distance that results in the black hole's entropy change being less than the box's entropy. Such a distance can be reached within a finite amount of time.

4.2. Entropy production in the lowering process

Hitherto, it has been assumed that the lowering process is isentropic. This assumption is commonly made in the literature [1–5] and, as demonstrated above, results in the violation of the generalised second law. We will now dispose of this assumption and instead assume that entropy is produced in the space-time surrounding the box during the lowering process. This entropy production will result in a loss of work to the surrounding space-time [64] which, by limiting the amount of work that the distant observer can extract, allows for the generalised second law to be preserved.

To demonstrate that the production of entropy results in a non-negative change in generalised entropy, the change in the black hole and common entropy must be determined. As in the isentropic lowering process, the change in black hole entropy can be found using the first law of thermodynamics and the black hole's corresponding change in energy, which itself can be found by calculating the work extracted by the distant observer. In this case, however, the work extracted by the distant observer will be limited by the loss of work to the surrounding space-time. Considering this, the maximum amount of work the distant observer can extract W_{ex} is given by

$$W_{\mathrm{ex}} = W_{\infty} - W_{\mathrm{lost}}, \tag{35}$$

where W_{∞} is the amount of work the distant observer would extract were there no entropy production and W_{lost} is the amount of work lost to the surrounding space-time which, in lowering the the box to a distance r_{f}, is itself given by [64]

$$W_{\mathrm{lost}} = -\int T \mathrm{d}S_{\mathrm{sur}}, \tag{36}$$

$$W_{\mathrm{lost}} = -\int_{r_{\mathrm{f}}}^{\infty} T \frac{\mathrm{d}S_{\mathrm{sur}}}{\mathrm{d}r} \mathrm{d}r, \tag{37}$$

where T is the locally measured temperature of the box's vacuum state, $\mathrm{d}S_{\mathrm{sur}}$ is the change in the surrounding space-time's entropy and $\frac{\mathrm{d}S_{\mathrm{sur}}}{\mathrm{d}r}$ is the rate of change of the surrounding space-time's entropy with respect to the box's distance r. Thus, the maximum amount of work the distant observer can extract by lowering the box to a distance r_{f} is given by

$$W_{\mathrm{ex}} = \int_{r_{\mathrm{f}}}^{\infty} F \mathrm{d}r + \int_{r_{\mathrm{f}}}^{\infty} T \frac{\mathrm{d}S_{\mathrm{sur}}}{\mathrm{d}r} \mathrm{d}r. \tag{38}$$

Taking into account the entropy produced in the surrounding space-time by lowering the box to a distance r_{f}, the change in the common entropy will be given by

$$\Delta S_m = -\int_{r_{\mathrm{f}}}^{\infty} \frac{\mathrm{d}S_{\mathrm{sur}}}{\mathrm{d}r} \mathrm{d}r - S_{\mathrm{box}}. \tag{39}$$

To determine the change in generalised entropy, a function $S_{\mathrm{sur}}(r)$ describing the production of entropy, which is dependent on the box's position r, must be identified. In the absence of such a function in the literature, we are left to postulate what the form of this function might be. To aid in this task, we impose two requirements on the change in black hole and common entropy following the lowering process. These requirements will provide two conditions on the form of $S_{\mathrm{sur}}(r)$. First, we require that the change in black hole entropy is finite. This is to ensure that the black hole's entropy remains finite following the lowering process [3, 48]. To satisfy this requirement, the black hole's energy change and hence, the work lost to the surrounding space-time, must also be finite. Using this requirement and equation (37) for the work lost to the surrounding space-time, we obtain our first condition: The function $S_{\mathrm{sur}}(r)$ must be of a form such that the integral

$$\int T \frac{\mathrm{d}S_{\mathrm{sur}}}{\mathrm{d}r} \mathrm{d}r \tag{40}$$

converges in both of the limits $r \to \infty$ and $r \to R_{\mathrm{s}}$. Second, we require that the change in common entropy is finite. This is to avoid an unsatisfactory resolution of the violation of the generalised second law on the basis of an infinite increase in common entropy. Using this requirement and equation (39) for the change in common entropy, we obtain our second condition: The function $S_{\mathrm{sur}}(r)$ must be of a form such that the integral

$$\int \frac{\mathrm{d}S_{\mathrm{sur}}}{\mathrm{d}r} \mathrm{d}r \tag{41}$$

converges in both of the limits $r \to \infty$ and $r \to R_s$.

Equipped with these two conditions, we can identify functions that might describe the production of entropy during the lowering process. There are a multitude of functions that satisfy both of these conditions. We begin by identifying one such function to demonstrate that the production of entropy allows for the generalised second law to not be violated via the lowering process. This is the function

$$S_{\text{sur}}(r) = \frac{\alpha}{r^2} + \text{const}, \tag{42}$$

which has the following derivative with respect to r:

$$\frac{dS_{\text{sur}}}{dr} = -\frac{2\alpha}{r^3}, \tag{43}$$

where α is some positive constant with units m^2JK^{-1} and const is an arbitrary integration constant. Taking equation (42) to describe the production of entropy, we now proceed with determining the change in generalised entropy following the lowering of the box arbitrarily close to the event horizon of the black hole. We begin by finding the black hole's change in entropy by calculating the work extracted by the distant observer. The maximum amount of work the distant observer can extract by lowering a box of entropy S_{box} to the event horizon is

$$W_{\text{ex}} = \int_{R_s}^{\infty} F dr + \int_{R_s}^{\infty} T \frac{dS_{\text{sur}}}{dr} dr. \tag{44}$$

Applying equation (22) for the locally measured temperature of the box's vacuum state, (24) for the force required to hold an observer in a stationary position and (43) for the derivative of our choice of $S_{\text{sur}}(r)$ yields

$$W_{\text{ex}} = GMm \int_{R_s}^{\infty} \frac{1}{r^2} \left(1 - \frac{R_s}{r}\right)^{-\frac{1}{2}} dr - \frac{\alpha \hbar GM}{\pi k_b c} \int_{R_s}^{\infty} \frac{1}{r^5} \left(1 - \frac{R_s}{r}\right)^{-\frac{1}{2}} dr. \tag{45}$$

Via the change of variables $u = 1 - \frac{R_s}{r}$, both of the above integrals can be solved to obtain

$$W_{\text{ex}} = mc^2 - \frac{32\alpha \hbar GM}{35\pi k_b c {R_s}^4}, \tag{46}$$

which, by applying equations (8) and (11) for the Schwarzschild radius and Hawking temperature, respectively, can be simplified to

$$W_{\text{ex}} = mc^2 - \frac{64\alpha T_H}{35{R_s}^2}. \tag{47}$$

Upon the box crossing the event horizon of the black hole, the resultant change in energy of the black hole will be

$$\Delta E_{\text{bh}} = mc^2 - W_{\text{ex}}, \tag{48}$$

$$\Delta E_{\text{bh}} = \frac{64\alpha T_H}{35{R_s}^2}. \tag{49}$$

Using the first law of Schwarzschild black hole thermodynamics, the corresponding change in the black hole's entropy will be

$$\Delta S_{\text{bh}} = \frac{64\alpha}{35{R_s}^2}. \tag{50}$$

Having obtained the black hole's entropy change, we now turn to calculating the change in common entropy. Considering the production of entropy during the lowering process, the change in common entropy following the lowering of the box to the event horizon will be

$$\Delta S_m = -\int_{R_s}^{\infty} \frac{dS_{\text{sur}}}{dr} dr - S_{\text{box}}, \tag{51}$$

Applying equation (43) for the derivative of our choice of $S_{\text{sur}}(r)$ yields

$$\Delta S_m = 2\alpha \int_{R_s}^{\infty} \frac{1}{r^3} dr - S_{\text{box}}, \tag{52}$$

$$\Delta S_m = \frac{\alpha}{{R_s}^2} - S_{\text{box}}. \tag{53}$$

Therefore, the resultant change in generalised entropy will be

$$\Delta S' = \Delta S_{\text{bh}} + \Delta S_m, \tag{54}$$

$$\Delta S' = \frac{99\alpha}{35{R_s}^2} - S_{\text{box}}, \tag{55}$$

which is non-negative if $\alpha \geqslant \frac{35{R_s}^2 S_{\text{box}}}{99}$. That is, taking $\alpha = {R_s}^2 S_{\text{box}}$ would give a positive change in generalised entropy. For this choice of α, the associated function describing entropy production in the surrounding space-time is

$$S_{\text{sur}}(r) = \frac{{R_s}^2 S_{\text{box}}}{r^2} + \text{const} = \frac{A_{\text{bh}} S_{\text{box}}}{4\pi r^2} + \text{const}. \tag{56}$$

The corresponding change in generalised entropy when the box is lowered arbitrarily close to the event horizon is $\Delta S' = \frac{99 S_{\text{box}}}{35}$.

4.3. A set of functions describing entropy production

The previous subsection's analysis can be generalised further by identifying a set of functions describing the production of entropy in the surrounding space-time that result in a non-negative change in generalised entropy. The reason for identifying a set of such functions is to make as few assumptions as possible about the mechanism by which entropy is produced during the lowering process. The first step in identifying such a set of functions is to notice that both of our conditions for $S_{\text{sur}}(r)$ are satisfied when $S_{\text{sur}}(r)$ is of the form

$$S_{\text{sur}}(r) = \frac{\alpha}{r^n} + \text{const}, \tag{57}$$

where $n \in \mathbb{Z}^+$ and α is a positive non-zero constant with units m^nJK^{-1} that will be chosen such that $\Delta S' \geqslant 0$. Here, *const* is once again an arbitrary integration constant. Next, for this form of $S_{\text{sur}}(r)$, the change in generalised entropy resulting from the lowering process must be calculated. This is done to identify the values of α for which the change in generalised entropy is non-negative. As before, to calculate the change in black hole entropy, the change in the black hole's energy and hence, the work extracted by the distant observer, must be calculated. In the process of lowering the box to the black hole's event horizon, the maximum amount of work extracted by the distant observer W_{ex} will be given by equation (44), where the derivative of $S_{\text{sur}}(r)$ with respect to the box's distance r is

$$\frac{\mathrm{d}S_{\text{sur}}}{\mathrm{d}r} = -\frac{\alpha n}{r^{n-1}}. \tag{58}$$

Applying equation (22) for the locally measured temperature of the box's vacuum state, (24) for the force required to hold the box in a stationary position and (58) for the derivative of $S_{\text{sur}}(r)$, equation (44) for the maximum amount of work extracted can be expressed as

$$W_{\text{ex}} = GMm \int_{R_s}^{\infty} \frac{1}{r^2}\left(1 - \frac{R_s}{r}\right)^{-\frac{1}{2}} \mathrm{d}r - \frac{\alpha n \hbar GM}{2\pi k_b c} \int_{R_s}^{\infty} \frac{1}{r^{n+3}}\left(1 - \frac{R_s}{r}\right)^{-\frac{1}{2}} \mathrm{d}r. \tag{59}$$

Via the change of variables $u = 1 - \frac{R_s}{r}$, the first integral in this expression can be solved to yield

$$W_{\text{ex}} = mc^2 - \frac{\alpha n \hbar GM}{2\pi k_b c} \int_{R_s}^{\infty} \frac{1}{r^{n+3}}\left(1 - \frac{R_s}{r}\right)^{-\frac{1}{2}} \mathrm{d}r. \tag{60}$$

The remaining integral can be solved for all $n \in \mathbb{Z}^+$ by making the same change of variables and then expanding the integrand using the binomial theorem. This is shown explicitly in appendix B. Doing so gives the maximum amount of work extracted as

$$W_{\text{ex}} = mc^2 - \frac{\alpha n \hbar GM \Sigma_n}{2\pi k_b c {R_s}^{n+2}}, \tag{61}$$

where

$$\Sigma_n = \frac{2^{2n+3}}{2n+3}\binom{2n+2}{n+1}^{-1}. \tag{62}$$

By applying equations (8) and (11) for the Schwarzschild radius and Hawking temperature, respectively, equation (61) for the maximum amount of work extracted can be simplified to

$$W_{\mathrm{ex}} = mc^2 - \frac{\alpha n T_{\mathrm{H}} \Sigma_n}{{R_{\mathrm{s}}}^n}. \tag{63}$$

Thus, once the box crosses the event horizon of the black hole, the energy change of the black hole will be

$$\Delta E_{\mathrm{bh}} = mc^2 - W_{\mathrm{ex}}, \tag{64}$$

$$\Delta E_{\mathrm{bh}} = \frac{\alpha n T_{\mathrm{H}} \Sigma_n}{{R_{\mathrm{s}}}^n}. \tag{65}$$

Using the first law of Schwarzschild black hole thermodynamics, we obtain the corresponding change in black hole entropy,

$$\Delta S_{\mathrm{bh}} = \frac{\alpha n \Sigma_n}{{R_{\mathrm{s}}}^n}. \tag{66}$$

Having found the change in the black hole's entropy, the change in common entropy must now be determined. For all $n \in \mathbb{Z}^+$, the change in common entropy resulting from the box being lowered to the event horizon of the black hole will be

$$\Delta S_m = -\int_{R_{\mathrm{s}}}^{\infty} \frac{\mathrm{d}S_{\mathrm{sur}}}{\mathrm{d}r} \mathrm{d}r - S_{\mathrm{box}}, \tag{67}$$

$$\Delta S_m = -\int_{R_{\mathrm{s}}}^{\infty} \frac{\alpha n}{r^{n+1}} \mathrm{d}r - S_{\mathrm{box}}, \tag{68}$$

$$\Delta S_m = \frac{\alpha}{{R_{\mathrm{s}}}^n} - S_{\mathrm{box}}. \tag{69}$$

Therefore, upon the box crossing the event horizon, the resultant change in generalised entropy $\Delta S'$ will be

$$\Delta S' = \Delta S_{\mathrm{bh}} + \Delta S_m, \tag{70}$$

$$\Delta S' = \frac{\alpha (n\Sigma_n + 1)}{{R_{\mathrm{s}}}^n} - S_{\mathrm{box}}. \tag{71}$$

To ensure that the change in generalised entropy is non-negative, such that the generalised second law is not violated as a result of the lowering process, we require

$$\alpha \geqslant \frac{{R_{\mathrm{s}}}^n S_{\mathrm{box}}}{n\Sigma_n + 1}. \tag{72}$$

Thus, provided that the entropy produced in the space-time surrounding the box is described by a function within the set of functions given by equation (57) and α satisfies this bound, the lowering process will not violate the generalised second law.

Even with the bound on α given in equation (72), there remains some ambiguity in identifying potential values for α. Ideally, we would like there to be some physical motivation for our choice in α. For the remainder of this section, we will discuss two natural choices for α that ensure that the generalised second law is not violated as a result of the lowering process.

First is the choice $\alpha = {R_{\mathrm{s}}}^n S_{\mathrm{box}}$, where $n \in \mathbb{Z}^+$. This choice of α corresponds to there being no change in common entropy. That is, the amount of entropy produced is equal to the entropy of the box. In this case, the change in generalised entropy is $\Delta S' = n\Sigma_n S_{\mathrm{box}}$ which, for some $n \in \mathbb{Z}^+$, is non-negative if $\Sigma_n \geqslant 0$. From the expression for Σ_n given in equation (62), it is evident that $\Sigma_n > 0$ for all $n \in \mathbb{Z}^+$. Thus, the choice $\alpha = {R_{\mathrm{s}}}^n S_{\mathrm{box}}$ will always give a non-negative change in generalised entropy. For this choice of α, The corresponding function describing the production of entropy during the lowering process, for a general $n \in \mathbb{Z}^+$, is

$$S_{\mathrm{sur}}(r) = \frac{{R_{\mathrm{s}}}^n S_{\mathrm{box}}}{r^n} + \mathrm{const}. \tag{73}$$

Second is the choice $\alpha = \frac{{R_{\mathrm{s}}}^n S_{\mathrm{box}}}{n\Sigma_n}$, where $n \in \mathbb{Z}^+$. This corresponds to imposing that the change in black hole entropy is equal to the entropy of the box. In this case, the change in generalised entropy is $\Delta S' = \frac{S_{\mathrm{box}}}{n\Sigma_n}$, which is non-negative and finite for all $n \in \mathbb{Z}^+$ as $\Sigma_n > 0$ for all $n \in \mathbb{Z}^+$. Here, the corresponding function describing the production of entropy during the lowering process, for a general $n \in \mathbb{Z}^+$, is

$$S_{\mathrm{sur}}(r) = \frac{{R_{\mathrm{s}}}^n S_{\mathrm{box}}}{n\Sigma_n r^n} + \mathrm{const}. \tag{74}$$

5. Conclusion

In this paper, we have argued that to ensure the generalised second law is not violated via the process in which a distant observer extracts work by lowering a box, of negligible volume, arbitrarily close to the event horizon of a black hole, entropy must be produced in the space-time surrounding the box. Using the Hawking temperature, Unruh effect, and concepts from quantum information, equations for the locally measured temperature of the box's vacuum state and the force required to hold this box in a stationary position relative to the black hole were derived. These equations form the framework for determining the change in generalised entropy as a result of this lowering process both in the case where the process is isentropic and in the case where entropy is produced during the process. In the former case, it was shown that the generalised second law is violated in the exact fashion as in the literature [1–5]. In the latter case, it was argued that the production of entropy in the surrounding space-time would limit the amount of work the distant observer can extract, allowing for the generalised second law to be preserved. To explicitly demonstrate this, a function describing entropy production in the space-time surrounding the box during the lowering process was required. In the absence of such a function in the literature, our imposition of two requirements: (1) that the resultant change in black hole entropy is finite; and (2) that the resultant change in common entropy is finite, was used to obtain two conditions that the form of such a function must satisfy. These conditions were in turn used to identify a set of possible functions describing the production of entropy during the lowering process. For this set of functions, the change in generalised entropy resulting from the lowering process was calculated and shown to be non-negative provided that a coefficient in this set of functions satisfies a given bound. Finally, two natural choices of this coefficient, which allow for the generalised second law to be preserved in the lowering process, were discussed. The first choice, which arose from imposing that the entropy produced be equal to the entropy of the box, corresponds to the following set of functions:

$$S_{\text{sur}}(r) = \frac{{R_{\text{s}}}^{n} S_{\text{box}}}{r^{n}} + \text{const.} \tag{75}$$

The second choice, which arose from imposing that the change in black hole entropy be equal to the entropy of the box, corresponds to the following set of functions:

$$S_{\text{sur}}(r) = \frac{{R_{\text{s}}}^{n} S_{\text{box}}}{n\Sigma_{n} r^{n}} + \text{const.} \tag{76}$$

The reason for identifying a set of possible functions describing the production of entropy during the lowering process was two-fold. First, it enabled us to make as few assumptions as possible about the mechanism by which entropy is produced during the lowering process. Second, it aided in demonstrating the flexibility of our argument. This flexibility being that the preservation of the generalised second law on the basis of entropy production during the lowering process is not reliant on the validity of any particular entropy production mechanism. This is particularly important as the set of functions presented in this paper does not exhaust all possible forms of such a function that allow for the preservation of the generalised second law. Thus, in the event that an alternative function is proposed, the framework presented in this paper provides a way by which the changes in black hole entropy, common entropy and generalised entropy can be found for the discussed lowering process.

In addition to providing a resolution to the discussed violation of the generalised second law, the framework presented in this paper can be applied to test and inform theories of gravity and quantum gravity. The reason for this is that any theory of gravity or quantum gravity must have entropy relations that satisfy the two conditions required by the form of an entropy production function, such that changes in black hole entropy, common entropy and generalised entropy remain finite, and the generalised second law is not violated, as a result of the discussed lowering process. Whilst we make no comment on its validity, an intriguing example of this is Verlinde's theory of entropy gravity [31], which postulates a linear relationship between a particle's position and entropy. This relationship violates the second condition required by an entropy production function, and consequently, if one were to take it to describe entropy production during the lowering process, one would find that the resultant change in common entropy is infinite.

Whilst the framework presented in this paper is specific to Schwarzschild black holes, it has the potential to be expanded to charged and massive Reissner–Nordström black holes. The first step towards this is deriving an equation for the locally measured temperature of a charged box's vacuum state. As shown in appendix C, by using the Hawking temperature and gravitational red-shift factor for a Reissner–Nordström black hole, this can be done by following the method presented in section 3. Next, an equation for the force required to hold the charged box in a stationary position relative to the black hole must be derived. It is not, however, immediately clear how the Unruh effect alone could be used to obtain an expression for this force that is dependent on the box's charge. In addition to this, whilst the production of entropy during the lowering process would limit the amount of work an external observer can extract, it is not clear whether the set of functions presented in this paper would apply to the case of a Reissner–Nordström black hole. Nevertheless, the requirements that both the change in the black hole entropy and the change in common entropy remain finite could once again be used to obtain two conditions on the form of such a function. It is not immediately clear how the framework of this paper could be expanded to rotating and massive Kerr black holes. The lack of an equivalent to the Unruh effect in the space-time surrounding a Kerr black hole [1, 65] would make this a challenging task.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. General relativity: stationary observer

In section 3.3, an equation for the force required to hold an observer in a stationary position relative to a Schwarzschild black hole is derived. This is done by deriving an equation for the locally measured temperature of the observer's vacuum state and then using the Unruh effect to relate this temperature to an acceleration. In this appendix, we derive the same equations for the force required to hold an observer in a stationary position relative to a Schwarzschild and the locally measured temperature of the observer's vacuum state using general relativity. This is done to verify that the force derived in section 3.3 is indeed the force required to hold the observer in a stationary position.

Consider an observer that is in a stationary position relative to the exterior of a Schwarzschild black hole with mass M. The curvature of this observer's space-time is described by the Schwarzschild metric line element $\mathrm{d}s^2$, which in spherical polar coordinates (t, r, θ, ϕ) is given by [66]

$$\mathrm{d}s^2 = \left(1 - \frac{R_\mathrm{s}}{r}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{R_\mathrm{s}}{r}\right)^{-1} \mathrm{d}r^2 - r^2 \mathrm{d}\theta^2 - r^2 \sin^2\theta \mathrm{d}\phi^2, \tag{A.1}$$

where t is the time coordinate, r is the observer's distance from the centre of the black hole, θ and ϕ are the observer's polar and azimuthal angles, respectively, and R_s is the Schwarzschild radius of the black hole.

As the observer is stationary, that is, r, θ and ϕ are all constant, $\mathrm{d}r = \mathrm{d}\theta = \mathrm{d}\phi = 0$. Therefore, for the stationary observer, the metric in equation (A.1) reduces to

$$\mathrm{d}s^2 = \left(1 - \frac{R_\mathrm{s}}{r}\right) c^2 \mathrm{d}t^2. \tag{A.2}$$

As the observer has mass, we can define the proper time τ as $\mathrm{d}s^2 = -c^2 \mathrm{d}\tau^2$. Substitution of the proper time's definition into equation (A.2) for the stationary observer's metric yields

$$\mathrm{d}\tau^2 = \left(1 - \frac{R_\mathrm{s}}{r}\right) \mathrm{d}t^2, \tag{A.3}$$

which can be rearranged to obtain

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(1 - \frac{R_\mathrm{s}}{r}\right)^{\frac{1}{2}}, \tag{A.4}$$

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \chi, \tag{A.5}$$

where we have identified the Schwarzschild metric gravitational red-shift factor $\chi = \left(1 - \frac{R_\mathrm{s}}{r}\right)^{\frac{1}{2}}$.

To find the force required to hold the observer in a stationary position, the magnitude of the observer's four-acceleration and therefore, the observer's four-velocity must be found. The four-velocity U^μ is defined as

$$U^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}, \tag{A.6}$$

where $x^\mu = (ct, r, \theta, \phi)$ and $\mu = (t, r, \theta, \phi)$ denotes space-time indices. The above equation for the four-velocity can be expressed as

$$U^\mu = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}x^\alpha}, \tag{A.7}$$

$$U^\mu = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}t} + \frac{\mathrm{d}r}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}r} + \frac{\mathrm{d}\phi}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\phi} + \frac{\mathrm{d}\theta}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\theta}, \tag{A.8}$$

which, as $\mathrm{d}r = \mathrm{d}\theta = \mathrm{d}\phi = 0$ for the stationary observer, reduces to

$$U^\mu = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}t}. \tag{A.9}$$

Hence, the four-velocity of the stationary observer is

$$U^\mu = \frac{\mathrm{d}t}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}t}(ct, r, \theta, \phi), \tag{A.10}$$

$$U^\mu = \chi^{-1}(c, 0, 0, 0). \tag{A.11}$$

As is to be expected for a stationary observer, the only non-zero component of the four-velocity is U^t.

Having obtained the stationary observer's four-velocity, we can now proceed with determining their four-acceleration. The stationary observer's four-acceleration A^μ can be defined using the geodesic equation as [66]

$$A^\mu = \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}, \tag{A.12}$$

where $\Gamma^\mu_{\alpha\beta}$ denotes the Christoffel symbols of the Schwarzschild metric. Equation (A.12) for the four-acceleration can be rearranged to obtain

$$A^\mu = \frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}, \tag{A.13}$$

$$A^\mu = \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}x^\gamma}\frac{\mathrm{d}^2x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}, \tag{A.14}$$

$$A^\mu = \frac{\mathrm{d}x^\gamma}{\mathrm{d}\tau}\frac{\mathrm{d}}{\mathrm{d}x^\gamma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau}, \tag{A.15}$$

$$A^\mu = U^\gamma U^\mu{}_{,\gamma} + \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta, \tag{A.16}$$

where $_{,\gamma}$ denotes the partial derivative with respect to x^γ. Expanding the summation of γ in the above expression yields

$$A^\mu = U^t U^\mu{}_{,t} + U^r U^\mu{}_{,r} + U^\theta U^\mu{}_{,\theta} + U^\phi U^\mu{}_{,\phi} + \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta, \tag{A.17}$$

which, as $U^r = U^\theta = U^\phi = 0$, reduces to

$$A^\mu = U^t U^\mu{}_{,t} + \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta. \tag{A.18}$$

As the four-velocity of the stationary observer is independent of t, $U^\mu{}_{,t} = 0$. The above expression therefore reduces further to

$$A^\mu = \Gamma^\mu_{\alpha\beta}U^\alpha U^\beta. \tag{A.19}$$

The summation of α in the above expression can be expanded to yield

$$A^\mu = \left(\Gamma^\mu_{t\beta}U^t + \Gamma^\mu_{r\beta}U^r + \Gamma^\mu_{\theta\beta}U^\theta + \Gamma^\mu_{\phi\beta}U^\phi\right)U^\beta, \tag{A.20}$$

which again, as $U^r = U^\theta = U^\phi = 0$, can be reduced to

$$A^\mu = \Gamma^\mu_{t\beta} U^t U^\beta. \tag{A.21}$$

This can be repeated for the summation of β to obtain

$$A^\mu = \Gamma^\mu_{tt} U^t U^t. \tag{A.22}$$

As the Christoffel symbols Γ^t_{tt}, Γ^θ_{tt} and Γ^ϕ_{tt} for the Schwarzschild metric are all zero [66], the only non-zero component of the four-acceleration will be

$$A^r = \Gamma^r_{tt} U^t U^t, \tag{A.23}$$

where [66]

$$\Gamma^r_{tt} = \frac{R_s}{2r^2}\left(1 - \frac{R_s}{r}\right), \tag{A.24}$$

$$\Gamma^r_{tt} = \frac{R_s \chi^2}{2r^2}. \tag{A.25}$$

Using the above expression for Γ^r_{tt}, $U^t = c\chi^{-1}$, and equation (A.23) for the non-zero component of the stationary observer's four-acceleration, we can express A^r as

$$A^r = \frac{R_s c^2}{2r^2}, \tag{A.26}$$

$$A^r = \frac{GM}{r^2}. \tag{A.27}$$

The magnitude of the stationary observer's four-acceleration a is given by $a = \sqrt{A_\mu A^\mu}$, where

$$A_\mu A^\mu = g_{\mu\nu} A^\nu A^\mu \tag{A.28}$$

and $g_{\mu\nu}$ is the Schwarzschild metric tensor, which is given in equation (7). Expanding the summation of μ and ν in the above expression gives

$$A_\mu A^\mu = g_{tt} A^t A^t + g_{rr} A^r A^r + g_{\theta\theta} A^\theta A^\theta + g_{\phi\phi} A^\phi A^\phi, \tag{A.29}$$

which as $A^t = A^\theta = A^\phi = 0$, reduces to

$$A_\mu A^\mu = g_{rr} A^r A^r, \tag{A.30}$$

where $g_{rr} = \chi^{-2}$ for the stationary observer. Applying equation (A.27) for the non-zero component of the stationary observer's four-acceleration yields

$$A_\mu A^\mu = \frac{1}{\chi^2}\left(\frac{GM}{r^2}\right)^2. \tag{A.31}$$

Hence, the magnitude of the stationary observer's acceleration a, in their reference frame, is

$$a = \frac{GM}{r^2 \chi}. \tag{A.32}$$

Newton's second law of motion and equation (A.32) for the magnitude of the stationary observer's acceleration can be used to obtain

$$F = \frac{GMm}{r^2 \chi}, \tag{A.33}$$

where F is the force required to hold the observer in a stationary position (as measured by the stationary observer) and m is the stationary observer's mass. Additionally, equation (A.32) for the magnitude of the stationary observer's acceleration can be equated to equation (5) for the Unruh temperature to obtain

$$T = \frac{\hbar GM}{2\pi k_b c r^2 \chi} \tag{A.34}$$

where T is the locally measured temperature of the stationary observer's vacuum state, k_b is the Boltzmann constant and $\hbar$ is the reduced Plank's constant. Both of the latter two equations are the same as those derived in section 3.3.

Appendix B. Integral of a set of functions

Consider the following integral:

$$I = \int_{R_s}^{\infty} \frac{1}{r^{n+3}} \left(1 - \frac{R_s}{r}\right)^{-\frac{1}{2}} \mathrm{d}r, \tag{B.1}$$

where $n \in \mathbb{Z}^+$ and R_s denotes the Schwarzschild radius of a black hole. To solve this integral $\forall n \in \mathbb{Z}^+$, we begin by making the change of variables $u = 1 - \frac{R_s}{r}$ such that $\mathrm{d}u = \frac{R_s}{r^2}\mathrm{d}r$. This yields

$$I = \int_{R_s}^{\infty} \frac{1}{r^{n+3}} \left(1 - \frac{R_s}{r}\right)^{-\frac{1}{2}} \mathrm{d}r, \tag{B.2}$$

$$I = \frac{1}{R_s} \int_{r\to R_s}^{r\to\infty} \frac{1}{r^{n+1}} u^{-\frac{1}{2}} \mathrm{d}u, \tag{B.3}$$

$$I = \frac{1}{{R_s}^{n+2}} \int_{r\to R_s}^{r\to\infty} (1-u)^{n+1} u^{-\frac{1}{2}} \mathrm{d}u. \tag{B.4}$$

Next, the term $(1-u)^{n+1}$ can be expanded using the binomial theorem to obtain

$$I = \frac{1}{{R_s}^{n+2}} \int_{r\to R_s}^{r\to\infty} \left(\sum_{k=0}^{n+1} \binom{n+1}{k} (-u)^k \right) u^{-\frac{1}{2}} \mathrm{d}u, \tag{B.5}$$

$$I = \frac{1}{{R_s}^{n+2}} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \int_{r\to R_s}^{r\to\infty} u^{k-\frac{1}{2}} \mathrm{d}u, \tag{B.6}$$

$$I = \frac{1}{{R_s}^{n+2}} \sum_{k=0}^{n+1} \binom{n+1}{k} (-1)^k \left[\frac{u^{k+\frac{1}{2}}}{k+\frac{1}{2}} \right]_{r\to R_s}^{r\to\infty}, \tag{B.7}$$

$$I = \frac{1}{{R_s}^{n+2}} \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{(-1)^k}{k+\frac{1}{2}} \left[\left(1 - \frac{R_s}{r}\right)^{k+\frac{1}{2}} \right]_{r\to R_s}^{r\to\infty}, \tag{B.8}$$

$$I = \frac{1}{{R_s}^{n+2}} \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{(-1)^k}{k+\frac{1}{2}} \left[\lim_{r\to\infty} \left(\left(1 - \frac{R_s}{r}\right)^{k+\frac{1}{2}} \right) - \lim_{r\to R_s} \left(\left(1 - \frac{R_s}{r}\right)^{k+\frac{1}{2}} \right) \right]. \tag{B.9}$$

In the limit $r \to \infty$, $\left(1 - \frac{R_s}{r}\right) \to 1$. Therefore, in this limit $\left(1 - \frac{R_s}{r}\right)^{k+\frac{1}{2}} \to 1\ \forall k$. Similarly, in the limit $r \to R_s$, $\left(1 - \frac{R_s}{r}\right) \to 0$. Therefore, in this limit $\left(1 - \frac{R_s}{r}\right)^{k+\frac{1}{2}} \to 0\ \forall k$. Consequently, taking these limits gives

$$I = \frac{1}{{R_s}^{n+2}} \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{(-1)^k}{k+\frac{1}{2}}. \tag{B.10}$$

We now wish to solve the remaining sum analytically $\forall n \in \mathbb{Z}^+$. This is done to show that this sum is non-negative $\forall n \in \mathbb{Z}^+$. In section 4.3, the non-negativity of this sum aids in finding a generic choice of α that results in a non-negative change in generalised entropy. To analytically solve the remaining sum, we begin by expressing I in the following form:

$$I = \frac{2}{{R_s}^{m+1}} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{2k+1}, \tag{B.11}$$

where $m = n + 1$. Expressing I in this form allows for the combinatorial identity [67]

$$\sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k}{2k+1} = \frac{2^{2m}}{2m+1} \binom{2m}{m}^{-1} \tag{B.12}$$

to be used to obtain

$$I = \frac{1}{{R_s}^{m+1}} \frac{2^{2m+1}}{2m+1} \binom{2m}{m}^{-1}, \tag{B.13}$$

$$I = \frac{1}{{R_s}^{n+2}} \frac{2^{2n+3}}{2n+3} \binom{2n+2}{n+1}^{-1}, \tag{B.14}$$

$$I = \frac{\Sigma_n}{{R_s}^{n+2}}, \tag{B.15}$$

where

$$\Sigma_n = \frac{2^{2n+3}}{2n+3} \binom{2n+2}{n+1}^{-1}. \tag{B.16}$$

Appendix C. Reissner–Nordström black holes

In this appendix, we derive an equation for the locally measured temperature of the vacuum state of an observer that is a finite distance from the event horizon of a charged and massive Reissner–Nordström black hole. The method for this follows that given in section 3 for the case of a Schwarzschild black hole, the only distinction being that expressions for the Hawking temperature and gravitational red-shift factor for a Reissner–Nordström black hole must be used.

Consider a Reissner–Nordström black hole of mass M and geometrized charge $Q' = Q/\sqrt{4\pi\,\epsilon_0 G}$, where Q denotes the charge of the black hole, G Newton's gravitational constant and ϵ_0 the permittivity of free space. The radii $R_\pm$ of the Reissner–Nordström black hole's event horizons are given by [61]

$$R_\pm = \frac{G\left(M \pm \sqrt{M^2 - Q'^2}\right)}{c^2}. \tag{C.1}$$

The existence of these event horizons is determined by the relative values of M^2 and Q'^2. When $M^2 > Q'^2$, both event horizons are real with the outermost event horizon being that with radius R_+. When $M^2 = Q'^2$, only the event horizon with radius R_+ is real. When $M^2 < Q'^2$, $R_\pm$ are both imaginary and therefore neither event horizon exists. We will not consider the latter-most case as, due to the non-existence of the event horizons, it is not considered to be physically realistic [61, 68, 69]. In the remaining two cases, the event horizon with radius R_+ is either the outermost or the only event horizon. We will therefore take R_+ to define the surface area of the black hole. Assuming that the event horizon with radius R_+ is spherically symmetric, its surface area A_{bh} will be [63]

$$A_{\mathrm{bh}} = 4\pi {R_+}^2, \tag{C.2}$$

$$A_{\mathrm{bh}} = \frac{4\pi G^2}{c^4} \left(M + \sqrt{M^2 - Q'^2}\right)^2. \tag{C.3}$$

For a Reissner–Nordström black hole, the Hawking temperature T_{H}, as defined by a distant observer, is given by [63]

$$T_{\mathrm{H}} = \frac{2G\hbar\sqrt{M^2 - Q'^2}}{k_{\mathrm{b}} c A_{\mathrm{bh}}}, \tag{C.4}$$

where $\hbar$ is the reduced Plank's constant and k_{b} is the Boltzmann constant.

For an observer at a distance r from the centre of the Reissner–Nordström black hole, the gravitational red-shift factor χ is given by [7]

$$\chi = \frac{\sqrt{(r - R_+)(r - R_-)}}{r}. \tag{C.5}$$

As in the Schwarzschild case, χ is normalised such that $\chi \to 0$ as $r \to R_+$ (or $r \to R_-$) and $\chi \to 1$ as $r \to \infty$ [7].

Following the argument given in section 3.2, we can associate an average thermal energy of $\frac{1}{2} k_{\mathrm{b}} T_{\mathrm{H}}$ with each bit of information stored on the event horizon of the Reissner–Nordström black hole. This way, the total energy E of the information stored on the event horizon can be given by

$$E = \frac{1}{2} N k_{\mathrm{b}} T_{\mathrm{H}}, \tag{C.6}$$

where N is the maximum number of bits of information stored on the event horizon. Assuming that the maximum amount of information stored on the event horizon is given by

$$N = \frac{A_{\mathrm{bh}} c^3}{G\hbar}, \tag{C.7}$$

and by making use of equation (C.4) for the Hawking temperature of a Reissner–Nordström black hole, equation (C.6) for the total energy of the information stored on the event horizon can be expressed as [63]

$$\frac{1}{2} N k_{\mathrm{b}} T_{\mathrm{H}} = c^2 \sqrt{M^2 - Q'^2}. \tag{C.8}$$

This is the corresponding equipartition rule for a Reissner–Nordström black hole.

We now proceed with deriving the equipartition rule for an observer that is at a finite distance from the black hole. Consider an observer at a distance $R \geqslant 0$ from the outermost event horizon of a Reissner–Nordström black hole. This observer will define a spherically symmetric surface $\mathcal{S}$ with a radius $r = R_+ + R$ that encloses the information contained on the event horizon. As in the Schwarzschild case, we assume that the maximum amount of information $N' \geqslant N$ enclosed by $\mathcal{S}$ will be

$$N' = \frac{4\pi r^2 c^3}{G\hbar}. \tag{C.9}$$

An average thermal energy of $\frac{1}{2} k_{\mathrm{b}} T_{\mathrm{H}}$ can once again be associated with each bit of information enclosed by $\mathcal{S}$ as each bit was erased at the event horizon. The total energy of the information contained within $\mathcal{S}$ is therefore related to the energy of the black hole via the following equation:

$$\frac{1}{2} N' k_{\mathrm{b}} T_{\mathrm{H}} = c^2 \sqrt{M^2 - Q'^2}. \tag{C.10}$$

As the observer is at a finite distance from the black hole's event horizon, by virtue of the Unruh effect in curved space-time, their vacuum state will be a thermal state with a temperature $T = T_{\mathrm{H}}/\chi$. This once again allows us to account for the gravitational red-shifting, which occurs because of the observer's finite distance from the black hole's event horizon, and to relate the energy of the black hole with the red-shifted energy of the information contained within $\mathcal{S}$ via the following equation:

$$\frac{1}{2} N' k_{\mathrm{b}} T = \frac{c^2 \sqrt{M^2 - Q'^2}}{\chi}, \tag{C.11}$$

This is the corresponding red-shifted equipartition rule for a Reissner–Nordström black hole. As in the Schwarzschild case, the substitution of equation (C.9) for the maximum number of bits contained within $\mathcal{S}$ into the red-shifted equipartition rule allows for an equation for the locally measured temperature of the vacuum state of the observer to be obtained. For an observer at a distance $r \geqslant R_+$ from the centre of a Reissner–Nordström black hole, the locally measured temperature T of their vacuum state is given by

$$T = \frac{G\hbar \sqrt{M^2 - Q'^2}}{2\pi k_{\mathrm{b}} c r^2 \chi}. \tag{C.12}$$

In the case where $Q' = 0$, $R_+ = R_{\mathrm{s}}$ and $R_- = 0$. Hence, when $Q' = 0$ the expression for the red-shift factor χ reduces to that of the Schwarzschild black hole and the above equation reduces to equation (22) for the locally measured temperature of an observer at a finite distance from a Schwarzschild black hole. It should be noted that the locally measured temperature T is imaginary when $Q'^2 > M^2$. This, however, is not problematic as, due to the non-existence of its event horizons, a black hole with $Q'^2 > M^2$ is not considered to be physically possible [61, 68, 69].

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References

[1] Wald R M 1994 *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press)
[2] Unruh W G and Wald R M 1982 Acceleration radiation and the generalized second law of thermodynamics *Phys. Rev.* D **25** 942–58
[3] Wald R M 2001 The thermodynamics of black holes *Living Rev. Relativ.* **4** 6
[4] Bekenstein J D 1973 Black holes and entropy *Phys. Rev.* D **7** 2333–46
[5] Bekenstein J D 1974 Generalized second law of thermodynamics in black-hole physics *Phys. Rev.* D **9** 3292–300
[6] Bekenstein J D 1981 universal upper bound on the entropy-to-energy ratio for bounded systems *Phys. Rev.* D **23** 287–98
[7] Bekenstein J D 1994 Entropy bounds and black hole remnants *Phys. Rev.* D **49** 1912–21
[8] Bekenstein J D 1999 Non-archimedean character of quantum buoyancy and the generalized second law of thermodynamics *Phys. Rev.* D **60** 124010
[9] Sorkin R D, Wald R M and Jiu Z Z 1981 Entropy of self-gravitating radiation *Gen. Relativ. Gravit.* **13** 1127–46
[10] Bekenstein J D 1982 Black holes and everyday physics *Gen. Relativ. Gravit.* **14** 355–9
[11] Bekenstein. J D 1983 Entropy bounds and the second law for black holes *Phys. Rev.* D **27** 2262–70
[12] Bekenstein J D 1984 Entropy content and information flow in systems with limited energy *Phys. Rev.* D **30** 1669–79
[13] Bekenstein J D and Guendelman E I 1987 Symmetry breaking induced by charge density and the entropy of interacting fields *Phys. Rev.* D **35** 716–31
[14] Schiffer M and Bekenstein J D 1989 Proof of the quantum bound on specific entropy for free fields *Phys. Rev.* D **39** 1109–15
[15] Mayo A E 1999 Optimal entropy bound and the self-energy of test objects in the vicinity of a black hole *Phys. Rev.* D **60** 104044
[16] Bekenstein J D 2014 Optimizing entropy bounds for macroscopic systems *Phys. Rev.* E **89** 052137
[17] Unwin S D 1982 Possible violations of the entropy-to-energy-ratio bound *Phys. Rev.* D **26** 944–6
[18] Page. D N 1982 Comment on a universal upper bound on the entropy-to-energy ratio for bounded systems *Phys. Rev.* D **26** 947–9
[19] Deutsch D 1982 Is there a fundamental bound on the rate at which information can be processed? *Phys. Rev. Lett.* **48** 286–8
[20] Unruh W G and Wald R M 1983 Entropy bounds, acceleration radiation and the generalized second law *Phys. Rev.* D **27** 2271–6
[21] Unruh W G 1990 Comment on proof of the quantum bound on specific entropy for free fields *Phys. Rev.* D **42** 3596–7
[22] Pelath M A and Wald R M 1999 Comment on entropy bounds and the generalized second law *Phys. Rev.* D **60** 104009
[23] Bousso R 1999 A covariant entropy conjecture *J. High Energy Phys.* JHEP07(1999)004
[24] Page. D N 2000 Subsystem entropy exceeding bekenstein's bound
[25] Bousso R 2002 The holographic principle *Rev. Mod. Phys.* **74** 825–74
[26] Ong Y C 2022 On black hole thermodynamics, singularity and gravitational entropy *Gen. Relativ. Gravit.* **54** 132
[27] Susskind L and Uglum J 1994 Black hole entropy in canonical quantum gravity and superstring theory *Phys. Rev.* D **50** 2700–11
[28] Smolin L 2001 The strong and weak holographic principles *Nucl. Phys.* B **601** 209–47
[29] Nishioka T, Ryu S and Takayanagi T 2009 Holographic entanglement entropy: an overview *J. Phys. A: Math. Theory* **42** 504008
[30] Padmanabhan T 2020 Gravity and quantum theory: domains of conflict and contact *Int. J. Mod. Phys.* D **29** 2030001
[31] Verlinde E 2011 On the origin of gravity and the laws of newton *J. High Energy Phys.* JHEP04(2011)029
[32] Jacobson T 1995 Thermodynamics of spacetime: the einstein equation of state *Phys. Rev. Lett.* **75** 1260–3
[33] Eling C, Guedens R and Jacobson T 2006 Nonequilibrium thermodynamics of spacetime *Phys. Rev. Lett.* **96** 121301
[34] Padmanabhan T 2002 Classical and quantum thermodynamics of horizons in spherically symmetric spacetimes *Class. Quantum Grav.* **19** 5387
[35] Padmanabhan T 2010 Thermodynamical aspects of gravity: new insights *Rep. Prog. Phys.* **73** 046901
[36] Lee J-W, Kim H-C and Lee J 2010 Gravity from quantum information *J. Korean Phys. Soc.* **63** 1094–98
[37] Davies P C W 1987 Cosmological horizons and the generalised second law of thermodynamics *Class. Quantum Grav.* **4** L225
[38] Cai Y-F, Liu J and Li H 2010 Entropic cosmology: a unified model of inflation and late-time acceleration *Phys. Lett.* B **690** 213–9
[39] Easson D A, Frampton P H and Smoot G F 2011 Entropic accelerating universe *Phys. Lett.* B **696** 273–7
[40] Easson D A, Frampton P H and Smoot G F 2012 Entropic inflation *Int. J. Mod. Phys.* A **27** 1250066
[41] Komatsu N and Kimura S 2013 Entropic cosmology for a generalized black-hole entropy *Phys. Rev.* D **88** 083534
[42] Komatsu N and Kimura S 2013 Non-adiabatic-like accelerated expansion of the late Universe in entropic cosmology *Phys. Rev.* D **87** 043531
[43] Tu F-Q, Chen Y-X, Sun B and Yang Y-C 2018 Accelerated expansion of the Universe based on emergence of space and thermodynamics of the horizon *Phys. Lett.* B **784** 411–6
[44] Zamora D J and Tsallis C 2022 Thermodynamically consistent entropic-force cosmology *Phys. Lett.* B **827** 136967
[45] Susskind L 1995 The world as a hologram *J. Math. Phys.* **36** 6377–96
[46] 't Hooft G 1993 Dimensional reduction in quantum gravity *Conf. Proc.* C **930308** 284–96
[47] Bekenstein J D 2003 Information in the holographic Universe *Sci. Am.* **289** 58–65
[48] Hawking S W 1976 Black holes and thermodynamics *Phys. Rev.* D **13** 191–7
[49] Davies P C W 1975 Scalar production in schwarzschild and rindler metrics *J. Phys. A: Math. Gen.* **8** 609
[50] Bardeen J M, Carter B and Hawking S W 1973 The four laws of black hole mechanics *Commun. Math. Phys.* **31** 161–70
[51] Hawking S W 1975 Particle creation by black holes *Commun. Math. Phys.* **43** 199–220
[52] Fulling S A 1973 Nonuniqueness of canonical field quantization in riemannian space-time *Phys. Rev.* D **7** 2850–62
[53] Unruh W G 1976 Notes on black-hole evaporation *Phys. Rev.* D **14** 870–92
[54] Bekenstein J D 2004 Black holes and information theory *Contemp. Phys.* **45** 31–43
[55] Kiefer C 2004 *Quantum Gravity International Series of Monographs in Physics.* 2nd edn (Clarendon)
[56] Hawking S W 1971 Gravitational radiation from colliding black holes *Phys. Rev. Lett.* **26** 1344–6
[57] Hawking S W 1972 Black holes in general relativity *Commun. Math. Phys.* **25** 152–66
[58] Christodoulou D 1970 Reversible and irreversible transformations in black-hole physics *Phys. Rev. Lett.* **25** 1596–7
[59] Israel W 1986 Third law of black-hole dynamics: a formulation and proof *Phys. Rev. Lett.* **57** 397–9
[60] Wald R M 1984 *General Relativity* (Chicago University Press)
[61] Hobson M P, Efstathiou G P and Lasenby A N 2006 *General Relativity: An Introduction for Physicists* (Cambridge University Press)

[62] Plenio M and Vitelli V 2001 The physics of forgetting: Landauer's erasure principle and information theory *Contemp. Phys.* **42** 25–60

[63] Wang T 2010 Coulomb force as an entropic force *Phys. Rev.* D **81** 104045

[64] Vanderslice J T, Schamp Jr. H W and Mason E A 1966 *Thermodynamics* (Prentice-Hall Inc.)

[65] Kay B S and Wald R M 1991 Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate killing horizon *Phys. Rep.* **207** 49–136

[66] Carroll S M 2019 *Spacetime and Geometry: An Introduction to General Relativity* (Cambridge University Press)

[67] Riordan J 1968 *Combinatorial Identities* (*Wiley Series in Probability and Mathematical Statistics)* vol 3 (Wiley)

[68] Wald R M 1974 Gedanken experiments to destroy a black hole *Ann. Phys., NY* **82** 548–56

[69] Sorce J and Wald R M 2017 Gedanken experiments to destroy a black hole. II. kerr-newman black holes cannot be overcharged or overspun *Phys. Rev.* D **96** 104014