OPEN ACCESS
IOP Publishing

European Journal of Physics

Eur. J. Phys. 45 (2024) 055802 (20pp)

https://doi.org/10.1088/1361-6404/ad61d3

Understanding the natural units and their hidden role in the laws of physics

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Received 8 March 2024, revised 24 June 2024
Accepted for publication 10 July 2024
Published 28 August 2024

Abstract

The *natural units of measure* lauded by Max Planck more than 100 years ago are underutilized today. Many physical constants, including the Planck constant, the gravitational constant, the speed of light, vacuum permittivity, and vacuum permeability consist of natural units in their unit dimensions. The natural units are present in all formulas containing these constants. The defining characteristic of the natural units is an alignment of unit values at *the Planck scale*. This alignment gives a computational basis of proportionality from which the correlated properties and dynamics of elementary particles, including wavelength, period, mass, momentum, and energy, manifest in equal or inversely proportional ratios of the Planck scale. These correlations explain many of the defining equations of quantum mechanics, classical gravity, and electromagnetism.

Supplementary material for this article is available online

Keywords: physical constants, dimensional analysis, Planck units, natural units, symmetry, electromagnetic unit systems, universal constants

1. Introduction

The present-day International System of Units (SI) defines units of length, mass, and time that were largely decided centuries ago. Base units of meters, seconds, and kilograms correspond with everyday physical phenomena, but the magnitudes of these units were selected arbitrarily

Table 1. The Planck Units.

Natural unit	Symbol	Std formula	Dimensional formula	Value
Planck length	l_P	$\sqrt{\frac{\hbar G}{c^3}}$	$\sqrt{\frac{l_P^2\,\cancel{m_P}}{\cancel{t_P}}\frac{\cancel{l_P^3}}{\cancel{m_P}\,\cancel{t_P^2}}\frac{\cancel{t_P^3}}{\cancel{l_P^3}}} = l_P$	$1.616\,255 \times 10^{-35}$ m
Planck mass	m_P	$\sqrt{\frac{\hbar c}{G}}$	$\sqrt{\frac{\cancel{l_P^2}\,m_P}{\cancel{t_P}}\frac{\cancel{l_P}}{\cancel{t_P}}\frac{m_P\,\cancel{t_P^2}}{\cancel{l_P^3}}} = m_P$	$2.176\,434 \times 10^{-8}$ kg
Planck time	t_P	$\sqrt{\frac{\hbar G}{c^5}}$	$\sqrt{\frac{\cancel{l_P^2}\,\cancel{m_P}}{\cancel{t_P}}\frac{\cancel{l_P^3}}{\cancel{m_P}\,\cancel{t_P^2}}\frac{t_P^2\,\cancel{t_P^3}}{\cancel{l_P^5}}} = t_P$	$5.391\,247 \times 10^{-44}$ s

and not from fundamental constants of nature. While the choice of units does not affect the underlying physics, natural units have certain advantages over other unit systems. Max Planck lauded the uniqueness of these units when he introduced formulas for calculating their values from the universal constants [1]

> ...it is possible to set up units for length, mass, time and temperature, which are independent of special bodies or substances, necessarily retaining their meaning for all times and for all civilizations, including extraterrestrial and non-human ones, which can be called 'natural units of measure.'

Planck combined the values and dimensions of c, $\hbar$, and G into ratios that isolate each unit dimension. His approach *presumes* that the universal constants contain natural units in their compound unit dimensions [2]. This is why the universal constants can be expressed in natural units as [3–7]

$$c = \frac{l_P}{t_P} = 2.997\,924 \times 10^8 \text{ m s}^{-1} \tag{1}$$

$$\hbar = \frac{m_P\,l_P^2}{t_P} = m_P\,l_P\,c = 1.054\,572 \times 10^{-34} \text{ kg m}^2\text{ s}^{-1} \tag{2}$$

$$G = \frac{l_P^3}{m_P\,t_P^2} = \frac{l_P}{m_P}c^2 = 6.674\,30 \times 10^{-11} \text{ m}^3\text{ kg}^{-1}\text{ s}^{-2}. \tag{3}$$

The dimensional structure of the universal constants in equations (1)–(3) allowed Planck to determine a natural unit value for each unit dimension. These units bear his name today (see table 1).

Because the natural units are calculated from ratios of three universal constants, their precision is limited by the precision of G which has a relative standard uncertainty of 2.2×10^{-5} [8, 9]. However, the intensive ratios of certain unit pairs, such as Planck length to Planck time, have been measured with much greater precision [10, 11]. Furthermore, the advantage of working in natural units is the correlations they reveal and not their utility in everyday experiments.

1.1. The natural unit scale

The characteristic feature of the natural units is an alignment of unit values at *the Planck scale*. Each unit dimension has a value of 1 and asymptotically approaches a limit of zero or infinity. Unit dimensions of length and time have minimum values of 1 and maximum values approaching infinity. With respect to discrete quanta, unit dimensions of mass and charge

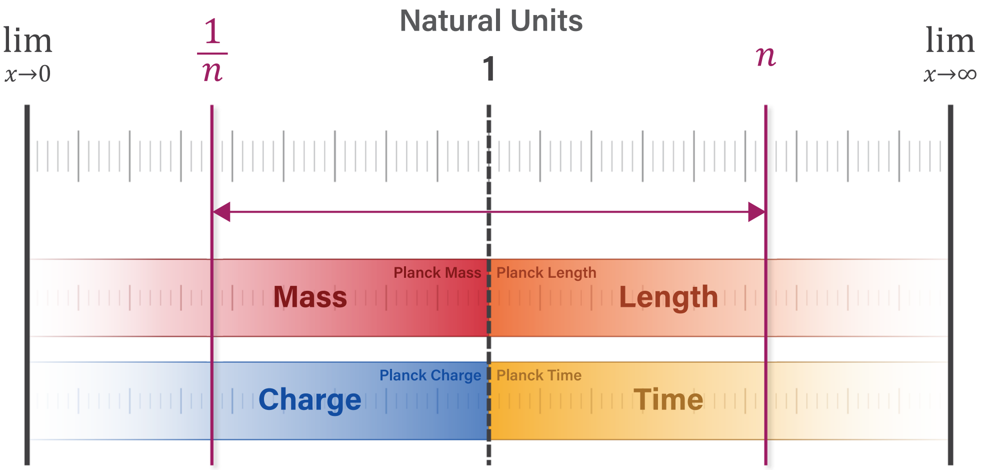

Figure 1. Unit dimensions have a defined limit at one end of their scale and asymptotically approach limits of zero or infinity at the other end. Mass and charge are inversely proportional to length and time.

have maximum values of 1 and minimum values approaching zero [12–14]. Figure 1 illustrates this natural structure.

The Planck scale acts as a basis of proportionality for quantifying physical phenomena in natural units. For certain correlated pairs, a quantity of length or time n on the right side of the scale corresponds with an inversely proportional quantity $1/n$ on the left side.

An example of this natural structure is the relationship between the electron mass and Compton wavelength. Restating these quantities in natural units reveals that the two electron properties are inversely proportional. The Compton wavelength in natural length units is the quotient of wavelength and Planck length, both given in meters

$$\frac{\lambda\!\!\!^{-}_C}{l_P} = \frac{3.861\ 593 \times 10^{-13}\ \text{m}}{1.616\ 255 \times 10^{-35}\ \text{m}} = 2.389\ 222 \times 10^{22}. \tag{4}$$

We can similarly calculate the electron mass in natural units as the quotient of mass and Planck mass in kilograms

$$\frac{m_0}{m_P} = \frac{9.109\ 384 \times 10^{-31}\ \text{kg}}{2.176\ 434 \times 10^{-08}\ \text{kg}} = \frac{1}{2.389\ 222 \times 10^{22}}. \tag{5}$$

The inversely proportional relationship between wavelength and mass yields a constant product of Compton wavelength and rest mass in each of the charged leptons, and which is equal to the product of Planck length and Planck mass

$$\lambda\!\!\!^{-}_C m_0 = l_P m_P = \frac{\hbar}{c} = 3.517\ 673 \times 10^{-43}\ \text{kgm}. \tag{6}$$

The natural structure shown in figures 1 and 2 produces proportionally meaningful insights by setting the Planck units—and certain collections of Planck units such as the universal constants—equal to 1. However, the occasional practice of setting universal constants equal to 1 gives an opaque view of this natural structure compared to the clarity obtained by stating the universal constants in natural units.

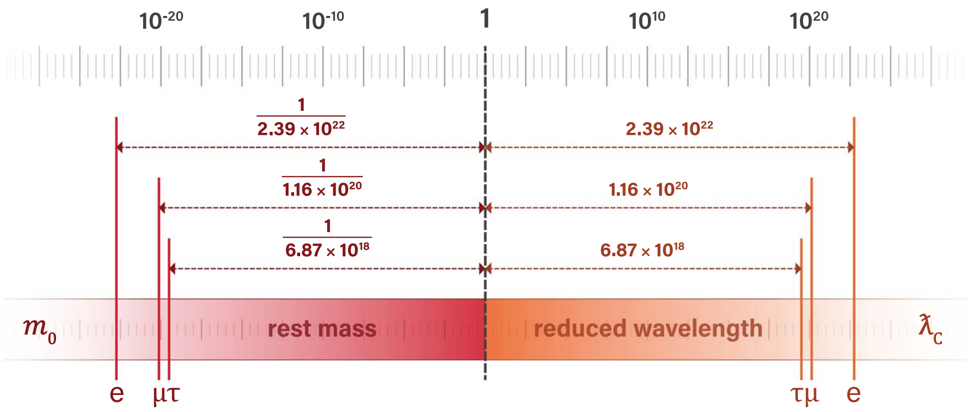

Figure 2. Natural units show that rest mass is inversely proportional to Compton wavelength.

The modern physics curriculum emphasizes the traditional approach to dimensional analysis which prefers the compound values and dimensions of universal constants over the unidimensional Planck units. The accompanying philosophy says that it does not matter how you write the equations as long as the computational results are the same. But students should have freedom to challenge philosophical assumptions and to explore the physical meaning of constants that offer a more granular view of the equations.

Furthermore, the Planck units are selectively employed within certain theoretical fields including general relativity and quantum gravity. Students should learn the natural unit forms of the equations if only to serve as a pedagogical tool.

The natural unit scale reveals meaningful correlations such as the inversely proportional relation between wavelength and mass shown in equations (4)–(6). But the natural scale is seldom identified because of the incongruence between SI unit values and the natural units. Figure 3 illustrates this misalignment. In the figure, limits of mass and charge pertain to discrete quanta and not to large-scale systems.

In figure 3, unit dimensions of meter, second, kilogram, and coulomb coincide arbitrarily at a value of one, obfuscating correlations that are present in natural phenomena. Restating the constants and equations in natural units illustrates how the universal constants re-quantify an arbitrary unit system into natural units where each dimension coincides at the Planck scale.

2. The laws of physics in natural units

The universal constants appear in many equations describing elementary structures of nature. Their contributions to the formulas include a single value and multiple unit dimensions—two, four, and six unit dimensions for c, $\hbar$, and G respectively. An impressive quality of the universal constants is that they consistently satisfy the following two requirements:

1. Formulas with universal constants produce the correct unit dimensions of the physical phenomenon you are solving for (i.e. momentum, energy, force, etc).
2. The formulas produce the correct magnitude of the phenomenon from the given inputs.

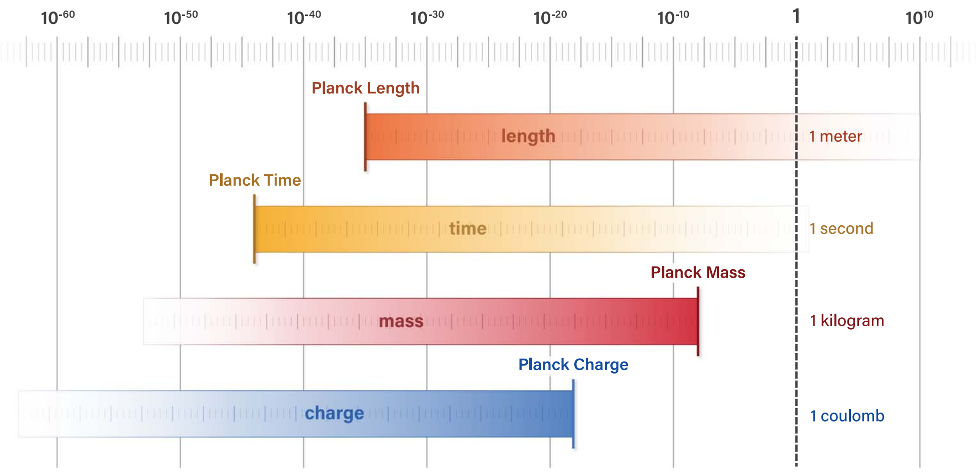

Figure 3. SI units of meter, kilogram, second, and coulomb are incongruent with the natural unit scale. Natural units of length, mass, time, and electric charge coincide at *the Planck scale* where each unit has a value of 1.

The Planck constant appears regularly in formulas describing the quantized behavior of matter and radiation, correctly predicting the mechanical properties and dynamics of elementary particles. Evaluating the formulas in each unit dimension reveals the hidden role of natural units in the mathematical transformations. For example, the Compton wavelength is determined using the Planck constant and inputs of mass and c

$$\lambda_{\mathrm{C}} = \frac{h}{m_0 c}. \tag{7}$$

The first requirement of the function is to reduce seven unit dimensions to one

$$\mathrm{L} = \frac{\mathrm{ML}^2}{\mathrm{T}} \times \frac{\mathrm{T}}{\mathrm{ML}}.$$

Three unit dimensions in the numerator and three in the denominator reduce the dimensionality to

$$\mathrm{L} = \left(\frac{\mathrm{M}}{\mathrm{M}}\right) \times \left(\frac{\mathrm{L}}{\mathrm{L}}\right) \times \left(\frac{\mathrm{T}}{\mathrm{T}}\right) \times \ \mathrm{L}.$$

The second requirement of the function is to obtain the correct magnitude of wavelength from the given inputs. Restating equation (7) in natural units accounts for the mathematical transformation in each unit dimension and shows how formula inputs produce the corresponding magnitude of Compton wavelength

$$\bar{\lambda}_{\mathrm{C}} = \frac{\hbar}{m_0 c} = \frac{l_{\mathrm{P}} m_{\mathrm{P}} c}{m_0 c} = \left(\frac{m_{\mathrm{P}}}{m_0}\right)\left(\frac{\cancel{c}}{\cancel{c}}\right) l_{\mathrm{P}}. \tag{8}$$

The physical significance of the dimensionless ratios in equation (8) becomes clearer as we evaluate the laws of physics in natural units. For now, we can summarize the role of natural units in quantifying the Compton wavelength in two parts:

1. The Planck length serves as a minimum limit or computational basis from which observable quantities of Compton wavelength are calculated.

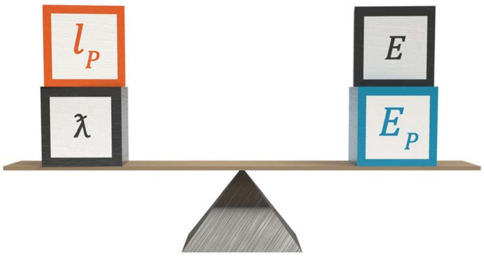

Figure 4. The physical properties and dynamics of elementary particles are correlated.

2. A matter particle's inverse-reduced Compton wavelength is the same ratio of the Planck scale as its mass.

Rearranging equation (8) emphasizes the second point

$$\frac{l_\mathrm{P}}{ƛ_\mathrm{C}} = \frac{m_0}{m_\mathrm{P}}. \tag{9}$$

The structure of equations (8) and (9) explains why the formula produces a proportionally significant result. It allows the discovery of one attribute or dynamic from a known quantity of a correlated attribute or dynamic. Another example of this natural structure is the formula for photon energy

$$E = \frac{hc}{\lambda} = \frac{l_\mathrm{P} m_\mathrm{P} c^2}{ƛ} = \left(\frac{l_\mathrm{P}}{ƛ}\right) E_\mathrm{P}. \tag{10}$$

Arranging the natural unit formula as an equality explains why the formula works

$$\frac{l_\mathrm{P}}{ƛ} = \frac{E}{E_\mathrm{P}}. \tag{11}$$

Natural unit formulas demonstrate that the reason we can calculate a photon's energy from its wavelength is because the ratio of Planck length to wavelength is equal to the ratio of energy to Planck energy (see figure 4).

These equalities suggest that the function of universal constants is to inject natural units into the formulas to exploit correlations between the physical properties and dynamics of elementary particles and systems. It is not simply that photon momentum and energy are proportional to the Planck constant. Rather, the momentum and energy of a photon are proportional to the Planck momentum and Planck energy. This pattern consistently explains the correlations, constants, and equations.

Tables 2, 3, and 5 present natural unit formulas for several defining equations of quantum mechanics, classical gravity, and electromagnetism. Each natural unit formula is constructed by restating the universal constants with natural units according to equations (1)–(3), (27), and (40). The column 'conserved quantity' shows the amount in each formula that remains constant as the individual terms change.

An important advantage of the natural unit formulas in tables 2, 3, and 5 is that they characterize the physical properties of natural phenomena in unit dimensions of length, mass, and time. For example, the Planck energy can be characterized by a Planck-length photon moving at the speed of light.

Table 2. Quantum mechanics in natural units.

Physical phenomenon	Standard formula	Natural unit formula	Conserved quantity
Photon wavelength	$\lambda = \frac{c}{f}$	$\boldsymbol{\lambda} = \left(\frac{T}{t_P}\right) l_P$	$\frac{l_P}{t_P} = \frac{\lambda}{T}$
Photon period	$T = \frac{\lambda}{c}$	$T = \left(\frac{\lambda}{l_P}\right) t_P$	$\frac{t_P}{l_P} = \frac{T}{\lambda}$
Momentum	$p = \frac{h}{\lambda}$	$p = \left(\frac{l_P}{\lambda\!\!\!^-}\right) m_P c$	$m_P c\, l_P = p\lambda\!\!\!^-$
Photon energy	$E = \frac{hc}{\lambda}$	$E = \left(\frac{l_P}{\lambda\!\!\!^-}\right) E_P$	$E_P l_P = E\lambda\!\!\!^-$
Compton wavelength	$\lambda_C = \frac{h}{m_0 c}$	$\lambda\!\!\!^-_C = \left(\frac{m_P}{m_0}\right)\left(\frac{c}{c}\right) l_P$	$l_P m_P = \lambda\!\!\!^-_C m_0$
de Broglie wavelength	$\lambda = \frac{h}{m_0 v}$	$\lambda\!\!\!^- = \left(\frac{m_P}{m_0}\right)\left(\frac{c}{v}\right) l_P$	$l_P m_P c = \lambda\!\!\!^- m_0 v$

l_P = Planck length, m_P = Planck mass, t_P = Planck time, E_P = Planck energy.

Table 3. Gravity in natural units.

Physical phenomenon	Standard formula	Natural unit formula	Conserved quantity
Schwarzschild radius	$R_S = \frac{2GM}{c^2}$	$R_S = 2\left(\frac{M}{m_P}\right) l_P$	$\frac{l_P}{m_P} = \frac{R_S}{2M}$
Escape velocity	$v_e = -\sqrt{\frac{2GM}{r}}$	$v_e = -\sqrt{2\left[\frac{l_P}{r}\frac{M}{m_P}\right]}\ c$	$\frac{c^2\, l_P}{m_P} = \frac{v_e^2\, r}{2M}$
Gravitational acceleration	$g = -\frac{GM}{r^2}$	$g = -\left[\frac{l_P}{r}\frac{M}{m_P}\right]\left(\frac{l_P}{r}\right) a_P$	$\frac{a_P\, l_P^2}{m_P} = \frac{gr^2}{M}$
Gravitational potential energy	$U = -\frac{GMm}{r}$	$U = -\left[\frac{l_P}{r}\frac{M}{m_P}\right]\left(\frac{m}{m_P}\right) E_P$	$\frac{E_P\, l_P}{m_P^2} = \frac{Ur}{Mm}$
Gravitational force	$F = \frac{GMm}{r^2}$	$F = \left[\frac{l_P}{r}\frac{M}{m_P}\right]\left[\frac{l_P}{r}\frac{m}{m_P}\right] F_P$	$\frac{F_P\, l_P^2}{m_P^2} = \frac{Fr^2}{Mm}$

l_P = Planck length, m_P = Planck mass, E_P = Planck energy, F_P = Planck force, a_P = Planck acceleration.

2.1. Independent units of measure

Natural unit quantities lose their dimensions when calculated in arbitrary unit systems, as shown in equations (4) and (5). To retain the dimensionality of these quantities, it is the practice throughout this study to label them with dimensional notation L, M, T, and Q for natural units of length, mass, time, and electric charge.

Natural unit quantities of length and time appear regularly in tables 2, 3, and 5. For example, the relationship between a photon's wavelength, oscillation period, and frequency is given by

$$\lambda = \frac{c}{f} = cT. \tag{12}$$

Restating c in natural units produces natural unit equations for wavelength and period

$$\lambda = \left(\frac{T}{t_P}\right) l_P \tag{13}$$

and

$$T = \left(\frac{\lambda}{l_P}\right)t_P. \tag{14}$$

Equation (13) finds the photon's wavelength by determining the equivalent number of natural time units and multiplying this by a single length unit. Equation (14) similarly finds the photon period from the number of natural length units. For a photon in the vacuum, the relationship between wavelength and period can be summarized as

$$\frac{\lambda}{l_P} = \frac{T}{t_P}. \tag{15}$$

While the speed of light is an intensive ratio that can be stated equivalently at different length and time scales (including meters per second), the natural units produce unique quantities of photon wavelength and period *that remain constant in any unit system*. This is because the natural unit values are calculated from SI unit ratios. Changing the SI unit scale would change the nominal values, but the ratios would remain the same.

To see that this is the case, consider the properties of a specific photon. The unperturbed ground-state hyperfine transition frequency of cesium-133 is a specific amount of radiation used in the SI definitions of the second, meter, and kilogram [15]. The photon has an oscillation period measured in seconds of

$$T = \frac{1}{f} = \frac{1}{9\ 192\ 631\ 770\ \mathrm{Hz}} = 1.087\ 827\ 757 \times 10^{-10}\ \mathrm{s} \tag{16}$$

which can be stated in natural time units as

$$\frac{T}{t_P} = \frac{1.087\ 827 \times 10^{-10}\ \mathrm{s}}{5.391\ 247 \times 10^{-44}\ \mathrm{s}} = 2.017\ 767 \times 10^{33}\ \mathrm{T}. \tag{17}$$

The photon also has a wavelength in meters

$$\frac{c}{\Delta\nu_{\mathrm{Cs}}} = \frac{299\ 792\ 458\ \mathrm{m\ s^{-1}}}{9\ 192\ 631\ 770\ \mathrm{Hz}} = 0.032\ 612\ 256\ \mathrm{m} \tag{18}$$

which can be stated in natural length units as

$$\frac{\lambda}{l_P} = \frac{0.032\ 612\ 2\ \mathrm{m}}{1.616\ 255 \times 10^{-35}\ \mathrm{m}} = 2.017\ 767 \times 10^{33}\ \mathrm{L}. \tag{19}$$

The natural unit values of photon wavelength and period are equivalent, and will remain equivalent in any unit system. If we were to redefine the meter as precisely one-half of its present-day value, we would measure the photon wavelength as twice its current nominal value, or 0.065 m'. But we would also measure the Planck length as twice its current nominal value, or 3.232×10^{-35} m'. So the natural unit value of the hyperfine transition radiation would remain unchanged

$$\lambda = \frac{0.065\ \mathrm{m'}}{3.232 \times 10^{-35}\ \mathrm{m'}} = 2.017\ 767 \times 10^{33}\ \mathrm{L}. \tag{20}$$

The natural length and time units in equations (17) and (19) are the *natural units of measure* Planck referred to which *retain their meaning for all times and all civilizations including extraterrestrial and non-human ones.* An alien civilization is unlikely to quantify length and time in units equivalent to meters and seconds, but any unit system they devise will produce the same natural unit value of wavelength and oscillation period for the hyperfine transition frequency of cesium-133.

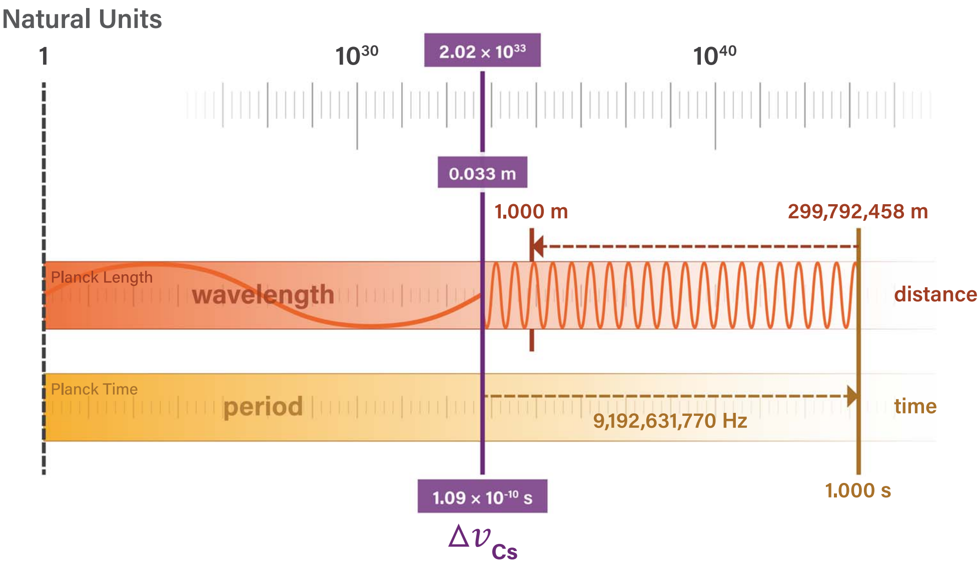

Figure 5. Natural unit measures of photon wavelength and period are equivalent. These values are independent of unit system.

Figure 5 illustrates the alignment of length and time units at the Planck scale, and the relationship between SI and natural units.

In the figure, the Planck scale is represented by a vertical line labeled "1," and the speed of light is a 1:1 ratio of length to time that can be represented anywhere by a vertical line.

The figure illustrates the relationship between the hyperfine transition radiation and the **second** which it defines. One unit of time is a precise integer multiple of the photon oscillation.

Given the defined value of one second, the **meter** is defined as a fraction of the distance that light travels in a vacuum over a period of one second. One meter is precisely 1/299 792 458 times this distance.

The natural unit value 2.018×10^{33} is an invariant measure of the photon's wavelength and period. It is the same regardless of how we define units of length and time.

The physical meaning of the universal constants is not found in the values acquired from a particular unit system, but from correlations of natural phenomena that remain constant in any unit system. A better understanding of the equations follows from a better understanding of the correlations driving the mathematical machinery.

3. Gravity in natural units

An investigation into the natural unit structure of the gravitational constant and the equations of classical gravity reveal similar correlations between the physical attributes of massive bodies and the gravitational field. In these formulas, the Planck scale plays a pivotal role in quantifying gravity.

The classical two-body gravity formulas stated in natural units show that G can be characterized in two parts:

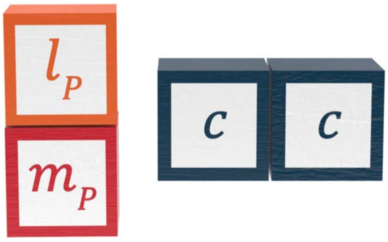

Figure 6. The gravitational constant is the ratio of Planck length to Planck mass and the speed of light squared.

1. The ratio $l_\mathrm{P}/m_\mathrm{P}$ gives a Planck scale basis of proportionality for quantifying the gravitational field generated by mass M and distance r.
2. The quantity c^2 included in the gravitational constant is the computational basis for stating gravitational field potentials in terms of the momentum and velocity of a test particle or second body (see figure 6).

Beneath the compound unit dimensions of G, the classical formulas compare a body's mass and radius with the Planck scale basis of Planck mass and Planck length. Two signature inputs into the formulas—mass in the numerator and radius in the denominator—produce a dimensionless ratio of *radial density*

$$\frac{l_\mathrm{P}}{r}\frac{M}{m_\mathrm{P}} \tag{21}$$

which is the correct ratio for quantifying gravitational field strength on a scale of 0 to 1, where 1 represents the Planck scale. Multiplying this dimensionless quantity by c^2 produces the correct momentum and velocity of a second body, as demonstrated by the natural unit equations in table 3.

Similar to the quantum mechanical formulas in table 2, classical gravity equations are characterized by equalities between the physical attributes of the system and the phenomena they induce.

A closer look at the equations in table 3 reveals an intimate relationship between the radial density quantified by equation (21) and the Schwarzschild radius, suggesting that the Planck scale quotient of length and mass constitutes a lower boundary on a massive body's radius. The Schwarzschild radius formula in natural units

$$\begin{aligned} R_\mathrm{S} &= 2G\frac{M}{c^2} \\ &= 2\left[\frac{l_\mathrm{P}}{m_\mathrm{P}}\cancel{c^2}\right]\frac{M}{\cancel{c^2}} \\ &= 2\left(\frac{M}{m_\mathrm{P}}\right)l_\mathrm{P}. \end{aligned} \tag{22}$$

gives a definition of the Schwarzschild radius as one-half the Planck scale limit

$$\frac{l_\mathrm{P}}{R_\mathrm{S}}\frac{M}{m_\mathrm{P}} = \frac{1}{2}. \tag{23}$$

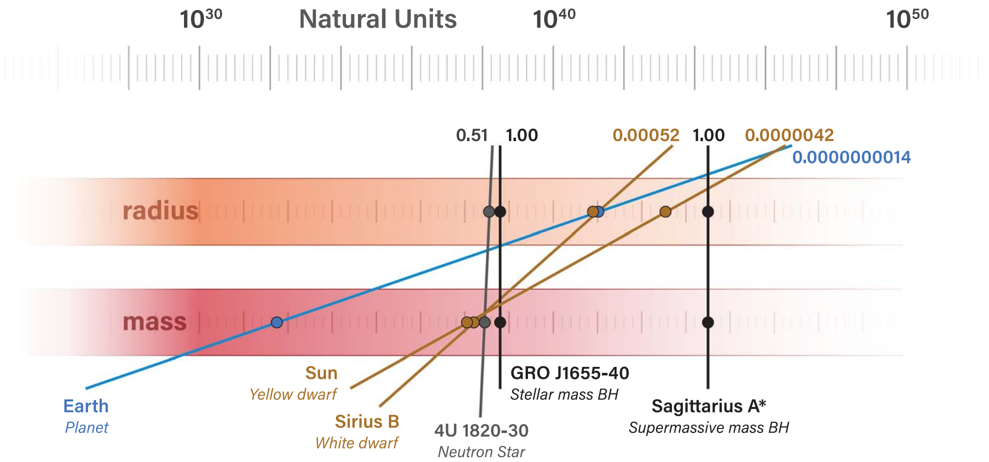

Figure 7. The mass and radius of several massive bodies are plotted in natural units (not drawn to scale).

Equation (23) reflects a known property of black holes. Because the relationship between the Schwarzschild radius and mass is constant, the volumetric density of a black hole decreases as its radius increases.

Table 4 compares the radial densities of several massive bodies. Radial density ratios for black holes are calculated using the Schwarzschild radius formula, while the ratios of other bodies are calculated using recent values of mass and radius [16–21].

A simple way to understand the ratio in the last column is to consider that each body's mass is constant. For the given mass of each body, the radial density ratio is equal to the ratio of its Schwarzschild radius to the surface radius. For example, multiplying the Earth's radius by this ratio produces its Schwarzschild radius

$$(637\ 1000\text{m}) \times (1.392\ 29 \times 10^{-9}) = 0.008\ 870\text{m}. \tag{24}$$

The dimensionless ratio of radial density accounts for both the size and density of a massive body. While the Earth and Sun have similar volumetric densities, the Sun's radial density is about four orders of magnitude larger. This is because the Sun's larger radius gives it greater volume accommodating more mass per unit of radial distance.

Figure 7 illustrates the relationship between natural units of radius and mass for the massive bodies in table 4.

The mass and radius of each body are plotted in natural units and normalized to a Schwarzschild radius value of one (rather than one-half). The slope of each line represents the ratio between the body's mass and radius in natural units. Black holes are characterized by a vertical line drawn between their mass and radius, and the slope decreases as the radius of the body increases with respect to mass. The slope of each line quantifies the massive body's gravitational field.

More massive bodies appear further to the right in the diagram, and different classes of massive bodies appear in different colors: black holes are shown in black, a neutron star in gray, dwarf stars in yellow, and planet Earth in blue.

Perhaps the most interesting insights gained from the natural unit formulas are descriptions of how the gravitational field is related to the mechanical properties induced on a second body. One example is gravitational potential energy. Setting the energy potential equal to the kinetic energy of a second body gives

Table 4. Massive bodies.

Massive body	Type	$\frac{l_P}{R}$ L^{-1}	$\frac{M}{m_P}$ M	$2\frac{l_P}{R}\frac{M}{m_P}$ L^{-1} M
Sagittarius A*	Supermassive black hole	1.317×10^{-45}	3.795×10^{44}	1.000
GRO J1655-40	Steller mass black hole	1.031×10^{-39}	4.851×10^{38}	1.000
4U 1820-30	Neutron star	1.776×10^{-39}	1.444×10^{38}	0.513
Sirius B	White dwarf star	2.764×10^{-42}	9.447×10^{37}	5.222×10^{-4}
Sun	Yellow dwarf star	2.322×10^{-44}	9.136×10^{37}	4.242×10^{-6}
Earth	Planet	2.537×10^{-42}	2.744×10^{32}	1.392×10^{-9}

l_P = Planck length, m_P = Planck mass.

$$\left[\frac{l_P}{r}\frac{M}{m_P}\right]mc^2 = \frac{1}{2}mv^2, \tag{25}$$

where v is equal to the escape velocity. At the Schwarzschild radius, potential energy is equal to $1/_2\ mc^2$ and decreases with distance. Simplifying equation (25) provides an insightful relationship between the radial density of a first body and the momentum and velocity of a second body

$$2\left[\frac{l_P}{r}\frac{M}{m_P}\right] = \frac{v^2}{c^2}. \tag{26}$$

4. Electromagnetism in natural units

The same structure underlying the mathematics of quantum mechanics and classical gravity is also found in the constants and equations of electromagnetism. These equations show that the Planck charge aligns with Planck units of length, mass, and time at the Planck scale.

4.1. Electrostatic force

Coulomb's law gives the electrostatic force acting between a pair of charged particles. The Coulomb constant transforms inputs of charge and distance into the resulting electrostatic force. Like the universal constants, Coulomb's constant contains natural units in its unit dimensions

$$\text{M L}^3\ \text{T}^{-2}\ \text{Q}^{-2}.$$

Inserting natural units of length, mass, time, and charge into the unit dimensions of Coulomb's constant reveals this natural unit structure

$$k_e = \frac{m_P\ l_P^3}{t_P^2\ q_P^2} = F_P\left(\frac{l_P}{q_P}\right)^2 = 8.988 \times 10^9\ \text{kgm}^3\ \text{s}^{-2}\ \text{C}^{-2}. \tag{27}$$

Evaluating the constant's role in the equations shows that the mathematics of electromagnetism are also based on correlations between the attributes and dynamics of charged

Table 5. Electromagnetism in natural units.

Phenomenon	Standard formula	Natural unit formula	Conserved quantity
Electrostatic force	$F = k_e \frac{q_1 q_2}{r^2}$	$F = \left(\frac{l_P}{r}\right)\left(\frac{l_P}{r}\right)\left(\frac{q_1}{q_P}\right)\left(\frac{q_2}{q_P}\right)F_P$	$\frac{F_P\, l_P^2}{q_P^2} = \frac{Fr^2}{q_1 q_2}$
Electric potential energy	$U_E = k_e \frac{q_1 q_2}{r}$	$U_E = \left(\frac{l_P}{r}\right)\left(\frac{q_1}{q_P}\right)\left(\frac{q_2}{q_P}\right)E_P$	$\frac{E_P\, l_P}{q_P^2} = \frac{U_E\, r}{q_1 q_2}$
Electric potential	$V_E = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$	$V_E = \left(\frac{l_P}{r}\right)\left(\frac{q}{q_P}\right)V_P$	$\frac{V_P\, l_P}{q_P} = \frac{V_E\, r}{q}$
Magnetic force	$\frac{F}{l} = \frac{\mu_0}{2\pi}\frac{I^2}{r}$	$F = 2\left(\frac{l}{l_P}\right)\left(\frac{I}{I_P}\right)F_P$	$\frac{F_P}{I_P^2} = \frac{F}{2I^2}$

l_P = Planck length, m_P = Planck mass, q_P = Planck charge, E_P = Planck energy, $I_P = q_P\ /\ t_P$, $V_P = E_P/q_P$.

particles and the Planck scale. For example, the standard formula for finding the electrostatic force

$$F = k_e \frac{q_1 q_2}{r^2} \tag{28}$$

can be expressed in natural units by combining equations (27) and (28)

$$F = F_P\left(\frac{l_P}{r}\right)^2\left(\frac{q_1}{q_P}\right)\left(\frac{q_2}{q_P}\right). \tag{29}$$

The corresponding equality is

$$\frac{F}{F_P} = \left(\frac{l_P}{r}\right)^2\left(\frac{q_1}{q_P}\right)\left(\frac{q_2}{q_P}\right). \tag{30}$$

Table 5 gives the natural unit equalities and formulas for several defining equations of electromagnetism.

The electrostatic force can be understood by considering the force between a pair of elementary charges

$$F = k_e \frac{e^2}{r^2} = F_P\left(\frac{l_P}{r}\right)^2\left(\frac{e}{q_P}\right)^2. \tag{31}$$

We can reduce equation (31) further using the relationship between the elementary charge and Planck charge

$$\frac{e}{q_P} = \frac{1.602\ 176\ 634 \times 10^{-19}\ \text{C}}{1.875\ 546\ 038 \times 10^{-18}\ \text{C}} = 0.085\ 424\ 543\ 132 = \sqrt{\alpha}. \tag{32}$$

The electrostatic force between a pair of elementary charges can therefore be reduced to

$$F = \alpha\left(\frac{l_P}{r}\right)^2 F_P. \tag{33}$$

In natural units, the electrostatic force is quantified by a dimensionless determinant on a scale of 0 to 1, where 1 is the Planck force

$$F = \frac{\alpha}{\mathrm{L}^2} F_P. \tag{34}$$

The natural unit formula indicates that two factors determine the strength of electrostatic force between a pair of electric charges. Given the basis of Planck force, the fine-structure

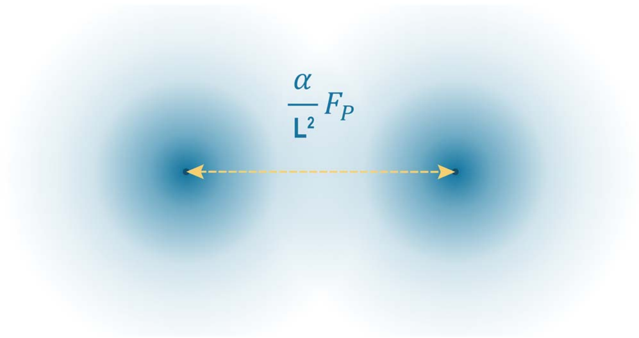

Figure 8. The electrostatic force between a pair of charged particles is proportional to the Planck force. The dimensionless fine-structure constant and distance squared in natural units determine the strength of the electrostatic force.

constant provides a fixed reduction in force between charges, and the inverse square of the distance provides a variable reduction in force (see figure 8).

4.2. Vacuum permittivity

The relationship between Coulomb's constant and the vacuum electric permittivity gives the natural unit definition of the electric constant as

$$\epsilon_0 = \frac{1}{4\pi F_\mathrm{P}}\left(\frac{q_\mathrm{P}}{l_\mathrm{P}}\right)^2 = 8.854\ 19 \times 10^{-12}\ \mathrm{C}^2\ \mathrm{s}^2\ \mathrm{kg}^{-1}\ \mathrm{m}^{-3}. \tag{35}$$

While the vacuum permittivity is conveniently structured for calculating force, it can also be used to calculate energy such as the potential energy between a pair of point charges

$$\begin{aligned} U_\mathrm{E} &= \frac{e^2}{4\pi\epsilon_0\, r} \\ &= \left[4\pi F_\mathrm{P}\,\frac{l_\mathrm{P}^2}{q_\mathrm{P}^2}\right]\frac{e^2}{4\pi r}. \end{aligned} \tag{36}$$

The equation restructures the Planck force into Planck energy and produces the fine-structure constant according to equation (32)

$$\begin{aligned} U_\mathrm{E} &= \frac{\cancel{4\pi}}{\cancel{4\pi}}\, F_\mathrm{P}\, l_\mathrm{P}\left(\frac{l_\mathrm{P}}{r}\right)\left(\frac{e^2}{q_\mathrm{P}^2}\right) \\ &= \alpha\left(\frac{l_\mathrm{P}}{r}\right)E_\mathrm{P}. \end{aligned} \tag{37}$$

The vacuum permittivity is a conserved quantity that remains constant as experimental inputs change. Re-arranging the natural unit equality in table 5 gives this conserved relationship

This is because the constant is the ratio of magnetic force and current squared, which are inversely proportional. From the natural unit equality and formula in table 5, we can state this relationship as

$$4\pi\epsilon_0 = \frac{q_P^2}{F_P\, l_P^2} = \frac{q^2}{Fr^2} = 1.11 \times 10^{-10}\ \mathrm{C^2\ N^{-1}\ \ m^{-2}} \tag{38}$$

This means that the ratio of charge squared to force times distance squared is the same at any scale, including the Planck scale and any distance between a pair of electric charges.

4.3. Vacuum permeability

The vacuum permeability is related to the vacuum permittivity by

$$\mu_0 = \frac{1}{\epsilon_0\, c^2} \tag{39}$$

which gives a natural unit equation for the magnetic constant

$$\mu_0 = 4\pi\ F_P\left(\frac{t_P}{q_P}\right)^2 = 1.256\ 637 \times 10^{-6}\ \mathrm{N\ A^{-2}}. \tag{40}$$

Similar to the electric constant, the magnetic constant has a basis of Planck force. However, units of Planck charge and Planck time make it more suitable for equations with inputs of electric current.

This difference in structure—the Planck length in the electric constant (equation (35)) and the Planck time in the magnetic constant (equation (40)), produces a value of c^2 from the product of the two.

$$\mu_0\ \epsilon_0 = \left[\cancel{4\pi F_P}\ \frac{t_P^2}{\cancel{q_P^2}}\right]\left[\frac{1}{\cancel{4\pi F_P}}\ \frac{\cancel{q_P^2}}{l_P^2}\right] = \frac{1}{c^2} \tag{41}$$

The vacuum permeability is a conserved quantity that remains constant as experimental inputs change. This is because the constant is the ratio of magnetic force to current squared, which are inversely proportional to each other. The natural unit equality in table 5 gives the relationship

$$\frac{\mu_0}{4\pi} = \frac{F_P}{I_P^2} = \frac{F}{2I^2} = 1 \times 10^{-7}\ \mathrm{NA^{-2}}. \tag{42}$$

This means that the ratio of force to current squared is the same at all scales, including the Planck scale and at scales used to define the ampere and coulomb. Table 6 lists the natural unit form of several electromagnetic constants in dimensions of length, mass, time, and charge. Each natural unit value of a constant is equivalent to the standard value within the uncertainty of the Planck units.

4.4. Electromagnetic unit systems

There is a long and somewhat confusing history surrounding electromagnetic unit systems including variations of the centimeter-gram-second (CGS) system of units. Re-quantifying MKS units is straightforward in any system, but early attempts at defining a unit of electric charge created confusion about the relationship between electrical properties of matter and mechanical phenomena in dimensions of length, mass, and time.

Many heuristic attempts were made at defining a unit of charge before it became clear that electric charge has an indivisible natural unit [22]. Different unit systems represent various attempts at deriving that unit from mechanical unit dimensions [23, 24].

Two important characteristics of each system include

Table 6. Electromagnetic constants.

Constant	Sym	Value	Std form	Natural units	Equivalent
Planck charge	q_P	1.876×10^{-18} C	$\sqrt{4\pi\epsilon_0 \hbar c}$	—	$\frac{e}{\sqrt{\alpha}}$
Elementary charge	e, q	1.602×10^{-19} C	$\sqrt{\frac{4\pi\hbar\alpha}{\mu_0 c}}$	$q_\mathrm{P}\sqrt{\alpha}$	—
Vacuum permeability	μ_0	1.257×10^{-6} N A^{-2}	$\frac{4\pi\alpha\hbar}{e^2 c}$	$4\pi F_\mathrm{P}\left(\frac{t_\mathrm{P}}{q_\mathrm{P}}\right)^2$	$4\pi\frac{l_\mathrm{P} m_\mathrm{P}}{q_\mathrm{P}^2}$
Vacuum permittivity	ϵ_0	8.854×10^{-12} F m^{-1}	$\frac{1}{\mu_0 c^2}$	$\frac{1}{4\pi F_\mathrm{P}}\left(\frac{q_\mathrm{P}}{l_\mathrm{P}}\right)^2$	$\frac{1}{4\pi F_\mathrm{P} c^2}\left(\frac{q_\mathrm{P}}{t_\mathrm{P}}\right)^2$
Coulomb constant	k_e	8.988×10^9 N m^2 C^{-2}	$\frac{1}{4\pi\epsilon_0}$	$F_\mathrm{P}\left(\frac{l_\mathrm{P}}{q_\mathrm{P}}\right)^2$	$F_\mathrm{P} c^2\left(\frac{t_\mathrm{P}}{q_\mathrm{P}}\right)^2$
von Klitzing constant	R_K	25 813 Ω	$\frac{2\pi\hbar}{e^2}$	$\frac{2\pi}{\alpha}\frac{E_\mathrm{P}}{t_\mathrm{P}}\left(\frac{t_\mathrm{P}}{q_\mathrm{P}}\right)^2$	$\frac{2\pi m_\mathrm{P}}{\alpha t_\mathrm{P}}\left(\frac{l_\mathrm{P}}{q_\mathrm{P}}\right)^2$
Josephson constant	K_J	483.6×10^{12} Hz V^{-1}	$\frac{2e}{h}$	$\frac{\sqrt{\alpha}}{\pi}\frac{1}{E_\mathrm{P}}\left(\frac{q_\mathrm{P}}{t_\mathrm{P}}\right)$	—
Bohr magneton	μ_B	9.274×10^{-24} J T^{-1}	$\frac{e\hbar}{2m_\mathrm{e}}$	$\frac{\sqrt{\alpha}}{2}\frac{m_\mathrm{P}}{m_\mathrm{e}} l_\mathrm{P}^2\left(\frac{q_\mathrm{P}}{t_\mathrm{P}}\right)$	—
Conductance quantum	G_0	7.748×10^{-5} S	$\frac{2e^2}{2\pi\hbar}$	$\frac{\alpha}{\pi}\frac{t_\mathrm{P}}{E_\mathrm{P}}\left(\frac{q_\mathrm{P}}{t_\mathrm{P}}\right)^2$	$\frac{\alpha}{\pi}\frac{t_\mathrm{P}}{m_\mathrm{P}}\left(\frac{q_\mathrm{P}}{l_\mathrm{P}}\right)^2$
Magnetic flux quantum	ϕ_0	2.068×10^{-15} Wb	$\frac{2\pi\hbar}{2e}$	$\frac{\pi}{\sqrt{\alpha}} E_\mathrm{P}\left(\frac{t_\mathrm{P}}{q_\mathrm{P}}\right)$	
Vacuum impedance	Z_0	376.7 Ω	$\mu_0 c$	$4\pi\frac{E_\mathrm{P}}{t_\mathrm{P}}\left(\frac{t_\mathrm{P}}{q_\mathrm{P}}\right)^2$	—

l_P = Planck length, m_P = Planck mass, t_P = Planck time, q_P = Planck charge, E_P = Planck energy, F_P = Planck force, m_e = electron mass.

1. The measurement units for electric charge. Units coulomb, statcoulomb (symbol: statC), and abcoulomb (symbol: abC) emerged from these systems.
2. The manner in which electric charge is derived from unit dimensions of length, mass, and time.

A lack of knowledge about the dimensional composition of physical constants contributed to the confusion, but this obstacle is easily overcome by evaluating the following relationship

$$\frac{\mu_0}{4\pi} = \frac{F_\mathrm{P}}{I_\mathrm{P}^2} = \frac{l_\mathrm{P}\, m_\mathrm{P}}{\cancel{t_\mathrm{P}^2}}\frac{\cancel{t_\mathrm{P}^2}}{q_\mathrm{P}^2} = \frac{l_\mathrm{P}\, m_\mathrm{P}}{q_\mathrm{P}^2} = 10^{-7}\mathrm{NA}^{-2} = 10^{-7}\mathrm{kg\ m\ C}^{-2}. \tag{43}$$

Equation (43) illustrates how early units of electric charge and current acquired their unit dimensions. In the electromagnetic unit system (emu), the abcoulomb has unit dimensions $\mathrm{M}^{1/2}\,\mathrm{L}^{1/2}$ and the abampere has dimensions $\mathrm{M}^{1/2}\,\mathrm{L}^{1/2}\,\mathrm{T}^{-1}$, the square root of force. Both of these units are derived from equation (43) as follows.

Setting the magnetic constant equal to 1 gives the following relation

$$\frac{l_\mathrm{P} m_\mathrm{P}}{q_\mathrm{P}^2} = 1. \tag{44}$$

We can then define electric charge in dimensions of length and mass as

$$q_{\rm P}^2 = l_{\rm P} m_{\rm P} \tag{45}$$

which yields the result

$$\begin{aligned} q_P = \sqrt{l_{\rm P} m_{\rm P}} &= 1.876 \times 10^{-19}\ {\rm cm}^{1/2}\ {\rm g}^{1/2} \\ &= 1.876 \times 10^{-19}\ {\rm abC}. \end{aligned} \tag{46}$$

The Planck charge is equal to 1.876×10^{-18} C reflecting a conversion factor of 10 coulomb per abcoulomb. The difference arises because we set the magnetic constant equal to 1 rather than its CGS value $10^{-2}\,{\rm cm\,g\,C^{-2}}$. The square root of this difference gives the conversion.

The abampere can be constructed as the ratio of abcoulomb per second, but it can also be derived directly from equation (43). Setting the magnetic constant equal to 1 gives

$$I_{\rm P}^2 = F_{\rm P} \tag{47}$$

yielding

$$\begin{aligned} I_{\rm P} = \sqrt{F_{\rm P}} &= 3.479 \times 10^{24}\ {\rm cm}^{1/2}\ {\rm g}^{1/2}\ {\rm s}^{-1} \\ &= 3.479 \times 10^{24}\ {\rm abA}. \end{aligned} \tag{48}$$

The same conversion factor gives a corresponding SI value $I_{\rm P} = 3.479 \times 10^{25}$ A.

The electrostatic unit system (esu) defines the statcoulomb using the electric constant rather than the magnetic constant, but everything else about the derivation is the same

$$\frac{1}{4\pi\epsilon_0} = \frac{l_{\rm P} m_{\rm P}}{t_{\rm P}^2} \frac{l_{\rm P}^2}{q_{\rm P}^2}. \tag{49}$$

Setting the constant equal to 1 gives

$$\frac{l_{\rm P} m_{\rm P}\ c^2}{q_{\rm P}^2} = 1 \tag{50}$$

yielding

$$\begin{aligned} q_{\rm P} = \sqrt{l_{\rm P} m_{\rm P}}\ c &= 5.623 \times 10^{-9}\ {\rm cm}^{3/2}\ {\rm g}^{1/2}\ {\rm s}^{-1} \\ &= 5.623 \times 10^{-9}\ {\rm statC}. \end{aligned} \tag{51}$$

In this case we set the electric constant equal to one rather than its CGS value $1.11 \times 10^{-19}\,{\rm C^2\,s^2\,cm^3\,g^{-1}}$. Taking the square root of this difference gives the conversion factor of 1 C to 2997 924 580 statC.

5. Reconciling the natural unit and SI unit scales

SI unit values of length, mass, time, and electric charge were not chosen with an understanding of the natural unit scale or the correlations between properties and dynamics of elementary particles and fields. However, the ratios of these properties to the Planck scale convert arbitrary units into natural units, and this is the role of physical constants in the equations of physics. Figure 9 realigns the four unit dimensions at the Planck scale and indicates where each SI unit value sits in relation to its Planck scale limit, and in relation to the other unit dimensions. A better understanding of this structure is important for students to understand the role of dimensional analysis.

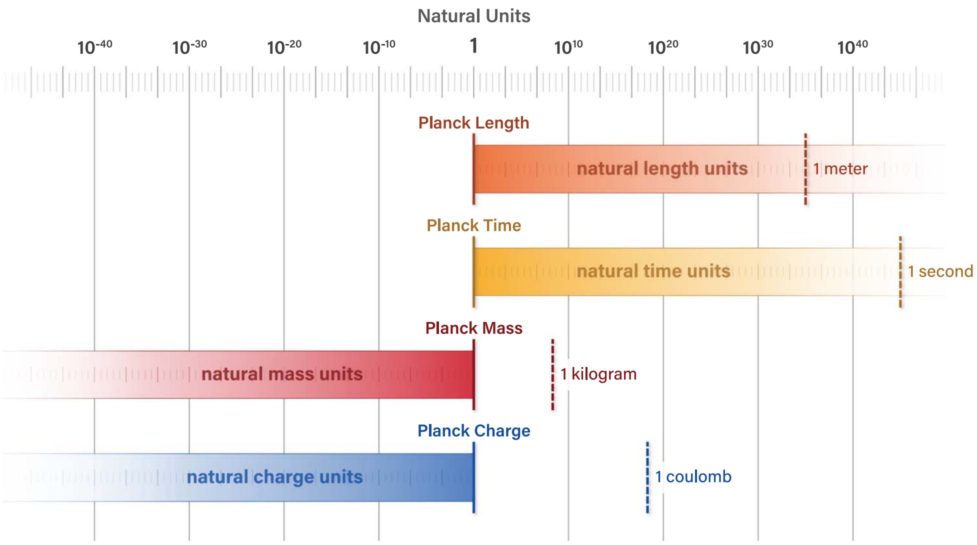

Figure 9. SI units of meter, kilogram, second, and coulomb are incongruent with the natural unit scale. Natural units of length, mass, time, and electric charge coincide at *the Planck scale* where each unit has a value of 1.

6. Conclusion

The *natural units of measure* which retain their meaning for all times and for all civilizations provide a single, unifying principle behind the values and dimensions of the constants and equations in tables 2, 3, 5, and 6. Restating compound-dimensional constants in natural units unveils a natural structure underpinning the mathematics, aligning each unit dimension at the Planck scale, and revealing the highly correlated properties and dynamics of elementary particles and fields.

Many compound-dimensional constants provide high-precision values that are important for experimental and applied physics. However, from a theoretical perspective, the natural units provide a more granular and insightful representation of the physical constants and equations.

The natural units are independent of unit systems. They reveal hidden correlations in the physical attributes and dynamics of natural phenomena that explain *why* the equations produce meaningful results—transforming formula inputs into the correct quantities and dimensions of formula outputs.

The natural units provide a rare opportunity to re-evaluate the physical meaning of the standard model. They challenge certain assumptions about the incompatibility of classical and quantum physics, they offer a common language for interpreting the equations, and they may illuminate a pathway towards a theory of quantum gravity.

The natural units have been integral to the constants and equations of physics since the beginning, but their presence has been largely overshadowed by the compound dimensional constants and by arbitrary unit scales. A more thorough examination of the dimensional structure of physical constants as part of the standard physics curriculum will yield a better understanding of the Planck scale and its role across the interdisciplinary physics landscape.

Acknowledgments

The author would like to thank the peer review referees for their valuable recommendations and insightful comments which significantly improved the quality of this paper.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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