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Equivariant theory of Lie groupoids from the perspective of non-commutative geometry

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For Bianca

Contents

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Introduction

In this thesis we treat the *equivariant theory of Lie groupoids* and try to describe aspects of it using the *non-commutative geometry* of the *convolution algebra*. The main goal is to present a general theory spanning two extremes:

- The treatment of geometry, locally and globally, on manifolds via differential forms, vector fields and deRham-cohomology. This leads to a framework in which a variety of disciplines in physics, for instance classical mechanics and electromagnetism, can be described. Mathematically more involved results, like the Atiyah-Singer Index Theorem, then help to obtain qualitative information about the physical systems described. As it turns out, this can be fully describing using algebraic properties of the algebra of smooth functions of the underlying manifold.

- The theory surrounding the convolution algebra of a Lie group, in particular the connections between the convolution algebra, the representation theory of the group, the adjoint representation and linear Poisson manifolds.

We first give a gentle introduction into the worlds of Lie groupoids and non-commutative geometry.

Symmetries and Lie groupoids

Classically, a symmetry on a space M is encoded by letting a Lie group G act on M. Symmetries are common in physics, where the presence of symmetries eases the treatment of the physical system by replacing the relevant tensors by those that are invariant under the action. Similar to how vector fields, differential forms and cohomology can be pulled out of the algebra of smooth functions by algebraic means, we shall see their siblings that are invariant under the action can be pulled algebraically out of the *convolution algebra*, which combines structure of the algebra of smooth functions on M and the convolution algebra of G.

Lie groupoids are a way to describe symmetries on a space which in some sense are point-dependent in nature, i.e. not globally defined. Heuristically, a Lie groupoid is a combination of two manifolds $\mathcal{G}$ and M, called the *space of arrows* and the *base* respectively, with maps

- $s, t\colon \mathcal{G} \to M$ called the *source* and *target*,
- $u\colon M \to \mathcal{G}$ called the *unit*,
- $i\colon \mathcal{G} \to \mathcal{G}$ called the *inversion*,
- $m\colon \{(g,h) \in \mathcal{G}^2 : s(g) = t(h)\} \to \mathcal{G}$, called the *multiplication*, defined when the second arrow begins where the first arrow ends[1].

These maps should satisfy group-like properties and should be smooth. The structure maps should be used to see elements of $\mathcal{G}$ as 'local symmetries' on M in the following sense:

- An arrow $g \in \mathcal{G}$ should be seen as a symmetry from $s(g)$ to $t(g)$;
- The symmetry $u(x)$ should be thought of as the trivial symmetry from x to itself;
- For $g \in \mathcal{G}$, the arrow $i(g)$ represents reversing the symmetry that g described;
- For $g, h \in \mathcal{G}$ such that $s(g) = t(h)$ the arrow $m(g, h)$ represents the symmetry that first applies h and then g.

The two extremes we alluded to in the beginning are present in this framework: any manifold can be interpreted as a Lie groupoid where $\mathcal{G} = M$ with all the structure maps the identity, while a Lie group is the same thing as a Lie groupoid with M consisting of only one point.

On top of this, the classical notion of a symmetry is contained in this framework: a manifold M with an action by a Lie group G can be encoded by a Lie groupoid $M \times G$ over M, where the arrow (x, g) should be thought of as going from x to xg.

In this, it becomes clear why Lie groupoids encode a more general notion of symmetry: while in the case of a global symmetry the 'collection of symmetries out of a point x' is independent of the point x, in a groupoid these can vary from point to point. A very down to earth slogan: 'Solving a Rubiks Cube is a group, solving a 15 Puzzle is a groupoid'.

As we shall see, Lie groupoids also have a convolution algebra attached to them, with the product being defined by

$$(f_1 * f_2)(g) := \int_{m(g_1,g_2)=g} f_1(g_1) f_2(g_2).$$

[1] Remark that composition is read from right to left in this context.

Here the elements f_1, f_2 of the convolution algebra are tensors which are like smooth functions, but can be canonically integrated. Details will be treated thoroughly in the text.

The notion of *equivariant theory of Lie groupoids* now makes sense, namely it is describing the geometry of M invariant under the point-dependent symmetries that $\mathcal{G}$ describe. The overarching philosophy, inspired by the noted examples, that we will try to convey in this thesis is that the equivariant theory of a Lie groupoid can be describing using the convolution algebra, in particular using the tools of *non-commutative geometry*.

Geometry beyond spaces

Non-commutative geometry, due to Connes [Co94], is the machinery that allows us to interpret statements in the world of algebras in 'geometrical' terms. It is what enables to distinguish between the real line $\mathbb{R}$ and the circle S^1 using only their algebras of smooth functions, which are the smooth functions $C^\infty(\mathbb{R})$ and the 1-periodic smooth functions $C^\infty_{\mathrm{per}}(\mathbb{R})$ respectively.

To wit, we look at the following maps

$$C^\infty(\mathbb{R}^3) \xrightarrow{b} C^\infty(\mathbb{R}^2) \xleftarrow{B} C^\infty(\mathbb{R})$$

where

$$(bF)(x, y) \coloneqq F(x, x, y) - F(x, y, y) + F(x, y, x)$$

and

$$(BF)(x, y) \coloneqq F(y).$$

One checks that

$$C^\infty(\mathbb{R}^2)/b(C^\infty(\mathbb{R}^3)) \cong C^\infty(\mathbb{R})$$

via a map that takes $F \in C^\infty(\mathbb{R}^2)$ to the map $x \mapsto \mathsf{D}_2(F)(x, x)$. In this way, the map B factors to the quotient $C^\infty(\mathbb{R}^2)/b(C^\infty(\mathbb{R}^3))$ to be simply the derivative

$$C^\infty(\mathbb{R}) \xleftarrow{\frac{d}{dx}} C^\infty(\mathbb{R}).$$

Using the Fundamental Theorem of Calculus, we know that this map is surjective, so that

$$\dim\left(\frac{C^\infty(\mathbb{R}^2)}{b(C^\infty(\mathbb{R}^3)) + B(C^\infty(\mathbb{R}))}\right) = 0.$$

Doing this for the circle, we can write down a similar collection of maps

$$C^\infty_{\mathrm{per}}(\mathbb{R}^3) \xrightarrow{b} C^\infty_{\mathrm{per}}(\mathbb{R}^2) \xleftarrow{B} C^\infty_{\mathrm{per}}(\mathbb{R})$$

where periodicity in higher dimensions means 1-periodic in every entry. The formulas for b and B are the same as before. The first step is now similar:

$$C^\infty_{\mathrm{per}}(\mathbb{R}^2)/b(C^\infty_{\mathrm{per}}(\mathbb{R}^3)) \cong C^\infty_{\mathrm{per}}(\mathbb{R})$$

with $C^\infty_{\text{per}}(\mathbb{R}) \overset{B}{\leftarrow} C^\infty_{\text{per}}(\mathbb{R})$ becoming the derivative. However, in the periodic case this map is not surjective, but has a 1-dimensional cokernel. In particular, we have

$$\dim\left(\frac{C^\infty_{\text{per}}(\mathbb{R}^2)}{b(C^\infty_{\text{per}}(\mathbb{R}^3)) + B(C^\infty_{\text{per}}(\mathbb{R}))}\right) = 1.$$

What we have calculated here very roughly, is the cyclic homology (in degree 1) of these two algebras, and their difference can be thought of as a reformulation of the fact that

$$\dim(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{R})) = 0 \qquad \text{and} \qquad \dim(\mathrm{H}^1_{\mathrm{dR}}(S^1)) = 1.$$

This is the central idea of non-commutative geometry, and we will discuss the proper interpretation.

One of the main tools in non-commutative geometry is to associate chain complexes to an associative algebra A, that have specific meaning when we plug in $A = C^\infty(M)$ for M a manifold. For instance, the Hochschild complex $\mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A)$ is defined by

$$\mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A) \coloneqq A^{\otimes(\bullet+1)}$$

with differential $b\colon \mathrm{C}^{\mathrm{Hoch}}_n(A, A) \to \mathrm{C}^{\mathrm{Hoch}}_{n+1}(A, A)$ set by

$$\begin{aligned} b(a_0 \otimes \cdots \otimes a_n) \coloneqq & \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ & + (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

By work of, among others, Connes [Co94], Feigin-Tsygan [FT87], Hochschild-Kostant-Rosenberg [HKR62] and Rinehart [Ri63], we obtain the following dictionary between algebraic invariants of an associative algebra, and geometric quantities associated to a manifold:

Algebra A	Manifold M
Hochschild homology $\mathrm{HH}_\bullet(A, A)$	Differential forms $\Omega^\bullet(M)$
Hochschild cohomology $\mathrm{HH}_\bullet(A, A)$	Multivector fields $\Lambda^\bullet\mathfrak{X}(M)$
(Periodic) cyclic homology $\mathrm{HC}_\bullet(A)$, $\mathrm{HP}_\bullet(A)$	deRham cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$
(Periodic) cyclic cohomology $\mathrm{HC}^\bullet(A)$, $\mathrm{HP}^\bullet(A)$	deRham homologhy $\mathrm{H}^{\mathrm{dR}}_\bullet(M)$.

This is the central idea of non-commutative geometry: the geometry of a space M is contained in in its algebra of smooth functions $C^\infty(M)$ by using the algebraic properties of this algebra. Applying this to arbitrary (non-commutative) algebras, we obtain a rich theory whose properties closely resembles geometric considerations for manifolds.

If we do this for the non-commutative algebra made out of a Lie groupoid, we end up with what we are looking for: a fair description of the quotient, even when this quotient is not a nice space. All together we end up with a coherent philosophy on how to deal with symmetries described by Lie groupoids: we calculate the algebraic properties of our non-commutative algebra and interpret them as equivariant geometric information on the base of the groupoid.

Contents of the text

In technical terms, the overarching philosophy can be summarized by saying that:

- The (cyclic (co)homology of the) convolution algebra of a Lie groupoid $\mathcal{G}$ is closely related to the equivariant theory of the groupoid, in particular to the geometry of the classifying space $B\mathcal{G}$ (for the case of an action groupoid we discuss this in Chapter 3).

- The symmetric powers of the adjoint representation, interpreted as higher order deformation elements, calculate the Hochschild cohomology of the relevant induced algebras, both locally and globally:

  - For a Lie groupoid, the symmetric powers of the adjoint are closely related to the Hochschild cohomology of the convolution algebra (Chapter 2);
  - For a Lie-Rinehart algebra, the symmetric powers of the adjoint calculate the Hochschild cohomology of the universal enveloping algebra (Chapter 4).

To expand the general philosophy into actual mathematics, this text covers three parts:

- The relation between the deformation theory of a Lie groupoid, the adjoint representation and the Hochschild cohomology of the convolution algebra, following [KP21].

- Equivariant characteristic classes for manifolds with a group action, and their relationship to the convolution algebra of the action groupoid, following [KP22].

- The relationship between the universal enveloping algebra of a Lie-Rinehart algebra, the symmetric powers of its adjoint representation, and inroads to understanding pseudodifferential calculus on Lie algebroids, following [KP23].

We now give a detailed overview of the text:

Chapter 1: Setting the stage

We start in Chapter 1 by properly defining the framework in which we will work. In particular, in Section 1.1 we define the main tools of non-commutative geometry: Hochschild (co)homology, and (periodic) cyclic (co)homology, and discuss their general properties, and relations to differential geometry. Then, in Section 1.2 we describe our main objects of interest: Lie groupoids. We discuss examples, their local equivalents in Lie algebroids and their algebraic counterparts in Lie-Rinehart algebras.

Chapter 2: Lie groupoid deformations and convolution algebras

Chapter 2 is devoted to treating the results of [KP21]. The goal is to relate the deformation theory of a Lie groupoid with the deformation theory of its convolution algebra $\mathcal{A}_\mathcal{G}$. Such a relation is not surprising heuristically, as changing the structure of a Lie groupoid in turn changes the structure of the convolution algebra. The deformation theory of a Lie groupoid $\mathcal{G}$ is governed the deformation complex $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ of Crainic, Mestre and Struchiner [CrMS20], while the deformation theory of the convolution algebra is governed by the Hochschild complex $\mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$.

We will describe this connection between the two via a cochain map

$$\Phi\colon \mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$$

from the deformation complex of a Lie groupoid $\mathcal{G}$ to the Hochschild complex of its convolution algebra $\mathcal{A}_\mathcal{G}$.

This chain map links into the known theory in the following ways:

- The deformation complex $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ consists of certain classes of vector fields on the nerve $\mathcal{G}^{(\bullet)}$ of the groupoid, with the differential incorporating the group-like properties of the groupoid. This is mirrored by the complex $\mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G})$ computing differential cohomology, which is given by smooth functions on the nerve. Work of Pflaum, Posthuma and Tang [PPT15] relate the differentialble cohomology $\mathrm{H}^\bullet_{\mathrm{diff}}(\mathcal{G})$ with the cyclic cohomology $\mathrm{HC}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G})$, and we show that our map Φ is compatible with this connection, in a way generalizing from 'smooth functions on the nerve' to 'vector fields on the nerve'.

- The deformation complex $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ is shown by Crainic, Mestre and Struchiner [CrMS20] to canonically incorporate *source-constant deformations* of the Lie groupoid $\mathcal{G}$, i.e. a smooth family of division maps $\overline{m}_\epsilon$, by associating to such a deformation the element $\xi \in \mathrm{C}^2_{\mathrm{def}}(\mathcal{G})$ given by

  $$\xi(g,h) \coloneqq \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \overline{m}_\epsilon(gh,h).$$

  By deforming the convolution product, we also obtain a Hochschild cochain $\beta \in \mathrm{C}^2_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ given by

  $$\beta(a_1,a_2)(g) \coloneqq \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_{h\in s^{-1}(s(g))} a_1(\overline{m}_\epsilon(g,h))a_2(h).$$

  We will show that our chain map links these deformation elements:

  $$\Phi(\xi) = \beta.$$

- There is a localization procedure linking the deformation theory of a Lie groupoid $\mathcal{G}$ to that of its algebroid $A(\mathcal{G})$. In the case of a Lie group G with Lie algebra $\mathfrak{g}$, this was extensively studied by van Est [vE53a, vE53b], and for the groupoid case the results were generalized by Crainic, Mestre and Struchiner [CrMS20] to obtain a van Est-map

$$\mathcal{V}\colon \mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{def}}(A(\mathcal{G})).$$

The manifold underlying the vector bundle $A(\mathcal{G})^*$ is canonically a linear Poisson manifold, and the complex $\mathrm{C}^\bullet_{\mathrm{def}}(A(\mathcal{G}))$ can canonically be interpreted as the linear Poisson complex of this Poisson manifold. Parallel to this, we can view the convolution algebra $\mathcal{A}_\mathcal{G}$ as a *strict deformation quantization* of the Poisson manifold $A(\mathcal{G})^*$ in the sense of Landsman and Ramazan [LR01] via quantization maps

$$q_t\colon \mathcal{S}_c(A(\mathcal{G})^*) \to \mathcal{A}_\mathcal{G}.$$

We show that the van Est-map $\mathcal{V}$ can be interpreted as a classical limit of our chain map Φ:

$$\mathcal{V}(c)(f_1,\ldots,f_k) = \mathcal{F}_\mu\left(\lim_{t\to 0}\left(\sum_{\sigma\in S_k}(-1)^\sigma \frac{1}{(it)^{k-1}}\Phi(c)(q_t(f_{\sigma(1)}),\ldots,q_t(f_{\sigma(k)}))\right)\right)$$

where $c \in \mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$ and we see $\mathcal{V}(c)$ as a Poisson cochain on $A(\mathcal{G})^*$ in which we plug in $f_1, ..., f_k \in \mathcal{S}_c(A(\mathcal{G})^*)$.

- Representation theory of Lie groupoids isn't as well-serving to the needs of the theory of Lie groupoids as the representation theory of Lie groups is well-serving to the needs of the theory of Lie groups. In particular, there is no proper definition of an adjoint *representation* for a Lie groupoid $\mathcal{G}$. This is solved by Abad and Crainic [AC13] by defining *representations up to homotopy* and in particular the *adjoint representation up to homotopy* for a Lie groupoid $\mathcal{G}$, which fulfils the same that the adjoint representation of a Lie group does. However, a priori the adjoint representation up to homotopy needs the choice of a connection on the Lie algebroid to be defined. Parallel to this, there are the symmetric powers of the adjoint representation up to homotopy, which was defined by Abad and Crainic [AC13] as a way to calculate the cohomology of the classifying space $B\mathcal{G}$, generalizing the Chern-Weil construction for Lie groups to the Lie groupoid setting.

  The deformation complex $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ is a way to give an intrinsic model of the adjoint representation up to homotopy, in that it does not need a choice of connection to be defined, but is isomorphic to the adjoint representation up to homotopy under the choice of a connection. We discuss in Chapter 2 how extending from smooth functions and vector fields to differential operators might be a way to combine our chain map Φ, the Gerstenhaber structure on the Hochschild complex $\mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ and the deformation complex $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ to obtain an intrinsic model for the symmetric powers of the adjoint representation up to homotopy.

The chapter starts in Section 2.1 by defining the convolution algebra of Lie groupoid as introduced by Connes [Co94], and seeing how it reduces to known algebras in specific examples. Then, in Section 2.2 we recall work by Crainic, Mestre and Struchiner [CrMS20] and treat the deformation complex of a Lie groupoid. In Section 2.3 we then define our chain map and discuss its properties, while in Section 2.4 we discuss what it means to take its classical limit and how it relates to the van Est-map. Lastly, in Section 2.5 we discuss some ideas on how to proceed with these results in the context of the symmetric powers of the adjoint.

Chapter 3: Action groupoids and equivariant characteristic classes

In Chapter 3 we discuss the results [KP22], in particular we study orientable manifolds M with an action of a unimodular group G and the relation between the convolution algebra of the underlying groupoid $M \times G \rightrightarrows M$ and the equivariant cohomology $\mathrm{H}_G(M)$. We generalize work of Connes [Co94], Gorokhovsky [Go99], Block-Getzler [BG94], Getzler [Ge94], Getzler-Jones [GJ93] and Ponge [Po18].

The equivariant cohomology $\mathrm{H}_G(M)$ is defined [Tu20] to be

$$\mathrm{H}^\bullet_G(M) \coloneqq \mathrm{H}^\bullet((EG \times M)/G)$$

i.e. the cohomology of the homotopy quotient. If the action of G on M is free and proper this calculates the cohomology of the quotient M/G, and generally if the group G is compact it is calculated by the Cartan model [Ca50], which is given by the complex

$$((\mathrm{Sym}^\bullet(\mathfrak{g}^*) \otimes \Omega^\bullet(M))^G, d^G_{\mathrm{dR}})$$

where the equivariant differential d^G_{dR} is given by

$$d^G_{\mathrm{dR}}(\alpha)(X) \coloneqq d_{\mathrm{dR}}(\alpha(X)) - \iota_{X_M}(\alpha(X)).$$

For the non-compact case, the equivariant cohomology presents a way to understand the 'smooth geometry of the quotient' even if the quotient is not a manifold. As such, it is the natural place where characteristic classes of equivariant vector bundles over M live, with the classical equivariant Chern character

$$\mathrm{Ch}_G \colon \mathrm{Vect}_G(M) \to \mathrm{H}^{\mathrm{ev}}_G(M)$$

defined by

$$\mathrm{Ch}_G(E) \coloneqq \mathrm{Ch}((\mathrm{pr}^* E)/G),$$

where $\mathrm{pr} \colon EG \times M \to M$ is the (equivariant) projection and $(\mathrm{pr}^*E)/G$ is the resulting (topological) vector bundle over the homotopy quotient.

The main goal of the chapter is to relate the equivariant cohomology $\mathrm{H}^\bullet_G(M)$ with the periodic cyclic cohomology $\mathrm{HP}^\bullet(G \ltimes C^\infty_c(M))$ of the convolution algebra and describe the resulting Chern character with values in $\mathrm{HP}^\bullet(G \ltimes C^\infty_c(M))$ in purely algebraic terms internal to the convolution algebra. This was already done in the case where the group G is discrete by Connes [Co94] and Gorokhovsky [Go99]. We do this in the following steps:

- The theory of (generalized) cycles, which was defined by Connes [Co94] and refined by Gorokhovsky [Go99], is a way to generlize the connection between (periodic) cyclic cohomology of $C_c^\infty(M)$, deRham currents, differential forms on M and deRham-cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(M)$ to obtain for any algebra A cyclic cohomology classes in $\mathrm{HC}^\bullet(A)$. This is done by embedding A into via a map $\rho\colon A \to \Omega^0$ into a differential graded algebra Ω with a closed graded trace $⨍\colon \Omega^n \to \mathbb{K}$ and the resulting generalization of a deRham current is a cyclic n-cochain given by

$$(a_0, ..., a_n) \mapsto ⨍ \rho(a_0) d(\rho(a_1)) \cdots d(\rho(a_n)).$$

  We exploit a version of this procedure involving curvature to associate to an equivariant vector bundle $E \to M$ with a connection a (curved) DGA Ω_E which is given by

$$\Omega_E \coloneqq C_c^\infty(G, \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M, \mathrm{End}(E))).$$

  with curvature related to the curvature of the connection, and differential given by a combination of the Cartan differential and a term measuring the defect of the connection to be G-invariant. The resulting cyclic class is invariant of the connection and hence we obtain a Chern character

$$\mathrm{Ch}_\Omega\colon \mathrm{Vect}_G(M) \to \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M)).$$

- Following work of Brylinski [Br87a] and Ponge [Po18] we use the Eilenberg-Zilber Theorem [EZ53] and its cyclic version due to Khalkali and Rangipour [KR04] to calculate the cyclic homology of $G \ltimes C_c^\infty(M)$ by a double complex $\mathrm{C}_{\bullet,\bullet}(G, M)$ given by

$$\mathrm{C}_{p,q}(G, M) \coloneqq C_c^\infty(G^{\times(q+1)} \times M^{\times(p+1)})$$

  with the structure in one direction given by a G-twisted variant of the Hochschild complex of $C_c^\infty(M)$ and the structure in the other direction given by the group homology complex for the G-module $C_c^\infty(M^{\times(p+1)})$.

  Using an equivariant HKR-map as defined by Block and Getzler [BG94], we refine this double complex to a double complex $\mathrm{C}_{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}(M))$ defined by

$$\mathrm{C}_{p,q}(G, \Omega_{\mathfrak{g}}(M)) \coloneqq C_c^\infty(G^{\times q}, C^\infty(\mathfrak{g}, \Omega_c^p(M))),$$

  which serves as our model for cyclic homology of the convolution algebra.

- A generalization of the Cartan model for equivariant cohomology to the non-compact case was defined by Getzler [Ge94]. Replacing G-invariants by the whole group cohomology complex, Getzler's model is a double complex $\mathrm{C}^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}(M))$ given by

$$\mathrm{C}^{p,q}(G, \Omega_{\mathfrak{g}}(M)) \coloneqq C^\infty(G^{\times q}, \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega^q(M)).$$

Combining integration of functions $G^{\times q}$ and the pairing between $\Omega_c(M)$ and $\Omega(M)$ we obtain a pairing

$$\mathrm{C}^{p,q}(G, \Omega_{\mathfrak{g}}(M)) \otimes \mathrm{C}_{\dim(M)-p,q}(G, \Omega_{\mathfrak{g}}(M)) \to \mathbb{R}.$$

- Apart from writing down his model for equivariant cohomology, Getzler also writes down an explicit cochain in his model representating the equivariant Chern character of an equivariant vector bundle E with connection ∇. Plugging this into the pairing we defined, we obtain a diagram relating the equivariant vector bundles, equivariant cohomology and periodic cyclic cohomology of the convolution algebra

$$\begin{array}{ccc} \mathrm{Vect}_G(M) & \longrightarrow & \mathrm{H}^{\mathrm{ev}}_G(M) \\ & \searrow & \downarrow \\ & & \mathrm{HP}^{\dim(M)}(G \ltimes C^\infty_c(M)) \end{array}$$

  We show that this diagram commutes on the level of chains for proper group actions, and discuss ideas to prove its commutativity on the level of cohomology for arbitrary orientation-preserving actions of unimodular groups.

The chapter starts with Section 3.1 where we discuss Connes' [Co94] and Gorokhovsky's [Go99] theory of generlized cycles. Then in Section 3.2 we write down our model for the cyclic homology of the convolution algebra using the Eilenberg-Zilber Theorem. We pair with equivariant cohomology in Section 3.3 and show in Section 3.4 that the resulting diagram (as drawn above) is commutative. In Section 3.5 we discuss how this procedure is a common generalization of known 'equivariant Chern characters' in several specific cases.

Chapter 4: Hochschild cohomology of Lie-Rinehart algebras

In Chapter 4 we discuss local analogues to the ideas from Chapter 2, taking place in the context of Lie algebroids and their algebraic analogues Lie-Rinehart algebras. In particular, we discuss modules over Lie-Rinehart algebras (L, R), discuss the cohomology associated to those in the form of Hochschild cohomology of modules over the universal enveloping algebra $\mathcal{U}(L, R)$ and derive ways to link this concept from the world of algebra to cohomology theories more akin to the Lie algebra-like nature of Lie-Rinehart algebras.

The central point of discussion is a result by Blom [Bl17] that relates Hochschild cohomology of the universal enveloping algebra $\mathcal{U}(\Gamma(A), C^\infty(M))$ and the linear Poisson cohomology of the Poisson manifold A^* for a Lie algebroid $A \to M$:

$$\mathrm{HH}^\bullet(\mathcal{U}(\Gamma(A), C^\infty(M)), \mathcal{U}(\Gamma(A), C^\infty(M))) \cong \mathrm{H}^\bullet_{\mathrm{Pois,poly}}(A^*).$$

The result stems from using the Poincare-Birkhoff-Witt Theorem of Rinehart [Ri63] to induce a Poisson structure on $\Gamma(\mathrm{Sym} A)$ from the commutator bracket of the universal enveloping algebra, which coincides with the Poisson bracket of fibrewise polynomial functions on the Poisson manifold A^*. This observation allows for a spectral sequence to be written down which has the Poisson cohomology $\mathrm{H}^\bullet_{\mathrm{Pois,poly}}(A^*)$ on its second page and which converges to the Hochschild cohomology $\mathrm{HH}^\bullet(\mathcal{U}(\Gamma(A), C^\infty(M)), \mathcal{U}(\Gamma(A), C^\infty(M)))$. Using Kontsevitch' formality for the Poisson manifold A^* Blom shows that this spectral sequence collapses and in turn establishes the isomorphism.

Perpendicular to this is a fully algebraic consideration of this phenomenon for a Lie algebra $\mathfrak{g}$, where using the fact that the Poincare-Birkhoff-Witt map pbw: $\mathrm{Sym}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is an intertwiner of $\mathfrak{g}$-representations one writes down an explicit chain map

$$\mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g})) \to \mathrm{C}^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$$

given by a combination of restriction to $\mathfrak{g}$, composition with pbw and antisymmetrization. An argument with a filtration and the associated graded quotient complexes (or equivalently, with the induced spectral sequence) then shows this map to be a quasi-isomorphism.

Recognizing that the Lie algebra cohomology complex $\mathrm{C}^\bullet_{\mathrm{CE}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$ is isomorphic to the polynomial Poisson complex of the Poisson manifold $\mathfrak{g}^*$, and more generally the polynomial Poisson complex associated to the dual of a Lie algebroid A is an intrinsic model for the symmetric powers of the adjoint representation up to homotopy of the algebroid of Abad and Crainic [AC12], we will in this chapter use the ideas of the Lie algebra-case to generalize the results of Blom to the Lie-Rinehart algebra-case, circumventing explicit arguments with the spectral sequence by establishing explicit chain maps calculating an isomorphism

$$\mathrm{HH}^\bullet(\mathcal{U}(L,R), \mathcal{U}(L,R)) \cong \mathrm{H}^\bullet(L, \mathrm{Sym}(\mathrm{ad}))$$

for the case where R is a smooth algebra and L is projective as an R-module.

We proceed in the following steps:

- Following Kordon-Lambre [KL21] and Lambre-le Meur [LLM18] we recognize that the functor $(-)^{\mathcal{U}}(L,R)$ that takes a $\mathcal{U}(L,R)$-bimodule and returns its $\mathcal{U}(L,R)$-invariants is a composition of functors, first taking the R-invariants of the underlying R-bimodule, and then inside the R-invariants taking the invariants of the diagonal Lie algebra-representation of L. In turn, it is argued by Kordon and Lambre that for a $\mathcal{U}(L,R)$-bimodule M there is a spectral sequence

  $$\mathrm{E}_2^{p,q} \cong \mathrm{H}^p_{\mathrm{CE}}(L, \mathrm{HH}^q_{\mathrm{Hoch}}(R,M)) \Rightarrow \mathrm{HH}^{p+q}(\mathcal{U}(L,R), M)$$

  which converges to the desired Hochschild cohomology.

  Using this as a starting point we write down a double complex $\mathrm{C}^{\bullet;\bullet}_{\mathrm{LR}}(L,R;M)$ given by

  $$\mathrm{C}^{p;q}_{\mathrm{LR}}(L,R;M) := \mathrm{Hom}(\Lambda^p L, \mathrm{Hom}(R^{\otimes q}, M))$$

  with the vertical differential given by the Hochschild differential of the complex $\mathrm{Hom}(R^{\otimes\bullet}, M)$ and the horizontal differential given by the Lie algebra differential associated to a certain L-representation on $\mathrm{Hom}(R^{\otimes q}, M)$. In the case where the anchor of (L, R) vanishes, this is a natural complex to consider in this context, with $\mathrm{Hom}(\Lambda^p L, -)$ replaced by $\mathrm{Hom}_R(\Lambda^p L, -)$. However, in the general case, the L-representation of $\mathrm{Hom}(R^{\otimes q}, M)$ is not R-linear. Writing down explicit homotopies which measure the defect of R-linearity, we write down a 'non-linear' Chevalley-Eilenberg complex $\mathrm{C}^\bullet_{\mathrm{nl}}(L, R; M)$ incorporating a 'symbol equation' relating the defect of the cochains to be R-linear and the homotopy. We show by exhibiting an explicit chain map

$$\mathfrak{s}\colon \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{U}(L, R), M) \to \mathrm{Tot}^\bullet(\mathrm{C}_{\mathrm{LR}}(L, R; M))$$

that the this non-linear complex is quasi-isomorphic to the Hochschild complex

$$\mathrm{HH}^\bullet(\mathcal{U}(L, R), M) \cong \mathrm{H}^\bullet_{\mathrm{nl}}(L, R; M)$$

in the case where R is a smooth algebra and L is projective over R.

- Using the Poincaré-Birkhoff-Witt Theorem of Rinehart in the Lie-Rinehart setting, we can see $\mathcal{U}(L, R)$ as a quantization of $\mathrm{Sym}_R L$ and in turn induce a Poisson bracket on $\mathrm{Sym}_R L$ using the commutator bracket of $\mathcal{U}(L, R)$. The resulting Poisson complex due to Huebschmann [Hu90] can be used as an intrinsic model for the symmetric powers of the adjoint representation up to homotopy of the Lie-Rinehart algebra (L, R) following Abad and Crainic [AC12], resulting in a complex $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad}))$.

  With the Lie algebra-case in mind, we can write down a chain map

  $$\Phi\colon \mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad})) \to \mathrm{Tot}^\bullet(\mathrm{C}_{\mathrm{LR}}(L, R; \mathcal{U}(L, R)))$$

  which is in essence given by composition with $\mathrm{pbw}^\nabla\colon \mathrm{Sym}_R L \to \mathcal{U}(L, R)$ under the choice of a connection. By making an argument with a filtration and the resulting graded quotient complexes, we obtain an isomorphism

  $$\mathrm{H}^\bullet(L, \mathrm{Sym}(\mathrm{ad})) \cong \mathrm{H}^\bullet_{\mathrm{nl}}(L, R; \mathcal{U}(L, R)),$$

  and in turn an isomorphism

  $$\mathrm{HH}^\bullet(\mathcal{U}(L, R), \mathcal{U}(L, R)) \cong \mathrm{H}^\bullet(L, \mathrm{Sym}(\mathrm{ad}))$$

  in the case when R is a smooth algebra and L is projective over R.

In Section 4.1 we discuss the work of Kordon, Lambre and le Meur an how to use its ideas to define the non-linear complex $\mathrm{C}^\bullet_{\mathrm{nl}}(L, R; M)$ calculating the Hochschild cohomology $\mathrm{HH}^\bullet(\mathcal{U}(L, R), M)$ of a $\mathcal{U}(L, R)$-bimodule. Section 4.2 treats the relationship between the non-linear complex $\mathrm{C}^\bullet_{\mathrm{nl}}(L, R; \mathcal{U}(L, R))$ with the symmetric powers of the adjoint representation of (L, R). The chapter has an appendix in Section 4.3 in which we discuss proofs of several Lemmata in the chapter whose proofs, while necessary to discuss, are of such high density of calculations that they would break up the pace of the text.

Appendix A: Constructions in homological algebra

In Appendix A we discuss the general machinery underlying most of the homological algebra we use in the text:

- Mixed complexes (Appendix A.1);
- Homological Pertubation Theory (Appendix A.2);
- Chain complexes and mixed complexes induced by simplicial and cyclic vector spaces (Appendix A.3);
- Cylindrical spaces and their induces mixed double complexes (Appendix A.4);
- The Eilenberg-Zilber Theorem relating the homology of a bisimplicial vector space and the homology of its simplicial diagonal (Appendix A.5).

Chapter 1

Setting the stage

This chapter is meant to give a concise introduction to the objects we will be working with. As such, this chapter is fully a review of known work. This chapter has two parts:

- Using the techniques described in the Appendix, we define the main notions in non-commutative geometry we are interested in. In particular, we discuss the simplicial and cyclic vector spaces that induce the complexes calculating Hochschild (co)homology and cyclic (co)homology.

- Then, we discuss the general theory of Lie groupoids and Lie algebroids: their definitions, classes of examples and going from groupoids to algebroids. We also describe the algebraic counterpart to Lie algebroids that are Lie-Rinehart algebras, and the universal enveloping algebra induced by such a Lie-Rinehart algebra.

1.1 Non-commutative geometry

Non-commutative geometry is the study of the 'geometry' of (possibly non-commutative) algebras. The general philosophy is that we describe 'geometry' of an algebra using algebraic invariants, strengthened by results which relate these very natural constructions to geometric quantities if we plug in the smooth functions on a manifold. We outline the concepts that we will use in this dissertation: Hochschild homology, Hochschild cohomology, cyclic homology and cyclic cohomology, which in the framework described correspond to differential forms, multi-vector fields, deRham cohomology and deRham homology, respectively. The main resources for this section are Connes [Co94], Loday [Lo98] and Nest-Tsygan [NT].

1.1.1 Hochschild (co)homology

Hochschild homology

Let $\mathbb{K}$ be a field of characteristic zero[1], and let A be an associative $\mathbb{K}$-algebra. Let M be a bimodule over A. Unless otherwise stated, in what follows the tensor product is simply the algebraic tensor product of $\mathbb{K}$-vector spaces.

Definition 1.1.1 (Hochschild homology) We define a complex $\mathrm{C}_\bullet^{\mathrm{Hoch}}(A, M)$ by

$$\mathrm{C}_k^{\mathrm{Hoch}}(A, M) \coloneqq M \otimes A^{\otimes k}$$

with differential $b\colon \mathrm{C}_k^{\mathrm{Hoch}}(A, M) \to \mathrm{C}_{k-1}^{\mathrm{Hoch}}(A, M)$ given by

$$\begin{aligned} b(m \otimes a_1 \otimes \cdots \otimes a_k) \coloneqq{}& ma_1 \otimes a_2 \otimes \cdots \otimes a_k \\ &+ \sum_{i=1}^{k-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k \\ &+ (-1)^k a_k m \otimes a_1 \otimes \cdots \otimes a_{k-1}. \end{aligned}$$

We define *Hochschild homology* to be the homology of this complex:

$$\mathrm{HH}_\bullet(A, M :== \mathrm{H}_\bullet(\mathrm{C}^{\mathrm{Hoch}}(A, M))$$

Remark 1.1.2 This complex originates from the machinery of simplicial vector spaces that we describe in detail in Appendix A. In particular, $V_k = M \otimes A^{\otimes k}$ has face maps given by

$$\begin{aligned} d_0(m \otimes a_1 \otimes \cdots \otimes a_k) &\coloneqq ma_1 \otimes a_2 \otimes \cdots \otimes a_k, \\ d_i(m \otimes a_1 \otimes \cdots \otimes a_k) &\coloneqq m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k, \qquad (0 < i < k) \\ d_k(m \otimes a_1 \otimes \cdots \otimes a_k) &\coloneqq a_k m \otimes a_1 \otimes \cdots \otimes a_{k-1}, \end{aligned}$$

which make it into a semi-simplicial vector space.

If A is unital, there are also degeneracy maps given by

$$s_i(m \otimes a_1 \otimes \cdots \otimes a_k) \coloneqq m \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_k,$$

such that the semi-simplicial vector space $M \otimes A^{\otimes \bullet}$ is simplicial.

From the discussion in the Appendix, this means that for a unital A we can also write down a normalized Hochschild-complex, quasi-isomorphic to the complex we wrote down above, where the chains in degree k are given by $M \otimes (A/\mathbb{K})^{\otimes k}$.

Remark 1.1.3 The Hochschild complex, and by extension Hochschild homology is functorial in two ways. First, if A and B are $\mathbb{K}$-algebras, M is a B-bimodule and $\varphi\colon A \to B$ is a morphism of $\mathbb{K}$-algebras, M can canonically be considered a A-bimodule, and φ induces a chain map $\mathrm{C}_\bullet^{\mathrm{Hoch}}(A, M) \to \mathrm{C}_\bullet^{\mathrm{Hoch}}(B, M)$. Second, if A is a $\mathbb{K}$-algebra, M and N are A-bimodules and $f\colon M \to N$ is a morphism of A-bimodules, we obtain a chain map $\mathrm{C}_\bullet^{\mathrm{Hoch}}(A, M) \to \mathrm{C}_\bullet^{\mathrm{Hoch}}(A, N)$.

[1] In practice this means $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$.

Remark 1.1.4 Define A^o to be opposite algebra of A and $A^e = A \otimes A^o$ to be the envelopping algebra. An A-bimodule is then the same thing as a left A^e-module. If A is unital, one can identify $\mathrm{HH}_\bullet(A, M) \cong \mathrm{Tor}_\bullet^{A^e}(M, A)$. Indeed, to compute Tor one needs to write down a projective resolution of A as an A^e-module, and the bar resolution is such a resolution. The bar-resolution is given by

$$\cdots \xrightarrow{d} A \otimes A \otimes A \xrightarrow{d} A \otimes A \xrightarrow{d} A \to 0$$

with differentials given by

$$d(a_1 \otimes \cdots \otimes a_k) \coloneqq \sum_{i=1}^{k-1} (-1)^{i+1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k.$$

Using the unit of A, we can write down a contraction $s\colon A^{\otimes k} \to A^{\otimes(k+1)}$ of this resolution by

$$s(a_1 \otimes \cdots \otimes a_k) \coloneqq 1 \otimes a_1 \otimes \cdots \otimes a_k.$$

With this resolution, one checks that $M \otimes_{A^e} A^{\otimes(k+1)} = M \otimes A^{\otimes k}$ with the induced differential given precisely by the Hochschild differential b.

In general, the bar-resolution may be acyclic even if there is not a unit, and in that case the algebra is called *H-unital* (cf. [Wo89]).

Example 1.1.5 We can interpret $\mathbb{K}$ as a $\mathbb{K}$-algebra. If A is any $\mathbb{K}$-algebra, and M is a bimodule over A, we can view M as a $\mathbb{K}$-bimodule. In this case

$$\mathrm{C}_\bullet^{\mathrm{Hoch}}(\mathbb{K}, M) \cong M,$$

with the differential under this isomorphism given by

$$b\colon \mathrm{C}_k^{\mathrm{Hoch}}(\mathbb{K}, M) \to \mathrm{C}_{k-1}^{\mathrm{Hoch}}(\mathbb{K}, M) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \mathrm{id} & \text{if } k \text{ is even} \end{cases}$$

and we conclude that

$$\mathrm{HH}_k(\mathbb{K}, M) = \begin{cases} M & \text{if } k = 0 \\ 0 & \text{else} \end{cases}$$

Remark 1.1.6 If A is not unital, we can adjoin a unit to obtain a unital $\mathbb{K}$-algebra $A^+ = A \oplus \mathbb{K}$ with product

$$(a_1, \lambda_1)(a_2, \lambda_2) \coloneqq (a_1 a_2 + \lambda_1 a_2 + \lambda_2 a_1, \lambda_1 \lambda_2).$$

Any A-bimodule M can be canonically seen as a A^+-bimodule by letting the $\mathbb{K}$-factor acting by scalar multiplication. There is a canonical short exact sequence of algebras

$$0 \to A \to A^+ \to \mathbb{K} \to 0$$

that takes a to $(a, 0)$ and (a, λ) to λ. Using this short exact sequence, one checks that the Hochschild homology can be recovered from the Hochschild homologies of the two unital algebras A^+ and $\mathbb{K}$ by

$$\mathrm{HH}_\bullet(A, M) \cong \ker(\mathrm{HH}_\bullet(A^+, M) \to \mathrm{HH}_\bullet(\mathbb{K}, M)).$$

We study Hochschild homology because of its connection to differential forms in the commutative case. So for now let A be a commutative $\mathbb{K}$-algebra. In this situation we can define objects akin to differential forms in the following way:

Definition 1.1.7 (Kähler differentials) We define $\Omega^1_{A/\mathbb{K}}$ to be the quotient of $A \otimes A$ where we identify $f_0 \otimes (f_1 f_2)$ with $(f_0 f_2) \otimes f_1 + (f_0 f_1) \otimes f_2$. We write $f_0 df_1$ for the element induced by $f_0 \otimes f_1$. In higher degree we define $\Omega^\bullet_{A/\mathbb{K}}$ to be the exterior algebra defined by $\Omega^1_{A/\mathbb{K}}$

$$\Omega^k_{A/\mathbb{K}} = \Lambda^k_A \Omega^1_{A/\mathbb{K}}.$$

There is a canonical surjective map $A^{\otimes(k+1)} \to \Omega^k_{A/\mathbb{K}}$ sending $f_0 \otimes \cdots \otimes f_k$ to $f_0 df_1 \wedge \cdots \wedge df_k$.

If we now look at the Hochschild complex $\mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A)$ we observe two things in degree 1. First, all elements are closed, since the differential out of degree 1 is just the commutator in this case. Second, the exact elements are spanned by elements of the form $f_0 f_1 \otimes f_2 - f_0 \otimes f_1 f_2 + f_0 f_2 \otimes f_1$. Together we see that there is an immediate isomorphism

$$\mathrm{HH}_1(A, A) \cong \Omega^1_{A/\mathbb{K}}.$$

The important result from the 60's due to Hochschild, Kostant and Rosenberg [HKR62] then fully identifies the Hochschild homology with the Kähler differentials.

Theorem 1.1.8 (Hochschild-Kostant-Rosenberg) If A is smooth, the canonical map $A^{\otimes(k+1)} \to \Omega^k_{A/\mathbb{K}}$ induces an isomorphism $\mathrm{HH}_k(A, A) \cong \Omega^k_{A/\mathbb{K}}$.

Note that for $A = C^\infty(M)$ the seemingly 'obvious' identification $\Omega^1_{C^\infty(M)/\mathbb{R}} = \Omega^1(M)$ is **not** true, since Kähler differentials only behave well with polynomial relations, not with any smooth functions[2]. In particular, for $M = \mathbb{R}$ we have that $d(e^x)$ does not equal $e^x dx$ in $\Omega^1_{C^\infty(\mathbb{R})/\mathbb{R}}$.

To counteract this, we can play the Hochschild-game with topological vector spaces and topological tensor products. As described by Connes [Co85] and Pflaum [Pf98], if $E \to M$ is a vector bundle we can endow $\Gamma(E \to M)$ with a Fréchet topology using the family of seminorms induced by partial derivatives over compact domains. Using the inductive tensor product $\widehat{\otimes}$ we then have the isomorphism

$$\Gamma(E \to M) \widehat{\otimes} \Gamma(E' \to M') \cong \Gamma(E \boxtimes E' \to M \times M'),$$

in particular we have the isomorphism

$$C^\infty(M) \widehat{\otimes} C^\infty(N) \cong C^\infty(M \times N).$$

With this, we can set up the bar-resolution of $C^\infty(M)$ with this new tensor product, and in turn obtain a continuous Hochschild complex

$$\cdots \to C^\infty(M^{\times 3}) \to C^\infty(M^{\times 2}) \to C^\infty(M) \to 0$$

[2]This is why these kind of algebraic constructions work so well in algebraic geometry, where everything is polynomial or rational.

Within this framework the canonical map $A^{\otimes \bullet+1} \to \Omega^\bullet_{A/\mathbb{K}}$ now takes the form of a map

$$C^\infty(M^{\times(k+1)}) \xrightarrow{\text{HKR}} \Omega^k(M)$$

which, as an example, for $k = 1$ is given by

$$\text{HKR}(f)_x(\dot{\gamma}) \coloneqq \frac{d}{dt}\bigg|_{t=0} f(x, \gamma(t)).$$

The HKR-theorem now takes the following form:

Theorem 1.1.9 (Continuous HKR) [Pf98, Thm 3.3] The continuous Hochschild complex $\mathrm{C}^{\text{Hoch,cont}}_\bullet(C^\infty(M), C^\infty(M))$ defined using the topological tensor product has homology

$$\mathrm{H}_k(C^{\text{Hoch,cont}}_\bullet(C^\infty(M), C^\infty(M))) \cong \Omega^k(M).$$

Hochschild cohomology

Dual to Hochschild homology, there is Hochschild cohomology. We return to the general case where A is a $\mathbb{K}$-algebra which is assumed to be neither nor commutative and M is an A-bimodule.

Definition 1.1.10 Define the *Hochschild cohomology complex* $\mathrm{C}^\bullet_{\text{Hoch}}(A, M)$ by

$$\mathrm{C}^k_{\text{Hoch}}(A, M) \coloneqq \text{Hom}_{\mathbb{K}}(A^{\otimes k}, M),$$

with differential $b\colon \mathrm{C}^k_{\text{Hoch}}(A, M) \to \mathrm{C}^{k+1}_{\text{Hoch}}(A, M)$ given by

$$\begin{aligned}(bf)(a_1, ..., a_{k+1}) \coloneqq & \, a_1 f(a_2, ..., a_{k+1}) \\ & + \sum_{i=1}^{k} (-1)^i f(a_1, ..., a_i a_{i+1}, ..., a_{k+1}) \\ & + (-1)^{k+1} f(a_1, ..., a_k) a_{k+1}.\end{aligned}$$

and we define the *Hochschild cohomology* of A with coefficients in M by

$$\mathrm{HH}^k(A, M) \coloneqq \mathrm{H}^k(C^\bullet_{\text{Hoch}}(A, M)).$$

Remark 1.1.11 Similar to the dual case, the Hochschild cochain complex, and in turn Hochschild cohomology, are functorial in both A and M. However, the functoriality in A is reversed, since an algebra map $A \to B$ induces a map $\mathrm{C}^\bullet_{\text{Hoch}}(B, M) \to \mathrm{C}^\bullet_{\text{Hoch}}(A, M)$.

Remark 1.1.12 Dually to writing Hochschild homology as Tor's, we can use the bar-resolution to obtain natural isomorphisms $\mathrm{HH}^\bullet(A, M) \cong \text{Ext}^\bullet_{A^e}(A, M)$.

In a certain sense, the cochain complex we have just written down comes from a 'cosimplicial vector space', a sequence of vector spaces with face maps increasing the degree and degenaracy maps decreasing the degree. Indeed, this structure is just the dual structure to the one for the Hochschild homology complex. In particular, also here we can normalize and find a quasi-isomorphic subcomplex, consisting of those maps $f\colon A^{\otimes k} \to M$ that satisfy $f(a_1, ..., a_k) = 0$ whenever $1 \in \{a_1, ..., a_k\}$.

Example 1.1.13 Starting with A and M, we can see M as a $\mathbb{K}$-bimodule and then $\mathrm{C}^k_{\mathrm{Hoch}}(\mathbb{K}, M) \cong M$ for every k with the differential given under this isomorphism by

$$b\colon \mathrm{C}^k_{\mathrm{Hoch}}(\mathbb{K}, M) \to \mathrm{C}^{k+1}_{\mathrm{Hoch}}(\mathbb{K}, M) = \begin{cases} 0 & \text{if } k \text{ is even} \\ \mathrm{id} & \text{if } k \text{ is odd} \end{cases}$$

and so we obtain the following calculation for the Hochschild cohomology

$$\mathrm{HH}^k(\mathbb{K}, M) \cong \begin{cases} M & k = 0 \\ 0 & \text{else} \end{cases}$$

Remark 1.1.14 If M is some A-bimodule, then $M^* = \mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K})$ is also naturally an A-bimodule, where the structure is given by

$$(a_1 f a_2)(m) \coloneqq f(a_2 m a_1).$$

From this, it is clear that the Hochschild cochain complex $\mathrm{C}^\bullet_{\mathrm{Hoch}}(A, M^*)$ is naturally isomorphic to the dual complex of $\mathrm{C}_\bullet^{\mathrm{Hoch}}(A, M)$.

Let us now look at the case where A is a bimodule over itself. In this case we can quite easily describe the closed and exact elements in degrees 0 and 1. In degree 0 we have that cochains are just elements of A with differential $ba = [-, a]$, so that

$$\mathrm{HH}^0(A, A) = Z(A),$$

the center of A.

In turn, the image of $b\colon \mathrm{C}^0_{\mathrm{Hoch}}(A, A) \to \mathrm{C}^1_{\mathrm{Hoch}}(A, A)$ is by definition all the maps $A \to A$ which are defined as commutating with a given element of A. These maps called inner derivations, and written as $\mathrm{Inn}(A)$.

The closed elements of degree 1 are those maps $f\colon A \to A$ such that

$$a_1 f(a_2) - f(a_1 a_2) + f(a_1) a_2 = 0$$

which by definition is the same thing as f being a derivation. In the end this gives us

$$\mathrm{HH}^1(A, A) = \mathrm{Der}(A)/\mathrm{Inn}(A).$$

When A is commutative, the center $Z(A)$ is the whole algebra itself, and there are no non-trivial inner derivations, so that in this case $\mathrm{HH}^0(A, A) = A$ and $\mathrm{HH}^1(A, A) = \mathrm{Der}(A)$. In that case, we can actually go further. Indeed we can embed $\Lambda^k_A \mathrm{Der}(A)$ into $\mathrm{C}^k_{\mathrm{Hoch}}(A, A)$, by seeing the wedge product of $D_1, ..., D_k \in \mathrm{Der}(A)$ as a k-cochain by the formula

$$(D_1 \wedge \cdots \wedge D_k)(a_1, ..., a_k) := \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma D_1(a_{\sigma(1)}) \cdots D_k(a_{\sigma(k)}).$$

One then checks that $D_1 \wedge \cdots \wedge D_k$ is always a closed cochain. The dual version of the HKR-theorem is then

Theorem 1.1.15 (HKR) If A is commutative and smooth, the map $\Lambda^\bullet_A \mathrm{Der}(A) \hookrightarrow \mathrm{C}^\bullet_{\mathrm{Hoch}}(A, A)$ induces an isomorphism

$$\Lambda^\bullet_A \mathrm{Der}(A) \cong \mathrm{HH}^\bullet(A, A).$$

Similar to before, we can play the same game for $A = C^\infty(M)$ with the topological tensor product, and we obtain the multi-vector fields as the (continuous) Hochschild cohomology of $C^\infty(M)$ [Pf98, Thm 3.3].

We can also calculate Hochschild cohomology of $C^\infty(M)$ with coefficients in the dual $C^\infty(M)^*$ and this turns out to consist of deRham-currents on M, i.e. the linear dual $\Omega^\bullet(M)$ [Co85, p.310].

Deformation theory side of Hochschild cohomology

Apart from the relations of Hochschild cohomology to differential geometry, the Hochschild cohomology also has plays a central role in deformation theory of associative algebras.

To wit, let A be a topological vector space with a continuous product $*$. Suppose we have a continuous deformation of A, that is the datum of a continuous map

$$\begin{aligned} I \times A \times A &\to A \\ (\epsilon, a_1, a_2) &\mapsto a_1 *_\epsilon a_2, \end{aligned}$$

where $I \subset \mathbb{R}$ is an open interval containing 0, that satisfies

- For every $\epsilon \in I$, the map $*_\epsilon$ defines an associative product on the underlying vector space of A;
- At $\epsilon = 0$ we have $*_0 = *$;
- For every $a_1, a_2 \in A$ the following limit is defined as an element of A

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_1 *_\epsilon a_2.$$

From this data we can construct a 2-cochain $\xi \in \mathrm{C}^2_{\mathrm{Hoch}}(A, A)$ in the (algebraic) Hochschild complex by the formula

$$\xi(a_1, a_2) := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} a_1 *_\epsilon a_2.$$

Lemma 1.1.16 The 2-cochain ξ is closed.

Proof. Writing out the explicit differential, we have

$$\begin{aligned}(b\xi)(a_1,a_2,a_3) =& a_1\xi(a_2,a_3) - \xi(a_1a_2,a_3) + \xi(a_1,a_2a_3) - \xi(a_1,a_2)a_3 \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} a_1 *_0 (a_2 *_\epsilon a_3) - \frac{d}{d\epsilon}\Big|_{\epsilon=0} (a_1 *_0 a_2) *_\epsilon a_3 \\ &+ \frac{d}{d\epsilon}\Big|_{\epsilon=0} a_1 *_\epsilon (a_2 *_0 a_3) - \frac{d}{d\epsilon}\Big|_{\epsilon=0} (a_1 *_\epsilon a_2) *_0 a_3 \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} a_1 *_\epsilon (a_2 *_\epsilon a_3) - \frac{d}{d\epsilon}\Big|_{\epsilon=0} (a_1 *_\epsilon a_2) *_\epsilon a_3 \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \mathrm{Ass}(*_\epsilon)(a_1,a_2,a_3).\end{aligned}$$

Here in the last line, for a map $f\colon A\otimes A \to A$, the associator $\mathrm{Ass}(f)$ measures the defect of f to be associative:

$$\mathrm{Ass}(f)(a_1,a_2,a_3) = f(a_1, f(a_2,a_3)) - f(f(a_1,a_2),a_3).$$

Then since $*_\epsilon$ is assumed to be an associative product for all ϵ, we see that $\mathrm{Ass}(*_\epsilon) = 0$ for every ϵ, and so ξ is closed. □

Even more specific, a deformation A_ϵ of A is trivial if $A_\epsilon \cong A$ as associative algebras for ϵ close enough to 0. This means that there is a linear automorphism $\varphi_\epsilon\colon A \to A$ such that $*_e = \varphi_\epsilon^{-1} \circ *_\epsilon \circ (\varphi_\epsilon \otimes \varphi_\epsilon)$. In this Hochschild-setting, trivial deformations are cohomologically trivial:

Lemma 1.1.17 If A_ϵ is a trivial deformation described by a family of automorphisms φ_ϵ. Then $\xi = b\xi'$ with

$$\xi'(a) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_\epsilon(a).$$

Proof. Doing the explicit calculation we have

$$\begin{aligned}(b\xi')(a_1,a_2) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} a_1 * \varphi_\epsilon(a_2) - \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_\epsilon(a_1 * a_2) + \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_\epsilon(a_1) * a_2 \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_\epsilon(a_1) * \varphi_\epsilon(a_2) - \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_\epsilon(a_1 * a_2) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_\epsilon^{-1}(\varphi_\epsilon(a_1) * \varphi_\epsilon(a_2)) \\ &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} a_1 *_\epsilon a_2 \\ &= \xi(a_1,a_2)\end{aligned}$$

which completes the proof. □

In the end the resulting philosophy is that $\mathrm{HH}^2(A, A)$ measures the non-trivial deformations of A as an associative algebra, as we see from this discussion that it calculates deformations modulo trivial deformations.

More algebraic structure on Hochschild cochains

An important point in later discussion is the existence of graded product and graded Lie algebra structures on the Hochschild cochain complex $\mathrm{C}^\bullet_{\mathrm{Hoch}}(A, A)$.

To wit, consider the operation $\cup\colon \mathrm{C}^p_{\mathrm{Hoch}}(A, A) \otimes \mathrm{C}^q_{\mathrm{Hoch}}(A, A) \to \mathrm{C}^{p+q}_{\mathrm{Hoch}}(A, A)$ given by

$$(f \cup g)(a_1, ..., a_{p+q}) \coloneqq f(a_1, ..., a_p)g(a_{p+1}, ..., a_q).$$

This is obviously associative and is compatible with b in the sense that

$$b(f \cup g) = (bf \cup g) + (-1)^p(f \cup bg).$$

On top of this, there is a generalization of composition in the form of an operation $\circ\colon \mathrm{C}^p_{\mathrm{Hoch}}(A, A) \otimes \mathrm{C}^q_{\mathrm{Hoch}}(A, A) \to \mathrm{C}^{p+q-1}_{\mathrm{Hoch}}(A, A)$ given by

$$(f \circ g)(a_1, ..., a_{p+q-1}) \coloneqq \sum_{i=1}^{p-1}(-1)^{q(p+i-1)} f(a_1, ..., g(a_i, ..., a_{i+p}), ..., a_{p+q-1}).$$

Even though it is not associative, we can construct from this operation the *Gerstenhaber bracket* $[-,-]\colon \mathrm{C}^p_{\mathrm{Hoch}}(A, A) \otimes \mathrm{C}^q_{\mathrm{Hoch}}(A, A) \to \mathrm{C}^{p+q-1}_{\mathrm{Hoch}}(A, A)$ by setting

$$[f, g] \coloneqq f \circ g + (-1)^{(p-1)(q-1)} g \circ f.$$

Again, this is compatible with b in the sense that

$$b([f, g]) = [bf, g] \pm [f, bg].$$

In particular both $\cup$ and $[-,-]$ descend down to Hochschild cohomology:

Proposition 1.1.18 On $\mathrm{HH}^\bullet(A, A)$ the operations $\cup$ and $[-,-]$ satisfy

- $a \cup b = (-1)^{pq} b \cup a$,
- $[a, b] = (-1)^{(p-1)(q-1)}[b, a]$,
- $(-1)^{(p-1)(r-1)}[a, [b, c]] + (-1)^{(q-1)(p-1)}[b, [c, a]] + (-1)^{(r-1)(q-1)}[c, [a, b]] = 0$,
- $[a, b \cup c] = [a, b] \cup c + (-1)^{(p-1)q} b[a, c]$.

In particular we see that $\mathrm{HH}^\bullet(A, A)$ exhibits the structure of a graded associative ring and a (shifted) graded Lie algebra, so that the Lie bracket is (graded) Poisson over the product. This structure is called a *Gerstenhaber algebra*, introduced by Gerstenhaber [Ge63].

1.1.2 Cyclic and periodic cyclic (co)homology

Cyclic homology

If we think about Hochschild homology as differential forms, we want a way to add the exterior differential into the picture. This is what cyclic homology will do for us. So we start with A an associative $\mathbb{K}$-algebra, for simplicity we assume that A is unital, and introduce an extra differential on the Hochschild complex $\mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A)$:

Definition 1.1.19 (Connes' cyclic differential) We define the map $B\colon \mathrm{C}^{\mathrm{Hoch}}_n(A, A) \to C^{\mathrm{Hoch}}_{n+1}(A, A)$ to be

$$\begin{aligned} B(a_0 \otimes \cdots \otimes a_n) = &\sum_{i=0}^{n} (-1)^{in} 1 \otimes a_{n-i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{n-i} \\ &+ \sum_{i=0}^{n} (-1)^{(i+1)n} a_{n-1} \otimes 1 \otimes a_{n-i+1} \otimes \cdots \otimes a_n \otimes a_0 \otimes \cdots \otimes a_{n-i}. \end{aligned}$$

With the cyclic map $t\colon \mathrm{C}^{\mathrm{Hoch}}_k(A, A) \to \mathrm{C}^{\mathrm{Hoch}}_k(A, A)$ defined by

$$t(a_0 \otimes \cdots \otimes a_k) \coloneqq a_k \otimes a_0 \otimes \cdots \otimes a_{k-1}$$

one recognizes that, together with the simplicial maps d and s from Remark 1.1.2, $(\mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A), d, s, t)$ is a cyclic vector space, with B the second differential obtain via Proposition A.3.6, so that we obtain:

Proposition 1.1.20 The maps

$$B\colon \mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A) \to \mathrm{C}^{\mathrm{Hoch}}_{\bullet+1}(A, A)$$

and

$$b\colon \mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A) \to C^{\mathrm{Hoch}}_{\bullet-1}(A, A)$$

together make $(\mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A), b, B)$ into a mixed chain complex.

Remark 1.1.21 Note that we can also write down the induced B on the normalized Hochschild complex $A \otimes (A/\mathbb{K})^{\otimes \bullet}$ where it is given by

$$B(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^{in} 1 \otimes a_{n-i+1} \otimes \cdots \otimes a_n \otimes \overline{a_0} \otimes \cdots \otimes a_{n-i}$$

where for $a_0 \in A$, $\overline{a_0} \in A/\mathbb{K}$ is the induced element in the quotient.

Definition 1.1.22 If A is unital, we define $\mathrm{HC}_\bullet(A)$ to be the homology of the complex $(\mathrm{CC}^{\mathrm{Hoch}}_\bullet(A, A), b + B)$ induced by the mixed complex $(\mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A), b, B)$.

Remark 1.1.23 Since the cyclic map t is natural in A, and the simplicial vector space $(\mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A), d, s)$ was already seen to be natural in A, we conclude that the association $A \mapsto \mathrm{HC}_\bullet(A)$ is functorial in A.

Example 1.1.24 If we see $\mathbb{K}$ as an algebra over itself, we obtain

$$HC_n(\mathbb{K}) \cong \begin{cases} \mathbb{K} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}.$$

Definition 1.1.25 If A is any associative $\mathbb{K}$-algebra, we define $\mathrm{HC}_\bullet(A)$ to be the kernel of the induced map $\mathrm{HC}_\bullet(A^+) \to \mathrm{HC}_\bullet(\mathbb{K})$. If A was unital to begin with this is consistent with the definition above.

If we now restrict ourselves to the case where A is commutative, we can easily see that the map $\mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A) \to \Omega^\bullet_{A/\mathbb{K}}$ fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{C}^{\mathrm{Hoch}}_p(A, A) & \xrightarrow{B} & \mathrm{C}^{\mathrm{Hoch}}_{p+1}(A, A) \\ \downarrow & & \downarrow \\ \Omega^p_{A/\mathbb{K}} & \xrightarrow{(p+1)d} & \Omega^{p+1}_{A/\mathbb{K}} \end{array}$$

where d is the map sending $f_0 df_1 \wedge \cdots df_p$ to $df_0 \wedge \cdots \wedge df_p$.

In particular, we can upgrade the statement of the HKR Theorem to say that the map $\mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A) \to \Omega^\bullet_{A/\mathbb{K}}$ as described is a quasi-isomorphism between the two with respect to b on the domain and 0 on the codomain, as well as with respect to B on the domain and d on the codomain.

Looking at the structure of the mixed complex $(\Omega^\bullet_{A/\mathbb{K}}, 0, d)$ we can conclude the following cyclic version of the HKR-theorem.

Corollary 1.1.26 [Lo98, Thm 3.4.12] If A is unital, commutative and smooth, then

$$\mathrm{HC}_n(A) \cong (\Omega^n_{A/\mathbb{K}})_{\mathrm{cl}} \oplus \left(\bigoplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} \mathrm{H}_{n-2i}(\Omega^\bullet_{A/\mathbb{K}}, d) \right)$$

Doing this for $A = C^\infty(M)$ with the topological tensor product $\widehat{\otimes}$ one can also calculate the cyclic homology of the topological Hochschild complex, to obtain $\mathrm{HC}^{\mathrm{cont}}_\bullet(C^\infty(M))$ which is then calculated ([Co85, Thm 46]) to be:

$$\mathrm{HC}^{\mathrm{cont}}_n(C^\infty(M)) \cong \Omega^n_{\mathrm{cl}}(M) \oplus \left(\bigoplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} H^{n-2i}_{\mathrm{dR}}(M) \right)$$

Cyclic cohomology

We can dualize the discussion from above to obtain the notion of cyclic cohomology. So for a unital algebra A, we start with the Hochschild complex $\mathrm{C}^\bullet_{\mathrm{Hoch}}(A, A^*)$, which under the adjunction between tensor products and duals we see as

$$\mathrm{C}^n_{\mathrm{Hoch}}(A, A^*) \coloneqq \mathrm{Hom}(A^{\otimes(n+1)}, \mathbb{K})$$

so that the Hochschild differential becomes

$$(bf)(a_0, ..., a_{n+1}) := \sum_{i=0}^{n} (-1)^i f(a_0, ..., a_i a_{i+1}, ..., a_{n+1}) \\ + (-1)^n f(a_{n+1} a_0, a_1, ..., a_n).$$

and we can define the cyclic differential $B\colon \mathrm{C}^n_{\mathrm{Hoch}}(A, A^*) \to \mathrm{C}^{n-1}_{\mathrm{Hoch}}(A, A^*)$ to be

$$(Bf)(a_0, ..., a_{n-1}) := \sum_{i=0}^{n} (-1)^{in} f(1, a_{n-i+1}, ..., a_n, a_0, ..., a_{n-i}) \\ + \sum_{i=0}^{n} (-1)^{(i+1)n} f(a_{n-i}, 1, a_{n-i+1}, ..., a_n, a_0, ..., a_{n-i-1}).$$

Since the resulting structure is obviously the dual to the mixed complex $(\mathrm{C}^{\mathrm{Hoch}}_\bullet(A, A), b, B)$, it is itself a mixed cochain complex $(\mathrm{C}^\bullet_{\mathrm{Hoch}}(A, A^*), b, B)$.

Definition 1.1.27 We define $\mathrm{HC}^\bullet(A)$ to be the cohomology of $(\mathrm{CC}^\bullet_{\mathrm{Hoch}}(A, A^*), b + B)$.

We can now in essence dualize the whole discussion we did for cyclic homology.

First, note that $\mathrm{HC}^\bullet(A)$ is also calculated by the *cyclic subcomplex* $\mathrm{C}^\bullet_\lambda(A)$ of $(\mathrm{C}^\bullet_{\mathrm{Hoch}}(A, A^*), b)$ consisting of those maps $f\colon A^{\otimes k+1} \to \mathbb{K}$ that satisfy

$$f(a_k, a_0, ..., a_{k-1}) = (-1)^k f(a_0, ..., a_k)$$

Second, we note that in the normalized case we obtain a quasi-isomorphic subcomplex given by chose maps $f\colon A^{\otimes(n+1)} \to \mathbb{K}$ such that $f(a_0, ..., a_n) = 0$ if $1 \in \{a_1, ..., a_n\}$. The B-differential reduces to

$$(Bf)(a_1, ..., a_n) = \sum_{i=0}^{n} (-1)^{in} f(1, a_{n-i+1}, ..., a_n, a_0, ..., a_{n-i})$$

and using Lemma A.1.14 we also know that $\mathrm{HC}^\bullet(A)$ is calculated by the cyclic complex coming out of the normalized complex.

Third, it is clear that a map $A \to B$ of $\mathbb{K}$-algebras induces a map $\mathrm{C}^\bullet_{\mathrm{Hoch}}(B, B^*) \to \mathrm{C}^\bullet_{\mathrm{Hoch}}(A, A^*)$ that intertwines both the Hochschild and cyclic differentials, and we obtain a map $\mathrm{HC}^\bullet(B) \to \mathrm{HC}^\bullet(A)$.

From this we can also define cyclic cohomology for non-unital algebras by setting $\mathrm{HC}^\bullet(A)$ to be the cokernel of the map $\mathrm{HC}^\bullet(\mathbb{K}) \to \mathrm{HC}^\bullet(A^+)$.

Lastly, if we do the whole construction for $A = C^\infty(M)$ together with the topological tensor product and the continuous Hom-functor, we obtain continuous cyclic cohomology $\mathrm{HC}^\bullet_{\mathrm{cont}}(C^\infty(M))$ which, similarly to the case for cyclic homology before, can be calculated to be

$$\mathrm{HC}^n_{\mathrm{cont}}(C^\infty(M)) \cong (\Omega_n(M))_{\mathrm{cl}} \oplus \left(\bigoplus_{i=1}^{\lfloor \frac{n}{2} \rfloor} \mathrm{H}^{\mathrm{dR}}_{n-2i}(M) \right)$$

where $\Omega_\bullet(M)$ is the coalgebra of deRham-currents on M, i.e. $\Omega_k(M) = (\Omega^k(M))^*$, endowed with the dual d^* of the deRham-differential $d\colon \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$ as homological differential, leading to the notion of deRham-homology $\mathrm{H}_\bullet^{\mathrm{dR}}(M)$ of M.

Periodic cyclic homology and cohomology

For a unital algebra A, starting with the mixed complexes $(\mathrm{C}_\bullet^{\mathrm{Hoch}}(A,A), b+B)$ and $(\mathrm{C}^\bullet_{\mathrm{Hoch}}(A,A^*), b+B)$, we can also write down the induced periodic cyclic complexes $(\mathrm{CP}_\bullet(A), b+B)$ and $(\mathrm{CP}^\bullet(A), b+B)$ respectively, leading to the definition of $\mathrm{HP}_\bullet(A)$ and $\mathrm{HP}^\bullet(A)$.

We will not fully repeat the whole discussion, but we will give the relevant calculations when for the commutative case.

When A is commutative and smooth we have

$$\mathrm{HP}_n(A) \cong \bigoplus_i \mathrm{H}_{n+2i}(\Omega^\bullet_{A/\mathbb{K}}, d)$$

where i ranges such that $n + 2i \geq 0$.

Similarly, when $A = C^\infty(M)$, using the topological tensor products we obtain

$$\mathrm{HP}_n^{\mathrm{cont}}(C^\infty(M)) \cong \bigoplus_i \mathrm{H}_{\mathrm{dR}}^{n+2i}(M),$$
$$\mathrm{HP}^n_{\mathrm{cont}}(C^\infty(M)) \cong \bigoplus_i \mathrm{H}^{\mathrm{dR}}_{n+2i}(M).$$

1.2 Lie groupoids and Lie algebroids

As is now mentioned a few times, Lie groupoids form are at the center of attention of this text. Groupoids and algebroids play an important role in the mathematical side of physics, as outlined by Landsman [La06]. We will now thoroughly discuss their definition, the definition of Lie algebroids and the procedure of going from a Lie groupoid to a Lie algebroid. We will also describe Lie-Rinehart algebras and the universal enveloping algebra of a Lie-Rinehart algebra. References for the parts about Lie groupoids and Lie algebroids are Mackenzie [Mack87, Mack05] and Moerdijk-Mrčun [MM03]. References for the parts about Lie-Rinehart algebras are Huebschmann [Hu04], Moerdijk-Mrčun [MM10] and Rinehart [Ri63].

1.2.1 Lie groupoids

We start this section with the definition of a Lie groupoid.

Definition 1.2.1 A *Lie groupoid* $\mathcal{G} \rightrightarrows M$ is given by:

- Two smooth manifolds $\mathcal{G}$, M;

- Two smooth submersions $s, t\colon \mathcal{G} \to M$;
- A smooth map $m\colon \mathcal{G}^{(2)} \colon = \{(g,h) \in \mathcal{G}; \times \mathcal{G} : s(g) = t(h)\} \to \mathcal{G}$;
- A smooth map $i\colon \mathcal{G} \to \mathcal{G}$;
- A smooth map $u\colon M \to \mathcal{G}$.

These are compatible in the following way

- $s(m(g,h)) = s(h)$,
- $t(m(g,h)) = t(g)$,
- $s(i(g)) = t(g)$,
- $t(i(g)) = s(g)$,
- $m(g, i(g)) = u(t(g))$,
- $m(i(g), g) = u(s(g))$,
- $s(u(x)) = x$,
- $t(u(x)) = x$,
- $m(u(t(g)), g) = g$,
- $m(g, u(s(g))) = g$.

for any $g, h \in \mathcal{G}$ such that $s(g) = t(h)$ and any $x \in M$.

Remark 1.2.2 The compatibility conditions in the previous definition is the same as saying that $(\mathcal{G}, M)$ has the structure of a category with M as objects and $\mathcal{G}$ as arrows, where every arrow is an isomorphism. Here s and t encode source and target, so that we see $g \in \mathcal{G}$ as an arrow $t(g) \xleftarrow{g} s(g)$, m encodes the composition of arrows, so that gh is the composed arrow $t(g) \xleftarrow{g} s(g) = t(h) \xleftarrow{h} s(h)$, i encodes inverses and u encodes units.

Remark 1.2.3 We ask for s and t to be submersions so that we are able to require m to be smooth. Indeed the fact that $\mathcal{G}^{(2)}$ is an embedded submanifold of $\mathcal{G}^{\times 2}$ follows from the fact that s and t are smooth submersions.

Example 1.2.4 (Trivial groupoids) For any manifold M, we can define the *trivial groupoid* over M, $M \rightrightarrows M$. Here s, t, u and i are the identity, $M^{(2)}$ is just the diagonal in $M \times M$ and we define $m\colon M^{(2)} \to M$ to be the map sending (x,x) to x.

Example 1.2.5 (Étale groupoids) A special class of Lie groupoids is that of groupoids where the dimension of $\mathcal{G}$ and M agree, or equivalently either (and then both) of s or t are local diffeomorphisms: those groupoids are called étale.

Example 1.2.6 (Fundamental groupoid) A generalization of the fundamental group, the fundamental groupoid $\Pi(M) \rightrightarrows M$ of a manifold M extends fundamental groups by going from loops to paths. The arrows of this groupoid are homotopy classes of paths in M, were source and target are the start and end points of paths, multiplication is concatenation, units are constants paths and inversion is given by tracing a path in the opposite direction. This set of arrows can be topologized and given the structure of a smooth manifold in the following way: given a homotopy class $[\gamma]$ of a path from x to y

and let U_x, U_y be contractible open neighbourhoods of x and y respectively. Then there is an injective map from $U_x \times U_y$ to $\Pi_1(M)$ which takes a point $x' \in U_x$ and $y' \in U_y$ and sends this to the homotopy class of the path that is the concaternation of the (homotopy unique) path through U_x from x' to x, the path γ and the (homotopy unique) path in U_y from y to y', and we can declare this map to a diffeomorphism onto an open submanifold.

Example 1.2.7 (Groupoids associated to foliations) Slightly altering the example of homotopy groupoids sketched above, let M be a manifold with $\mathcal{F}$ a foliation on M. Then we can define a groupoid called the *monodromy groupoid of the folation*, denoted by $\mathrm{Mon}(M, \mathcal{F})$ of which the arrows are homotopy classes of paths parallel to the foliation. Again, source and target are the start and end points of paths and multiplication is concatenation. Note that the homotopy groupoid of a manifold M is just the monodromy of the folation $\mathcal{F} = TM$, while the trivial groupoid over M is the monodromy of the foliation $\mathcal{F} = 0$.

Example 1.2.8 (Lie groups) A Lie group is easily seen to the same thing as a Lie groupoid where $M = \{\mathrm{pt}\}$.

Example 1.2.9 (Action groupoids) When M is a manifold that is acted upon (from the left) by a Lie group G, there is a groupoid $G \times M \rightrightarrows M$, called the action groupoid, that encodes this action. The structure maps are given by

- $s(g, x) = x$
- $t(g, x) = gx$
- $m((h, gx), (g, x)) = (gh, x)$
- $u(x) = (e, x)$
- $i(g, x) = (g^{-1}, gx)$

For right actions we can either transform it into a left action by setting $g \cdot x = x \cdot g^{-1}$, or we can write down a slightly twisted variant of the construction above by giving a groupoid structure on $M \times G \rightrightarrows M$.

Example 1.2.10 (Pair groupoids) When M is any manifold, we see $M \times M$ as a Lie groupoid over M. Here, we set the structure maps by

- $s(x, y) = y$
- $t(x, y) = x$
- $m((x, y), (y, z)) = (x, z)$
- $i(x, y) = (y, x)$
- $u(x) = (x, x)$

In particular, one has to think as (x, y) as the unique arrow $x \xleftarrow{(x,y)} y$.

Example 1.2.11 (Groupoids associated to a submersion) When $f\colon M \to N$ is any submersion, we can define a variant on the pair groupoid. Writing $M\, {}^f\!\times^f M = \{(x, y) \in M \times M : f(x) = f(y)\}$ we can define the groupoid associated to f, $M\, {}^f\!\times^f M \rightrightarrows M$ with the structure maps

- $s(x, y) = y$
- $t(x, y) = x$
- $m((x, y), (y, z)) = (x, z)$
- $i(x, y) = (y, x)$
- $u(x) = (x, x)$

Note that the pair groupoid is this construction applied to the trivial map $M \to \{\mathrm{pt}\}$, while the trivial groupoid over M is this construction applied to the identity $M \xrightarrow{=} M$.

In full generality the relevant observation is that $M\ {}^f\!\times^f M$ can be written as $(f, f)^{-1}(\Delta)$ and hence is the pre-image of a submanifold under a submersion, and hence itself is a submanifold. Its tangent space can be described as

$$T_{(x,y)} M\ {}^f\!\times^f M = \{(v, w) \in T_x M \times T_y M : df(v) = df(w)\},$$

from which it is immediately clear that s and t are submersions, since their derivatives are given by $(v, w) \mapsto w$ and $(v, w) \mapsto v$ respectively.

Example 1.2.12 (Bundle of Lie groups) If $\mathcal{G} \rightrightarrows M$ is a groupoid such that the source and target maps agree, every source fiber $s^{-1}(x)$ is a Lie group, and the groupoid is then nothing more than a family of Lie groups, parametrized by M. The easiest example of this is taking the constant fibre $M \times G \rightrightarrows M$, but to see that the fibres can be different, we consider the $\mathcal{G} = S^1 \times (\mathbb{R}\backslash\{0\}) \sqcup \mathbb{R} \times \{0\}$. We can topologize this set and give it a smooth structure if we require that $S^1 \times (\mathbb{R}\backslash\{0\})$ is an open submanifold, and that the map

$$\begin{aligned} \{(x, t) \in \mathbb{R} : 4x^2 < t^2\} &\to \mathcal{G} \\ (x, 0) &\mapsto (x, 0) \\ (x, t) &\mapsto (e^{2\pi i \frac{x}{t}}, t)\ (t \neq 0) \end{aligned}$$

is a diffeomorphism onto an open submanifold.

We can then make a groupoid $\mathcal{G} \rightrightarrows \mathbb{R}$ by taking the source and target to be the projection onto the second factor, with multiplication given by $m((x, 0), (y, 0)) = (x + y, 0)$ and $m((z, t), (w, t)) = (zw, t)$. Clearly we see that the source fibres outside of $t = 0$ are isomorphic to S^1, while at $t = 0$ the fibre is isomorphic to $\mathbb{R}$.

In Chapter 2 we will discuss the adiabatic groupoid associated to a groupoid. The example here is the adiabatic groupoid associated to the Lie group S^1.

Example 1.2.13 (Vector bundles) As a special case of the previous example, any vector bundle $\pi \colon E \to M$ can canonically be made into a Lie groupoid $E \rightrightarrows M$, in the following way: We set the source and target maps to be $s = t = \pi$, the multiplication will be fibrewise addition in E, taking inverses will be multiplying with -1 and the units consist of the zero-section.

Example 1.2.14 (Tangent groupoid) As a kind of meta-example, we can make a new Lie groupoid out of a given Lie groupoid $\mathcal{G} \rightrightarrows M$, called the tangent groupoid. It is given by $T\mathcal{G} \rightrightarrows TM$, with the all the structure maps replaced by their derivatives. We remark that this works since as submanifolds of $T\mathcal{G}^{\times 2}$ the following are equal

$$T\mathcal{G}^{(2)} = \{(v, w) \in T\mathcal{G}^{\times 2} : ds(v) = dt(w)\}.$$

Representations of Lie groupoids

A representation of a Lie group is the group acting linearly on a vector space. If we want to define an equivalent concept for a Lie groupoid $\mathcal{G} \rightrightarrows M$, we need to replace a vector space with a collection of vector spaces indexed by M: a vector bundle.

Definition 1.2.15 A *representation of a Lie groupoid* $\mathcal{G} \rightrightarrows M$ is a vector bundle $E \to M$, together with for every $g \in \mathcal{G}$ a linear isomorphism $A_g \colon E_{s(g)} \to E_{t(g)}$ such that

- $A_{u(x)} = \mathrm{id}_{E_x}$,
- $A_g A_h = A_{gh}$.

We denote the set of $\mathcal{G}$-representations by $\mathrm{Rep}(\mathcal{G})$.

Definition 1.2.16 From a representation $E \in \mathrm{Rep}(\mathcal{G})$ we can define *groupoid cohomology with coefficients in* E via the complex $\mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G}, E)$ which is set by

$$\mathrm{C}^0_{\mathrm{diff}}(\mathcal{G}, E) \coloneqq \Gamma(E \to M)$$

and for $n \geq 1$ by

$$\mathrm{C}^n_{\mathrm{diff}}(\mathcal{G}, E) \coloneqq \Gamma(t^* E \to \mathcal{G}^{(n)})$$

where $t \colon \mathcal{G}^{(n)} \to M$ is given in terms of the target map t of $\mathcal{G}$ by $t(g_1, ..., g_n) = t(g_1)$. The differential is given by

$$(dc)(g) \coloneqq A_g(c(s(g)) - c(t(g))$$

for $c \in \mathrm{C}^0_{\mathrm{diff}}(\mathcal{G}, E)$ and

$$\begin{aligned}(dc)(g_1, ..., g_{n+1}) \coloneqq & A_{g_1}(c(g_2, ..., g_{n+1})) \\ & + \sum_{i=1}^{n} (-1)^i c(g_1, ..., g_i g_{i+1}, ..., g_{n+1}) \\ & + (-1)^{n+1} c(g_1, ..., g_n)\end{aligned}$$

for $c \in \mathrm{C}^n_{\mathrm{diff}}(\mathcal{G}, E)$ where $n \geq 1$.

Example 1.2.17 For any Lie groupoid $\mathcal{G} \rightrightarrows M$, the trivial bundle $\underline{\mathbb{R}} \to M$ can be canonically given the structure of a representation of $\mathcal{G}$ by setting $A_g = \mathrm{id}_{\mathbb{R}}$ for any $g \in \mathcal{G}$. We denote the resulting Lie groupoid cohomology complex by $\mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G})$ and call it the *differentiable cohomology of* $\mathcal{G}$. Notice that on the level of chains it is given by $\mathrm{C}^n_{\mathrm{diff}}(\mathcal{G}) = C^\infty(\mathcal{G}^{(n)})$.

Example 1.2.18 For a Lie group G, seen as a groupoid $G \rightrightarrows \{\mathrm{pt}\}$, a representation is the same thing as a Lie group representation: a vector space V and a Lie group morphism $G \to \mathrm{GL}(V)$.

Example 1.2.19 If M is a manifold acted upon by a Lie group G, a representation of the action groupoid is the same thing as a G-equivariant vector bundle over M.

Example 1.2.20 For the pair groupoid $M \times M \rightrightarrows M$, the only representations are the trivial vector bundles. Indeed, fixing a base point $x_0 \in M$ the isomorphism $E_{x_0} \to E_y$ induced by the arrow (y, x_0) induces an isomorphism between E and the trivial vector bundle with fibre E_{x_0}.

Remark 1.2.21 As seen by the previous example, in general a Lie groupoid does not have many representations. In particular, there is no analogue of the adjoint representation. We will discuss this further in Section 2.5.

Example 1.2.22 If $\mathcal{G} \rightrightarrows M$ is an étale groupoid, we can make $TM \to M$ into a representation of $\mathcal{G}$ by setting $A_g \colon T_{s(g)}M \to T_{t(g)}M$ to be

$$A_g := dt_g \circ (ds_g)^{-1}$$

The fact that this is indeed is a representation follows from looking at composable pair of arrows (g, h), a vector $v \in T_{s(h)}M$ and noticing that

$$ds_g(((ds_g)^{-1} \circ dt_h \circ (ds_h)^{-1})(v)) = dt_h((ds_h)^{-1}(v))$$

so that

$$dm_{(g,h)}(((ds_g)^{-1} \circ dt_h \circ (ds_h)^{-1})(v), (ds_h)^{-1}(v)) \in T_{gh}\mathcal{G}$$

is defined. Using the fact relations between the source-, target- and multiplication-maps, we have that

$$dt_{gh}(dm_{(g,h)}(((ds_g)^{-1} \circ dt_h \circ (ds_h)^{-1})(v), (ds_h)^{-1}(v))) = (dt_g \circ (ds_g)^{-1} \circ dt_h \circ (ds_h)^{-1})(v)$$

and

$$ds_{gh}(dm_{(g,h)}(((ds_g)^{-1} \circ dt_h \circ (ds_h)^{-1})(v), (ds_h)^{-1}(v))) = v.$$

All in all we see that

$$\begin{aligned} A_{gh}(v) &= dt_{gh}((ds)^{-1}_{gh}(v)) \\ &= dt_{gh}(dm_{(g,h)}(((ds_g)^{-1} \circ dt_h \circ (ds_h)^{-1})(v), (ds_h)^{-1}(v))) \\ &= (dt_g \circ (ds_g)^{-1} \circ dt_h \circ (ds_h)^{-1})(v) \\ &= A_g(A_h(v)). \end{aligned}$$

1.2.2 Lie algebroids

Definition 1.2.23 A *Lie algebroid* is a vector bundle $A \to M$ together with the following two points of data:

- A Lie bracket $[-,-]$ on $\Gamma(A)$
- A vector bundle map $\rho\colon A \to TM$

that are compatible in the sense that ρ encodes how $[-,-]$ fails to be $C^\infty(M)$-linear, i.e. for $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$ it holds that

$$[\alpha, f\beta] = f[\alpha,\beta] + \rho(\alpha)(f)\beta.$$

The main class of examples consists of Lie algebroids induced by Lie groupoids, following a procedure similar to how a Lie group induces a Lie algebra.

Proposition 1.2.24 Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and let $A(\mathcal{G})$ be the vector bundle given by $\ker ds|_M \to M$. Write $\mathfrak{X}(\mathcal{G})_{\text{inv}}$ for the space of right-invariant vector fields on $\mathcal{G}$: vector fields $X \in \mathfrak{X}(\mathcal{G})$ which are tangent to the s-fibres and satisfy $dR_g(X(h)) = X(hg)$.

1. A section $\alpha \in \Gamma(A(\mathcal{G}))$ can be extended to a right-invariant vector field $\overrightarrow{\alpha} \in \mathfrak{X}_{\text{inv}}(\mathcal{G})$ by the formula $\overrightarrow{\alpha}(g) = dR_g(\alpha(t(g)))$,
2. The map $\alpha \mapsto \overrightarrow{\alpha}$ is a linear isomorphism between $\Gamma(A(\mathcal{G}))$ and $\mathfrak{X}_{\text{inv}}(\mathcal{G})$,
3. The Lie bracket of two right-invariant vector fields on $\mathcal{G}$ is right-invariant,
4. The vector bundle $A(\mathcal{G})$ together with the bracket induced by the previous points and $\rho = dt$ is a Lie algebroid.

Remark 1.2.25 For Lie groups and algebras, Lie's Third Theorem states that every finite dimensional Lie algebra is induced by some Lie group. As discussed in [CF03] the equivalent statement is false for Lie groupoids and Lie algebroids.

Using the Lie algebroid associated to a Lie groupoid, we can write down a Lie groupoid dual to the tangent groupoid.

Example 1.2.26 (Cotangent groupoid) Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, we have the cotangent groupoid $T^*\mathcal{G} \rightrightarrows A(\mathcal{G})^*$. Its source map takes $\xi \in T^*_g\mathcal{G}$ and maps it to $s(\xi) \in A(\mathcal{G})^*_{s(g)}$ given by the formula

$$s(\xi)(v) = -\xi(d(L_g \circ \iota)_{u(s(g))}v)$$

where $v \in \ker(ds)_{u(s(g))}$. Similarly, the target $t(\xi) \in A(\mathcal{G})^*_{t(g)}$ is given by

$$t(\xi)(w) = \alpha(d(R_g)_{u(t(g))}w).$$

The multiplication of two composable elements $\xi \in T^*_g\mathcal{G}$ and $\zeta \in T^*_h\mathcal{G}$, is given by $m(\xi,\zeta) \in T^*_{gh}\mathcal{G}$ determined by the formula

$$m(\xi,\zeta)(dm_{(g,h)}(v,w)) = \xi(v) + \zeta(w).$$

The fact that this is well-defined, i.e. $\xi(v) = -\zeta(w)$ whenever $dm_{(g,h)}(v,w) = 0$ is precisely the fact that $s(\xi) = t(\zeta)$.

Next, the unit associated to an element $\alpha \in A(\mathcal{G})^*_x$ is given by $u(\alpha) \in T^*_{u(x)}\mathcal{G}$ given by the formula

$$u(\alpha)(v) = \alpha(v - d(u \circ s)v).$$

Lastly, the inversion is given by $\iota(\xi) = \iota^*\xi$.

Using Proposition 1.2.24, we can take the examples of the previous section to produce examples of Lie algebroids.

Example 1.2.27 (Trivial algebroids) The trivial vector bundle $\underline{0} \to M$ canonically has the structure of a Lie algebroid with the zero bracket and the zero map $\underline{0} \to TM$ as anchor. This example is the Lie algebroid associated to any étale groupoid over M, in particular trivial groupoids.

Example 1.2.28 (Action algebroids) When $\xi\colon \mathfrak{g} \to TM$ is any Lie algebra-morphism (i.e. a Lie algebra action), we can induce the structure of a Lie algebroid on the trivial vector bundle $M \times \mathfrak{g} \to M$. Sections of this vector bundle are smooth functions from M to $\mathfrak{g}$ and elements of that are finite sums of functions of the form fc_v for $f \in C^\infty(M)$ and $v \in \mathfrak{g}$, where by this notation we mean $(fc_v)(x) = f(x)v$. The bracket on $C^\infty(M,\mathfrak{g})$ is then given by

$$[fc_v, gc_w] = fgc_{[v,w]} + f\xi(v)(g)c_w - g\xi(w)(f)c_v.$$

If M is a manifold with a smooth left G-action, then we can induce an action on M by the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of G by the formula

$$\xi(v)(x) = \left.\frac{d}{dt}\right|_{t=0} (e^{tv} \cdot x).$$

The action algebroid associated to this action is then the Lie algebroid associated to the original action groupoid.

Example 1.2.29 (Tangent bundle) When M is any manifold, the tangent bundle $TM \to M$ canonically exhibits the structure of a Lie algebroid, with the bracket given by the Lie bracket of vector fields, and the anchor $TM \to TM$ given by the identity.

This is the Lie algebroid associated to both the pair groupoid $M \times M \rightrightarrows M$ and the homotopy groupoid $\Pi_1(M) \rightrightarrows M$.

Example 1.2.30 (Foliation algebroids) A foliation $\mathcal{F}$ on M is the same thing as a Lie algebroid on M with injective anchor, since both can be described as a vector subbundle of TM whose sections are closed under the Lie bracket of vector fields. Foliation algebroids have been studied extensively by Winkelnkemper [Wi83].

In general, this algebroid is the algebroid associated to the monodromy groupoid of the foliation. If the foliation is given by the fibres of a submersion, then the algebroid associated to the submersion-groupoid is also the algebroid associated to the foliation. This is the generalization of the fact that the pair groupoid and the fundamental groupoid have the same algebroid associated to them.

Example 1.2.31 (Bundle of Lie algebras) If $A \to M$ is a Lie algebroid such that the anchor $A \to TM$ is zero, the Lie bracket is $C^\infty(M)$-bilinear. In particular, using bump-functions we can localize it to any fibre to obtain fibrewise Lie brackets. This presents Lie algebroids with vanishing anchor as bundles of Lie algebras, i.e. vector bundles $A \to M$ with a smooth family of Lie brackets $[-,-]_x$ on A_x for $x \in M$ with the bracket of $\Gamma(A)$ given by

$$[\alpha, \beta](x) = [\alpha(x), \beta(x)]_x.$$

If $\mathcal{G} \rightrightarrows M$ is a bundle of Lie groups, the algebroid associated to it is simply the bundle of corresponding Lie algebras.

In particular, looking at the case where all the brackets themselves also vanish, we have a commuting triangle where a vector bundle $E \to M$ induces a Lie groupoid $E \rightrightarrows M$, and Lie algebroid $E \to M$ with vanishing bracket and anchor, and the Lie algebroid associated to the Lie groupoid $E \rightrightarrows M$ is precisely the Lie algebroid $E \to M$ with vanishing bracket and anchor.

Apart from the examples coming from Lie groupoids, there are also examples which are internal to the theory of Lie algebroids.

Example 1.2.32 (Poisson manifolds) Let M be a manifold with a Poisson bivector $\pi \in \Lambda^2 TM$. Then there is a Lie algebroid structure on T^*M due to Coste, Dazord and Weinstein [CDW87] in the following way: the anchor is given by the induced map $\pi^\sharp\colon T^*M \to TM$ and the bracket is given for $\alpha, \beta \in \Omega^1(M)$ by

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp\alpha}(\beta) - \mathcal{L}_{\pi^\sharp\beta}(\alpha) - d(\pi(\alpha, \beta)).$$

1.2.3 Lie-Rinehart algebras

There is a way to lift the definition of a Lie algebroid from the world of geometry to the world of algebra, replacing a vector bundle with an associative algebra, leading to the definition of a Lie-Rinehart algebra.

Definition 1.2.33 A *Lie-Rinehart algebra* is the combination of a commutative algebra R, a Lie algebra L, a R-module structure on the underlying vector space of L and a Lie algebra map $\rho\colon L \to \mathrm{Der}(R)$ such that

$$[\alpha, f\beta] = f[\alpha, \beta] + \rho(\alpha)(f)\beta$$

for $\alpha, \beta \in L$ and $f \in R$.

Remark 1.2.34 If $R = C^\infty(M)$ and L is a finitely generated R-module, the Serre-Swan Theorem states that there is a vector bundle $A \to M$ such that $L = \Gamma(A)$, and the rest of the structure can be translated in such a way that we recover the definition of a Lie algebroid over M.

Apart from this there are certain examples which will come back later in our consideration.

Example 1.2.35 For any associative algebra R, the space of derivations $\mathrm{Der}(R)$, together with the commutator bracket as Lie bracket and the identity as anchor, describes a Lie-Rinehart algebra.

Example 1.2.36 If M is any R-module, we may see (M, R) as a Lie-Rinehart algebra by setting the bracket and anchor to 0.

There are also examples which are explicitly associated to physics-phenomena, for instance there is a Lie-Rinehart algebra constructed by Blohmann, Schiavina and Weinstein [BSW22], relating to constraints of initial value problems in General Relativity.

1.2.4 The universal enveloping algebra of a Lie-Rinehart algebra

Let (L, R) be a Lie-Rinehart algebra. Out of it, we can create a Lie algebra-structure on $L \oplus R$, where we define the bracket by

$$[(\alpha, f), (\beta, g)] \coloneqq ([\alpha, \beta], \rho(\alpha)(f) - \rho(\beta)(g)).$$

Then, from this Lie algebra, we can make the universal enveloping algebra $\mathcal{U}(L \oplus R)$, and from this we can make the universal enveloping algebra of the Lie-Rinehart algebra.

Definition 1.2.37 The universal enveloping algebra is defined by

$$\mathcal{U}(L, R) = \mathcal{U}(L \oplus R)/\langle fX - f \cdot X, fg - f \cdot g,\ f, g \in R,\ X \in L\rangle$$

the quotient of the universal enveloping algebra of the Lie algebra $L \oplus R$ by the ideal generated by the elements $fX - f \cdot X$ and $fg - f \cdot g$ for $f, g \in R$ and $X \in L$.

Remark 1.2.38 The universal enveloping algebra $\mathcal{U}(L, R)$ is the algebra linearly generated by L and R, with the relations

$$\begin{aligned} fX &= f \cdot X, & XY - YX &= [X, Y], \\ Xf - fX &= \rho(X)f, & fg &= f \cdot g. \end{aligned}$$

As such, it is the universal algebra with respect to having maps $\iota_R \colon R \to A$ and $\iota_L \colon L \to A$ to an algebra A such that

- ι_R is a algebra map,
- ι_L is a Lie algebra map with respect to the commutator bracket on A,
- $\iota_L(fX) = \iota_R(f)\iota_L(X)$ for $f \in R$ and $X \in L$,
- $\iota_L(X)\iota_R(f) - \iota_R(f)\iota_L(X) = \iota_R(\rho(X)f)$ for $f \in R$ and $X \in L$.

The main example to have in mind when taking the universal enveloping algebra is for the case of vector fields over a manifold:

Example 1.2.39 For the Lie-Rinehart algebra coming from the algebroid $TM \to M$ we have $\mathcal{U}(\mathfrak{X}(M), C^\infty(M)) = \mathrm{Diff}(M)$, the algebra of differential operators on M.

Example 1.2.40 More generally, for a Lie algebroid $A(\mathcal{G}) \to M$ associated to a Lie groupoid, there is a canonical identification between the universal enveloping algebra $\mathcal{U}(\Gamma(A(\mathcal{G})), C^\infty(M))$ and right-invariant differential operators on $\mathcal{G}$.

In general the universal enveloping algebra is not commutative, but it admits the structure of an almost commutative algebra.

Definition 1.2.41 An *almost commutative algebra* A is an associative filtered algebra, where the filtration satisfies that if $a_1 \in A^{\leq k}$ and $a_2 \in A^{\leq l}$ then $a_1a_2 - a_2a_1 \in A^{\leq k+l-1}$.

Lemma 1.2.42 The filtration on $\mathcal{U}(L, R)$ defined by setting that R lives in filtered degree 0 and L lives in filtered degree L makes $\mathcal{U}(L, R)$ into an almost commutative algebra.

Proof. Investigating the commutation relations of the generators R and L, and seeing that for $X, Y \in L$ and $f, g \in R$ we have

$$\begin{aligned} XY - YX &= [X, Y] \in \mathcal{U}(L, R)^{\leq 1}, \\ Xf - fX &= \rho(X)f \in \mathcal{U}(L, R)^{\leq 0}, \\ fg - gf &= 0 \in \mathcal{U}(L, R)^{\leq -1}. \end{aligned}$$

Which shows that $\mathcal{U}(L, R)$ is an almost commutative algebra. □

Since the algebra is almost commutative, the graded quotient is commutative, and so since the filtered quotient is a commutative graded algebra generated on R in degree 0 and L on degree 1, it is quite immediate to try and relate $\mathcal{U}(L, R)$ to $\mathrm{Sym}_R(L)$. Indeed, this is also what one does in the case of a Lie algebra, where the Poincaré-Birkhoff-Witt theorem stipulates that the symmetrization map $\mathrm{Sym}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is a linear isomorphism of filtered vector spaces, whose induced map between the graded quotients is an isomorphism of graded commutative algebras.

For a general Lie-Rinehart algebra, the symmetrization map will not make sense as a map $\mathrm{Sym}_R L \to \mathcal{U}(L, R)$. Indeed, already in degree 2 the map $L^{\times 2} \to \mathcal{U}(L, R)$ given by

$$(X, Y) \mapsto \frac{1}{2}(XY + YX)$$

is not R-bilinear. To see this, we look at the images of (X, fY) and (fX, Y) in $\mathcal{U}(L, R)$ and see that

$$(X, fY) \mapsto \frac{1}{2}(XfY + fYX),$$

$$(fX, Y) \mapsto \frac{1}{2}(fXY + YfX) = \frac{1}{2}(XfY + fYX - \rho(X)(f)Y + \rho(Y)(f)X).$$

However, we see that the deficiency is in lower order terms. By work of Laurent-Gengoux, Stiénon and Xu [LSX21], the deficiency can be resolved using a *connection* on L.

Definition 1.2.43 If (L, R) is a Lie-Rinehart algebra, an *L-connection on L* is an $\mathbb{K}$-bilinear map $\nabla\colon L \times L \to L$ satisfying

$$\nabla_{fX}Y = f\nabla_X Y$$
$$\nabla_X(fY) = f\nabla_X Y + \rho(X)(f)Y$$

for all $X, Y \in L$ and $f \in R$.

Remark 1.2.44 If L is projective as an R-module, then it admits an L-connection (see [AF74]).

Using a connection ∇, we can write down a variant of the PBW-map, following Laurent-Gengoux, Stiénon and Xu [LSX21].

Definition 1.2.45 Under the choice of an L-connection ∇ on L, the PBW-map

$$\mathrm{pbw}^{\nabla}\colon \mathrm{Sym}_R L \to \mathcal{U}(L, R)$$

is recursively defined by

$$\mathrm{pbw}^{\nabla}(f) = f, \qquad (f \in R)$$

$$\mathrm{pbw}^{\nabla}(X) = X, \qquad (X \in L)$$

and for $X_1, ..., X_n \in L$ by

$$\begin{aligned} \mathrm{pbw}^{\nabla}(X_1 \odot \cdots \odot X_n) =& \frac{1}{n}\sum_{i=1}^{n} X_i \mathrm{pbw}^{\nabla}(X_1 \odot \cdots \widehat{X_i} \cdots \odot X_n) \\ &- \frac{1}{n}\sum_{1 \le i \ne j \le n} \mathrm{pbw}^{\nabla}(\nabla_{X_i}X_j \odot X_1 \odot \cdots \widehat{X_i}\widehat{X_j} \cdots \odot X_n). \end{aligned}$$

We remark that this is indeed well-defined, and also R-linear. To see this, we note that it is clearly symmetric and $\mathbb{K}$-multilinear, so that we only need to see what happens when we replace X_1 by fX_1, and we get

$$\begin{aligned}
\mathrm{pbw}^\nabla((fX_1)\odot X_2\cdots\odot X_n) =&\frac{1}{n}(fX_1)\mathrm{pbw}^\nabla(X_2\odot\cdots\odot X_n) && \text{(a)}\\
&+\frac{1}{n}\sum_{i=2}^n X_i\mathrm{pbw}^\nabla((fX_1)\odot X_2\odot\cdots\widehat{X_i}\cdots\odot X_n) && \text{(b)}\\
&-\frac{1}{n}\sum_{j=2}^n \mathrm{pbw}^\nabla(\nabla_{fX_1}X_j\odot X_2\odot\cdots\widehat{X_j}\cdots\odot X_n) && \text{(c)}\\
&-\frac{1}{n}\sum_{i=2}^n \mathrm{pbw}^\nabla(\nabla_{X_i}(fX_1)\odot X_2\odot\cdots\widehat{X_j}\cdots\odot X_n) && \text{(d)}\\
&-\frac{1}{n}\sum_{2\le i\neq j\le n} \mathrm{pbw}^\nabla(\nabla_{X_i}X_j\odot(fX_1)\odot X_2\odot\cdots\widehat{X_i}\widehat{X_j}\cdots\odot X_n) && \text{(e)}
\end{aligned}$$

Now we can use the commutation relations in $\mathcal{U}(L,R)$, the properties of the connection, and the inductive assumption that pbw^∇ is well-defined and R-linear when restricted to $\mathrm{Sym}_R^{<n}L$ to work the f forward, and we obtain

$$\begin{aligned}
\text{(a)} =&\frac{1}{n}f(X_1\mathrm{pbw}^\nabla(X_2\odot\cdots\odot X_n))\\
\text{(b)} =&\frac{1}{n}\sum_{i=2}^n f(X_i\mathrm{pbw}^\nabla(X_1\odot\cdots\widehat{X_i}\cdots\odot X_n))\\
&+\frac{1}{n}\sum_{i=2}^n \rho(X_i)(f)\mathrm{pbw}^\nabla(X_1\odot\cdots\widehat{X_i}\cdots\odot X_n) && (*)\\
\text{(c)} =&-\frac{1}{n}\sum_{j=2}^n f\mathrm{pbw}^\nabla(\nabla_{X_1}X_j\odot X_2\odot\cdots\widehat{X_j}\cdots\odot X_n)\\
\text{(d)} =&-\frac{1}{n}\sum_{i=2}^n f\mathrm{pbw}^\nabla(\nabla_{X_i}X_1\odot X_2\odot\cdots\widehat{X_i}\cdots\odot X_n)\\
&-\frac{1}{n}\sum_{i=2}^n \rho(X_i)(f)\mathrm{pbw}^\nabla(X_1\odot\cdots\widehat{X_i}\cdots\odot X_n) && (**)\\
\text{(e)} =&-\frac{1}{n}\sum_{2\le i\neq j\le n} f\mathrm{pbw}^\nabla(\nabla_{X_i}X_j\odot X_1\odot\cdots\widehat{X_i}\widehat{X_j}\cdots\odot X_n)
\end{aligned}$$

We see that $(*)$ and $(**)$ cancel each other, and the remaining terms precisely reconstruct $f(\mathrm{pbw}^\nabla(X_1\odot\cdots\odot X_n))$. So we can indeed conclude that pbw^∇ is a well-defined R-linear map $\mathrm{Sym}_R L\to\mathcal{U}(L,R)$. The important result is then:

Theorem 1.2.46 [LSX21, Thm 2.2] For any connection ∇, the map $\mathrm{pbw}^\nabla\colon \mathrm{Sym}_R L\to\mathcal{U}(L,R)$ is an isomorphism of filtered R-modules and induces an isomorphism of R-algebras between $\mathrm{Sym}_R L$ and the graded quotient of $\mathcal{U}(L,R)$.

Chapter 2

Lie groupoid deformations and convolution algebras

In this chapter we discuss the results obtained [KP21]. Apart from the last section of this chapter, most is a transcription of this paper, with some extra remarks or exposition which were deemed appropriate for this text.

The main results of this chapter are a relationship between the deformation theory of a Lie groupoid and the deformation theory of its convolution algebra, and showing that a localization principle in connection with the deformation theory of the Lie algebroid can be interpreted as a 'classical limit' of our connection with the convolution algebra.

The deformation complex $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ for a Lie groupoid $\mathcal{G} \rightrightarrows M$ was introduced by Crainic, Mestre and Struchiner [CrMS20] to encode the deformation theory of the Lie groupoid. We link it to the Hochschild complex $\mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ of the groupoid. The convolution algebra, introduced first by Renault [Re80] using counterparts to Haar measures and defined intrinsically using half-densities by Connes [Co82], forms the main algebra encoding the group-like properties of the groupoid. Slightly changing the construction of Connes to make it more digestible in connection to the deformation complex, we define the convolution algebra using s-fibred densities. This allows us to properly define the convolution product by the formula

$$(a_1 * a_2)(g) = \int_{h \in s^{-1}(s(g))} a_1(gh^{-1}) a_2(h).$$

In the end, we define a cochain map

$$\Phi \colon \mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$$

roughly by exhibiting deformation elements as vector fields (with parameters) and then interpreting those as Hochschild cochains by applying a Lie-derivative-like procedure. We will show that one of the main properties this chain map possesses is that it links groupoid deformations to their induced algebra deformations. Following [CrMS20], we encounter a class of deformations of a Lie groupoid $\mathcal{G}$ which deform the division map $\overline{m}$ on $\mathcal{G}\,{}^s\!\times^s \mathcal{G}$ to a family $\overline{m}_\epsilon$ of division maps, inducing an element $\xi \in \mathrm{C}^2_{\mathrm{def}}(\mathcal{G})$ by

$$\xi(g,h) \coloneqq \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \overline{m}_\epsilon(gh,h),$$

and also a deformation element $\beta \in \mathrm{C}^2_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ describing the induced deformation of the convolution product

$$\beta(a_1,a_2)(g) \coloneqq \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \int_{h\in s^{-1}(s(g))} a_1(\overline{m}_\epsilon(g,h))a_2(h).$$

We will show that our chain map Φ satisfies

$$\Phi(\xi) = \beta$$

establishing the map Φ as a key connection between the deformation theory of the Lie groupoid and the deformation theory of the convolution algebra.

Furthermore, there is a localization procedure to relate the deformation theory of $\mathcal{G}$ and that of its algebroid $A(\mathcal{G})$. In the case where $\mathcal{G}$ is a Lie group G, this goes back to the work of van Est [vE53a, vE53b], and in the general context of this deformation complex the van Est map

$$\mathcal{V}\colon \mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{def}}(A(\mathcal{G}))$$

was defined by Crainic, Mestre and Struchiner [CrMS20]. By seeing the convolution algebra $\mathcal{A}_\mathcal{G}$ as a deformation quantization of the Poisson manifold $A(\mathcal{G})^*$ induced by the algebroid, following Landsman and Ramazan [LR01], we will show that we can see the convolution algebra $\mathcal{A}_\mathcal{G}$ as a kind of deformation quantization of the Poisson manifold $A(\mathcal{G})^*$ by quantization maps

$$q_t\colon \mathcal{S}(A(\mathcal{G})^*) \to \mathcal{A}_\mathcal{G}.$$

We will use this to show that the van Est-map can be interpreted as a 'classical limit' of our map Φ:

$$\mathcal{V}(c)(f_1,\dots,f_k) = \mathcal{F}_\mu\left(\lim_{t\to 0}\left(\sum_{\sigma\in S_k}(-1)^\sigma \frac{1}{(it)^{k-1}}\Phi(c)(q_t(f_{\sigma(1)}),\dots,q_t(f_{\sigma(k)}))\right)\right)$$

where $c \in \mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$, $f_1, ..., f_k \in \mathcal{S}(A(\mathcal{G})^*)$ and $\mathcal{F}_\mu$ is a Fourier transform.

Lastly, we discuss the relationship the deformation complex has with the adjoint representation up to homotopy as defined by Abad and Crainic [AC13], and sketch ideas on how to use our chain map, the Gerstenhaber structure on the Hochschild complex and differential operators to obtain intrinsic models for the tensor powers of this representation.

2.1 The convolution algebra of a groupoid

Convolution algebras form a central tool for the study of the structure of given groups, as they are in some sense the 'easiest' associative algebras encoding the group structure. However, between certain classes of groups, there are many variants on the precise, or common, definition of the convolution algebra, as we see in the following examples.

Example 2.1.1 (Finite groups) For a finite group G the space of all maps $f\colon G \to \mathbb{C}$ constitutes the convolution algebra, spanned as a linear space by the functions δ_g for $g \in G$ which are 1 at g and 0 elsewhere. The product is then simply given by $\delta_g * \delta_h = \delta_{gh}$, or, if one writes it internal to the definition with functions, as

$$(f_1 * f_2)(g) := \sum_{h \in G} f_1(gh^{-1}) f_2(h).$$

It is a classical result of Webberburn [St12, Thm 5.5.6] that the convolution algebra $\mathcal{A}_G = \{f\colon G \to \mathbb{C}\}$ contains all the information about the irreducible representations of G:

$$\mathcal{A}_G \cong \bigoplus_{\pi \in \widehat{G}} \mathrm{End}(V_\pi).$$

Example 2.1.2 (Compact Lie groups) When G is a compact Lie group, the group algebra does not convey the right information, indeed one loses the information about G being a manifold, indeed here is it more natural to consider smooth functions $C^\infty(G, \mathbb{C})$. Then the sum from the previous example needs to be replaced by an integral. To do this, we choose a Haar measure $d\lambda$ and define the convolution product $*$ by

$$(f_1 * f_2)(g) := \int_G f_1(gh^{-1}) f_2(h) d\lambda(h).$$

Square integrating against $d\lambda$ induces a space of L^2-functions on G with a convolution product, and similar to the finite case the Peter-Weyl Theorem [Kn86, Thm 1.12] in essence states that all the relevant information about the group is contained in the convolution algebra

$$L^2(G) \cong \widehat{\bigoplus}_{\pi \in \widehat{G}} \mathrm{End}(V_\pi).$$

Restricting this to smooth functions, Fourier inversion exhibits the smooth convolution algebra as a subalgebra of the right hand side above, consisting of those functions which are rapidly decreasing in a suitable fashion (see [NN90] for details).

However, even though Haar measures are well-defined up to a constant, picking a Haar measure still involves making a choice. To counteract this, we need objects that are canonically integrable, and this what densities do. To see how this makes the convolution algebra of a Lie group a canonical object, and also how they help define the convolution algebra of a Lie groupoid, we discuss densities along the fibres of a submersion.

2.1.1 Densities along the fibres of a submersion

Definition 2.1.3 Let V be an n-dimensional vector space. A *density* on V is a map $a\colon \Lambda^n V \to \mathbb{R}$ such that for every invertible linear map $A \in \mathrm{GL}(n,\mathbb{R})$ it holds that

$$a(Av_1, ..., Av_n) = |\det(A)| a(v_1, ..., v_n).$$

Remark 2.1.4 The collection of densities on V forms a linear space that is 1-dimensional.

More generally, from a vector bundle $E \to M$, one constructs a line bundle of densities $\mathcal{D}_E \to M$. Then if one has a vector bundle isomorphism $\Psi\colon E \to E$ covering a diffeomorphism $\Phi\colon M \to M$, one obtains an action on the sections of $\mathcal{D}_E$, defined by

$$(\Psi^* a)_x(v_1, ..., v_n) := a_{\Phi(x)}(\Psi v_1, ..., \Psi v_n).$$

Remark 2.1.5 If $E \to M$ is a rank n vector bundle and $U_1, U_2 \subset M$ are opens over which E trivializes with transition function $\varphi\colon U_1 \cap U_2 \to \mathrm{GL}(n,\mathbb{R})$, then $\mathcal{D}_E$ also trivializes over U_1 and U_2 with transition function $|\det| \circ \varphi\colon U_1 \cap U_2 \to \mathbb{R}^\times$. As such, we can always find a cover of local trivializations for $\mathcal{D}_E$ such that all the transition functions are positive, so that we can conclude that for every vector bundle $E \to M$ the density bundle $\mathcal{D}_E \to M$ admits a nowhere-vanishing section.

Remark 2.1.6 There is a clear connection between the bundle of densities of E and the bundle of top forms $\Lambda^{\mathrm{top}} E^*$, most canonically obtained by the absolute-linear map sending a form ω to its absolute value $|\omega|$.

Importantly, over domains where $\Lambda^{\mathrm{top}} E^*$ trivializes, i.e. domains over which E is oriented, there is an isomorphism between both bundles, canonical after choosing an orientation. In this case one sends a top-form to its absolute value if it is positively oriented and to minus its absolute value if it is negatively oriented.

Example 2.1.7 The case $E = TM$ is of particular interest, because for a compactly supported section a of $\mathcal{D}_{TM}$ the integral $\int_M a$ is canonically defined. Indeed, differential top-forms transform like the determinant, while coordinate transformations of the Lesbegue integral introduce a factor of the absolute value of the determinant of the Jacobian. To make integral of a top-form well-defined this results in the need of an atlas where all determinants of the all Jacobians are positive, i.e. an orientation of M. Since we define a density to transform with the absolute value of the determinant, this problem does not arise, and hence integrals are canonically defined, even for non-orientable manifolds.

Further in the case $E = TM$, one obtains an action of a vector field $X \in \mathfrak{X}(M)$ on the densities on TM, namely:

$$Xa := \left.\frac{d}{dt}\right|_{t=0} (\Phi_X^t)^* a, \tag{2.1}$$

where Φ_X^t denotes the flow of X.

We will be mostly interested in densities along the fibers of a submersion. For this, let $f\colon M \to N$ be a submersion, and denote by $\mathcal{D}_f$ the bundle of densities of the vector bundle $\ker df \subset TM$. In this case, by doing the density-integration along every f-fiber $f^{-1}(x)$ for $x \in N$ the fiber integral

$$\int_f \colon \Gamma_c(M, \mathcal{D}_f) \to C_c^\infty(N)$$

is canonically defined. Now for a vector field to act on $\mathcal{D}_f$ in a canonical way, we need to make sure that the flow of the vector field preserves the distribution $\ker(df)$.

Definition 2.1.8 Let $f\colon M \to N$ be a submersion. A vector field $X \in \mathfrak{X}(M)$ is called *f-projectable* if there is a vector field $\sigma(X) \in \mathfrak{X}(N)$ such that $df \circ X = \sigma(X) \circ f$. We denote the space of f-projectable vector fields by $\mathfrak{X}_f(M)$. The projection $\sigma(X)$ (which is unique if X is f-projectable) will also be called the *symbol* of X.

Lemma 2.1.9 Let $X \in \mathfrak{X}_f(M)$ be a complete vector field, the following hold:

- The flow of X preserves the fibres of f, and in turn the distribution $\ker(df)$.
- The formula (2.1) induces a well defined action of X on $\Gamma(\mathcal{D}_f)$.

Proof. The first item is directly checked by noting that $\Phi_X^t \circ f = f \circ \Phi_{\sigma(X)}^t$, and in turn we see that if $\gamma\colon \mathbb{R} \to M$ maps into a single fibre of f, so does $\Phi_X^t \circ \gamma$ and hence $d\Phi_X^t$ preserves $\ker(df)$ for every t.

The second item is then immediate since (2.1) is well-defined in this case, as $(\Phi_X^t)^*$ is well-defined as an operator on $\Gamma(\mathcal{D}_f)$ for all $t \in \mathbb{R}$. $\square$

Remark 2.1.10 We can extend the action of complete vector fields to that of non-complete vector fields. Indeed, while $(\Phi_X^t)^* a$ may not be globally defined on any open neighbourhood of $t = 0$, fixing a point $p \in M$, $((\Phi_X^t)^* a)_p$ can be defined on an open neighbourhood of $t = 0$.

In the next Lemma, we describe the locality of the action of projectable vector fields on densities along the fibres. This is essentially the same statement and proof as the properties of the Lie-derivative on differential forms.

Lemma 2.1.11 Let $a \in \Gamma(\mathcal{D}_f)$, $y \in N$ and $x \in f^{-1}(y)$, and let $X \in \mathfrak{X}_f(M)$ be an f-projective vector field. If X vanishes on an open neighbourhood of x in $f^{-1}(y)$, then $(Xa)_x = 0$.

Proof. Let $U \subset f^{-1}(y)$ be an open neighbourhood of x such that $X(p) = 0$ for all $p \in U$. Then $\Phi_X^t(p) = p$ for all $p \in U$. This means that for all $v \in \ker(df)_x$ we have $d\Phi_X^t(v) = v$ so that for all t it holds that $((\Phi_X^t)^* a)_x = a_x$, so that $(Xa)_x = 0$. $\square$

Remark 2.1.12 The previous Lemma allows us to define $(Xa)_x$ for $x \in f^{-1}(y)$, $a \in \Gamma(\mathcal{D}_f)$ and $X \in \mathfrak{X}_f(M)|_{f^{-1}(y)}$. Indeed, we can choose $Y \in \mathfrak{X}_f(M)$ to be an extension of X to a global vector field and define $(Xa)_x = (Ya)_x$. The previous Lemma then implies that this definition is independent of the choice of Y.

2.1.2 The convolution algebra

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Heuristically, we want to write down a convolution product which looks like

$$(f_1 * f_2)(g) = \int_{g_1 g_2 = g} f_1(g_1) f_2(g_2)$$

where f_1, f_2 are objects which are like functions $\mathcal{G} \to \mathbb{C}$. Since not all arrows in $\mathcal{G}$ can be multiplied, the rewriting to an integral over a single explicit variable changes a bit, into the following

$$(f_1 * f_2)(g) = \int_{h \in s^{-1}(s(g))} f_1(gh^{-1}) f_2(h)$$

From this heuristic picture we see that to define such a product, we need to have objects which can be integrated along source fibres. This is precisely why we discussed densities before, and so we set the underlying space of the convolution algebra to be compactly supported smooth densities along the source fibres of the groupoid.

To define the convolution product, we first note that there is a canonical isomorphism between $\ker ds$ and $t^*A(\mathcal{G})$ using right translations. In particular, for $g \in \mathcal{G}$ the differential of the map $dR_g \colon s^{-1}(t(g)) \to s^{-1}(s(g))$ at the unit $u(t(g))$ induces an isomorphism between $\ker(ds)_{u(t(g))} = A_{t(g)}$ and $\ker(ds)_g$. Using this identification, we define the convolution product for two compactly supported densities $a_1, a_2 \in \Gamma_c(\mathcal{D}_s)$ by

$$(a_1 * a_2)_g(v_1, ..., v_n) := \int_{h \in s^{-1}(s(g))} (a_1)_{gh^{-1}}(v_1, ..., v_n)(a_2)_h.$$

In this notation $v_1, ..., v_n \in A_{t(g)} = A_{t(gh^{-1})}$ so that the product in the integrand yields a well-defined compactly supported density along $s^{-1}(s(g))$ that can be integrated. Colloquially this product will be written as:

$$(a_1 * a_2)(g) = \int_{g_1 g_2 = g} a_1(g_1) a_2(g_2) = \int_{h \in s^{-1}(s(g))} a_1(gh^{-1}) a_2(h)$$

We define the convolution algebra $\mathcal{A}_{\mathcal{G}}$ of $\mathcal{G}$ to be $\mathcal{A}_{\mathcal{G}} = (\Gamma_c(\mathcal{D}_s), *)$. This definition of the convolution algebra differs slightly (but is isomorphic as a complex algebra) from the more usual one in e.g. [Co82] using 1/2-densities along source *and* target fibres. The usage of half-densities has the advantage that it allows for a C*-algebra to be made out of a completion of the convolution algebra, with the differential of the inverse map $\iota \colon \mathcal{G} \to \mathcal{G}$ inducing the involution. We do not need a C*-structure in the discussion that follows, so we stick to the construction using source-fibred densities.

2.1.3 Haar systems

If we want to align our convolution algebra more with known examples, where the underlying space of the algebra is the space of smooth functions, we can backtrack the path we took in the example of compact Lie groups (2.1.2). In particular we can replace integrating tensor-like objects with integrating functions against a measure. This leads to the notion of (right) Haar-systems.

Definition 2.1.13 A *right Haar-system* on a groupoid $\mathcal{G} \rightrightarrows M$ is a collection of measures $\{\lambda^x\}_{x\in M}$, where λ^x is a Borel measure on $s^{-1}(x)$, that satisfies three properties:

- (Positivity) For every $x \in M$ and every non-empty open subset $U \subset s^{-1}(x)$ it holds that $\lambda^x(U) > 0$;
- (Right-invariance) For every $g \in \mathcal{G}$ and every $f \in C_c^\infty(\mathcal{G})$ it holds that
$$\int_{s^{-1}(t(g))} f(hg)d\lambda^{t(g)}(h) = \int_{s^{-1}(s(g))} f(h)d\lambda^{s(g)}(h);$$
- (Smoothness) For every $f \in C_c^\infty(\mathcal{G})$ it holds that the function $\lambda(f)$ on M defined by
$$\lambda(f)(x) := \int_{s^{-1}(x)} f(h)d\lambda^x(h)$$
is smooth.

Remark 2.1.14 The second property is essentially the same as asking for the right translation by g to be a measure preserving diffeomorphism

$$R_g\colon (s^{-1}(t(g)), \lambda^{t(g)}) \to (s^{-1}(s(g)), \lambda^{s(g)}).$$

If we have chosen a Haar system we can define a convolution product on the space of compactly supported functions $C_c^\infty(\mathcal{G})$ by

$$(f_1 * f_2)(g) := \int_{s^{-1}(s(g))} f_1(gh^{-1})f_2(h)d\lambda^{s(g)}(h)$$

Now given a positive section $a \in \mathcal{D}_s$, we can cook up a collection of Borel measures on the source fibres by setting

$$\lambda_a^x(U) := \int_U a|_U$$

With this definition, the last property of a Haar system is automatic since a is a smooth section. The first property is taken care of if the section a has no zeroes. The second property is taken care of by this Lemma of which the proof is clear:

Lemma 2.1.15 Given $a \in \Gamma(\mathcal{D}_s)$, the collection of measures λ_a is right-invariant if and only if $a(h)$ depends only on $t(h)$ under the right trivialization $\ker(ds) \cong t^*A(\mathcal{G})$, i.e. is determined by a section of $\mathcal{D}_{A(\mathcal{G})} \to M$.

This process can essentially be reversed, so that given a Haar system λ we find a density a inducing it. It is then clear that the map $f \mapsto fa$ is an isomorphism between $C_c^\infty(\mathcal{G})$ with the λ-convolution product and $\Gamma_c(\mathcal{D}_s)$ with the intrinsic convolution product.

Note however, that this isomorphism does *not* preserve the action of s-projectable vector fields, since for any $X \in \mathfrak{X}_s(\mathcal{G})$, $f \in C_c^\infty(\mathcal{G})$ and $a \in \Gamma(\mathcal{D}_s)$ we have

$$X(fa) = X(f)a + fXa.$$

In the end we see that any nowhere vanishing section of $\mathcal{D}_{A(\mathcal{G})} \to M$ induces a Haar system. As any density bundle has a nowhere vanishing section, we obtain the following result, originally due to Westman [We67].

Proposition 2.1.16 Any Lie groupoid $\mathcal{G} \rightrightarrows M$ admits a right Haar system.

2.1.4 Examples

We discuss some examples.

Example 2.1.17 (Trivial groupoids) For a trivial groupoid $M \rightrightarrows M$, the source map is a diffeomorphism, so $\ker(ds)$ is the zero bundle. By convention $\Lambda^0 V = \mathbb{R}$ for any vector space V, and so $\mathcal{D}_s$ is a trivializable line bundle, with canonical trivialization given by the section that sends every point to the identity $\mathbb{R} \to \mathbb{R}$. This trivialization induces an isomorphism $\Gamma_c(\mathcal{D}_s) \cong C_c^\infty(M)$. Under this isomorphism, the convolution product becomes simply the commutative product

$$(f_1 * f_2)(x) = f_1(x)f_2(x).$$

Note that a Haar system on this groupoid is a collection of non-zero measures on the point spaces $\{x\}$, that 'vary smoothly' with x, i.e. nothing more than a strictly positive smooth map on M. Consequently, we see that the isomorphism above is the same as the isomorphism induced by the Haar system that is constant equal to 1.

Example 2.1.18 (Étale groupoids) Let $\mathcal{G} \rightrightarrows M$ be an étale groupoid. Similar to the example above, the source-fibres density bundle canonically trivializes, and so again $\Gamma_c(\mathcal{D}_s) \cong C_c^\infty(G)$, with the convolution product in this case given by

$$(f_1 * f_2)(g) = \sum_{h \in s^{-1}(s(g))} f_1(gh^{-1})f_2(h)$$

where the integral is replaced by a sum because source fibres are discrete, and furthermore this sum is finite since f_1 and f_2 have compact support (and hence finite support along every source or target fibre).

Similar to the example above, we can identify Haar systems with strictly positive continuous functions on $\mathcal{G}$ that are right invariant, and the canonical isomorphism is associated to the function that is constant equal to 1.

Example 2.1.19 (Lie groups) Since for a Lie group G the source map is the trivial map, the source-fibred densities are simply the densities of G as a manifold.

As for the convolution product the discussion is generally the same as for compact Lie groups discussed previously. Here, the choice of a Haar system is nothing but a Haar measure, which is equivalent to a left-invariant section of the density bundle. The convolution algebra is in this case normally represented by compactly supported functions under the choice of a Haar measure λ with convolution product

$$(f_1 * f_2)(g) = \int_G f_1(gh^{-1})f_2(h)d\lambda(h).$$

Example 2.1.20 (Action groupoids) For the action groupoid $M \times G \rightrightarrows M$ of a manifold M with a right G-action, it holds that $\ker(ds)_{(x,g)} = T_gG$, and so source-fibred densities are simply densities on G, parametrized by M.

For this case, Haar systems are simply choices of measures λ^x on G for every $x \in M$ such that $\lambda^{xg} = \lambda^x \cdot g$. Of course, simply choosing $\lambda^x = \lambda$ for λ some Haar measure on G suffices, and we get a quasi-canonical isomorphism between the convolution algebra $\mathcal{A}_{M\times G}$ of $M \times G$ and $C_c^\infty(M \times G)$ where the convolution product is then given by

$$(f_1 * f_2)(x, g) = \int_G f_1(x, h)f_2(xh, h^{-1}g)d\lambda(h).$$

Example 2.1.21 (Pair groupoids) For a pair groupoid $M \times M \rightrightarrows M$ we have that $\ker(ds)_{(x,y)} = T_xM$ so that $\Gamma_c(\mathcal{D}_s) = \mathcal{D}_c(M)\widehat{\otimes}C_c^\infty(M)$. Now to calculate the convolution product we pick $a_1, a_2 \in \mathcal{D}_c(M)$ and $f_1, f_2 \in C_c^\infty(M)$ and see that

$$((a_1 \otimes f_1) * (a_2 \otimes f_2))(x, y)(v_1, ..., v_n) = \int_M a_1(x)(v_1, ..., v_n)f_1(z)a_2(z)f_2(y)dz$$

which simplifies to saying that

$$(a_1 \otimes f_1) * (a_2 \otimes f_2) = \left(\int_M f_1a_2\right)(a_1 \otimes f_2).$$

By the Schwarz Kernel Theorem, we can see elements of this convolution algebra as kernels of the smoothing operators on M. Under this identification the convolution product is precisely given by composition of smoothing operators, see also [vEY19].

2.2 The deformation complex of a Lie groupoid

The deformation complex of a groupoid, as defined by [CrMS20], is the model for defining cohomology classes associated to deformation problems. We discuss a slightly altered variant of this complex (c.f. Remark 2.2.2 below) that fits into the picture we are sketching between deformations of groupoids and that of convolution algebras.

Definition 2.2.1 For $k \geq 1$ define $\mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$ to be the set of sections of the vector bundle $\mathrm{pr}_1^* T\mathcal{G}$ over $\mathcal{G}^{(k)}$, that are s-projectable in the sense that there is a section s_c of the vector bundle t^*TM over $\mathcal{G}^{(k-1)}$ satisfying

$$ds(c(g_1, ..., g_k)) = s_c(g_2, ..., g_k).$$

The differential $\delta\colon \mathrm{C}^k_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^{k+1}_{\mathrm{def}}(\mathcal{G})$ is defined by setting:

$$\begin{aligned}(\delta c)(g_1, ..., g_{k+1}) \coloneqq &- d\overline{m}(c(g_1 g_2, g_3, ..., g_{k+1}), c(g_2, ..., g_{k+1})) \\ &+ \sum_{i=2}^{k} (-1)^i c(g_1, ..., g_i g_{i+1}, ..., g_{k+1}) + (-1)^{k+1} c(g_1, ..., g_k).\end{aligned}$$

The *deformation complex* is defined by the graded vector space

$$\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) \coloneqq \bigoplus_{k \geq 1} \mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$$

equipped with the differential δ, its cohomology is denoted by $\mathrm{H}^\bullet_{\mathrm{def}}(\mathcal{G})$.

Remark 2.2.2 It is possible, as in [CrMS20], to extend the deformation complex in degree zero by putting $\mathrm{C}^0_{\mathrm{def}}(\mathcal{G}) = \Gamma(M, A(\mathcal{G}))$ with differential defined for $\alpha \in \Gamma(M, A(\mathcal{G}))$ by

$$(\delta\alpha)(g) \coloneqq (dR_g)(\alpha(t(g)) + (d(l_g \circ \iota))(\alpha(s(g))$$

We exclude these elements in degree 0 because, as we will see, these elements cannot correspond to Hochschild 0-cochains.

Remark 2.2.3 The fact that the differential is well-defined (i.e. that δc also has a symbol), follows from the fact that one immediately checks that the following describes a symbol for δc:

$$\begin{aligned}s_{\delta c}(g_2, ..., g_{k+1}) = &- dt(c(g_2, ..., g_{k+1})) \\ &+ \sum_{i=2}^{k} (-1)^i s_c(g_2, ..., g_i g_{i+1}, ..., g_{k+1}) \\ &+ (-1)^{k+1} s_c(g_2, ..., g_k).\end{aligned}$$

Remark 2.2.4 It is shown in [CrMS20, Lem 2.2] that the map δ is indeed a differential, i.e. $\delta^2 = 0$, by using the associativity of the multiplication.

Remark 2.2.5 It follows from the definition above that the closed elements in degree 1 are exactly the multiplicative vector fields, c.f. [CrMS20, §4.3]. These are vector fields $X \in \mathfrak{X}(\mathcal{G})$ that are s and t-projectable to the same image in $\mathfrak{X}(M)$, satisfying the following equation:

$$dm_{(g,h)}(X(g), X(h)) = X(gh),$$

where we used that the tangent space to the nerve $\mathcal{G}^{(2)}$ is given by

$$T_{(g,h)}\mathcal{G}^{(2)} = \{(v, w) \in T_g\mathcal{G} \times T_h\mathcal{G} : ds(v) = dt(w)\}.$$

For certain purposes, most importantly applying the Van Est map, it is often necessary to impose more strict relations on elements $c \in \mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$ and their symbol s_c. To this end we also introduce the normalized deformation complex:

Definition 2.2.6 The *normalized deformation complex* is the subcomplex $\widehat{\mathrm{C}}^\bullet_{\mathrm{def}}(\mathcal{G})$ of $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ consisting of those elements $c \in \mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$ which satisfy

$$c(1_x, g_2, ..., g_k) = du(s_c(g_2, ..., g_k))$$

and

$$s_c(g_2, ..., 1_x, ..., g_k) = 0,$$

where the unit is put in any of the $k-1$ slots.

It is shown in [CrMS20, Prop 11.8] that the inclusion of the normalized deformation complex into the whole deformation complex is a quasi-isomorphism.

2.2.1 Examples

Example 2.2.7 (Trivial groupoids) For a trivial groupoid $\mathcal{G} = M \rightrightarrows M$, the k-nerve $\mathcal{G}^{(k)}$ is always canonically diffeomorphic to M (indeed there are only identities, so every chain of composable arrows is a chain of identities). From this it follows that $\mathrm{pr}_1^* T\mathcal{G} \to \mathcal{G}^{(k)}$ is simply $TM \to M$ with every section having a symbol (namely itself). We conclude that

$$\mathrm{C}^k_{\mathrm{def}}(\mathcal{G}) = \mathfrak{X}(M)$$

for all $k \geq 1$.

As for the differential we have that $d\overline{m}(v, v) = v$ for every $v \in T_{\mathcal{G}}$, so that the differential $\delta\colon \mathrm{C}^k_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^{k+1}_{\mathrm{def}}(\mathcal{G})$ is given by

$$\delta = \begin{cases} \mathrm{id} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

and we conclude that the deformation cohomology of $M \rightrightarrows M$ is concentrated in degree 1 where it is given by $\mathfrak{X}(M)$.

Example 2.2.8 (Étale groupoids) More generally for étale groupoids, every chain in the deformation complex is uniquely determined by its symbol, since

$$s(g_1, ..., g_k) = (ds_{g_1})^{-1}(s_c(g_2, ..., g_k))$$

As such, for an étale groupoid $\mathcal{G} \rightrightarrows M$ we have an equivalent description of the deformation complex as

$$\mathrm{C}^k_{\mathrm{def}}(\mathcal{G}) = \Gamma(t^* TM \to \mathcal{G}^{(k-1)})$$

where $t^*TM \to \mathcal{G}^{(0)}$ means $TM \to M$. The formula for the differential in this description follows from the formula for the symbol of the differential in the general deformation complex and is given for $a \in \Gamma(t^*TM \to \mathcal{G}^{(k-1)})$ by

$$\begin{aligned}(\delta a)(g_1, ..., g_k) = &- dt_{g_1}((ds_{g_1})^{-1}(a(g_2, ..., g_k)) \\ &- \sum_{i=i}^{k-1}(-1)^i a(g_1, ..., g_i g_{i+1}, ..., g_k) \\ &- (-1)^k a(g_1, ..., g_{k-1}),\end{aligned}$$

so we see that -up to a sign in the differential and a shift of 1 in degrees- the deformation complex of $\mathcal{G}$ is given by the groupoid cohomology of $\mathcal{G}$ with coefficients in $TM \to M$, where TM is the representation of $\mathcal{G}$ as given in Example 1.2.22, with g acting $T_{s(g)}M \to T_{t(g)}M$ by

$$g \cdot v = dt_g(ds_{g^{-1}}(v)).$$

In particular we have

$$\mathrm{H}^\bullet_{\mathrm{def}}(\mathcal{G}) \cong \mathrm{H}^{\bullet-1}_{\mathrm{diff}}(\mathcal{G}, TM).$$

We also note that for this and the previous example, the not-considered degree 0 part mentioned in Remark 2.2.3 is trivial since the Lie algebroid is trivial.

Example 2.2.9 (Lie groups) For Lie groups, we note that the symbol-property is void, since $ds = 0$, so the deformation complex consists of all sections of the bundle $\mathrm{pr}_1^*TG \to G^k$. Using the right-trivialization of TG, we again rewrite the complex to something that we already know

$$\mathrm{C}^k_{\mathrm{def}}(G) = C^\infty(G^{\times k}, \mathfrak{g})$$

with differential given by

$$\begin{aligned}(\delta f)(g_1, ..., g_{k+1}) =& \mathrm{Ad}_{g_1}(f(g_2, ..., g_{k+1})) \\ &+ \sum_{i=1}^{k}(-1)^i f(g_1, ..., g_i g_{i+1}, ..., g_{k+1}) \\ &+ (-1)^{k+1} f(g_1, ..., g_k)\end{aligned}$$

and so we see that the deformation complex is nothing more than the group cohomology complex with values in the adjoint representation, in particular

$$\mathrm{H}^\bullet_{\mathrm{def}}(G) \cong \mathrm{H}^\bullet(G, \mathrm{Ad}).$$

Example 2.2.10 (Action groupoids) For the action groupoid $\mathcal{G} = G \times M \rightrightarrows M$ of a manifold M with a left G-action, we have natural inverse diffeomorphisms between $G^k \times M$ and $\mathcal{G}^{(k)}$ given by

$$(g_1, ..., g_k, x) \mapsto \left(g_1 \cdots g_k x \xleftarrow{(g_1, g_2 \cdots g_k x)} g_2 \cdots g_k x \leftarrow \cdots \leftarrow g_k x \xleftarrow{(g_k, x)} x\right),$$

$$\left(x_0 \xleftarrow{(g_1,x_1)} x_1 \leftarrow \cdots \leftarrow x_{k-1} \xleftarrow{(g_k,x_k)} x_k\right) \mapsto (g_1,...,g_k,x_k).$$

Using this description, and using the right-trivializations of the previous example, the fibre of the bundle $\mathrm{pr}_1^*T\mathcal{G} \to \mathcal{G}^{(k)}$ over $(g_1,...,g_k,x)$ is given by $\mathfrak{g}\oplus T_{g_2\cdots g_k x}M$. The symbol-property now means that the TM-factor is only dependent on $(g_2,...,g_k,x)$, while the $\mathfrak{g}$-factor is completely free. In the end we have the description of the deformation complex

$$\mathrm{C}^k_{\mathrm{def}}(\mathcal{G}) \cong C^\infty(G^k, C^\infty(M,\mathfrak{g})) \oplus C^\infty(G^{k-1}, \mathfrak{X}(M)),$$

where for $F_1 \in C^\infty(G^k, C^\infty(M,\mathfrak{g}))$, $F_2 \in C^\infty(G^{k-1},\mathfrak{X}(M))$, the deformation element $c(F_1,F_2)$ associated to it is given by

$$c(F_1,F_2)(g_1,...,g_k,x) = (dR_{g_1}(F_1(g_1,...,g_k,x)), F_2(g_2,...,g_k,x)(g_2\cdots g_k x))$$

as elements of $T_{g_1}G \oplus T_{g_2\cdots g_k x}M$.

Then we have

$$\delta(F_1,F_2) = (\delta^{1,1}F_1, \delta^{1,2}F_1 + \delta^{2,2}F_2)$$

where the parts of the differential are given by

$$\begin{aligned}(\delta^{1,1}F_1)(g_1,...,g_{k+1}) =& g_1 \cdot F_1(g_2,...,g_{k+1}) \\ &+ \sum_{i=1}^{k}(-1)^i F(g_1,...,g_ig_{i+1},...,g_{k+1}) \\ &+ (-1)^{k+1}F(g_1,...,g_k),\end{aligned}$$

$$(\delta^{1,2}F_1)(g_1,...,g_k,x) = \xi_{F_1(g_1,...,g_k)(x)}(x),$$

$$\begin{aligned}(\delta^{2,2}F_2)(g_1,...,g_k) =& -g_1 \cdot F_2(g_2,...,g_k) \\ &+ \sum_{i=1}^{k-1}(-1)^{i+1} F_2(g_1,...,g_ig_{i+1},...,g_k) \\ &+ (-1)^{k+1}F_2(g_1,...,g_{k-1}).\end{aligned}$$

So we see that the deformation complex in this case inherits a lot of structure from the group cohomology of G with coefficients in both $\mathfrak{X}(M)$ and $C^\infty(M,\mathfrak{g})$.

Example 2.2.11 (Pair groupoids) Next we consider a pair groupoid $\mathcal{G} = M\times M \rightrightarrows M$ with $s(x,y) = y$. In this case the k-nerve $\mathcal{G}^{(k)}$ is canonically isomorphic to $M^{\times(k+1)}$ and we may write a section of $\mathrm{pr}_1^*T\mathcal{G} \to \mathcal{G}^{(k)}$ as

$$c(x_1,...,x_{k+1}) = (X_1(x_1,...,x_{k+1}), X_2(x_1,...,x_{k+1}))$$

with $X_1(x_1,...,x_{k+1}) \in T_{x_1}M$ and $X_2(x_1,...,x_{k+1}) \in T_{x_2}M$.

The symbol-propery in this case translates to X_2 not depending on x_1, and so we can write the deformation complex as

$$\mathrm{C}^k_{\mathrm{def}}(\mathcal{G}) \cong \Gamma(\mathrm{pr}_1^* TM \to M^{\times(k+1)}) \oplus \Gamma(\mathrm{pr}_1^* TM \to M^{\times k})$$

and using this description, the differential becomes

$$\delta(X_1, X_2)(x_1, ..., x_{k+2}) = \left(\sum_{i=2}^{k+2} (-1)^{i+1} X_1(x_1, ...\widehat{x_i}..., x_{k+2}), \right.$$
$$\left. X_1(x_2, ..., x_{k+2}) + \sum_{i=3}^{k+2} (-1)^{i+1} X_2(x_2, ...\widehat{x_i}..., x_{k+2}) \right).$$

One checks that if $(X_1, X_2) \in \mathrm{C}^1_{\mathrm{def}}(\mathcal{G})$ then $\delta(X_1, X_2) = 0$ if and only if $X_1(x, y)$ only depends on x and $X_2(x) = X_1(x, y)$. Furthermore, if $(X_1, X_2) \in \mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$ for $k > 1$ with $\delta(X_1, X_2) = 0$, then $(X_1, X_2) = \delta(X_2, 0)$. In particular

$$\mathrm{H}^k_{\mathrm{def}}(\mathcal{G}) \cong \begin{cases} \mathfrak{X}(M) & k = 1 \\ 0 & \text{else} \end{cases}.$$

2.2.2 Deformations of groupoids

We briefly discuss the results of Crainic, Mestre and Struchiner relating actual deformations of groupoids to elements of the deformation complex we have just written down.

As written in [Me16, Prop 5.67], the structure of a Lie groupoid on a pair of spaces $(\mathcal{G}, M)$ can also be seen as a tuple $(s, \overline{m})$ of a smooth submersion $s\colon \mathcal{G} \to M$ and a smooth map $\overline{m}\colon \mathcal{G}\ {}^s\!\times^s \mathcal{G} \to \mathcal{G}$ taking the rôle of source and division, satisfying the obvious relations.

Definition 2.2.12 An *s-constant deformation* of a groupoid $(\mathcal{G}, M, s, \overline{m})$ is a family of division maps $(\overline{m}_\epsilon)_{\epsilon \in I}$ for I an open interval containing 0 that is smooth in the sense that the induced map $\mathcal{G}\ {}^s\!\times^s \mathcal{G} \times I \to \mathcal{G} \times I$ is smooth, such that $(s, \overline{m}_\epsilon)$ defines a groupoid structure for every $\epsilon \in I$, and such that $\overline{m}_0 = \overline{m}$.

The relevant result of [Me16, Lem 5.31] is then

Proposition 2.2.13 For an s-constant deformation $(\overline{m}_\epsilon)_{\epsilon \in I}$, the element $\xi \in \mathrm{C}^2_{\mathrm{def}}(\mathcal{G})$ given by

$$\xi(g, h) := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \overline{m}_\epsilon(gh, h)$$

defines a closed element of $\mathrm{C}^2_{\mathrm{def}}(\mathcal{G})$, whose cohomology class $[\xi] \in \mathrm{H}^2_{\mathrm{def}}(\mathcal{G})$ only depends on the equivalence class of the deformation.

Remark 2.2.14 There is a way to assign deformation classes to deformations of the groupoid that are not s-constant, as is done in [Me16, Prop 5.38]. The construction there uses certain choices to reduce to the s-constant case in a way that is invariant under choices after passing to cohomology.

2.3 From deformation elements to Hochschild chains

As we saw in the last section, the deformation complex of a groupoid $\mathcal{G} \rightrightarrows M$ indeed encodes deformations of a Lie groupoid. On the other hand, the Hochschild complex $\mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ essentially encodes the deformation theory of the associative algebra that is the convolution algebra of said Lie groupoid.

Now, an s-constant deformation of a Lie groupoid is nothing more than a deformation of the division map, and in turn quite canonically induces a deformation of the convolution product (indeed, the fact that the deformation is s-constant means that the underlying vector space $\Gamma_c(\mathcal{D}_s)$ does not change).

This suggests a close connection between both complexes, and in this section we make this precise by defining a cochain map from the deformation complex of $\mathcal{G}$ to the Hochschild complex of the convolution algebra $\mathcal{A}_\mathcal{G}$.

As a first hint that the correspondence is indeed encoded by a chain map, we make the following observation:

Proposition 2.3.1 Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid. Multiplicative vector fields on $\mathcal{G}$ act as derivations on the convolution algebra.

Proof. Recall the definition of a multiplicative vector field from Remark 2.2.5. Since a multiplicative vector field on $\mathcal{G}$ is by definition s-projectable to M, its action on an s-density is well-defined by the discussion in Section 2.1.1, c.f. equation (2.1).

The key ingredient in the proof is the observation that the flow of a multiplicative vector field is a groupoid map, that is if $X \in \mathfrak{X}(\mathcal{G})$ is a multiplicative vector field then $\Phi^t_X(gh^{-1}) = \Phi^t_X(g)\Phi^t_X(h)^{-1}$ whenever defined. A simple calculation when X is complete then shows

$$\begin{aligned}
X(a_1 * a_2)(g) &= \frac{d}{dt}\bigg|_{t=0} (a_1 * a_2)(\Phi^t_X g) \\
&= \frac{d}{dt}\bigg|_{t=0} \int_{h \in s^{-1}(s(\Phi^t_X g))} a_1((\Phi^t_X g)h^{-1})a_2(h) \\
&= \frac{d}{dt}\bigg|_{t=0} \int_{h \in s^{-1}(s(g))} a_1((\Phi^t_X g)(\Phi^t_X h)^{-1})a_2(\Phi^t_X h) \\
&= \frac{d}{dt}\bigg|_{t=0} \int_{h \in s^{-1}(s(g))} a_1(\Phi^t_X(gh^{-1}))a_2(\Phi^t_X h) \\
&= \frac{d}{dt}\bigg|_{t=0} \int_{h \in s^{-1}(s(g))} a_1(\Phi^t_X(gh^{-1}))a_2(h) \\
&\qquad + \frac{d}{dt}\bigg|_{t=0} \int_{h \in s^{-1}(s(g))} a_1(gh^{-1})a_2(\Phi^t_X h) \\
&= (Xa_1 * a_2)(g) + (a_1 * Xa_2)(g)
\end{aligned}$$

which proves the proposition.

Note that since we take the derivative at $t = 0$ and since a_2 has compact support we may pretend that X is complete along the source fibers (note that if $\mathcal{G}$ is not source-proper this may fail). Indeed, fixing g, we first find ϵ so that $\Phi_X^t(g)$ is defined for all $-\epsilon \leq t \leq \epsilon$. Then let K be the compact set which is the support of a_2 along $s^{-1}(s(\Phi_X^{[-\epsilon,\epsilon]}(g))$ (this is the intersection of a compact with a closed and hence compact). Next fix $\epsilon' \leq \epsilon$ such that Φ_X^t is defined on K for all $-\epsilon' \leq t \leq \epsilon'$ (this is possible since K is compact). Then, in the second line of the calculation above, for every $t \in [-\epsilon', \epsilon']$ the only h that contribute to the integral are h inside K, for which Φ_X^t is defined, and hence we may change variables. From this point on the calculation is well-defined and the abuse of notation is resolved. □

2.3.1 Defining a chain map

In the following we write $\mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ for the Hochschild complex of the convolution algebra $\mathcal{A}_\mathcal{G}$ with values in the bimodule $\mathcal{A}_\mathcal{G}$ with differential b_{Hoch}. Even though the theory below is well-defined for the algebraic Hochschild complex (using algebraic tensor products and Hom), it is more natural to consider $\mathcal{A}_\mathcal{G}$ as an inductive limit of Fréchet spaces and use completed tensor products and continuous homomorphism to define the continuous version of the Hochschild cochain complex. We refer to [NPPT06] for more details, the cochain map below naturally extends to this topological Hochschild complex.

We now describe the cochain map $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$.

Definition 2.3.2 The map $\Phi\colon \mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ is defined for $c \in C^k_{\mathrm{def}}(\mathcal{G})$ by

$$(\Phi c)(a_1, ..., a_k)(g) := \int_{g_1 \cdots g_k = g} (c(-, g_2, ..., g_k)a_1)(g_1)a_2(g_2) \cdots a_k(g_k).$$

This formula should be read as an inductive convolution (first over $g_1 g_2 = h_1$, then over $h_1 g_3 = h_2$, et cetera).

Remark 2.3.3 The formula for Φ above is justified by Lemma 2.1.11: $c \in \mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$ is s-projectable and therefore, if we keep $(g_2, ..., g_k) \in \mathcal{G}^{(k-1)}$ fixed, the action of $c(-, g_2, ..., g_k)$ on $a \in \mathcal{A}_\mathcal{G}$ along $s^{-1}(t(g_2))$ is well-defined. In particular $(c(-, g_2, ..., g_k)a)(g_1)$ is a well-defined density at g_1 for $(g_1, ..., g_k) \in \mathcal{G}^{(k)}$.

Showing that Φ is a chain map is done by a calculation similar to the one in Theorem 2.3.1. In particular we need to deal with the term $\Phi_X^t(g)(\Phi_X^t(h))^{-1}$ for divisible g and h when t goes to 0. In the mulitplicative case this is precisely $\Phi_X^t(gh^{-1})$, but for general deformation elements we need a more general description.

Streamlining this, we introduce the notation $\overline{m}c$ for $c \in \mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ for the term involving $d\overline{m}$ in the formula for δc. That is, we set:

$$(\overline{m}c)(g_1, ..., g_{k+1}) := d\overline{m}(c(g_1 g_2, ..., g_{k+1}), c(g_2, ..., g_{k+1})).$$

We notice that this formula also makes sense along s-fibres. In particular, if $X \in \mathfrak{X}_s(\mathcal{G})_{s^{-1}(x)}$ is an s-projectable vector field along a single s-fibre, the formula

$$(\overline{m}X)(g_1, g_2) = d\overline{m}(X(g_1 g_2), X(g_2))$$

defines an s-projectable vector field defined along $\{(g_1, g_2) \in \mathcal{G}^{(2)} : s(g_2) = x\}$. With this in mind, we have the following relation:

$$\overline{m}(c(-, g_3, ..., g_{k+1}))(g_1, g_2) = (\overline{m}c)(g_1, ..., g_{k+1}),$$

which makes sure that for dealing with $\overline{m}c$, we only need to deal with the first two entries g_1 and g_2 as variables, while the other entries $g_3, ..., g_k$ can be seen as parameters.

The key Lemma is then as follows:

Lemma 2.3.4 Let $x \in M$, $X \in \mathfrak{X}_s(\mathcal{G})|_{s^{-1}(x)}$ and $a_1, a_2 \in \mathcal{A}_{\mathcal{G}}$. Then for all $h \in s^{-1}(x)$ we have $\overline{m}X(-, h) \in \mathfrak{X}_s(\mathcal{G})|_{s^{-1}(t(h))}$ and for $g \in s^{-1}(x)$ we have

$$X(a_1 * a_2)(g) = (a_1 * Xa_2)(g) + \int_{h \in s^{-1}(x)} ((\overline{m}X(-, h))a_1)(gh^{-1})a_2(h).$$

Proof. By definition we have:

$$\overline{m}X(gh^{-1}, h) = d\overline{m}(X(g), X(h)) \in T_{gh^{-1}}\mathcal{G}$$

with s-projection

$$ds(\overline{m}X(gh^{-1}, h)) = dt(X(h))$$

so indeed $\overline{m}X(-, h) \in \mathfrak{X}_s(\mathcal{G})|_{s^{-1}(t(h))}$.

Next we assume that X is a globally defined s-projectable vector field (otherwise, we choose an extension at this point). Then we know that $\overline{m}X(-, h)$ is generated by the path Φ_t through s-fibres preserving diffeomorphisms of $\mathcal{G}$, which along $s^{-1}(t(h))$ looks like

$$\Phi_t(gh^{-1}) = \Phi_X^t(g)(\Phi_X^t(h))^{-1},$$

so that we see that:

$$\int_{h \in s^{-1}(x)} ((\overline{m}X(-, h))a_1)(gh^{-1})a_2(h) = \left.\frac{d}{dt}\right|_{t=0} \int_{h \in s^{-1}(x)} a_1(\Phi_X^t(g)(\Phi_X^t(h))^{-1})a_2(h).$$

Using this we calculate $X(a_1 * a_2)(g)$:

$$\begin{aligned}
X(a_1 * a_2)(g) &= \frac{d}{dt}\Big|_{t=0} (a_1 * a_2)(\Phi^t_X g) \\
&= \frac{d}{dt}\Big|_{t=0} \int_{h \in s^{-1}(s(\Phi^t_X g))} a_1((\Phi^t_X g)h^{-1})a_2(h) \\
&= \frac{d}{dt}\Big|_{t=0} \int_{h \in s^{-1}(x)} a_1((\Phi^t_X g)(\Phi^t_X h)^{-1})a_2(\Phi^t_X(h)) \\
&= \frac{d}{dt}\Big|_{t=0} \int_{h \in s^{-1}(x)} a_1((\Phi^t_X g)(\Phi^t_X h)^{-1})a_2(h) \\
&\quad + \frac{d}{dt}\Big|_{t=0} \int_{h \in s^{-1}(x)} a_1(gh^{-1})a_2(\Phi^t_X h) \\
&= \int_{h \in s^{-1}(x)} ((\overline{m}X(-,h))a_1)(gh^{-1})a_2(h) \\
&\quad + (a_1 * Xa_2)(g),
\end{aligned}$$

which finishes the proof.

Note that we run into the same problem as in Theorem 2.3.1 as before, namely that Φ^t_X need not be defined on the whole s-fiber. However, we use change of variables using Φ^t_X on an integral where one of the terms is $a_2(h)$. Furthermore, we are only interested in the behaviour for small t, so we only need a find an $\epsilon > 0$ such that for all $-\epsilon \le t \le \epsilon$ we have that $\Phi^t_X(h)$ is defined whenever h is in the support of a_2. Similar to Proposition 2.3.1, since the support of a_2 is compact, such an ϵ can be found, and we may carry the abuse of notation with Φ^t_X as if it is globally defined. □

Proposition 2.3.5 The map $\Phi\colon \mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ is a morphism of cochain complexes.

Proof. This proof is essentially writing out all the parts of the Hochschild differential and applying some bookkeeping. We start with $c \in \mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$ for $k \ge 1$, and write down the definition of the various parts of $b_{\mathrm{Hoch}}(\Phi c)$.

$$\begin{aligned}
&(a_1 * (\Phi c)(a_2, ..., a_{k+1}))(g) = \\
&\qquad = \int_{g_1 \cdots g_{k+1} = g} a_1(g_1)(c(-, g_3, ..., g_{k+1})a_2)(g_2)a_3(g_3) \cdots a_{k+1}(g_{k+1}), \quad (\star)
\end{aligned}$$

$$\begin{aligned}
&-(\Phi c)(a_1 * a_2, a_3, ..., a_{k+1})(g) = \\
&\qquad = -\int_{h \cdot g_3 \cdots g_{k+1} = g} (c(-, g_3, ..., g_{k+1})(a_1 * a_2))(h)a_3(g_3) \cdots a_{k+1}(g_{k+1}), \quad (\star\star)
\end{aligned}$$

$$\sum_{i=2}^{k}(-1)^i(\Phi c)(a_1,...,a_i * a_{i+1},...,a_{k+1})(g) =$$
$$= \sum_{i=2}^{k}\int_{g_1\cdots g_{k+1}=g}(-1)^i(c(-,g_2,...,g_ig_{i+1},...,g_{k+1})a_1)(g_1)a_2(g_2)\cdots a_{k+1}(g_{k+1}),$$

$$(-1)^{k+1}((\Phi c)(a_1,...,a_k) * a_{k+1})(g) =$$
$$= (-1)^{k+1}\int_{g_1\cdots g_{k+1}=g}(c(-,g_2,...,g_k)a_1)(g_1)a_2(g_2)\cdots a_{k+1}(g_{k+1}).$$

The latter two terms we recognize from the differential of the deformation complex, while the first two terms can be rewritten to:

$$(\star)+(\star\star) = \int_{hg_3\cdots g_{k+1}=g}\left((a_1 * (c(-,g_3,...,g_{k+1})a_2) - c(-,g_3,...,g_{k+1})(a_1 * a_2)\right)(h)$$
$$a_3(g_3)\cdots a_{k+1}(g_{k+1}).$$

Then by Lemma 2.3.4 we can rewrite this to

$$(\star)+(\star\star) = -\int_{g_1\cdots g_{k+1}=g}((\overline{m}c)(-,g_2,...,g_{k+1})a_1)(g_1)a_2(g_2)\cdots a_{k+1}(g_{k+1})$$

Putting this all together we conclude that:

$$(b_{\text{Hoch}}(\Phi c))(a_1,...,a_{k+1})(g) = (\Phi(\delta c))(a_1,...,a_{k+1})(g),$$

so we see that Φ is indeed a chain-map. □

2.3.2 Properties of the chain map

Comparing deformation classes

As we described before, an s-constant deformation $\overline{m}_\epsilon$ of a Lie groupoid $\mathcal{G}$ induces a closed deformation element $\xi \in \mathrm{C}^2_{\mathrm{def}}(\mathcal{G})$ given by

$$\xi(g,h) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\overline{m}_\epsilon(gh,h).$$

As mentioned, such a family of division maps also induces a deformation of the convolution product. Indeed, for every ϵ we have that $\overline{m}_\epsilon$ is a division map to the original s, so that the convolution algebra of groupoid $\mathcal{G}_\epsilon$ has the same underlying vector space $\Gamma_c(\mathcal{D}_s)$. The deformation of the product then naturally takes its form as a Hochschild 2-cycle for $\mathcal{A}_\mathcal{G}$ given by

$$\beta(a_1,a_2)(g) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0}\int_{h\in s^{-1}(s(g))}a_1(\overline{m}_\epsilon(g,h))a_2(h).$$

So out of one s-constant deformation of our groupoid, we end up with canonical closed elements of the domain and codomain of our chain map respectively. They are related as follows:

Proposition 2.3.6 The chain map Φ sends ξ to β.

Proof. This follows from the observation that if $s(h) = s(g)$, then

$$\xi(gh^{-1}, h) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \overline{m}_e(g, h).$$

With this we see that

$$\beta(a_1, a_2)(g) = \int_{h \in s^{-1}(s(g))} (\xi(-, h)a_1)(gh^{-1})a_2(h) = \Phi(\xi)(a_1, a_2)(g),$$

exactly as needed. □

Remark 2.3.7 In [CrMS20, Prop 5.12] a deformation cocycle $\xi \in \mathrm{C}^2_{\mathrm{def}}(\mathcal{G})$ is assigned to any deformation (in particular those who are not s-constant), such that the cohomology class is canonical. Then $\Phi(\xi)$ induces a Hochschild cohomology class of degree 2, which is not immediately linked to a deformation of the convolution product, since if the source map changes the underlying space of the convolution algebra also changes as it consists of densities along the s-fibers. Indeed, in [CrMS20] the authors need an auxillary choice of a vector field on the larger deformation space to define the cocycle. This choice of an auxillary vector field is precisely what is needed to compare the various convolution algebras when the source map varies, and in this way Φ maps $[\xi] \in \mathrm{H}^2_{\mathrm{def}}(\mathcal{G})$ to the Hochschild class of the deformation of the convolution product thus defined.

The case $k = 0$

For the chain map between $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ and $\mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ we have just defined, a natural question is whether it can be extended to degree $k = 0$, c.f. Theorem 2.2.2. For this, one must find a a map $\Phi^0 \colon \Gamma(A) \to \mathcal{A}_\mathcal{G}$ which extends the chain map Φ. This is only possible if $\Phi(\delta(\alpha)) \in \mathrm{Der}(\mathcal{A}_\mathcal{G})$ is an inner derivation for every $\alpha \in \Gamma(A)$.

Intuitively it is clear that is should not be always possible, since the derivation $\Phi(\delta(\alpha))$ includes taking derivatives, while an inner derivation $\partial_H(a)$ only includes integrations. The following example presents a concrete counterexample:

Example 2.3.8 Consider the pair groupoid $\mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}$. For this groupoid, a bundle of densities is trivialized by $|dx|$, so that every compactly supported density is of the form $f|dx|$ for a compactly supported smooth function f. Furthermore, a section of the algebroid is simply a vector field $X \in \mathfrak{X}(\mathbb{R})$ and for this example we take $X = \frac{\partial}{\partial x}$. We have

$$\delta(X)(x, y) = (X(x), X(y))$$

so that in this case $\delta(X) = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. This vector field has flow

$$\Phi^t_{\delta(X)}(x,y) = (x+t, y+t).$$

Next we consider $\Phi\left(\delta\left(\frac{\partial}{\partial x}\right)\right)$, so we look at the action of $\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ on a density $f(x,y)|dx| \in \mathcal{A}_{\mathbb{R}\times\mathbb{R}}$. We see

$$(\Phi^t_{\delta(X)})^*(f(x,y)|dx|) = f(x+t,y+t)|d(x+t)| = f(x+t,y+t)|dx|,$$

so that:

$$\Phi(\delta(X))(f|dx|) = \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right)|dx|.$$

Now suppose that there is some $g|dx| \in \mathcal{A}_{\mathbb{R}\times\mathbb{R}}$, such that $\Phi(\delta(X)) = \partial_H(g|dx|)$. Then since always $\partial_H(g|dx|)(g|dx|) = 0$, we see that:

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} = 0$$

so that

$$g(x+t,y+t) = g(x,y).$$

Since g has to be compactly supported, the only possibility is that $g = 0$, which is obviously not a solution to $\Phi(\delta(X)) = \partial_H(g|dx|)$. We conclude that $\Phi(\delta(X))$ is not an inner derivation.

In fact, using the fact that Lie derivatives preserve support, we can deduce that $\Phi(X)$ can never be an inner derivation for any $X \in \mathfrak{X}_s(\mathcal{G})$.

Proposition 2.3.9 Let $D \in \mathrm{Hom}(\mathcal{A}_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}})$ be a non-zero Hochschild-1-cochain. If D satisfies $\mathrm{supp}(Da) \subset \mathrm{supp}(a)$, then there is no $b \in \mathcal{A}_{\mathcal{G}}$ such that $D = [-, b]$.

Proof. Suppose by contrary that there is a b such that $D = [-, b]$. Let $g \in \mathcal{G}$ and let $a \in \mathcal{A}_{\mathcal{G}}$ be supported arbitrarily close to g. For $h \in t^{-1}(s(g))$ outside of the isotropy of $s(g)$ we obtain:

$$(a * b)(gh) = \int_k a(gk^{-1})b(kh)$$

which, using that is chosen to be supported arbitrarily close to g behaves like $a(g)b(h)$. where we use that a is only non-zero close enough to g. For the other part of the commutator we have

$$(b * a)(gh) = \int_k b(gk^{-1})a(kh),$$

which is 0, since there is no way to let kh come arbitrarily close to g since h is not in the isotropy of $s(g)$.

Since $\mathrm{supp}(Da) \subset \mathrm{supp}(a)$ we see that $(a*b)(gh)$ also has to be supported arbitrarily close to g, so that b is identically zero outside of the isotropy of $\mathcal{G}$.

If we look at an isotropy element h of $\mathcal{G}$ we see that the second term acts like $b(ghg^{-1})a(g)$, so that we see that b is invariant under conjugation. However, if b is invariant under conjugation we conclude that $b \in Z(\mathcal{A}_\mathcal{G})$, which is in contradiction to the fact that D is non-zero. We conclude that there is no b that solves $D = [-, b]$. □

Corollary 2.3.10 If $X \in \mathrm{C}^1_{\mathrm{def}}(\mathcal{G})$ is non-zero, then $\Phi(X)$ can never be an inner derivation.

Proof. This follows from the previous proposition by the observation that $\Phi(X)$ is local since it involves taking derivatives and the fact that Φ is easily observed to be injective. □

Compatibility with the characteristic map to cyclic cohomology

Recall the differentiable cohomology complex $(\mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G}), \delta)$ of the groupoid $\mathcal{G}$ given by $\mathrm{C}^k_{\mathrm{diff}}(\mathcal{G}) := C^\infty(\mathcal{G}^{(k)})$ with differential

$$\begin{aligned}\delta\varphi(g_1, \dots, g_{k+1}) =& \varphi(g_2, \dots, g_{k+1}) \\ &+ \sum_{i=1}^{k} (-1)^i \varphi(g_1, \dots, g_i g_{i+1}, \dots, g_k) \\ &+ (-1)^{k+1} \varphi(g_1, \dots, g_k).\end{aligned}$$

We can turn this cochain complex into a DGA by introducing the product $\cup\colon \mathrm{C}^k_{\mathrm{diff}}(\mathcal{G}) \times \mathrm{C}^l_{\mathrm{diff}}(\mathcal{G}) \to \mathrm{C}^{k+l}_{\mathrm{diff}}(\mathcal{G})$ given by

$$(\varphi \cup \psi)(g_1, \dots, g_{k+l}) := \varphi(g_1, \dots, g_k)\psi(g_{l+1}, \dots, g_{k+l}).$$

In [CrMS20] it is shown that by replacing φ by a deformation cochain $c \in \mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$ in the above formula, $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ becomes a right module over $\mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G})$. On the other hand, in [PPT15], the smooth groupoid cohomology was used to construct cyclic cocycles. In this section we shall see that these two structures are compatible with each other under the cochain map Φ to Hochschild cohomology of Section 2.3.1. We start by rewriting the map to cyclic cohomology of [PPT15] in the way described below.

First recall that the Hochschild cochain complex $\mathrm{C}^\bullet(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ can be given a DGA structure by introducing the product $\cup\colon \mathrm{C}^k(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G}) \times \mathrm{C}^l(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G}) \to \mathrm{C}^{k+l}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$

$$(D \cup E)(a_1, \dots, a_{k+l}) := D(a_1, \dots, a_k) * E(a_{k+1}, \dots, a_{k+l}).$$

Construct a map $\Phi_0\colon \mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G}) \to \mathrm{C}^\bullet(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ by

$$\Phi_0(\varphi)(a_1, \dots, a_k)(g) := \int_{g_1 \cdots g_k = g} \varphi(g_1, \dots, g_k) a_1(g_1) \cdots a_k(g_k). \tag{2.2}$$

Lemma 2.3.11 The map $\Phi_0\colon (\mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G}), \delta, \cup) \to (\mathrm{C}^\bullet(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G}), b_{\mathrm{Hoch}}, \cup)$ is a morphism of DGA's.

Proof. This is a straightforward computation. □

With this Lemma we can also equip the Hochschild complex $\mathrm{C}^\bullet(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ with a module structure over $\mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G})$ by using the cup product on Hochschild cochains. Explicitly, this module structure is given by

$$D \cdot \varphi := D \cup \Phi_0(\varphi).$$

We then have:

Proposition 2.3.12 The cochain map $\Phi\colon \mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ defined in Theorem 2.3.2 is a morphism of $\mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G})$-modules.

Proof. Let us start with the following case: For $c \in \mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$ and $f \in C^\infty(\mathcal{G}) = C^1_{\mathrm{diff}}(\mathcal{G})$ we have

$$\Phi(c \cup f)(a_1, ..., a_{k+1}) = \Phi(c)(a_1, ..., a_k) * (f \cdot a_{k+1}).$$

The claim follows by carefully writing out the definition

$$\begin{aligned}
\Phi(c \cup f)(a_1, ..., a_{k+1})(g) &= \int_{g_1 \cdots g_{k+1} = g} ((c \cup f)(-, g_2, ..., g_{k+1})a_1)(g_1)a_2(g_2) \cdots a_{k+1}(g_{k+1}) \\
&= \int_{g_1 \cdots g_{k+1} = g} (f(g_{k+1})c(-, g_2, ..., g_k)a_1)(g_1)a_2(g_2) \cdots a_{k+1}(g_{k+1}) \\
&= \int_{g_1 \cdots g_{k+1} = g} (c(-, g_2, ..., g_k)a_1)(g_1)a_2(g_2) \cdots \\
&\qquad\qquad \cdots a_k(g_k)\left(f(g_{k+1})a_{k+1}(g_{k+1})\right) \\
&= \int_{hg_{k+1} = g} \int_{g_1 \cdots g_k = h} (c(-, g_2, ..., g_k)a_1)(g_1)a_2(g_2) \cdots \\
&\qquad\qquad \cdots a_k(g_k)\left(f(g_{k+1})a_{k+1}(g_{k+1})\right) \\
&= \int_{hg_{k+1} = g} \Phi(c)(a_1, ..., a_k)(h)\left(f(g_{k+1})a_{k+1}(g_{k+1})\right) \\
&= \left(\Phi(c)(a_1, ..., a_k) * (f \cdot a_{k+1})\right)(g).
\end{aligned}$$

Hence by induction we obtain

$$\Phi(c \cup (f_1 \otimes \cdots \otimes f_l))(a_1, ..., a_{k+l}) = \Phi(c)(a_1, ..., a_k) * (f_1 \cdot a_{k+1}) * \cdots * (f_l \cdot a_{k+l}).$$

Writing $fa = \Phi_0(f)(a)$, we can rewrite this as

$$\Phi(c \cup (f_1 \otimes \cdots \otimes f_l)) = \Phi(c) \cup \Phi_0(f_1) \cup \ldots \cup \Phi_0(f_l).$$

From this the general statement of the proposition for $\varphi \in C^k_{\mathrm{diff}}(\mathcal{G}) = C^\infty(\mathcal{G}^{(k)})$ follows. □

Now, analogous to the action of vector fields on differential forms in geometry, the Hochschild cochains act on Hochschild chains by contraction [NT, Sec 2.5]

$$\mathrm{C}^k(\mathcal{A}_\mathcal{G},\mathcal{A}_\mathcal{G})\times \mathrm{C}_l(\mathcal{A}_\mathcal{G})\longrightarrow \mathrm{C}_{l-k}(\mathcal{A}_\mathcal{G}),\qquad (D,a)\mapsto \iota_D a,$$

given explicitly by

$$\iota_D(a_0\otimes\ldots\otimes a_l) := a_0 D(a_1,\ldots,a_k)\otimes a_{k+1}\otimes\ldots\otimes a_l.$$

This action satisfies the properties

$$\begin{aligned}\iota_D\circ\iota_E &= \iota_{D\cup E}\\ [b,\iota_D] &= \iota_{\delta D}.\end{aligned}$$

Furthermore we can define a "Lie derivative" using the analogue of the Cartan formula: $L_D := B\circ\iota_D+\iota_D\circ B$.

Dualizing, we obtain a contraction of cochains

$$\mathrm{C}^k(\mathcal{A}_\mathcal{G},\mathcal{A}_\mathcal{G})\times \mathrm{C}^{l-k}(\mathcal{A}_\mathcal{G})\to \mathrm{C}^l(\mathcal{A}_\mathcal{G}),\qquad (D,\varphi)\mapsto \iota_D\varphi$$

given explicitly by

$$(\iota_D\varphi)(a_0,...,a_l)=\varphi(a_0D(a_1,...,a_k),a_{k+1},...,a_l). \tag{2.3}$$

Next, recall from [PPT15] that when $\mathcal{G}$ is unimodular we can define a trace on the convolution algebra $\mathcal{A}_\mathcal{G}$ by

$$\tau(a) := \int_M a\Omega,$$

with on the right hand side Ω a $\mathcal{G}$-invariant section of the bundle $\mathcal{D}_{A^*}\otimes\mathcal{D}_{TM}$, and we use the duality $\mathcal{D}_A\times\mathcal{D}_{A^*}\to\mathbb{R}$ together with the isomorphism $\mathcal{D}_s|_M=\mathcal{D}_A$, to obtain a density on M that can be integrated. With this trace (a degree 0 cyclic cocycle), the cochain map

$$\Psi_\tau\colon (\mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G}),\delta)\longrightarrow(\mathrm{C}^\bullet(\mathcal{A}_\mathcal{G}),b_{\mathrm{Hoch}}), \tag{2.4}$$

constructed in [PPT15] is simply given by $\Psi_\tau(c) := \iota_{\Phi_0(c)}\tau$.

Corollary 2.3.13 Let $c\in \mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$ and $\varphi\in\mathrm{C}^l_{\mathrm{diff}}(\mathcal{G})$. Then the following identity is true:

$$\iota_{\Phi(c\cup\varphi)}\tau=\iota_{\Phi(c)}\Psi_\tau(\varphi).$$

With this Corollary, we can construct new cyclic cocycles on the convolution algebra. First of all, if we start with a smooth groupoid cocycle $\varphi\in\mathrm{C}^k_{\mathrm{diff}}(\mathcal{G})$, we obtain a Hochschild cocycle by applying Ψ_τ as in (2.4). A small computation shows that this cocycle is closed under the B-differential, i.e., $B\Psi_\tau(\varphi)=0$, when φ is cyclic:

$$\varphi(g_1,\ldots,g_k)=(-1)^k\varphi((g_1\cdots g_k)^{-1},g_1,\ldots,g_{k-1})$$

We can work out similar conditions for an element $c\in\mathrm{C}^k_{\mathrm{def}}(\mathcal{G})$ to satisfy $B(\Phi(c))=0$, but they are more involved. For example, for $k=2$ we find

$$(d\iota)(c(g,g^{-1}))=-c(g^{-1},g).$$

2.3.3 Examples

In this section we discuss how the chain map Φ links the deformation cohomology of $\mathcal{G}$ and the Hochschild cohomology of $\mathcal{A}_\mathcal{G}$ in certain examples, referring back to the examples of both sides of the equation we gave before.

Example 2.3.14 (Trivial groupoid) We consider the trivial groupoid $\mathcal{G} = M \rightrightarrows M$. On the density side we simply have $(\mathcal{A}_\mathcal{G}, *) = (C_c^\infty(M), \cdot)$, with $\mathrm{H}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G}) = \Lambda^\bullet \mathfrak{X}(M)$. At the side of the deformation complex we note that the k-nerve of the trivial groupoid is M for every k and s-projectability is a void property, so that for $k > 0$ we have $\mathrm{C}^k_{\mathrm{def}}(\mathcal{G}) = \mathfrak{X}(M)$, with differential alternating between the identity and the zero map:

$$C^\bullet_{\mathrm{def}}(\mathcal{G}) = \Big[0 \to \mathfrak{X}(M) \xrightarrow{0} \mathfrak{X}(M) \xrightarrow{\mathrm{id}} \mathfrak{X}(M) \to \cdots\Big].$$

So the deformation cohomology equals:

$$\mathrm{H}^k_{\mathrm{def}}(\mathcal{G}) \cong \begin{cases} \mathfrak{X}(M) & \text{if } k = 1 \\ 0 & \text{else} \end{cases}$$

The chain map $\Phi\colon \mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ simply becomes:

$$\Phi(X)(f_1, ..., f_k) = (X f_1) \cdot f_2 \cdots f_k.$$

and we see that under the isomorphism $\mathrm{HH}^1(C^\infty(M), C^\infty(M)) \cong \mathfrak{X}(M)$ provided by the Hochschild–Kostant–Rosenberg Theorem, we have

$$\mathrm{H}^k(\Phi) = \begin{cases} \mathrm{id} & \text{if } k = 1 \\ 0 & \text{else} \end{cases}$$

We should also remark for this example that using, the classical Hochschild–Kostant–Rosenberg theorem, we see that taking exterior powers of deformation elements we retrieve the whole Hochschild cohomology of $C_c^\infty(M)$.

Example 2.3.15 (Étale groupoids) In the case of an étale groupoid $\mathcal{G} \rightrightarrows M$, we have $\mathcal{A}_\mathcal{G} = C_c^\infty(\mathcal{G})$, since the distribution $\ker(ds)$ is the trivial distribution. The convolution product in this case is commonly written as

$$(f_1 * f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2).$$

In this case the action of vector fields on densities is just the normal action of vector fields on functions, and the map Φ reduces to

$$\Phi(c)(f_1, ..., f_k)(g) = \sum_{g_1 \cdots g_k = g} (c(g_1, ..., g_k) f_1) \cdot f_2(g_2) \cdots f_k(g_k).$$

In the case that we have a proper étale groupoid (over a connected base M) we can calculate the cohomologies in both sides of the equation. On the side of the deformation complex we use [CrMS20, Thm 6.1] to obtain:

$$\begin{aligned}
&\mathrm{H}^0_{\mathrm{def}}(\mathcal{G}) \cong \{0\}\\
&\mathrm{H}^1_{\mathrm{def}}(\mathcal{G}) \cong \mathfrak{X}(M)_{\mathrm{inv}}\\
&\mathrm{H}^k_{\mathrm{def}}(\mathcal{G}) \cong \{0\} \qquad (k \geq 2)
\end{aligned}$$

For the Hochschild cohomology of the convolution algebra we refer to [NPPT06, Thm 3.11] to obtain

$$\mathrm{H}^k(\mathcal{A}_{\mathcal{G}}, \mathcal{A}_{\mathcal{G}}) \cong \bigoplus_{\mathcal{O} \in \mathrm{Sec}(\mathcal{G})} \Gamma_{\mathrm{inv}}(\Lambda^{k-\mathrm{codim}(\mathcal{O})} T\mathcal{O})$$

where the sum is over the sectors $\mathcal{O}$ of $\mathcal{G}$. The action of the chain map Φ on the cohomology of degree 1 is the inclusion of $\mathfrak{X}(M)_{\mathrm{inv}}$ into this sum as the term for the sector $\mathcal{O} = M$.

2.4 Relationship to deformation quantization and the van Est-map

Classically, the theory of Lie group deformations is only one half of the coin, and the theory of Lie algebra deformations completes this picture. As we saw in Example 2.2.9, the deformation cohomology of a Lie group is calculated by the group cohomology with coefficients in the adjoint representation. To recall in general, if V is a representation of G, then the group cohomology complex with coefficients in V is given by

$$\mathrm{C}^k(G, V) \coloneqq C^\infty(G^{\times k}, V)$$

with differential

$$\begin{aligned}
(\delta f)(g_1, ..., g_{k+1}) \coloneqq\; & g_1 \cdot f(g_2, ..., g_{k+1}) \\
&+ \sum_{i=1}^{k} (-1)^i f(g_1, ..., g_i g_{i+1}, ..., g_{k+1}) \\
&+ (-1)^{k+1} f(g_1, ..., g_k).
\end{aligned}$$

Local to this, we can see V as a $\mathfrak{g}$-representation by setting $X \cdot v = \frac{d}{dt}\big|_{t=0} e^{tX} \cdot v$, and write down the Lie algebra cohomology complex

$$\mathrm{C}^k(\mathfrak{g}, V) \coloneqq \Lambda^k \mathfrak{g}^* \otimes V$$

with differential

$$(\delta f)(X_1, ..., X_{k+1}) := \sum_{i=1}^{k+1} (-1)^{i+1} X_i f(X_1, ..., \widehat{X_i}, ..., X_{k+1}) + \sum_{i<j} (-1)^{i+j} f([X_i, X_j], X_1, ..., \widehat{X_i}, \widehat{X_j}, ..., X_{k+1}).$$

There are clear connections between these two cohomology theories. For instance, in degree 0 we have that the group cohomology equals

$$\mathrm{H}^0(G, V) = V^G = \{v \in V : gv = v \,\forall g \in G\}$$

the invariants under the action, while in the algebra case we have

$$\mathrm{H}^0(\mathfrak{g}, V) = V^{\mathfrak{g}} = \{v \in V : \, Xv = 0 \,\forall X \in \mathfrak{g}\}.$$

It is clear from the definition of the $\mathfrak{g}$-action on V that $\mathrm{H}^0(G, V) \subset H^0(\mathfrak{g}, V)$ and if the group G is connected, this is even an equality.

The work of van Est [vE53a, vE53b] then gives a precise connection between these two theories. First we 'normalize' the group cohomology complex by requiring chains $f \in \mathrm{C}^k(G, V)$ to satisfy $f(1, g_2, ..., g_k) = 0$, and we obtain a subcomplex $\mathrm{C}^k_{\mathrm{norm}}(G, V)$, which one can show to be quasi-isomorphic. Van Est then writes down a chain map $\mathcal{V}\colon \mathrm{C}^k_{\mathrm{norm}}(G, V) \to \mathrm{C}^k(\mathfrak{g}, V)$, nowadays called the van Est-map, as

$$\mathcal{V}(f)(X_1, ..., X_k) = \sum_{\sigma \in S_k} (-1)^{\sigma} \left.\frac{d}{d\epsilon_1}\right|_{\epsilon_1=0} \cdots \left.\frac{d}{d\epsilon_k}\right|_{\epsilon_k=0} f(e^{\epsilon_1 X_{\sigma(1)}}, ..., e^{\epsilon_k X_{\sigma(k)}}).$$

and the importance of this map is the following result, known as the van Est-theorem

Theorem 2.4.1 [vE53b] If G is q-connected then $\mathcal{V}$ induces isomorphisms

$$\mathrm{H}^k(G, V) \xrightarrow{\cong} \mathrm{H}^k(\mathfrak{g}, V)$$

for $0 \leq k \leq q$.

If one plugs in $V = \mathbb{R}$, this is a statement about differentiable cohomology of G, while for $V = \mathfrak{g}$ with the adjoint action, this relates deformation cohomology of G with that of $\mathfrak{g}$. There is a similar story for Lie groupoids and Lie algebroids, in the differentiable case given by a generalization of the van Est-map by Weinstein and Xu [WX91] and in the deformation case with a generalization to a van Est-map $\mathcal{V}\colon \widehat{\mathrm{C}}^{\bullet}_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^{\bullet}_{\mathrm{def}}(A(\mathcal{G}))$ as defined for Lie groupoids by [CrMS20], which have much the same properties as the classical van Est map under assumptions on the connectivity of the s-fibres. Here $\mathrm{C}^{\bullet}_{\mathrm{def}}(A(\mathcal{G}))$, defined below, can be thought of as the linear Poisson complex of the linear Poisson manifold $A(\mathcal{G})^*$.

The aim of this section is to weave our map Φ from Definition 2.3.2 into this story by exhibiting the van Est map $\mathcal{V}$ as a 'classical limit' of our map Φ. The final result in Theorem 2.4.23 is the following equation, which holds for $k \geq 1$, $c \in \widehat{\mathrm{C}}^k_{\mathrm{def}}(\mathcal{G})$ and $f_1, \dots, f_k \in \mathcal{S}_c(A^*)$:

$$\mathcal{V}(c)(f_1,\dots,f_k) = \mathcal{F}_\mu\left(\lim_{t\to 0}\left(\sum_{\sigma\in S_k}(-1)^\sigma \frac{1}{(it)^{k-1}}\Phi(c)(q_t(f_{\sigma(1)}),\dots,q_t(f_{\sigma(k)}))\right)\right)$$

Here $\mathcal{F}_\mu\colon \mathscr{S}_c(A(\mathcal{G})) \to \mathcal{S}_c(A(\mathcal{G})^*)$ is the Fourier transform with respect to some Haar system μ, the maps $q_t\colon \mathcal{S}_c(A(\mathcal{G})^*) \to C_c^\infty(\mathcal{G})$ are a certain deformation quantization of the Poisson manifold $A(\mathcal{G})^*$, and the limit is some kind of normal derivative. All of this will be properly treated in this section, starting with Lie algebroid deformations.

2.4.1 Lie algebroid deformations

Let $A \to M$ be a Lie algebroid. Following [CM08] we define the deformation complex $\mathrm{C}^\bullet_{\mathrm{def}}(A)$ of A, by setting $\mathrm{C}^k_{\mathrm{def}}(A)$ to be those antisymmetric $\mathbb{R}$-multilinear maps $D\colon \Gamma(A)^{\times k} \to \Gamma(A)$ that have a symbol, i.e. a map $\sigma_D\colon \Gamma(A)^{\times(k-1)} \to TM$ such that

$$D(\alpha_1,...,\alpha_{k-1}, f\alpha_k) = fD(\alpha_1,...,\alpha_k) + \sigma_D(\alpha_1,...,\alpha_{k-1})(f)\alpha_k$$

for $f \in C^\infty(M)$. The differential is then set by

$$\begin{aligned}(\delta D)(\alpha_1,...,\alpha_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}[\alpha_i, D(\alpha_1,...,\widehat{\alpha_i},...,\alpha_{k+1})]\\ &+\sum_{i<j}(-1)^{i+j}D([\alpha_i,\alpha_j],\alpha_1,...,\widehat{\alpha_i},\widehat{\alpha_j},...,\alpha_{k+1}).\end{aligned}$$

We note that this complex is a form of Poisson cohomology. Recall that for a Poisson manifold (M,π), the Poisson complex $\mathrm{C}^\bullet_{\mathrm{Pois}}(M,\pi)$ is given by

$$\mathrm{C}^k_{\mathrm{Pois}}(M,\pi) \coloneqq \Gamma(\Lambda^k TM).$$

Note that we can see multivector fields as maps

$$D\colon \Lambda^\bullet C^\infty(M) \to C^\infty(M)$$

that have a symbol

$$\sigma_D\colon \Lambda^{\bullet-1}C^\infty(M) \to \mathfrak{X}(M).$$

The differential on this complex is given by taking the Schouten-Nijenhuis bracket with the Poisson bivector π, or, if we write it using the (D,σ_D)-notation, we have

$$\begin{aligned}(\delta D)(f_1,...,f_{k+1}) \coloneqq &\sum_{i=1}^{k+1}(-1)^{i+1}\left\{f_i, D(f_1,...,\widehat{f_i},...,f_{k+1})\right\}\\ &+\sum_{i<j}(-1)^{i+j}D(\{f_i,f_j\},f_1,...,\widehat{f_i},\widehat{f_j},...,f_{k+1})\end{aligned}$$

where $\{-,-\}$ is the Poisson bracket on $C^\infty(M)$ induced by π.

Also recall that A^* is canonically a Poisson manifold, with the Poisson bracket on $C^\infty(A^*)$ determined by what is does with functions that are constant on fibres, and those that are fibrewise linear. Setting notation, we write $p\colon A \to M$ for the projection, and for $a \in \Gamma(A)$ we denote by $\widehat{a} \in C^\infty(A^*)$ the induced smooth map. The Poisson bracket is now determined by setting

$$\begin{aligned} \{p^*f, p^*g\} &= 0, \\ \{\widehat{a}, p^*f\} &= p^*(\rho(a)f), \\ \{\widehat{a}, \widehat{b}\} &= \widehat{[a,b]}. \end{aligned}$$

Using this description we see that the deformation complex of a Lie algebroid is the *linear* Poisson complex, i.e. the subcomplex that consists of those multivector fields that preserve fibrewise linear smooth functions. To see this equivalence, note that fibrewise linear functions on A^* are the same thing as sections of A. In that way, if we have an element $D \in \mathrm{C}^k_{\mathrm{Pois}}(A^*)$ of the Poisson complex that preserves fibrewise linear functions, we can induce a map $\widehat{D}\colon \Lambda^k\Gamma(A) \to \Gamma(A)$ by the requirement that

$$D(\widehat{a_1}, ..., \widehat{a_k}) = \widehat{\widehat{D}(a_1, ..., a_k)}.$$

Notice that $\widehat{D}$ has a symbol, precisely since D has one. So we obtain an element $\widehat{D} \in \mathrm{C}^k_{\mathrm{def}}(A)$.

Conversely, if we have an element $D \in \mathrm{C}^k_{\mathrm{def}}(A)$ of the Lie algebroid deformation complex, we can induce an element $\check{D} \in \mathrm{C}^k_{\mathrm{Pois}}(A^*)$ by reversing this process. First, by reversing the previous equation, we can define $\check{D}$ on fibrewise linear functions. Then using the fact that $\check{D}$ should be a multivector field, and hence have derivational properties, we can define its values for any fibrewise polynomial function. By a Taylor approximation argument, the value on any function on A^* is now fixed. Notice that by construction, $\check{D}$ preserves fibrewise linear functions.

To see that the isomorphisms $\mathrm{C}^\bullet_{\mathrm{Pois,lin}}(A^*) \leftrightarrow \mathrm{C}^\bullet_{\mathrm{def}}(A)$ are isomorphisms of cochain complexes, we note that the differentials get intertwined precisely because

$$\{\widehat{a}, \widehat{b}\} = \widehat{[a,b]}$$

for $a, b \in \Gamma(A)$.

So we see that the Lie algebroid deformation complex is a kind of Poisson cohomology complex. This is not surprising, since the Poisson cohomology complex encodes deformations of the Poisson bivector, with linear Poisson cohomology encoding linear deformations of the linear Poisson bivector, and since linear Poisson bivectors on A^* are equivalent to Lie algebroid structures on A, the connection is clear.

2.4.2 The van Est-map

Next, we recall the van Est-map for Lie groupoids as defined by [CrMS20]. First, we restrict ourselves to the normalized deformation complex $\widehat{\mathrm{C}}^k_{\mathrm{def}}(\mathcal{G})$ of Definition 2.2.6[1]. Given a section $\alpha \in \Gamma(A(\mathcal{G}))$ we define maps $R_\alpha \colon \widehat{\mathrm{C}}^k_{\mathrm{def}}(\mathcal{G}) \to \widehat{\mathrm{C}}^{k-1}_{\mathrm{def}}(\mathcal{G})$ for $k > 1$ given by

$$(R_\alpha(c)(g_1, ..., g_{k-1}) := (-1)^{k-1} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} c(g_1, ..., g_{k-1}, \Phi^{\epsilon}_{\overrightarrow{\alpha}}(s(g_{k-1}))^{-1})$$

and a map $R_\alpha \colon \widehat{\mathrm{C}}^1_{\mathrm{def}}(\mathcal{G}) \to \Gamma(A(\mathcal{G}))$ given by

$$R_\alpha(c) := [c, \overrightarrow{\alpha}]|_M$$

The van Est-map is now a map $\mathcal{V} \colon \widehat{\mathrm{C}}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{def}}(A(\mathcal{G}))$ given by

$$(\mathcal{V}c)(\alpha_1, ..., \alpha_k) = \sum_{\sigma \in S_k} (-1)^\sigma (R_{\alpha_{\sigma(k)}} \circ \cdots \circ R_{\alpha_{\sigma(1)}})(c),$$

and the important generalization of the classical van Est Theorem is then:

Theorem 2.4.2 [CrMS20, Thm 10.1] The van Est-map is a chain map $\mathcal{V} \colon \widehat{\mathrm{C}}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet(A(\mathcal{G}))$ with the property that if $\mathcal{G}$ has k-connected s-fibres, the van Est-map induces isomorphisms

$$\mathrm{H}^k_{\mathrm{def}}(\mathcal{G}) \xrightarrow{\cong} \mathrm{H}^k_{\mathrm{def}}(A(\mathcal{G}))$$

for $0 \leq k \leq q$.

2.4.3 Deformation quantization

We want to see the van Est map as some kind of 'classical limit' of our chain map, in the following sense. We note that the van Est map takes values in the Poisson cohomology complex of A^* (in particular in the linear Poisson complex), so we can also think of $\mathcal{V}(c)$ as something that eats functions on A^* and spits out a function on A^*. Suppose now that f is a smooth function on $A^*(\mathcal{G})$ and we have a family $\{q_t(f) \in \mathcal{A}_\mathcal{G}\}_{t \in \mathbb{R}\setminus\{0\}}$ that has f as their limit is some sense. We may want then to look at the limiting behaviour of

$$\Phi(c)(q_t(f_1), ..., q_t(f_n))$$

as t approaches 0. The consideration of this kind of behaviour is what the idea of a 'classical limit' is.

To make this precise, we take a step into the world of deformation quantization.

Definition 2.4.3 Let $(A, \cdot)$ be a commutative $\mathbb{K}$-algebra with a unit e. A *$\star$-product* on A is a product $\star$ on the power series ring $A[[\hbar]]$ satisfying

[1]With the $(k = 0)$-part attached!

- It is bilinear over $\mathbb{K}[[\hbar]]$.
- It is associative.
- For $a, b \in A$, writing
$$a \star b = \sum_{n=0}^{\infty} B_n(a,b)\hbar^n$$
for bilinear maps $B_n \colon A \times A \to A$, we have $B_0(a,b) = a \cdot b$.
- It has e as a unit.

Remark 2.4.4 If A has any kind of topology, we can assign a numerical value to $\hbar$ and see whether the induced power series defining $a \star b$ converges in A. If it does, we may think of the $\star$-product as proper deformation of the product $\cdot$, by a family of products $\star_\epsilon$ obtained by setting $\hbar = \epsilon$. Even if the power series does not converge, or if A indeed has no topology, we may still think of the $\star$-product as a 'formal deformation'.

From a $\star$-product we can extract certain pieces of information, describing pieces of 'geometry' of A.

- Using the associativity of $\star$ we find that
$$a_1 B_1(a_2,a_3) - B_1(a_1a_2,a_3) + B_1(a_1,a_2a_3) - B_1(a_1,a_2)a_3 = 0$$
as this is the order-$\hbar$-term of the associator of $\star$. In particular B_1 is a closed element of $\mathrm{C}^2_{\mathrm{Hoch}}(A,A)$. This is similar in spirit, although different in origin, to the construction of Theorem 1.1.16, since if we assign a value ϵ to $\hbar$ to obtain a formal product $\star_\epsilon$ on A we see that B_1 is the deformation element $\frac{d}{dt}\big|_{\epsilon=0} \star_\epsilon$ we described in Theorem 1.1.16.
- We can induce a Poisson bracket on A out of a $\star$-product by the formula
$$\{a,b\} = B_1(a,b) - B_1(b,a) = \lim_{\hbar \to 0} \frac{a \star b - b \star a}{\hbar}.$$
That this is indeed a Poisson bracket on A, follows similar to the previous point from the associativity of $*$.

From these constructions, the following definition arises.

Definition 2.4.5 If $(A, \cdot, \{-,-\})$ is a Poisson algebra, then a *formal deformation quantization of* A is a $\star$-product on A whose induced Poisson bracket is $\{-,-\}$.

In particular we can ponder on the question whether for a Poisson manifold (M,π), there is a deformation quantization of $(C^\infty(M), \cdot, \{-,-\}_\pi)$. The question is answered positively by Kontsevich [Ko03] for the formal case.

This formalism, even though it is well-studied and gives the heuristic picture of what we're trying to achieve, does not cover the philosophy that we want to see $\mathcal{A}_\mathcal{G}$ as deformation of the Poisson manifold $A(\mathcal{G})^*$. For this we want to have a notion of deformations where the underlying space changes with the time parameter. To this end, we make the following definition, which is due to Landsman [LR01, Def 4.3] and Rieffel [Ri90].

Definition 2.4.6 A *strict deformation quantization* of a Poisson algebra $(A, \cdot, \{-,-\})$ consists of

- An interval $I \subset \mathbb{R}$ containing 0
- A continuous field A_I of C^*-algebras with evaluations $(A_t)_{t\in I}$ where $A_0 = A$.
- A linear map $q\colon A \to A_I$ with evaluations $q_t\colon A \to A_t$ such that $q_0 = \mathrm{id}_A$, and
  - $\lim_{t\to 0}(q_t(a_1)q_t(a_2) - q_t(a_1 \cdot a_2)) = 0$ for every $a_1, a_2 \in A$
  - $\lim_{t\to 0} \frac{1}{it}([q_t(a_1), q_t(a_2)] - q_t(\{a_1, a_2\})) = 0$ for every $a_1, a_2 \in A$.

Remark 2.4.7 The notion of a continuous field of C^*-algebras more or less means that an element of A_I can be thought of as a collection $\{a(t) \in A_t\}_{i\in I}$ that is *continuous* in a certain sense. Saying that $\lim_{t\to 0} a(t) = a_0$ then means that setting $a(0) := a_0$ makes sure that $\{a(t)\}_{t\in I}$ is an element of A_I.

The idea is now to define such a strict deformation quantization of the Poisson manifold $A(\mathcal{G})^*$ using $A_t = \mathcal{A}_\mathcal{G}$ for every $t \neq 0$. In such a setting we can make sense of a classical limit of derivations on $\mathcal{A}_\mathcal{G}$. For instance,suppose we have a family D_t of derivations of A_t such that for every $a \in A$ we have that $\lim_{t\to 0} D_t(q_t(a))$ is defined. We then see that

$$D_0(a) = \lim_{t\to 0} D_t(q_t(a))$$

satisfies

$$D_0(\{a_1, a_2\}) = \{a_1, D_0(a_2)\} + \{D_0(a_1), a_2\}$$

by taking the limit of $\frac{1}{it}\left((bD_t)(q_t(a_1), q_t(a_2)) - (bD_t)(q_t(a_2), q_t(a_1))\right) = 0$ as $t \to 0$.

2.4.4 The adiabatic groupoid

In the theory of deformation quantizations and applications thereof, there is an inherent place for replacing a groupoid with its adiabatic groupoid, as first described in [Co94]. Furthermore, in the context of deformation quantization as discussed in the last part, the adiabatic groupoid allows us to define a notion of taking limits that is especially suited to treatment of the van Est-map. In this subsection we review the theory of the adiabatic groupoid in a way tailored to our construction of the cochain map in the previous section. We therefore start by describing the adiabatic groupoid using the division map:

Definition 2.4.8 Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid, with Lie algebroid $A \xrightarrow{\pi} M$. We define the *adiabatic groupoid* $\mathcal{G}_{\text{ad}} \to M \times \mathbb{R}$ by:

$$\mathcal{G}_{\text{ad}} = A \times \{0\} \sqcup \mathcal{G} \times \mathbb{R}^*$$

The source and target are defined by:

$$\begin{aligned} s(v,0) &= (\pi(v),0), \\ s(g,\tau) &= (s(g),\tau), \\ t(v,0) &= (\pi(v),0), \\ t(g,\tau) &= (t(g),\tau). \end{aligned}$$

Then we define the inversion map by

$$\begin{aligned} \iota(v,0) &= (-v,0), \\ \iota(g,\tau) &= (\iota(g),\tau). \end{aligned}$$

Lastly, to define the division map, we note that pairs of divisible arrows come in 2 shapes, namely pairs $(v,0)$ and $(w,0)$ with $\pi(v)=\pi(w)$, and pairs (g,τ) and (h,τ) where g and h are divisible. We then define the division map by:

$$\begin{aligned} \overline{m}((v,0),(w,0)) &= (v-w,0), \\ \overline{m}((g,\tau),(h,\tau)) &= (\overline{m}(g,h),\tau). \end{aligned}$$

This is just the set-theoretical description, but the remarkable feature is that the adiabatic groupoid can be given a smooth structure. Here we briefly recall this smooth structure and show how to extend normalized deformation elements to deformation elements of the adiabatic groupoid. Both will be done in the context of the procedure known as the *deformation to the normal cone*.

Deformation to the normal cone

The part of the discussion below concerning the smooth structure and the smooth maps on the deformation to the normal cone is after [Hi10, §4] and [DS17, §1.1].

Definition 2.4.9 Let $S \hookrightarrow M$ be a submanifold with normal bundle $N \to S$. The *deformation to the normal cone* $N(M,S)$ is the manifold defined by:

$$N(M,S) = N \times \{0\} \sqcup M \times \mathbb{R}^*.$$

The deformation to the normal cone can be given a topology and smooth structure in two ways, leading to the same result. Either it is characterized by the fact that the following two types of maps are smooth:

- The map $N(M,S) \to M \times \mathbb{R}$ that sends (x,τ) for $\tau \neq 0$ to (x,τ) and sends $(v,0)$ with $v \in N_x$ to $(x,0)$.

- For every $f \in C^\infty(M)$ such that $f|_S = 0$, the map $\delta f\colon N(M,S) \to \mathbb{R}$ defined by

$$(\delta f)(x,\tau) = \frac{f(x)}{\tau} \ \ (x \in M,\ \tau \neq 0),$$
$$(\delta f)(v,0) = d_n f(v) \ \ (v \in N).$$

Here by $d_n f$ we mean the smooth map on N that for $v \in TM|_S$ sends $[v]$ to $df(v)$. This map is well-defined since $f|_S = 0$.

Equivalently, one uses an exponential map, that is a map $\theta\colon U \to M$ from an open neighbourhood $U \subset N$ of the zero-section, with the property that for all $p \in S$ and $v \in N_p$ it holds that

$$\theta(0_p) = p, \qquad \left.\frac{d}{d\tau}\right|_{\tau=0} \theta(\tau v) = v \mod T_p S.$$

The smooth structure on $N(M,S)$ can then also be characterized by the fact that the maps

$$i_1\colon M \times \mathbb{R}^* \to N(M,S)\colon\ (x,\tau) \mapsto (x,\tau)$$
$$i_2\colon U' = \{(v,\tau) \in N \times \mathbb{R} : \tau v \in U\} \to N(M,S)\colon\ \begin{array}{ccc} (v,\tau) & \mapsto & (\theta(\tau v),\tau) \\ (v,0) & \mapsto & (v,0) \end{array}$$

are smooth open embeddings.

We remark that deformations to normal cones have the action of $\mathbb{R}^\times$, given on $N(M,S)$ by:

$$\lambda \cdot (x,\tau) = (x,\lambda\tau),$$
$$\lambda \cdot (v,0) = \left(\frac{v}{\lambda},0\right),$$

where $\lambda,\tau \in \mathbb{R}^*$, $x \in M$ and $v \in N$.

Using this smooth structure, we have the following functoriality result.

Lemma 2.4.10 Let (M_1,S_1) and (M_2,S_2) be two pairs of a manifold with a submanifold, with normal bundles N_1 and N_2. If $F\colon (M_1,S_1) \to (M_2,S_2)$ is a smooth map, the map $NF\colon N(M_1,S_1) \to N(M_2,S_2)$ defined by

$$NF(x,\tau) = (F(x),\tau) \ \ (x \in M,\ \tau \neq 0),$$
$$NF(v,0) = (d_n F(v),0) \ \ (v \in N_1)$$

is smooth. Here $d_n F\colon N_1 \to N_2$ is the normal derivative, which is well-defined since $F(S_1) \subset S_2$.

We will describe how to extend a vector field on M that is parallel to S to a vector field on $N(M,S)$ that is invariant under the $\mathbb{R}^\times$-action.

This will be done by writing down a vector field on the normal bundle and combining it with a vector field over $M \times (\mathbb{R}\backslash\{0\})$ to a discrete vector field on $N(M,S)$, and using an explicit description of the smooth functions on $N(M,S)$ to show that this is in fact a *smooth* vector field.

Definition 2.4.11 [Hi10] Let X be a set and let $\mathcal{F} = \{f_\alpha : X \to V_\alpha\}$ be a family of functions from X into smooth manifolds. We say that a function $f\colon X \to \mathbb{R}$ is *smoothly composed from the family* $\mathcal{F}$ if there is a finite collection $(f_{\alpha_1}, ..., f_{\alpha_n}) \in \mathcal{F}^{\times n}$ and a smooth map $h\colon V_{\alpha_1} \times \cdots \times V_{\alpha_n} \to \mathbb{R}$ such that

$$f(x) = h(f_{\alpha_1}(x), ..., f_{\alpha_n}(x)).$$

The smooth structure of $N(M,S)$ then means that all smooth functions on $N(M,S)$ are smoothly composed of type of functions as described after Theorem 2.4.9. If we then apply Taylors theorem we conclude the following.

Lemma 2.4.12 A discrete vector field X on $N(M,S)$ is smooth if and only if for every $f \in C^\infty(M)$ with $f|_S = 0$ and every $g \in C^\infty(M\times\mathbb{R})$ the maps δf and $\widetilde{g} \in C^\infty(N(M,S))$ defined by:

$$\begin{aligned} (\delta f)(x,\tau) &= \tfrac{f(x)}{\tau} && (\tau \neq 0) \\ (\delta f)(v,0) &= d_n f(v) && (v \in N) \end{aligned}$$

$$\begin{aligned} \widetilde{g}(x,\tau) &= g(x,\tau) && (\tau \neq 0) \\ \widetilde{g}(v,0) &= g(x,0) && (v \in N_x) \end{aligned}$$

satisfy that $X(\delta f), X(\widetilde{g}) \in C^\infty(N(M,S))$.

We start with writing down the vector field over N. This is the *linearization*, as also described in [AZ14, §4.1], that we describe in detail below:

Proposition 2.4.13 Let $S \hookrightarrow M$ be a submanifold with normal bundle $\pi\colon N \to S$ and $X \in \mathfrak{X}(M)$ a vector field that is parallel to S. Then:

a) The map that sends a smooth function $f \in C^\infty(M)$ satisfying $f|_S = 0$ to the map $d_n f \in C^\infty_{\text{lin}}(N)$ is a surjection onto $C^\infty_{\text{lin}}(N)$.

b) If $f \in C^\infty(M)$ satisfies $f|_S = 0$ and $d_n f = 0$, then Xf satisfies $d_n(Xf) = 0$.

c) The maps $(X_N)_{\text{lin}}\colon C^\infty_{\text{lin}}(N) \to C^\infty(N)$ and $(X_N)_{\text{cst}}\colon C^\infty(S) \to C^\infty(N)$ defined by

$$(X_N)_{\text{lin}}(d_n f) = d_n(Xf)$$

$$(X_N)_{\text{cst}}(g) = X|_S(g) \circ \pi$$

define a smooth vector field $X_N \in \mathfrak{X}(N)$.

Proof. Working down the list:

a) By using a partition of unity this reduces to the local case $M = \mathbb{R}^m \times \mathbb{R}^n$ with $S = \mathbb{R}^m \times \{0\}$. In this local case there is a canonical diffeomorphism between M and N and pushing a linear map on N through this canonical diffeomorphism yields a smooth map on M whose normal derivative equals the linear map on N we started with.

b) This is again a computation in the local case $M = \mathbb{R}^m \times \mathbb{R}^n$ with $S = \mathbb{R}^m \times \{0\}$. Write

$$X = \sum_{i=1}^{m} \alpha_i(x,y)\frac{\partial}{\partial x_i} + \sum_{j=1}^{n} \beta_j(x,y)\frac{\partial}{\partial y_j}.$$

The fact that X is parallel to S means that $\beta_j(x,0) = 0$ for all $j = 1, ..., n$. The fact that $d_n f = 0$ is equivalent to the fact $\frac{\partial f}{\partial y_j}(x,0) = 0$ for all $j = 1, ..., n$. Then we have

$$Xf = \sum_{i=1}^{m} \alpha_i \frac{\partial f}{\partial x_i} + \sum_{j=1}^{n} \beta_j \frac{\partial f}{\partial y_j},$$

so that for $k = 1, ..., n$ we have

$$\frac{\partial(Xf)}{\partial y_k} = \sum_{i=1}^{m} \frac{\partial \alpha_i}{\partial y_k}\frac{\partial f}{\partial x_i} + \sum_{i=1}^{m} \alpha_i \frac{\partial^2 f}{\partial y_k \partial x_i} + \sum_{j=1}^{n} \frac{\partial \beta_j}{\partial y_k}\frac{\partial f}{\partial y_j} + \sum_{j=1}^{n} \beta_j \frac{\partial^2 f}{\partial y_k \partial y_j}.$$

Then since respectively we have $\frac{\partial f}{\partial x_i}(x,0) = 0$ (since $f(x,0) = 0$), $\frac{\partial^2 f}{\partial y_k \partial x_i}(x,0) = \left(\frac{\partial}{\partial x_i}\frac{\partial f}{\partial y_k}\right)(x,0) = 0$ (since $\frac{\partial f}{\partial y_k}(x,0) = 0$), $\frac{\partial f}{\partial y_j}(x,0) = 0$ (by assumption) and $\beta_j(x,0) = 0$ (by assumption), we see that

$$\frac{\partial(Xf)}{\partial y_k}(x,0) = 0$$

which implies that $d_n(Xf) = 0$.

c) First note that (by restriction) a smooth vector field $Y \in \mathfrak{X}(E)$ on a vector bundle $\pi\colon E \to M$ is the same as a pair of maps $Y_{\text{lin}}\colon C^\infty_{\text{lin}}(E) \to C^\infty(E)$ and $Y_{\text{cst}}\colon C^\infty(M) \to C^\infty(E)$ such that for all $f, g \in C^\infty(M)$ and $h \in C^\infty_{\text{lin}}(E)$ it holds that

$$Y_{\text{cst}}(fg) = (f \circ \pi) \cdot Y_{\text{cst}}(g) + (g \circ \pi) \cdot Y_{\text{cst}}(f)$$
$$Y_{\text{lin}}((f \circ \pi) \cdot h) = (f \circ \pi) \cdot Y_{\text{lin}}(h) + h \cdot Y_{\text{cst}}(f)$$

We show that these properties hold for the maps $(X_N)_{\text{cst}}$ and $(X_N)_{\text{lin}}$.

First we note that $(X_N)_{\text{lin}}$ is well-defined by parts a) and b). To show that they define a smooth vector field we see for $f, g \in C^\infty(S)$ that

$$\begin{aligned}(X_N)_{\text{cst}}(fg) &= (X|_S(fg)) \circ \pi = (f \cdot X|_S(g) + g \cdot X|_S(f)) \circ \pi \\ &= (f \circ \pi) \cdot (X|_S(g) \circ \pi) + (g \circ \pi) \cdot (X|_S(f) \circ \pi) \\ &= (f \circ \pi) X_{\text{cst}}(g) + (g \circ \pi) X_{\text{cst}}(f).\end{aligned}$$

Secondly let $f \in C^\infty(S)$ and $h \in C^\infty_{\text{lin}}(N)$ given by $h = d_n g$ with $g \in C^\infty(M)$ such that $g|_S = 0$. Then first we need to find $g' \in C^\infty(M)$ with $g'|_S = 0$ such that $fh = d_n(g')$. This can be done by choosing an extension of f which is 'constant in the normal direction', which is only well-defined locally or if we choose an exponential map.

We resort to the local case $M = \mathbb{R}^m \times \mathbb{R}^n$ with $S = \mathbb{R}^m \times \{0\}$. Then the map $g'(x,y) = f(x)g(x,y)$ clearly satisfies that $d_n g' = fh$. Then writing X in coordinates as

$$X = \sum_{i=1}^{m} \alpha_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial y_j},$$

we have

$$(Xg')(x,y) = \sum_{i=1}^{m} \alpha_i(x,y) \frac{\partial f}{\partial x_i}(x) g(x,y) + f(x)(Xg)(x,y)$$

so that we see

$$\begin{aligned} \frac{\partial (Xg')}{\partial y_k}(x,0) = \sum_{i=1}^{m} \alpha_i(x,0) \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial y_k}(x,0) + \\ + \sum_{i=1}^{m} \frac{\partial \alpha_i}{\partial y_k}(x,0) \frac{\partial f}{\partial x_i}(x) g(x,0) + f(x) \frac{\partial (Xg)}{\partial y_k}(x,0) \end{aligned}$$

Notice that $g(x,0) = 0$ so that the middle term vanishes. Then by recognizing the relevant terms, we obtain

$$d(Xg')_{(x,0)} \left(\frac{\partial}{\partial y_k} \right) = X|_S(f)(x) \cdot (dg)_x \left(\frac{\partial}{\partial y_k} \right) + f(x) \cdot d(Xg)_{(x,0)} \left(\frac{\partial}{\partial y_k} \right)$$

so that globally we have

$$\begin{aligned} (X_N)_{\text{lin}}((f \circ \pi) \cdot d_n g) &= (X_N)_{\text{lin}}(d_n g') \\ &= d_n(Xg') \\ &= (X|_S(f) \circ \pi) d_n g + (f \circ \pi) d(Xg) \\ &= (X_N)_{\text{cst}}(f) d_n g + (f \circ \pi)(X_N)_{\text{lin}}(d_n g), \end{aligned}$$

so we see that we obtain a smooth vector field $X_N \in \mathfrak{X}(N)$.

This completes the proof. □

We are now ready to define the $\mathbb{R}^*$-invariant extension of the vector field X.

Proposition 2.4.14 Let $S \hookrightarrow M$ be a submanifold with normal bundle $N \to S$. Let $X \in \mathfrak{X}(M)$ be a vector field that is parallel to S. Then the discrete vector field X_{inv} on $N(M,S)$ defined by

$$\begin{aligned} X_{\text{inv}}(x,\tau) &= X(x), \quad (\tau \neq 0) \\ X_{\text{inv}}|_{N \times \{0\}} &= X_N \end{aligned}$$

is a smooth vector field $X_{\text{inv}} \in \mathfrak{X}(N(M,S))$, which is the unique vector field on $N(M,S)$ which equals X on $M \times \mathbb{R} \backslash \{0\}$ and the unique $\mathbb{R}^\times$-invariant vector field on $N(M,S)$ which equals X along $M \times \{1\}$.

Proof. The invariance and uniqueness is clear once we know that X_{inv} is smooth. To show that it is smooth, by Theorem 2.4.12 the only thing we have to check is that $X_{\text{inv}}(\delta f)$ and $X_{\text{inv}}(\widetilde{g})$ are smooth for $f \in C^\infty(M)$ with $f|_S = 0$ and $g \in C^\infty(M \times \mathbb{R})$. The definition of X_N makes sure that

$$X_{\text{inv}}(\delta f) = \delta(Xf),$$
$$X_{\text{inv}}(\widetilde{g}) = \widetilde{Xg},$$

where in the second equation X acts on $C^\infty(M \times \mathbb{R})$ as the vector field $X(x,\tau) = X(x)$ on $M \times \mathbb{R}$. By definition $\delta(Xf)$ and $\widetilde{Xg}$ are smooth and so the result follows. □

The adiabatic groupoid as a deformation to the normal cone

We can now apply this to the case $M \hookrightarrow \mathcal{G}$ with normal bundle $A = \ker ds|_M$, and we obtain the underlying set of the adiabatic groupoid $\mathcal{G}_{\text{ad}}$ from Definition 2.4.8. The fact that the source, target and division maps are smooth, follows from the fact that away from $\tau = 0$ they are just the respective maps of the original groupoid, while along $\tau = 0$ they are the normal derivatives of the respective maps. By Lemma 2.4.10 they are smooth. We note that an exponential map can be obtained by choosing a connection on A, see [NWX99] and [La98].

Next we want to describe the nerve of the adiabatic groupoid. As a set it equals $(\mathcal{G}_{\text{ad}})^{(k)} = \mathcal{G}^{(k)} \times \mathbb{R}^* \sqcup A^{\oplus k} \times \{0\}$. From the viewpoint of trying to define vector fields on the nerve of the adiabatic groupoid, this set-theoretic description leads to searching for a connection between $A^{\oplus k}$ and the normal bundle of M inside $\mathcal{G}^{(k)}$ seen as the diagonal of units.

Lemma 2.4.15 Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid with $\Delta\colon M \to \mathcal{G}^{(k)}$ the diagonal inclusion via the units. The vector bundle map $\nu\colon A^{\oplus k} \to \Delta^* T\mathcal{G}^{(k)}$ given by

$$\nu(v_1, ..., v_k) = \left(v_1 + \sum_{i=2}^{k} du(dt(v_i)), v_2 + \sum_{i=3}^{k} du(dt(v_i)), ..., v_{k-1} + du(dt(v_k)), v_k \right)$$

induces an isomorphism between $A^{\oplus k}$ and the normal bundle of M inside $\mathcal{G}^{(k)}$.

Proof. First one checks that ν indeed maps into the tangent space of $\mathcal{G}^{(k)} \subset \mathcal{G}^{\times k}$, which is a simple calculation. Next to show that it induces an isomorphism to the normal bundle to Δ, we first use the decomposition $T_{1_x}\mathcal{G} = A_x \oplus T_x M$ to see that if $\nu(v_1, ..., v_k) \in T_x M \subset T_{\Delta(x)}\mathcal{G}^{(k)}$ then $(v_1, ..., v_k) = 0$, so that the map into the normal bundle is injective. A simple case of dimension counting then implies that it the induced map is an isomorphism. □

Corollary 2.4.16 There is a natural isomorphism between $N(\mathcal{G}^{(k)}, M)$ and $\mathcal{G}_{\text{ad}}^{(k)}$ which away from $\tau = 0$ links $((g_1, ..., g_k), \tau)$ and $((g_1, \tau), ..., (g_k, \tau))$.

Haar systems on the adiabatic groupoid

We intend to link deformation quantizations of the Poisson manifold A^* with the Van Est map $\mathcal{V}\colon \widetilde{\mathrm{C}}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{def}}(A)$. To make the syntax line up, we need the convolution algebra to be given in terms of functions, not densities. Hence as discussed before, we need to use Haar systems. We therefore briefly discuss Haar systems on the adiabatic groupoid.

In particular, we describe how to make a Haar system on $\mathcal{G}_{\mathrm{ad}}$ out of a Haar system λ on $\mathcal{G}$. We do this by fixing a section $a \in \Gamma(\mathcal{D}_s)$ of the density bundle inducing the Haar system λ. From this we make a section of the the density bundle of $\mathcal{G}_{\mathrm{ad}}$. For this we write down the canonical identifications of the distribution $\ker(ds) \to \mathcal{G}_{\mathrm{ad}}$:

$$\begin{aligned} \ker(ds)_{(g,\tau)} &\cong \ker(ds)_g \quad (\tau \neq 0) \\ \ker(ds)_{(v,0)} &\cong \ker(ds)_{u(\pi(v))} \end{aligned}$$

where $\pi\colon A(\mathcal{G}) \to M$ is the projection.

Using this identification, we have a definition of a Haar system $\widehat{\lambda}$ on $\mathcal{G}_{\mathrm{ad}}$ due to Landsman [LR01, p.19], induced by the density $\widehat{a}$ on $\mathcal{G}_{\mathrm{ad}}$ given by the formula

$$\begin{aligned} \widehat{a}(g,\tau) &= |\tau|^d a(g) \quad (\tau \neq 0), \\ \widehat{a}(v,0) &= a(\pi(v)), \end{aligned}$$

where d is the dimension of the s-fibres of $\mathcal{G}$.

Note that in particular we obtain a Haar system on the vector bundle $A \to M$, seen as a groupoid in the canonical way.

We can also write down the convolution product on the adiabatic groupoid under the isomorphism given by this Haar system. If we have two compactly supported functions f_1, f_2 on $\mathcal{G}_{\mathrm{ad}}$ we obtain:

$$(f_1 * f_2)(g,\tau) = |\tau|^{-d} \int_{s^{-1}(s(g))} f_1(gh^{-1},\tau) f_2(h,\tau) \lambda_{s(g)}(h) \quad (\tau \neq 0),$$

$$(f_1 * f_2)(v,0) = \int_{A_{\pi(v)}} f_1(v-w,0) f_2(w,0) \lambda_{\pi(v)}(w).$$

At this point we notice that the convolution at $\tau = 0$ does not require the functions to be compactly supported on A_x, being Schwartz is enough (c.f. the usual theory of Fourier transform in $\mathbb{R}^n$). This allows us, in the case of $\mathcal{G}_{\mathrm{ad}}$, to enlarge the type of functions/densities on which we let the deformation complex act.

To this end we refer to the work of Carrillo-Rouse [C-R08], where a Fréchet algebra $\mathcal{S}_c(\mathcal{G}_{\mathrm{ad}})$ is constructed with evaluations

$$\mathcal{S}_c(\mathcal{G}_{\mathrm{ad}})_t = \begin{cases} \mathcal{S}_c(A) & t = 0 \\ C_c^\infty(\mathcal{G}) & t \neq 0. \end{cases}$$

Here $\mathcal{S}_c(A)$ denotes the space of functions that are Schwartz along the fibers of the Lie algebroid and have compact support along M. This Schwartz type algebra is roughly defined as consisting of those functions $f \in C^\infty(\mathcal{G}_{\mathrm{ad}})$ that have the property that

- $f(-,0)$ is a Schwartz function on $A(\mathcal{G})$;
- For every $t \neq 0$, $f(-,t)$ has compact support on $\mathcal{G}$;
- The behaviour of f in the $\mathbb{R}$-direction of $\mathcal{G}_{\mathrm{ad}}$ is of Schwartz type.

It is topologized using the usual seminorms to obtain a Fréchet topology and, as such, it should be thought of as a dense subalgebra of the reduced C^*-algebra $C^*_r(\mathcal{G}_{\mathrm{ad}})$.

By the discussion above, the convolution product is perfectly well-defined on $\mathcal{S}_c(\mathcal{G}_{\mathrm{ad}})$ and we can extend our viewpoint of the map $\Phi\colon \mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}_{\mathrm{ad}}) \to \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_{\mathcal{G}_{\mathrm{ad}}})$ to let $\Phi(c)$ (for $c \in \mathrm{C}^k_{\mathrm{def}}(\mathcal{G}_{\mathrm{ad}})$) act on functions in $\mathcal{S}_c(\mathcal{G}_{\mathrm{ad}})$. At this point it should be remarked that the isomorphism between functions and densities induced by a Haar system does not preserve the action of vector fields (indeed on the level of densities one also needs to compare $\mathcal{L}_X\lambda$ with λ!). So really we should introduce in parallel to $\mathscr{S}_c(\mathcal{G}_{\mathrm{ad}})$ the notion of densities with are of Schwartz-type along $\tau = 0$, but for the sake of not being overly pedantic we will not do this and just be careful when writing down the action of $\Phi(c)$.

This algebra also allows us to define the suitable notion of a limit of a family of compactly supported functions on $\mathcal{G}$.

Definition 2.4.17 A smooth family $\{f_t\}_{t\neq 0}$ of compactly supported functions on $\mathcal{G}$ converges to $f' \in \mathcal{S}_c(A)$, with notation

$$\lim_{t\to 0} f_t = f' \tag{2.5}$$

if the function $F\colon \mathcal{G}_{\mathrm{ad}} \to \mathbb{R}$ given by

$$F(g,t) = f_t(g)$$

$$F(v,0) = f'(v)$$

is an element of $\mathcal{S}_c(\mathcal{G}_{\mathrm{ad}})$.

Remark 2.4.18 The limit as defined above has the very desirable property that if it exists, it is unique. Indeed, if $F \in C^\infty(\mathcal{G}_{\mathrm{ad}})$ is such that $F(g,t) = 0$ for every $g \in \mathcal{G}$ and $t \neq 0$, it follows that $F(v,0) = 0$ for every $v \in A(\mathcal{G})$.

2.4.5 Fourier transform on vector bundles

We briefly discuss the notion of Fourier transform on a vector bundle $E \to M$ under the choice of a Haar system on E. This discussion follows the results of Landsman and Ramazan [LR01, §7]. Recall that a vector bundle $\pi\colon E \to M$ can be seen as a groupoid over M where both the source and the target map are the projection π and the multiplication is the fiberwise addition. Since $\ker(d\pi) \cong \pi^* E$ a choice of a Haar system is at every $v \in E$ a choice of a density on $E_{\pi(v)}$ that is invariant, where invariance in this case means that the choice is constant along the fiber.

If we choose such a Haar system $\{\mu_x\}_{x\in M}$, in [LR01] the Fourier transform $\mathcal{F}_\mu \colon \mathcal{S}(E) \to \mathcal{S}(E^*)$ was defined by

$$(\mathcal{F}_\mu f)(\xi_x) = \int_{E_x} f(v) e^{-i\langle \xi_x, v\rangle} d\mu_x(v).$$

Furthermore, it was shown that this map is a linear isomorphism which intertwines the μ-convolution product on E and the pointwise product on E^*, and when (x, v) are coordinates on E induced by a frame with dual coordinates (x, ξ), we have for $f \in \mathcal{S}(E)$, $g \in \mathcal{S}(E^*)$ and $a \in C^\infty(M)$ that

$$\begin{aligned}
\mathcal{F}_\mu((a\circ\pi)f) &= (a\circ\pi)\mathcal{F}_\mu(f),\\
\frac{\partial \mathcal{F}_\mu(f)}{\partial x_j} &= \mathcal{F}_\mu\left(\frac{\partial f}{\partial x_j}\right) + \left(\frac{\partial \log(\mu_e)}{\partial x_j}\circ\pi\right)\mathcal{F}_\mu(f),\\
\frac{\partial \mathcal{F}_\mu(f)}{\partial \xi_j} &= -i\mathcal{F}_\mu(v_j f),\\
\frac{\partial \mathcal{F}_\mu^{-1}(g)}{\partial v_j} &= i\mathcal{F}_\mu^{-1}(\xi_j g).
\end{aligned}$$

Note that after the choice of a Haar system μ we obtain an isomorphism between the algebra of functions $C_c^\infty(E)$ with the μ-convolution product and the convolution algebra $\mathcal{A}_E$ of densities with the (intrinsic) convolution product. In particular if $X \in \mathfrak{X}(E)$, we can see $\Phi(X)$ as defined on functions (which is, again, not equal to the usual action of vector fields on functions), and we can extend the action to Schwartz functions.

Now using the Fourier transform, we can transport the action on the convolution algebra of E to an action on the usual algebra with the pointwise product on E^*.

Proposition 2.4.19 Let X be a linear vector field on E. Then the map $\widehat{X} \colon \mathcal{S}(E^*) \to \mathcal{S}(E^*)$ given by

$$\widehat{X}(f) = \mathcal{F}_\mu(\Phi(X)(\mathcal{F}_\mu^{-1}(f))),$$

defines a linear vector field on E^*. Here Φ is the natural chain map from Definition 2.4.8, applied to the vector bundle E seen as a groupoid.

Proof. First we show that $\widehat{X}$ is indeed a vector field, i.e. a derivation with respect to the pointwise product. Since $\widehat{X}$ is the conjugation of $\Phi(X)$ with an isomorphism which intertwines the convolution product on $\mathcal{S}(E)$ and the pointwise product on $\mathcal{S}(E^*)$ this is equivalent to showing that $\Phi(X)$ is a derivation for the convolution product. When we see $E \to M$ as a groupoid, this is equivalent to showing that X is a multiplicative vector field, and it is easy to see that on a vector bundle the multiplicative vector fields are precisely the linear vector fields.

To see that $\widehat{X}$ is a linear vector field we do a local computation on a trivial vector bundle $E = \mathbb{R}^m_x \times \mathbb{R}^n_v \to \mathbb{R}^m_x$ with Haar system $f(x)dv_1 \wedge \cdots \wedge dv_n$. Using the properties of the Fourier transform stated before it follows that if

$$X(x,v) = \sum_{i=1}^m X_i(x)\frac{\partial}{\partial x_i} + \sum_{j=1}^n \sum_{k=1}^n Y_{jk}(x) v_j \frac{\partial}{\partial v_k},$$

then

$$\widehat{X}(x,\xi) = \sum_{i=1}^m X_i(x)\frac{\partial}{\partial x_i} - \sum_{j=1}^n \sum_{k=1}^n Y_{jk}(x) \xi_k \frac{\partial}{\partial \xi_j},$$

which indeed shows that $\widehat{X}$ is a linear vector field. □

Recall that a linear vector field $X \in \mathfrak{X}(E)$ is the same as a linear map $X\colon \Gamma(E^*) \to \Gamma(E^*)$ with a symbol $s_X \in \mathfrak{X}(M)$ such that

$$X(f\alpha) = fX(\alpha) + s_X(f)\alpha \qquad (f \in C^\infty(M),\ \alpha \in \Gamma(E)).$$

Furthermore, recall the canonical pairing $\langle -, - \rangle \colon \Gamma(E^*) \times \Gamma(E) \to C^\infty(M)$. Then for a linear vector field X, the local calculation from the proof above generalizes to the following.

Proposition 2.4.20 Let $X \in \mathfrak{X}(E)$ be a linear vector field, then the linear vector field $\widehat{X} \in \mathfrak{X}(E^*)$ is uniquely determined by the fact that for $\beta \in \Gamma(E^*)$ and $\alpha \in \Gamma(E)$

$$\langle \beta, \widehat{X}(\alpha)\rangle + \langle X(\beta), \alpha\rangle = s_X(\langle \beta, \alpha \rangle).$$

We can play a similar game, albeit slightly more involved in notation, for higher order deformation elements of the vector bundle. So consider an element $X \in \widehat{\mathrm{C}}^k_{\mathrm{def}}(E)$ in the normalized deformation complex given by

$$X(v_1, ..., v_n) = X_1(v_1)\langle \beta_2, v_2\rangle \cdots \langle \beta_k, v_k \rangle$$

where X_1 is a linear vector field on E and $\beta_2, ..., \beta_k \in \Gamma(E^*)$. One immediately checks that this is a closed element of $\widehat{C}^k_{\mathrm{def}}(E)$, so that the Fourier transform

$$\widehat{X}(f_1, ..., f_k) = \mathcal{F}_\mu(\Phi(X)(\mathcal{F}_\mu^{-1}(f_1), ..., \mathcal{F}_\mu^{-1}(f_k)))$$

is a closed element of the Hochschild complex of $C^\infty(E^*)$. From the specific form of X it is easy to see that

$$\Phi(X)(a_1, ..., a_k) = \Phi(X_1)(a_1) * (\beta_2 a_2) * \cdots * (\beta_k a_k)$$

where we see the a_i as fiberwise linear maps on E^*. In particular we see that

$$\widehat{X} = \widehat{X_1} \otimes \widehat{\beta}_2 \otimes \cdots \otimes \widehat{\beta}_k$$

where for $\beta \in \Gamma(E^*)$, $\widehat{\beta}$ is the vector field on E^* given by

$$\widehat{\beta}(f) = \mathcal{F}_\mu(\beta \mathcal{F}_\mu^{-1}(f)).$$

A local computation shows that $\widehat{\beta}$ is identically zero on fiberwise constant maps and that for the map induced by a section $\alpha \in \Gamma(E)$ we have

$$\widehat{\beta}(\alpha) = \frac{1}{i}\langle \beta, \alpha \rangle.$$

In particular, we see that if we anti-symmetrize, we obtain the linear multivectorfield $\widehat{X_1} \wedge \widehat{\beta}_2 \wedge \cdots \wedge \widehat{\beta}_k$ on E^*.

2.4.6 Deformation quantization of A^* and the van Est-map

In this section we apply the theory we recalled above to show that the Van Est map is in some sense a classical limit of the chain map we defined in the previous chapter. This also allows us to give an alternative proof that the van Est-map itself is a chain map.

Now, fix a choice of a Haar system of $\mathcal{G}$, which by the discussion above induces a Haar system on $\mathcal{G}_{\mathrm{ad}}$ and a Haar system μ on $A \to M$. The last one makes sure that we can talk about a Fourier transform $\mathcal{F}_\mu \colon \mathcal{S}(A) \to \mathcal{S}(A^*)$. Using the construction of [LR01] in the context of the Schwartz algebra of [C-R08], we obtain the following result.

Proposition 2.4.21 The maps $q_t \colon \mathcal{S}_c(A^*) \to C_c^\infty(\mathcal{G}),\ t \neq 0$ given by

$$q_t(f)(g) := \chi(g)\mathcal{F}_\mu^{-1}(f)\left((\frac{1}{t}\exp^{-1}(g) \right),$$

make $\mathcal{S}_c(\mathcal{G}_{\mathrm{ad}})$ into a strict deformation quantization of $A(\mathcal{G})^*$ in the sense of Definition 2.4.6. In particular

$$\lim_{t\to 0}(q_t(f_1 f_2) - q_t(f_1) * q_t(f_2)) = 0, \quad \lim_{t\to 0}(\frac{1}{it}[q_t(f_1), q_t(f_2)] - q_t(\{f_1, f_2\})) = 0. \qquad (2.6)$$

In the formula for q_t here $\chi \in C_c^\infty(\mathcal{G})$ is a cut-off function that equals 1 in a neighborhood of $M \subset \mathcal{G}$ with support inside an open neighbourhood of the units onto which the exponential map is a diffeomorphism.

Proof. The bulk of this is in done in [LR01] in a slightly different setting, but for the parts that are relevant to our consideration the proofs can still be applied. Explicitely, one of the differences with the results of [LR01] is that we do not need the property $q_t(f^*) = q_t(f)^*$ for which the Weyl exponential map $\exp^{\mathrm{W}}$ is used, and instead we can use the normal exponential map. Secondly, we do not need to restrict to Paley-Wiener functions, as we allow for Schwarz-type functions at $t = 0$ and use the cut-off function on the level of $\mathcal{G}$ instead of A, the deviation vanishing as t approaches 0. Lastly, as the relevant calculations on the local forms in A and A^* are valid for all Schwarz functions and not just Paley-Wiener functions, the relevant propositions in [LR01] still hold in this situation. □

Remark 2.4.22 The variety of quantizations by using different types of exponential maps is also reflected on the more algebraic level in [NW09] by using different orderings in the Fedosov construction of *formal* deformation quantizations of A^*.

Using this strict deformation quantization of $A(\mathcal{G})^*$, the relation between the van Est-map and our chain map $\Phi\colon C^\bullet_{\mathrm{def}}(\mathcal{G}) \to C^\bullet_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ is now as follows:

Theorem 2.4.23 Let $k \geq 1$ and $c \in \widehat{\mathrm{C}}^k_{\mathrm{def}}(\mathcal{G})$ and suppose we have chosen a Haar system on $\mathcal{G}$ inducing a Haar system μ on the algebroid A. Given $f_1, \ldots, f_k \in \mathcal{S}_c(A^*)$, the following equality is true:

$$\mathcal{V}(c)(f_1, \ldots, f_k) = \mathcal{F}_\mu \left(\lim_{t\to 0} \left(\sum_{\sigma \in S_k} (-1)^\sigma \frac{1}{(it)^{k-1}} \Phi(c)(q_t(f_{\sigma(1)}), \ldots, q_t(f_{\sigma(k)})) \right) \right).$$

Remark 2.4.24 The limit in the above equation is the limit as defined in Definition 2.4.17. In particular, if it exists, the limit is a Schwartz function on A, and its Fourier transform is a Schwartz function on A^*. The proof of the Theorem shows that the function made out of the content of the RHS, with the LHS at $t = 0$, is an element of $\mathcal{S}_c(\mathcal{G}_{\mathrm{ad}})$, and hence the limit exists and is equal to the LHS.

Proof. We start with the case $k = 1$. First note that for $f \in \mathcal{S}_c(A^*)$ the map $q(f)\colon \mathcal{G}_{\mathrm{ad}} \to \mathbb{R}$ given by

$$q(f)(g, t) = q_t(f)(g)$$
$$q(f)(v, 0) = \mathcal{F}_\mu^{-1}(f)(v)$$

is an element of $\mathcal{S}_c(\mathcal{G}_{\mathrm{ad}})$. Then note that the family $\{c\}_{t\neq 0}$ is a family of vector fields on $\mathcal{G}$ which can be extended to a vector field on $\mathcal{G}_{\mathrm{ad}}$, namely to the vector field c_{inv} obtained by Theorem 2.4.14. Then notice that $\Phi(c_{\mathrm{inv}})(q(f))$ is an element of $\mathcal{S}_c(\mathcal{G}_{\mathrm{ad}})$ consisting of

$$\Phi(c_{\mathrm{inv}})(q(f))_t = \Phi(c)(q_t(f)), \qquad (t \neq 0)$$
$$\Phi(c_{\mathrm{inv}})(q(f))_0 = \Phi(c_0)(\mathcal{F}_\mu^{-1}(f))$$

where c_0 is the linear vector field that is the restriction of c_{inv} to $t = 0$. Note that it is linear, since it is the application of Theorem 2.4.13 to the vector field c on $\mathcal{G}$. In particular we see that

$$\lim_{t\to 0} \Phi(c)(q_t(f)) = \Phi(c_0)(\mathcal{F}_\mu^{-1}(f))$$

and so we need to show that $\mathcal{V}(c) = \widehat{c_0}$.

By Theorem 2.4.20 this means that we need to show that for $\beta \in \Gamma(A^*)$ and $\alpha \in \Gamma(A)$ we have

$$\langle \beta, \mathcal{V}(c)(\alpha) \rangle + \langle c_0(\beta), \alpha \rangle = s_c(\langle \beta, \alpha \rangle).$$

We note two things. First that every $\beta \in \Gamma(A^*) = C^\infty_{\mathrm{lin}}(A)$ can be written as $d_n h$ for $h \in C^\infty(\mathcal{G})$ with $h|_M = 0$. Second that, since we have an explicit inclusion of A into the tangent bundle of $\mathcal{G}$, this means that:

$$\langle d_n h, \alpha \rangle(x) = \alpha(x)(h).$$

We are now ready to show the equality. First we have

$$\begin{aligned}\langle d_n h, \mathcal{V}(c)(\alpha)\rangle(x) &= [c, \overrightarrow{\alpha}](1_x)(h) \\ &= c(1_x)(\overrightarrow{\alpha}(h)) - \alpha(x)(c(h)),\end{aligned}$$

$$\langle c_0(d_n h), \alpha\rangle(x) = \langle d_n(ch), \alpha\rangle(x) = \alpha(x)(c(h)),$$

and since $c_{1_x} = du(s_c(x))$ combined with $\overrightarrow{\alpha}|_M = \alpha$ we have

$$s_c(\langle d_n h, \alpha\rangle)(x) = s_c(\overrightarrow{\alpha}(h)|_M)(x) = c(1_x)(\overrightarrow{\alpha}(h)).$$

For $k > 1$ we restrict to the case where $c = c_1 \otimes h_2 \otimes \cdots \otimes h_k$ with $c_1 \in \mathfrak{X}(\mathcal{G})$ and $h_2, ..., h_k \in C^\infty(\mathcal{G})$. For c to be an element of $\widehat{\mathrm{C}}^k_{\mathrm{def}}(\mathcal{G})$ it is necessary and sufficient to have $c_1 \in \widehat{\mathrm{C}}^1_{\mathrm{def}}(\mathcal{G})$ and $h_i|_M = 0$. Similar to the case $k = 1$ we note that

$$\frac{1}{(it)^{k-1}}\Phi(c)(q_t(f_1), ..., q_t(f_k)) = \Phi\left((\frac{1}{(it)^{k-1}}c\right)(q_t(f_1), ..., q_t(f_k))$$

which, as $t \to 0$, converges to

$$\Phi(c_0)(\mathcal{F}_\mu^{-1}(f_1), ..., \mathcal{F}_\mu^{-1}(f_k))$$

if we find a vector field c_0 on A that together with the family $\{\frac{1}{(it)^{k-1}}c\}_{t\neq 0}$ defines a smooth deformation element of $\mathcal{G}_{\mathrm{ad}}$.

To calculate this localization we remark that we can do the calculation in $\mathcal{G}^k$ using the cartesian product of the exponential map $A \to \mathcal{G}$, instead of working in $\mathcal{G}^{(k)}$ and using the machinery of the previous section. This is for two reasons: firstly our definition of c extends to $\mathcal{G}^k$. Secondly the difference of $(v_1, ..., v_k) \in A^{\oplus k}$ seen as tangent vectors on $\mathcal{G}^k$ and $(v_1, ..., v_k) \in A^{\oplus k}$ seen as tangent vectors in $\mathcal{G}^{(k)}$ which are normal to the units, using the isomorphism of Theorem 2.4.15, are tangent vectors in $\mathcal{G}^k$ which are along the units. Since c vanishes along the units, we can neglect this.

Now to do the actual calculation we consider the chart $\theta\colon A^{\oplus k} \times \mathbb{R}^* \to \mathcal{G}^k \times \mathbb{R}^*$ given by

$$\theta(v_1, ..., v_k, t) = (\exp(tv_1), ..., \exp(tv_k), t).$$

Then if we look at the family $\{\frac{1}{(it)^{k-1}}c\}_{t\neq 0}$, we see that if we take the pullback along θ we obtain:

$$\theta^*(\{\frac{1}{(it)^{k-1}}c\}_{t\neq 0})(v_1, ..., v_k, t) = \frac{1}{(it)^k}c_1(\exp(tv_1))h_2(\exp(tv_2)) \cdots h_k(\exp(tv_k))$$

Distributing the k powers of $\frac{1}{t}$ over the k different terms we see that

$$c_0(v_1, ..., v_k) = \frac{1}{i^{k-1}}(c_1)_0(v_1)d_n h_2(v_2) \cdots d_n h_2(v_k)$$

since

$$\frac{1}{t}c_1(\exp(tv_1)) \to (c_1)_0(v_1)$$

$$\frac{1}{t}h(\exp(tv)) \to d_n h(v)$$

as $t \to 0$, so we see that $c_0 = \frac{1}{i^{k-1}}(c_1)_0 \otimes d_n h_2 \otimes \cdots \otimes d_n h_k$, which is a linear deformation element, and we want to show that $\mathcal{V}(c)$ is the anti-symmetrization of the Fourier transform $\widehat{c_0}$. By the discussion at the end of the previous subsection we see that $\widehat{c_0}$ is determined for $\alpha_1, ..., \alpha_k \in \Gamma(A)$ by

$$\widehat{c_0}(\alpha_1, ..., \alpha_k) = \frac{1}{i^{2(k-1)}}\widehat{(c_1)_0}(\alpha_1)\langle d_n h_2, \alpha_2\rangle \cdots \langle d_n h_k, \alpha_k\rangle$$

Next we investigate $R_\alpha(c)$, we obtain:

$$\begin{aligned} R_\alpha(c)(g_1, ..., g_{k-1}) &= (-1)^{k-1}\frac{d}{d\epsilon}|_{\epsilon=0} c_1(g_1)h_2(g_2)\cdots h_{k-1}(g_{k-1})h_k(\Phi^\epsilon_{\overrightarrow{\alpha}}(s(g_k))^{-1}) \\ &= (-1)^{k-1}c_1(g_1)h_2(g_2)\cdots h_{k-1}(g_{k-1})dh_k(d\iota(\alpha(s(g_k)))). \end{aligned}$$

Then since $f_k|_M = 0$ and for $v \in A_x$ we have $d\iota(v) = -v + d(u \circ t)(v)$ we obtain

$$R_\alpha(c)(g_1, ..., g_{k-1}) = (-1)^k c_1(g_1)h_2(g_2)\cdots h_{k-1}(g_{k-1})d_n h_k(\alpha(s(g_{k-1}))).$$

Doing this inductively, and using that the flow of $\overrightarrow{\alpha}$ preserves source fibers, we see

$$(R_{\alpha_2} \circ \cdots \circ R_{\alpha_k})(c)(g) = (-1)^{\frac{(k-1)(k-2)}{2}} c_1(g)d_n h_2(\alpha_2(s(g)))\cdots d_n h_k(\alpha_k(s(g))).$$

Since this is simply c_1 multiplied with a function that is constant along the s-fibers, we then obtain:

$$\begin{aligned} (R_{\alpha_1} \circ \cdots \circ R_{\alpha_k})(c) &= (-1)^{\frac{(k-1)(k-2)}{2}}\mathcal{V}(c_1)(\alpha_1)\langle d_n h_2, \alpha_2\rangle \cdots \langle d_n h_k, \alpha_k\rangle \\ &= i^{(k-1)(k-2)}\mathcal{V}(c_1)(\alpha_1)\langle d_n h_2, \alpha_2\rangle \cdots \langle d_n h_k, \alpha_k\rangle. \end{aligned}$$

Since already know by the calculation in the case $k = 1$ that $\mathcal{V}(c_1)(\alpha_1) = \widehat{(c_1)_0}(\alpha_1)$ we see that

$$(R_{\alpha_1} \circ \cdots \circ R_{\alpha_k})(c) = i^{k(k-1)}\widehat{c_0}(\alpha_1, ..., \alpha_k).$$

Then note that there is a mismatch in the summation over S_k in $\mathcal{V}(c)$ and in the right hand side of the theorem. In particular the right hand side in the last equation corresponds to the identity permutation in the statement of the theorem, while the right hand side corresponds to the permutation in the definition of $\mathcal{V}(c)$ that sends j to $k-j$. The sign of this permutation is $(-1)^{\frac{k(k-1)}{2}}$, for which we have to correct, so that we obtain

$$\begin{aligned} \mathcal{V}(c)(\alpha_1, ..., \alpha_k) &= \sum_{\sigma \in S_k} (-1)^\sigma (R_{\alpha_{\sigma(k)}} \circ \cdots R_{\alpha_{\sigma(1)}})(c) \\ &= \sum_{\sigma \in S_k} (-1)^\sigma i^{k(k-1)} (R_{\alpha_{\sigma(1)}} \circ \cdots \circ R_{\alpha_{\sigma(k)}})(c) \\ &= \sum_{\sigma \in S_k} (-1)^\sigma i^{2k(k-1)}\widehat{c_0}(\alpha_{\sigma(1)}, ..., \alpha_{\sigma(k)}) \\ &= \sum_{\sigma \in S_k} (-1)^\sigma \widehat{c_0}(\alpha_{\sigma(1)}, ..., \alpha_{\sigma(k)}). \end{aligned}$$

So we see that $\mathcal{V}(c)$ equals the linear multivector field that is the antisymmetrization of $\widehat{c_0}$. In particular this means that for $f_1, ..., f_k \in \mathcal{S}_c(A^*)$ we have

$$\begin{aligned}\mathcal{V}(c)(f_1, ..., f_k) &= \sum_{\sigma \in S_k} (-1)^\sigma \widehat{c_0}(f_{\sigma(1)}, ..., f_{\sigma(k)}) \\ &= \frac{1}{i^{k-1}} \mathcal{F}_\mu \left(\lim_{t\to 0} \left(\sum_{\sigma \in S_k} (-1)^\sigma \frac{1}{(it)^{k-1}} \Phi(c)(q_t(f_{\sigma(1)}), \ldots, q_t(f_{\sigma(k)})) \right) \right).\end{aligned}$$

This completes the proof. □

Remark 2.4.25 This theorem, restricted to multiplicative vector fields, can be viewed as a statement about the 'classical limit' of certain derivations of the convolution algebra, and looks very similar to certain aspects of the proof of the Atiyah–Singer index theorem given in [ENN96]. Indeed, it would be interesting to investigate its use in index theory for Lie groupoids, as it exactly fits into the framework of relating the van Est map to the classical limit, as shown in the index theorem of [PPT15] for smooth groupoid cohomology $\mathrm{H}^\bullet_{\mathrm{diff}}(\mathcal{G})$, by replacing the characteristic map $\mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G}) \to \mathrm{C}^\bullet(\mathcal{A}_\mathcal{G})$ of [PPT15] by the map

$$\begin{aligned}\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) &\to \mathrm{C}^\bullet(\mathcal{A}_\mathcal{G}) \\ c &\mapsto \iota_{\Phi(c)}(\tau)\end{aligned}$$

obtained by using the contraction from Equation (2.3).

In the previous proof we have only used the fact that $q_t(f)$ converges to $\mathcal{F}_\mu^{-1}(f)$ in $\mathcal{S}_c(\mathcal{G}_{\mathrm{ad}})$ as t goes to 0, we have not used the properties which makes the family $\{q_t\}_{t\neq 0}$ a family of quantization maps, namely their compatibility with the Poisson bracket. However, we have not introduced these specific maps without reason, since we will use the fact that

$$\lim_{t\to 0} \left(\frac{1}{it}[q_t(f_1), q_t(f_2)] \right) = \lim_{t\to 0} q_t(\{f_1, f_2\}))$$

to give an alternative proof of the fact that the Van Est map is a *chain map*, i.e, compatible with the differentials:

Corollary 2.4.26 The van Est map $\mathcal{V}\colon \widehat{\mathrm{C}}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{Pois,lin}}(A^*)$ is a chain map.

Proof. Let $c \in \widehat{\mathrm{C}}^k_{\mathrm{def}}(\mathcal{G})$ for $k \geq 1$ and we start by dissecting $\mathcal{V}(\delta c)$. Using the previous theorem we obtain

$$\begin{aligned}\mathcal{V}(\delta c)(f_1,...,f_{k+1}) =&\mathcal{F}_\mu\left(\lim_{t\to 0}\left(\sum_{\sigma\in S_{k+1}}(-1)^\sigma\frac{1}{(it)^k}\Phi(\delta c)(q_t(f_{\sigma(1)}),\dots,q_t(f_{\sigma(k+1)}))\right)\right)\\ =&\mathcal{F}_\mu\left(\lim_{t\to 0}\left(\sum_{\sigma\in S_{k+1}}(-1)^\sigma\frac{1}{(it)^k}(b(\Phi(c)))(q_t(f_{\sigma(1)}),\dots,q_t(f_{\sigma(k+1)}))\right)\right)\\ =&\mathcal{F}_\mu\left(\lim_{t\to 0}\left(\sum_{\sigma\in S_{k+1}}(-1)^\sigma\frac{1}{(it)^k}[q_t(f_{\sigma(1)}),\Phi(c)(q_t(f_{\sigma(2)}),...,q_t(f_{\sigma(k+1)}))]\right)\right)\\ &+\mathcal{F}_\mu\left(\lim_{t\to 0}\left(\sum_{j=1}^k\sum_{\substack{\sigma\in S_{k+1}\\ \sigma^{-1}(j)<\sigma^{-1}(j+1)}}(-1)^\sigma(-1)^j\frac{1}{(it)^k}\Phi(c)(q_t(f_{\sigma(1)}),...\right.\right.\\ &\left.\left. ...,[q_t(f_{\sigma(j)}),q_t(f_{\sigma(j+1)})],...,q_t(f_{\sigma(k)}))\right)\right).\end{aligned}$$

By the relation between the commutator, the Poisson bracket and the quantization maps, we can now use 1 power of $\frac{1}{it}$ to turn the commutators into Poisson brackets. Also using the fact that $q_t(f)\to\mathcal{F}_\mu^{-1}(f)$ as $t\to 0$ this results in

$$\begin{aligned}\mathcal{V}(\delta c)(f_1,...,f_{k+1}) =&\sum_{\sigma\in S_{k+1}}(-1)^\sigma\left\{f_{\sigma(1)},\mathcal{F}_\mu\left(\lim_{t\to 0}\left(\frac{1}{(it)^{k-1}}\Phi(c)(q_t(f_{\sigma(2)}),...,q_t(f_{\sigma(k+1)}))\right)\right)\right\}\\ &+\mathcal{F}_\mu\left(\lim_{t\to 0}\left(\sum_{j=1}^k\sum_{\substack{\sigma\in S_{k+1}\\ \sigma^{-1}(j)<\sigma^{-1}(j+1)}}(-1)^\sigma(-1)^j\frac{1}{(it)^{k-1}}\Phi(c)(q_t(f_{\sigma(1)}),...\right.\right.\\ &\left.\left. ...,q_t(\{f_{\sigma(j)},f_{\sigma(j+1)}\}),...,q_t(f_{\sigma(k)}))\right)\right).\end{aligned}$$

Then using the previous Theorem in reverse order we see that this leads to

$$\begin{aligned}\mathcal{V}(\delta c)(f_1,...,f_{k+1}) =&\sum_{j=1}^{k+1}(-1)^{j+1}\left\{f_j,\mathcal{V}(c)(f_1,...,\widehat{f_j},...,f_{k+1})\right\}\\ &+\sum_{j_1<j_2}(-1)^{j_1+j_2}\mathcal{V}(c)(\{f_{j_1},f_{j_2}\},f_1,...,\widehat{f_{j_1}},\widehat{f_{j_2}},...,f_{k+1}),\end{aligned}$$

which shows that the Van Est map is a chain map. □

2.5 Higher order elements

2.5.1 Adjoint representation of to homotopy

The theory of Lie groupoids and algebroids has some distinct features that makes it more difficult than the theory of Lie groups and algebras. For one, not every Lie algebroid comes from a Lie groupoid ([CF03]), but more importantly for what we'll discuss here, in general the Lie algebroid $A(\mathcal{G}) \to M$ of a groupoid $\mathcal{G} \rightrightarrows M$ cannot be interpreted as a representation of the groupoid. Indeed, if that would be the case, any arrow $g \in \mathcal{G}$ would induce a linear map $A_{s(g)} \to A_{t(g)}$. However, since $A_{s(g)} = \ker(ds_{u(s(g))})$, the only natural action g has on this space is by right multiplication, which does not send $A_{s(g)}$ to $A_{t(g)}$.

In specific classes of examples, there are representations which can take the place of the adjoint representation [Me16, Ch 5], but in general, something more involved is needed. In particular we need to consider the complex $A \xrightarrow{\rho} TM$ of vector bundles as some kind of representation. The concept that we need is a 'representation up to homotopy', a notion introduced by Abad and Crainic in [AC13].

Definition 2.5.1 A *representation up to homotopy* of a Lie groupoid $\mathcal{G} \rightrightarrows M$ is a graded vector bundle $E = \oplus_{k\in\mathbb{Z}} E^k \to M$ together with a differential $D\colon \mathrm{C}^\bullet(\mathcal{G}, E) \to \mathrm{C}^{\bullet+1}(\mathcal{G}, E)$ on the complex

$$\mathrm{C}^n(\mathcal{G}, E) \coloneqq \bigoplus_{p+q=n} \Gamma(t^*E^q \to \mathcal{G}^{(p)})$$

that is a *super connection*, i.e. D satisfies

$$D(\eta \star f) = D(\eta) \star f + (-1)^n \eta \star \delta(f)$$

for $\eta \in \mathrm{C}^n(\mathcal{G}, E)$ and $f \in \mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G})$. Here $t\colon \mathcal{G}^{(k)} \to M$ is the map sending $(g_1, ..., g_k)$ to $t(g_1)$, and $\star$ takes an element η of $\Gamma(t^*E^q \to G^{(p)})$ and an element f of $\mathrm{C}^{p'}_{\mathrm{Diff}}(\mathcal{G})$ and spits out an element $\eta \star f$ of $\Gamma(t^*E^q \to \mathcal{G}^{(p+p')})$ given by the formula

$$(\eta \star f)(g_1, ..., g_{p+p'}) = \eta(g_1, ..., g_p) f(g_{p+1}, ..., g_{p+p'}).$$

Definition 2.5.2 A morphism of two representations up to homotopy is a $\mathrm{C}^\bullet_{\mathrm{Diff}}(\mathcal{G})$-linear chain map

$$\varphi\colon (\mathrm{C}^\bullet(\mathcal{G}, E), D) \to (\mathrm{C}^\bullet(\mathcal{G}, E'), D')$$

Remark 2.5.3 By [AC13, Prop 3.2] we can see a representation up to homotopy as the following data:

- A cochain complex of vector bundles with differentials $\partial\colon E^\bullet \to E^{\bullet+1}$.
- For every $g \in \mathcal{G}$ maps $\lambda_g\colon E^n_{s(g)} \to E^n_{t(g)}$ that satisfy $\partial \circ \lambda_g = \lambda_g \circ \partial$, called the quasi-action.

- Homotopies $R_2(g_1, g_2)\colon E^\bullet \to E^{\bullet-1}$ for $(g_1, g_2) \in \mathcal{G}^{(2)}$ giving homotopies between $\lambda_{g_1} \circ \lambda_{g_2}$ and $\lambda_{g_1 g_2}$:

$$\lambda_{g_1} \circ \lambda_{g_2} - \lambda_{g_1 g_2} = \partial \circ R_2(g_1, g_2) - R_2(g_1, g_2) \circ \partial.$$

- Higher homotopies R_i for $i \geq 3$ describing homotopies between homotopies in an A_∞ kind of sense.

In this way the differential can be understood as

$$D = \partial + \lambda + \sum_{i \geq 2} R_i.$$

This clarifies the word 'homotopy' in the name, and shows that a representation can be thought of as a representation up to homotopy where all the higher homotopies are identically zero.

Example 2.5.4 Using this definition, in [AC13, Def 3.13] the adjoint representation up to homotopy was defined. The main input for this is an Ehresmann connection σ on $\mathcal{G}$, i.e. a section $\sigma\colon s^*TM \to T\mathcal{G}$ of the map ds that equals du along the units. With this at hand we can describe the representation by the following data:

- The cochain complex of vector bundles $A(\mathcal{G}) \xrightarrow{dt} TM$ with $A(\mathcal{G})$ in degree 0 and TM in degree 1.

- The quasi-actions on $A(\mathcal{G})$ and TM given by

  - $\lambda_g(X) := (dt)_g(\sigma_g(X))$
  - $\lambda_g(\alpha) := dR_{g^{-1}}(\sigma_g(dt_{u(s(g))}(\alpha)) - dL_g(d\iota_{u(s(g))}(\alpha)))$

  for $X \in T_{s(g)}M$ and $\alpha \in A(\mathcal{G})_{s(g)}$.

- A homotopy $R_2 = K_\sigma^{\mathrm{bas}}$ for the quasi-action λ defined using the 'basic curvature' of σ. Explicitly, for $(g_1, g_2) \in \mathcal{G}^{(2)}$, the map $R_2(g_1, g_2)\colon T_{s(g_1)}M \to A(\mathcal{G})_{t(g_2)}$ is given by

$$R_2(g_1, g_2)(v) := dR_{(g_1 g_2)^{-1}}(\sigma_{g_1 g_2}(v) - dm_{(g_1, g_2)}(\sigma_{g_1}(\lambda_{g_2}(v)), \sigma_{g_2}(v))).$$

In the end, the resulting complex is given by

$$\mathrm{C}^n(\mathcal{G}, \mathrm{Ad}_\sigma) = \Gamma(\mathcal{G}^{(n)}, t^*A) \oplus \Gamma(\mathcal{G}^{(n-1)}, t^*TM)$$

with differential defined using σ. Also, it is shown in [AC13] that the isomorphism class of Ad_σ does not depend on the connection σ.

Now what does this adjoint representation have to do with deformation cohomology of the Lie groupoid? It turns out that the deformation complex is an intrinsic model for the adjoint representation, in the sense that it is a cochain complex that is defined without any choices and isomorphic to the complex associated to the adjoint representation with an explicit isomorphism obtained by the choice of a connection. This should be thought of as a generalization of the equivalent statements for Lie groups, where the group cohomology complex with values in the adjoint representation is isomorphic to the deformation complex of the group and as a justification for naming this specific representation up to homotopy the adjoint representation.

Proposition 2.5.5 [Me16, Lem 5.53] Given an Ehresmann connection, the map

$$I_\sigma \colon \mathrm{C}^n(\mathcal{G}, \mathrm{Ad}_\sigma) \to \mathrm{C}^n_{\mathrm{def}}(\mathcal{G})$$

associating to $u \in \Gamma(\mathcal{G}^{(n)}, t^*A(\mathcal{G}))$ and $v \in \Gamma(\mathcal{G}^{(n-1)}, t^*TM)$ the map

$$I_\sigma(u, v)(g_1, ..., g_n) := dR_{g_1}(u(g_1, ..., g_n)) - \sigma_{g_1}(v(g_2, ..., g_n))$$

is an isomorphism of cochain complexes.

Note that we did not spell out the differential on $\mathrm{C}^\bullet(\mathcal{G}, \mathrm{Ad}_\sigma)$ precisely, but in a roundabout way we can interpret the differential as the differential induced by this isomorphism and the differential on the deformation complex.

Remark 2.5.6 From the adjoint representation up to homotopy we can make a canonical representation that every Lie groupoid $\mathcal{G} \rightrightarrows M$ possesses. This is the representation of 'transverse densities'

$$Q := \Lambda^{\mathrm{top}} T^*M \otimes \Lambda^{\mathrm{top}} A(\mathcal{G}).$$

The appropriate combinations of the quasi-action of Example 2.5.4 turn this into an actual representation of the groupoid. If there is a $\mathcal{G}$-invariant section on this representation, i.e. if this is simply the trivial representation of $\mathcal{G}$, we call $\mathcal{G}$ *unimodular.*

Tensor powers of the adjoint representation

In classical Lie theory, the story about the adjoint representation of a Lie group on its algebra is extended by consideration of the symmetric powers of the adjoint. In particular, if a Lie group G is compact the invariant polynomials $\mathbb{C}[\mathfrak{g}]^G$ on its Lie algebra are isomorphic, via the Chern-Weil Construction [Bo73], to the cohomology of the classifying space BG. On the other hand, the invariant polynomials $\mathbb{C}[\mathfrak{g}]^G$ are also isomorphic to the group cohomology with coefficients in the symmetric powers of the adjoint representation.

For representations up to homotopy, there is a similar operation, but just as defining representation up to homotopy is a tricky business, more so is defining tensor powers.

Indeed, as described in a paper by Abad, Crainic and Dherin [ACD10], taking the k'th symmetric power of a representation up to homotopy involves the following:

- The graded vector bundle E is replaced by its k'th graded symmetric power $\mathrm{Sym}^k E$. Writing out $E = \oplus_{n\in\mathbb{Z}} E^n$ we have that

  $$\mathrm{Sym}^k E = \bigoplus_{p+q=k} \mathrm{Sym}^p E^{\mathrm{ev}} \otimes \Lambda^q E^{\mathrm{odd}}$$

  where a term

  $$\left(\bigotimes_{m \text{ even}} \mathrm{Sym}^{k_m} E^m\right) \otimes \left(\bigotimes_{n \text{ odd}} \Lambda^{k_n} E^n\right)$$

  lives in degree $\sum_{n\in\mathbb{Z}} nk_n$. The differential is then applied derivation-wise to the tensor products.

- The maps $\lambda_g \colon E^n_{s(g)} \to E^n_{t(g)}$ are replaced by what they induce on the respective tensor products.

- The higher homotopies are complicated, in that one needs to make a choice of combinatorics to define the higher homotopies out of the original higher homotopies.

In the end, there is always a coherent choice to make, and the resulting representations up to homotopy are always isomorphic, so that taking the symmetric powers (and indeed any tensor power) is well-defined as isomorphism classes.

In particular we obtain SymAd as an equivalence class of a representation up to homotopy, and associated to that an isomorphism class of cochain complexes calculating Lie groupoid cohomology with coefficients in the symmetric powers of the adjoint. Parallel to classical Lie theory, one is interested in this cohomology because of its relations to the cohomology of the classifying space $B\mathcal{G}$. In particular, it is shown in [AC13, Thm 4.3] that there is a spectral sequence converging to the cohomology of $B\mathcal{G}$ which on its first page is calculated by $\mathrm{H}^\bullet(\mathcal{G}, \mathrm{SymAd})$.

Now, we would be interested in intrinsic models of this complex, akin to the deformation complex for the adjoint representation. In this context, we are interested in investigating the the relation between the chain map Φ from Definition 2.3.2, the symmetric powers of the adjoint, and the following conjecture:

Conjecture 2.5.7 There is a natural injection

$$\mathrm{H}^\bullet(\mathcal{G}, \mathrm{Sym}(\mathrm{Ad})) \to \mathrm{H}^\bullet(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G}).$$

In particular we want to see whether we can use the Gerstenhaber structure of the Hochschild complex to induce 'higher order' elements out of the deformation complex.

2.5.2 From vector fields to differential operators

As a baby case for upgrading the deformation complex to something describing the symmetric powers of the adjoint representation, we look at what happens in cohomological degree 1. Indeed, if the deformation complex exhibits a structure in the same vein as a DGLA, this degree should be kept fixed by the higher order structures.

In degree 1 we have that $\mathrm{C}^1_{\mathrm{def}}(\mathcal{G})$ consists of s-projectable vector fields on $\mathcal{G}$, which are a module over $C^\infty(M)$ by setting

$$f \cdot X = (f \circ s)X. \tag{2.7}$$

Now if we want to go to something alike a symmetric powers operation, we would need to go to $\mathrm{Sym}_{C^\infty(M)}\mathfrak{X}_s(\mathcal{G})$, but there is a procedure more aligned with the philosophy that the symmetric powers of the adjoint should come as some kind of graded quotient: we want to replace vector fields with differential operators, in a way as described in Theorem 1.2.40. To be able to do this, we shoehorn our situation into the framework of Lie-Rinehart algebras.

Lemma 2.5.8 The pair $(\mathfrak{X}_s(\mathcal{G}), C^\infty(M))$, with the bracket given by the Lie bracket of vector fields, module structure given by (2.7), and the anchor $\rho\colon \mathfrak{X}_s(\mathcal{G}) \to \mathfrak{X}(M)$ given by $\rho(X) = \sigma_X$ is a Lie-Rinehart algebra.

Proof. We only need to check that

$$[X, f \cdot Y] = f \cdot [X, Y] + \rho(X)(f) \cdot Y$$

for $X, Y \in \mathfrak{X}_s(\mathcal{G})$ and $f \in C^\infty(M)$. For this we use the ordinary rules we have for dealing with Lie brackets of vector fields:

$$\begin{aligned} [X, f \cdot Y] &= [X, (f \circ s)Y] \\ &= (f \circ s)[X, Y] + X(f \circ s)Y \\ &= f \cdot [X, Y] + (\sigma_X(f) \circ s)Y \\ &= f \cdot [X, Y] + \rho(X)(f) \cdot Y. \end{aligned}$$

This finishes the proof. □

Definition 2.5.9 We define the *s-projectable differential operators on $\mathcal{G}$* to be

$$\mathrm{Diff}_s(\mathcal{G}) := \mathcal{U}(\mathfrak{X}_s(\mathcal{G}), C^\infty(M)).$$

Lemma 2.5.10 There is a canonical injection

$$\mathrm{Diff}_s(\mathcal{G}) \hookrightarrow \mathrm{Diff}(\mathcal{G})$$

induced by the inclusion $\mathfrak{X}_s(\mathcal{G}) \hookrightarrow \mathfrak{X}(\mathcal{G})$ and the injection $C^\infty(M) \xrightarrow{s^*} s^*C^\infty(M) \hookrightarrow C^\infty(\mathcal{G})$. The image of this injection is given by

$$\mathrm{Diff}_s(\mathcal{G}) = \{D \in \mathrm{Diff}(\mathcal{G})\colon\ D(s^*C^\infty(M)) \subset s^*C^\infty(M)\}.$$

Proof. The first statement about the existence of the injection is clear. As for the characterization of the image, notice that the whole procedure can be done for any submersion $f\colon M \to N$, and by a partition of unity argument this lemma can be reduced to the projection $\pi\colon \mathbb{R}^{n+k} \to \mathbb{R}^n$, where it is easy to see by some extensive bookkeeping.

In particular, choosing coordinates $(x_1, ..., x_n, y_1, ..., y_k)$, we see the following:

$$\pi^*C^\infty(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^{n+k}) : f(x_1, ..., x_n, y_1, ..., y_k) = f(x_1, ..., x_n)\},$$
$$\mathfrak{X}_\pi(\mathbb{R}^{n+k}) = \left\{ \sum_{i=1}^n f_i \frac{\partial}{\partial x_i} + \sum_{j=1}^k g_j \frac{\partial}{\partial y_j} : f_i \in \pi^*C^\infty(\mathbb{R}^n), g_j \in C^\infty(\mathbb{R}^{n+k}) \right\},$$

so that $\mathrm{Diff}_\pi(\mathbb{R}^{n+k})$ consists of those combinations

$$D = \sum_{p,q\geq 0} \sum_{1\leq i_1 \leq \cdots \leq i_p \leq n} \sum_{1 \leq j_1 \leq \cdots \leq j_q \leq k} f_{i_1,...,i_p,j_1,...,j_q} \frac{\partial}{\partial x_{i_1}} \cdots \frac{\partial}{\partial x_{i_p}} \frac{\partial}{\partial y_{j_1}} \cdots \frac{\partial}{\partial y_{j_q}}$$

where $f_{i_1,...,i_p}$ has to be an element of $\pi^*C^\infty(\mathbb{R}^n)$ whenever $q = 0$. It is easily seen that those are precisely the differential operators that preserve $\pi^*C^\infty(\mathbb{R}^n)$. $\square$

Using this formalism, we can define an action of projectable differential operators on the convolution algebra. Indeed, we have two maps

$$\Phi_0\colon\ C^\infty(M) \to C^1_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G}), \qquad \Phi_1\colon\ \mathfrak{X}_s(M) \to C^1_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G}),$$

where $\Phi_0(f)(a) = (f \circ s)a$ and Φ_1 is our chain map Φ from before. Since Φ_1 is defined using Lie derivatives of tensors which are locally fibred top-forms, we have

$$[\Phi_1(X), \Phi_1(Y)] = \Phi_1([X, Y]), \qquad \Phi_0(f)\Phi_0(g) = \Phi_0(fg),$$
$$\Phi_1(f \cdot X) = \Phi_0(f)\Phi_1(X), \qquad [\Phi_1(X), \Phi_0(f)] = \Phi_0(\rho(X)(f)).$$

Here, the product in $\mathrm{C}^1_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G})$ is simply composition of maps. By Remark 2.1.6, elements of the convolution algebra are locally fibred differential forms, so that this follows from the well-known relations for the Lie derivative of differential forms:

$$[\mathcal{L}_X, \mathcal{L}_Y]\omega = \mathcal{L}_{[X,Y]}\omega, \qquad f(g\omega) = (fg)\omega,$$
$$\mathcal{L}_{fX}\omega = df \wedge \iota_X\omega + f\mathcal{L}_X\omega, \qquad \mathcal{L}_X(f\omega) - f\mathcal{L}_X\omega = X(f)\omega,$$

combined with the observation that densities are locally the same as differential forms, and hence behave similarly under these kinds of local statements. The only equation which not immediately reduces to our desired relation is the one involving $\mathcal{L}_{fX}\omega$, where the term $df \wedge \iota_X\omega$ seems to spoil the fun. However, we need this relation for f of the form $\widetilde{f} \circ s$, we we obtain

$$\mathcal{L}_{fX}\omega = s^*d\widetilde{f} \wedge \iota_X\omega + f\mathcal{L}_X\omega.$$

Furthermore, we only need this relation for an s-fibred differential form of top degree, i.e. we apply this relation to tangent vectors that lie in the distribution $\ker(ds)$, and we need to fill all the available slots. This means that the term $s^*d\widetilde{f} \wedge \iota_X$ always vanishes, and we indeed end up with our desired relation.

In the end, by the universal property of $\mathcal{U}(\mathfrak{X}_s(\mathcal{G}), C^\infty(M))$ we obtain a map

$$\Phi\colon \mathrm{Diff}_s(\mathcal{G}) \to \mathrm{C}^1_{\mathrm{Hoch}}(\mathcal{A}_\mathcal{G}, \mathcal{A}_\mathcal{G}).$$

This map leads to the following interesting class of differential operators.

Definition 2.5.11 A differential operator $D \in \mathrm{Diff}_s(\mathcal{G})$ is called *multiplicative* if $b(\Phi(D)) = 0$.

Remark 2.5.12 Since Φ_1 is an injective map, this definition of multiplicative reduces to the usual definition Theorem 2.2.5 for s-projectable vector fields.

Next, it is natural to look at the symbols of these differential operators, and we wonder about how multiplicity behaves with respect to taking the symbol. For this, we turn to the work of Bursztyn and Drummond [BD19], which describes the notion of multiplicative tensors in terms of the tangent groupoids $\oplus^\bullet T^*\mathcal{G}$. We think of it in the following way: the symbol of a differential operator induces a fibrewise polynomial function on $T^*\mathcal{G}$ (about which we can also think as a symmetric function on $\oplus^k T^*\mathcal{G}$ if the differential operator is of order k), and in the vein of Bursztyn and Drummond, we define

Definition 2.5.13 For a groupoid $\mathcal{G}$, we call a function $f \in C^\infty(\mathcal{G})$ *multiplicative* if it satisfies

$$f(m(g,h)) = f(g) + f(h)$$

and if D is a differential operator on $\mathcal{G}$, we call a symbol $\sigma(D)$ *multiplicative* if it is multiplicative as a smooth function on $T^*\mathcal{G}$, on which we defined a groupoid structure in Example 1.2.26.

We then expect the connection between multiplicative differential operators and differential symbols to be as follows.

Conjecture 2.5.14 The symbol of a multiplicative differential operator is a multiplicative symbol. Reversely, every symbol that is multiplicative is the symbol of some multiplicative differential operator.

We can investigate this connection in a few examples.

Example 2.5.15 (Trivial groupoids) For a trivial groupoid $\mathcal{G} = M \rightrightarrows M$, we have $\mathrm{Diff}_s(\mathcal{G}) = \mathrm{Diff}(M)$, of which the multiplicative differential operators are by definition precisely the vector fields $\mathfrak{X}(M)$ by definition. In turn we need to show that the multiplicative polynomials on $T^*\mathcal{G}$ are precisely the linear functions.

To see this, note that for the trivial groupoid $\mathcal{G} = M \rightrightarrows M$, the cotangent groupoid $T^*\mathcal{G}$ is simply the vector bundle $T^*M \to M$ seen as a groupoid. In particular, $f \in C^\infty(T^*M)$ is multiplicative if and only if

$$f(v + w) = f(v) + f(w)$$

for $v, w \in T_p^*M$ for some $p \in M$. So, we see quite easily that the multiplicative symbols are precisely the fibrewise linear functions on T^*M.

Example 2.5.16 (Pair groupoid over $\mathbb{R}$) For the pair groupoid $\mathcal{G} = \mathbb{R}^2 \rightrightarrows \mathbb{R}$, a projectable differential operator D of order k is of the form

$$D = \sum_{i+j\leq k} \alpha_{i,j} \frac{\partial^{i+j}}{\partial x^i \partial y^j}$$

where $\alpha_{0,j} \in C^\infty(\mathbb{R}^2)$ only depends on y. Here we give $\mathbb{R}^2$ the coordinates (x, y) with the source map being the projection of (x, y) onto y.

Identifying $\mathcal{A}_\mathcal{G}$ with $C^\infty(\mathbb{R}^2)$ by ways of the density $|dx|$, we see that D acts on the convolution algebra via

$$Df = \sum_{i+j\leq k} \alpha_{i,j} \frac{\partial^{i+j} f}{\partial x^i \partial y^j} + \sum_{\substack{i+j\leq k \\ i>0}} \frac{\partial \alpha_{i,j}}{\partial x} \frac{\partial^{i+j-1} f}{\partial x^{i-1} \partial y^j}.$$

Then, we look at what it means for D to act as a derivation. For that, we note that under the identification we have

$$(f_1 * f_2)(x, z) = \int_\mathbb{R} f_1(x, z) f_2(z, y) dz$$

so that

$$\frac{\partial(f_1 * f_2)}{\partial x} = \left(\frac{\partial f_1}{\partial x}\right) * f_2, \qquad \frac{\partial(f_1 * f_2)}{\partial y} = f_1 * \left(\frac{\partial f_2}{\partial y}\right).$$

Repeated partial integration then yields that D acts as a derivation if and only if the following identities are true:

- $\alpha_{k,0}(x, y) = (-1)^{k+1} \alpha_{0,k}(x)$,
- $\alpha_{i,j} \equiv 0$ whenever $i > 0$ and $j > 0$,
- $\alpha_{i,0}(x, y) = \sum_{n=0}^{k-i} (-1)^{i+n-1} \binom{i+n}{n} \frac{\partial^n \alpha_{0,i+n}}{\partial x^n}(x) - \frac{\partial \alpha_{i+1,0}}{\partial x}(x, y)$ for $1 \leq i \leq k - 1$ (in particular $\alpha_{i,0}$ is only dependent on x for $i = 1, ..., k$),
- $\alpha_{0,0}(y) = -\frac{d\alpha_{1,0}}{dy}(y) + \sum_{n=1}^{k} (-1)^{n-1} \frac{d^n \alpha_{0,n}}{dy^n}(y)$.

Next, we look at the cotangent groupoid $T^*\mathbb{R}^2 \rightrightarrows T^*\mathbb{R}$. Sifting through the definitions we see that, writing the coordinates (x,y,a,b) for $T^*\mathbb{R}^2$ with a dual to x and b dual to y, we have

$$s(x,y,a,b) = (y,-b), \qquad\qquad t(x,y,a,b) = (x,a),$$

$$m((x,y,a,b),(y,z,-b,c)) = (x,z,a,c).$$

Then, if we have a map $f \in C^\infty(T^*\mathbb{R}^2)$ of the form

$$f(x,y,a,b) = \sum_{i=0}^{k} \alpha_i(x,y)a^i b^{k-i},$$

we can write out the multiplicity equation

$$f(x,z,a,c) = f(x,y,a,b) + f(y,z,-b,c)$$

to obtain

$$\begin{aligned}\alpha_0(x,z)c^k + \sum_{i=1}^{k-1} \alpha_i(x,z)a^i c^{k-1} + \alpha_k(x,z)a^k = \\ = \alpha_0(x,y)b^k + \sum_{i=1}^{k-1} \alpha_i(x,y)a^i b^{k-i} + \alpha_k(x,y)a^k + \\ + \alpha_0(y,z)c^k + \sum_{i=1}^{k-1} (-1)^i \alpha_i(y,z)b^i c^{k-i} + (-1)^k \alpha_k(y,z)b^k.\end{aligned}$$

So we see that f being multiplicative is equivalent to

- $\alpha_i \equiv 0$ for $i = 1, ..., k-1$,
- $\alpha_0(x,z) = \alpha_0(y,z) =: \alpha_0(z)$,
- $\alpha_k(x,z) = \alpha_k(x,y) =: \alpha_k(x)$,
- $\alpha_0(y) = (-1)^{k+1}\alpha_k(y)$,

so that we see that multiplicative symbols are those of the form

$$f(x,y,a,b) = \alpha(x)a^k + (-1)^{k+1}\alpha(y)b^k$$

and we indeed see that symbols of multiplicative differential operators are multiplicative and all multiplicative symbols are obtained in this way.

Example 2.5.17 (General pair groupoids) For a general pair groupoid $\mathcal{G} = M \times M \rightrightarrows M$, we fix a $D \in \mathrm{Diff}_s(\mathcal{G})$, and then we can choose opens $U_1, U_2, U_3 \subset M$ which are all coordinate domains, and look at the derivation-equation $D(f_1 * f_2) = f_1 * D(f_2) + D(f_1) * f_2$ where f_1 is supported inside $U_1 \times U_2$ and f_2 is supported inside $U_2 \times U_3$. A similar argument with partial integration shows that if D is multiplicative, its symbol $\sigma(D) \in C^\infty(T^*M^{\times 2})$ is of the form

$$\sigma(D)(x, y, \xi, \zeta) = P(x)(\xi) + (-1)^{k+1} P(y)(\zeta)$$

for $P \in \Gamma(\mathrm{Sym}^k TM)$ a map that sends a point $x \in M$ to a degree k polynomial on T_x^*M.

Similarly, we see that functions $f \in C^\infty(T^*M^{\times 2})$ that are fibrewise degree k homogeneous polynomials are multiplicative if and only if they are of the same form

$$f(x, y, \xi, \zeta) = P(x)(\xi) + (-1)^{k+1} P(y)(\zeta)$$

and so we again see that multiplicative symbols and multiplicative differential operators are linked.

Example 2.5.18 (Lie groups) For a Lie group G, seen as a Lie groupoid $G \rightrightarrows \{\mathrm{pt}\}$, we have that $\mathrm{Diff}_s(G)$ consists of those differential operators on G whose degree 0-term is constant. To see the effect on the convolution algebra, we first choose a left-invariant volume form ω on G. For $\alpha \in C^\infty(G)$ and $v \in \mathfrak{g}$ we have

$$\mathcal{L}_{\alpha \overrightarrow{v}} \omega = \overrightarrow{v}(\alpha)\omega + \alpha \mathcal{L}_{\overrightarrow{v}} \omega = \overrightarrow{v}(\alpha)\omega. \tag{2.8}$$

Here, we remark that $\mathcal{L}_{\overrightarrow{v}}\omega = 0$ since ω is left-invariant.

Now, let $D \in \mathrm{Diff}_s(G)$ be defined by

$$D = \sum_{i \in I} \alpha_i \overrightarrow{v_{i_1}} \cdots \overrightarrow{v_{i_k}} + \mathcal{O}(k-1)$$

for $\alpha_i \in C^\infty(G)$ and $v \in \mathfrak{g}$. Here, by $\mathcal{O}(k-1)$ we mean differential operators of orders $k-1$ and less. We then see that D acts on the convolution algebra via

$$Df = \sum_{i \in I} \left(\alpha_i (\overrightarrow{v_{i_1}} \cdots \overrightarrow{v_{i_k}})(f) + \overrightarrow{v_{i_1}}(\alpha_i)(\overrightarrow{v_{i_2}} \cdots \overrightarrow{v_{i_k}})(f) \right) + \mathcal{O}(k-1).$$

Here, the second term comes from the Lie derivative of $\alpha\omega$ (2.8). Of course, this term is also of order $k-1$, so that

$$Df = \sum_{i \in I} \alpha_i (\overrightarrow{v_{i_1}} \cdots \overrightarrow{v_{i_k}})(f) + \mathcal{O}(k-1).$$

By a partial integration argument, we can explicitely write out this formula (up to lower order terms) to obtain:

$$\begin{aligned} b(D)(f_1, f_2)(g) = \sum_{i \in I} \frac{d}{dt_1}\bigg|_{t_1=0} \cdots \frac{d}{dt_k}\bigg|_{t_k=0} \int_G \big(\alpha_i(h) f_1(gh^{-1} e^{t_1 v_{i_1}} \cdots e^{t_k v_{i_k}}) + \\ + \alpha_i(gh^{-1}) f_1(e^{t_1 v_{i_1}} \cdots e^{t_k v_{i_k}} gh^{-1}) - \alpha_i(g) f_1(e^{t_1 v_{i_1}} \cdots e^{t_k v_{i_k}} gh^{-1}) \big) \, f_2(h)\omega(h). \end{aligned}$$

So, if D is a derivation, we see that

$$\sum_{i\in I} \frac{d}{dt_1}\bigg|_{t_1=0} \cdots \frac{d}{dt_k}\bigg|_{t_k=0} \big(\alpha_i(h) f(gh^{-1}e^{t_1 v_{i_1}} \cdots e^{t_k v_{i_k}}) + \alpha_i(gh^{-1}) f(e^{t_1 v_{i_1}} \cdots e^{t_k v_{i_k}} gh^{-1}) - \alpha_i(g) f(e^{t_1 v_{i_1}} \cdots e^{t_k v_{i_k}} gh^{-1}\big) = 0$$

for any $f \in C^\infty(G)$, and any $g, h \in G$. Rewriting this, we get

$$\sum_{i\in I} \big(\alpha_i(h)\mathrm{Hes}(f)_{gh^{-1}}(\mathrm{Ad}_{gh^{-1}}(v_1), ..., \mathrm{Ad}_{gh^{-1}}(v_k)) + +\alpha_i(gh^{-1})\mathrm{Hes}(f)_{gh^{-1}}(v_1, ..., v_k) + \alpha_i(g)\mathrm{Hes}(f)_{gh^{-1}}(v_1, ..., v_k\big) = 0 \quad (2.9)$$

where $\mathrm{Hes}(f)_{gh^{-1}}$ is the Hessian of f:

$$\mathrm{Hes}(f)_{gh^{-1}}(v_1, ..., v_k) = \frac{1}{k!} \sum_{\sigma\in S_k} \frac{d}{dt_1}\bigg|_{t_1=0} \cdots \frac{d}{dt_k}\bigg|_{t_k=0} f(e^{t_1 v_{\sigma(1)}} \cdots e^{t_k v_{\sigma(k)}} gh^{-1}).$$

Now to see that the symbol is multiplicative, notice that using right translations of the tangent bundle $TG \cong G \times \mathfrak{g}$ we have that the cotangent groupoid is given by

$$G \times \mathfrak{g}^* \rightrightarrows \mathfrak{g}^*$$

with source and target given by

$$s(g, \xi) = \mathrm{Ad}^*_g(\xi), \qquad t(g, \xi) = \xi,$$

and multiplication given by

$$m((g, \xi), (h, \xi')) = (gh, \xi).$$

In particular a multiplicative map is a map $F \in C^\infty(G \times \mathfrak{g}^*)$ such that

$$F(gh, \xi) = F(g, \xi) + F(h, \mathrm{Ad}^*_g(\xi)).$$

Note that we can also derive this fact for symmetric multilinear maps $F \in C^\infty(G \times \mathrm{Sym}^k\mathfrak{g}^*)$, and the equation (2.9) is precisely the equation

$$\sigma(D)(gh^{-1}, \mathrm{Hes}(f)_{gh^{-1}}) + \sigma(D)(h, \mathrm{Ad}^*_{gh^{-1}}\mathrm{Hes}(f)_{gh^{-1}}) = \sigma(D)(g, \mathrm{Hes}(f)_{gh^{-1}})$$

where we see $\sigma(D)$ as a function on $G \times \mathrm{Sym}^k(\mathfrak{g}^*)$. So since every symmetric map $\xi \in \mathrm{Sym}^k(\mathfrak{g}^*)$ can be written as $\xi = \mathrm{Hes}(f)_{gh^{-1}}$ for a suitable $f \in C^\infty_c(G)$ (this can be very easily seen by using the local coordinates that $\mathfrak{g}$ gives around gh^{-1} by using the exponential map), we see that if D is a derivation on the convolution algebra, then its symbol is multiplicative.

Remark 2.5.19 For the general case, we have, at the moment of writing, not found a proof of this fact. It is reasonable to expect that a partial integration argument should go a long way, when applied to the integral over the s-fibres that are present in the convolution product in the general case. Similarly, the characterization of the multiplicative symbols should be greatly eased by explicitly dividing the cotangent space $T^*\mathcal{G}$ in the direction parallel to the source fibres, and the normal direction. As such, it would seem that by choosing a reasonably invariant splitting of the short exact sequence

$$t^*A \to T\mathcal{G} \to TM$$

similar types of arguments as above would yield a proof.

So what is the general philosophy behind this consideration of the cohomological degree 1, and how does this connect to the story we sketched around Conjecture 2.5.7? The following conjecture arises as a routemap to research the relations that are present:

Conjecture 2.5.20 On the deformation complex $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ there is the structure of a 'L_∞-algebroid over $\mathrm{C}^\bullet_{\mathrm{diff}}(\mathcal{G})$', with the following properties:

- In cohomological degree 1, the L_∞-structure is given by the Lie-bracket of s-projectable vector fields;
- A universal enveloping procedure induces a complex which in degree n is given by a certain class of differential operators on $\mathcal{G}^{(n)}$;
- The chain map $\Phi\colon \mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G}) \to \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_G, \mathcal{A}_G)$ from 2.3.2 is compatible with this the L_∞-structure on $\mathrm{C}^\bullet_{\mathrm{def}}(\mathcal{G})$ and the Gerstenhaber structure on $\mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{A}_G, \mathcal{A}_G)$, leading to a map between the universal enveloping complex and the Hochschild complex;
- The symbol map from the universal enveloping complex to $\mathrm{C}^\bullet_{\mathrm{diff}}(T^*\mathcal{G})$ is a chain map;
- The graded quotient of the universal enveloping complex is isomorphic to the chain complex associated to the symmetric powers of the adjoint representation up to homotopy.

Chapter 3

Action groupoids and equivariant characteristic classes

In this chapter we discuss the contents of [KP22]. We specifically study action groupoids $M \times G \rightrightarrows M$ induced by right group actions with the main goal being to relate the equivariant cohomology $\mathrm{H}^\bullet_G(M)$ of M with the periodic cyclic cohomology $\mathrm{HP}^\bullet(G \ltimes C_c^\infty(M))$ of the convolution algebra. In this specific case the convolution algebra is given by $C_c^\infty(M \times G)$ with product

$$(f_1 * f_2)(x,g) = \int_G f_1(x,h) f_2(xh, h^{-1}g)dh \qquad (x \in M,\, g \in G).$$

The non-commutative geometry of this algebra has been studied extensively, specifically since its product can be interpreted as a twisted combination of the convolution product of G and the commutative product on $C_c^\infty(M)$. The main tool, following Brylinski [Br87a], is to exploit this specific form of the product to split the Hochschild complex of the convolution algebra into a double complex $\{C_c^\infty(M^{\times(p+1)} \times G^{\times(q+1)})\}_{p,q\geq 0}$, where the behaviour in the M-direction is a twisted version of the usual Hochschild complex of $C_c^\infty(M)$ and in the behaviour in the G-direction uses the group cohomology associated to the G-module $C_c^\infty(M^{\times\bullet})$.

Using these facts, calculations for the Hochschild and cyclic homology of the convolution algebras were obtained in the case where G is discrete in works of for instance Baum-Connes [BC88], Brodzki-Dave-Nistor [BDN17], Connes [Co94], Feigin-Tsygan [FT87] and Getzler-Jones [GJ93] and recently Ponge [Po18], either directly or by a spectral sequence argument. In the compact case, these ideas have been generalized by Block, Getzler and Jones [BG94, BGJ95] using an equivariant form of the Hochschild-Kostant-Rosenberg map to obtain a model using the G-module of functions on $\mathfrak{g}$ with values in the differential forms on M. In this chapter, we will build upon this work by writing down a model $\mathrm{C}_{\bullet,\bullet}(G, \Omega_{\mathfrak{g},c}(M))$ for the cyclic homology of the convolution algebra $G \ltimes C_c^\infty(M)$ (in the case when G is unimodular) inspired by Getzlers model for equivariant cohomology [Ge94], and which is roughly given by

$$\mathrm{C}_{p,q}(G, \Omega_{\mathfrak{g},c}(M)) = C_c^\infty(G^{\times q}, C^\infty(\mathfrak{g}, \Omega_c^p(M))).$$

We will derive this model by combining the double complex written down by Brylinski [Br87a], the Eilenberg-Zilber Theorem and the equivariant HKR-map as defined by Block and Getzler [BG94]. Using this one can write down spectral sequences, which reduce quite significantly, for instance in the case that G is compact or when the action of G on M is proper, giving connections between the Hochschild and cyclic homology of the convolution algebra $G \ltimes C_c^\infty(M)$ and the invariant de Rham cohomology $\mathrm{H}^\bullet(\Omega_c(M)^G, d_{\mathrm{dR}})$ of M.

Apart from defining this model and exhibiting its properties, we want to use this to describe, internal to algebraic considerations of the convolution algebra, an equivariant Chern character in the non-commutative setting for equivariant vector bundles.

The equivariant cohomology $\mathrm{H}_G^\bullet(M)$ is defined [Tu20] to be

$$\mathrm{H}_G^\bullet(M) \coloneqq H^\bullet((EG \times M)/G),$$

the singular cohomology of the homotopy quotient $(EG \times M)/G$. If the action of G on M is free and proper, this cohomology is isomorphic to the (deRham) cohomology $\mathrm{H}^\bullet(M/G)$ of the quotient. In general, it is calculated using the Bott-Shulman complex of the simplicial manifold $G^{\times\bullet} \times M$ of [BSS76], which is given by

$$\Omega^{p,q}(G^{\times\bullet} \times M) \coloneqq \Omega^p(G^{\times q} \times M)$$

with the horizontal differential being the deRham differential, and the vertical differential being the alternating sums of the pullbacks under the maps ∂_i defined by

$$\partial_i(g_1, ..., g_q, x) \coloneqq \begin{cases} (g_1, ..., g_i g_{i+1}, ..., g_q, x), & 0 \le i \le q-1 \\ (g_1, ..., g_{q-1}, g_q x), & i = q \end{cases}.$$

Work by Getzler [Ge94] allows us to calculate equivariant cohomology by looking at a much smaller complex $\mathrm{C}^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}(M))$, which is inspired by the Cartan model and is roughly given by

$$\mathrm{C}^{p,q}(G, \Omega_{\mathfrak{g}}(M)) \coloneqq C^\infty(G^{\times q}, \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega^p(M)).$$

For equivariant vector bundles over M, there is the notion of the equivariant Chern character $\mathrm{Ch}_G \colon \mathrm{Vect}_G(M) \to \mathrm{H}_G^{\mathrm{ev}}(M)$ given by the concatenation

$$\mathrm{Vect}_G(M) \to \mathrm{Vect}_G(EG \times M) \to \mathrm{Vect}((EG \times M)/G) \to \mathrm{H}^{\mathrm{ev}}((EG \times M)/G).$$

Here, the map $\mathrm{Vect}_G(M) \to \mathrm{Vect}_G(EG \times M)$ is given by pullback along the equivariant projection $EG \times M \to M$, and $\mathrm{Vect}_G(EG \times M) \to \mathrm{Vect}((EG \times M)/G)$ is defined using the remark that if $E \to X$ is an equivariant vector bundle then $E/G \to X/G$ is a (topological) vector bundle. Lastly, the map $\mathrm{Vect}((EG \times M)/G) \to \mathrm{H}^{\mathrm{ev}}((EG \times M)/G)$ is the ordinary (topological) Chern character.

In this chapter we shed light on the structure of this Chern character, internal to the underlying action groupoid, in the following steps:

- In Section 3.1, we define cyclic cohomology classes for the convolution algebra $G \ltimes C_c^\infty(M)$ by constructing generalized cycles following Connes [Co94] and Gorokhovsky [Go99].

- In Section 3.2 and Section 3.3, we describe a way to obtain a map from the equivariant cohomology $\mathrm{H}_G^\bullet(M)$ to the cyclic cohomology $\mathrm{HP}^\bullet(G \ltimes C_c^\infty(M))$ by pairing our model $\mathrm{C}_{\bullet,\bullet}(G, \Omega_{\mathfrak{g},c}(M))$ for cyclic homology with Getzler's model $\mathrm{C}^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}(M))$ for equivariant cohomology.

- Using these two points, we obtain a diagram

$$\begin{array}{ccc} \mathrm{Vect}_G(M) & \longrightarrow & \mathrm{H}_G^{\mathrm{ev}}(M) \\ & \searrow & \downarrow \\ & & \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M)) \end{array}$$

  which we show to be commutative when the action of G on M is proper in Section 3.4.

This diagram has already been studied in the discrete case by Connes [Co94] and Gorokhovsky [Go99] and we generalize their results to the unimodular case.

To be able to do our calculations, we restrict ourselves to the case where M is orientable and the case where G is unimodular and the action of G on M is orientation-preserving. This means that we are able to integrate top forms over M, and that left and right Haar measures on G agree and inversion is measure preserving. In particular, for every $f \in C_c^\infty(G)$ and every $g \in G$, we have

$$\int_G f(gh)dh = \int_G f(h)dh,$$
$$\int_G f(hg)dh = \int_G f(h)dh,$$
$$\int_G f(h^{-1})dh = \int_G f(h)dh.$$

3.1 Cyclic cohomology classes through generalized cycles

We start by recalling the formalism of cycles of Connes [Co94] and its associated characters, as well as the JLO-type of Gorokhovsky [Go99] extending this formalism to more general situations. We then apply it to the case of a Lie group G acting on M by studying the convolution algebra $G \ltimes C_c^\infty(M)$.

3.1.1 Characters of generalized cycles

In this section, we outline the character of a cycle as defined by Connes, an important (and in essence the only) way to inducing cyclic cohomology classes for an algebra A. To motivate the definition, suppose M is a compact, oriented n-dimensional manifold. If $\omega \in \Omega^{n-k}(M)$ is a differential form, we can write down a Hochschild k-cochain $[\omega] \in \mathrm{C}^k_{\mathrm{Hoch}}(C^\infty(M), (C^\infty(M))^*)$ by the formula

$$[\omega](f_0, ..., f_k) := \int_M f_0 df_1 \wedge \cdots \wedge df_k \wedge \omega.$$

One then simply checks that

$$b[\omega] = 0 \qquad \text{and} \qquad B[\omega] = [(k+1)(-1)^k d\omega].$$

Indeed,

$$\begin{aligned}
b[\omega](f_0, ..., f_{k+1}) &= \sum_{i=0}^{k} (-1)^i [\omega](f_0, ..., f_i f_{i+1}, ..., f_{k+1}) \\
&\quad + (-1)^{k+1} [\omega](f_{k+1} f_0, f_1, ..., f_k) \\
&= \int_M f_0 f_1 df_2 \wedge \cdots \wedge df_{k+1} \wedge \omega \\
&\quad + \sum_{i=1}^{k} (-1)^i \int_M f_0 \wedge df_1 \wedge \cdots \wedge d(f_i f_{i+1}) \wedge \cdots \wedge df_{k+1} \wedge \omega \\
&\quad + (-1)^{k+1} \int_M f_{k+1} f_0 df_1 \wedge \cdots \wedge df_k \wedge \omega \\
&= \int_M f_0 f_1 df_2 \wedge \cdots \wedge df_{k+1} \wedge \omega \\
&\quad + \sum_{i=1}^{k} (-1)^i f_i f_0 \wedge df_1 \wedge \cdots \wedge \widehat{df_i} \wedge \cdots \wedge df_{k+1} \wedge \omega \\
&\quad + \sum_{i=1}^{k} (-1)^i f_{i+1} f_0 \wedge df_1 \wedge \cdots \wedge \widehat{df_{i+1}} \wedge \cdots \wedge df_{k+1} \wedge \omega \\
&\quad + (-1)^{k+1} \int_M f_{k+1} f_0 df_1 \wedge \cdots \wedge df_k \wedge \omega.
\end{aligned}$$

By a telescoping argument on the two sums, we see that this vanishes. For the other relation we see

$$\begin{aligned}
B[\omega](f_0, ..., f_{k-1}) &= \sum_{i=0}^{k} (-1)^{ik} \int_M df_{k-i+1} \wedge \cdots \wedge df_k \wedge df_0 \wedge \cdots \wedge df_{k-i} \wedge \omega \\
&= \sum_{i=0}^{k} \int_M df_0 \wedge \cdots \wedge df_{k-1} \wedge \omega \\
&= (k+1) \int_M d(f_0 df_1 \wedge \cdots \wedge df_{k-1} \wedge \omega) \\
&\quad + (k+1)(-1)^k \int_M f_0 df_1 \wedge \cdots \wedge df_{k-1} \wedge d\omega \\
&= 0 + [(k+1)(-1)^k d\omega](f_0, ..., f_{k-1}).
\end{aligned}$$

So we see that -up to a factor- this procedure gives a chain map $(\Omega^{n-\bullet}(M), d) \to (\mathrm{CC}^\bullet_{\mathrm{Hoch,cont}}(C^\infty(M)), b + B)$.

There is a more general picture behind this, extending this example to arbitrary algebras. So, let us start again with $\omega \in \Omega^{n-k}(M)$ closed, and set $\Omega = \Omega^{\leq k}(M)$ the differential graded algebra that is obtained by truncating at degree k. Clearly this DGA comes with an algebra map $\rho\colon C^\infty(M) \to \Omega^0$. Furthermore, we have a 'trace' $⨍ \colon \Omega^k \to \mathbb{R}$ given by

$$⨍ \alpha := \int_M \alpha \wedge \omega \quad (\alpha \in \Omega^k),$$

which satisfies

$$⨍ \alpha \wedge \beta = (-1)^{|\alpha||\beta|} ⨍ \beta \wedge \alpha \quad (\alpha, \beta \in \Omega^{\leq k},\ |\alpha| + |\beta| = k)$$

and, since ω is closed, also

$$⨍ d\alpha = 0.$$

The cyclic cocycle $[\omega]$ from before can then also be written as

$$(f_0, ..., f_k) \mapsto ⨍ \rho(f_0) \wedge d(\rho(f_1)) \wedge \cdots \wedge d(\rho(f_k)).$$

This leads to the following definition.

Definition 3.1.1 Let A be an associative algebra. A *cycle of dimension n over A* is the data

- A differential graded algebra $(\Omega, *, D)$ with $\Omega = \bigoplus_{i=0}^{n} \Omega^i$.
- A linear map $⨍ \colon \Omega^n \to \mathbb{K}$ satisfying

  - $\fint \alpha * \beta = (-1)^{|\alpha||\beta|} \fint \beta * \alpha$ for every $\alpha, \beta \in \Omega^{\leq n}$ with $|\alpha| + |\beta| = n$.
  - $\fint D\alpha = 0$ for every $\alpha \in \Omega^{n-1}$.

- An algebra map $\rho\colon A \to \Omega^0$.

These cycles are important because of the following:

Proposition 3.1.2 [Co94, Prop 3.4] If $(\Omega, \fint, \rho)$ is a cycle of dimension n over A, then the map $\mathrm{Ch}_\Omega\colon A^{\otimes(n+1)} \to \mathbb{K}$ given by

$$\mathrm{Ch}_\Omega(a_0, ..., a_k) = \fint \rho(a_0) * D(\rho(a_1)) * \cdots * D(\rho(a_k))$$

satisfies $b\mathrm{Ch}_\Omega = 0$ and $B\mathrm{Ch}_\Omega = 0$.

The proof of the proposition above is essentially the same calculation as we did before, with the calculations with differential forms replaced with formal manipulations allowed by the properties that a cycle has by definition.

However, we can do better. For this we need a few more definitions.

Definition 3.1.3 [Bu68, Def 2.1] If $(\Omega, *)$ is a graded algebra, then a *multiplier of degree k on Ω* is a pair of linear maps $\Theta_l, \Theta_r\colon : \Omega^\bullet \to \Omega^{\bullet+k}$ satisfying

- $\Theta_l(\alpha * \beta) = \Theta_l(\alpha) * \beta$,
- $\Theta_r(\alpha * \beta) = \alpha * \Theta_r(\beta)$,
- $\Theta_r(\alpha) * \beta = \alpha * \Theta_l(\beta)$.

If Θ_l, Θ_r form such a pair, we may also write $\Theta * \alpha$ and $\alpha * \Theta$ for $\Theta_l(\alpha)$ and $\Theta_r(\alpha)$ respectively, and by extension $[\Theta, -]$ for $\Theta_l - \Theta_r$.

Remark 3.1.4 Any element $\alpha \in \Omega^k$ defines a multiplier of degree k by setting $\Theta_l(\beta) = \alpha * \beta$ and $\Theta_r(\beta) = \beta * \alpha$. If Ω is unital, any multiplier is of this form, induced by $\alpha = \Theta_l(1) = \Theta_r(1)$.

Example 3.1.5 If $\Omega \subset \widetilde{\Omega}$ is an inclusion of Ω into a larger graded algebra, any element $\Theta \in \widetilde{\Omega}$ with the property that

$$\Theta * \Omega \subset \Omega \qquad \text{and} \qquad \Omega * \Theta \subset \Omega$$

induces a multiplier of Ω using multiplication inside $\widetilde{\Omega}$. The three properties of a multiplier are then satisfied because of associativity of the product of $\widetilde{\Omega}$.

Definition 3.1.6 An *externally curved DGA* is the datum $(\Omega, *, D, \Theta)$ of:

- A graded algebra $(\Omega, *)$,

- A graded derivation $D\colon \Omega^* \to \Omega^{*+1}$,
- A multiplier Θ of degree 2 on Ω such that
  - $D^2 = [\Theta, -]$,
  - $D(\alpha * \Theta) = (D\alpha) * \Theta$.

If Θ is a multiplier induced by an element of Ω^2, we call Ω a *curved DGA*.

Remark 3.1.7 The last property in the previous definition is called the (right) Bianchi identity. In the unital case where Θ simply is an element of Ω^2, this equation is equivalent to $D\Theta = 0$. There is also a left Bianchi identity $D(\Theta * \alpha) = \Theta * (D\alpha)$, but under the assumption that $D^2\alpha = [\Theta, \alpha]$, the left and right Bianchi identities are equivalent since

$$D(\Theta * \alpha - \alpha * \Theta) = D(D^2\alpha) = D^2(D\alpha) = \Theta * (D\alpha) - (D\alpha) * \Theta.$$

Now we can make the definition of a cycle over A, but using externally curved DGAs instead of DGAs.

Definition 3.1.8 [Go99, §2] A *generalized cycle of dimension n over A* is the data $(\Omega, ⨍, \rho)$, where

- $(\Omega, *, D, \Theta)$ is an externally curved DGA
- $⨍\colon \Omega^n \to \mathbb{K}$ a linear map satisfying
  - $⨍ \alpha * \beta = (-1)^{|\alpha||\beta|} ⨍ \beta * \alpha$ for $\alpha, \beta \in \Omega^{\leq n}$ s.t. $|\alpha| + |\beta| = n$, (trace property)
  - $⨍ D\alpha = 0$ for every $\alpha \in \Omega^{n-1}$ (closedness)
  - $⨍ \Theta * \alpha = ⨍ \alpha * \Theta$ for every $\alpha \in \Omega^{n-2}$,
- An algebra map $\rho\colon A \to \Omega^0$.

With this definition at hand we can also amend our definition of the character of a cycle. In what follows we use this definition for the standard simplex Δ^k:

$$\Delta^k = \{t_0, ..., t_k \geq 0 : t_0 + \cdots + t_k = 1\}$$

which has measure $dt_1 \cdots dt_k$.

Theorem 3.1.9 [Go99, Thm 2.1] Let A be a unital, associative algebra and let $(\Omega, ⨍, \rho)$ be a generalized cycle of dimension n over A. The maps $\mathrm{Ch}^k_\Omega\colon A^{\otimes(k+1)} \to \mathbb{K}$, defined for $0 \leq k \leq n$ such that $k \equiv n \bmod 2$, by the formula

$$\mathrm{Ch}^k_\Omega(a_0, ..., a_k) = \int_{\Delta^k} ⨍ \rho(a_0) * e^{-t_0\Theta} * D\rho(a_1) * e^{-t_1\Theta} * \cdots * D(\rho(a_k)) * e^{-t_k\Theta} dt_1 \cdots dt_k$$

satisfy $b\mathrm{Ch}^k_\Omega = B\mathrm{Ch}^{k-2}\Omega$ and hence define a cocycle Ch_Ω in the (b, B)-bicomplex $\mathrm{CC}^\bullet(A)$.

Remark 3.1.10 By the terms $e^{-t\Theta}$ we mean the power series $\sum_{i\geq 0} \frac{(-t)^i}{i!}\Theta^{*i}$. If we put this under $⨍$ this is well-defined, since $⨍$ only picks up terms of total degree n. As such, in Ch^k_Ω we need a total of $\frac{n-k}{2}$ applications of Θ. In particular we can rewrite the integral as

$$\mathrm{Ch}^k_\Omega(a_0, ..., a_k) = \frac{(-1)^{\frac{n-k}{2}}}{\left(\frac{n+k}{2}\right)!} \sum_{i_0+\cdots+i_k=\frac{n-k}{2}} ⨍ \rho(a_0) * \Theta^{*i_0} D\rho(a_1) * \Theta^{*i_1} * \cdots * D\rho(a_k) * \Theta^{*i_k},$$

where we have also used the integral formula

$$\int_{\Delta^k} \prod_{j=0}^{k} t_j^{i_j} dt_1 \cdots dt_k = \frac{i_0! \cdots i_k!}{(i_0 + \cdots + i_k + k)!}.$$

Example 3.1.11 Let M be a compact, oriented manifold of dimension n, and let $E \to M$ be a complex vector bundle with a connection ∇. Then we can look at the graded algebra

$$\Omega^\bullet = \Omega^\bullet(M, \mathrm{End}(E))$$

with the product given by the braiding of the wedge product in $\Omega(M)$ and the composition in $\mathrm{End}(E)$. The connection ∇ induces a graded derivation $d_\nabla \colon \Omega^\bullet \to \Omega^{\bullet+1}$ which satisfies

- $d_\nabla^2 \omega = F(\nabla) \wedge \omega - \omega \wedge F(\nabla)$,
- $d_\nabla(F(\nabla)) = 0$.

In particular we see that it is a curved DGA with curvature $F(\nabla) \in \Omega^2$.

If we set $⨍ \colon \Omega^n \to \mathbb{C}$ by the formula

$$⨍ \omega = \int_M \mathrm{tr}_E(\omega),$$

we obtain a cyclic cocycle $\mathrm{Ch}_{\Omega,E} \in \mathrm{HC}^n(C^\infty(M))$. Under the isomorphism $\mathrm{HP}^n(C^\infty(M)) \cong \bigoplus_i H^{2i}_{\mathrm{dR}}(M)$ this cocycle corresponds to the Chern character of E.

Remark 3.1.12 By work of Connes [Co94] any cyclic cohomology class can be induced by a generalized cycle.

Remark 3.1.13 If A is not unital and we have a generalized cycle Ω over A, we still want to make sense of the character of this cycle as a cyclic cohomology class of A. By definition, we need to induce a cyclic cohomology class for A^+ out of this. The first ansatz to curing this would be to adjoin a unit to Ω and go from there. However this means that the curvature Θ will also have to become an element of the bigger algebra and we end up with a lot more elements on which to define the trace $⨍$. In general, this does not work and we have to do something more involved. Note that we have the formula for Ch_Ω when we only plug in elements of A, and we only need to see what happens when we plug in the adjoined unit into the formula. While Ω has no unit in general, there is always the unit as a multiplier, in a phantom way satisfying $D(1) = 0$, and this is what helps us out. Indeed, we can extend the definition of Ch_Ω to $(A^+)^{\otimes(k+1)}$ by requiring

$$\mathrm{Ch}^k_\Omega(1, a_1, ..., a_k) = \int_{\Delta^k} ⨍ e^{-t_0\Theta} * D\rho(a_1) * e^{-t_1\Theta} * \cdots * D(\rho(a_k)) * e^{-t_k\Theta} dt_1 \cdots dt_k$$

and $$\mathrm{Ch}^k_\Omega(a_0, a_1, ..., 1...., a_k) = 0.$$

This is well-defined and, by the same arguments as before, yields an n-cocycle in the (b, B)-bicomplex $\mathrm{CC}(A^+)$, which in turn induces a cyclic cohomology class of A.

3.1.2 The fundamental cycle over the convolution algebra

We can now write down generalized cycles for the convolution algebra of a manifold M with a right G-action. We start with the *fundamental cycle*. The constructions are inspired by the Cartan model for equivariant cohomology. For this we start by writing down our algebra

$$\Omega := C^\infty_c(G, \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M)),$$

elements $\alpha \in \Omega$ of which we regard as functions

$$\begin{array}{rcl} \mathfrak{g} \times G & \to & \Omega_c(M) \\ (X, g) & \mapsto & \alpha(X, g) \end{array}$$

which are polynomial in X, and smooth and compactly supported in g.

To define the multiplier that will be the curvature, we can embed Ω into a larger algebra $\Omega^{-\infty}$ given by those distributions

$$T\colon C^\infty_c(G, \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M)) \to \mathrm{Sym}(\mathfrak{g}^*) \times \Omega_c(M)$$

that are $C^\infty(M)$-linear and continuous with respect to the Fréchet topologies.

We embed Ω into $\Omega^{-\infty}$ by sending $\alpha \in \Omega$ to $T_\alpha \in \Omega^{-\infty}$ given by

$$\langle T_\alpha, \varphi\rangle(X) = \int_G \alpha(g, X) \wedge \varphi(g, X) dg.$$

On this space of distributions $\Omega^{-\infty}$ we have an associative convolution product that is defined by Lescure-Manchon-Vassout [LMV17, Thm 20] as follows:

$$\langle T_1 * T_2, \varphi\rangle = \langle T_1(g_1), g_1 \cdot \langle T_2(g_2), g_1^{-1}(\varphi(g_1 g_2))\rangle\rangle. \tag{3.1}$$

Here, g_1 and g_2 are dummy variables, and the formula has to be read as follows. First, with g_1 fixed, we have a map

$$g_2 \mapsto g_1^{-1}(\varphi(g_1 g_2))$$

that is an element of $C_c^\infty(G, \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M))$, so we can pair it with T_2 to obtain the element we write as

$$\langle T_2(g_2), g_1^{-1}(\varphi(g_1 g_2))\rangle \in \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M).$$

Now letting g_1 run as a variable we obtain a function

$$g_1 \mapsto g_1 \cdot \langle T_2(g_2), g_1^{-1}(\varphi(g_1 g_2))\rangle$$

which is again an element of $C_c^\infty(G, \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M))$, and so we can pair it with T_1 to obtain the element

$$\langle T_1 * T_2, \varphi\rangle = \langle T_1(g_1), g_1 \cdot \langle T_2(g_2), g_1^{-1}(\varphi(g_1 g_2))\rangle\rangle.$$

With this tool in hand we define four structures on Ω: a grading, a multiplication $*$, a differential D and a curvature Θ.

- The grading on Ω is given by usual one in the Cartan model: the sum of the degree of the differential form factor and twice the degree of the polynomial on $\mathfrak{g}$;
- The multiplication $*$ is given by

$$(\alpha * \beta)(g, X) := \int_G \alpha(h, X) \wedge h^*\beta(h^{-1}g, \mathrm{Ad}_{h^{-1}}(X))dh; \tag{3.2}$$

- The differential D is given by $d_{\mathrm{dR}} + \iota$ with d_{dR} and ι given by

$$(d_{\mathrm{dR}}\alpha)(g, X) := d_{\mathrm{dR}}(\alpha(g, X)), \qquad (\iota\alpha)(g, X) := \iota_{X_M}(\alpha(g, X));$$

- The curvature Θ is given as an element of $\Omega^{-\infty}$ by

$$\langle \Theta, \varphi\rangle(X) := \left.\frac{d}{dt}\right|_{t=0} \varphi(e^{tX}, X). \tag{3.3}$$

First we investigate how elements of Ω, the convolution product in Ω and the element $\Theta \in \Omega^{-\infty}$ interact in $\Omega^{-\infty}$.

Lemma 3.1.14 The following identities hold true.

i) $T_\alpha * T_\beta = T_{\alpha*\beta}$ for all $\alpha, \beta \in \Omega$;

ii) $T_\alpha * \Theta = T_{\Theta_r(\alpha)}$ for all $\alpha \in \Omega$ where

$$\Theta_r(\alpha)(g, X) = \left.\frac{d}{dt}\right|_{t=0} \alpha(e^{-tX}g, X);$$

iii) $\Theta * T_\alpha = T_{\Theta_l(\alpha)}$ for all $\alpha \in \Omega$ where

$$\Theta_l(\alpha)(g,X) = \frac{d}{dt}\Big|_{t=0} (e^{tX})^*\alpha(e^{-tX}g,X).$$

Proof. Let us fix a test function $\varphi \in C_c^\infty(G, \mathrm{Sym}(\mathfrak{g}^*), \Omega_c(M))$ and let g_1 be fixed. Then the map $g_2 \mapsto g_1^{-1}(\varphi(g_1g_2))$ is given by

$$(g_2, X) \mapsto (g_1^{-1})^*\varphi(g_1g_2, \mathrm{Ad}_{g_1}(X)).$$

Then for the first point we have

$$\langle T_\beta(g_2), g_1^{-1}(\varphi(g_1g_2))\rangle(X) = \int_G \beta(g_2,X) \wedge (g_1^{-1})^*\varphi(g_1g_2, \mathrm{Ad}_{g_1}(X))dg_2,$$

and in particular

$$(g_1 \cdot \langle T_\beta(g_2), g_1^{-1}(\varphi(g_1g_2))\rangle)(X) = \int_G (g_1^*\beta(g_2, \mathrm{Ad}_{g_1^{-1}}X))) \wedge \varphi(g_1g_2, X)dg_2.$$

Therefore,

$$\langle T_\alpha * T_\beta, \varphi\rangle(X) = \int_G\int_G \alpha(g_1,X) \wedge (g_1^*\beta(g_2, \mathrm{Ad}_{g_1^{-1}}(X))) \wedge \varphi(g_1g_2,X)dg_2dg_1,$$

and by setting $h = g_1$, $g = g_1g_2$, this becomes

$$\langle T_\alpha * T_\beta, \varphi\rangle(X) = \int_G \left(\int_G \alpha(h,X) \wedge h^*\beta(h^{-1}g, \mathrm{Ad}_{h^{-1}}(X))dh\right) \wedge \varphi(g,X)dg.$$

This does indeed equal $\langle T_{\alpha*\beta}, \varphi\rangle(X)$.

For the second point, we have

$$\langle \Theta(g_2), g_1^{-1}(\varphi(g_1g_2))\rangle(X) = \frac{d}{dt}\Big|_{t=0} (g_1^{-1})^*\varphi(g_1e^{tX}, \mathrm{Ad}_{g_1}(X))$$

and so

$$g_1 \cdot \langle \Theta(g_2), g_1^{-1}(\varphi(g_1g_2))\rangle(X) = \frac{d}{dt}\Big|_{t=0} \varphi(g_1e^{t\mathrm{Ad}_{g_1^{-1}}(X)}, X) = \frac{d}{dt}\Big|_{t=0} \varphi(e^{tX}g_1, X).$$

Pairing this with T_α we obtain

$$\langle T_\alpha(g_1), g_1 \cdot \langle \Theta(g_2), g_1^{-1}(\varphi(g_1g_2))\rangle\rangle(X) = \int_G \frac{d}{dt}\Big|_{t=0} \alpha(g_1,X) \wedge \varphi(e^{tX}g_1, X)dg_1,$$

which by a change of variables $g = e^{tX}g_1$ yields

$$\langle T_\alpha * \Theta, \varphi\rangle(X) = \int_G \left(\frac{d}{dt}\Big|_{t=0} \alpha(e^{-tX}g, X)\right) \wedge \varphi(g,X)dg.$$

The third point is proven analogously. □

This Lemma allows us to interpret Θ as a multiplier on Ω (cf. Example 3.1.5), and it is of degree 2 since it raises the polynomial degree by 1. Similarly, D is of degree 1 since the d_{dR}-part increases the differential form degree by 1 and leaves the polynomial degree constant, while the ι-part lowers the differential form degree by 1 and increases the polynomial degree by 1. All together, we reach the following conclusion:

Proposition 3.1.15 The data $(\Omega, *, D, \Theta)$ defines an externally curved DGA in the sense of Definition 3.1.6.

Proof. It is easy to see that $*$ is associative, since it is the convolution product induced by the G-module $\mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M)$. Then to see that D is a graded derivation, we first investigate the d_{dR}-part:

$$\begin{aligned}(d_{\mathrm{dR}}(\alpha * \beta))(g, X) &= \int_G d_{\mathrm{dR}}(\alpha(h, X) \wedge h^*\beta(h^{-1}g, \mathrm{Ad}_{h^{-1}}(X)))dh \\ &= \int_G (d_{\mathrm{dR}}(\alpha(h, X))) \wedge h^*\beta(h^{-1}g, \mathrm{Ad}_{h^{-1}}(X))dh \\ &\quad + (-1)^{|\alpha|} \int_G \alpha(h, X) \wedge h^*(d_{\mathrm{dR}}(\beta(h^{-1}g, \mathrm{Ad}_{h^{-1}}(X)))dh \\ &= ((d_{\mathrm{dR}}\alpha) * \beta)(g, X) + (-1)^{|\alpha|}(\alpha * (d_{\mathrm{dR}}\beta))(g, X).\end{aligned}$$

For the ι-part we have

$$\begin{aligned}(\iota(\alpha * \beta))(g, X) &= \int_G \iota_{X_M}(\alpha(h, X) \wedge h^*\beta(h^{-1}g, \mathrm{Ad}_{h^{-1}}(X)))dh \\ &= \int_G (\iota_{X_M}(\alpha(h, X)) \wedge h^*\beta(h^{-1}g, \mathrm{Ad}_{h^{-1}}(X))dh \\ &\quad + (-1)^{|\alpha|} \int_G \alpha(h, X) \wedge \iota_{X_M}(h^*\beta(h^{-1}g, \mathrm{Ad}_{h^{-1}}(X)))dh \\ &= \int_G (\iota_{X_M}(\alpha(h, X)) \wedge h^*\beta(h^{-1}g, \mathrm{Ad}_{h^{-1}}(X))dh \\ &\quad + (-1)^{|\alpha|} \int_G \alpha(h, X) \wedge h^*(\iota_{(\mathrm{Ad}_{h^{-1}}X)_M}\beta(h^{-1}g, \mathrm{Ad}_{h^{-1}}(X)))dh \\ &= ((\iota\alpha) * \beta)(g, X) + (-1)^{|\alpha|}(\alpha * (\iota\beta))(g, X).\end{aligned}$$

Next we show that

$$D^2 = [\Theta, -].$$

To see that this is true, note that we have

$$D^2 = d_{\mathrm{dR}}^2 + \{d_{\mathrm{dR}}, \iota\} + \iota^2.$$

Clearly $d_{\mathrm{dR}}^2 = \iota^2 = 0$ while, by Cartan's Magic formula, we have that the anti-commutator equals

$$\{d_{\mathrm{dR}}, \iota\} = \mathcal{L},$$

where $\mathcal{L}\colon \Omega^\bullet \to \Omega^{\bullet+2}$ is given by

$$(\mathcal{L}\alpha)(g,X) = \mathcal{L}_{X_M}(\alpha(g,X)).$$

Then so see that this equals $[\Theta,\alpha]$, we simply calculate

$$\begin{aligned}[\Theta,\alpha](g,X) =& \left.\frac{d}{dt}\right|_{t=0}(e^{tX})^*\alpha(e^{-tX}g,X) - \left.\frac{d}{dt}\right|_{t=0}\alpha(e^{-tX}g,X)\\ =& \left.\frac{d}{dt}\right|_{t=0}(e^{tX})^*\alpha(g,X)\\ =&\mathcal{L}_{X_M}(\alpha(g,X)).\end{aligned}$$

Lastly, we show that

$$D(\alpha * \Theta) = (D\alpha) * \Theta.$$

This simply follows from the fact that D plays with the output of α, while Θ plays with the input, explicitely:

$$\begin{aligned}D(\alpha*\Theta)(g,X) &= (d_{\mathrm{dR}} + \iota_{X_M})((\alpha*\Theta)(g,X))\\ &= (d_{\mathrm{dR}} + \iota_{X_M})\left(\left.\frac{d}{dt}\right|_{t=0}\alpha(e^{-tX}g,X)\right)\\ &= \left.\frac{d}{dt}\right|_{t=0}(d_{\mathrm{dR}} + \iota_{X_M})\alpha(e^{-tX}g,X)\\ &= (\Theta*(D\alpha))(g,X).\end{aligned}$$

This finishes the proof. □

With an externally curved DGA at hand, we set to decorate it more, so that it becomes a generalized cycle over $G \ltimes C_c^\infty(M)$.

Proposition 3.1.16 Define $⨍\colon \Omega^n \to \mathbb{K}$ and $\rho\colon G \ltimes C_c^\infty(M) \to \Omega^0$ by

$$⨍ \alpha = \int_M \alpha(e,0), \tag{3.4}$$

$$\rho(f)(g,X) = f(g) \in C_c^\infty(M) = \Omega_c^0(M).$$

Then $(\Omega, ⨍, \rho)$ is a generalized cycle over $G \ltimes C_c^\infty(M)$.

Proof. We start by showing

$$⨍ \alpha*\beta = (-1)^{|\alpha||\beta} ⨍ \beta*\alpha.$$

To do this, we begin on the left:

$$\begin{aligned}⨍ \alpha*\beta &= \int_M (\alpha*\beta)(e,0)\\ &= \int_M\int_G \alpha(h,0)\wedge h^*\beta(h^{-1},0)dh.\end{aligned}$$

Then, using that the action of h^{-1} defines an orientation-preserving automorphism of M, we can do a change of variables in M:

$$\begin{aligned}\fint \alpha * \beta &= \int_M \int_G (h^{-1})^*\alpha(h,0) \wedge \beta(h^{-1},0)dh \\ &= (-1)^{|\alpha||\beta|} \int_M \int_G \beta(h^{-1},0) \wedge (h^{-1})^*\alpha(h,0)dh.\end{aligned}$$

Doing a change of variables in G by $h \leftrightarrow h^{-1}$ we obtain[1]

$$\begin{aligned}\fint \alpha * \beta &= (-1)^{|\alpha||\beta|} \int_M \int_G \beta(h,0) \wedge h^*\alpha(h^{-1},0)dh \\ &= (-1)^{|\alpha||\beta|} \fint \beta * \alpha.\end{aligned}$$

Next, we show that

$$\fint D\alpha = 0.$$

This follows from the fact that

$$(D\alpha)(e,0) = d_{\mathrm{dR}}(\alpha(e,0))$$

so the result follows from Stokes' Theorem.

Next, to show that

$$\fint \Theta * \alpha = \fint \alpha * \Theta,$$

we simply remark that

$$(\Theta * \alpha)(e,0) = 0 \qquad \text{and} \qquad (\alpha * \Theta)(e,0) = 0.$$

So we see that $\fint$ is indeed a closed graded trace on Ω, and we only need to show that ρ is an algebra map. This is clear since it is induced by the map of G-algebras $C_c^\infty(M) \to \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M)$ sending f to $1 \otimes f$. □

By Theorem 3.1.9, the triple $(\mathcal{A}, \Omega, \fint)$ gives rise to a cyclic cocycle Ch_Ω of degree $n = \dim M$ which is given in the (b, B)-complex by the components

$$\mathrm{Ch}^k_\Omega(a_0, \dots, a_k) := \int_{\Delta^k} \fint a_0 * e^{-t_0\Theta} * Da_1 * e^{-t_1\Theta} * \cdots * Da_k * e^{-t_k\Theta} dt_1 \dots dt_k,$$

where $k \equiv n \bmod 2$.

Using the fact that $(\alpha * \Theta)(g,0) = 0$ we see that this cocycle only has contributions for $k = n$, where it can be written explicitly as

$$\begin{aligned}\mathrm{Ch}^n_\Omega(a_0, \dots, a_n) = \frac{1}{n!} \int_M \int_{G^{\times n}} & a_0(h_1)h_1^* da_1(h_2) \wedge \cdots \\ & \cdots \wedge (h_1 \cdots h_{k-1})^* da_{k-1}(h_k) \wedge (h_1 \cdots h_k)^* da_k((h_1 \cdots h_k)^{-1}) dh_1 \cdots dh_k.\end{aligned}$$

[1] This is an example where we really need G to be unimodular.

Using that G is unimodular, then we can also write this as

$$\mathrm{Ch}^n_\Omega(a_0,\ldots,a_n)=\frac{1}{n!}\int_M\int_{G^{\times n}}a_0((h_1\cdots h_k)^{-1})((h_1\cdots h_k)^{-1})^*da_1(h_1)\wedge\cdots \\ \cdots\wedge((h_{k-1}h_k)^{-1})^*da_{k-1}(h_{k-1})\wedge(h_k^{-1})^*da_k(h_k)dh_1\cdots dh_k.$$

Remark 3.1.17 Inspired by equivariant cohomology one might be tempted to define a $\mathrm{Sym}(\mathfrak{g}^*)^G$-valued functional by

$$\alpha\mapsto\int_M\alpha(e,-),$$

but this fails to be a trace for the convolution product (3.2). The problem is the adjoint action of G on $X\in\mathfrak{g}$ in formula (3.2) for the product, and this explains why we put $X=0$ in the definition (3.4) of the trace above. To capture the higher degree polynomial terms of $\alpha\in\Omega$, one can twist the trace by an element $\gamma\in\mathrm{Sym}(\mathfrak{g})^G$, viewed as an invariant differential operator D_γ on the Lie algebra $\mathfrak{g}$.

It will follow that if one sets

$$⨍_\gamma\alpha:=\int_M\mathsf{D}_\gamma(\alpha)(e,0),$$

this also defines a closed graded trace on Ω (see below). Remark that in combination with evaluation at $0\in\mathfrak{g}$, the invariants $\mathrm{Sym}(\mathfrak{g})^G$ can be identified as the algebra of distributions supported at 0 in the form of derivatives of the δ-distribution via $\gamma\mapsto\mathsf{D}_\gamma(\delta_0)$.

Proposition 3.1.18 For $\gamma\in(\mathrm{Sym}^q\mathfrak{g})^G$, the functional $⨍_\gamma$ defines a closed graded trace on Ω of degree $\dim(M)+2q$.

Proof. The degree of $⨍_\gamma$ follows from the fact that to obtain a top-form on M after applying D_γ and applying $0\in\mathfrak{g}$ to $\mathsf{D}_\gamma(\alpha)$ we need α to be of degree $\dim(M)$ in the differential form part and of polynomial degree q, i.e. we need α to be of degree $\dim(M)+2q$.

To see that $⨍_\gamma$ vanishes on graded commutators, we compare $⨍_\gamma\alpha*\beta$ and $(-1)^{|\alpha||\beta|}⨍_\gamma\beta*\alpha$ for the case $\gamma=v_1\odot\cdots\odot v_q\in(\mathrm{Sym}\mathfrak{g})^G$ by an explicit calculation. The expression $⨍_\gamma\alpha*\beta$ will now look like

$$⨍_\gamma\alpha*\beta=\frac{d}{dt_1}\bigg|_{t_1=0}\cdots\frac{d}{dt_q}\bigg|_{t_q=0}\int_M\int_G\alpha(g,\sum_{i=1}^q t_iv_i)\wedge\beta(g^{-1},g^*\mathrm{Ad}_{g^{-1}}(\sum_{i=1}^q t_iv_i))dg.$$

After applying the same manipulations as in the proof of Proposition 3.1.16, this will equal

$$⨍_\gamma\alpha*\beta=(-1)^{|\alpha||\beta|}\frac{d}{dt_1}\bigg|_{t_1=0}\cdots\frac{d}{dt_q}\bigg|_{t_q=0}\int_M\int_G\beta(g,\sum_{i=1}^q t_i\mathrm{Ad}_g(v_i))\wedge g^*\alpha(g^{-1},\sum_{i=1}^q t_iv_i)dg.$$

Now since $v_1 \odot \cdots \odot v_q$ is G-invariant we may replace $\{v_i\}_{i=1,\dots,q}$ by $\{\mathrm{Ad}_{g^{-1}}(v_i)\}_{i=1,\dots,q}$ at no cost, to see that

$$\fint_\gamma \alpha * \beta = (-1)^{|\alpha||\beta|} \frac{d}{dt_1}\Big|_{t_1=0} \cdots \frac{d}{dt_q}\Big|_{t_q=0} \int_M \int_G \beta(g, \sum_{i=1}^q t_i v_i) \wedge g^* \alpha(g^{-1}, \sum_{i=1}^q t_i \mathrm{Ad}_{g^{-1}}(v_i)) dg$$

and this precisely equals $(-1)^{|\alpha||\beta|} \fint_\gamma \beta * \alpha$.

The argument that $\fint_\gamma$ is closed is the same as the argument for $\fint$ in the proof of the previous Proposition, since the d-part of D does not contribute to $\fint_\gamma \circ D$ by deRham's Theorem, while the ι-part of D does not contribute since the only top-form of the type $\iota\omega$ is the 0-form. □

Remark 3.1.19 When writing out the JLO-cocycle of Theorem 3.1.9 for this closed graded trace, one recognizes, just like in the case that $\gamma = 0$, that it only has contributions in degree equal to $\dim(M)$. Indeed, looking at the contribution for a given k, we need to consider $(f_0, ..., f_k) \in C_c^\infty(G \times M)^{\times(k+1)}$ and argue with

$$f_0 * e^{-t_0\Theta} * Df_1 * e^{-t_1\Theta} * \cdots * Df_k * e^{-t_k\Theta}.$$

However, one notices that this function will eat $g \in G$ and $X \in \mathfrak{g}$ and spit out a k-form on M, since $(Df_i)(g, X) = d(f_i(g))$ and applying Θ does not change the degree of the differential form.

So, no matter the application of D_γ or evaluating at $g = e$ and $X = 0$, we see that integrating over M only gives a non-trivial contribution when $k = \dim(M)$.

Therefore, the resulting cocycle lives in the image of the shift map

$$S^q \colon \mathrm{HC}^{\dim(M)}(G \ltimes C_c^\infty(M)) \to \mathrm{HC}^{\dim(M)+2q}(G \ltimes C_c^\infty(M)).$$

3.1.3 Twisting by an equivariant vector bundle

For an equivariant vector bundle $E \to M$ with a (not necessarily G-invariant) connection ∇, there is a variant of the construction of the previous section. In the case that the group is discrete, this construction is due to Gorokhovsky [Go99, §3] and here we generalize it to the case of a unimodular Lie group. In this case $\Omega_E := C_c^\infty(G, \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M, \mathrm{End}(E)))$ and we change the differential to

$$(D_\nabla \alpha)(g, X) := d_{\nabla^{\mathrm{End}}}(\alpha(g, X)) + (-1)^{|\alpha|} \alpha(g, X) \wedge \delta(g) + \iota_{X_M} \alpha(g, X),$$

where ∇^{End} is the induced connection on $\mathrm{End}(E)$ and

$$\delta(g) := \nabla - g^* \nabla \in \Omega^1(M, \mathrm{End}(E)).$$

The curvature is now given by

$$\Theta_\nabla := \Theta + (F(\nabla) + \mu)\delta_e,$$

where $F(\nabla) \in \Omega^2(M, \mathrm{End}(E))$ is the ordinary curvature of the connection ∇ and $\mu \in \mathfrak{g}^* \otimes \mathrm{End}(E)$ is the moment of ∇ (cf. [BG94, p.518], [BGV92, Def 7.5]) given by

$$\mu(X) := \nabla_{X_M} - \mathcal{L}_X$$

for $X \in \mathfrak{g}$.

The multiplication in Ω_E has the same formula from equation (3.2), induced by the G-action on $\mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M, \mathrm{End}(E))$ given by

$$(g^*\alpha)(X)(v_1, ..., v_k)(s) = g \cdot (\alpha(\mathrm{Ad}_{g^{-1}}(X))(g^{-1} \cdot v_1, ..., g^{-1} \cdot v_k)(g^{-1} \cdot s))$$

for $X \in \mathfrak{g}$, $v_1, ..., v_k \in \mathfrak{X}(M)$ and $s \in \Gamma(E)$.

To make sense of the curvature we again embed Ω_E into a bigger algebra $\Omega_E^{-\infty}$ consisting of distributions, in this case distributions

$$T\colon C_c^\infty(G, \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M, \mathrm{End}(E))) \to \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M, \mathrm{End}(E))$$

that, similar to the case of the fundamental cycle, are continuous with respect to the Fréchet topologies. The product in $\Omega_E^{-\infty}$ is given by the same procedure as in the untwisted case (3.1). Note that by using the same formula as in (3.3) we can also give meaning to the element Θ in $\Omega_E^{-\infty}$, and the term $(F(\nabla) + \mu)\delta_e$ is the distribution given by

$$((F(\nabla) + \mu)\delta_e)(\varphi)(X) = (F(\nabla) + \mu(X)) \wedge \varphi(e, X).$$

The results of Lemma 3.1.14 translate verbatim to this situation, together with the following calculations regarding the $(F(\nabla) + \mu(X))\delta_e$-term.

Lemma 3.1.20 The element $(F(\nabla) + \mu(X))\delta_e \in \Omega_E^{-\infty}$ defines a multiplier of Ω_E given by the formulae

$$((F(\nabla) + \mu)\delta_e * \alpha)(g, X) = (F(\nabla) + \mu(X)) \wedge \alpha(g, X),$$
$$(\alpha * (F(\nabla) + \mu)\delta_e)(g, X) = \alpha(g, X) \wedge (g^*F(\nabla) + g^*(\mu(\mathrm{Ad}_{g^{-1}}(X)))).$$

Proof. We use the description of the convolution product as in (3.1). So we pick a test function $\varphi \in C_c^\infty(G, \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M, \mathrm{End}(E)))$. Fixing the placeholder variable g_1 we need to test the function $g_2 \mapsto g_1^{-1} \cdot \varphi(g_1 g_2)$ on T_α. Adding the $\mathfrak{g}$-component explicitly, this is the function sending (g_2, X) to $(g_1^{-1})^*\varphi(g_1 g_2, \mathrm{Ad}_{g_1}(X))$, so pairing with T_α we obtain

$$\langle T_\alpha(g_2), g_1^{-1} \cdot \varphi(g_1 g_2)\rangle(X) = \int_G \alpha(g_2, X) \wedge (g_1^{-1})^*(\varphi(g_1 g_2, \mathrm{Ad}_{g_1}(X)))dg_2$$

and so the function to pair with $(F(\nabla) + \mu)\delta_e$ sends (g_1, X) to

$$(g_1 \cdot \langle T_\alpha(g_2), g_1^{-1} \cdot \varphi(g_1 g_2)\rangle)(X) = g_1^*(\langle T_\alpha(g_2), g_1^{-1} \cdot \varphi(g_1 g_2)\rangle(\mathrm{Ad}_{g_1^{-1}}(X)))$$
$$= \int_G (g_1^*(\alpha(g_2, \mathrm{Ad}_{g_1^{-1}}(X))) \wedge \varphi(g_1 g_2, X)dg_2$$

Pairing this with $(F(\nabla)+\mu)\delta_e$ we plug in $g_1 = e$ and take the wedge product to obtain

$$\langle (F(\nabla)+\mu)\delta_e * T_\alpha, \varphi\rangle(X) = \int_G (F(\nabla)+\mu(X)) \wedge \alpha(g_2, X) \wedge \varphi(g_2, X) dg_2$$

and we see that this distribution is precisely of the form T_β for

$$\beta(g, X) = (F(\nabla)+\mu(X)) \wedge \alpha(g, X),$$

which proves the first equation.

For the second equation, we again let φ be a test function and fix the placeholder variable g_1. Now we pair the function $(g_2, X) \mapsto (g_1^{-1})^*\varphi(g_1g_2, \mathrm{Ad}_{g_1}(X))$ with $(F(\nabla)+\mu)\delta_e$ to obtain:

$$\langle (F(\nabla)+\mu)\delta_e(g_2), g_1 \cdot \varphi(g_1g_2)\rangle(X) = (F(\nabla)+\mu(X)) \wedge (g_1^{-1})^*\varphi(g_1, \mathrm{Ad}_{g_1}(X)).$$

From this we get

$$(g_1 \cdot \langle (F(\nabla)+\mu)\delta_e(g_2), g_1 \cdot \varphi(g_1g_2)\rangle)(X) = (g_1^*F(\nabla) + g_1^*(\mu(\mathrm{Ad}_{g_1^{-1}}(X)))) \wedge \varphi(g_1, X).$$

Pairing this with $T_\alpha(g_1)$ results in

$$\langle T_\alpha * (F(\nabla)+\mu)\delta_e, \varphi\rangle(X) = \int_G \alpha(g_1, X) \wedge (g_1^*F(\nabla) + g_1^*(\mu(\mathrm{Ad}_{g_1^{-1}}(X)))) \wedge \varphi(g_1, X) dg_1$$

and this is precisely of the form T_β for

$$\beta(g, X) = \alpha(g, X) \wedge (g^*F(\nabla) + g^*(\mu(\mathrm{Ad}_{g^{-1}}(X)))),$$

which proves the Lemma. □

From this one can infer the following:

Proposition 3.1.21 The quadruple $\Omega_{E,\nabla} = (\Omega_E, *, D_\nabla, \Theta_\nabla)$ is an externally curved DGA.

Proof. This is an explicit calculation in the same vein as Proposition 3.1.15. We skip the details. □

Next, we can introduce a closed graded trace on $\Omega_{E,\nabla}$, combining the ideas of Proposition 3.1.16 and Example 3.1.11. The result is the functional

$$⨍ \alpha := \int_M \mathrm{tr}_E \alpha(e, 0)$$

where $\mathrm{tr}_E \colon \Omega^\bullet(M, \mathrm{End}(E)) \to \Omega^\bullet(M)$ is the application of the matrix trace.

Proposition 3.1.22 The functional $⨍$ is a closed graded trace on $\Omega_{E,\nabla}$.

Proof. This is similar to the treatment of the untwisted case in Proposition 3.1.16. Performing the same steps as in the untwisted case, we obtain:

$$⨍(\alpha * \beta) = (-1)^{|\alpha||\beta|} ⨍(\beta * \alpha) + \int_M \int_G \mathrm{tr}_E([\alpha(g,0), g^*\beta(g^{-1},0)])dg$$

and then remark that $\mathrm{tr}_E([\alpha(g,0), g^*\beta(g^{-1},0)]) = 0$. To see that it is closed we have

$$⨍ D_\nabla \alpha = \int_M \mathrm{tr}_E(d_{\nabla^{\mathrm{End}}}(\alpha(e,0))) = \int_M d(\mathrm{tr}_E(\alpha(e,0))) = 0.$$

To check that $⨍ \Theta_\nabla * \alpha = ⨍ \alpha * \Theta_\nabla$ we first note that

$$(\Theta_\nabla * \alpha)(e,0) = F(\nabla) \wedge \alpha(e,0)$$
$$(\alpha * \Theta_\nabla)(e,0) = \alpha(e,0) \wedge F(\nabla),$$

and then using the fact that taking the trace over $\mathrm{End}(E)$ is cyclically invariant we obtain

$$⨍(\Theta_\nabla * \alpha) = \int_M \mathrm{tr}_E(F(\nabla) \wedge \alpha(e,0)) = \int_M \mathrm{tr}_E(\alpha(e,0) \wedge F(\nabla)) = ⨍(\alpha * \Theta_\nabla)$$

and hence $⨍$ is a closed graded trace. □

Plugging this into the machinery of Theorem 3.1.9 we obtain, for every pair (E, ∇) of an equivariant vector bundle with connection, a cyclic cohomology class $\mathrm{Ch}_{\Omega,\nabla} \in \mathrm{HC}^{\dim(M)}(G \ltimes C_c^\infty(M))$.

Mirroring results of Gorokhovsky [Go99] we see that the resulting cohomology class is intrinsic to E.

Proposition 3.1.23 If ∇ and ∇' are two connections on E, then the cyclic cohomology classes $\mathrm{Ch}_{E,\nabla}$ and $\mathrm{Ch}_{E,\nabla'}$ are equal.

Proof. Write $\eta = \nabla' - \nabla \in \Omega^1(M, \mathrm{End}(E))$. Then for $t \in [0,1]$, $\nabla_t = \nabla + t\eta$ is a family of connections on E connecting ∇ and ∇'. In turn Ω_{E,∇_t} is a family of externally curved DGA's and in turn $(\Omega_{E,\nabla_t}, ⨍, \rho)$ gives a family of generalized cycles. In this way, we see that the cycles $(\Omega_{E,\nabla}, ⨍, \rho)$ and $(\Omega_{E,\nabla'}, ⨍, \rho)$ are cobordant, and so by [Go99, Cor 2.3], the resulting cyclic cohomology classes are equal. □

The conclusion of all this is the definition of a characteristic class of an equivariant vector bundle in the form of a cyclic cohomology class over the convolution algebra.

Theorem 3.1.24 The association $(E, \nabla) \mapsto \mathrm{Ch}_{\Omega_{E,\nabla}}$ defines a map

$$\mathrm{Ch}_\Omega \colon \mathrm{Vect}_G(M) \to \mathrm{HC}^{\dim(M)}(G \ltimes C_c^\infty(M))$$

that is independent of the connection ∇. Here $\mathrm{Vect}_G(M)$ is the set of isomorphism classes of G-vector bundles over M.

3.2 Understanding the convolution algebra via a double complex

In this section we use strategies created by Brylinski [Br87a] to replace the Hochschild complex of the convolution algebra $G \ltimes C_c^\infty(M)$ with a double complex which in some sense splits the behaviour in the G-direction and the behaviour in the $C_c^\infty(M)$-direction.

To this end we look at the case where A is a smooth algebra that has a smooth left G-action of a Lie group G. Following Brylinski [Br87a], we mean this to be an algebra map $\rho\colon G \to \mathrm{Aut}(A)$, $\rho(g)(a) = g \cdot a$ that has the following smoothness properties:

- For $K \subset G$ a compact subset, the family $\rho(K)$ is equicontinuous;
- For every $a \in A$ the map $g \mapsto g \cdot a$ is differentiable
- The map $A \to C^\infty(G, A)$ sending a to the map $g \mapsto g \cdot a$ is continuous.

The convolution algebra $G \ltimes A$, given by $C_c^\infty(G, A)$ with product given by

$$(f_1 * f_2)(g) = \int_G f_1(h)(h \cdot f_2(h^{-1}g))dh,$$

is then canonically a topological algebra, and we are interested in the Hochschild homology calculated using the topological tensor porduct. Notice that this means that we have an identification between the degree k-part of the Hochschild complex and $C_c^\infty(G^{\times k}, A^{\otimes k})$. In particular, for our main example $A = C_c^\infty(M)$ we have that the degree k-part of the Hochschild complex is given by $C_c^\infty(G^{\times k} \times M^{\times k})$.

Let us first investigate the case where A is unital and G is discrete, following a classic story as told by for instance Ponge [Po18]. The associative algebra $G \ltimes A$ induces a cyclic vector space $(G \ltimes A)^{\otimes \bullet+1}$, which in this case is just $G^{\times(n+1)} \times A^{\otimes(n+1)}$ as vector spaces. We want to understand more of the underlying structure present in the cyclic structure here. Writing $(g_0, ..., g_n | a_0, ..., a_n)$ for the element corresponding to $(g_0, ..., g_n) \times (a_0 \otimes \cdots \otimes a_n)$, we can write down the simplicial maps as

$$\begin{aligned}
d_i(g_0, ..., g_n|a_0, ..., a_n) &= (g_0, ..., g_i g_{i+1}, ..., g_n|a_0, ..., a_i(g_i a_{i+1}), ..., a_n), \qquad (i < n)\\
d_n(g_0, ..., g_n|a_0, ..., a_n) &= (g_n g_0, g_1, ..., g_{n-1}|a_n(g_n a_0), a_1, ..., a_{n-1}),\\
s_i(g_0, ..., g_n|a_0, ..., a_n) &= (g_0, ..., g_i, 1, g_{i+1}, ..., g_n|a_0, ..., a_i, 1, a_{i+1}, ..., a_n),\\
t(g_0, ..., g_n|a_0, ..., a_n) &= (g_n, g_0, ..., g_{n-1}|a_n, a_0, ..., a_{n-1}).
\end{aligned}$$

This suggests that the simplicial structure is in a certain sense composed of the obvious simplicial structure on the group algebra of G and a kind of twisted version of the simplicial structure on $A^{\otimes(\bullet+1)}$. To run this point home further, we look at the automorphism φ on $G^{\times(\bullet+1)} \times A^{\otimes(\bullet+1)}$ given by

$$\varphi(g_0, ..., g_n|a_0, ..., a_n) = (g_0, ..., g_n|(g_0 \cdots g_n)^{-1}a_0, ..., g_n^{-1}a_n).$$

Conjugating with this map φ we obtain an isomorphic simplicial structure on $G^{\times(\bullet+1)} \times A^{\otimes(\bullet+1)}$ given by

$$\begin{aligned}
d_i(g_0,...,g_n|a_0,...,a_n) &= (g_0,...,g_ig_{i+1},...,g_n|a_0,...,a_ia_{i+1},...,a_n), && (i<n)\\
d_n(g_0,...,g_n|a_0,...,a_n) &= (g_ng_0,g_1,...,g_{n-1}|g_n((g_0\cdots g_n)^{-1}a_n)a_0),g_na_1,...,g_na_{n-1}),\\
s_i(g_0,...,g_n|a_0,..,a_n) &= (g_0,...,g_i,1,g_{i+1},...,g_n|a_0,...,a_i,1,a_{i+1},...,a_n),\\
t(g_0,...,g_n|a_0,...,a_n) &= (g_n,g_0,...,g_{n-1}|g_n(g_0\cdots g_n)^{-1}a_n,g_na_0,...,g_na_{n-1}).
\end{aligned}$$

These formulae really allow us to separate the simplicial structure into a simplicial structure on the group algebra and a simplicial structure on $A^{\times(\bullet+1)}$. Indeed, if we write $\mathrm{C}_{p,q} = G^{\times(q+1)} \times A^{\otimes(p+1)}$, we can write maps

$$\begin{aligned}
&d_i^h\colon \mathrm{C}_{p,q}\to \mathrm{C}_{p-1,q} \quad (0\le i\le p) && d_i^v\colon sC_{p,q}\to \mathrm{C}_{p,q-1} \quad (0\le i\le q)\\
&s_i^h\colon \mathrm{C}_{p,q}\to \mathrm{C}_{p+1,q} \quad (0\le i\le p) && s_i^v\colon \mathrm{C}_{p,q}\to \mathrm{C}_{p,q+1} \quad (0\le i\le q)\\
&t^h\colon \mathrm{C}_{p,q}\to \mathrm{C}_{p,q} && t^v\colon \mathrm{C}_{p,q}\to \mathrm{C}_{p,q}
\end{aligned}$$

defined by

$$\begin{aligned}
d_i^h(g_0,...,g_q|a_0,...,a_p) &= (g_0,...,g_q|a_0,...,a_ia_{i+1},...,a_p), && (i<p)\\
d_p^h(g_0,...,g_q|a_0,...,a_p) &= (g_0,...,g_q|((g_0\cdots g_q)^{-1}a_p)a_0,a_1,...,a_{p-1}),\\
s_i^h(g_0,...,g_q|a_0,...,a_p) &= (g_0,...,g_q|a_0,...,a_i,1,a_{i+1},...,a_p),\\
t^h(g_0,...,g_q|a_0,...,a_p) &= (g_0,...,g_q|(g_0\cdots g_q)^{-1}a_p,a_0,...,a_{p-1}),\\
d_i^v(g_0,...,g_q|a_0,...,a_p) &= (g_0,...,g_ig_{i+1},...,g_q|a_0,...,a_p), && (i<q)\\
d_q^v(g_0,,...,g_q|a_0,...,a_p) &= (g_qg_0,g_1,...,g_{q-1}|g_qa_0,...,g_qa_p),\\
s_i^v(g_0,...,g_q|a_0,...,a_p) &= (g_0,...,g_i,1,g_{i+1},...,g_q|a_0,...,a_p),\\
t^v(g_0,...,g_q|a_0,...,a_p) &= (g_q,g_0,...,g_{q-1}|g_qa_0,...,g_qa_p),
\end{aligned}$$

which, as shown in Getzler-Jones [GJ93], together give $G^{\times(\bullet+1)} \times A^{\otimes(\bullet+1)}$ the structure of a cylindrical vector space.

3.2.1 The cylindrical space associated to the convolution algebra

Let us now investigate the case where G is a unimodular Lie group and A is a smooth G-algebra (not necessarily unital) by describing the cylindrical space introduced by Block-Getzler-Jones [BGJ95] for this situation. We first remark that adjoining a unit to $G \ltimes A$ is the same as adjoining $\delta_e \otimes 1$ where δ_e is the Dirac delta-distribution at the identity of G and 1 is the adjoined unit of A. So we may understand $(G \ltimes A)^+$ as those distributions on G with values in A^+ whose singular behaviour is limited to $\delta_e(g)$.

We can then write $((G \ltimes A)^+)^{\otimes(k+1)}$ as $(C_c^\infty(G)^+)^{\otimes(k+1)} \otimes (A^+)^{\otimes(k+1)}$, where $C_c^\infty(G)^+$ denotes $C_c^\infty(G)$ with the Dirac delta δ_e adjoined. Since we use the inductive tensor product with the property that $C_c^\infty(G)\otimes C_c^\infty(G)\cong C_c^\infty(G\times G)$, we may think of $(C_c^\infty(G)^+)^{\otimes(k+1)}$ as distributions on $G^{\times(k+1)}$ whose singular behaviour is limited to products of $\delta_e(g_0)$,...,$\delta_e(g_k)$. We can now write down the cyclic strucutre on $((G\ltimes A)^+)^{\otimes(\bullet+1)}$ by investigating it for elements of the form $F(g_0,...,g_k)=f(g_0,...,g_k)\otimes a_0\otimes\cdots\otimes a_k$. The cyclic structure maps are then given by

$$\begin{aligned}
d_i(F)(g_0,...,g_{k-1}) &= \int_G (f(g_0,...,\gamma,\gamma^{-1}g_i,...,g_{k-1})\otimes a_0\otimes\cdots\otimes a_i(\gamma a_{i+1})\otimes\cdots\otimes a_k d\gamma, \\
&\qquad\qquad (0\le i\le k-1)\\
d_k(F)(g_0,...,g_{k-1}) &= \int_G (f(\gamma^{-1}g_0,...,g_{k-1},\gamma)\otimes a_k(\gamma a_0)\otimes a_1\otimes\cdots\otimes a_{k-1}d\gamma,\\
s_i(F)(g_0,...,g_{k+1}) &= \delta(g_{i+1})f(g_0,...,g_i,g_{i+2},...,g_{k+1})\otimes a_0\otimes\cdots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\cdots\otimes a_k,\\
t(F)(g_0,...,g_k) &= f(g_1,...,g_k,g_0)\otimes a_k\otimes a_0\otimes\cdots\otimes a_{k-1}.
\end{aligned}$$

Mimicking the procedure we described before when G is discrete, we can now write down the cylindrical space introduced in [BGJ95], which splits the cyclic structure on $((G\ltimes A)^+)^{\otimes(\bullet+1)}$ into the cyclic structure on $(C_c^\infty(G)^+)^{\otimes(\bullet+1)}$ and that of $A^{\otimes(\bullet+1)}$. To this end, we define $\mathrm{L}^+(G,A)_{p,q}$ to be compactly supported distributions from $G^{\times(q+1)}$ to $(A^+)^{\otimes(p+1)}$ whose singular behaviour is restricted to $\delta_e(g_i)$ for $i=0,...,q$. For elements $F\in\mathrm{L}^+(G,A)_{p,q}$ of the form $F(g_0,...,g_q)=f(g_0,...,g_q)\otimes a_0\otimes\cdots\otimes a_p$ we can write down structure maps by the formulae

$$\begin{aligned}
d_i^h(F)(g_0,...,g_q) &= f(g_0,...,g_q)\otimes a_0\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_p, \qquad (0\le i\le p-1)\\
d_p^h(F)(g_0,...,g_q) &= f(g_0,...,g_q)\otimes((g_0\cdots g_q)^{-1}a_p)a_0\otimes a_1\otimes\cdots\otimes a_{p-1},\\
s_i^h(F)(g_0,...,g_q) &= f(g_0,...,g_q)\otimes a_0\otimes\cdots\otimes a_i\otimes 1\otimes a_{i+1}\otimes\cdots\otimes a_p,\\
t^h(F)(g_0,...,g_q) &= f(g_0,...,g_q)\otimes(g_0\cdots g_q)^{-1}a_p\otimes a_0\otimes\cdots\otimes a_{p-1},\\
d_i^v(F)(g_0,...,g_{q-1}) &= \int_G F(g_0,...,\gamma,\gamma^{-1}g_i,...,g_{q-1})d\gamma, \qquad (0\le i\le q-1)\\
d_q^v(F)(g_0,...,g_{q-1}) &= \int_G \gamma\cdot F(\gamma^{-1}g_0,...,g_{q-1},\gamma)d\gamma,\\
s_i^v(F)(g_0,...,g_{q+1)} &= \delta(g_{i+1})F(g_0,...,g_i,g_{i+2},...,g_{q+1}),\\
t^v(F)(g_0,...,g_q) &= g_0\cdot F(g_1,...,g_q,g_0).
\end{aligned}$$

Using these maps, we have the following result.

Proposition 3.2.1 With the structure maps as above, $(\mathrm{L}^+(G,A)_{\bullet,\bullet},d^h,s^h,t^h,d^v,s^v,t^v)$ is a cylindrical space.

Proof. We argue why this is true, and skip the explicit calculations. We need to check four things

- For every q, $(L^+(G,A)_{\bullet,q},d^h,s^h,t^h)$ is an Λ_∞-space;

- For every p, $(L^+(G,A)_{p,\bullet}, d^v, s^v, t^v)$ is an Λ_∞-space;
- The horizontal and vertical structures commute;
- As maps from $L^+(G,A)_{p,q} \to L^+(G,A)_{p,q}$ we have $(t^h)^{p+1}(t^v)^{q+1}$ is the identity.

The first two points follow from the fact that we recognize known Λ_∞-spaces in the horizontal and vertical structures: in the horizontal case the structure on the Hochschild complex of A (with a G-twist), and in the vertical case the Λ_∞-space underlying the group homology complex for the G-module $(A^+)^{\otimes(p+1)}$.

The fact that they commute follows largely from the fact that the horizontal structure has effect on the output of functions, while the vertical structure has effect on the input of functions. The only complication is the action of $(g_0 \cdots g_q)^{-1}$ in d^h_p and t^h. However, this is counteracted by how the vertical structure changes the input. In particular we see:

- For d^v_i when $0 \leq i \leq q-1$ we have
$$(g_0 \cdots g_{i-1}\gamma\gamma^{-1}g_i \cdots g_{q-1})^{-1} = (g_0 \cdots g_{q-1})^{-1};$$
- For d^v_q we have
$$\gamma \cdot (\gamma^{-1}g_0 \cdots g_{q-1}\gamma)^{-1} = (g_0 \cdots g_{q-1})^{-1}\gamma;$$
- For s^v_i we have
$$(g_0 \cdots g_i g_{i_2} \cdots g_{q+1})^{-1} = (g_0 \cdots g_{q+1})^{-1}$$
whenever $g_{i+1} = e$;
- For t^v we have
$$g_0(g_1 \cdots g_q g_0^{-1})^{-1} = (g_0 \cdots g_q)^{-1} g_0.$$

Lastly, to see that the maps t^h and t^v satisfy the cyclindricity equation we simply note that
$$(t^h)^{p+1}(F)(g_0, ..., g_q) = (g_0 \cdots g_q)^{-1} \cdot F(g_0, ..., g_q)$$
and
$$(t^v)^{q+1}(F)(g_0, ..., g_q) = (g_0 \cdots g_q) \cdot F(g_0, ..., g_q).$$
Together this shows that $L^+(G,A)$ is a cylindrical space. □

On top of this we can write down a map $\Psi_1 \colon (G \ltimes A)^{\otimes(k+1)} \to \mathrm{L}^+(G,A)_{k,k}$ following [Br87a] by the formula

$$\Psi_1(F)(g_0, ..., g_k) = ((g_0 \cdots g_k)^{-1} \otimes \cdots \otimes g_k^{-1})F(g_0, ..., g_k).$$

This map yields the connection between the simplicial space induced by the convolution algebra and this cylindrical space, via the following Lemma whose proof is again an explicit calculation.

Lemma 3.2.2 [Br87a, p.14] The map Ψ_1 is an isomorphism of cyclic spaces

$$((G \ltimes A)^+)^{\otimes(\bullet+1)} \to \mathrm{diag}(\mathrm{L}^+(G,A))_\bullet.$$

This means that we can use the Eilenberg-Zilber Theorem to understand the cyclic homology of the convolution algebra in terms of this cylindrical space.

To start, let use the cylindrical structure to induce the four differentials. First we can write down the contractions c^h, c^v, which in this case are given by

$$c^h(F)(g_0,...,g_q) = 1 \otimes F(g_0,...,g_q),$$
$$c^v(F)(g_0,...,g_{q+1}) = \delta(g_0)F(g_1,...,g_q).$$

From this we can write down the differentials

$$b^h\colon \mathrm{L}^+(G,A)_{p,q} \to \mathrm{L}^+(G,A)_{p-1,q}, \qquad b^v\colon \mathrm{L}^+(G,A)_{p,q} \to \mathrm{L}^+(G,A)_{p,q-1},$$
$$B^h\colon \mathrm{L}^+(G,A)_{p,q} \to \mathrm{L}^+(G,A)_{p+1,q}, \qquad B^v\colon \mathrm{L}^+(G,A)_{p,q} \to \mathrm{L}^+(G,A)_{p,q+1},$$

which by the general machinery of Lemma A.3.2 and Proposition A.3.6 are given by the formulae

$$b^h = \sum_{i=0}^{p}(-1)^i d_i^h,$$
$$b^v = \sum_{i=0}^{q}(-1)^{i+p},$$
$$B^h = (1+(-1)^p t^h)c^h\left(\sum_{j=0}^{p} d_j^h\right)c^h\left(\sum_{i=0}^{p}(-1)^{ip}(t^h)^i\right),$$
$$B^v = (-1)^p(t^h)^{p+1}(1+(-1)^q t^v)c^v\left(\sum_{j=0}^{q} d_j^v\right)c^v\left(\sum_{i=0}^{q}(-1)^{iq}(t^v)^i\right).$$

Remark 3.2.3 Following Brylinski [Br87a], we can also avoid using delta-distributions on G by defining a contraction of $(b')^v$ that stays inside the world of smooth functions. For this we use an approximate unit, i.e. a smooth function $u \in C_c^\infty(G)$ such that

$$\int_G u(g)dg = 1.$$

Using this approximate unit we can write down a map $c^v\colon C_c^\infty(G^n, A^{\otimes\bullet}) \to C_c^\infty(G^{n+1}, A^{\otimes\bullet})$ with the formula

$$(c^v F)(g_0,...,g_n) = u(g_0g_1)F(g_1,...,g_n)$$

and one checks that this indeed contracts $(b')^v$. We do not pursue this path, since, while this contraction is compatible with the semi-simplicial structure given by the maps d_i^v, there are no simplicial degeneracy maps s_i^v complementing this to a simplicial structure of which c^v is the induced contraction of $(b')^v$. In particular, we can not use the full machinery acquired by the cyclic Eilenberg-Zilber map, since this explicitly needs the degeneracies to define the homotopy inverse of the Eilenberg-Zilber map.

To see that the contraction with the approximate unit is not induced by a full simplicial structure, we note that such a simplicial structure would have to look like

$$(s_i^v F)(g_0, ..., g_n) = u(g_i g_{i+1}) F(g_0, ..., g_{i-1}, g_{i+1}, ..., g_n),$$

but these maps do not satisfy all the simplicial identities. We do remark that using this contraction, the convolution algebra $G \ltimes C_c^\infty(M)$ is H-unital.

We will write down explicit forms of the four differentials b^h, b^v, B^h and B^v in the normalized complex $\mathrm{L}(G, A)_{\bullet,\bullet}$. Since taking normalized chains kills $\delta_e(g_i)$ for $i = 1, ..., q$ and the adjoined unit of A^+ in all but the first entries, we may write the normalized chains $\mathrm{L}(G, A)_{p,q}$ as distributions on $G^{\times(q+1)}$ with values in $A^+ \otimes A^{\otimes(p+1)}$ such that the singular behaviour is restricted to $\delta_e(g_0)$.

For a normalized chain $F(g_0, ..., g_q) = f(g_0, ..., f_q) \otimes a_0 \otimes \cdots \otimes a_p$, the differentials in the normalized complex are now given by

$$\begin{aligned}
b^h(F)(g_0, ..., g_q) =& \sum_{i=0}^{p-1} (-1)^i f(g_0, ..., g_q) \otimes a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p \\
&+ (-1)^p f(g_0, ..., g_q) \otimes ((g_0 \cdots g_q)^{-1} a_p) a_0 \otimes a_1 \otimes \cdots \otimes a_{p-1}, \\
b^v(F)(g_0, ..., g_{q-1}) =& \sum_{i=0}^{q-1} (-1)^{i+p} \int_G F(g_0, ..., \gamma, \gamma^{-1} g_i, ..., g_{q-1}) d\gamma \\
&+ (-1)^{p+q} \int_G \gamma \cdot F(\gamma^{-1} g_0, ..., g_{q-1}, \gamma) d\gamma, \\
B^h(F)(g_0, ..., g_q) =& \sum_{i=0}^{p} (-1)^{ip} f(g_0, ..., g_q) \otimes 1 \otimes (g_0 \cdots g_q)^{-1} (a_{p-i+1} \otimes \cdots \otimes a_p) \otimes \\
&\otimes \overline{a_0} \otimes \cdots \otimes a_{p-i}, \\
B^v(F)(g_0, ..., g_{q+1}) =& \sum_{i=0}^{q} (-1)^{iq+p} \delta_e(g_0) (g_{i+1} \cdots g_{q+1})^{-1} \cdot \overline{F}(g_{i+1}, ..., g_{q+1}, g_1, ..., g_i),
\end{aligned}$$

where in this notation $\overline{a_0}$ is the application of the map $\bar{\cdot}\colon A^+ \to A$ that kills the adjoined unit, and $\overline{F}$ is the application of the map $\mathrm{L}(G, A)_{p,q} \to C_c^\infty(G^{\times(q+1)}, A^{\otimes(p+1)})$ that kills any factors of $\delta_e(g_0)$.

The Eilenberg-Zilber Theorem discussed in Appendix A.5 now gives us the following chain of quasi-isomorphisms

$$(\mathrm{CP}_{\bullet,\mathrm{norm}}((G \ltimes A)^+), b + B) \xrightarrow[\cong]{\Psi_1} (\mathrm{CPdiag}_{\bullet,\mathrm{norm}}(L^+(G, A)), b_{\mathrm{diag}} + B_{\mathrm{diag}}) \xrightarrow[\simeq]{\mathrm{EZ}^{\mathrm{pert}}}$$

$$\xrightarrow[\simeq]{\mathrm{EZ}^{\mathrm{pert}}} (\mathrm{CPTot}_\bullet(L(G, A)), b^h + b^v + B^h + B^v)$$

Remark 3.2.4 By the ordinary Eilenberg-Zilber Theorem, the Hochschild homology $\mathrm{HH}_\bullet(G \ltimes A, G \ltimes A)$ is calculated by the double complex $(\mathrm{Tot}(\mathrm{L}(G, A)), b^h + b^v)$. We can write down the spectral sequence associated to this double complex, where we obtain on the first page

$$\mathrm{E}^1_{p,q} = \mathrm{H}_q(G, A^{\otimes(p+1)}).$$

When G is compact, we know that this homology is concentrated in degree 0, where it is given by the coinvariants

$$\mathrm{H}_0(G, A^{\otimes(p+1)}) \cong (A^{\otimes(p+1)})_G := A^{\otimes(p+1)}/\langle x - gx : g \in G, x \in A^{\otimes(p+1)}\rangle.$$

In particular, in case only the first row of the first page is filled, and we see that the spectral sequence collapses on the second page where it is given by

$$\mathrm{E}^2_{p,q} = \begin{cases} \mathrm{H}_p((A^{\otimes(\bullet+1)})_G, b_{\mathrm{Hoch}}) & \text{if } q = 0 \\ 0 & \text{else} \end{cases}.$$

3.3 Pairing with equivariant cohomology

When M is a manifold with a right G-action and $A = C_c^\infty(M)$, we want to use the double complex constructed above to pair the equivariant cohomology $\mathrm{H}^\bullet_G(M)$ with the cyclic homology $\mathrm{HP}_\bullet(G \ltimes C_c^\infty(M))$. To achieve this we need to do two things. First, we introduce differential forms in the picture we started to sketch in the previous section. Secondly, we discuss Getzler's model of equivariant cohomology. Together, these will induce a pairing between equivariant cohomology and periodic cyclic homology.

Getzler's model of equivariant cohomology makes use of differential forms on M, instead of compactly supported differential forms. To mimic this situation algebraically, in this section we will be in the situation where $A \subset \mathbf{A}$ are two smooth G-algebras with A an ideal in $\mathbf{A}$ such that ΩA is an ideal in $\Omega\mathbf{A}$. There are two extremes of this situation: on the one hand we can take $\mathbf{A} = A$; on the other hand (and more importantly) we have the example $(A, \mathbf{A}) = (C_c^\infty(M), C^\infty(M))$ for a manifold M.

3.3.1 Equivariant differential forms and equivariant HKR

Starting with the double complex $\mathrm{L}(G, A)_{\bullet,\bullet}$, we want to impose some kind of HKR-like procedure to replace $A^+ \otimes A^{\otimes\bullet}$ with $\Omega^\bullet A$. Since there is still a G-action around, and since this action also twists the simplicial structure on $A^+ \otimes A^{\otimes\bullet}$ we need to do this in an equivariant fashion. This leads to the a definition of an equivariant Hochschild–Kostant–Rosenberg map, following Block and Getzler [BG94], which we recall here. Following their conventions, we now use a different definition for the standard simplex than we did before Section 3.1. From this point on we use the definition

$$\Delta^k = \{(t_1, ..., t_k) \in \mathbb{R}^k : 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}.$$

Since in the result of Theorem 3.1.9 we integrated out the standard simplex, this should not lead to confusion.

Definition 3.3.1 [BG94, §3] Let A be a smooth G-algebra. For $X \in \mathfrak{g}$, the *equivariant HKR-map* $\mathrm{HKR}_X \colon A^+ \otimes A^{\otimes k} \to \Omega^k A$ is given by

$$\mathrm{HKR}_X(a_0 \otimes \cdots \otimes a_k) = \int_{\Delta^k} a_0 d(e^{-t_1 X} a_1) \wedge \cdots \wedge d(e^{-t_k X} a_k) dt_1 \cdots dt_k.$$

Note that the action of G on A extends to an action of G on $\Omega^p A$ given by

$$g \cdot (a_0 da_1 \wedge \cdots \wedge da_p) = (g \cdot a_0) d(g \cdot a_1) \wedge \cdots d(g \cdot a_p).$$

Similarly, mimicking what we know for $A = C_c^\infty(M)$, we can contract a form in $\Omega^p A$ with an element $X \in \mathfrak{g}$ by the formula

$$\iota_X(a_0 da_1 \wedge \cdots \wedge da_p) = \sum_{i=1}^{p} (-1)^{i+1} \left.\frac{d}{dt}\right|_{t=0} a_0 (e^{tX} \cdot a_i) da_1 \wedge \cdots \wedge \widehat{da_i} \wedge \cdots \wedge da_p.$$

Next, we want to understand how this equivariant HKR-map behaves with respect to the G- and $\mathfrak{g}$-actions on $\Omega^\bullet A$, and the (G-twisted) Hochschild differential on $A^+ \otimes A^{\otimes \bullet}$. For $g \in G$, we write $b_g \colon A^+ \otimes A^{\otimes k} \to A^+ \otimes A^{\otimes (k-1)}$ for the map

$$\begin{aligned} b_g(a_0 \otimes \cdots \otimes a_k) = &\sum_{i=0}^{k-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_k \\ &+ (-1)^k (g a_k) a_0 \otimes \cdots \otimes a_{k-1}. \end{aligned}$$

Note that with this notation, the differential b^h in $\mathrm{L}(G, A)$ simply becomes

$$b^h(F)(g_0, ..., g_q) = b_{(g_0 \cdots g_q)^{-1}}(F(g_0, ..., g_q)).$$

We then have the following Lemma.

Lemma 3.3.2 [BG94, 2.3] The map HKR_X satisfies the following three equations

$$\begin{aligned} \mathrm{HKR}_{\mathrm{Ad}_g(X)}(g \cdot (a_0 \otimes \cdots \otimes a_p)) &= g \cdot \mathrm{HKR}_X(a_0 \otimes \cdots \otimes a_p), \\ \iota_X(\mathrm{HKR}_X(a_0 \otimes \cdots \otimes a_p)) &= \mathrm{HKR}_X\left(b_{e^X}(a_0 \otimes \cdots \otimes a_p)\right), \\ \int_{\Delta^1} e^{-tX} \cdot d(\mathrm{HKR}_X(a_0 \otimes \cdots \otimes a_p)) dt &= \mathrm{HKR}_X \Bigg(\sum_{i=0}^{p} (-1)^{ip} 1 \otimes e^{-X} a_{p-i+1} \otimes \cdots \\ &\qquad \cdots \otimes e^{-X} a_p \otimes \overline{a_0} \otimes \cdots \otimes a_{p-i} \Bigg). \end{aligned}$$

With these properties in mind we can write down a new complex, comparable with $\mathrm{L}(G, A)$, but then with differential forms.

Definition 3.3.3 We define the mixed complex $\mathrm{C}_{\bullet,\bullet}(G, \Omega_{\mathfrak{g}} A)$ by setting $\mathrm{C}_{p,q}(G, \Omega_{\mathfrak{g}} A) \subset C^{-\infty}(\mathfrak{g} \times G^{\times q}, \Omega^p A)$ to be those distributions which are compactly supported in $G^{\times q}$ and whose singular behaviour is restricted to $\delta(e^X(g_1\cdots g_q)^{-1})$. On this we introduce four differentials

$$\widetilde{b^h}\colon \mathrm{C}_{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}A)\to \mathrm{C}_{\bullet-1,\bullet}(G,\Omega_{\mathfrak{g}}A),\qquad \widetilde{B^h}\colon \mathrm{C}_{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}A)\to \mathrm{C}_{\bullet+1,\bullet}(G,\Omega_{\mathfrak{g}}A),$$
$$\widetilde{b^v}\colon \mathrm{C}_{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}A)\to \mathrm{C}_{\bullet,\bullet-1}(G,\Omega_{\mathfrak{g}}A),\qquad \widetilde{B^v}\colon \mathrm{C}_{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}A)\to \mathrm{C}_{\bullet,\bullet+1}(G,\Omega_{\mathfrak{g}}A),$$

given by

$$\begin{aligned}
\widetilde{b^h}(F)(X,g_1,...,g_q) &= \iota_X(F(X,g_1,...,g_q))\\
\widetilde{b^v}(F)(X,g_1,...,g_{q-1}) &= \int_G (-1)^p F(X,\gamma^{-1},g_1,...,g_q)d\gamma\\
&\quad+\sum_{i=1}^{q-1}(-1)^{i+p}\int_G F(X,g_1,...,\gamma,\gamma^{-1}g_j,...,g_q)d\gamma\\
&\quad+(-1)^{p+q}\int_G \gamma\cdot F(\mathrm{Ad}_{\gamma^{-1}}(X),g_1,...,g_{q-1},\gamma)d\gamma\\
\widetilde{B^h}(F)(X,g_1,...,g_q) &= \int_{\Delta^1} e^{-tX}\cdot d(F(X,g_1,...,g_q))dt\\
\widetilde{B^v}(F)(X,g_1,...,g_{q+1}) &= \sum_{i=0}^{q}(-1)^{iq+p}\delta_e(e^X(g_1\cdots g_{q+1})^{-1})(g_{i+1}\cdots g_{q+1})^{-1}\cdot\\
&\qquad\cdot\overline{F}(\mathrm{Ad}_{g_{i+1}\cdots g_{q+1}}(X),g_{i+2},...,g_{q+1},g_1,...,g_i)
\end{aligned}$$

We define a map $\Psi_2\colon \mathrm{L}_{p,q}(G,A)\to \mathrm{C}_{p,q}(G,\Omega_{\mathfrak{g}}A)$ by

$$\Psi_2(F)(X,g_1,...,g_q)=\mathrm{HKR}_X(F(e^X(g_1\cdots g_q)^{-1},g_1,...,g_q)).$$

This map is inspired by the map α from Block-Getzler [BG94], for the case when G is compact.

Using the properties of HKR_X, we can investigate $\Psi_2\circ b^h$, $\Psi_2\circ b^v$, $\Psi_2\circ B^h$ and $\Psi_2\circ B^v$ to obtain differentials $\widetilde{b^h}$, $\widetilde{b^v}$, $\widetilde{B^h}$ and $\widetilde{B^v}$ on $\mathrm{C}_{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}A)$ and obtain the following result:

Proposition 3.3.4 With the four differentials, $(\mathrm{C}_{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}A),\widetilde{b^h}+\widetilde{b^v},\widetilde{B^h}+\widetilde{B^v})$ is a mixed double complex and Ψ_2 is a map of mixed double complexes.

In particular we have

Corollary 3.3.5 The following composition is a chain map of cyclic complexes:

$$\mathrm{CP}((G\ltimes A)^+)\xrightarrow{\Psi_1}\mathrm{CP}(\mathrm{diag}_{\mathrm{norm}}\mathrm{L}(G,A))\xrightarrow{\mathrm{EZ}^{\mathrm{pert}}}\mathrm{CPTot}(\mathrm{L}(G,A))\xrightarrow{\Psi_2}\mathrm{CPTot}(\mathrm{C}_{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}A))$$

We will denote the composition of these maps by

$$\Psi\colon \mathrm{CP}^\bullet((G\ltimes A)^+)\to \mathrm{CPTot}(\mathrm{C}_{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}A)).$$

3.3.2 Getzler's model for equivariant cohomology

We now recall the model for equivariant cohomology obtained by [Ge94], inspired by a Cartan-like construction, and in turn our inspritation for our model for cyclic homology of Definition 3.3.3.

The cochain complex of Getzler is given by

$$\mathrm{C}^{p,q}(G,\Omega_{\mathfrak{g}}\mathbf{A}) = \left\{ F \in C^\infty(\mathfrak{g}\times G^{\times q}, \Omega^p\mathbf{A}) : \begin{matrix} F \text{ is polynomial in } \mathfrak{g} \\ F(X,g_1,...,g_q)=0 \text{ if } e\in\{g_1,...,g_q\} \end{matrix} \right\}$$

endowed with 4 differentials

$$\begin{aligned}
\iota\colon\ & \mathrm{C}^{p,q}(G,\Omega_{\mathfrak{g}}\mathbf{A}) \to \mathrm{C}^{p-1,q}(G,\Omega_{\mathfrak{g}}\mathbf{A}),\\
\bar{\iota}\colon\ & \mathrm{C}^{p,q}(G,\Omega_{\mathfrak{g}}\mathbf{A}) \to \mathrm{C}^{p,q-1}(G,\Omega_{\mathfrak{g}}\mathbf{A}),\\
d\colon\ & \mathrm{C}^{p,q}(G,\Omega_{\mathfrak{g}}\mathbf{A}) \to \mathrm{C}^{p+1,q}(G,\Omega_{\mathfrak{g}}\mathbf{A}),\\
\overline{d}\colon\ & \mathrm{C}^{p,q}(G,\Omega_{\mathfrak{g}}\mathbf{A}) \to \mathrm{C}^{p,q+1}(G,\Omega_{\mathfrak{g}}\mathbf{A}),
\end{aligned}$$

given by

$$\begin{aligned}
(\iota F)(X,g_1,...,g_q) &= (-1)^q \iota_X(F(X,g_1,...,g_k)),\\
(\bar{\iota}F)(X,g_1,...,g_{q-1}) &= \sum_{i=0}^{q-1}(-1)^i \left.\frac{d}{dt}\right|_{t=0} F(X,g_1,...,g_i,e^{t\mathrm{Ad}_{g_{i+1}\cdots g_{q-1}}(X)},g_{i+1},...,g_{q-1}),\\
(dF)(X,g_1,...,g_q) &= (-1)^q d(F(X,g_1,...,g_q)),\\
(\overline{d}F)(X,g_1,...,g_{q+1}) &= F(X,g_2,...,g_{q+1})\\
&\quad + \sum_{i=1}^{q}(-1)^i F(X,g_1,...,g_ig_{i+1},...,g_{q+1})\\
&\quad + (-1)^{q+1} g_{q+1}^{-1}\cdot F(\mathrm{Ad}_{g_{q+1}}(X),g_1,...,g_q).
\end{aligned}$$

As shown in [Ge94, 1.2.3], if M is a manifold with a right G-action, this complex calculates the equivariant cohomology $\mathrm{H}^\bullet_G(M)$ of M if we plug in $\mathbf{A}=C^\infty(M)$ into it.

In Getzler's work the grading of this complex is the sum of whose degree as a group cochain (q), its degree as an element of $\Omega\mathbf{A}$ (p) and twice the polynomial degree (not denoted above). With this grading $\iota+\bar{\iota}+d+\overline{d}$ is a differential of degree 1. For our deliberations it will be more natural to disregard the polynomial degree and see $\mathrm{C}^{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}\mathbf{A})$ as a mixed double cochain complex with differentials $(d+\overline{d},\iota+\bar{\iota})$.

Within $\mathrm{C}^{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}\mathbf{A})$ we emphasize two specific kinds of cochains:

Definition 3.3.6 An element $\alpha_{p,q}\in\mathrm{C}^{p,q}(G,\Omega_{\mathfrak{g}}\mathbf{A})$ is called *cyclically normalized* if $\alpha_{p,q}(0,g_1,...,g_q)=0$ whenever $g_1\cdots g_q=e$ and *cyclic* if it satisfies

$$\alpha_{p,q}(X,g_1,...,g_q) = (-1)^q g_q^{-1}\cdot\alpha_{p,q}(\mathrm{Ad}_{g_q}(X),(g_1\cdots g_q)^{-1},g_1,...,g_{q-1})$$

for any $X\in\mathfrak{g}$ and $g_1,...,g_q\in G$.

Lemma 3.3.7 Any cyclic cochain is cyclically normalized.

Proof. Let α be a cyclic cochain, and let $g_1, ..., g_q$ be such that $g_1 \cdots g_q = e$, then

$$\alpha(0, g_1, ..., g_q) = (-1)^q g_q^{-1} \cdot \alpha(0, e, g_1, ..., g_{q-1}) = 0 \tag{3.5}$$

by the definition of $\mathrm{C}^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}\mathbf{A})$. □

Lemma 3.3.8 The space of cyclic cochains in $\mathrm{C}^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}\mathbf{A})$ is preserved by all four differentials ι, $\bar{\iota}$, d and $\overline{d}$.

In particular we have that the cyclic cochains form a subcomplex $\mathrm{C}_\lambda^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}\mathbf{A})$ of $(\mathrm{C}^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}\mathbf{A}), d + \overline{d}, \iota + \bar{\iota})$.

Proposition 3.3.9 The inclusion of the subcomplex $\mathrm{C}_\lambda^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}\mathbf{A}) \subset \mathrm{C}^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}\mathbf{A})$ is a quasi-isomorphic.

Proof. We describe the reasoning for $\mathbf{A} = C^\infty(M)$, following the work by Getzler [Ge94] where this complex is defined for this case. The complex $\mathrm{C}^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}\mathbf{A})$ arises by constructing the reduced cobar resolution $\Omega^\bullet_{\mathrm{red}}(\mathbb{C}, \Omega^\bullet(G), \Omega\mathbf{A})$ and then applying a quasi-isomorphism $\mathcal{J} \colon \Omega^\bullet_{\mathrm{red}}(\mathbb{C}, \Omega^\bullet(G), \Omega\mathbf{A}) \to C^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}\mathbf{A})$.

The reduced cobar resolution is defined by $\Omega^\bullet_{\mathrm{red}}(\mathbb{C}, \Omega^\bullet(G), \Omega\mathbf{A}) = \overline{\Omega^\bullet(G)} \otimes \Omega\mathbf{A}$ where $\overline{\Omega^\bullet(G)}$ denotes the kernel of the counit $\Omega^\bullet(G) \to \mathbb{C}$ which is evaluation at the identity. The quasi-isomorphism $\mathcal{J}$ is then given by the formula

$$\mathcal{J}(\omega_1 \otimes \ldots \otimes \omega_k \otimes \gamma)(g_1, \ldots, g_\ell, X) = (-1)^l \sum_{\sigma \in \mathrm{Sh}_{\ell,k-\ell}} \left(\prod_{i=1}^{\ell} \omega_{\sigma(i)}(g_i) \right) \left(\prod_{j=\ell+1}^{k} \omega_{\sigma(j)}(e)(X_j) \right) \gamma,$$

where $X_j = \mathrm{Ad}_{g_i \cdots g_l} X$ with $i \leq l$ minimal such that $\sigma(j) < \sigma(i)$ (and $X_j = X$ if such an i does not exist). In this formula the terms $\omega(g)$ only contribute when ω is a zero-form on G, and the terms $\omega(e)(X)$ only contribute when ω is a one-form on G.

For $\mathbf{A} = C^\infty(M)$ we can identify elements $\Omega^\bullet_{\mathrm{red}}(\mathbb{C}, \Omega^\bullet(G), \Omega\mathbf{A})$ with differential forms on $G^{\times\bullet} \times M$, and the differential on the cobar resolution is then given by the sum of the deRham-differential and the differential coming from the underlying simplicial structure on $G^{\times\bullet} \times M$ given as usual by the face operator $\partial_i \colon G^{\times k} \times M \to G^{\times(k-1)} \times M$ defined as

$$\partial_i(g_1, \ldots, g_k, x) = \begin{cases} (g_1, \ldots, g_i g_{i+1}, \ldots, g_k, x), & 0 \leq i \leq k-1, \\ (g_1, \ldots, g_{k-1}, g_k x), & i = k. \end{cases}$$

This simplicial space also has a cyclic structure given by

$$t(g_1, \ldots, g_k, x) = ((g_1 \cdots g_k)^{-1}, g_1, \ldots, g_{k-1}, g_k x)$$

and so by general machinery of cyclic vector spaces we discuss in the appendix, the subspace

$$\Omega^\bullet_{\mathrm{red}}(\mathbb{C}, \Omega G, \Omega M)_\lambda := \bigoplus_{q \in \mathbb{N}} \{\alpha \in \Omega_{\mathrm{red}}(G^{\times q} \times M),\ t^*\alpha = (-1)^q \alpha\}$$

is a quasi-isomorphic subcomplex, and $\mathrm{C}_\lambda^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}\mathbf{A})$ can be checked to be precisely the image of $\Omega^\bullet_{\mathrm{red}}(\mathbb{C}, \Omega G, \Omega M)_\lambda$ under $\mathcal{J}$. □

3.3.3 Pairing equivariant differential forms

We are now in the position to define a pairing between the cyclic homology of $G \ltimes A$ and the equivariant cohomology of $\mathbf{A}$. To this end, assume we have an $n \in \mathbb{N}$ and a functional $⨍: \Omega^n A \to \mathbb{C}$ such that

- $\Omega^{>n} A = 0$,
- $⨍ g \cdot \omega = ⨍ \omega$,
- $⨍ d\omega = 0$.

For the case $A = C_c^\infty(M)$, we can get this situation for $n = \dim(M)$ and $⨍ = \int_M$.

Using this we write down a map $\langle -, - \rangle : \mathrm{C}^{n-p,q}(G, \Omega_{\mathfrak{g}} A) \times \mathrm{C}_{p,q}(G, \Omega_{\mathfrak{g}} A)$ by

$$\langle \alpha, \beta \rangle = (-1)^{p(n+q)+\frac{1}{2}p(p+1)} \int_{G^{\times q}} ⨍ \alpha(0, g_1, ..., g_q) \wedge \beta(0, g_1, ..., g_q) dg_1 \cdots dg_q.$$

This pairing is cohomological in the following sense:

Lemma 3.3.10 The following identities hold for all $\alpha_{i,j} \in \mathrm{C}^{i,j}(G, \Omega_{\mathfrak{g}} \mathbf{A})$ and $\beta_{i,j} \in \mathrm{C}_{i,j}(G, \Omega_{\mathfrak{g}} A)$:

$$\begin{aligned}
\langle \iota \alpha_{n-p+1,q}, \beta_{p,q} \rangle &= \langle \alpha_{n-p+1,q}, \widetilde{b^h} \beta_{p,q} \rangle = 0, \\
\langle d \alpha_{n-p-1,q}, \beta_{p,q} \rangle &= \langle \alpha_{n-p-1,q}, \widetilde{B^h} \beta_{p,q} \rangle, \\
\langle \overline{d} \alpha_{n-p,q-1}, \beta_{p,q} \rangle &= \langle \alpha_{n-p,q-1}, \widetilde{b^v} \beta_{p,q} \rangle,
\end{aligned}$$

If furthermore $\alpha_{n-p,q+1}$ is cyclically normalized, the following also holds for all $\beta_{p,q}$:

$$\langle \bar{\iota} \alpha_{n-p,q+1}, \beta_{p,q} \rangle = \langle \alpha_{n-p,q+1}, \widetilde{B^v} \beta_{p,q} \rangle = 0.$$

Proof. For the first equation we note that $(\iota \alpha_{n-p+1,q})(0, g_1, ..., g_q) = 0$ since $\iota_0 \omega = 0$ for any $\omega \in \Omega \mathbf{A}$. Similarly, $(\widetilde{b^h} \beta_{p,q})(0, g_1, ..., g_q) = 0$.

For the second line note that for $\omega_1 \in \Omega^{n-p-1} \mathbf{A}$ and $\omega_2 \in \Omega^p A$ we have

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{n-p-1} \omega_1 \wedge d\omega_2.$$

Since $⨍$ vanishes on exact forms, we obtain

$$⨍ d\omega_1 \wedge \omega_2 = (-1)^{n-p} ⨍ \omega_1 \wedge d\omega_2.$$

Furthermore we have

$$\begin{aligned}
(\widetilde{B^h} \beta_{p,q})(0, g_1, ..., g_q) &= \int_{\Delta^1} e^{-t \cdot 0} \cdot d(\beta_{p,q}(0, g_1, ..., g_q)) dt \\
&= \int_{\Delta^1} d(\beta_{p,q}(0, g_1, ..., g_q)) dt \\
&= d(\beta_{p,q}(0, g_1, ..., g_q)).
\end{aligned}$$

This results in

$$\begin{aligned}\langle d\alpha_{n-p-1,q}, \beta_{p,q}\rangle &= (-1)^{p(n+q)+\frac{1}{2}p(p+1)+q} \int_{G^q} \fint d(\alpha_{n-p-1,q}(0,g_1,...,g_q))\wedge \\ &\qquad \wedge \beta_{p,q}(0,g_1,...,g_q)dg_1\cdots dg_q \\ &= (-1)^{(p-1)(n+q)+\frac{1}{2}p(p-1)} \int_{G^q} \fint \alpha_{n-p-1,q}(0,g_1,...,g_q)\wedge \\ &\qquad \wedge d(\beta_{p,q}(0,g_1,...,g_q))dg_1\cdots dg_q \\ &= (-1)^{(p-1)(n+q)+\frac{1}{2}p(p-1)} \int_{G^q} \fint \alpha_{n-p-1,q}(0,g_1,...,g_q)\wedge \\ &\qquad \wedge (\widetilde{B^h}\beta_{p,q})(0,g_1,...,g_q)dg_1\cdots dg_q \\ &= \langle \alpha_{n-p-1,q}, \widetilde{B^h}\beta_{p,q}\rangle.\end{aligned}$$

Then for the third part of the Lemma we do an explicit calculation:

$$\begin{aligned}\langle \overline{d}\alpha_{n-p,q-1}, \beta_{p,q}\rangle &= (-1)^{p(n+q)+\frac{1}{2}p(p+1)} \int_{G^{\times q}} \fint \alpha_{n-p,q-1}(0,g_2,...,g_q)\wedge \\ &\qquad \wedge \beta_{p,q}(0,g_1,...,g_q)dg_1\cdots dg_q \\ &+ (-1)^{p(n+q)+\frac{1}{2}p(p+1)} \sum_{i=1}^{q-1}(-1)^i \int_{G^{\times q}} \fint \alpha_{n-p,q-1}(0,g_1,...,g_ig_{i+1},...,g_q)\wedge \\ &\qquad \wedge \beta_{p,q}(0,g_1,...,g_q)dg_1\cdots dg_q \\ &+ (-1)^{p(n+q)+\frac{1}{2}p(p+1)}(-1)^q \int_{G^{\times q}} \fint (g_q^{-1}\cdot \alpha_{n-p,q-1}(0,g_1,...,g_{q-1}))\wedge \\ &\qquad \wedge \beta_{p,q}(0,g_1,...,g_q)dg_1\cdots dg_q.\end{aligned}$$

Now using a few changes of variables for the integrals of $G^{\times q}$ (and using the fact that G is unimodular for the first line) and using that $\fint$ is G-invariant, we obtain:

$$\begin{aligned}\langle d\overline{\alpha}_{n-p,q-1}, \beta_{p,q}\rangle =&(-1)^{p(n+q-1)+\frac{1}{2}p(p+1)}(-1)^p \int_{G^{\times(q-1)}} \int_G \fint \alpha_{n-p,q-1}(0,g_1,...,g_{q-1})\wedge \\ &\qquad \wedge \beta_{p,q}(0,\gamma^{-1},g_1,...,g_{q-1})d\gamma dg_1\cdots dg_q \\ &+ (-1)^{p(n+q-1)+\frac{1}{2}p(p+1)} \sum_{i=1}^{q-1}(-1)^{p+i} \int_{G^{\times(q-1)}} \int_G \fint \alpha_{n-p,q-1}(0,g_1,...,g_{q-1})\wedge \\ &\qquad \wedge \beta_{p,q}(0,g_1,...,\gamma,\gamma^{-1}g_i,...,g_{q-1})dg_1\cdots dg_{q-1} \\ &+ (-1)^{p(n+q-1)+\frac{1}{2}p(p+1)}(-1)^{p+q} \int_{G^{\times(q-1)}} \int_G \fint \alpha_{n-p,q-1}(0,g_1,...,g_{q-1})\wedge \\ &\qquad \wedge (\gamma\cdot \beta_{p,q}(0,g_1,...,g_{q-1},\gamma))d\gamma dg_1\cdots dg_{q-1} \\ =&(-1)^{p(n+q-1)+\frac{1}{2}p(p+1)} \int_{G^{\times(q-1)}} \fint \alpha_{n-p,q-1}(0,g_1,...,g_{q-1})\wedge \\ &\qquad \wedge (\widetilde{b^v}\beta_{p,q})(0,g_1,...,g_{q-1})dg_1\cdots dg_{q-1} \\ =&\langle \alpha_{n-p,q-1}, \widetilde{b^v}\beta_{p,q}\rangle.\end{aligned}$$

For the last part we note that the $\delta(g_1 \cdots g_{q+1})$ in the definition of $\widetilde{B^v}$ cancels against the cyclic normalization of $\alpha_{n-p,q+1}$, as we only need to integrate over $\{(g_1, ..., g_{q+1}) : g_1 \cdots g_{q+1} = e\}$ and we know that $\alpha_{n-p,q+1}$ vanishes there. In particular, we have

$$\langle \alpha_{n-p,q+1}, \widetilde{B^v}\beta_{p,q} \rangle = 0.$$

On the other hand, it is easy to see that $\bar{\iota}\alpha_{n-p,q+1}(0, g_1, ..., g_q) = 0$ for every $(g_1, ..., g_q) \in G^{\times q}$. □

With this pairing, we can write down a map from equivariant cohomology to periodic cyclic cohomology.

Definition 3.3.11 We define cochain complexes $\mathrm{CP}^\bullet(G, \Omega_\mathfrak{g}\mathbf{A}), \mathrm{CP}^\bullet_\lambda(G, \Omega_\mathfrak{g}\mathbf{A})$ by

$$\mathrm{CP}^k(G, \Omega_\mathfrak{g}\mathbf{A}) = \bigoplus_{p+q \equiv n+k \bmod 2} \mathrm{C}^{p,q}(G, \Omega_\mathfrak{g}\mathbf{A}),$$
$$\mathrm{CP}^k_\lambda(G, \Omega_\mathfrak{g}\mathbf{A}) = \bigoplus_{p+q \equiv n+k \bmod 2} \mathrm{C}^{p,q}_\lambda(G, \Omega_\mathfrak{g}\mathbf{A}),$$

with the differential given by $\iota + d + \bar{\iota} + \bar{d}$.

Lemma 3.3.12 The complex $\mathrm{CP}^\bullet(G, \Omega_\mathfrak{g}\mathbf{A})$ calculates the following cohomology

$$\mathrm{H}^k(\mathrm{CP}^\bullet(G, \Omega_\mathfrak{g}\mathbf{A})) = \bigoplus_{p \equiv n+k \bmod 2} \mathrm{H}^p(\mathrm{Tot}(\mathrm{C}^{\bullet,\bullet}(G, \Omega_\mathfrak{g}\mathbf{A}))).$$

The previous discussion can then be summarized by

Proposition 3.3.13 The map $\Phi\colon \mathrm{CP}^\bullet(G, \Omega_\mathfrak{g}\mathbf{A}) \to (\mathrm{CP}(G, \Omega_\mathfrak{g}A)^*)^\bullet$ given by

$$\Phi(\sum_{p',q'} \alpha_{p',q'})(\beta_{p,q}) = \langle \alpha_{n-p,q}, \beta_{p,q} \rangle$$

becomes a cochain map when restricted to $\mathrm{CP}^\bullet_\lambda(G, \Omega_\mathfrak{g}\mathbf{A})$.

Combining the map Φ with the map Ψ of Section 3.2 we arrive at the second main result of this chapter, which is a corollary of the previous proposition and the fact that Ψ is a chain map.

Corollary 3.3.14 The map $c\colon \mathrm{CP}^\bullet(G, \Omega_\mathfrak{g}\mathbf{A}) \to \mathrm{CP}^\bullet((G \ltimes A)^+)$ given by

$$c\left(\sum_{p,q} \alpha_{p,q}\right)(a_0, ..., a_k) = \Phi\left(\sum_{p,q} \alpha_{p,q}\right)(\Psi(a_0 \otimes \cdots \otimes a_k))$$

becomes a cochain map when restricted to $\mathrm{CP}^\bullet_\lambda(G, \Omega_\mathfrak{g}\mathbf{A})$, and in turn induces a map

$$\bigoplus_{p \equiv n+k \bmod 2} \mathrm{H}^p(\mathrm{Tot}(C^{\bullet,\bullet}(G, \Omega_\mathfrak{g}\mathbf{A}))) \to \mathrm{HP}^k(G \ltimes A) \tag{3.6}$$

Remark 3.3.15 Applying this to $(A, \mathbf{A}) = (C_c^\infty(M), C^\infty(M))$ for M an oriented manifold with an oriented right G-action, this construction yields maps

$$\mathrm{H}_G^{\mathrm{ev}}(M) \to \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M)), \qquad \mathrm{H}_G^{\mathrm{odd}}(M) \to \mathrm{HP}^{\dim(M)+1}(G \ltimes C_c^\infty(M)),$$

since the complex $\mathrm{CP}^\bullet(G, \Omega_{\mathfrak{g}} C^\infty(M))$ calculates the even- and odd-degree terms of the equivariant cohomology $\mathrm{H}_G^\bullet(M)$.

3.3.4 The equivariant Chern character in equivariant cohomology

The main point of this chapter is understanding the equivariant Chern character through the convolution algebra. In particular, we want to show that our Chern character from Theorem 3.1.24 agrees with the known equivariant Chern character living in $\mathrm{H}_G^{\mathrm{ev}}(M)$, under the map $\mathrm{H}_G^{\mathrm{ev}}(M) \to \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M))$. So we describe the work of Getzler [Ge94] explaining how this Chern character resides in his model for equivariant cohomology.

So let $E \to M$ be an G-equivariant vector bundle, together with a connection ∇ on E. From these data, we can write down a connection ∇_t on the vector bundle $\mathrm{pr}_M^* E \to \Delta^q \times G^q \times M$ by the formula

$$\nabla_t = t_1(\nabla_1 - \nabla_2) + \cdots t_q(\nabla_q - \nabla) + \nabla,$$

where $\nabla_i := (g_i \cdots g_q)^* \nabla$ and $(t_1, ..., t_q) \in \Delta^q$.

From this we obtain an differential form on $G^q \times M$ by taking the fibred integral over Δ^q of the Chern character of ∇_t:

$$\int_{\Delta^q} \mathrm{tr} \exp(F(\nabla_t)) dt_1 \cdots dt_q \in \Omega^\bullet(G^{\times q} \times M).$$

Note that this differential form lives in even degree if q is even, and in odd degree if q is odd. Indeed, it is always of even degree over $\Delta^q \times G^q \times M$ and the fibred integral reduces the degree by q . From this, Getzler [Ge94, Thm 3.2.1] defines the equivariant Chern character $\mathrm{Ch}_G(E, \nabla)$ in $\mathrm{C}^{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}(M))$ by

$$\mathrm{Ch}_G(E, \nabla) = \sum_{q \geq 0} (-1)^q \mathcal{J} \int_{\Delta^q} \mathrm{tr} \exp(F(\nabla_t)),$$

where $\mathcal{J}$ is the quasi-isomorphism we saw in the proof of Proposition 3.3.9. Getzler shows [Ge94, Thm 3.2.1] that this class represents the equivariant Chern character of E in equivariant cohomology as defined at the start of this chapter. Notice that our remark about the specific degrees where the Chern character lives in $\Omega^\bullet(G^{\times \bullet} \times M)$ implies the resulting chain is an element of $\mathrm{CP}^{\mathrm{ev}}(G, \Omega_{\mathfrak{g}}(M))$.

Now, in the end we want to look at $c(\mathrm{Ch}_G(E,\nabla)) \in \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M))$ and hence we are interested in the functions $\mathrm{Ch}_G(E,\nabla)_0^q \colon G^p \to \Omega^\bullet(M)$ given by

$$\mathrm{Ch}_G(E,\nabla)_0^q(g_1,...,g_q) = \mathrm{Ch}_G(E,\nabla)(0,g_1,...,g_q). \tag{3.7}$$

For this, we quickly delve into the classical Chern-Simons forms. Let $\nabla_0,...,\nabla_q$ be connections on E. Then

$$\begin{aligned}\nabla_t &= t_1(\nabla_0 - \nabla_1) + \cdots + t_q(\nabla_{q-1} - \nabla_q) + \nabla_q \\ &= -t_1(\nabla_q - \nabla_0) + \sum_{i=1}^{q-1}(t_i - t_{i+1})(\nabla_q - \nabla_i) + \nabla_q\end{aligned}$$

is a connection on $\mathrm{pr}_M^* E \to \Delta^q \times M$. Fiberwise-integration of the Chern character of this connection over Δ^q produces the Chern-Simons form $\mathrm{cs}(\nabla_0,...,\nabla_q) \in \Omega^{\mathrm{even}-q}(M)$ given by

$$\mathrm{cs}(\nabla_0,...,\nabla_q) = \int_{\Delta^q} \mathrm{tr}\exp(F(\nabla_t)).$$

We can write this out explicitly

$$\begin{aligned}\mathrm{cs}(\nabla_0,...,\nabla_q) &= \int_{\Delta^q} \mathrm{tr}\left(\exp\left(-dt_1 \wedge (\nabla_q - \nabla_0) + \right.\right. \\ &\qquad \left.\left. + \sum_{i=1}^{q-1}(dt_i - dt_{i+1}) \wedge (\nabla_q - \nabla_i) + F(t)\right)\right), \\ &= \int_{\Delta^q} \mathrm{tr}\left(\exp\left(\sum_{i=1}^{q} dt_i \wedge (\nabla_{i-1} - \nabla_i) + F(t)\right)\right),\end{aligned}$$

where $F(t) \in \Omega^2(M, \mathrm{End}(E))$ is the curvature of the connection $\nabla(t)$.

From this discussion we obtain a relation to and an explicit formula for $\mathrm{Ch}_G(E,\nabla)_0^q$:

$$\mathrm{Ch}_G(E,\nabla)_0^q(g_1,...,g_q) = (-1)^{p+q}\mathrm{cs}(\gamma_1^*\nabla,...,\gamma_q^*\nabla,\nabla) \tag{3.8}$$

where $\gamma_i = g_i \cdots g_q$. Investigating the specific form of the Chern-Simons forms then gives the following Lemma:

Lemma 3.3.16 The functions $\mathrm{Ch}_G(E,\nabla)_q^0$, defined by (3.7), satisfy the following:

i) $\mathrm{Ch}_G(E,\nabla)_0^q(g_1,...,g_q) \in \bigoplus_{i=q}^n \Omega^i(M)$,

ii) $\mathrm{Ch}_G(E,\nabla)_0^q(g_1,...,g_q) = 0$ if either one of the g_i's or their product $g_1 \cdots g_q$ is the identity element of the group.

Proof. For i) note that the only terms under the exponent in the Chern-Simons form that persist in $\Omega(M)$ after taking the integral over Δ^q are the terms where all the parts $dt_i \wedge (\nabla_{i-1} - \nabla_i)$ appear exactly once. This means in particular that the term that results from $\int_{\Delta^q}$ is a form on M of degree at least q.

For ii) note that $g_i = 1$ for $i = 1, ..., q-1$ corresponds to $\gamma_i^* \nabla = \gamma_{i+1}^* \nabla$, $g_q = 1$ corresponds to $\gamma_q^* \nabla = \nabla$ and $g_1 \cdots g_q = 1$ corresponds to $\gamma_1^* \nabla = \nabla$. To obtain the statement of the Lemma, we now argue that $\mathrm{cs}(\nabla_0, ..., \nabla_q) = 0$ if either $\nabla_{i-1} = \nabla_i$ for some $i = 1, ..., q$ or $\nabla_0 = \nabla_q$. Indeed, if one of these holds we look at the form on $\Delta^q \times M$ of which we take the exponent and look at the forms on Δ^q which come out of taking the exponential. If we have one the equalities of the ∇_i's we see that only $q-1$ different one forms on Δ^q remain: for the case $\nabla_{i-1} = \nabla_i$ we look at the second way of writing $\mathrm{cs}(\nabla_0, ..., \nabla_q)$ and only the terms $\{dt_j \wedge (\nabla_{j-1} - \nabla_j)\}_{j \neq i}$ remain, while for the case $\nabla_q = \nabla_0$ we use the first way of writing $\mathrm{cs}(\nabla_0, ..., \nabla_q)$ to see that only $\{(dt_i - dt_{i+1}) \wedge (\nabla_q - \nabla_i)\}_{i=1,...,q-1}$ remain. In that case, the only way to obtain a q-form on Δ^q is to take a q-fold wedge product with at least one repeating one-form, and hence all the terms vanish. □

Remark 3.3.17 i) The first part of the Lemma says that the periodic cyclic cochain in $\mathrm{CP}^{\dim(M)}((G \ltimes C_c^\infty(M))^+)$ is actually given by an explicit cyclic cochain in $\mathrm{CC}^{\dim(M)}((G \ltimes C_c^\infty(M))^+)$.

ii) The second part of the Lemma says that $\mathrm{Ch}_G(E, \nabla)$ is a cyclically normalized cochain. Moreover, it is in fact cyclic by looking at the effect of cyclically permuting the $(g_1, ..., g_q)$ in $\Delta^q \times G^q \times M$.

In particular, for every equivariant vector bundle $E \to M$ we now obtain a periodic cohomology class

$$c(\mathrm{Ch}_G(E, \nabla)) \in \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M)),$$

which is independent of the connection ∇.

3.4 Compatibility with the Chern character

From the previous sections we have obtained a diagram

$$\begin{array}{ccc} \mathrm{Vect}_G(M) & \xrightarrow{\mathrm{Ch}_G(-)} & \mathrm{H}_G^{\mathrm{ev}}(M) \\ & \searrow^{\mathrm{Ch}_\Omega} & \downarrow c \\ & & \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M)) \end{array} \tag{3.9}$$

The main result of this section and indeed this chapter is the following.

Theorem 3.4.1 The diagram (3.9) is commutative for any unimodular Lie group G and any oriented manifold M with a right, oriented proper G-action.

We will split this section in two parts, starting with the proof of the Theorem above in the proper case, and then sketching ideas how to reduce to the proper case, which would lead to a proof in the general case.

3.4.1 The proper case

The proof for the proper case is supported by the following Lemma, which makes sure that we can choose a specific connection ∇ on an equivariant vector bundle E to calculate $\Omega_{E,\nabla}$ and $\mathrm{Ch}_G(E,\nabla)$.

Lemma 3.4.2 [Pf01, Thm 4.2.4] If the action of G on M is proper, then any G-equivariant vector bundle $E \to M$ possesses a G-invariant connection ∇.

Proof of Theorem 3.4.1. Let $E \to M$ be a G-equivariant vector bundle, and let ∇ be a G-invariant connection, the existence of which is assured by the previous Lemma. As ∇ is G-invariant, we see by (3.8) that

$$\mathrm{Ch}_G(E,\nabla)^q_0(g_1, ..., g_q) = \pm\mathrm{cs}(\nabla, ..., \nabla)$$

Since we already remarked in the proof of Lemma 3.3.16 that plugging in repeating arguments into a Chern-Simons form makes it vanish, we see that $\mathrm{Ch}_G(E,\nabla)^q_0 = 0$ for $q > 0$, while at $q = 0$ we have

$$\mathrm{Ch}_G(E,\nabla)^{p,q}_0 = \frac{(-1)^{p/2}}{(p/2)!}\mathrm{tr}(F(\nabla)^{\wedge p/2})$$

if p is even and

$$\mathrm{Ch}_G(E,\nabla)^{p,q}_0 = 0$$

if p is odd. So we see that $\mathrm{Ch}_G(E,\nabla)_0$ is concentrated in degrees $(2k, 0)$. In turn we need only to look at the contributions of

$$\Psi\colon \mathrm{CP}_{\dim(M)}((G \ltimes C^\infty(M))^+) \to \mathrm{CP}_{\dim(M)}(\mathrm{Tot}(\mathrm{C}_{\bullet,\bullet}(G, \Omega_{\mathfrak{g}}(M))$$

that land in $\mathrm{C}_{\dim(M)-2k,0}(G, \Omega_{\mathfrak{g}}(M))$.

Recalling the formula for Ψ,

$$\Psi = \sum_{i\geq 0} \Psi_2 \mathrm{EZ}(Bh)^i \Psi_1, \tag{3.10}$$

we first look at the contributions of the term $i = 0$ and argue that the contributions of the terms $i > 0$ vanish. The only contribution of $i = 0$ that lands in the degree $(n-2k, 0)$ is

$$\Psi_2\mathrm{EZ}_{n-2k,0}\Psi_1\colon \mathrm{C}_{n-2k}((G \ltimes C_c^\infty(M))^+) \to \mathrm{C}_{n-2k,0}(G, \Omega_{\mathfrak{g}}(M)).$$

To see what this map does, let us pick $a_0, ..., a_{n-2k} \in G \ltimes C_c^\infty(M)$. First applying Ψ_1 to this, we end up with the following element $C_c^\infty(G^{\times(n-2k+1)}, C_c^\infty(M^{\times(n-2k+1)}))$:

$$\Psi_1(a_0 \otimes \cdots \otimes a_{n-2k})(g_0, ..., g_{n-2k}) = (g_0 \cdots g_{n-2k})^{-1}(a_0(g_0)) \otimes \cdots \otimes g^{-1}_{n-2k}(a_{n-2k}(g_{n-2k})).$$

Then, looking at the Eilenberg-Zilber-term, we have

$$\mathrm{EZ}_{n-2k,0} = (d_0^v)^{n-2k}$$

and, starting with a map $F \in \mathrm{L}_{n-2k,n-2k}(G, C_c^\infty(M))$, we see that

$$(d_0^v)^{n-2k}(F)(g) = \int_{G^{n-2k}} F(\gamma_1, ..., \gamma_{n-2k}, (\gamma_1 \cdots \gamma_{n-2k})^{-1} g) d\gamma_1 \cdots d\gamma_{n-2k}. \qquad (*)$$

Going a few steps ahead, we are only interested in

$$\Psi_2(\mathrm{EZ}_{n-2k,0}(\Psi_1(a_0 \otimes \cdots \otimes a_{n-2k})))(0) = \mathrm{HKR}_0(\mathrm{EZ}_{n-2k,0}(\Psi_1(a_0 \otimes \cdots \otimes a_{n-2k}))(e))$$

so in what follows we only need the formula $(*)$ for $g = e$. In particular, we have

$$\begin{aligned}\mathrm{EZ}_{n-2k,0}(\Psi_1(a_0 \otimes \cdots \otimes a_{n-2k}))(e) = \int_{G^{n-2k}} & a_0(\gamma_1) \otimes \gamma_1(a_1(\gamma_2)) \otimes \cdots \\ & \cdots \otimes (\gamma_1 \cdots \gamma_{n-2k}) a_{n-2k}((\gamma_1 \cdots \gamma_{n-2k})^{-1}) d\gamma_1 \cdots d\gamma_{n-2k},\end{aligned}$$

or, if we rewrite the integral a bit,

$$\begin{aligned}\mathrm{EZ}_{n-2k,0}(\Psi_1(a_0 \otimes \cdots \otimes a_{n-2k}))(e) = \int_{G^{n-2k}} & a_0((\gamma_1 \cdots \gamma_{n-2k})^{-1}) \otimes \\ & \otimes (\gamma_1 \cdots \gamma_{n-2k})^{-1}(a_1(\gamma_1)) \otimes \cdots \otimes \gamma_{n-2k}^{-1} a_{n-2k}(\gamma_{n-2k}) d\gamma_1 \cdots d\gamma_{n-2k}.\end{aligned}$$

So, in the end, we obtain

$$\begin{aligned}\Psi_2(\mathrm{EZ}_{n-2k,0}(\Psi_1(a_0 \otimes \cdots \otimes a_{n-2k})))(0) = \frac{1}{(n-2k)!} \int_{G^{n-2k}} & a_0((\gamma_1 \cdots \gamma_{n-2k})^{-1}) \otimes \\ & \otimes (\gamma_1 \cdots \gamma_{n-2k})^{-1} d(a_1(\gamma_1)) \otimes \cdots \otimes \gamma_{n-2k}^{-1} d(a_{n-2k}(\gamma_{n-2k})) d\gamma_1 \cdots d\gamma_{n-2k}.\end{aligned}$$

Here, the $\frac{1}{(n-2k)!}$ comes from the fact that if we plug in $X = 0$ we get an integral over Δ^{n-2k} that is independent of $t_1, ..., t_{n-2k}$ and hence we pick up a factor $\mathrm{vol}(\Delta^{n-2k}) = \frac{1}{(n-2k)!}$. Pairing with $\mathrm{Ch}_G(E, \nabla)$, this yields the following formula for the $i = 0$-term in (3.10):

$$\begin{aligned}\frac{(-1)^k}{k!(n-2k)!} \int_M \mathrm{tr}(F(\nabla)^{\wedge k}) \int_{G^{n-2k}} & a_0((\gamma_1 \cdots \gamma_{n-2k})^{-1}) \otimes \\ & \otimes (\gamma_1 \cdots \gamma_{n-2k})^{-1} d(a_1(\gamma_1)) \otimes \cdots \otimes \gamma_{n-2k}^{-1} d(a_{n-2k}(\gamma_{n-2k})) d\gamma_1 \cdots d\gamma_{n-2k}.\end{aligned}$$

Now, for the terms where $i > 0$, we plug B-exact elements into this formula, which means that we need to look at what the formula above reduces to when $a_0 = \delta_e \otimes 1$, where 1 is the adjoined unit in $C_c^\infty(M)$, i.e. the constant function 1 on M. Then this formula becomes

$$\begin{aligned}\frac{(-1)^k}{k!(n-2k)!} \int_M \mathrm{tr}(F(\nabla)^{\wedge k} \int_{G^{n-2k-1}} & d(a_1((\gamma_1 \cdots \gamma_{n-2k-1})^{-1})) \otimes \\ & \otimes (\gamma_1 \cdots \gamma_{n-2k-1})^{-1} d(a_2(\gamma_1)) \otimes \cdots \otimes \gamma_{n-2k-1}^{-1} d(a_{n-2k-1}(\gamma_{n-2k})) d\gamma_1 \cdots d\gamma_{n-2k-1}.\end{aligned}$$

Since $\mathrm{tr}(F(\nabla)^{\wedge k})$ is a closed $2k$-form (this is a consequence of the classical Bianchi identity) the above becomes

$$\frac{(-1)^k}{k!(n-2k)!}\int_M d\left(\mathrm{tr}(F(\nabla)^{\wedge k})\int_{G^{n-2k-1}} a_1((\gamma_1\cdots\gamma_{n-2k-1})^{-1})\otimes\right.$$
$$\left.\otimes(\gamma_1\cdots\gamma_{n-2k-1})^{-1}d(a_2(\gamma_1))\otimes\cdots\otimes\gamma_{n-2k-1}^{-1}d(a_{n-2k-1}(\gamma_{n-2k}))d\gamma_1\cdots d\gamma_{n-2k-1}\right),$$

which is an integral over an exact form, and hence 0. We see that the contributions for $i>0$ in (3.10) vanish, and hence we conclude that

$$c(\mathrm{Ch}_G(E,\nabla))(a_0,...,a_{n-2k}) = \frac{(-1)^k}{k!(n-2k)!}\int_M \mathrm{tr}(F(\nabla)^{\wedge k})\int_{G^{n-2k}} a_0((\gamma_1\cdots\gamma_{n-2k})^{-1})\otimes$$
$$\otimes(\gamma_1\cdots\gamma_{n-2k})^{-1}d(a_1(\gamma_1))\otimes\cdots\otimes\gamma_{n-2k}^{-1}d(a_{n-2k}(\gamma_{n-2k}))d\gamma_1\cdots d\gamma_{n-2k}.$$

Next we investigate the externally curved DGA $\Omega_{E,\nabla}$ from Section 3.1.3. Here, using that fact that ∇, and hence $F(\nabla)$, are G-invariant, and the facts that $\mu(0)=0$ and $(\Theta*\alpha)(g,0)=(\alpha*\Theta)(g,0)=0$ we have

$$(\Theta_\nabla*\alpha)(g,0)=F(\nabla)\wedge\alpha(g,0) \qquad \text{and} \qquad (\alpha*\Theta_\nabla)(g,0)=\alpha(g,0)\wedge F(\nabla).$$

Furthermore, since $\delta\equiv 0$ (again since ∇ is G-invariant) we have

$$(D_\nabla\alpha)(g,0)=d_{\nabla^{\mathrm{End}}}(\alpha(g,0)),$$

and if $\alpha\in C_c^\infty(G,\mathrm{Sym}(\mathfrak{g}^*)\otimes\Omega_c(M,\mathrm{End}(E)))$ is of the form

$$\alpha(g,X)=a(g)\mathrm{id}_E$$

for $a\in C_c^\infty(G\times M)$, this simply means

$$(D_\nabla\alpha)(g,0)=d(a(g,0)).$$

From this we see that the repeated convolution in the formula for $\mathrm{Ch}_{\Omega_{E,\nabla}}$ equals

$$(a_0*\Theta_\nabla^{*i_0}*D_\nabla a_1*\cdots*D_\nabla a_{n-2k}*\Theta_\nabla^{*i_{n-2k}})(e,0)=\int_{G^{\times(n-2k)}} a_0(h_1)\wedge$$
$$\wedge F(\nabla)^{\wedge i_0}\wedge h_1^*d(a_1(h_1^{-1}h_2))\wedge\cdots\wedge h_{n-2k}^*d(a_{n-2k}(h_{n-2k}^{-1})\wedge F(\nabla)^{\wedge i_{n-2k}}dh_1\cdots dh_{n-2k}$$

which, after rearranging integrals and noticing that $a_0(h_1)$ and $d(a_i(h_i^{-1}h_{i+1}))$ commute with $F(\nabla)$ -since they are scalar differential forms- equals

$$F(\nabla)^{\wedge(i_0+\cdots+i_{n-2k})}\int_{G^{\times(n-2k)}} a_0((\gamma_1\cdots\gamma_{n-2k})^{-1})\wedge$$
$$\wedge((\gamma_1\cdots\gamma_{n-2k})^{-1})^*d(a_1(\gamma_1))\wedge\cdots\wedge(\gamma_{n-2k}^{-1})^*d(a_{n-2k}(\gamma_{n-2k}))d\gamma_1\cdots d\gamma_{n-2k},$$

so that in the end we have

$$\mathrm{Ch}^{n-2k}_{\Omega_{E,\nabla}}(a_0,...,a_{n-2k}) = \frac{(-1)^k}{(n-k)!}\sum_{i_0+\cdots+i_{n-2k}=k}\int_M \mathrm{tr}\left(F(\nabla)^{\wedge(i_0+\cdots+i_{n-2k})}\right)$$
$$\int_{G^{\times(n-2k)}} a_0((\gamma_1\cdots\gamma_{n-2k})^{-1})\wedge((\gamma_1\cdots\gamma_{n-2k})^{-1})^* d(a_1(\gamma_1))\wedge\cdots$$
$$\cdots\wedge(\gamma_{n-2k}^{-1})^* d(a_{n-2k}(\gamma_{n-2k}))d\gamma_1\cdots d\gamma_{n-2k}.$$

Then, since the summand does not depend on $i_0,...,i_{n-2k}$, we can just replace the sum by the size of the index set, which is $\binom{n-k}{n-2k}$, to obtain

$$\mathrm{Ch}^{n-2k}_{\Omega_{E,\nabla}}(a_0,...,a_{n-2k}) = \frac{(-1)^k}{k!(n-k)!}\sum_{i_0+\cdots+i_{n-2k}=k}\int_M \mathrm{tr}\left(F(\nabla)^{\wedge(i_0+\cdots+i_{n-2k})}\right)$$
$$\int_{G^{\times(n-2k)}} a_0((\gamma_1\cdots\gamma_{n-2k})^{-1})\wedge((\gamma_1\cdots\gamma_{n-2k})^{-1})^* d(a_1(\gamma_1))\wedge\cdots$$
$$\cdots\wedge(\gamma_{n-2k}^{-1})^* d(a_{n-2k}(\gamma_{n-2k}))d\gamma_1\cdots d\gamma_{n-2k}.$$

We conclude that the cyclic chains $\mathrm{Ch}_{\Omega_{E,\nabla}}, c(\mathrm{Ch}_G(E,\nabla)) \in \mathrm{CP}^n(G \ltimes C_c^\infty(M))$ agree on the nose , so certainly their classes in $\mathrm{HP}^n(G \ltimes C_c^\infty(M))$ do. □

3.4.2 Outlook: the non-proper case

The last part of this section is devoted to the equivalent of 3.4.1 in the case that the action of G on M is not proper. We expect this to be true, but at the time of writing we are not sure about all of the details. However, it would be remiss not to include this discussion.

Reduction to maximal compact subgroups

To show Theorem 3.4.1 for also the non-proper case, we need more structure. We start by recalling the following ring from [Ni93].

Definition 3.4.3 We define $C^\infty_{\mathrm{inv}}(G)$ to be the ring of smooth functions on G that are invariant under conjugation, i.e. if f is a smooth function then $f \in C^\infty_{\mathrm{inv}}(G)$ if and only if

$$f(hgh^{-1}) = f(g)$$

Lemma 3.4.4 [Ni93, Lem 4.1]The maximal ideals of $C^\infty_{\mathrm{inv}}(G)$ are in a one-to-one correspondence with the conjugation classes of G, by associating to a point $g \in G$ the ideal

$$\mathfrak{m}_g = \{f \in C^\infty_{\mathrm{inv}}(G) : f(g) = 0\}$$

The cyclic space $\mathrm{C}_\bullet((G \ltimes A)^+)$, the cylindrical space $\mathrm{L}^+(G,A)_{\bullet,\bullet}$ and the double complex $\mathrm{C}_{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}A)$ are modules over $C^\infty_{\mathrm{inv}}(G)$, by the following formulas respectively

$$\begin{aligned}(f\cdot F_1)(g_0,...,g_k) &= f(g_0\cdots g_k)F_1(g_0,...,g_k)\\ (f\cdot F_2)(g_0,...,g_q) &= f(g_0\cdots g_q)F_2(g_0,...,g_q)\\ (f\cdot F_3)(X,g_1,...,g_q) &= f(e^X)F_3(X,g_1,...,g_q)\end{aligned}$$

where in the first line $F_1 \in \mathrm{C}_k((G\ltimes A)^+) = \mathrm{L}^+(G,A)_{k,k}$, in the second line $F_2 \in \mathrm{L}^+(G,A)_{p,q}$ and in the last line $F_3 \in \mathrm{C}_{p,q}(G,\Omega_{\mathfrak{g}}A)$. For $\mathrm{C}_\bullet((G\ltimes A)^+)$ and $\mathrm{L}^+(G,A)_{\bullet,\bullet}$ all the simplicial and cyclic structure maps are $\mathrm{C}^\infty_{\mathrm{inv}}(G)$-module maps, as are the differentials in $\mathrm{C}_{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}A)$, in particular their homologies are also $C^\infty_{\mathrm{inv}}(G)$-modules. Furthermore the map Ψ is then a map of chain complexes of $C^\infty_{\mathrm{inv}}(G)$-modules.

This means that we can look at the localizations of the homologies at the maximal ideals of $C^\infty_{\mathrm{inv}}(G)$. In particular we are interested in the localization at the identity, which we will write as $\mathrm{HP}^\bullet(G\ltimes C^\infty_c(M))_e$.

To do this, we refer to work of Nistor, who used localizations to reduce the cyclic cohomologies of convolution algebras to the maximal compact subgroup.

Theorem 3.4.5 [Ni93, Cor 4.10] Let G be a Lie group with a maximal compact subgroup K, with $q = \dim(G/K)$. Let M be a manifold with a G-action, then there is an isomorphism

$$\mathrm{HP}^k(G\ltimes C^\infty_c(M))_e \cong \mathrm{HP}^{k+q}(K\ltimes C^\infty_c(M))_e$$

In particular, this allows us to do the following. If we look at M as a K-manifold, we may take the product with the space G/K which carries a trivial K-action. In particular we have

$$K\ltimes C^\infty_c(M\times G/K) \cong (K\ltimes C^\infty_c(M))\widehat{\otimes} C^\infty_c(G/K)$$

The action of $C^\infty_{\mathrm{inv}}(G)$ is then fully on the left factor, so that we conclude by a Künneth-like theorem that

$$\mathrm{HP}^\bullet(K\ltimes C^\infty_c(M\times G/K))_e \cong \mathrm{HP}^\bullet(K\ltimes C^\infty_c(M))_e\widehat{\otimes}\mathrm{H}^\bullet_{\mathrm{dR},c}(G/K)$$

Next, since G/K is diffeomorphic to $\mathbb{R}^q$, we have that $\mathrm{H}^\bullet_{\mathrm{dR},c}(G/K)$ is concentrated in degree q where it is spanned by the orientation form τ.

In particular we have an isomorphism with we will call $\cup\tau$

$$\mathrm{HP}^\bullet(G\ltimes C^\infty_c(M))_e \xrightarrow{\cup\tau} \mathrm{HP}^{\bullet+q}(G\ltimes C^\infty_c(M\times G/K))_e$$

which is defined as the composition of the following maps

$$\begin{array}{ccc}
\mathrm{HP}^\bullet(G\ltimes C^\infty_c(M))_e & \xrightarrow{\ \cong\ } & \mathrm{HP}^{\bullet+q}(K\ltimes C^\infty_c(M))_e \\
 & & \downarrow{\scriptstyle \otimes\tau} \\
\Big\downarrow{\scriptstyle \cup\tau} & & \mathrm{HP}^{\bullet+q}(K\ltimes C^\infty_c(M))_e\otimes \mathrm{H}^q_{\mathrm{dR},c}(G/K) \\
 & & \downarrow{\scriptstyle \cong} \\
\mathrm{HP}^{\bullet+q}(G\ltimes C^\infty_c(M\times G/K))_e & \xleftarrow[\ \cong\ ]{} & \mathrm{HP}^{\bullet+2q}(K\ltimes C^\infty_c(M\times G/K))_e
\end{array}$$

Reduction to the proper case

We now sketch a way to prove Theorem 3.4.1 in the general case. For this, given an equivariant vector bundle $E \to M$ with connection ∇, we will localize both $\mathrm{Ch}_{\Omega_{E,\nabla}}$ and $c(\mathrm{Ch}_G(E,\nabla)) \in \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M))$ at the different maximal ideals of $C^\infty_{\mathrm{inv}}(G)$.

We start with the following Lemma regarding localizations at maximal ideals that are not determined by the identity:

Lemma 3.4.6 Let $g \in G\backslash\{e\}$ and let $\mathfrak{m} = \{f \in C^\infty_{\mathrm{inv}}(G) : f(g) = 0\}$. Then

$$(\mathrm{Ch}_{\Omega_{E,\nabla}})_{\mathfrak{m}} = c(\mathrm{Ch}_G(E,\nabla))_{\mathfrak{m}} = 0 \in \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M))_{\mathfrak{m}}$$

Proof. For $\mathrm{Ch}_{\Omega_{E,\nabla}}$ this is because it is defined using the functional $⨍\colon C_c^\infty(G, \mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega_c(M, \mathrm{End}(E)))$ given by

$$⨍ \alpha = \int_M \mathrm{tr}(\alpha(e,0))$$

which is distinctly seen to be localized at the identity, in particular taking chains which are ever closely supported around g, we see that they will vanish under $⨍$.

Similarly for $c(\mathrm{Ch}_G(E,\nabla))$, we see that it involves pairing $\mathrm{Ch}_G(E,\nabla)$ against evaluations of chains at $X = 0 \in \mathfrak{g}$, also an operation localized at the identity e. □

Remark 3.4.7 This Lemma states that our Chern classes are really localized at the identity. Comparing this to the 'bouquet of Chern characters' of Duflo and Vergne [DV93] we again arrive at the conclusion that for non-discrete groups there should a more involved way of defining these Chern characters taking into account the behaviour in the $\mathfrak{g}$-direction compatible with the ideas of the bouquet of Chern characters.

After this, we can try to understand the localizations at the identity, for which we have the following conjecture.

Conjecture 3.4.8 Writing e for the maximal ideal $\{f \in C^\infty_{\mathrm{inv}}(G) : f(e) = 0\}$ we have that

- $(\mathrm{Ch}_{\Omega_{\mathrm{pr}^*E,\mathrm{pr}^*\nabla}})_e = (\mathrm{Ch}_{\Omega_{E,\nabla}})_e \cup \tau$
- $(c(\mathrm{Ch}_G(\mathrm{pr}^*E, \mathrm{pr}^*\nabla)))_e = (c(\mathrm{Ch}_G(E,\nabla))_e \cup \tau$

Ideas for a proof. The ideas to prove this follow from the observation that $(\mathrm{pr}^*E, \mathrm{pr}^*\nabla) = (E \boxtimes \underline{\mathbb{C}}, \nabla \boxtimes d)$, and the remark that when we look at everything from the K-equivariant perspective, $\underline{\mathbb{C}} \to G/K$ is the trivial line bundle with no action present. In particular, one checks that the equivariant Chern character is just $1 \in \mathrm{H}_K^{\mathrm{ev}}(G/K)$, which, ones plugged into $c\colon \mathrm{H}_K^{\mathrm{ev}}(G/K) \to \mathrm{HP}^q(C_c^\infty(G/K))$ simply yields the orientation form

$$c(1)(f_0, ..., f_q) = \int_{G/K} f_0 df_1 \wedge \cdots df_q$$

Indeed, this is essentially the Poincaré-duality statement, exhibiting the orientation class as a Poincaré-dual to the constant function 1.

Simiarly, one checks that $\mathrm{Ch}(\Omega_{\underline{\mathbb{C}},d})$ also equals the orientation form.

In particular, we see that in the K-equivariant setting taking the pullback picks up precisely the cupproduct with τ, and one hopes that by translating through Nistor's isomorphisms and by definition of our map $\cup\tau$ the result follows. □

These Lemmas would then give a proof of Theorem 3.4.1 in the non-proper.

Ideas for a proof of Theorem 3.4.1 in the non-proper case. Fix $E \to M$ an equivariant vector bundle and ∇ a connection on E. By Lemma 3.4.6 we know that

$$(\mathrm{Ch}_{\Omega_{E,\nabla}})_{\mathfrak{m}} = (c(\mathrm{Ch}_G(E,\nabla))_{\mathfrak{m}} \in \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M))_{\mathfrak{m}}$$

For every maximal ideal $\mathfrak{m}$ that is not the maximal ideal determined by $e \in G$.

Looking at the localization at $e \in G$, if Conjecture 3.4.8 is true, we'd have

$$((\mathrm{Ch}_{\Omega_{E,\nabla}})_e - (c(\mathrm{Ch}_G(E,\nabla))_e) \cup \tau = ((\mathrm{Ch}_{\Omega_{\mathrm{pr}^*E,\mathrm{pr}^*\nabla}})_e - (c(\mathrm{Ch}_G(\mathrm{pr}^*E,\mathrm{pr}^*\nabla))_e$$

as elements of $\mathrm{HP}^{\dim(M)+q}(G \ltimes C_c^\infty(M \times G/K))_e$. Since G acts properly on $M \times G/K$ we know that the right hand side vanishes by Theorem 3.4.1, so we conclude that

$$((\mathrm{Ch}_{\Omega_{E,\nabla}})_e - (c(\mathrm{Ch}_G(E,\nabla))_e) \cup \tau = 0$$

Since $\cup\tau$ is an isomorphism, we conclude

$$(\mathrm{Ch}_{\Omega_{E,\nabla}})_e = (c(\mathrm{Ch}_G(E,\nabla))_e \in \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M))_e$$

Seeing that their localizations at every maximal ideal of $C^\infty_{\mathrm{inv}}(G)$ would agree, we'd conclude that

$$\mathrm{Ch}_{\Omega_{E,\nabla}} = c(\mathrm{Ch}_G(E,\nabla)) \in \mathrm{HP}^{\dim(M)}(G \ltimes C_c^\infty(M))$$

□

3.5 Comparison with known cases

3.5.1 Trivial group actions

If the group G is the trivial group, the convolution algebra $G \ltimes A$ is simply the commutative algebra A. Our double complex in this case becomes

$$\mathrm{L}(G,A)_{p,q} = (A^+)^{\otimes(p+1)}.$$

However, the vertical differential $d^v \colon \mathrm{L}(G,A)_{\bullet,q} \to \mathrm{L}(G,A)_{\bullet,q-1}$ satisfies

$$d^v = \begin{cases} \mathrm{id} & q \text{ is even,} \\ 0 & q \text{ is odd.} \end{cases}$$

In particular, we may replace the total complex $\mathrm{Tot}_\bullet(\mathrm{L}(G,A))$ by the first row $\mathrm{L}(G,A)_{\bullet,0}$ and we see that our complex is simply the Hochschild complex of A^+.

Similarly, the complex $\mathrm{C}_{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}A)$ can be replaced by the complex $\mathrm{C}_{\bullet,0}(G,\Omega_{\mathfrak{g}}A)$, which in this case is the 'deRham complex of A', i.e. the mixed complex $(\Omega^\bullet A, 0, d)$. The map $\Psi_2\colon \mathrm{L}(G,A)\to \mathrm{C}(G,\Omega_{\mathfrak{g}}A)$ now simply becomes the ordinary HKR map $(A^+)^{\otimes(\bullet+1)}\to\Omega^\bullet A$, and since the map $\mathrm{EZ}\circ\Psi_1\colon \mathrm{C}^{\mathrm{Hoch}}_\bullet(G\ltimes A, G\ltimes A)\to \mathrm{L}_{\bullet,0}(G,A)$ is just the identity, we see that our chain of maps Ψ is simply the HKR-morphism

$$(\mathrm{C}^\bullet_{\mathrm{Hoch}}(A,A),b,B)\to(\Omega^\bullet A,0,d).$$

Looking at the case $A=C^\infty_c(M)$ for M a manifold, we note that the equivariant cohomology $\mathrm{H}_G(M)$ is the deRham cohomology and that Getzler's model $\mathrm{C}^{\bullet,\bullet}(G,\Omega_{\mathfrak{g}}C^\infty(M))$ is concentrated in degree $q=0$, where it is given by the deRham-complex of M. All in all, the maps

$$c\colon \mathrm{H}^{\mathrm{ev}}_G(M)\to \mathrm{HP}^{\dim(M)}(G\ltimes C^\infty_c(M)),\ \mathrm{H}^{\mathrm{odd}}_G(M)\to \mathrm{HP}^{\dim(M)+1}(G\ltimes C^\infty_c(M))$$

are in this case induced (up to some signs) by the map

$$[-]\colon \Omega^\bullet(M)\to \mathrm{Hom}(C^\infty_c(M)^{\dim(m)+1-\bullet},\mathbb{R})$$

that takes a differential form $\omega\in\Omega^n(M)$ and sends it so

$$[\omega](f_0,\dots,f_{\dim(M)-n})=\int_M \omega\wedge f_0 df_1\wedge\cdots\wedge df_{\dim(M)-n}.$$

In particular it is the concatenation of the isomorphisms

$$\mathrm{H}^\bullet_{\mathrm{dR}}(M)\xrightarrow{\cong}\mathrm{H}^{\mathrm{dR},c}_{\dim(M)-\bullet}(M)\xleftarrow{\cong}\mathrm{HP}^{\dim(M)-\bullet}(C^\infty_c(M)).$$

Here the first map is an instance of Poincaré duality which associates to a k-form $\omega\in\Omega^k(M)$, the density $\widetilde{\omega}\in\Omega^{n-k}_c(M)^\times$ given by

$$\widetilde{\omega}(\alpha)=\int_M\omega\wedge\alpha,$$

and the second map is a consequence of the Hochschild-Kostant-Rosenberg Theorem in the continuous setting.

That the Chern characters are compatible in this case is clear from an immediate determination of the generalized cycle associated to a vector bundle $E\to M$ with a connection ∇. As there is no group action, we see that the underlying curved DGA is the curved DGA

$$\Omega=\Omega(M,\mathrm{End}(E))$$

of Example 3.1.11 (in the non-compact case), with differential $d_{\nabla^{\mathrm{End}}}$ and curvature $F(\nabla)$. The resulting generalized cycle is then seen to be equal (up to a sign) to the current induced by the differential

$$\sum_{i\geq 0}\frac{1}{i!}\mathrm{tr}(F(\nabla)^{\wedge i}),$$

which is exactly the differential form inducing the ordinary Chern character $\mathrm{Ch}(E)\in\mathrm{H}^{\mathrm{ev}}_{\mathrm{dR}}(M)$.

3.5.2 Compact groups

If we plug in $M = \{\mathrm{pt}\}$, we recover the convolution algebra of the group G itself. If G is compact, the periodic cyclic cohomology of $C^\infty(G)$ has been computed by Natsume and Nest [NN90, 1.II]. In this case, it is concentrated in even degrees, where it is represented by traces τ_φ for functions φ on the spectrum $\widehat{G}$ of G which are slowly increasing in a certain sense. Here, for $f \in C^\infty(G)$ this trace τ_φ is given by

$$\tau_\varphi(f) = \sum_{\pi \in \widehat{G}} \frac{\varphi(\pi)}{\dim(V_\pi)} \int_G f(g)\chi_\pi(g)dg$$

Looking at the equivariant cohomology $\mathrm{H}^\bullet_G(\mathrm{pt})$, we obtain from Getzler's model that it is contained in even degrees, where it is given by

$$\mathrm{H}^{2q}_G(\mathrm{pt}) \cong (\mathrm{Sym}^q(\mathfrak{g}^*))^G,$$

the invariant degree q polynomials on $\mathfrak{g}$.

Dissecting the map $c\colon \mathrm{H}^{\mathrm{ev}}_G(M) \to \mathrm{HP}^{\mathrm{ev}}(C^\infty(G))$ in this case we notice that the invariant polynomials live in $\mathrm{C}^{0,0}(G, \Omega_{\mathfrak{g}}(\{\mathrm{pt}\}))$, so that the only interesting pairing is with $\mathrm{C}_{0,0}(G, \Omega_{\mathfrak{g}}(\{\mathrm{pt}\}))$. Next, notice that pairing between $\mathrm{C}^{0,0}(G, \Omega_{\mathfrak{g}}(\{\mathrm{pt}\}))$ and $\mathrm{C}_{0,0}(G, \Omega_{\mathfrak{g}}(\{\mathrm{pt}\}))$ kills off polynomials of strictly positive degree as the pairing takes a polynomial $P \in \mathrm{Sym}(\mathfrak{g}^*)$ and a function $f \in C^\infty(\mathfrak{g})$ and pairs them by

$$\langle P, f \rangle = P(0)f(0).$$

We conclude the following:

Proposition 3.5.1 The map $c\colon \mathrm{H}^{\mathrm{ev}}_G(\mathrm{pt}) \to \mathrm{HP}^{\mathrm{ev}}(C^\infty(G))$ takes an invariant polynomial $P \in \mathrm{Sym}(\mathfrak{g}^*) \cong \mathrm{H}^{\mathrm{ev}}_G(\mathrm{pt})$ and sends it to the trace $c(P) \in \mathrm{HP}^0(C^\infty(G))$ given by

$$c(P)(f) = f(e)P(0).$$

Remark 3.5.2 The fact that $\tau(f) = f(e)$ is even a trace on the convolution algebra $C^\infty(G)$ is a consequence of the fact that any compact group is unimodular. Indeed, one checks that

$$\tau([f_1, f_2]) = \int_G f_1(g)f_2(g^{-1})dg - \int_G f_2(g)f_1(g^{-1})dg,$$

and these two integrals are equal because of unimodularity.

Remark 3.5.3 Under the isomorphism of Natsume and Nest, the trace $c(P)$ corresponds to τ_φ for

$$\varphi(\pi) = P(0)\dim(V_\pi)^2.$$

This follows from the fact that the character of the regular representation $L^2(G) \cong \widehat{\bigoplus}_{\pi \in \widehat{G}} V(\pi)^{\oplus \dim(V_\pi)}$ acts as the Dirac delta distribution at $e \in G$ on the space $C^\infty(G)$.

Recall that by Proposition 3.1.18 we have a map $\mathrm{Ch}_{\Omega,-}\colon \mathrm{Sym}(\mathfrak{g})^G \to \mathrm{HP}^{\mathrm{ev}}(\mathcal{A}_G)$ which sends an invariant polynomial γ to the character of the cycle with closed graded trace $⨏_\gamma$. A simple calculation with Fourier inversion shows that under the isomorphism of Natsume and Nest, $\mathrm{Ch}_{\Omega,\gamma}$ is the trace associated to the map

$$\varphi(\pi) = \dim(V_\pi)^2 \mathsf{D}_\gamma(\mathrm{tr}(\pi))(0).$$

3.5.3 Compact group actions

In Block-Getzer [BG94], a model for equivariant cyclic homology was presented for when the group G is compact using sheaves over G (with the topology defined by conjugacy-invariant opens) where stalks at $g \in G$ sketch the picture of $M_g = \{p \in M : pg = p\}$ and the Lie algebra $\mathfrak{g}^g$ of the centralizer of g using germs of G^g-invariant forms on M_g with polynomial coefficients in $\mathfrak{g}^g$. Using the $C^\infty_{\mathrm{inv}}(G)$-module structure on our complexes, we see that localizing at the identity in our models correspond to the stalk at the identity in the models of Block-Getzler, since we can go from G-cohomology to G-invariants at no cost by compactness of the group.

As such, we precisely recover the map α_e of [BG94, 3.3], from which we infer that

Corollary 3.5.4 [BG94, Thm 3.3] When the group G is compact, the map $\Psi\colon \mathrm{CC}(G \ltimes A) \to \mathrm{Tot}(\mathrm{CC}(G, \Omega_\mathfrak{g} A))$ is a quasi-isomorphism when localized at the identity.

As such, understanding the effect of the map $c\colon \mathrm{H}^\bullet_G(M) \to \mathrm{HP}^\bullet(G \ltimes C^\infty_c(M))$ is tantamount to understanding the pairing between $\mathrm{C}^{\bullet,\bullet}(G, \Omega_\mathfrak{g} C^\infty(M))$ and $\mathrm{C}_{\bullet,\bullet}(G, \Omega_\mathfrak{g} C^\infty_c(M))$. Since the group G is compact we can use the procedure outlined in Remark 3.2.4 to replace the double complexes with the concentration of G-cohomology and G-homology respectively in their first rows. In particular:

$$\mathrm{Tot}(\mathrm{C}^{\bullet,\bullet}(G, \Omega_\mathfrak{g} C^\infty(M)) \simeq (\mathrm{Sym}(\mathfrak{g}^*) \otimes \Omega^\bullet(M))^G$$

and

$$\mathrm{Tot}(\mathrm{C}_{\bullet,\bullet}(G, \Omega_\mathfrak{g} C^\infty_c(M)) \simeq (C^\infty(\mathfrak{g}) \otimes \Omega^\bullet_c(M))_G.$$

The pairing between these two starts with evaluating at $X = 0$ in $\mathfrak{g}$, and the rest is an equivariant instance of the Poincaré pairing between differential forms:

$$\begin{array}{ccc} \Omega^{\dim(M)-p}(M)^G \otimes \Omega^p_c(M)_G & \to & \mathbb{R} \\ \langle \omega, \eta \rangle & \mapsto & \int_M \omega \wedge \eta, \end{array}$$

which is a (homologically) perfect pairing by Poincaré duality. Again we see the story that we almost have a homologically perfect pairing, apart from the fact that we first kill all the behaviour in the $\mathfrak{g}$-direction.

3.5.4 Actions of discrete groups

When the group G is discrete, our constructions and results directly generalize parts of the work done in [Go99]. In particular, the curved DGA defined in Section 3.1.3 is precisely the curved DGA defined in [Go99, Sect 3] when the group is discrete (of course in this case the moment μ vanishes).

We also note that when the group is discrete, we obviously overcome the problem where we lose information in the $\mathfrak{g}$-direction when pairing between equivariant cohomology and cyclic homology: indeed, as the group is discrete the Lie algebra $\mathfrak{g}$ is trivial. Combining this with the remarks about compact groups before, we arrive at the following conclusion in the case of finite group actions (cf. [BC88]).

Theorem 3.5.5 If a finite group Γ acts orientation-preservingly on an oriented manifold M, then the maps

$$c\colon \mathrm{H}^{\mathrm{ev}}_{\Gamma}(M) \to \mathrm{HP}^{\dim(M)}(\Gamma \ltimes C^{\infty}_c(M)), \qquad \mathrm{H}^{\mathrm{odd}}_{\Gamma}(M) \to \mathrm{HP}^{\dim(M)+1}(\Gamma \ltimes C^{\infty}_c(M))$$

are isomorphisms.

In the general discrete case we notice that the convolution algebra $\Gamma \ltimes C^{\infty}_c(M)$ is a twisted tensor product of the group algebra of Γ and the Γ-algebra $C^{\infty}_c(M)$. In particular, elements are normally written as sums of elements $U_g f$ for $g \in \Gamma$ and $f \in C^{\infty}_c(M)$. The product is then given by

$$(U_g f)(U_h f') = U_{gh} f(g \cdot f').$$

In his book, Connes [Co94, III.2.δ] describes a model for the equivariant cohomology $\mathrm{H}_{\Gamma}(M)$. In this model, he makes use of maps

$$\gamma\colon \Gamma^{\times \bullet} \to \Omega_{\bullet}(M)$$

between products of the group and deRham-currents on M. He also gives a map pairing the chains of the (b, B)-bicomplex of the convolution algebra $\Gamma \ltimes C^{\infty}_c(M)$, which -up to signs and combinatorial factors- pairs a map γ as above with chains of the convolution algebra by the formula

$$\langle \gamma, (U_{g_0} f_0, ..., U_{g_n} f_n) \rangle \sim \gamma(g_0, ..., g_n)(f_0 df_1 \wedge \cdots \wedge df_n).$$

Furthermore, he shows [Co94, Thm III.2.14] that this procedure gives an isomorphism between equivariant cohomology $\mathrm{H}^{\bullet}_{\gamma}(M)$ and the periodic cyclic cohomology $\mathrm{HP}^{\bullet}(\Gamma \ltimes C^{\infty}_c(M))$.

Using a Poincaré duality argument, we can replace currents with differential forms, and we recover Getzler's model for equivariant cohomology. Translating the pairing to this situation and working through the calculations with the Eilenberg-Zilber map, one concludes that our map c between equivariant cohomology $\mathrm{H}^{\bullet}_{\Gamma}(M)$ and periodic cyclic cohomology $\mathrm{HP}^{\bullet}(\Gamma \ltimes C^{\infty}_c(M))$ is precisely the map written down by Connes.

Using this, [Go99, Thm 3.1] directly translates to a proof for Theorem 3.4.1 in the general case.

Corollary 3.5.6 [Go99, Thm 3.1] When a discrete group Γ acts on an oriented manifold M, then for any equivariant vector bundle E, the two Chern classes $c(\mathrm{Ch}_\Gamma(E)), \mathrm{Ch}_{\Omega_\Gamma(E)} \in \mathrm{HP}^{\mathrm{even}}(\Gamma \ltimes C_c^\infty(M))$ agree.

It is noteworthy how the argument simplifies for discrete groups, as opposed to the ideas for a proof we describe in the general case. So let us quickly discuss Gorokhovsky's argument to reduce to the proper case. Again, the main point in reducing to the proper case is replacing M by $M \times X$ such that Γ acts properly on X. Next, in [Go99, Prop 3.3], the standard cup product in cyclic cohomology is used to obtain a map

$$\mathrm{HP}^{\dim(M)}(\Gamma \ltimes C_c^\infty(M)) \otimes \mathrm{HP}^{\dim(X)}(\Gamma \ltimes C_c^\infty(X)) \xrightarrow{\cup} \mathrm{HP}^{\dim(M\times X)}((\Gamma \times \Gamma) \ltimes C_c^\infty(M \times X))$$

and it is shown that $\mathrm{Ch}_{\Omega(E)} \cup \tau = \mathrm{Ch}_{\Omega(\mathrm{pr}^* E)}$. Here τ is the equivariant orientation class that is associated to the trivial equivariant line bundle over X and pr^*E is seen as a $\Gamma \times \Gamma$-equivariant vector bundle. Then, using a map

$$\Delta\colon \Gamma \ltimes C_c^\infty(M \times X) \to (\Gamma \times \Gamma) \ltimes C_C^\infty(M \times X)$$

defined by $\Delta(U_g f) = U_{g,g} f$, one obtains a map

$$\mathrm{HP}^{\dim(M\times X)}((\Gamma \times \Gamma) \ltimes C_c^\infty(M \times X)) \to \mathrm{HP}^{\dim(M\times X)}(\Gamma \ltimes C_c^\infty(M \times X))$$

and it is shown that this total reduction sends $\mathrm{Ch}_{\Omega(E)} \otimes \tau$ to $\mathrm{Ch}_{\Omega(\mathrm{pr}^* E)}$. From that point on, the argument proceeds along roughly the same lines.

Of course, the big difference with the case where G is a non-discrete Lie group is that a map like Δ does not exist, and there is no obvious way to reduce the cyclic cohomology of $(G \times G) \ltimes A$ (for A an $G \times G$-algebra) to the cyclic cohomology of $G \ltimes A$ where we see A as an G-algebra by the diagonal action.

Chapter 4

Hochschild cohomology of Lie-Rinehart algebras

In this last chapter we turn the local theory of Lie algebroids. The main result of this chapter is a generalization to the Lie-Rinehart setting of a result of Blom [Bl17], relating the Hochschild cohomology of the universal enveloping algebra $\mathcal{U}(L, R)$ of a Lie-Rinehart algebra (L, R) with the symmetric powers of the adjoint representation of (L, R).

Theorem: Let (L, R) be a Lie-Rinehart algebra. If R is smooth and L is projective as an R-module, then there is a natural isomorphism

$$\mathrm{HH}^{\bullet}(\mathcal{U}(L, R), \mathcal{U}(L, R)) \cong \mathrm{H}^{\bullet}(L, \mathrm{Sym}(\mathrm{ad})). \tag{4.1}$$

These results should be thought of as the local analogues of the conjectural connections we alluded to in Section 2.5. We improve Blom's results by writing down a zig-zag of quasi-isomorphism defined purely in algebraic terms, inspired by the recent calculations of Kordon and Lambre [KL21].

The chapter is divided into two parts:

- In Section 4.1 we exploit the structure of $\mathcal{U}(L, R)$ as generated by a Lie algebra and a commutative algebra, to write down a 'non-linear' complex that combines Lie algebra cohomology of L and Hochschild cohomology of R and show that it is quasi-isomorphic to the Hochschild cohomology complex associated to $\mathcal{U}(L, R)$.
- In Section 4.2 we use this non-linear complex to write down a chain map from the complex associated to the symmetric powers of the adjoint representation of (L, R) to the non-linear complex. We show that the resulting map between the cohomology of the symmetric powers of the adjoint and the Hochschild cohomology of $\mathcal{U}(L, R)$ is an isomorphism.

This chapter has an appendix in Section 4.3, where we give the proof of certain results whose proofs would break up the rhythm of the text too much to give immediately.

4.1 The Hochschild cohomology of the universal enveloping algebra

In this section we calculate the Hochschild cohomology of the universal enveloping algebra $\mathcal{U}(L, R)$ of a Lie-Rinehart algebra (L, R). In the end we will show that it is isomorphic to the Lie-Rinehart cohomology with values in the symmetric powers of the adjoint representation. For the case where (L, R) arises from a Lie algebroid $A \to M$ this was already known by work of Blom [Bl17], who used Kontsevitch Formality to relate the Hochschild cohomology of the universal enveloping algebra with the polynomial Poisson cohomology of the Poisson manifold A^*. Since the polynomial Poisson complex is the same complex as the Lie algebroid cohomology complex with values in the symmetric powers of the adjoint representation on the nose, this does the trick.

Parallel to this, for a Lie algebra $\mathfrak{g}$, one can write down a direct chain map between the Hochschild complex of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and the Lie algebra cohomology complex with values in the symmetric powers of the adjoint, using the Poincaré–Birkhoff–Witt map.

This Lie algebra cohomology complex is defined as follows:

Definition 4.1.1 Given a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-representation M, the Lie algebra cohomology (or Chevalley-Eilenberg) complex $\mathrm{C}^\bullet_{\mathrm{CE}}(\mathfrak{g}, M)$ is given by

$$\mathrm{C}^k_{\mathrm{CE}}(\mathfrak{g}, M) \coloneqq \mathrm{Hom}(\Lambda^k \mathfrak{g}, M)$$

with differential $\partial_{\mathrm{CE}} \colon \mathrm{C}^k_{\mathrm{CE}}(\mathfrak{g}, M) \to \mathrm{C}^{k+1}_{\mathrm{CE}}(\mathfrak{g}, M)$ given by

$$\begin{aligned}(\partial_{\mathrm{CE}} F)(X_1, ..., X_{k+1}) \coloneqq & \sum_{i=1}^{k+1} (-1)^{i+1} X_i \cdot F(X_1, ...\widehat{X_i}..., X_{k+1}) \\ & + \sum_{i<j} (-1)^{i+j} F([X_i, X_j], X_1, ...\widehat{X_i}\widehat{X_j}..., X_{k+1}).\end{aligned}$$

It is now quite direct to write down an isomorphism between Hochschild cohomology of $\mathcal{U}(\mathfrak{g})$ and the Lie algebra cohomology of $\mathfrak{g}$. The resulting calculation is dual to the calculations on Hochschild homology by Kassel [Ka88] and Loday [Lo98, Thm 3.3.2]:

Example 4.1.2 (Lie algebras) If $\mathfrak{g}$ is a Lie algebra, the adjoint action induces a representation of $\mathfrak{g}$ on $\mathrm{Sym}(\mathfrak{g})$ by

$$[X, Y_1 \odot \cdots \odot Y_n] \coloneqq \sum_{i=1}^{n} Y_1 \odot \cdots \odot [X, Y_i] \odot \cdots \odot Y_n.$$

Similarly, using the canonical inclusion of $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra also becomes a representation of $\mathfrak{g}$ via the commutator action. Importantly, one checks that the Poincaré-Birkhoff-Witt map pbw: $\mathrm{Sym}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ is an isomorphism *of* $\mathfrak{g}$-*representations*. Using this fact, one checks that the map

$$\Phi\colon \mathrm{C}^{\bullet}_{\mathrm{Hoch}}(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g})) \to \mathrm{C}^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$$

from the Hochschild cohomology complex of $\mathcal{U}(\mathfrak{g})$ to the Chevalley-Eilenberg cohomology of $\mathfrak{g}$ with coefficients in $\mathrm{Sym}(\mathfrak{g})$ given by

$$\Phi(c)(X_1, ..., X_n) \coloneqq \sum_{\sigma\in S_n} (-1)^{\sigma}\mathrm{pbw}^{-1}(c(X_{\sigma(1)}, ..., X_{\sigma(n)}))$$

is a chain map.

Next, we recall that $\mathcal{U}(\mathfrak{g})$ is a filtered algebra where $\mathfrak{g}$ is put in filtration degree 1, and that pbw: $\mathrm{Sym}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ induces an isomorphism between $\mathrm{Sym}(\mathfrak{g})$ and the graded quotient of $\mathcal{U}(\mathfrak{g})$. Using this filtration we put a filtration on the Hochschild complex by

$$F^pC^n(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g})) = \{c\colon \mathcal{U}(\mathfrak{g})^{\otimes n} \to \mathcal{U}(\mathfrak{g}) : c((\mathcal{U}(\mathfrak{g})^{\otimes n})^{\leq k}) \subset \mathcal{U}(\mathfrak{g})^{\leq p+k} \text{ for all } k \geq 0\}$$

Similarly we have a filtration of $C^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$ given by

$$F^pC^n_{\mathrm{CE}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g})) = \mathrm{Hom}(\Lambda^n\mathfrak{g}, \mathrm{Sym}^{\leq p+n}(\mathfrak{g})).$$

The map Φ is then a map of filtered chain complexes. Again using the fact that pbw induces an isomorphism of graded algebras from $\mathrm{Sym}(\mathfrak{g})$ to the graded quotient of $\mathcal{U}(\mathfrak{g})$ we see that the graded quotient complex of the Hochschild complex equals

$$\mathrm{GC}^{\bullet}_{\mathrm{Hoch}}(\mathcal{U}(\mathfrak{g}),\mathcal{U}(\mathfrak{g})) \cong \mathrm{C}^{\bullet}_{\mathrm{Hoch}}(\mathrm{Sym}(\mathfrak{g}), \mathrm{Sym}(\mathfrak{g})).$$

Similarly, we can calculate the graded quotient of $\mathrm{C}^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))$, where due to the fact that the differential lowers the filtration degree by 1 we have

$$\mathrm{GC}^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g})) \cong (\mathrm{Hom}(\Lambda^{\bullet}\mathfrak{g}, \mathrm{Sym}(\mathfrak{g})), 0)$$

The map $\mathrm{G}\Phi$ induced by Φ between the graded quotients is simply given by

$$\mathrm{G}\Phi(c)(X_1, ..., X_n) = \sum_{\sigma\in S_n} (-1)^{\sigma} c(X_{\sigma(1)}, ..., X_{\sigma(n)}).$$

Now, as $\mathrm{Sym}(\mathfrak{g})$ is generated by $\mathfrak{g}$ we have inverse isomorphisms

$$\begin{aligned} \mathrm{Der}(\mathrm{Sym}(\mathfrak{g})) &\to \mathrm{Hom}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g})) \\ D &\mapsto (X \mapsto D(X)), \end{aligned}$$

and

$$\begin{aligned} \mathrm{Hom}(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g})) &\to \mathrm{Der}(\mathrm{Sym}(\mathfrak{g})) \\ c &\mapsto (X_1 \odot \cdots \odot X_k \mapsto \sum_{i=1}^{k} X_1 \odot \cdots \odot c(X_i) \odot \cdots \odot X_k). \end{aligned}$$

Similarly we have an isomorphism

$$\Lambda^\bullet_{\mathrm{Sym}(\mathfrak{g})}\mathrm{Der}(\mathrm{Sym}(\mathfrak{g})) \cong \mathrm{Hom}(\Lambda^\bullet\mathfrak{g}, \mathrm{Sym}_R\mathfrak{g})$$

where the map from left to right is given by restriction. As $\mathrm{Sym}(\mathfrak{g})$ is a commutative algebra, we can use Hochschild–Kostant–Rosenberg Theorem to see that the inclusion

$$\Lambda^\bullet_{\mathrm{Sym}(\mathfrak{g})}\mathrm{Der}(\mathrm{Sym}(\mathfrak{g})) \hookrightarrow \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathrm{Sym}(\mathfrak{g}), \mathrm{Sym}(\mathfrak{g}))$$

induces an isomorphism

$$\Lambda^\bullet_{\mathrm{Sym}(\mathfrak{g})}\mathrm{Der}(\mathrm{Sym}(\mathfrak{g})) \cong \mathrm{HH}^\bullet(\mathrm{Sym}(\mathfrak{g}), \mathrm{Sym}(\mathfrak{g})).$$

Combining these two facts, we see that $G\Phi$ is a quasi-isomorphism, and by the Spectral Sequence Comparison Theorem [Ze57], we see that Φ is also a quasi-isomorphism.

We remark that this proof does not explicitely use a formality argument involving the linear Poisson manifold $\mathfrak{g}^*$. As such, we use this example in this section as a starting point for obtaining a fully algebraic proof of the connection (4.1) between Hochschild cohomology of $\mathcal{U}(L, R)$ and the cohomology of the symmetric powers of the adjoint, focused on using the Poincare-Birkhoff-Witt map. For this, we restrict to the case that L is projective as an R-module, so that it admits an L-connection ∇, and we can write down a Poincare-Birkhoff-Witt map pbw^∇.

4.1.1 Understanding $\mathcal{U}(L, R)$-modules

As we have seen in Example 4.1.2, we can use the fact that $\mathcal{U}(\mathfrak{g})$ is generated by $\mathfrak{g}$ to relate the Hochschild homology of $\mathcal{U}(\mathfrak{g})$ with the Lie algebra cohomology of $\mathfrak{g}$. In this section, we try to generlize this to the case of a Lie-Rinehart algebra, and try to relate $\mathrm{HH}^\bullet(\mathcal{U}(L, R), M)$ to certain Lie algebra cohomology-like complexes associated to L. To mimick the filtration-argument from before, we restrict ourselves to the class of $\mathcal{U}(L, R)$-bimodules that behave the same as $\mathcal{U}(L, R)$ behaves as a bimodule over itself.

Definition 4.1.3 A *filtered* $\mathcal{U}(L, R)$*-bimodule* is a $\mathcal{U}(L, R)$-bimodule M with a filtration $\{M^{\leq k}\}_{k\geq 0}$ such that

- For every $f \in R$ and $m \in M^{\leq k}$ it holds that $fm, mf \in M^{\leq k}$ and $fm - mf \in M^{\leq k-1}$;
- For every $X \in L$ and $m \in M^{\leq k}$ it holds that $Xm, mX \in M^{\leq k+1}$ and $Xm - mX \in M^{\leq k}$.

Remark 4.1.4 We make three remarks on the definition above.

- Clearly $\mathcal{U}(L, R)$ is a filtered $\mathcal{U}(L, R)$-bimodule.

- The definition is equivalent to requiring that for every $D \in \mathcal{U}(L,R)^{\leq n}$ and $m \in M^{\leq k}$ it holds that $Dm, mD \in M^{\leq k+m}$ and $Dm - mD \in M^{\leq k+m-1}$.
- From the definition it follows immediately that if M is a filtered $\mathcal{U}(L,R)$-bimodule, then its graded quotient $\mathrm{G}M$ is canonically a (graded) $\mathrm{Sym}_R L$-bimodule.

We first note that for any unital algebra A, the Hochschild cohomology functor $HH^\bullet(A,-)$ is the right derived functor of the functor which takes an A-bimodule M and spits out the invariants

$$M^A = \{m \in M : am = ma \, \forall a \in A\}$$

This follows from Remark 1.1.12 and the natural isomorphism

$$M^A \cong \mathrm{Hom}_{A^e}(A, M)$$

Now, when M is a $\mathcal{U}(L,R)$-bimodule, the whole structure is of course defined by how $R \subset \mathcal{U}(L,R)$ acts upon M and how $L \subset \mathcal{U}(L,R)$ does. Indeed, dissecting the universal properties of $\mathcal{U}(L,R)$ we obtain the following characterizations of $\mathcal{U}(L,R)$-modules:

Lemma 4.1.5 A left $\mathcal{U}(L,R)$-module structure on M is the same as

- A left R-module structure on M,
- A left action of the Lie algebra L on M

that are compatible in the following sense

- $(fX) \cdot m = f \cdot (X \cdot m)$
- $X \cdot (f \cdot m) = f \cdot (X \cdot m) + \rho(X)(f) \cdot m$

Similarly a right $\mathcal{U}(L,R)$-module structure on M is the same as

- A right R-module strucutrce on M.
- A right action of the Lie algebra L on M

that are compatible in the following sense

- $m \cdot (fX) = (m \cdot f) \cdot X$
- $(m \cdot X) \cdot f = (m \cdot f) \cdot X + m \cdot \rho(X)(f)$

Now if M is a $\mathcal{U}(L,R)$-bimodule, we see that the invariants can also be described using L and R:

$$M^{\mathcal{U}(L,R)} = \{m \in M : fm = mf, \, \forall f \in R \, , \, Xm = mX, \, \forall X \in L\} \tag{4.2}$$

This means that the $\mathcal{U}(L,R)$-invariants are elements which are both R-invariants, and invariant under the diagonal L-action

$$[X, m] = Xm - mX$$

We will now show that, using this observation, we can write the functor $(-)^{\mathcal{U}(L,R)}$ as a composition of two functors. The resulting discussion can be summarized as follows:

Notation 4.1.6 For M a $\mathcal{U}(L,R)$-bimodule, write

$$M^R := \{m \in M : \ fm = mf, \ \forall f \in R\},$$

and for $\widetilde{M}$ a L-representation, write

$$\widetilde{M}^L := \{m \in \widetilde{M} : \ Xm = 0, \ \forall X \in L\}.$$

Lemma 4.1.7 Let M be a $\mathcal{U}(L,R)$-bimodule. The space of R-invariants M^R is invariant under the diagonal L-action on M.

Proof. Fixing $m \in M^R$ and $X \in L$, we need to show that $Xm - mX \in M^R$. For this, take $f \in R$ and calculate, using the fact that $m \in M^R$ and in $\mathcal{U}(L,R)$ we have $Xf - fX = \rho(X)(f)$:

$$\begin{aligned} f(Xm - mX) &= fXm - fmX \\ &= Xfm - \rho(X)(f)m - fmX \\ &= Xmf - m\rho(X)(f) - mfX \\ &= Xmf - mXf \\ &= (Xm - mX)f. \end{aligned}$$

This finishes the proof. □

Theorem 4.1.8 Consider the following two functors:

- The functor $(-)^R \colon \mathcal{U}(L,R)\text{-bmod} \to L\text{-rep}$ that takes a bimodule M and sends it to its R-invariants M^R endowed with the diagonal L-action;
- The functor $(-)^L \colon L\text{-rep} \to \mathrm{Vect}_{\mathbb{K}}$ that takes an L-representation $\widetilde{M}$ and sends it to its L-invariants $\widetilde{M}^L$.

These functors are well-defined and there is the following equality of functors:

$$(-)^L \circ (-)^R = (-)^{\mathcal{U}(L,R)} \colon \mathcal{U}(L,R)\text{-bimod} \to \mathrm{Vect}_{\mathbb{K}}.$$

Proof. By the previous Lemma, we can indeed lift the functor $(-)^R$ to make objects land in L-rep. One needs to check that a map of $\mathcal{U}(L,R)$-bimodules then also restricts to a map of L-representations, but this is an easy check. The equality of functors[1] then follows from the discussion leading to equation (4.2). □

This result will give us a starting point to understanding the Hochschild cohomology of $\mathcal{U}(L,R)$ in terms of R- and L-modules.

[1]Remark that using the explicit definitions of invariants as specific subsets of the original spaces, this is indeed an *equality*, not just an *equivalence*, of functors.

4.1.2 Combining Lie algebra and Hochschild cohomology

From Theorem 4.1.8 we want to take inspiration to write down a complex calculating the Hochschild cohomology of $\mathcal{U}(L,R)$ using the hyper-derived functors of both $(-)^R\colon \mathcal{U}(L,R)$-bimod $\to L$-rep and $(-)^L\colon L$-rep $\to \mathrm{Vect}_{\mathbb{K}}$.

One approach is the one used by Kordon and Lambre [KL21], who use the fact that $(-)^{\mathcal{U}}(L,R)$ is a composition of functors to write down a spectral sequence converging to $\mathrm{HH}^\bullet(\mathcal{U}(L,R),M)$ whose second page is given by

$$\mathrm{E}_2^{p,q} = \mathrm{H}^p_{\mathrm{CE}}(\mathrm{HH}^q(R,M)) \Rightarrow \mathrm{HH}^{p+q}(\mathcal{U}(L,R),M),$$

and then argue about the higher differentials in this spectral sequence. We take a different path, using the chain complexes calculating the derived functors.

Recall that (models for) the hyper-derived functors are given as follows: for $(-)^R$ we have that the left derived functors are calculated by chain complex $C^\bullet_{\mathrm{Hoch}}(R,-)$, while for $(-)^L$ the left derived functors are calculated by the Lie algebra cohomology complex $C^\bullet_{\mathrm{CE}}(L,-)$.

From this observation, the following plan arises:

- For a $\mathcal{U}(L,R)$-bimodule M, write down spaces $\mathrm{C}^{p,q}_{\mathrm{LR}}(L,R;M) \coloneqq \mathrm{C}^p_{\mathrm{CE}}(L,\mathrm{C}^q_{\mathrm{Hoch}}(R,M))$.
- Define a differential $d_v\colon \mathrm{C}^{p,q}_{\mathrm{LR}}(L,R;M) \to \mathrm{C}^{p,q+1}_{\mathrm{LR}}(L,R;M)$ using the Hochschild differential on $\mathrm{C}^\bullet_{\mathrm{Hoch}}(R,M)$.
- Define a differential $d_h\colon \mathrm{C}^{p,q}_{\mathrm{LR}}(L,R;M) \to \mathrm{C}^{p+1,q}_{\mathrm{LR}}(L,R;M)$ using the Chevalley-Eilenberg differential associated to some L-representation on $\mathrm{C}^q_{\mathrm{Hoch}}(R,M)$.
- Check that this describes the structure of a double complex $\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L,R;M)$.
- See how the cohomology that this complex calculates compares with $\mathrm{HH}^\bullet(\mathcal{U}(L,R),M)$.

The only questions arises, what is the L-module structure on $\mathrm{C}^q_{\mathrm{Hoch}}(R,M)$? After all, the abstract nonsense of derived functors stipulates that since the derived functors of $(-)^R$ (as a functor from $\mathcal{U}(L,R)$-bimodules to L-modules) are $\mathrm{HH}^\bullet(R,-)$, these Hochschild cohomology groups need to have some L-module structure. Following the works of Kordon and Lambre [KL21] we see that this L-module structure is already given on the level of Hochschild chains by, for $X \in L$, the map $L_X\colon \mathrm{C}^q_{\mathrm{Hoch}}(R,M) \to \mathrm{C}^q_{\mathrm{Hoch}}(R,M)$ with the formula

$$(L_X F)(f_1,...,f_q) = [X, F(f_1,...,f_q)] - \sum_{i=1}^q F(f_1,...,\rho(X)f_i,...,f_q), \tag{4.3}$$

where the notation $[X,m]$ again denotes the diagonal action $[X,m] = Xm - mX$ induced on M by the $\mathcal{U}(L,R)$-module structure.

From this we arrive at the following definition.

Definition 4.1.9 For M a $\mathcal{U}(L,R)$-bimodule, we define the Lie-Rinehart double complex $(\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L,R;M), d_h, d_v)$ by

- The spaces $\mathrm{C}^{p,q}_{\mathrm{LR}}(L,R;M)$ are given by

$$\mathrm{C}^{p,q}_{\mathrm{LR}}(L,R;M) = \mathrm{Hom}(\Lambda^p L, \mathrm{Hom}(R^{\otimes q}, M))$$

- The horizontal differential $d_h\colon \mathrm{C}^{p,q}_{\mathrm{LR}}(L,R;M) \to \mathrm{C}^{p+1,q}_{\mathrm{LR}}(L,R;M)$ is given by

$$\begin{aligned}(d_h F)(X_1, ..., X_{p+1}) = &\sum_{i=1}^{p+1}(-1)^{i+1} L_{X_i}(F(X_1, ...\widehat{X_i}..., X_{p+1})\\ &+ \sum_{i<j}(-1)^{i+j} F([X_i, X_j], X_1, ...\widehat{X_i}\widehat{X_j}..., X_{p+1})\end{aligned}$$

- The vertical differential $d_v\colon \mathrm{C}^{p,q}_{\mathrm{LR}}(L,R;M) \to \mathrm{C}^{p,q+1}_{\mathrm{LR}}(L,R;M)$ is given by

$$(d_v F)(X_1, ..., X_p) = (-1)^p b(F(X_1, ..., X_p))$$

where $b\colon \mathrm{Hom}(R^{\otimes q}, M) \to \mathrm{Hom}(R^{\otimes(q+1)}, M)$ is the usual Hochschild differential.

Remark 4.1.10 Using the $\otimes$-Hom-adjunction we can also rewrite this as

$$\mathrm{C}^{p,q}_{\mathrm{LR}}(L,R;M) = \mathrm{Hom}(\Lambda^p L \otimes R^{\otimes q}, M)$$

with differentials

$$\begin{aligned}(d_h F)(X_1, ..., X_{p+1}, f_1, ..., f_q) = &\sum_{i=1}^{p+1}(-1)^{i+1}[X_i, F(X_1, ...\widehat{X_i}..., X_{p+1}, f_1, ...f_q)]\\ &+ \sum_{i=1}^{p+1}\sum_{j=1}^{q}(-1)^i F(X_1, ...\widehat{X_i}..., X_{p+1}, f_1, ..., \rho(X_i)f_j, ..., f_q)\\ &+ \sum_{i<j}(-1)^{i+j} F([X_i, X_j], X_1, ...\widehat{X_i}\widehat{X_j}..., X_{p+1}, f_1, ..., f_q)\end{aligned}$$

and

$$\begin{aligned}(d_v F)(X_1, ..., X_p, f_1, ..., f_{q+1}) = &(-1)^p f_1 F(X_1, ..., X_p, f_2, ..., f_{q+1})\\ &+ \sum_{i=1}^{q}(-1)^{i+p} F(X_1, ..., X_p, f_1, ..., f_i f_{i+1}, ..., f_{q+1})\\ &+ (-1)^{p+q} F(X_1, ..., X_p, f_1, ..., f_q) f_{q+1}\end{aligned}$$

To see that d^h and d^v indeed describe the structure of a double complex we have the following properties of the action of L on $\mathrm{C}^{\bullet}_{\mathrm{Hoch}}(R,M)$:

Lemma 4.1.11 For every $X, Y \in L$ we have

$$[L_X, L_Y] = L_{[X,Y]}, \qquad b \circ L_X = L_X \circ b.$$

Proof. This is an explicit calculation that we skip. □

Using this, we have the following result:

Proposition 4.1.12 The maps

$$d_h\colon \mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L, R; M) \to \mathrm{C}^{\bullet+1,\bullet}_{\mathrm{LR}}(L, R; M)$$

and

$$d_v\colon \mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L, R; M) \to \mathrm{C}^{\bullet,\bullet+1}_{\mathrm{LR}}(L, R; M)$$

satisfy

$$d_h^2 = 0, \qquad d_v^2 = 0, \qquad d_v d_h + d_h d_v = 0.$$

Proof. The fact that $d_v^2 = 0$ follows from the fact that $b^2 = 0$ on $\mathrm{C}^{\bullet}_{\mathrm{Hoch}}(R, M)$, the fact that $d_h^2 = 0$ follows from the first part of the previous Lemma, since d_h is simply the Chevalley-Eilenberg differential associated to L_X. The relation $d_v d_h + d_h d_v$ follows from the fact that the action L_X commutes with b. □

So what does $\mathrm{Tot}(\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L, R; M), d_h + d_v)$ calculate, and how does it compare to $\mathrm{HH}^{\bullet}(\mathcal{U}(L, R), M)$? As we see in the following two examples, the result is close, but not quite what we are looking for.

Example 4.1.13 (Degree 0) In degree 0 we have $\mathrm{HH}^0(\mathcal{U}(L, R), M) = M^{\mathcal{U}(L,R)}$ and by the discussion before we see that this consists precisely of those elements of M which are both R- and L-invariant. Now in our complex $\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L, R; M)$ we have the differentials

$$d^h\colon \mathrm{C}^{0,0}_{\mathrm{LR}}(L, R; M) \to \mathrm{C}^{1,0}_{\mathrm{LR}}(L, R; M) \qquad d^v\colon \mathrm{C}^{0,0}_{\mathrm{LR}}(L, R; M) \to \mathrm{C}^{0,1}_{\mathrm{LR}}(L, R; M)$$

given by

$$(d^h m)(X) = Xm - mX \qquad (d^v m)(f) = fm - mf$$

So we see that $\mathrm{H}^0(\mathrm{Tot}(\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L, R; M))$ is precisely those element of M that are L- and R-invariant. In particular we see that in degree 0 we obtain precisely the Hochschild cohomology.

Example 4.1.14 (Degree 1) In degree 1 we are searching for

$$\mathrm{HH}^1(\mathcal{U}(L, R), M) = \mathrm{Der}(\mathcal{U}(L, R), M)/\mathrm{Inn}(\mathcal{U}(L, R), M).$$

In this case we start by dissecting what it means for a map $D\colon \mathcal{U}(L, R) \to M$ to be a derivation. Since $\mathcal{U}(L, R)$ is generated by L and R we can take D apart by restriction into maps $D_0\colon L \to M$ and $D_1\colon R \to M$. With this, we can check how D_0 and D_1 interact, by looking at products in $\mathcal{U}(L,R)$ of the form $X \cdot Y$, $X \cdot f$, $f \cdot X$ and $f \cdot g$ for $X, Y \in L$ and $f, g \in R$.

In particular, we get

$$\begin{aligned} D(X \cdot Y) &= X \cdot D_0(Y) + D_0(X) \cdot Y, \\ D(X \cdot f) &= X \cdot D_1(f) + D_0(X) \cdot f, \\ D(f \cdot X) &= f \cdot D_0(X) + D_1(f) \cdot X, \\ D(f \cdot g) &= f \cdot D_1(g) + D_1(f) \cdot g. \end{aligned}$$

Now, we have the following relations in $\mathcal{U}(L,R)$:

$$\begin{aligned} X \cdot Y - Y \cdot X &= [X,Y], \\ X \cdot f &= fX + \rho(X)(f), \\ f \cdot X &= fX, \\ f \cdot g &= fg. \end{aligned}$$

So that we obtain the following relations between D_0 and D_1:

$$D_0([X,Y]) = [X, D_0(Y)] - [Y, D_0(X)], \tag{D1}$$
$$D_1(\rho(X)(f)) = [X, D_1(f)] - f \cdot D_0(X) + D_0(X) \cdot f, \tag{D2}$$
$$D_0(fX) = f \cdot D_0(X) + D_1(f) \cdot X, \tag{D3}$$
$$D_1(fg) = f \cdot D_1(g) + D_1(f) \cdot g. \tag{D4}$$

Conversely, given $D_0\colon L \to M$ and $D_1\colon R \to M$ subject to these relations, we can define a derivation $D\colon \mathcal{U}(L,R) \to M$ which restricts to D_0 and D_1: the important remark is that due to the relations imposed on D_0 and D_1 the defining equation

$$D(fX_1 \cdots X_k) = D_1(f)X_1 \cdots X_k + \sum_{i=1}^{k} fX_1 \cdots X_{k-1} D_0(X_i) X_{i+1} \cdots X_k$$

is well-defined. We conclude that

$$\mathrm{Der}(\mathcal{U}(L,R), M) = \{D_0\colon L \to M,\ D_1\colon R \to M \,\text{subject to D1-D4}\}.$$

Within this framework, the inner-derivation associated to an element $m \in M$ is simply given by

$$D_0(X) = Xm - mX, \qquad\qquad D_1(f) = fm - mf.$$

Now, we look at what happens in degree 1 in $\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L,R;M)$. From the definition 4.1.9 of the differentials it is easy to see that

$$Z^1(\mathrm{Tot}(\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L,R;M))) = \{D_0\colon L \to M, D_1\colon R \to M \,\text{subject to D1, D2 and D4}\}$$

since D1 is equivalent to $d_h(D_0) = 0$, D2 is equivalent to $d_v(D_0) + d_h(D_1) = 0$ and D4 is equivalent to $d_v(D_1) = 0$. On the side of the boundaries, we see that the exact element in $B^1(\mathrm{Tot}(\mathrm{C}^{\bullet;\bullet}_{\mathrm{LR}}(L, R; M)))$ associated to an element $m \in C^{0,0}_{\mathrm{LR}}(L, R; M) = M$ is given by

$$D_0(X) = [X, m], \qquad D_1(f) = fm - mf.$$

So we see that that in $\mathrm{Tot}(\mathrm{C}^{\bullet;\bullet}_{\mathrm{LR}}(L, R; M))$ we obtain exactly the correct 1-boundaries, but not the correct 1-cocycles, since we miss the relation D3.

As we see, the complex $\mathrm{C}^{\bullet;\bullet}_{\mathrm{LR}}(L, R; M)$ does not incorporate the R-module structure that L has. Indeed, this is already present in the fact that the action L_X on $\mathrm{C}^q_{\mathrm{Hoch}}(R, M)$ is not R-linear in X. However, it turns out it is 'R-linear up to homotopy' in the following sense:

Definition 4.1.15 For $f \in R$ and $X \in L$, define the operator $h_{f,X} \colon \mathrm{C}^q_{\mathrm{Hoch}}(R, M) \to \mathrm{C}^{q-1}_{\mathrm{Hoch}}(R, M)$ by

$$\begin{aligned}(h_{f,X}F)(f_1, ..., f_{q-1}) = &\sum_{i=1}^{q} (-1)^{i+1} F(f_1, ..., f_{i-1}, f, f_i, ..., f_{q-1})X \\ &+ \sum_{1 \le i \le j \le q-1} (-1)^{i+1} F(f_1, ..., f_{i-1}, f, f_i, ..., \rho(X)(f_j), ..., f_{q-1})\end{aligned}$$

Proposition 4.1.16 The following equation holds true for every $f \in R$ and $X \in L$:

$$L_{fX} - fL_X = b \circ h_{f,X} + h_{f,X} \circ b.$$

The proof of this Proposition is by an explicit calculation which nevertheless gives some insight in the underlying structure. We defer the proof to Proposition 4.3.1.

With this homotopy at hand, it makes sense to impose a kind of symbol equation in our double complex to encode the failure of R-linearity and define the following 'non-linear' complex:

Definition 4.1.17 We define $\mathrm{C}^{\bullet}_{\mathrm{nl}}(L, R; M) \subset \mathrm{Tot}(\mathrm{C}^{\bullet;\bullet}_{\mathrm{LR}}(L, R; M))$ by

$$\begin{aligned}\mathrm{C}^n_{\mathrm{nl}}(L, R; M) = \{&(\varphi_0, ..., \varphi_n), \varphi_i \in \mathrm{C}^{n-i,i}_{\mathrm{LR}}(L, R; M) : \\ &\varphi_i(X_1, ..., fX_{n-i}) = f\varphi_i(X_1, ..., X_{n-i}) + h_{f,X_{n-i}}(\varphi_{i+1}(X_1, ..., X_{n-i-1}))\}.\end{aligned}$$

Remark 4.1.18 We will refer to the equation

$$\varphi_i(X_1, ..., fX_{n-i}) = f\varphi_i(X_1, ..., X_{n-i}) + h_{f,X_{n-i}}(\varphi_{i+1}(X_1, ..., X_{n-i-1})) \tag{4.4}$$

as the *symbol equation*.

Lemma 4.1.19 The following equations hold true for all $X, Y \in L$, $f \in R$ and $\varphi \in \mathrm{C}^{\bullet}_{\mathrm{Hoch}}(R, M)$.

- $L_X(f\varphi) = fL_X(\varphi) + \rho(X)(f)\varphi$
- $L_Y \circ h_{f,X} = h_{f,X} \circ L_Y + h_{\rho(Y)f,X} + h_{f,[Y,X]}$

Proof. This is an explicit calculation that we skip. □

From this Lemma one infers by an explicit calculation:

Proposition 4.1.20 The spaces $\mathrm{C}^\bullet_{\mathrm{nl}}(L,R;M)$ define a subcomplex of the total complex $\mathrm{Tot}(\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L,R;M), d^h + d^v)$.

We postpone this proof to Proposition 4.3.2.

We see that this non-linear complex solves the problem we encountered in Example 4.1.14.

Example 4.1.21 In degree 0 we have $\mathrm{C}^0_{\mathrm{nl}}(L,R;M) = \mathrm{C}^{0,0}_{\mathrm{LR}}(L,R;M)$, as there is no symbol equation. In degree 1 we have

$$\mathrm{C}^1_{\mathrm{nl}}(L,R;M) = \{\varphi_0\colon R \to M, \varphi_1\colon L \to M : \varphi_1(fX) = f\varphi_1(X) + \varphi_0(f)X\}$$

because for $\varphi_0\colon R \to M$ we have $h_{f,X}\varphi_0 = \varphi_0(f)X$. In particular we see that the non-linear subcomplex recoveres the missing relation D3 from Example 4.1.14, in particular we see that $\mathrm{H}^1(C^\bullet_{\mathrm{nl}}(L,R;M))$ is on the nose isomorphic to $\mathrm{HH}^1(\mathcal{U}(L,R),M)$.

In the next subsection, we will prove the general statement that

$$\mathrm{H}^\bullet(C^\bullet_{\mathrm{nl}}(L,R;M)) \cong \mathrm{HH}^\bullet(\mathcal{U}(L,R),M)$$

We do this by ways of a chain map $\mathrm{C}^\bullet(\mathcal{U}(L,R),M) \to \mathrm{Tot}(\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L,R;M))$.

4.1.3 The shuffle map

We want to write down a map between the Hochschild complex of $\mathcal{U}(L,R)$ with values in the filtered $\mathcal{U}(L,R)$-bimodule M and our Lie-Rinehart complex of Definition 4.1.9, using the philosophy that we are combining the structure of R as an algebra and L as a Lie algebra that together generate $\mathcal{U}(L,R)$. As such we want a 'shuffle map', a map that shuffles L- and R-inputs amongst each other, while also anti-symmetrizing the L-inputs.

Definition 4.1.22 We define the set of *semi-symmetrized (p,q)-shuffles* $S_{p,q}S_p \subset S_{p+q}$ is defined by

$$S_{p,q}S_p = \{\sigma \in S_{p+q}; \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(p+q)\}$$

Using these shuffles, we define the shuffle map

$$\mathfrak{s}\colon \mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{U}(L,R),M) \to \mathrm{Tot}(\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L,R;M)).$$

Definition 4.1.23 We define the map $\mathfrak{s}^{p,q}\colon \mathrm{C}^{p+q}_{\mathrm{Hoch}}(\mathcal{U}(L,R),M) \to \mathrm{C}^{p,q}_{\mathrm{LR}}(L,R;M)$ by

$$\mathfrak{s}^{p,q}(\varphi)(D_1,...,D_{p+q}) = \sum_{\sigma\in S_{p,q}S_p} (-1)^{\sigma}\varphi(D_{\sigma^{-1}(1)},...,D_{\sigma^{-1}(p+q)})$$

here $D_1,...,D_p \in L$ and $D_{p+1},...,D_{p+q} \in R$.

Lemma 4.1.24 The map $\mathfrak{s}$ satisfies

$$\mathfrak{s}^{p,q}\circ b = d_h \circ \mathfrak{s}^{p-1,q} + d_v \circ \mathfrak{s}^{p,q-1}$$

The proof of this statement is postponed to Lemma 4.3.3.

In particular we see that $\mathfrak{s}$ induces a chain map between the Hochschild complex $\mathrm{C}^{\bullet}_{\mathrm{Hoch}}(\mathcal{U}(L,R),M)$ and the total complex $\mathrm{Tot}(\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L,R;M))$.

Theorem 4.1.25 If M is a filtered $\mathcal{U}(L,R)$-bimodule, R is smooth and L is a projective R-module, the chain map $\mathfrak{s}$ induces a zig-zag of quasi-isomorphisms

$$\mathrm{C}^{\bullet}_{\mathrm{Hoch}}(\mathcal{U}(L,R),M) \xleftarrow{\iota} \mathfrak{s}^{-1}(\mathrm{C}^{\bullet}_{\mathrm{nl}}(L,R;M)) \xrightarrow{\mathfrak{s}} \mathrm{C}^{\bullet}_{\mathrm{nl}}(L,R;M)$$

We will prove this theorem by setting up spectral sequences using filtrations and showing that the induced maps between the graded quotients are quasi-isomorphisms. Principally, we will use the filtration on $\mathcal{U}(L,R)$ which is defined by putting L in filtration degree 1 and R in filtration degree 0 and extending in such a way that $\mathcal{U}(L,R)$ is a filtered algebra.

Definition 4.1.26 We put filtrations on $\mathcal{U}(L,R)^{\otimes n}$, $C^n_{\mathrm{Hoch}}(\mathcal{U}(L,R),M)$, $C^{p,q}_{\mathrm{LR}}(L,R;M)$ and $C^n_{\mathrm{nl}}(L,R;M)$ by:

$$(\mathcal{U}(L,R)^{\otimes n})^{\leq k} := \sum_{i_1+\cdots+i_n\leq k} \mathcal{U}(L,R)^{\leq i_1}\otimes\cdots\otimes\mathcal{U}(L,R)^{\leq i_n},$$

$$F^k(C^n_{\mathrm{Hoch}}(\mathcal{U}(L,R),M)) = \{\varphi\in \mathrm{Hom}(\mathcal{U}(L,R)^{\otimes n},M) : \varphi((\mathcal{U}(L,R)^{\otimes n})^{\leq m}) \subset M^{\leq m+k}\,\forall m\},$$
$$F^k(C^{p,q}_{\mathrm{LR}}(L,R;M)) = \{\varphi\in\mathrm{Hom}(\Lambda^p L\otimes R^{\otimes q},M) : \mathrm{im}(\varphi)\subset M^{\leq p+k}\},$$

and

$$F^k(C^n_{\mathrm{nl}}(L,R;M)) = \{(\varphi_0,...,\varphi_n)\in C^n_{\mathrm{LR}}(L,R;M) : \varphi_i\in F^k(C^{i,n-i}_{\mathrm{LR}}(L,R;M))\}.$$

Remark 4.1.27 These filtrations are not exhaustive necessarily (not every map is of finite filtration degree), this turns out to be not a problem cohomologically, as finite degree maps contain all the cohomological information.

Proof of Theorem 4.1.25. Clearly, the shuffle map is a filtered map, so it induces a morphism between the spectral sequences on all sides induced by the filtration. On the Hochschild complex we use that $\mathrm{Sym}_R L$ is the graded quotient algebra of $\mathcal{U}(L,R)$ via the map pbw^∇ under the choice of a connection ∇. In turn, the graded quotient complex of $\mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{U}(L,R),M)$ is $\mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathrm{Sym}_R L, \mathrm{G}M)$ where $\mathrm{G}M$ is the graded quotient of M.

On the side of $\mathrm{C}^\bullet_{\mathrm{nl}}(L,R;M)$ we can play the same trick and obtain that the filtered quotient of this complex is $\mathrm{C}^\bullet_{\mathrm{nl}}(\overline{L},R;\mathrm{G}M)$ where $(\overline{L},R)$ is the Lie-Rinehart algebra with the same underlying algebra R and R-module L, but with vanishing bracket and anchor. This is due to the fact that the Chevalley-Eilenberg differential decreases the filtered degree by 1 and hence vanishes in the filtered quotient. Notice that $\mathcal{U}(\overline{L},R) = \mathrm{Sym}_R L$. Furthermore, the map induced by the shuffle map on the filtered quotient is the shuffle map for the Lie-Rinehart algebra $(\overline{L},R)$.

So we see that by the Spectral Sequence Comparison Theorem [Ze57], we only need to prove this theorem for the case where the bracket and anchor vanish, and the diagonal L-represention on GM is trivial. In this case we see that in the non-linear complex only the horizontal differential, induced by the Hochschild differential on $\mathrm{C}^\bullet_{\mathrm{Hoch}}(R,\mathrm{G}M)$, survives. Investigating the definition of the non-linear complex, the homotopy h in the case where the anchor vanishes, and noting that we know the cohomology of $(\mathrm{C}^\bullet_{\mathrm{Hoch}}(R,\mathrm{G}M),b)$ we obtain that the spectral sequence of $\mathrm{C}^\bullet_{\mathrm{nl}}(L,R;M)$ on the first page looks like

$$E_1 C^\bullet_{\mathrm{nl}}(L,R;M) \cong \left\{ \begin{matrix} (\varphi_0,\dots,\varphi_n), \varphi_i\colon \Lambda^i L \to \Lambda^{\bullet - i}\mathrm{Der}(R)\otimes \mathrm{G}M : \\ \varphi_i(X_1,\dots,fX_i) = f\varphi_i(X_1,\dots,X_i) + (\iota_f(\varphi_{i-1}(X_1,\dots,X_{i-1}))X_i \end{matrix} \right\}$$

On the Hochschild side we remark that we also know what the cohomology of the filtered quotient is, because $\mathrm{Sym}_R L$ is a commutative algebra, so due to the Hochschild-Kostant-Rosenberg Theorem we see that

$$\mathrm{E}_1\mathrm{C}^\bullet_{\mathrm{Hoch}}(L,R;M) \cong \Lambda^\bullet \mathrm{Der}(\mathrm{Sym}_R L)\otimes \mathrm{G}M$$

We remark that on the first page the differential is the Koszul differential of the Lie algebra cohomology associated to the Lie algebra $\mathrm{Der}(\mathrm{Sym}_R L)$. In this formalism, on the first page, the shuffle map takes a multiderivation $\varphi\colon \Lambda^n(\mathrm{Sym}_R L) \to \mathrm{G}M$ and sends it to the sequence $(\varphi_0,\dots,\varphi_n)$ defined by

$$\varphi_i(X_1,\dots,X_i)(f_1,\dots,f_{n-i}) \coloneqq \varphi(X_1,\dots,X_i,f_1,\dots,f_{n-i}).$$

In particular, noting the structure of $\mathrm{Sym}_R L$ as being freely, commutatively and R-linearly generated by the vector space L, we see that the shuffle map induces an isomorphism between the E^2-pages of both spectral sequences, which proves the theorem. □

4.2 Relationship with the adjoint representation

We now turn to the case where $M = \mathcal{U}(L,R)$. As we described at the start of this chapter, we know that for the case where (L,R) arises from a Lie algebroid $A \to M$, the Hochschild cohomology $\mathrm{HH}^\bullet(\mathcal{U}(L,R),\mathcal{U}(L,R))$ is calculated by the symmetric powers of the adjoint representation of $A \to M$. We now give an algebraic proof for this fact, making use of the non-linear complex $\mathrm{C}^\bullet_{\mathrm{nl}}(L,R;\mathcal{U}(L,R))$ we defined before.

In this section, we will make extensive use of different kinds of connections.

Definition 4.2.1 An *L-connection on L* is a map

$$\nabla\colon L \times L \to L$$

satisfying for all $X, Y \in L$ and $f \in R$:

$$\begin{aligned} \nabla_{fX} Y &= f \nabla_X Y, \\ \nabla_X (fY) &= f \nabla_X Y + \rho(X)(f) Y. \end{aligned}$$

A *Der*(R)*-connection on* L is a map

$${}^R\nabla\colon \mathrm{Der}(R) \times L \to L$$

satisfying that for all $X \in L$, $D \in \mathrm{Der}(R)$ and $f \in R$:

$$\begin{aligned} {}^R\nabla_{fD} X &= f\, {}^R\nabla_D X, \\ {}^R\nabla_D (fX) &= f\, {}^R\nabla_D X + D(f) X. \end{aligned}$$

A *L-connection on Der*(R) is a map

$${}^L\nabla\colon L \times \mathrm{Der}(R) \to \mathrm{Der}(R)$$

satisfying that for all $X \in L$, $D \in \mathrm{Der}(R)$ and $f \in R$:

$$\begin{aligned} {}^L\nabla_{fX} D &= f\, {}^L\nabla_X D, \\ {}^L\nabla_X (fD) &= f\, {}^L\nabla_X D + \rho(X)(f) D. \end{aligned}$$

Remark 4.2.2 If (L, R) is induced by a Lie algebroid $A \to M$, then an Der(R)-connection on L is simply a vector bundle connection on the underlying vector bundle $A \to M$.

Lemma 4.2.3 [AC12] If ${}^R\nabla$ is an Der(R)-connection on L, then the map $\nabla\colon L \times L \to L$ defined by

$$\nabla_X Y = \nabla_{\rho(Y)} X + [X, Y]$$

is an L-connection on L. Similarly the map ${}^L\nabla\colon L \times \mathrm{Der}(R) \to \mathrm{Der}(R)$ defined by

$${}^L\nabla_X D = \rho({}^R\nabla_D X) + [\rho(X), D]$$

is an L-connection on Der(R). Both of these will we call the *basic connection*.

Remark 4.2.4 In what follows, we choose an Der(R)-connection ${}^R\nabla$ on L and let ∇ and ${}^L\nabla$ be the induced basic connections. Furthermore, we do recall again at this point that if L is projective as an R-module, then there is a Der(R)-connection on L.

4.2.1 The symmetric powers of the adjoint representation of (L, R)

If (L, R) is a Lie-Rinehart algebra induced by a Lie algebroid $A \to M$, Abad and Crainic [AC12] introduced the adjoint representation up to homotopy of (L, R). This is a local analogue to the adjoint representation up to homotopy for Lie groupoids we discussed in Section 2.5.1. In this case, we interpret

$$A \xrightarrow{\rho} TM$$

as the adjoint complex. A differential is then induced by a choice of a connection on A.

A similar construction can be done for a general Lie-Rinehart algebra (L, R), when L is projective over R. In this case the adjoint complex is

$$L \xrightarrow{\rho} \mathrm{Der}(R).$$

Under the choice of a connection ${}^R\nabla$, the induced complex is then as follows:

Definition 4.2.5 For (L, R) a Lie-Rinehart algebra with $\mathrm{Der}(R)$-connection ${}^R\nabla$ on L with induced basic connection ∇, the adjoint complex $\mathrm{C}^\bullet(L, \mathrm{ad}_\nabla)$ is defined by

$$\mathrm{C}^n(L, \mathrm{ad}_\nabla) = \mathrm{Hom}_R(\Lambda^n L, L) \oplus \mathrm{Hom}_R(\Lambda^{n-1} L, \mathrm{Der}(R))$$

with differential for $c_0 \in \mathrm{Hom}_R(\Lambda^n L, L)$ given by

$$\begin{aligned} d(c_0)_0(X_1, ..., X_{n+1}) =& \sum_{i=1}^{n+1} (-1)^{i+1} \nabla_{X_i}(c_0(X_1, ...\widehat{X_i}..., X_{n+1})) \\ &+ \sum_{i<j} (-1)^{i+j} c_0([X_i, X_j], X_1, ...\widehat{X_i}\widehat{X_j}..., X_{n+1}) \end{aligned}$$

$$d(c_0)_1(X_1, ..., X_n) = \rho(c_0(X_1, ..., X_n))$$

and for $c_1 \in \mathrm{Hom}_R(\Lambda^{n-1} L, \mathrm{Der}(R))$ by

$$d(c_1)_0(X_1, ..., X_{n+1}) = \sum_{i<j} K^{\mathrm{bas}}_{{}^R\nabla}(X_i, X_j)(c_1(X_1, ...\widehat{X_i}\widehat{X_j}..., X_{n+1})$$

$$\begin{aligned} d(c_1)_1(X_1, ..., X_n) =& \sum_{i=1}^{n} (-1)^{i+1}\, {}^L\nabla_{X_i}(c_1(X_1, ...\widehat{X_i}..., X_n)) \\ &+ \sum_{i<j} (-1)^{i+j} c_1([X_i, X_j], X_1, ...\widehat{X_i}\widehat{X_j}..., X_n) \end{aligned}$$

Here $K^{\mathrm{bas}}_{{}^R\nabla} \in \mathrm{Hom}_R(\Lambda^2_R L, \mathrm{Hom}_R(\mathrm{Der}(R), L))$ is the basic curvature of ${}^R\nabla$ defined by

$$K^{\mathrm{bas}}_{{}^R\nabla}(X, Y)(D) := {}^R\nabla_D([X, Y]) - [{}^R\nabla_D X, Y] - [X, {}^R\nabla_D Y] - {}^R\nabla_{{}^L\nabla_Y D} X + {}^R\nabla_{{}^L\nabla_X D} Y.$$

Remark 4.2.6 Similarly to the discussion in the groupoid case, different choices of connections yield isomorphic complexes, see [AC12, Thm 3.11].

Similarly, we can look at

$$\mathrm{Sym}_R^k L \xrightarrow{P} \mathrm{Sym}_R^{k-1} L \otimes \mathrm{Der}(R) \xrightarrow{P} \mathrm{Sym}_R^{k-2} L \otimes \Lambda_R^2 \mathrm{Der}(R) \xrightarrow{P} \cdots \xrightarrow{P} \Lambda_R^k \mathrm{Der}(R)$$

as the complex associated to the k'th symmetric power of the adjoint. Here

$$P\colon \mathrm{Sym}_R^p L \otimes \Lambda_R^q \mathrm{Der}(R) \to \mathrm{Sym}_R^{p-1} L \otimes \Lambda_R^{q+1} \mathrm{Der}(R)$$

is the map

$$P(X_1 \odot \cdots \odot X_p \otimes D_1 \wedge \cdots \wedge D_q) = \sum_{i=1}^{p} X_1 \odot \cdots \widehat{X_i} \cdots \odot X_p \otimes D_1 \wedge \cdots \wedge D_q \wedge \rho(X_i)$$

Again, under the choice of a connection, we can make a complex $\mathrm{C}^\bullet(L, \mathrm{Sym}^k(\mathrm{ad}_\nabla))$ out of this given by

$$\mathrm{C}^n(L, \mathrm{Sym}^k(\mathrm{ad}_\nabla)) = \bigoplus_{i=0}^{k} \mathrm{Hom}_R(\Lambda_R^{n-i} L, \mathrm{Sym}_R^{k-i} L \otimes \Lambda^i \mathrm{Der}(R))$$

with differential induced by ∇.

Similar to the story we saw for the adjoint representation up to homotopy for Lie groupoids in Proposition 2.5.5, there is also a connection independent interpretation of $\mathrm{C}^\bullet(L, \mathrm{ad}_\nabla)$.

Definition 4.2.7 We write $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{ad})$ for the complex

$$\mathrm{C}^n_{\mathrm{def}}(L, \mathrm{ad}L) = \{c_0\colon \Lambda^n L \to L,\ c_1\colon \Lambda^{n-1} L \to \mathrm{Der}(R) : \\ c_0(X_1, ..., fX_n) = fc_0(X_1, ..., X_n) + c_1(X_1, ..., X_{n-1})(f)X_n\}$$

where the differential $d\colon \mathrm{C}^n_{\mathrm{def}}(L, \mathrm{ad}) \to \mathrm{C}^{n+1}_{\mathrm{def}}(L, \mathrm{ad})$ is determined by

$$d(c_0, c_1)_0(X_1, ..., X_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} [X_i, c_0(X_1, ...\widehat{X_i}..., X_{n+1}] \\ + \sum_{1 \le i < j \le n+1} (-1)^{i+j} c_0([X_i, X_j], X_1, ...\widehat{X_i}\widehat{X_j}..., X_{n+1})$$

Proposition 4.2.8 Given a connection ${}^R\nabla$ there is an isomorphism

$$\mathrm{C}^n_{\mathrm{def}}(L, \mathrm{ad}) \to \mathrm{Hom}_R(\Lambda_R^n L, L) \oplus \mathrm{Hom}_R(\Lambda_R^{n-1} L, \mathrm{Der}(R))$$

sending (c_0, c_1) to $(c_L, -c_1)$ where $c_L \in \mathrm{Hom}_R(\Lambda_R^n L, L)$ is given by

$$c_L(X_1, ..., X_n) = c_0(X_1, ..., X_n) + (-1)^{n-1} \sum_{i=1}^{n} (-1)^i \, {}^R\nabla_{c_1(X_1, ...\widehat{X_i}..., X_n)} X_i$$

which intertwines the differential on $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{ad})$ and $\mathrm{C}^\bullet(L, \mathrm{ad}_\nabla)$.

Remark 4.2.9 In the case where the Lie-Rinehart algebra (L, R) is determined by a Lie algebroid $A \to M$, the complex $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{ad})$ is exactly the same as the deformation complex $\mathrm{C}^\bullet_{\mathrm{def}}(A)$ of Section 2.4.1.

Similar to the case $k = 1$, there is also a connection invariant definition of the complex $\mathrm{C}^\bullet(L, \mathrm{Sym}^k(\mathrm{ad}_\nabla))$ for the symmetric powers of the adjoint:

Definition 4.2.10 [AC12, Ex 4.5] We define the complex $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}^k(\mathrm{ad}))$ by

$$\mathrm{C}^n_{\mathrm{def}}(L, \mathrm{Sym}^k(\mathrm{ad})) = \left\{ \begin{matrix} c = (c_0, c_1, ..., c_n); c_i \in \mathrm{Hom}(\Lambda^{n-i}L, \mathrm{Sym}^{k-i}_R L \otimes \Lambda^i_R \mathrm{Der}(R)) \\ c_i(Y_1, ..., fY_{n-i}) - fc_i(Y_1, ..., Y_{n-i}) = \iota_f(c_{i+1}(Y_1, ..., Y_{n-i-1}) \odot Y_{n-i}) \end{matrix} \right\}$$

with differential determined by

$$\begin{aligned} d(c_0, ..., c_n)_0(X_1, ..., X_{n+1}) = &\sum_{i=1}^{n+1} (-1)^{i+1} [X_i, c_0(X_1, ...\widehat{X_i}..., X_{n+1})] \\ &+ \sum_{1 \le i < j \le n} c_0([X_i, X_j], X_1, ...\widehat{X_i}\widehat{X_j}..., X_{n+1}) \end{aligned}$$

Remark 4.2.11 In this context, we will refer to the equation

$$c_i(Y_1, ..., fY_{n-i}) - fc_i(Y_1, ..., Y_{n-i}) = \iota_f(c_{i+1}(Y_1, ..., Y_{n-i-1}) \odot Y_{n-i}) \tag{4.5}$$

as the *symbol equation.* Notice that, under the isomorphism $\mathrm{pbw}^\nabla \colon \mathrm{Sym}_R L \to \mathcal{U}(L, R)$ and the associated inclusion $\mathrm{Sym}_R L \otimes \Lambda^\bullet_R \mathrm{Der}(R) \hookrightarrow C^\bullet_{\mathrm{Hoch}}(R, \mathcal{U}(L, R))$, it can be interpreted as the 'top order term' of the symbol equation (4.4).

Lemma 4.2.12 [Ab08, Thm 2.3.9] Under the choice of a connection ∇ there is an isomorphism between $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}^k(\mathrm{ad}))$ and $\mathrm{C}^\bullet(L, \mathrm{Sym}^k(\mathrm{ad}_\nabla))$.

Remark 4.2.13 To give a full description of the differential in this last complex, we remark that there is a L-module structure on $\mathrm{Sym}^p_R L \otimes \Lambda^q_R \mathrm{Der}(R)$ by the formula

$$\begin{aligned} &[Y, X_1 \odot \cdots \odot X_p \otimes D_1 \wedge \cdots \wedge D_q] = \\ &\qquad = \sum_{i=1}^{p} X_1 \odot \cdots \odot [Y, X_i] \odot \cdots \odot X_p \otimes D_1 \wedge \cdots \wedge D_q \\ &\qquad\qquad + \sum_{i=1}^{q} X_1 \odot \cdots \odot X_p \otimes D_1 \wedge \cdots \wedge [\rho(Y), D_i] \wedge \cdots \wedge D_q \end{aligned} \tag{4.6}$$

With this module structure in hand we obtain a Chevalley Eilenberg differential

$$\partial_{\mathrm{CE}} \colon \mathrm{Hom}(\Lambda^\bullet L, \mathrm{Sym}^p_R L \otimes \Lambda^q_R \mathrm{Der}(R)) \to \mathrm{Hom}(\Lambda^{\bullet+1} L, \mathrm{Sym}^p_R L \otimes \Lambda^q_R \mathrm{Der}(R))$$

A simple, but tedious, calculation with the symbol equation (4.5) then gives the following:

Lemma 4.2.14 The differential in the complex $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}^k(\mathrm{ad}))$ is given by

$$(d(c_0, ..., c_n))_i = \partial_{\mathrm{CE}}(c_i) + (-1)^{n+1} P \circ c_{i-1}.$$

We postpone the proof to Lemma 4.3.4.

4.2.2 From the adjoint representation to the non-linear complex

Next, we want to write down a chain map

$$\Phi_k\colon \mathrm{C}^{\bullet}_{\mathrm{def}}(L, \mathrm{Sym}^k(\mathrm{ad})) \to \mathrm{Tot}(\mathrm{C}^{\bullet;\bullet}_{\mathrm{LR}}(L, R; \mathcal{U}(L, R))).$$

We want to think of this map as implementing pbw^{∇}, in that it should be determined by

$$\Phi_k^{n,0}(c_0, ..., c_n)(X_1, ..., X_n) = \mathrm{pbw}^{\nabla}(c_0(X_1, ..., X_n)) \mod \mathcal{U}(L, R)^{<k}.$$

To write an actual map from this idea, we need to investigate how the Poincaré–Birkhoff–Witt map communicates with L- and R-modules on both $\mathrm{Sym}_R L$ and $\mathcal{U}(L, R)$. Central to this discussion will be the following.

Definition 4.2.15 If ${}^R\nabla$ is a $\mathrm{Der}(R)$-connection on L, then the *friction*

$$\mathrm{Fri}({}^R\nabla)\colon \mathrm{Der}(R) \otimes_R L \to \mathrm{Hom}(L, L)$$

is the R-linear map defined by

$$\mathrm{Fri}({}^R\nabla)(D, Y)(X) = [X, {}^R\nabla_D Y] - {}^R\nabla_{[\rho(X), D]} Y - {}^R\nabla_D [X, Y].$$

Similarly, if ∇ is an L-connection on L, the *friction*

$$\mathrm{Fri}(\nabla)\colon L \otimes_R L \to \mathrm{Hom}(L, L)$$

is the R-linear map defined by

$$\mathrm{Fri}(\nabla)(Y, Z)(X) = [X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X, Z].$$

In both cases, the R-module structure of $\mathrm{Hom}(L, L)$ is given by

$$(f\varphi)(X) \coloneqq f(\varphi(X)).$$

Lemma 4.2.16 Let ${}^R\nabla$ be a $\mathrm{Der}(R)$-connection on L with induced basic connection ∇. If $\mathrm{Fri}({}^R\nabla) = 0$, then $\mathrm{Fri}(\nabla) = 0$.

Proof. This is an explicit calculation:

$$\begin{aligned}
\mathrm{Fri}(\nabla)(Y, Z)(X) =& [X, \nabla_Y Z] - \nabla_{[X,Y]} Z - \nabla_Y [X, Z] \\
=& [X, {}^R\nabla_{\rho(Z)} Y] + [X, [Y, Z]] - {}^R\nabla_{\rho(Z)} [X, Y] \\
& - [[X, Y], Z] - {}^R\nabla_{[\rho(X), \rho(Z)]} Y - [Y, [X, Z]] \\
=& \mathrm{Fri}({}^R\nabla)(\rho(Z), Y)(X)
\end{aligned}$$

□

Remark 4.2.17 As we shall see below, the friction of the connection ∇ is an obstruction to pbw^∇ being an intertwiner of L-representations[2]. In a previous version, we mistakenly recognized the torsion of the connection in the defining formula. Requiring that the connection would be torsion-free would not be too much of a stretch, as they always exist. Whether friction-free connections exists is something we have not yet had the time to consider.

Now, in Example 4.1.2 we used the fact that the PBW-map in the context of Lie algebras was an isomorphism of Lie algebra representations to write down an explicit quasi-isomorphism. We discuss how this generalizes to the Lie-Rinehart setting. First, we remark that both $\mathrm{Sym}_R L$ and $\mathcal{U}(L,R)$ carry the structure on an L-representation. On $\mathrm{Sym}^k_R L$ it is given by

$$[X, Y_1 \odot \cdots \odot Y_k] := \sum_{i=1}^{k} Y_1 \odot \cdots \odot [X, Y_i] \odot \cdots \odot Y_k,$$

for $k > 0$ and

$$[X, f] = \rho(X)(f)$$

for $k = 0$, while on $\mathcal{U}(L,R)$ it is given by

$$[X, D] = XD - DX.$$

The role of torsion is then enlightened by the following proposition:

Proposition 4.2.18 If the connection ∇ is friction-free, the map $\mathrm{pbw}^\nabla\colon \mathrm{Sym}_R L \to \mathcal{U}(L,R)$ is an isomorphism of L-representations.

Proof. If ∇ is friction-free, we show that pbw^∇ is an isomorphism of L-representations by induction on the degree k in $\mathrm{Sym}^k_R L$. For $k = 0, 1$ this is immediate (irrespective of the connection) by the definition of pbw^∇. Then, for $k \geq 2$ we assume that $\mathrm{pbw}^\nabla\colon \mathrm{Sym}^{<k}_R L \to \mathcal{U}(L,R)$ is a map of L-representations. Then to see that pbw^∇ is a map of L-representations when restricted to $\mathrm{Sym}^{\leq k}_R L$ is a map of L-representation, we check it for a homogeneous element $Y_1 \odot \cdots \odot Y_k$ of $\mathrm{Sym}^k_R L$. Starting with the recursive

[2]Hence the name friction, which is an obstruction to the physical process of intertwining (or knotting) pieces of rope.

definition of pbw^∇ and the induction hypothesis we obtain:

$$\begin{aligned}[X, \mathrm{pbw}^\nabla(Y_1 \odot \cdots \odot Y_k)] =& \frac{1}{k}\sum_{i=1}^{k}[X, Y_i \mathrm{pbw}^\nabla(Y_1 \odot \cdots \widehat{Y_i} \cdots \odot Y_k)] \\ &- \frac{1}{k}\sum_{1\leq i\neq j\leq k}[X, \mathrm{pbw}^\nabla(\nabla_{Y_i}Y_j \odot Y_1 \odot \cdots \widehat{Y_i}\widehat{Y_j} \cdots \odot Y_k)] \\ =& \frac{1}{k}\sum_{i=1}^{k}[X, Y_i]\mathrm{pbw}^\nabla(Y_1 \odot \cdots \widehat{Y_i} \cdots \odot Y_k) \\ &+ \frac{1}{k}\sum_{1\leq i\neq j\leq k} Y_j \mathrm{pbw}^\nabla([X, Y_i] \odot Y_1 \odot \cdots \widehat{Y_i}\widehat{Y_j} \cdots \odot Y_k) \\ &- \frac{1}{k}\sum_{1\leq i\neq j\leq k} \mathrm{pbw}^\nabla([X, \nabla_{Y_i}Y_j] \odot Y_1 \odot \cdots \widehat{Y_i}\widehat{Y_j} \cdots \odot Y_k) \\ &- \frac{1}{k}\sum_{1\leq i\neq j\neq n\leq k} \mathrm{pbw}^\nabla(\nabla_{Y_j}Y_n \odot [X, Y_i] \odot Y_1 \odot \cdots \widehat{Y_i}\widehat{Y_j}\widehat{Y_n} \cdots \odot Y_k)\end{aligned}$$

Then reversing the recursive definition of pbw^∇ we obtain

$$\begin{aligned}[X, \mathrm{pbw}^\nabla(Y_1 \odot \cdots \odot Y_k)] =& \mathrm{pbw}^\nabla([X, Y_1 \odot \cdots \odot Y_k]) \\ &- \frac{1}{k}\sum_{1\leq i\neq j\leq k} \mathrm{pbw}^\nabla(\mathrm{Fri}(\nabla)(Y_i, Y_j)(X) \odot Y_1 \odot \cdots \widehat{Y_i}\widehat{Y_j} \cdots \odot Y_k).\end{aligned}$$

□

In the case where ${}^R\nabla$ is not torsion-free and ∇ is its basic connection, it does hold that for pbw^∇ is a map of L-representations up to lower order terms, and there is recursive definition of the correction terms.

Using this fact, we can give a proof of the following Theorem in the torsion-free case. This Theorem is true in any case, but the proof simplifies significantly in the torsion-free case.

Theorem 4.2.19 Let ${}^R\nabla$ be a $\mathrm{Der}(R)$-connection on L with basic connection ∇. There is a collection of maps

$$\Phi_k^{p,q} \colon \mathrm{C}^{p+q}_{\mathrm{def}}(L, \mathrm{Sym}^k(\mathrm{ad})) \to \mathrm{C}^{p,q}_{\mathrm{LR}}(L, R; \mathcal{U}(L, R))$$

with the following properties

- $\Phi_k^{p,q} \circ d = d^h \circ \Phi_k^{p-1,q} + d^v \circ \Phi^{p,q-1}$
- $\Phi^{0,0} \colon \mathrm{Sym}^k_R L \to \mathcal{U}(L, R)$ equals pbw^∇
- $\Phi_k^{p,q}(c_0, ..., c_{p+q})$ only depends on $(c_q, ..., c_{p+q})$

- $(\Phi_k^{p,q}c)(X_1, ..., X_p, f_1, ..., f_q) - \frac{1}{q!}\mathrm{pbw}^\nabla(\iota_{(f_1,...,f_q)}c_q(X_1, ..., X_p)) \in \mathcal{U}(L,R)^{<k-q}$

Proof in the friction-free case. In the friction-free case, this boils down to the fact that there exist maps

$$\eta_k^i \colon \mathrm{Sym}_R^{k-i} L \otimes_R \Lambda_R^i \mathrm{Der}(R) \to \mathrm{Hom}(R^{\otimes i}, \mathcal{U}(L,R)^{\leq k-i})$$

satisfying

1. $\eta_k^0 = \mathrm{pbw}^\nabla$;
2. η_k^i is an intertwiner of the L-representations on $\mathrm{Sym}_R^{k-i} L \otimes \Lambda_R^i \mathrm{Der}(R)$ (as given by (4.6)) and $\mathrm{Hom}(R^{\otimes i}, \mathcal{U}(L,R))$ (as given in (4.3));
3. $b \circ \eta_k^i = (-1)^{i+1}\eta_k^{i+1} \circ P$;
4. For every $\mathcal{X} \in \mathrm{Sym}_R^{k-i} L$ and $\mathcal{D} \in \Lambda_R^i \mathrm{Der}(R)$ it holds that

$$\eta_k^i(\mathcal{X} \otimes \mathcal{D})(f_1, .., .f_i) - \frac{1}{i!}\mathrm{pbw}^\nabla((\iota_{(f_1,...,f_i)}\mathcal{D})\mathcal{X}) \in \mathcal{U}(L,R)^{<k-i}.$$

Note that in particular, we have that η_k^1 describes the deficiency of pbw^∇ to preserve the right R-module structure:

$$[\mathrm{pbw}^\nabla(\mathcal{X}), f] = \eta_k^1(P(\mathcal{X}))(f).$$

The existence of such η's can be shown recursively by the following relations:

$$\eta_k^k(D_1 \wedge \cdots \wedge D_k)(f_1, ..., f_k) = \frac{1}{k!}\sum_{\sigma \in S_k}(-1)^\sigma D_1(f_{\sigma(1)}) \cdots D_k(f_{\sigma(k)});$$

and for $i > k$:

$$\eta_k^i(Y_1 \odot \cdots \odot Y_{k-i} \otimes D_1 \wedge \cdots \wedge D_i)(f_1, ..., f_i) =$$

$$\begin{aligned}
= \frac{1}{k}\sum_{m=1}^{i}(-1)^{m+1}D_m(f_1)\eta_{k-1}^{i-1}(Y_1 \odot \cdots \odot Y_{k-i} \otimes D_1 \wedge \cdots \widehat{D_m} \cdots \wedge D_i)(f_2, ..., f_i) \\
+ \frac{1}{k}\sum_{m=1}^{k-i} Y_m \eta_{k-1}^i(Y_1 \odot \cdots \widehat{Y_m} \cdots \odot Y_{k-i} \otimes D_1 \wedge \cdots \wedge D_i)(f_1, ..., f_i) \\
- \frac{1}{k}\sum_{m=1}^{k-1}\sum_{n=1}^{i}\eta_{k-1}^i(Y_1 \odot \cdots \widehat{Y_m} \cdots \odot Y_{k-1} \otimes D_1 \wedge \cdots \wedge \rho({}^R\nabla_{D_n}(Y_m)) \wedge \cdots \wedge D_i)(f_1, ..., f_i) \\
- \frac{1}{k}\sum_{m \neq n}\eta_{k-1}^i(\nabla_{Y_m}Y_n \odot Y_1 \odot \cdots \widehat{Y_m}\widehat{Y_n} \cdots \odot Y_{k-i} \otimes D_1 \wedge \cdots \wedge D_i)(f_1, ..., f_i).
\end{aligned}$$

Using the η's, we write down our chain map $\Phi_k^{\bullet,\bullet}$ by

$$\Phi_k^{p,q}(c_0, ..., c_{p+q}) = \eta_k^q(c_q(X_1, ..., X_p)).$$

The fact that this is a chain map is a simple corollary of the properties of the η's. Indeed, the second property of the η's means that the $d^h \circ \Phi$-terms correspond to the $\Phi \circ \partial_{\mathrm{CE}}$-terms and the third property of the η's make sure that the $d^v \circ \Phi$-terms corresponds to the $\Phi \circ P$-terms (including the correct signs). □

Remark 4.2.20 If ${}^R\nabla$ is not friction-free, the second property of the η's only works up to lower order terms, and we need to correct for that. The correction terms lead to mixing of terms, so that $\Phi^{p,q}(c_0, ..., c_{p+q})$ not only depends on c_q but also on $c_{q+1}, ..., c_{p+q}$.

Using the Φ_k's we constructed in the proof of the previous theorem, we can also obtain one chain map $\Phi\colon \mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad})) \to \mathrm{C}^\bullet_{\mathrm{nl}}(L, R; \mathcal{U}(L, R))$, simply by setting $\Phi = \sum_{k\geq 0} \Phi_k$

Theorem 4.2.21 If R is smooth and L is a projective R-module, the map

$$\Phi\colon \mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad})) \to \mathrm{C}^\bullet_{\mathrm{LR}}(L, R; \mathcal{U}(L, R))$$

defines a chain of quasi-isomorphisms

$$\mathrm{C}^\bullet_{\mathrm{nl}}(L, R; \mathcal{U}(L, R)) \xleftarrow{\Phi} \Phi^{-1}(\mathrm{C}^\bullet_{\mathrm{nl}}(L, R; \mathcal{U}(L, R))) \xrightarrow{\iota} \mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad})).$$

Proof. Similar to Theorem 4.1.25, we prove this with a filtration argument. In this case we put the following, slightly unintuitive, filtration on $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad}))$:

$$F^k(\mathrm{C}^n_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad}))) = \{(c_0, ..., c_n) : c_i(\Lambda^{n-i} L) \subset \mathrm{Sym}^{n-i+k}_R L \otimes \Lambda^i_R \mathrm{Der}(R)\}.$$

We note that this is essentially the filtration induced by the grading $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad})) = \bigoplus_{k\geq 0} \mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}^k(\mathrm{ad}))$, but we shift it with a filtration degree n in cohomological degree n. This has the effect that in the graded quotient, the differential vanishes (indeed, it only shows up when going to the second page, and the spectral sequence collapses on the third), and we have the following first page:

$$\mathrm{E}_1 \mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad})) = \mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad}))$$

We also recall the first page of $\mathrm{C}^\bullet_{\mathrm{nl}}(L, R; \mathcal{U}(L, R))$ as we saw in the proof of Theorem 4.1.25.

$$\mathrm{E}_1 \mathrm{C}^\bullet_{\mathrm{nl}}(L, R; \mathcal{U}(L, R)) = \left\{ \begin{matrix} c = (\varphi_0, \varphi_1, ..., \varphi_n); \varphi_i \in \mathrm{Hom}(\Lambda^i L, \mathrm{Sym}_R L \otimes \Lambda^{\bullet - i}_R \mathrm{Der}(R)) \\ \varphi_i(X_1, ..., fX_i) - f\varphi_i(X_1, ..., X_i) = \iota_f(\varphi_{i-1}(X_1, ..., X_{i-1}) \odot X_{n-i}) \end{matrix} \right\}$$

Also we note that since the map Φ is up to top order given by applying pbw^∇ and the inclusion of $\Lambda^i_R \mathrm{Der}(R)$ into $\mathrm{Hom}(R^{\otimes \bullet}, R)$, we see that under these isomorphisms, Φ induces the identity between the two second pages. By the Spectral Sequence Comparison Theorem, this shows our theorem. □

Combining Theorem 4.1.25 and Theorem 4.2.21, we infer the main result of this section.

Theorem 4.2.22 If (L, R) be a Lie-Rinehart algebra, where L is projective as an R-module and R is smooth, then the complexes $\mathrm{C}^\bullet_{\mathrm{Hoch}}(\mathcal{U}(L,R), \mathcal{U}(L,R))$ and $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad}))$ are quasi-isomorphic. In particular there is a natural isomorphism

$$\mathrm{HH}^\bullet(\mathcal{U}(L,R), \mathcal{U}(L,R)) \cong \mathrm{H}^\bullet(L, \mathrm{Sym}(\mathrm{ad})).$$

Remark 4.2.23 When (L, R) arises from a Lie algebroid $A \to M$, there is another intrinsic way to think about $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad}))$. Indeed, for $k = 1$, we have seen in Section 2.4.1 that the complex $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{ad})$ is isomorphic to the linear Poisson complex of the linear Poisson manifold A^*. In similar vein the complex $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad}))$ calculates the polynomial Poisson complex of the A^*, i.e. the Poisson complex of those multivector fields which preserve fibrewise polynomial functions on A^*. The grading k here corresponds to those multivector fields that eat fibrewise linear functions on A^* and spit out fibrewise homogeneous polynomials of degree k.

Remark 4.2.24 Due to Lambre and Le Meur [LLM18], the universal enveloping algebra $\mathcal{U}(L, R)$ exhibits Van den Bergh-duality in the sense of [VdB98], with dualising module

$$C = \Lambda_R^{\mathrm{top}} L \otimes_R \Lambda_R^{\mathrm{top}} \mathrm{Der}(R).$$

In particular, there is a Van den Bergh-isomorphism

$$\mathrm{HH}^\bullet(\mathcal{U}(L,R), M) \cong \mathrm{HH}_{n-\bullet}(\mathcal{U}(L,R), C \otimes_{\mathcal{U}(L,R)} M).$$

This means that we can alternatively understand Hochschild cohomology via understanding Hochschild homology. In the unimodular case, where C is the trivial representation, this means that we can directly calculate Hochschild cohomology from Hochschild homology and vice versa.

Using Van den Bergh-duality we can resolve the problem with the exhaustiveness of the filtration we remarked upon in 4.1.27, since the dual filtrations on tensor powers of $\mathcal{U}(L, R)$ are exhaustive.

4.2.3 Examples

We discuss some examples to see how our calculations relate to known cases.

Example 4.2.25 (Lie algebras) If $R = \mathbb{K}$ is a field, then L is simply a Lie algebra over $\mathbb{K}$. In this case, the double complex $\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L, \mathbb{K}; \mathcal{U}(L))$ is given by

$$\mathrm{C}^{p,q}_{\mathrm{LR}}(L, \mathbb{K}, \mathcal{U}(L)) = \mathrm{Hom}(\Lambda^p L, \mathcal{U}(L))$$

with the differential in the vertical direction being alternatively zero and the identity. In particular we may replace the total complex simply by its first row $\mathrm{C}^{\bullet,0}_{\mathrm{LR}}(L, \mathbb{K}, \mathcal{U}(L))$.

Also note that as everything is automatically R-linear, the non-linear complex is the whole total complex.

Similarly, the complex $\mathrm{C}^{\bullet}_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad}))$ is simply the Lie algebra cohomology complex for the symmetric powers of the adjoint representation of the Lie algebra. In turn, the map $\Phi\colon \mathrm{C}^{\bullet}_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad})) \to \mathrm{C}^{\bullet,0}_{\mathrm{LR}}(L, \mathbb{K}, \mathcal{U}(L))$ is easily seen to be invertible (it is simply given by composition with pbw).

We find that the whole picture fits nicely into the picture of Example 4.1.2 in that the following diagram commutes

$$
\begin{array}{ccccc}
 & & \mathrm{C}^{\bullet,0}_{\mathrm{LR}}(L, \mathbb{K}, \mathcal{U}(L)) & & \\
 & \overset{\simeq}{\nearrow} & & \underset{\Phi_{(4.2.19)}}{\overset{\cong}{\nwarrow}} & \\
\mathrm{C}^{\bullet}_{\mathrm{Hoch}}(\mathcal{U}(L), \mathcal{U}(L)) & & \xrightarrow{\quad \Phi_{(4.1.2)} \quad} & & \mathrm{C}^{\bullet}_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad}))
\end{array}
$$

In particular, our construction is an honest generalization of the construction in Example 4.1.2.

Example 4.2.26 (Abelian Lie-Rinehart algebras) If L is an R-module with vanishing bracket and vanishing anchor, we already saw the calculation of our procedure in the proofs of Theorem 4.1.25 and Theorem 4.2.21. In this case the universal enveloping algebra $\mathcal{U}(L, R)$ is isomorphic as an algebra to $\mathrm{Sym}_R L$, so that by the Hochschild-Kostant-Rosenberg Theorem we know that the canonical map

$$\Lambda^{\bullet}_{\mathrm{Sym}_R L}\mathrm{Der}(\mathrm{Sym}_R L) \to \mathrm{HH}^{\bullet}(\mathrm{Sym}_R L, \mathrm{Sym}_R L)$$

is an isomorphism. Furthermore, it is clear that by restriction one obtains an isomorphism

$$\Lambda^{\bullet}_{\mathrm{Sym}_R L}\mathrm{Der}(\mathrm{Sym}_R L) \xrightarrow{\cong} \mathrm{C}^{\bullet}_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad})).$$

Since in this case the differential on the complex $\mathrm{C}^{\bullet}_{\mathrm{def}}(L, \mathrm{Sym}(\mathrm{ad}))$ vanishes, we obtain a chain of isomorphisms

$$\mathrm{HH}^{\bullet}(\mathrm{Sym}_R L, \mathrm{Sym}_R L) \xleftarrow{\cong} \Lambda^{\bullet}_{\mathrm{Sym}_R L}\mathrm{Der}(\mathrm{Sym}_R L) \xrightarrow{\cong} \mathrm{H}^{\bullet}(L, \mathrm{Sym}(\mathrm{ad})).$$

It is easy to see that by the explicit forms of the maps involved, the isomorphisms we obtain via the non-linear complex fit into a commutative diagram

$$
\begin{array}{ccccc}
 & & \mathrm{H}^{\bullet}_{\mathrm{nl}}(L, R; \mathrm{Sym}_R L) & & \\
 & \overset{\cong}{\nearrow} & & \overset{\cong}{\nwarrow} & \\
\mathrm{HH}^{\bullet}(\mathrm{Sym}_R L, \mathrm{Sym}_R L) & & & & \mathrm{H}^{\bullet}(L, \mathrm{Sym}(\mathrm{ad})) \\
 & \overset{\cong}{\nwarrow} & & \overset{\cong}{\nearrow} & \\
 & & \Lambda^{\bullet}_{\mathrm{Sym}_R L}\mathrm{Der}(\mathrm{Sym}_R L) & &
\end{array}
$$

so that we see that the isomorphism we obtain is exactly the isomorphism induced by the Hochschild-Kostant-Rosenberg Theorem.

Example 4.2.27 (Differential operators) In the case where we look at the Lie-Rinehart algebra induced by the Lie algebroid $TM \to M$, we have already remarked a few times that the universal enveloping algebra $\mathcal{U}(\mathfrak{X}(M), C^\infty(M))$ is isomorphic to the algebra $\mathrm{Diff}(M)$ of differential operators on M. In this case, we complex $C^\bullet_{\mathrm{def}}(L, \mathrm{SymAd}L)$ calculates the polynomial Poisson cohomology of T^*M, in particular it calculates the Poisson cohomology of the space of symbols on M.

In turn, our procedure induces an isomorphism

$$HH^\bullet(\mathrm{Diff}(M), \mathrm{Diff}(M)) \cong H^\bullet_{\mathrm{Pois}}(\mathrm{Poly}(T^*M)).$$

Using the fact that the graded quotient of $\mathrm{Diff}(M)$ is $\mathrm{Poly}(T^*M)$ via the principle symbol map, we see that the isomorphism associates to a Hochschild cocycle $c \in C^n_{\mathrm{Hoch}}(\mathrm{Diff}(M), \mathrm{Diff}(M))$ a Poisson cocycle $\widetilde{c}\colon \Lambda^n \mathrm{Poly}(T^*M) \to \mathrm{Poly}(T^*M)$ in such a way that up to lower terms we have

$$\widetilde{c}(\sigma(D_1), ..., \sigma(D_n)) = \sum_{\tau \in S_n} (-1)^\tau \sigma(c(D_{\tau(1)}, ..., D_{\tau(n)}))$$

where $\sigma\colon \mathrm{Diff}(M) \to \mathrm{Poly}(T^*M)$ is the principal symbol map.

Remark 4.2.28 Continuing on the previous example, there is a way in which one can think of this result (and in particular they way it is proven) as a starting point to calculate the Hochschild cohomology of the full symbol algebra of a Lie algebroid. In the case where (L, R) is induced by a Lie groupoid $\mathcal{G} \rightrightarrows M$ via its Lie algebroid $A(\mathcal{G}) \to M$, we can, following Nistor, Weinstein and Xu [NWX99], see $\mathcal{U}(L, R)$ as the algebra of invariant differential operators on $\mathcal{G}$. Symbols of such differential operators can, by invariance, seen as polynomial functions on $A(\mathcal{G})^*$, i.e. elements of $\mathrm{Sym}_R L$ and the PBW-map can be thought of as the inverse of the symbol map.

Nistor, Weinstein and Xu also define a class of invariant pseudodifferential operators on $\mathcal{G}$, leading to an algebra $\Psi^\infty(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})$ called the full symbol algebra. Here $\Psi^\infty(\mathcal{G})$ stands for (invariant) pseudodifferential operators on $\mathcal{G}$ of any order, and $\Psi^{-\infty}(\mathcal{G})$ for the invariant smoothing operators. Via their symbol, this algebra can be thought of as a deformation quantization of the Poisson algebra $\mathcal{S}(A(\mathcal{G})^*) \subset C^\infty(A(\mathcal{G}) \backslash M)$ of symbols. We remark that the Poisson bracket on this algebra of symbols $\mathcal{S}(A(\mathcal{G})^*)$ is induced by the Poisson structure on $A(\mathcal{G})^*$.

An extension of our work here to this case would be the result, alluded to in work of Benameur and Nistor [BN03]:

Conjecture 4.2.29 There is a natural isomorphism

$$\mathrm{HH}^\bullet(\Psi^\infty(\mathcal{G})/\Psi^{-\infty}(\mathcal{G}), \Psi^\infty(\mathcal{G})/\Psi^{-\infty}(\mathcal{G})) \cong \mathrm{H}_{\mathrm{Pois}}(\mathcal{S}(A(\mathcal{G})^*)).$$

4.3 Remaining proofs

Proposition 4.3.1 (4.1.16) The following equation holds true for every $f \in R$ and $X \in L$:

$$L_{fX} - fL_X = b \circ h_{f,X} + h_{f,X} \circ b$$

Proof. For the sake of clarity, let us define $h^1_{f,X,i}\colon \mathrm{C}^q_{\mathrm{Hoch}}(R,M) \to \mathrm{C}^{q-1}_{\mathrm{Hoch}}(R,M)$ for $1 \leq i \leq q$ by

$$(h^1_{f,X,i}\varphi)(f_1,...,f_{q-1}) = \varphi(f_1,...,f_{i-1},f,f_i,...,f_q)X$$

and $h^2_{f,X,i,j}\colon \mathrm{C}^q_{\mathrm{Hoch}}(R,M) \to \mathrm{C}^{q-1}_{\mathrm{Hoch}}(R,M)$ for $1 \leq i \leq j \leq q-1$ by

$$(h^2_{f,X,i,j}\varphi)(f_1,...,f_{q-1}) = \varphi(f_1,...,f_{i-1},f,f_i,...,\rho(X)f_j,...,f_{q-1})$$

so that

$$h_{f,X} = \sum_{i=1}^{q}(-1)^{i+1}h^1_{f,X,i} + \sum_{1\leq i\leq j\leq q-1}(-1)^{i+1}h^2_{f,X,i,j}$$

We also remind ourselves of the simplicial maps $d_i\colon \mathrm{C}^q_{\mathrm{Hoch}}(R,M) \to \mathrm{C}^{q+1}_{\mathrm{Hoch}}(R,M)$ for $0 \leq i \leq q+1$ defined by

$$\begin{aligned}(d_0\varphi)(f_1,...,f_{q+1}) &= f_1\varphi(f_2,...,f_{q+1})\\ (d_i\varphi)(f_1,...,f_{q+1}) &= \varphi(f_1,...,f_if_{i+1},...,f_{q+1})\\ (d_{q+1}\varphi)(f_1,...,f_{q+1}) &= \varphi(f_1,...,f_q)f_{q+1}\end{aligned}$$

so that

$$b = \sum_{i=0}^{q}(-1)^i d_i$$

In the end we are interested in

$$bh_{f,X} = \sum_{k=0}^{q}\sum_{i=1}^{q}(-1)^{k+i+1}d_kh^1_{f,X,i} + \sum_{k=0}^{q}\sum_{1\leq i\leq j\leq q-1}(-1)^{k+i+1}d_kh^2_{f,X,i,j}$$

and

$$h_{f,X}b = \sum_{i=1}^{q+1}\sum_{k=0}^{q+1}(-1)^{k+i+1}h^1_{f,X,i}d_k + \sum_{1\leq i\leq j\leq q}\sum_{k=0}^{q+1}(-1)^{k+i+1}h^2_{f,X,i,j}d_k$$

Writing out all the terms we have, for $bh_{f,X}$

- For $1 \leq i \leq q$

$$(d_0h^1_{f,X,i}\varphi)(f_1,...,f_q) = f_1\varphi(f_2,...,f_i,f,f_{i+1},...,f_q)X \tag{1}$$

- For $2 \leq i \leq q$ and $1 \leq k \leq i-1$

$$(d_kh^1_{f,X,i}\varphi)(f_1,...,f_q) = \varphi(f_1,...,f_kf_{k+1},...,f_i,f,f_{i+1},...,f_q)X \tag{2}$$

- For $1 \leq i \leq q-1$ and $i \leq k \leq q-1$

$$(d_kh^1_{f,X,i}\varphi)(f_1,...,f_q) = \varphi(f_1,...,f_{i-1},f,f_i,...,f_kf_{k+1},...,f_q)X \tag{3}$$

- For $1 \leq i \leq q$

$$(d_q h^1_{f,X,i}\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., f_{q-1})Xf_q \tag{4}$$

- For $1 \leq i \leq q-1$ and $i \leq j \leq q-1$

$$(d_0 h^2_{f,X,i,j}\varphi)(f_1, ..., f_q) = f_1\varphi(f_2, ..., f_i, f, f_{i+1}, ..., \rho(X)f_{j+1}, ..., f_q) \tag{5}$$

- For $2 \leq i \leq q-1$, $i \leq j \leq q-1$ and $1 \leq k \leq i-1$

$$(d_k h^2_{f,X,i,j}\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_k f_{k+1}, ..., f_i, f, f_{i+1}, ..., \rho(X)(f_{j+1}), ..., f_q) \tag{6}$$

- For $1 \leq i \leq q-2$, $i+1 \leq j \leq q-1$, $i \leq k \leq j-1$

$$(d_k h^2_{f,X,i,j}\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., f_k f_{k+1}, ..., \rho(X)(f_{j+1}), ..., f_q) \tag{7}$$

- For $1 \leq i \leq q-1$, $i \leq k \leq q-1$

$$(d_k h^2_{f,X,i,k}\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., \rho(X)(f_k f_{k+1}), ..., f_q) \tag{8}$$

- For $1 \leq i \leq q-2$, $i \leq j \leq q-2$, $j+1 \leq k \leq q-1$

$$(d_k h^2_{f,X,i,j}\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., \rho(X)f_j, ..., f_k f_{k+1}, ..., f_q) \tag{9}$$

- For $1 \leq i \leq q-1$, $i \leq j \leq q-1$

$$(d_q h^2_{f,X,i,j}\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., \rho(X)f_j, ..., f_{q-1})f_q \tag{10}$$

and for $h_{f,X}b$

-

$$(h^1_{f,X,0}d_0\varphi)(f_1, ..., f_q) = f\varphi(f_1, ..., f_q)X \tag{11}$$

- For $2 \leq i \leq q+1$

$$(h^1_{f,X,i}d_0\varphi)(f_1, ..., f_q) = f_1\varphi(f_2, ..., f_{i-1}, f, f_i, ..., f_q)X \tag{12}$$

- For $3 \leq i \leq q+1$ and $1 \leq k \leq i-2$

$$(h^1_{f,X,i}d_k\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_k f_{k+1}, ..., f_{i-1}, f, f_i, ..., f_q)X \tag{13}$$

- For $2 \leq i \leq q+1$

$$(h^1_{f,X,i}d_{i-1}\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}f, ..., f_q)X \tag{14}$$

- For $1 \leq i \leq q$
$$(h^1_{f,X,i} d_i \varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f f_i, ..., f_q)X \tag{15}$$
- For $1 \leq i \leq q-1$ and $i+1 \leq k \leq q$
$$(h^1_{f,X,i} d_k \varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., f_{k-1} f_k, ..., f_q)X \tag{16}$$
- For $1 \leq i \leq q$
$$(h^1_{f,X,i} d_{q+1} \varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., f_{q-1}) f_q X \tag{17}$$
- $$(h^1_{f,X,q+1} d_{q+1} \varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_q) f X \tag{18}$$
- For $1 \leq j \leq q$
$$(h^2_{f,X,1,j} d_0 \varphi)(f_1, ..., f_q) = f \varphi(f_1, ..., \rho(X) f_j, ..., f_q) \tag{19}$$
- For $2 \leq i \leq q$ and $i \leq j \leq q$
$$(h^2_{f,X,i,j} d_0 \varphi)(f_1, ..., f_q) = f_1 \varphi(f_2, ..., f_{i-1}, f, f_i, ..., \rho(X) f_j, ..., f_q) \tag{20}$$
- For $3 \leq i \leq q$, $i \leq j \leq q$ and $1 \leq k \leq i-2$
$$(h^2_{f,X,i,j} d_k \varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_k f_{k-1}, ..., f_{i-1}, f, f_i, ..., \rho(X) f_j, ..., f_q) \tag{21}$$
- For $2 \leq i \leq q$ and $i \leq j \leq q$
$$(h^2_{f,X,i,j} d_{i-1} \varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1} f, f_i, ..., \rho(X) f_j, ..., f_q) \tag{22}$$
- For $1 \leq i \leq q$
$$(h^2_{f,X,i,i} d_i \varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f \rho(X)(f_i), ..., f_q) \tag{23}$$
- For $1 \leq i \leq q-1$ and $i+1 \leq j \leq q$
$$(h^2_{f,X,i,j} d_i \varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f f_i, \rho(X) f_j, ..., f_q) \tag{24}$$
- For $1 \leq i \leq q-2$, $i+2 \leq j \leq q$ and $i+1 \leq k \leq j-1$
$$(h^2_{f,X,i,j} d_k \varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., f_{k-1} f_k, ..., \rho(X) f_j, ..., f_q) \tag{25}$$
- For $1 \leq i \leq q-1$ and $i+1 \leq j \leq q$
$$(h^2_{f,X,i,j} d_j \varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., f_{j-1} \rho(X) f_j, ..., f_q) \tag{26}$$

- For $1 \leq i \leq q-1$ and $i \leq j \leq q-1$

$$(h^2_{f,X,i,j} d_{j+1}\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., (\rho(X)f_j)f_{j+1}, ..., f_q) \tag{27}$$

- For $1 \leq i \leq q-2$, $i \leq j \leq q-2$ and $j+2 \leq k \leq q$

$$(h^2_{f,X,i,j} d_k\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., \rho(X)f_j, ..., f_{k-1}f_k, ..., f_q) \tag{28}$$

- For $1 \leq i \leq q-1$, $i \leq j \leq q-1$

$$(h^2_{f,X,i,j} d_{q+1}\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., \rho(X)f_j, ..., f_{q-1})f_q \tag{29}$$

- For $1 \leq i \leq q$

$$(h^2_{f,X,i,q} d_{q+1}\varphi)(f_1, ..., f_q) = \varphi(f_1, ..., f_{i-1}, f, f_i, ..., f_{q-1})\rho(X)f_q \tag{30}$$

Now playing a game of match, we see the following:

- Matching (1) and (12), for $1 \leq i \leq q$:

$$d_0 h^1_{f,X,i} = h^1_{f,X,i+1} d_0;$$

- Matching (2) and (13), for $2 \leq i \leq q$ and $1 \leq k \leq i-1$:

$$d_k h^1_{f,X,i} = h^1_{f,X,i+1} d_k;$$

- Matching (3) and (16), for $1 \leq i \leq q-1$ and $i \leq k \leq q-1$:

$$d_k h^1_{f,X,i} = h^1_{f,X,i} d_{k-1};$$

- Matching (4) with (17) and (30), using that in $\mathcal{U}(L,R)$ we have $[X, f_q] = \rho(X)f_q$, for $1 \leq i \leq q$:

$$d_q h^1_{f,X,i} = h^1_{f,X,i} d_{q+1} + h^2_{f,X,i,q} d_{q+1};$$

- Matching (5) and (20), for $1 \leq i \leq q-1$ and $i \leq j \leq q-1$:

$$d_0 h^2_{f,X,i,j} = h^2_{f,X,i-1,j-1} d_0;$$

- Matching (6) and (21), for $2 \leq i \leq q-1$, $i \leq j \leq q-1$ and $1 \leq k \leq i-1$:

$$d_k h^2_{f,X,i,j} = h^2_{f,X,i+1,j+1} d_k;$$

- Matching (7) and (25), for $1 \leq i \leq q-2$, $i+1 \leq j \leq q-1$ and $i \leq k \leq j-1$:

$$d_k h^2_{f,X,i,j} = h^2_{f,X,i,j+1} d_{k+1};$$

- Matching (8) with (26) and (27), using that $\rho(X)$ is a derivation, for $1 \leq i \leq q-1$ and $i \leq k \leq q-1$:

$$d_k h^2_{f,X,i,k} = h^2_{f,X,i,k+1} d_{k+1} + h^2_{f,X,i,k} d_{k+1};$$

- Matching (9) and (28), for $1 \leq i \leq q-2$, $i \leq j \leq q-2$ and $j+1 \leq k \leq q-1$:

$$d_k h^2_{f,X,i,j} = h^2_{f,X,i,j} d_{k+1};$$

- Matching (10) and (29), for $1 \leq i \leq q-1$ and $i \leq j \leq q-1$:

$$d_q h^2_{f,X,i,j} = h^2_{f,X,i,j} d_{q+1};$$

- Matching (14) and (15), for $1 \leq i \leq q$:

$$h^1_{f,X,i} d_i = h^1_{f,X,i+1} d_i;$$

- Matching (22) and (24), for $1 \leq i \leq q-1$ and $i+1 \leq j \leq q$:

$$h^2_{f,X,i,j} d_i = h^2_{f,X,i+1,j} d_i.$$

Combining all this, we are left with (11), (18), (19), (23), so that we infer

$$\begin{aligned}(b \circ h_{f,X} + h_{f,X} \circ b)(\varphi)(f_1, ..., f_q) = & - \varphi(f_1, ..., f_q) f X && (18)\\ & - \sum_{i=1}^{q} \varphi(f_1, ..., f \rho(X)(f_i), ... f_q) && (23)\\ & - f \varphi(f_1, ..., f_q) X && (11)\\ & + \sum_{i=1}^{q} f \varphi(f_1, ..., \rho(X) f_i, ..., f_q) && (19)\end{aligned}$$

Using that $f\rho(X) = \rho(fX)$ and referring back to the definition of L_X, we see that we obtain:

$$\begin{aligned}(b \circ h_{f,X} + h_{f,X} \circ b)(\varphi)(f_1, ..., f_q) = & (L_{fX}\varphi)(f_1, ..., f_q) - (f L_X \varphi)(f_1, ..., f_q)\\ & + f X \varphi(f_1, ..., f_q)\\ & - f X \varphi(f_1, ..., f_q)\end{aligned}$$

and so we conclude that

$$b \circ h_{f,X} + h_{f,X} \circ b = L_{fX} - f L_X$$

which finishes the proof. □

Proposition 4.3.2 (4.1.20) The spaces $\mathrm{C}^{\bullet}_{\mathrm{nl}}(L,R;M)$ define a subcomplex of the total complex $\mathrm{Tot}(\mathrm{C}^{\bullet,\bullet}_{\mathrm{LR}}(L,R;M),d_h+d_v)$.

Proof. Let $c=(c_0,...,c_n)\in\mathrm{C}^k_{\mathrm{nl}}(L,R;M)$. We show that $((dc)_0,...,(dc)_{k+1})\in\mathrm{C}^{k+1}_{\mathrm{nl}}(L,R;M)$ by checking explicitely for $1\le i\le k-1$ that the symbol equation

$$\begin{aligned}(d_vc_{i-1})(X_1,...,fX_{k+1-i})+(d_hc_i)(X_1,..,fX_{k+1-i})=\\ =f(d_vc_{i-1})(X_1,...,X_{k+1-i})+f(d_hc_i)(X_1,...,X_{k+1-i})\\ +h_{f,X_{k+1-i}}((d_vc_i)(X_1,...,X_{k-i}))+h_{f,X_{k+1-i}}((d_hc_{i+1})(X_1,...,X_{k-i}))\end{aligned}$$

The edge-cases $i=0,k,k+1$ follow from similar arguments by systematically not writing down the terms that are not present.

Starting on the LHS we use the fact that $(c_0,...,c_k)$ satisfies the symbol equation to obtain

$$\begin{aligned}(d_vc_{i-1})(X_1,...,fX_{k+1-i})=&(-1)^{k-i+1}b(c_{i-1}(X_1,...,fX_{k+1-i}))\\ =&(-1)^{k-i+1}b(fc_{i-1}(X_1,...,X_{k+1-i})) \qquad \text{(a.1)}\\ &+(-1)^{k-i+1}b(h_{f,X_{k+1-i}}(c_i(X_1,...,X_{k-i}))), \qquad \text{(a.2)}\end{aligned}$$

and

$$\begin{aligned}(d_hc_i)(X_1,...,fX_{k+1-i})=&\sum_{m=1}^{k-i}(-1)^{m+1}L_{X_i}(c_i(X_1,...\widehat{X_i}...,fX_{k+1-i})\\ &+(-1)^{k-i}L_{fX_{k+1-i}}(c_i(X_1,...,X_{k-i}))\\ &+\sum_{1\le m<n\le k-i}(-1)^{m+n}c_i([X_m,X_n],X_1,...\widehat{X_m}\widehat{X_n}...,fX_{k-i+1})\\ &+\sum_{m=1}^{k-i}(-1)^{m+k-i+1}c_i([X_m,fX_{k-i+1}],X_1,...\widehat{X_m}...,X_{k-i}).\end{aligned}$$

Applying the symbol equation to the first, third and fourth line this results in

$$\begin{aligned}(d_h c_i)(X_1, ..., fX_{k+1-i}) = &\sum_{m=1}^{k-i}(-1)^{m+1}L_{X_m}(fc_i(X_1, ...\widehat{X_m}..., X_{k+1-i})) && \text{(b.1)}\\
&+\sum_{m=1}^{k-i}(-1)^{m+1}L_{X_m}(h_{f,X_{k+1-i}}(c_{i+1}(X_1, ...\widehat{X_m}..., X_{k-i}))) && \text{(b.2)}\\
&+(-1)^{k-i}L_{fX_{k+1-i}}(c_i(X_1, ..., X_{k-i})) && \text{(b.3)}\\
&+\sum_{1\le m<n\le k+1-i}(-1)^{m+n}fc_i([X_m, X_n]X_1, ...\widehat{X_m}\widehat{X_n}..., X_{k+1-i}) && \text{(b.4)}\\
&+\sum_{1\le m<n\le k-i}(-1)^{m+n}h_{f,X_{k-i+1}}(c_{i+1}([X_m, X_n], X_1, ...\widehat{X_m}\widehat{X_n}..., X_{k-i})) && \text{(b.5)}\\
&+\sum_{m=1}^{k-i}(-1)^{m}h_{f,[X_m,X_{k-i+1}]}(c_{i+1}(X_1, ...\widehat{X_m}..., X_{k-i})) && \text{(b.6)}\\
&+\sum_{m=1}^{k-i}(-1)^{m}\rho(X_m)(f)c_i(X_1, ...\widehat{X_m}..., X_{k-i+1}) && \text{(b.7)}\\
&+\sum_{m=1}^{k-i}(-1)^{m}h_{\rho(X_m)(f),X_{k-i+1}}(c_{i+1}(X_1, ...\widehat{X_m}..., X_{k-i}) && \text{(b.8)}\end{aligned}$$

Now using the homotopy equation from Proposition 4.1.16 we can see that (b.3) equals:

$$\begin{aligned}\text{(b.3)} =&(-1)^{k-i}fL_{X_{k+1-i}}(c_i(X_1, ..., X_{k-i}))\\
&+(-1)^{k-i}b(h_{f,X_{k+1-i}}(c_i(X_1, ..., X_{k-i})))\\
&+(-1)^{k-i}h_{f,X_{k+1-i}}(b(c_i(X_1, ..., X_{k-i}))),\end{aligned}$$

so that combining terms we have

$$\begin{aligned}\text{(a.1)} + \text{(a.2)} + \text{(b.3)} =&f(d_v c_{i-1})(X_1, ..., X_{k+1-i})\\
&+h_{f,X_{k+1-i}}((d_v c_i)(X_1, ..., X_{k-i}))\\
&+(-1)^{k-i}fL_{X_{k+1-i}}(c_i(X_1, ..., X_{k-i})) && \text{(b.3*)}\end{aligned}$$

Next, we use the first point of Lemma 4.1.19 to dissect (b.1) into

$$\begin{aligned}\text{(b.1)} =&\sum_{m=1}^{k-i}(-1)^{m+1}fL_{X_m}(c_i(X_1, ...\widehat{X_m}..., X_{k+1-i})\\
&+\sum_{m=1}^{k-i}(-1)^{m+1}\rho(X_m)(f)c_i(X_1, ...\widehat{X_m}..., X_{k+1-i}).\end{aligned}$$

Combining this with other terms we have

$$(\text{b.1}) + (\text{b.3}*) + (\text{b.4}) + (\text{b.7}) = f(d_h c_i)(X_1, ..., X_{k+1-i})$$

Lastly we use the second point of Lemma 4.1.19 to dissect (b.2) into

$$\begin{aligned}(\text{b.2}) = &\sum_{m=1}^{k-i}(-1)^{m+1}h_{f,X_{k+1-i}}(L_{X_m}(c_{i+1}(X_1, ...\widehat{X_m}..., X_{k-i}))) \\ &+\sum_{m=1}^{k-i}(-1)^{m+1}h_{\rho(X_m)f,X_{k+1-i}}(c_{i+1}(X_1, ...\widehat{X_m}..., X_{k-i})) \\ &+\sum_{m=1}^{k-i}(-1)^{m+1}h_{f,[X_m,X_{k+1-i}]}(c_{i+1}(X_1, ...\widehat{X_m}..., X_{k-i})),\end{aligned}$$

so that combining terms we have

$$(\text{b.2}) + (\text{b.5}) + (\text{b.6}) + (\text{b.8}) = h_{f,X_{k+1-i}}((d_h c_{i+1})(X_1, ..., X_{k-i})).$$

This finishes the proof. □

Lemma 4.3.3 (4.1.24) The map $\mathfrak{s}$ satisfies

$$\mathfrak{s}^{p,q} \circ b = d_h \circ \mathfrak{s}^{p-1,q} + d_v \circ \mathfrak{s}^{p,q-1}$$

Proof. We start by writing out $(*) = (\mathfrak{s}^{p,q} \circ b)(\varphi)(X_1, ..., X_p, f_1, ..., f_q)$:

$$\begin{aligned}(*) = &\sum_{\sigma\in S_{p,q}S_p}(-1)^\sigma D_{\sigma^{-1}(1)}\varphi(D_{\sigma^{-1}(2)}, ..., D_{\sigma^{-1}(p+q)}) &\text{(a)}\\ &+\sum_{i=1}^{p+q-1}\sum_{\sigma\in S_{p,q}S_p}(-1)^i(-1)^\sigma\varphi(D_{\sigma^{-1}(1)}, ..., D_{\sigma^{-1}(i)}D_{\sigma^{-1}(i+1)}, ..., D_{\sigma^{-1}(p+q)}) &\text{(b)}\\ &+\sum_{\sigma\in S_{p,q}S_p}(-1)^{p+q}(-1)^\sigma\varphi(D_{\sigma^{-1}(1)}, ..., D_{\sigma^{-1}(p+q-1)})D_{\sigma^{-1}(p+q)} &\text{(c)}\end{aligned}$$

Now we can split (a) by distinguishing the cases where $\sigma^{-1}(1) \in \{1, ..., p\}$ and the cases where $\sigma^{-1}(1) \in \{p+1, ..., p+q\}$. Note that the second case means that $\sigma(p+1) = 1$, since we do not permute that last q indices. We obtain:

$$\begin{aligned}(\text{a}) = &\sum_{\substack{\sigma\in S_{p,q}S_p\\ \sigma^{-1}(1)\in\{1,...,p\}}}(-1)^\sigma X_{\sigma^{-1}(1)}\varphi(D_{\sigma^{-1}(2)}, ..., D_{\sigma^{-1}(p+q)})\\ &+\sum_{\substack{\sigma\in S_{p,q}S_p\\ \sigma(p+1)=1}}(-1)^\sigma f_1\varphi(D_{\sigma^{-1}(2)}, ..., D_{\sigma^{-1}(p+q)})\end{aligned}$$

Similarly we can split (c) by distinguishing the cases where $\sigma^{-1}(p+q) \in \{1, ..., p\}$ and $\sigma(p+q) = p+q$ to get

$$\begin{aligned}(c) = & \sum_{\substack{\sigma\in S_{p,q}S_p \\ \sigma^{-1}(p+q)\in\{1,...,p\}}} (-1)^{p+q}(-1)^{\sigma}\varphi(D_{\sigma^{-1}(1)}, ..., D_{\sigma^{-1}(p+q-1)})X_{\sigma^{-1}(p+q)} \\ & + \sum_{\substack{\sigma\in S_{p,q}S_p \\ \sigma(p+q)=p+q}} (-1)^{p+q}(-1)^{\sigma}\varphi(D_{\sigma^{-1}(1)}, ..., D_{\sigma^{-1}(p+q-1)})f_q\end{aligned}$$

Next we notice that in the first term in (c) we can rearrange the σ's such that $\sigma^{-1}(p+q)$ becomes $\sigma^{-1}(1)$. This comes at a cost of a sign $(-1)^{p+q-1}$ and we see:

$$\begin{aligned}(a) + (c) = & \sum_{i=1}^{p} \sum_{\substack{\sigma\in S_{p,q}S_p \\ \sigma^{-1}(1)=i}} (-1)^{\sigma}[X_i, \varphi(D_{\sigma^{-1}(2)}, ..., D_{\sigma^{-1}(p+q)})] && \text{(a.1)} \\ & + \sum_{\substack{\sigma\in S_{p,q}S_p \\ \sigma(p+1)=1}} (-1)^{\sigma} f_1\varphi(D_{\sigma^{-1}(2)}, ..., D_{\sigma^{-1}(p+q)}) && \text{(a.2)} \\ & + \sum_{\substack{\sigma\in S_{p,q}S_p \\ \sigma(p+q)=p+q}} (-1)^{p+q}(-1)^{\sigma}\varphi(D_{\sigma^{-1}(1)}, ..., D_{\sigma^{-1}(p+q-1)})f_q && \text{(a.3)}\end{aligned}$$

Now when we look at (a.1) we see that $\{\sigma \in S_{p,q}S_p;\ \sigma^{-1}(1) = i\}$ is in bijective correspondence with $S_{p-1,q}S_{p-1}$ by taking out $\sigma(i) = 1$. If we plug in this bijection we recognize a term of $\mathfrak{s}^{p-1,q}$. This all comes at a cost of a sign $(-1)^{i+1}$ so we obtain:

$$(a.1) = \sum_{i=1}^{p} (-1)^{i+1}[X_i, (\mathfrak{s}^{p-1,q}\varphi)(X_1, ...\widehat{X_i}..., X_p, f_1, ..., f_q)]$$

In similar fashion we get

$$(a.2) = (-1)^p f_1(\mathfrak{s}^{p,q-1}\varphi)(X_1, ..., X_p, f_2, ..., f_q)$$

$$(a.3) = (-1)^{p+q}(\mathfrak{s}^{p,q-1}\varphi)(X_1, ..., X_p, f_1, ..., f_{q-1})f_q$$

Now to tackle (b) we want to switch products $D_{\sigma^{-1}(i)}D_{\sigma^{-1}(i+1)}$ for commutators of the form $[D_{\sigma^{-1}(i)}, D_{\sigma^{-1}(i+1)}]$. Indeed inside our double complex we can only plug in elements of L and R, but elements of $L \cdot L \subset \mathcal{U}(L,R)$ or $L \cdot R \subset \mathcal{U}(L,R)$ do not fit this. Instead we use that $[L,L] \subset L$ and $[L,R] \subset R$. Note that $R \cdot R \subset R$ so this does not pose a problem.

So at a cost of a sign -1 we switch $D_{\sigma^{-1}(i)}$ and $D_{\sigma^{-1}(i+1)}$ whenever $\sigma^{-1}(i) > \sigma^{-1}(i+1)$. Note that this is only possible if at least one of $\sigma^{-1}(i)$ and $\sigma^{-1}(i+1)$ lie in $\{1, ..., p\}$.

We obtain:

$$(b) = \sum_{i=1}^{p+q-1} \sum_{\substack{\sigma \in S_{p,q}S_p \\ \sigma^{-1}(i)\in\{p+1,...,p+q\} \\ \sigma^{-1}(i+1)\in\{p+1,...,p+q\}}} (-1)^i(-1)^\sigma \varphi(D_{\sigma^{-1}(1)}, ..., f_{\sigma^{-1}(i)-p} f_{\sigma^{-1}(i+1)-p}, ..., D_{\sigma^{-1}(p+q)}) \quad (b.1)$$

$$+ \sum_{m<n}^{p} \sum_{i=1}^{p+q-1} \sum_{\substack{\sigma \in S_{p,q}S_p \\ \sigma^{-1}(i)=m \\ \sigma^{-1}(i+1)=n}} (-1)^i(-1)^\sigma \varphi(D_{\sigma^{-1}(1)}, ..., [X_m, X_n], ..., D_{\sigma^{-1}(p+q)}) \quad (b.2)$$

$$+ \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{i=1}^{p+q-1} \sum_{\substack{\sigma \in S_{p,q}S_p \\ \sigma^{-1}(i)=m \\ \sigma^{-1}(i+1)=n}} (-1)^i(-1)^\sigma \varphi(D_{\sigma^{-1}(1)}, ..., \rho(X_m)f_n, ..., D_{\sigma^{-1}(p+q)}) \quad (b.3)$$

First looking at (b.1), we note that since the f's should stay ordered we have $f_{\sigma^{-1}(i+1)-p} = f_{\sigma^{-1}(i)-p+1}$ so we have

$$(b.1) = \sum_{i=1}^{p+q-1} \sum_{m=1}^{q} \sum_{\substack{\sigma \in S_{p,q}S_p \\ \sigma^{-1}(i)=m+p \\ \sigma^{-1}(i+1)=m+p+1}} (-1)^t(-1)^\sigma \varphi(D_{\sigma^{-1}(1)}, ..., f_m f_{m+1}, ..., D_{\sigma^{-1}(p+q)})$$

Now we again play the trick where we see the subsets of $S_{p,q}S_p$ as either $S_{p-1,q}S_{p-1}$ or $S_{p,q-1}S_p$ up to a sign shift. In the case of (b.1) we have to merge i and $i+1$, so in practice we delete $\sigma(m+1) = i+p+1$ from the permutation, which comes at a sign $m+p+i$. We then recognize a term of $\mathfrak{s}^{p,q-1}$ and we see:

$$(b.1) = \sum_{i=1}^{q} (-1)^{i+p} (\mathfrak{s}^{p,q-1}\varphi)(X_1, ..., X_p, f_1, ..., f_i f_{i+1}, ..., f_q)$$

Then in (b.2) we first delete $\sigma(i+1) = n$ at a cost of a sign $(-1)^{m+i+1}$ and then make sure that $[X_m, X_n]$ is actually the first term in the input, which comes at another sign $(-1)^{m+1}$. Then we obtain:

$$(b.2) = \sum_{m<n}^{p} (-1)^{m+n} (\mathfrak{s}^{p-1,q}\varphi)([X_m, X_n], X_1, ...\widehat{X_m}\widehat{X_n}..., X_p, f_1, ..., f_q)$$

For (b.3) we delete $\sigma(m) = i$ at a a cost of a sign $(-1)^{m+i}$ and we obtain:

$$(b.3) = \sum_{m=1}^{p} \sum_{n=1}^{q} (-1)^m (\mathfrak{s}^{p-1,q}\varphi)(X_1, ...\widehat{X_m}..., X_p, f_1, ..., \rho(X_m)f_n, ..., f_q)$$

This proves the Lemma. □

Lemma 4.3.4 (4.2.14) The differential in the complex $\mathrm{C}^\bullet_{\mathrm{def}}(L, \mathrm{Sym}^k \mathrm{Ad} L)$ is given by

$$(d(c_0, ..., c_n))_i = \partial_{\mathrm{CE}}(c_i) + (-1)^{n+1} P \circ c_{i-1}.$$

Proof. The top order term

$$(d(c_0, ..., c_n))_0 = \partial_{\mathrm{CE}}(c_0)$$

is clearly the correct term, so we need to show that with this formula $d(c_0, ..., c_n)$ indeed satisfies the symbol equation, i.e.

$$\begin{aligned}(d(c_0, ..., c_n))_i(X_1, ..., fX_{n+1-i}) =& f(d(c_0, ..., c_n))_i(X_1, ..., X_{n+1-i}) \\ &+ \iota_f((d(c_0, ..., c_n))_{i+1}(X_1, ..., X_{n-i}) \odot X_{n-i+1}).\end{aligned}$$

We again show this for $1 \leq i \leq n-1$ with the edge cases $i = 0, n, n+1$ being similar by keeping track of which terms are not present. Starting on the LHS we have

$$\begin{aligned}(d(c_0, ..., c_n))_i(X_1, ..., fX_{n+1-i}) =& \sum_{s=1}^{n-i} (-1)^{s+1} [X_s, c_i(X_1, ...\widehat{X_s}..., fX_{n+1-i})] && \text{(a)} \\ &+ (-1)^{n-i} [fX_{n+1-i}, c_i(X_1, ..., X_{n-i})] && \text{(b)} \\ &+ \sum_{1 \leq s < t \leq n-i} (-1)^{s+t} c_i([X_s, X_t], X_1, ...\widehat{X_s}\widehat{X_t}..., fX_{n+1-i}) && \text{(c)} \\ &+ \sum_{s=1}^{n-i} (-1)^{s+n-i+1} c_i([X_s, fX_{n+1-i}], X_1, ...\widehat{X_s}..., X_{n-i}) && \text{(d)} \\ &+ (-1)^{n+1} P(c_{i-1}(X_1, ..., fX_{n+1-i})) && \text{(e)}\end{aligned}$$

Using the symbol equation on (a), (c), (d) and (e) we obtain

$$\begin{aligned}\text{(a)} =& \sum_{s=1}^{n-i} (-1)^{s+1} [X_s, fc_i(X_1, ...\widehat{X_s}..., X_{n+1-i})] \\ &+ \sum_{s=1}^{n-i} (-1)^{s+1} [X_s, \iota_f(c_{i+1}(X_1, ...\widehat{X_s}..., X_{n-i}) \odot X_{n+1-i})] \\ =& \sum_{s=1}^{n-i} (-1)^{s+1} \rho(X_s)(f) c_i(X_1, ...\widehat{X_s}..., X_{n+1-i}) && \text{(a.1)} \\ &+ \sum_{s=1}^{n-i} (-1)^{s+1} f[X_s, c_i(X_1, ...\widehat{X_s}..., X_{n+1-i})] && \text{(a.2)} \\ &+ \sum_{s=1}^{n-i} (-1)^{s+1} [X_s, \iota_f(c_{i+1}(X_1, ...\widehat{X_s}..., X_{n-i}))] \odot X_{n+1-i} && \text{(a.3)} \\ &+ \sum_{s=1}^{n-i} (-1)^{s+1} \iota_f(c_{i+1}(X_1, ...\widehat{X_s}..., X_{n-i}) \odot [X_s, X_{n+1-i}]) && \text{(a.4)}\end{aligned}$$

$$(\mathrm{c}) = \sum_{1\leq s<t\leq n-i} (-1)^{s+t} f c_i([X_s, X_t], X_1, ...\widehat{X_s}\widehat{X_t}..., X_{n+1-i}) \quad (\mathrm{c.1})$$
$$+ \sum_{1\leq s<t\leq n-i} (-1)^{s+t} \iota_f(c_{i+1}([X_s, X_t], X_1, ...\widehat{X_s}\widehat{X_t}..., X_{n-i}) \odot X_{n-i+1}) \quad (\mathrm{c.2})$$

$$(\mathrm{d}) = \sum_{s=1}^{n-i} (-1)^{s+n+1-i} f c_i([X_s, X_{n-i+1}], X_1, ...\widehat{X_s}..., X_{n-i}) \quad (\mathrm{d.1})$$
$$+ \sum_{s=1}^{n-i} (-1)^s \rho(X_s)(f) c_i(X_1, ...\widehat{X_i}..., X_{n-i+1}) \quad (\mathrm{d.2})$$
$$+ \sum_{s=1}^{n-i} (-1)^s \iota_f(c_{i+1}(X_1, ...\widehat{X_s}..., X_{n-i}) \odot [X_s, X_{n-i+1}]) \quad (\mathrm{d.3})$$
$$+ \sum_{s=1}^{n-i} (-1)^s \iota_{\rho(X_s)(f)}(c_{i+1}(X_1, ...\widehat{X_s}..., X_{n-i}) \odot X_{n-i+1}) \quad (\mathrm{d.4})$$

$$(\mathrm{e}) = (-1)^{n+1} f P(c_{i-1}(X_1, ..., X_{n+1-i})) \quad (\mathrm{e.1})$$
$$+ (-1)^{n+1} P(\iota_f(c_i(X_1, ..., X_{n-i}) \odot X_{n+1-i})) \quad (\mathrm{e.2})$$

Immediately, we spot that (a.1) and (d.2) and (a.4) and (d.3) cancel against each other. Next we see that

$$(\mathrm{a.2}) + (\mathrm{c.1}) + (\mathrm{d.1}) + (\mathrm{e.1}) = f d(c_0, ..., c_n)_i(X_1, ..., X_{n+1-i})$$
$$+ (-1)^{n-i+1} f [X_{n+1-i}, c_i(X_1, ..., X_{n-i})] \quad (\tilde{\mathrm{b}})$$

and similarly

$$(\mathrm{c.2}) = \iota_f(d(c_0, ..., c_n)_{i+1}(X_1, ..., X_{n-i}) \odot X_{n-i+1})$$
$$+ \sum_{s=1}^{n-i} (-1)^s \iota_f([X_s, c_{i+1}(X_1, ...\widehat{X_s}..., X_{n-i})] \odot X_{n+1-i}) \quad (\tilde{\mathrm{a}}.3)$$
$$+ (-1)^n \iota_f(P(c_i(X_1, ..., X_{n-i})) \odot X_{n+1-i}) \quad (\tilde{\mathrm{e}}.2)$$

So we need to cancel the trio (a.3), (ã.3) and (d.4) against each other and the total of (b), ($\tilde{\mathrm{b}}$), (e.2) and (ẽ.2). This can be summarized in two claims:

Claim 1: For every $X \in L, f \in R$ and $\overrightarrow{Y} \otimes \overrightarrow{D} \in \mathrm{Sym}_R^{k-i-1} L \otimes_R \Lambda_R^{i+1} \mathrm{Der}(R)$ it holds that

$$[X, \iota_f(\overrightarrow{Y} \otimes \overrightarrow{D})] = \iota_f([X, \overrightarrow{Y} \otimes \overrightarrow{D}]) + \iota_{\rho(X)f}(\overrightarrow{Y} \otimes \overrightarrow{D}).$$

Claim 2: For every $X \in L$, $f \in R$ and $\overrightarrow{Y} \otimes \overrightarrow{D} \in \mathrm{Sym}_R^{k-i} L \otimes_R \Lambda_R^i \mathrm{Der}(R)$ it holds that

$$(-1)^i [fX, \overrightarrow{Y} \otimes \overrightarrow{D}] + (-1)^{i+1} f [X, \overrightarrow{Y} \otimes \overrightarrow{D}] - P\iota_f(\overrightarrow{Y} \otimes \overrightarrow{D} \odot X) + \iota_f(P(\overrightarrow{Y} \otimes \overrightarrow{D}) \odot X) = 0.$$

The first claim follows from the fact that

$$[X, D(f)] = [\rho(X), D](f) + D(\rho(X)(f))$$

since both sides are equal to $\rho(X)(D(f))$.

The second claim follows from an explicit calculation. Manipulating the first terms, we have:

$$\begin{aligned}(-1)^i[fX, \overrightarrow{Y} \otimes \overrightarrow{D}] =&(-1)^i \sum_{s=1}^{k-i} Y_1 \odot \cdots \odot [fX, Y_s] \odot \cdots \odot Y_{k-i} \otimes \overrightarrow{D} \\ &+ (-1)^i \sum_{s=1}^{i} \overrightarrow{Y} \otimes D_1 \wedge \cdots \wedge [\rho(fX), D_s] \wedge \cdots \wedge D_i \\ =&(-1)^i \sum_{s=1}^{k-i} fY_1 \odot \cdots \odot [X, Y_s] \odot \cdots \odot Y_{k-i} \otimes \overrightarrow{D} \\ &+ (-1)^{i+1} \sum_{s=1}^{k-i} \rho(Y_s)(f) X \odot Y_1 \odot \cdots \widehat{Y_s} \cdots \odot Y_{k-i} \otimes \overrightarrow{D} \\ &+ (-1)^i \sum_{s=1}^{i} f\overrightarrow{Y} \otimes D_1 \wedge \cdots \wedge [\rho(X), D_s] \wedge \cdots \wedge D_i \\ &+ \sum_{s=1}^{i} (-1)^{s+1} D_s(f) \overrightarrow{Y} \otimes D_1 \wedge \cdots \widehat{D_s} \cdots \wedge D_i \wedge \rho(X) \\ =&(-1)^i f[X, \overrightarrow{Y} \otimes \overrightarrow{D}] \\ &+ (-1)^{i+1} \sum_{s=1}^{k-i} \rho(Y_s)(f) X \odot Y_1 \odot \cdots \widehat{Y_s} \cdots \odot Y_{k-i} \otimes \overrightarrow{D} \\ &+ \iota_f(\overrightarrow{Y} \otimes \overrightarrow{D}) \wedge \rho(X)\end{aligned}$$

Manipulating the third term we have:

$$\begin{aligned}-P(\iota_f(\overrightarrow{Y} \otimes \overrightarrow{D}) \odot X) =& - \iota_f(\overrightarrow{Y} \otimes \overrightarrow{D}) \wedge \rho(X) \\ &- X \odot P(\iota_f(\overrightarrow{Y} \otimes \overrightarrow{D}))\end{aligned}$$

Recognizing all the terms, we are reduced to showing that

$$\begin{aligned}(-1)^{i+1} \sum_{s=1}^{k-i} \rho(Y_s)(f) Y_1 \odot \cdots \widehat{Y_s} \cdots \odot Y_{k-i} \otimes \overrightarrow{D} =&P(\iota_f(\overrightarrow{Y} \otimes \overrightarrow{D})) \\ &- \iota_f(P(\overrightarrow{Y} \otimes \overrightarrow{D}))\end{aligned}$$

which is clear by looking at the case where there is only one Y and one D and we have

$$\begin{aligned} P(\iota_f(Y \otimes D)) - \iota_f(P(Y \otimes D)) =& P(D(f)Y) - \iota_f(D \wedge \rho(Y)) \\ =& D(f)\rho(Y) - D(f)\rho(Y) + \rho(Y)(f)D \\ =& \rho(Y)(f)D. \end{aligned}$$

This finishes the proof. □

Appendix A

Constructions in homological algebra

In this appendix we outline results in homological algebra that allow us to effectively deal with the Hochschild complexes of various algebras. The main points of business are:

- Defining chain complexes out of simplicial vector spaces, used in Section 1.1.1 to define the Hochschild homology complex;
- Defining mixed complexes out of cyclic vector spaces, used in Section 1.1.2 to define cyclic and periodic cyclic homology;
- The notion of cylindrical spaces and the Eilenberg-Zilber Theorem to deal with the Hochschild complex of the convolution algebra $G \ltimes A$ in Section 3.2.

The main resources for this section are Loday [Lo98] and Nest-Tsygan [NT], with Crainic [Cr04] being a specific reference for the subsection about the Homological Pertubation Lemma, Getzler-Jones [GJ93] a specific reference for the part about cylindrical spaces and Khalkali-Rangipour [KR04] a specific reference for the part about the cyclic Eilenberg-Zilber Theorem.

A.1 Mixed complexes

Let $\mathbb{K}$ be a field.

Definition A.1.1 A *mixed chain complex* is a collection of $\mathbb{K}$-vector spaces $\{\mathrm{C}_k\}_{k\geq 0}$ together with collections of linear maps $b\colon \mathrm{C}_k \to \mathrm{C}_{k-1}$ for $k \geq 1$ and $B\colon \mathrm{C}_k \to \mathrm{C}_{k+1}$ for $k \geq 0$ such that

$$\begin{aligned} b^2 &= 0, \\ B^2 &= 0, \\ bB + Bb &= 0. \end{aligned}$$

A morphism $f\colon (\mathrm{C}_\bullet, b, B) \to (\mathrm{C}'_\bullet, b', B')$ is a degree 0 linear map that intertwines b with b' and B with B'.

Definition A.1.2 A *mixed cochain complex* is a collection of $\mathbb{K}$-vector spaces $\{\mathrm{C}^k\}_{k\geq 0}$ together with collections of linear maps $b\colon \mathrm{C}^k \to \mathrm{C}^{k+1}$ for $k \geq 0$ and $B\colon \mathrm{C}^k \to \mathrm{C}^{k-1}$ for $k \geq 0$ such that

$$\begin{aligned} b^2 &= 0, \\ B^2 &= 0, \\ bB + Bb &= 0. \end{aligned}$$

A morphism $f\colon (\mathrm{C}^\bullet, b, B) \to (\mathrm{C}'^\bullet, b', B')$ is a degree 0 linear map that intertwines b with b' and B with B'.

Remark A.1.3 Clearly mixed chain complexes and mixed cochain complexes are the same thing, by replacing C_k and C^k and switching the roles of b and B. In practice mixed complexes come in the form of a chain complex $(\mathrm{C}_\bullet, b)$ or a cochain complex $(\mathrm{C}^\bullet, b)$ with an extra differential B going 'in the wrong direction'. In what follows the 'leading role' of b has some notational importance, so we make two separate definitions.

Now, from a mixed (co)chain complex one wants to make an ordinary (co)chain complex, using the information of both b and B. Of course, b and B go in different directions, but as they change the grading of the complex by increments of 1 in either direction it is clear that the mod 2 grading is *increased* by 1 by both differentials.
This leads to the following two constructions for both mixed chain complexes and mixed cochain complexes.

Definition A.1.4 For $(\mathrm{C}_\bullet, b, B)$ a mixed chain complex, we define a chain complex $\mathrm{CC}_\bullet$ by

$$\mathrm{CC}_n = \bigoplus_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathrm{C}_{n-2i}$$

where $d\colon \mathrm{CC}_n \to \mathrm{CC}_{n-1}$ is determined by setting the value for $x_k \in \mathrm{C}_k \subset \mathrm{CC}_n$ by

$$dx_k = \begin{cases} bx_k & \text{if } k = n \\ bx_k + Bx_k & \text{if } 0 < k < n \\ Bx_k & \text{if } k = 0 \end{cases}$$

Note that the last case can only happen if n is even. A map of mixed complexes $f\colon (\mathrm{C}_\bullet, b, B) \to (\mathrm{C}'_\bullet, b, B)$ induces a chain map $\mathrm{C}f\colon (\mathrm{CC}_\bullet, d) \to (\mathrm{C}'\mathrm{C}_\bullet, d')$ by sending $x_k \in \mathrm{C}_k \subset \mathrm{CC}_n$ to $f(x_k) \in \mathrm{C}'_k \subset \mathrm{C}'\mathrm{C}_n$.

Definition A.1.5 For $(\mathrm{C}_\bullet, b, B)$ a mixed chain complex, we define a chain complex $\mathrm{CP}_\bullet$ by

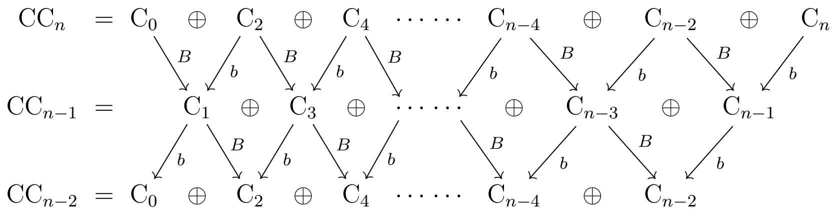
$$\mathrm{CP}_n = \bigoplus_{\substack{k \geq 0 \\ n \equiv k \bmod 2}} \mathrm{C}_k$$

where $d\colon \mathrm{CP}_n \to \mathrm{CP}_{n-1}$ is determined by setting the value for $x_k \in \mathrm{C}_k \subset \mathrm{CC}_n$ by

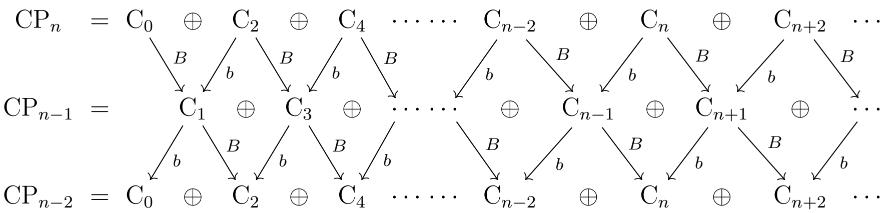
$$dx_k = \begin{cases} bx_k + Bx_k & \text{if } k \neq 0 \\ Bx_k & \text{if } k = 0 \end{cases}$$

A map of mixed complexes $f\colon (\mathrm{C}_\bullet, b, B) \to (\mathrm{C}'_\bullet, b, B)$ induces a chain map $\mathrm{P}f\colon (\mathrm{CP}_\bullet, d) \to (\mathrm{C}'\mathrm{P}_\bullet, d')$ by sending $\{x_k \in \mathrm{C}_k\}_k \in \mathrm{CP}_n$ to $\{f(x_k) \in \mathrm{C}'_k\} \in \mathrm{C}'\mathrm{P}_n$.

Remark A.1.6 One can think of $\mathrm{CC}_\bullet$ and $\mathrm{CP}_\bullet$ and their differentials in the form of pictures. For $\mathrm{CC}_\bullet$ the picture looks like (for n even):



while for $\mathrm{CP}_\bullet$ the picture looks like (again for n even):



From these pictures it is also clear that $\mathrm{CC}_\bullet$ and $\mathrm{CP}_\bullet$ are indeed chain complexes, i.e. $d^2 = 0$, since all the terms in d^2 either involve b^2, B^2 or $bB + Bb$.

Remark A.1.7 There is a shift map $S\colon \mathrm{CC}_\bullet \to \mathrm{CC}_{\bullet+2}$ that sends $(x_n, x_{n-2}, ...)$ to $(0, x_n, x_{n-2}, ...)$. This map induces a chain map $(\mathrm{CC}_\bullet, b + B) \to (\mathrm{CC}_\bullet[2], b + B)$. Using this map, we can also see $\mathrm{CP}_\bullet$ and its homology as inverse limits of $CC_\bullet$ by constructing a formal inverse to S.

We can do the dual picture for mixed cochain complexes

Definition A.1.8 For $(\mathrm{C}^\bullet, b, B)$ a mixed cochain complex, we define a chain complex $\mathrm{CC}^\bullet$ by

$$\mathrm{CC}^n = \prod_{i=0}^{\lfloor \frac{n}{2} \rfloor} \mathrm{C}_{n-2i}$$

where $d\colon \mathrm{CC}^n \to \mathrm{CC}^{n+1}$ is determined by setting the value for $x_k \in \mathrm{C}^k \subset \mathrm{CC}^n$ by

$$dx_k = \begin{cases} bx_k + Bx_k & \text{if } k \neq 0 \\ bx_k & \text{if } k = 0 \end{cases}$$

Note again that the last case can only happen if n is even. A map of mixed complexes $f\colon (\mathrm{C}^\bullet, b, B) \to (\mathrm{C}'^\bullet, b, B)$ induces a chain map $\mathrm{C}f\colon (\mathrm{CC}^\bullet, d) \to (\mathrm{C}'\mathrm{C}^\bullet, d')$ by sending $x_k \in \mathrm{C}^k \subset \mathrm{CC}^n$ to $f(x_k) \in \mathrm{C}'^k \subset \mathrm{C}'\mathrm{C}^n$.

Definition A.1.9 For $(\mathrm{C}^\bullet, b, B)$ a mixed cochain complex, we define a cochain complex $\mathrm{CP}^\bullet$ by

$$\mathrm{CP}^n = \prod_{\substack{k\geq 0\\ n\equiv k \bmod 2}} \mathrm{C}_k$$

where $d\colon \mathrm{CP}^n \to \mathrm{CP}^{n-1}$ is determined by sending $\{x_k \in \mathrm{C}^k\} \in \mathrm{CP}^n$ to $\sum_k dx_k \in \mathrm{CP}^{n+1}$ where dx_k has contributions in C^{k+1} and C^{k-1} given by

$$dx_k = \begin{cases} bx_k + Bx_k & \text{if } k \neq 0 \\ bx_k & \text{if } k = 0 \end{cases}$$

A map of mixed complexes $f\colon (\mathrm{C}^\bullet, b, B) \to (\mathrm{C}'^\bullet, b, B)$ induces a chain map $\mathrm{P}f\colon (\mathrm{CP}^\bullet, d) \to (\mathrm{C}'\mathrm{P}^\bullet, d')$ by sending $\{x_k \in \mathrm{C}_k\} \in \mathrm{CP}_n$ to $\{f(x_k) \in \mathrm{C}'_k\} \in \mathrm{C}'\mathrm{P}_n$.

Remark A.1.10 We can also draw pictures for $\mathrm{CC}^\bullet$ and $\mathrm{CP}^\bullet$, and these are essentially the same pictures as before, but then with all arrows reversed (and $\oplus$ replaced by $\times$).

Remark A.1.11 For notational purposes, in all cases we may also write '$b + B$' for d.

In the cochain case we use direct products in stead of direct sums, since the direct product is the dual of the direct sum. In particular the following holds:

Lemma A.1.12 If $(\mathrm{C}_\bullet, b, B)$ is a mixed chain complex, then the dual $(\mathrm{C}^*)^k = (\mathrm{C}_k)^*$ together with b^* and B^* makes $((\mathrm{C}^*)^\bullet, b^*, B^*)$ into a mixed cochain complex. Furthermore $(\mathrm{C}^*)\mathrm{C}^\bullet$ is the dual complex to $\mathrm{CC}_\bullet$ and $(\mathrm{C}^*)\mathrm{P}^\bullet$ is the dual complex to $\mathrm{CP}_\bullet$.

Remark A.1.13 Since in a mixed chain complex $(\mathrm{C}_\bullet, b, B)$, the two differentials b and B commute, there is a map induced by B on the b-homology of $C_\bullet$[1]. One can make good use of this by inducing the following filtration on $\mathrm{CC}_\bullet$:

$$F_p\mathrm{CC}_n = \bigoplus_{i=0}^{p} \mathrm{C}_{n-2i}.$$

With this filtration we have

$$F_p\mathrm{CC}_\bullet / F_{p-1}\mathrm{CC}_\bullet \cong \mathrm{C}_{n-2p}$$

with the induced differential

$$\mathrm{C}_{n-2p} \cong F_p\mathrm{CC}_n / F_{p-1}\mathrm{CC}_n \to F_p\mathrm{CC}_{n-1}/F_{p-1}\mathrm{CC}_{n-1} \cong \mathrm{C}_{n-1-2p}$$

[1]Also in reverse, but we are respectful of the leading role b has in the dance

given by b. In particular, setting up the spectral sequence associated to this filtration we have

$$\mathrm{E}^1_{p,q} \cong \mathrm{H}_{q-p}(\mathrm{C}_\bullet, b)$$

Then on this page the differential is given by the map induced by B on $\mathrm{H}_\bullet(\mathrm{C}_\bullet, b)$, so that on the second page of the spectral sequence we have

$$\mathrm{E}^2_{p,q} \cong \mathrm{H}^{q-p}(\mathrm{H}_\bullet(\mathrm{C}_\bullet, b), B).$$

From this remark, the following is an easy consequence of the Spectral Sequence Comparison Theorem [Ze57]:

Proposition A.1.14 If $f\colon (\mathrm{C}_\bullet, b, B) \to (\mathrm{C}'_\bullet, b', B')$ is a map of mixed complexes such that the underlying map $f\colon (\mathrm{C}_\bullet, b) \to (\mathrm{C}'_\bullet, b')$ is a quasi-isomorphism, so is $\mathrm{C}f\colon (\mathrm{CC}_\bullet, d) \to (\mathrm{C}'\mathrm{C}_\bullet, d')$.

A.2 Homological perturbation theory

The main aim of this subsection is discussing the Homological Perturbation Lemma, which turns out to be an effective tool to understand for a mixed complex $(\mathrm{C}_\bullet, b, B)$ the differences between $\mathrm{H}(\mathrm{C}_\bullet, b)$, $\mathrm{H}(\mathrm{CC}_\bullet, b+B)$ and $\mathrm{H}(\mathrm{CP}_\bullet, b+B)$.

We start with two filtered chain complexes $(\mathrm{C}_\bullet, b)$ and $(\mathrm{C}'_\bullet, b')$ that are bounded below by 0. This means that for every $p \in \mathbb{Z}$ we have subcomplexes $F^p\mathrm{C}_\bullet$ and $F^p\mathrm{C}'_\bullet$ such that $F^p\mathrm{C} \subset F^{p+1}\mathrm{C}$, $F^{-1}\mathrm{C} = \{0\}$, $\cup_{p\geq 0} F^p\mathrm{C} = \mathrm{C}$ and similarly for C'. Together with this data we have two chain maps $f\colon \mathrm{C} \to \mathrm{C}'$ and $g\colon \mathrm{C}' \to \mathrm{C}$ and a chain homotopy h between gf and the identity on C' (i.e. $gf = 1 + bh + hb$) satisfying the following requirements

- f and g are quasi isomorphisms
- f, g and h respect the filtrations

Define a perturbation δ of the differential b to be a map $\delta\colon \mathrm{C}_\bullet \to \mathrm{C}_{\bullet-1}$ such that

- $(b+\delta)^2 = 0$
- $\delta(F^p\mathrm{C}_\bullet) \subset F^{p-1}\mathrm{C}_\bullet$

Note that since δh decreases the filtration degree by 1, the power series $\sum_{i\geq 0}(\delta h)^i$ is well-defined, as $(\delta h)^i(x) = 0$ for i big enough for every $x \in \mathrm{C}_\bullet$.

The point of the Homological Perturbation Lemma to perturb the differential b on $\mathrm{C}_\bullet$, and the maps f and g to obtain quasi-isomorphisms between $(\mathrm{C}_\bullet, b+\delta)$ and $(\mathrm{C}'_\bullet, b'+\delta')$, and in particular do this with very explicit formulas.

Theorem A.2.1 (Homological Perturbation Lemma) [Cr04, 2.4] Given a pertubation δ of $(\mathrm{C}_\bullet, b)$, the operators

$$\delta' = f\left(\sum_{i\geq 0}(\delta h)^i\right)\delta g \qquad \widetilde{f} = f + f\left(\sum_{i\geq 0}(\delta h)^i\right)\delta h$$

$$\widetilde{g} = g + h\left(\sum_{i\geq 0}(\delta h)^i\right)\delta g \qquad \widetilde{h} = h + h\left(\sum_{i\geq 0}(\delta h)^i\right)\delta h$$

satisfy

- $(b' + \delta')^2 = 0$ (i.e. $b' + \delta'$ is a differential on $C'_\bullet$)
- $\widetilde{f}(b + \delta) = (b' + \delta')\widetilde{f}$ (i.e. $\widetilde{f}$ is a chain map $(C_\bullet, b + \delta) \to (C'_\bullet, b' + \delta')$)
- $\widetilde{g}(b' + \delta') = (b + \delta)\widetilde{g}$ (i.e. $\widetilde{g}$ is a chain map $(C'_\bullet, b' + \delta') \to (C_\bullet, b + \delta)$)
- $\widetilde{g}\widetilde{f} = 1 + (b + \delta)\widetilde{h} + \widetilde{h}(b + \delta)$ (i.e. $\widetilde{h}$ is a chain homotopy from $\widetilde{g}\widetilde{f}$ to the identity on $(C_\bullet, b + \delta)$
- $\widetilde{f}$ and $\widetilde{g}$ are quasi-isomorphisms

We discuss one important application of Homological Perturbation, namely the case where if $(\mathrm{C}_\bullet, b, B)$ is a mixed complex and $(\mathrm{C}'_\bullet, b')$ is a chain complex which is suitably quasi-isomorphic to $(\mathrm{C}_\bullet, b)$ we can use Homological Pertubation to obtain a second differential B' on C' to make $(\mathrm{C}'_\bullet, b', B')$ in a mixed complex so that $\mathrm{CC}_\bullet$ and $\mathrm{C}'\mathrm{C}_\bullet$ are quasi-isomorphic. Tracing back the assumptions of the Homological Pertubation Lemma, the following example arises.

Example A.2.2 Let $(\mathrm{C}_\bullet, b, B)$ be a mixed complex, let $(\mathrm{C}'_\bullet, b')$ be a chain complex, let $f\colon (\mathrm{C}_\bullet, b) \to (\mathrm{C}'_\bullet, b')$ and $g\colon (\mathrm{C}'_\bullet, b') \to (\mathrm{C}_\bullet, b)$ be inverse quasi-isomorphisms and let $h\colon \mathrm{C}_\bullet \to \mathrm{C}_{\bullet+1}$ be a homotopy from gf to the identity. We can extend f, g and h to maps between the induced spaces $\mathrm{CC}_\bullet$ and $\mathrm{C}'\mathrm{C}_\bullet$ in the obvious way. I.e. $f\colon \mathrm{CC}_\bullet \to \mathrm{C}'\mathrm{C}_\bullet$ becomes

$$f(x_n, x_{n-2}, ...) = (f(x_n), f(x_{n-2}), ...)$$

and similarly for g and h. In this way f becomes a chain map $(\mathrm{CC}_\bullet, b) \to (\mathrm{C}'\mathrm{C}_\bullet, b')$, g a chain map $(\mathrm{C}'\mathrm{C}_\bullet, b') \to (\mathrm{CC}_\bullet, b)$ and h becomes a homotopy between gf and the identity of $\mathrm{CC}_\bullet$. The complexes $(\mathrm{CC}_\bullet, b)$ and $(\mathrm{C}'\mathrm{C}_\bullet, b')$ have filtrations, by setting

$$F^p\mathrm{CC}_n = \bigoplus_{i=0}^{p} C_{n-2i}$$

The maps f, g and h respect this filtration, and this filtration is especially useful, as we can see B as a perturbation of the differential b on $\mathrm{CC}_\bullet$. We now assume that

$$f(Bh)^i Bg = 0$$

for any $i \geq 1$. If we now run the Homological Perturbation Lemma, we obtain perturbation δ' to the differential b' of $\mathrm{C'C}_\bullet$ given by

$$\delta'(x_n, x_{n-2}, ...) = (fBg(x_{n-2}), fBg(x_{n-4}), ...)$$

In particular we see that $B' = fBg$ becomes a second differential on $\mathrm{C}'_\bullet$ such that $(\mathrm{C}'_\bullet, b', B')$ is a mixed complex.

Furthermore the maps $\widetilde{f}\colon \mathrm{CC}_\bullet \to \mathrm{C'C}_\bullet$ and $\widetilde{g}\colon \mathrm{C'C}_\bullet \to \mathrm{CC}\bullet$ given by

$$\widetilde{f} = f + f\left(\sum_{i\geq 1}(Bh)^i\right)h \qquad \widetilde{g} = g + h\left(\sum_{i\geq 1}(Bh)^i\right)g$$

become inverse quasi-isomorphisms $(\mathrm{CC}_\bullet, b+B) \leftrightarrows (\mathrm{C'C}_\bullet, b'+B')$.

A.3 Chain complexes associated to simplicial vector spaces

In this section we outline a very general way to construct chain complexes, namely via simplicial vector spaces. This is of particular importance, since our main examples of complexes will be made out of this framework, and their properties are closely related to 'general abstract non-sense' associated to their simplicial origin. If not otherwise mentioned the definitions and results of this section can be found in [We94, Ch. 8].

Definition A.3.1 A *semi-simplicial vector space* is a sequence $\{V_n\}_{n\geq 0}$ of vector spaces, together with maps $d_i\colon V_n \to V_{n-1}$ $(0 \leq i \leq n)$, called face maps, that satisfy

$$d_i d_j = d_{j-1} d_i \qquad (i < j).$$

A *simplicial vector space* is a semi-simplicial vector space $(V_\bullet, d)$ with maps $s_j\colon V_n \to V_{n+1}$ $(0 \leq j \leq n)$, called degeneracy maps, satisfying the following relations

$$\begin{aligned} d_i s_j &= s_{j-1} d_i && (i < j),\\ d_i s_j &= \mathrm{id} && (i = j, \text{ or } i = j+1),\\ d_i s_j &= s_j d_{i-1} && (i > j+1),\\ s_i s_j &= s_{j+1} s_i. \end{aligned}$$

The reason we care about these objects is the following construction we can do on them:

Lemma A.3.2 If $(V_\bullet, d)$ is a semi-simplicial vector space we can define chain complexes $(\mathrm{C}_\bullet(V), b')$ and $(\mathrm{C}_\bullet(V), b)$, where $\mathrm{C}_k(V) = V_k$ and the maps $b'\colon V_n \to V_{n-1}$, $b\colon V_n \to V_{n-1}$ are given by

$$b' = \sum_{i=0}^{n-1}(-1)^i d_i,$$
$$b = b' + (-1)^n d_n.$$

While in the case of a simplicial vector space it is clear that the degeneracy maps do not contribute to the chain complex $\mathrm{C}_\bullet(V)$, they can be used to make the complex 'smaller'. Indeed, we can define the space of degeneracies to be $D_n = s_0 V_{n-1} + \cdots + s_{n-1}V_{n-1}$. It turns out that $D_\bullet$ is an acyclic subcomplex of $(\mathrm{C}_\bullet(V), b)$, so that the quotient

$$\mathrm{C}_\bullet(V)_{\mathrm{norm}} = \mathrm{C}_\bullet(V)/D_\bullet$$

is canonically a chain complex and the quotient map becomes a quasi-isomorphism. We call this resulting complex the *normalized complex*. Remark that while the individual face maps d_i do not descend to normalized complex, the differential b does.

Now we can slowly add more structure on $V_\bullet$ and look what kind of structure this induces on $\mathrm{C}_\bullet(V)$. The most important thing we add is a cyclic operator.

Definition A.3.3 A Λ_∞-*vector space* is a simplicial vector space $(V_\bullet, d, s)$ together with maps $t\colon V_n \to V_n$ such that

$$\begin{aligned} d_i t &= t d_{i-1} && i > 0 \\ d_0 t &= d_n && \\ s_i t &= t s_{i-1} && i > 0 \\ s_0 t &= t^2 s_n && \end{aligned}$$

If furthermore we have $t\colon V_n \to V_n$ satisfies $t^{n+1} = 1$ we call the resulting structure a *cyclic vector space*.

It is immediate that for any Λ_∞-vector space $(V_\bullet, d, s)$ we have that

$$(1 - (-1)^{n-1}t)b' = b(1 - (-1)^n t),$$

so that b descends to the quotient $V_\bullet^\lambda$ defined by

$$V_n^\lambda := V_n/(1 - (-1)^n t).$$

This leads to the following definition of a new chain complex:

Definition A.3.4 If $(V_\bullet, d, t)$ is part of the structure of an Λ_∞-vector space, then we define the cyclic complex $C_\bullet^\lambda(V)$ to be the given by $\mathrm{C}_n^\lambda(V) = V_n/(1 - (-1)^n t)$, with the differential the map induced by $b\colon V_n \to V_{n-1}$.

Now we want to understand the homology of $(\mathrm{C}_\bullet^\lambda(V), b)$ in terms of the homology of $(\mathrm{C}_\bullet(V), b)$. The idea here is that we can enrich $\mathrm{C}_\bullet(V)$ with the structure of a mixed complex so that the induced complex $\mathrm{CC}_\bullet(V)$ is quasi isomorphic to $\mathrm{C}_\bullet^\lambda(V)$.

We start with the following Lemma, where finally the degeneracies enter the stage. We include the proof of this and the next statement, since generally this is only treated for cyclic vector spaces, and we were not able to find a precise proof of this statement for Λ_∞-vector spaces.

Lemma A.3.5 Let $(V_\bullet, d, s, t)$ be a Λ_∞-vector space, then the map $c = ts_n : V_n \to V_{n+1}$ defines a contraction for the differential b' and satisfies $t^{n+2}c = ct^{n+1}$

Proof. This is again an exercise in bookkeeping and all the relevant commutation relations, but we do spell this one out. We want to show that

$$cb' + b'c = \mathrm{id}$$

Starting with cb' we have

$$\begin{aligned} cb' &= \sum_{i=0}^{n-1} (-1)^i t s_{n-1} d_i \\ &= \sum_{i=0}^{n-1} (-1)^i t d_i s_n \\ &= \sum_{i=0}^{n-1} (-1)^i d_{i+1} t s_n \\ &= -\sum_{i=1}^{n} (-1)^i d_i t s_n \\ &= -b'c + d_0 t s_n \\ &= -b'c + d_n s_n \\ &= -b'c + \mathrm{id} \end{aligned}$$

For the second statement we have

$$\begin{aligned} t^{n+2}c &= t^{n+3}s_n \\ &= t^{n+1}s_0 t \\ &= ts_n t^{n+1} \end{aligned}$$

□

Now from this contraction, or really any contraction of b', we can define an extra differential on $\mathrm{C}_\bullet(V)$:

Proposition A.3.6 If $(V_\bullet, d, t)$ is part of the structure of a Λ_∞-vector space and $c : V_n \to V_{n+1}$ is a contraction of b' such that $t^{n+2}c = ct^{n+1}$, then the map $B : V_n \to V_{n+1}$ given by

$$B = (1 - (-1)^{n+1}t)cb'c \left(\sum_{i=0}^{n} (-1)^{in} t^i \right)$$

satisfies

$$\begin{aligned} B^2 &= 0 \\ bB + Bb &= 1 - t^{n+1} \end{aligned}$$

Proof. To show that $B^2 = 0$ we have first note that

$$\left(\sum_{i=0}^{n+1}(-1)^{i(n+1)}t^i\right) b = b' \left(\sum_{i=0}^{n}(-1)^{in}t^i\right)$$

so that

$$b'(1 - t^{n+1}) = (1 - t^{n+2})b'$$

Then calculating B^2 we obtain

$$B^2 = (1 - (-1)^{n+2}t)cb'c(1 - t^{n+2})cb'c\left(\sum_{i=0}^{n}(-1)^{in}t^i\right)$$

using that both b' and c commute with the cyclic powers of t, we get

$$B^2 = (1 - (-1)^{n+2}t)cb'ccb'c(1 - t^{n+1})\left(\sum_{i=0}^{n}(-1)^{in}t^i\right)$$

Now zeroing in on the term $b'ccb'$ in the middle, using that c is a contraction of b' and that b' is a differential, we have

$$\begin{aligned} b'ccb' &= cb' - cb'cb' \\ &= cb' - cb' + ccb'b' \\ &= 0 \end{aligned}$$

and we conclude that $B^2 = 0$. Before we start with $bB + Bb$, we can so a similar trick

$$\begin{aligned} b'cb'c &= -b'b'cc + b'c \\ &= b'c \\ &= \mathrm{id} - cb' \\ &= \mathrm{id} - cb'cb' - cb'b'c \\ &= \mathrm{id} - cb'cb' \end{aligned}$$

Then we start with bB, where we have

$$\begin{aligned}
bB =& b(1-(-1)^{n+1}t)cb'c\left(\sum_{i=0}^{n}(-1)^{in}t^i\right)\\
=&(1-(-1)^n t)b'cb'c\left(\sum_{i=0}^{n}(-1)^{in}t^i\right)\\
=&(1-(-1)^n t)\left(\sum_{i=0}^{n}(-1)^{in}t^i\right)\\
&-(1-(-1)^n t)cb'cb'\left(\sum_{i=0}^{n}(-1)^{in}t^i\right)\\
=&1-t^{n+1}-(1-(-1)^n t)cb'c\left(\sum_{i=0}^{n-1}(-1)^{i(n-1)}t^i\right)b\\
=&1-t^{n+1}-Bb
\end{aligned}$$

which concludes the proof. □

The important result is then the following

Theorem A.3.7 If $(V_\bullet, d, s, t)$ is a cyclic vector space, then the map $\mathrm{CC}_\bullet(V) \to \mathrm{C}^\lambda_\bullet(V)$ which is given in degree n by

$$(x_n, x_{n_2}, \ldots) \mapsto [x_n]$$

is a quasi isomorphism $(\mathrm{CC}_\bullet(V), b+B) \to (\mathrm{C}^\lambda_\bullet(V), b)$.

Remark A.3.8 If (V, d, s, t) is a cyclic vector space, and we take the contraction $c = ts_n$ of b', then B descends to the normalized complex where it is given by

$$B_{\text{norm}} = \sum_{i=0}^{n}(-1)^{in}ts_n t^i$$

Remark A.3.9 If (V, d, s, t) is a cyclical vector space, we may also define $B\colon V_n \to V_{n+1}$ to be

$$B = (1-(-1)^{n+1}t)c\left(\sum_{i=0}^{n}(-1)^{in}t^i\right)$$

This operator satisfies $B^2 = 0$ because if one writes down $B \circ B$ one gets a term $\left(\sum_{i=0}^{n+1}(-1)^{i(n+1)}t^i\right)(1-(-1)^{n+1}t) = (1-(-1)^{n(n+1)}t^{n+2}) = 0$ in the middle. Furthermore, it satisfies $bB + Bb = 0$ similarly to the calculation above, and so we end up with an a priori different structure of a mixed complex on $\mathrm{C}_\bullet(V)$. However, it is easy to see that on the normalized complex this B is *equal* to the original B_{norm} and hence the two definitions of B induce quasi-isomorphic structures.

A.4 Cylindrical spaces

In this section we discuss cylindrical spaces. These are a type of bisimplicial vector spaces that allow us to put the double complex of Section 3.2 in an efficient framework.

Definition A.4.1 A *bisimplicial vector space* is a collection of vector spaces $V_{p,q}$ for $p, q \in \mathbb{N}$ and maps

$$\begin{aligned} &d_i^h \colon V_{p,q} \to V_{p-1,q} \quad (0 \le i \le p) &&d_i^v \colon V_{p,q} \to V_{p,q-1} \quad (0 \le i \le q)\\ &s_i^h \colon V_{p,q} \to V_{p+1,q} \quad (0 \le i \le p) &&s_i^h \colon V_{p,q} \to V_{p,q+1} \quad (0 \le i \le q)\end{aligned}$$

satisfying the following properties

- For every q, $(V_{\bullet,q}, d^h, s^h)$ is a simplicial vector space
- For every p, $(V_{p,\bullet}, d^v, s^v)$ is a simplicial vector space
- For every suitable choice of i and j we have

$$d_i^h d_j^v = d_j^v d_i^h \qquad d_i^h s_j^v = s_j^v d_i^h \qquad s_i^h d_j^v = d_j^v s_i^h \qquad s_i^h s_j^v = s_j^v s_i^h$$

Definition A.4.2 A $\Lambda_\infty \times \Lambda_\infty$*-vector space* is a collection of vector spaces $V_{p,q}$ for $p, q \in \mathbb{N}$ and maps

$$\begin{aligned} &d_i^h \colon V_{p,q} \to V_{p-1,q} \quad (0 \le i \le p) &&d_i^v \colon V_{p,q} \to V_{p,q-1} \quad (0 \le i \le q)\\ &s_i^h \colon V_{p,q} \to V_{p+1,q} \quad (0 \le i \le p) &&s_i^h \colon V_{p,q} \to V_{p,q+1} \quad (0 \le i \le q)\\ &t^h \colon V_{p,q} \to V_{p,q} &&t^v \colon V_{p,q} \to V_{p,q}\end{aligned}$$

satisfying the following

- For every q, $(V_{\bullet,q}, d^h, s^h)$ is a Λ_∞-vector space
- For every p, $(V_{p,\bullet}, d^v, s^v)$ is a Λ_∞-vector space
- For every suitable choice of i and j we have

$$\begin{aligned} d_i^h d_j^v &= d_j^v d_i^h & d_i^h s_j^v &= s_j^v d_i^h & d_i^h t^v &= t^v d_i^h\\ s_i^h d_j^v &= d_j^v s_i^h & s_i^h s_j^v &= s_j^v s_i^h & s_i^h t^v &= t^v s_i^h\\ t^h d_j^v &= d_j^v t^h & t^h s_j^v &= s_j^v t^h & t^h t^v &= t^v t^h\end{aligned}$$

We call a $\Lambda_\infty \times \Lambda_\infty$-vector space *cylindrical* if $(t^h)^{p+1}(t^v)^{q+1}$ equals the identity on $V_{p,q}$ for every p and q, and we call a $\Lambda_\infty \times \Lambda_\infty$-vector space *bicyclic* if $(t^h)^{p+1}$ and $(t^v)^{q+1}$ both equal the identity on $V_{p,q}$ for every p and q.

Now, from these bisimplicial vector spaces we can construct simplicial vector spaces.

Lemma A.4.3 If $(V_{\bullet,\bullet}, d^h, s^h, d^v, s^v)$ is a bisimplicial vector space, then $(\mathrm{diag}(V)_\bullet, d, s)$ is a simplicial vector space, where

$$\begin{aligned}\mathrm{diag}(V)_n &= V_{n,n}\\ d_i &= d_i^h d_i^v\\ s_i &= s_i^h s_i^v\end{aligned}$$

We can also crank this up to the $\Lambda_\infty \times \Lambda_\infty$-environment, where we recognize the rationale about the definition of a cylindrical space.

Lemma A.4.4 If $(V_{\bullet,\bullet}, d^h, s^h, t^h, d^v, s^v, t^v)$ is a $\Lambda_\infty \times \Lambda_\infty$-vector space, then $(\mathrm{diag}(V)_\bullet, d, s, t)$ is a Λ_∞-vector space, where

$$\begin{aligned}\mathrm{diag}(V)_n &= V_{n,n}\\ d_i &= d_i^h d_i^v\\ s_i &= s_i^h s_i^v\\ t &= t^h t^v\end{aligned}$$

Furthermore, $\mathrm{diag}(V)$ is a cyclic vector space if and only if V is a cylindrical vector space.

The raison d'être of bisimplicial vector spaces is that just as one can go from simplicial vector spaces to chain complexes, we can go from bisimplicial vector spaces to double complexes.

Proposition A.4.5 Let $(V_{\bullet,\bullet}, d^h, d^v)$ be part of a bisimplicial vector space structure. We define the operators $b^h \colon V_{p,q} \to V_{p-1,q}$, $b^v \colon V_{p,q} \to V_{p,q-1}$ by

$$b^h = \sum_{i=0}^{p} (-1)^i d_i^h \qquad\qquad b^v = \sum_{i=0}^{q} (-1)^{i+p} d_i^v$$

These operators satisfy

$$(b^h)^2 = 0 \qquad\qquad (b^v)^2 = 0 \qquad\qquad b^h b^v + b^v b^h = 0$$

i.e. they induce a double complex $\mathrm{C}_{\bullet,\bullet}(V) = (V_{\bullet,\bullet}, b^h, b^v)$.

Again we may jazz this up to the Λ_∞-setting, where again we recover the importance of cylindrical spaces.

Proposition A.4.6 Let $(V_{\bullet,\bullet}, d^h, t^h, d^v, t^v)$ be part of a $\Lambda_\infty \times \Lambda_\infty$-structure. Let $c^h \colon V_{p,q} \to V_{p+1,q}$ and $c^v \colon V_{p,q} \to V_{p,q+1}$ be maps satisfying

- The family of maps c^h commutes with the families of maps d^v, t^v, the map c^v and the collection of maps $(t^h)^{p+1} \colon V_{p,q} \to V_{p,q}$.

- The family c^h contracts the differential $(b')^h = \sum_{i=0}^{p-1} d_i^h$ on $V_{\bullet,q}$.
- The family of maps c^v commutes with the families of maps d^h, t^h and the collection of maps $(t^v)^{q+1}\colon V_{p,q} \to V_{p,q}$.
- The family c^v contracts the differential $(b')^v = \sum_{i=0}^{q-1} d_i^v$ on $V_{p,\bullet}$.

We define operators $b^h\colon V_{p,q} \to V_{p-1,q}$, $b^v\colon V_{p,q} \to V_{p,q-1}$, $B^h\colon V_{p,q} \to V_{p-1,q}$, $B^v\colon V_{p,q} \to V_{p,q+1}$ by the formulae

$$b^h = \sum_{i=0}^{p}(-1)^i d_i^h \qquad B^h = (1-(-1)^{p+1}t^h)c^h(b')^h c^h\left(\sum_{i=0}^{p}(-1)^{ip}(t^h)^i\right)$$

$$b^v = \sum_{i=0}^{q}(-1)^{i+p} d_i^v \qquad B^v = (-1)^p (t^h)^{p+1}(1-(-1)^{q+1}t^v)c^v(b')^v c^v\left(\sum_{i=0}^{q}(-1)^{iq}(t^v)^i\right)$$

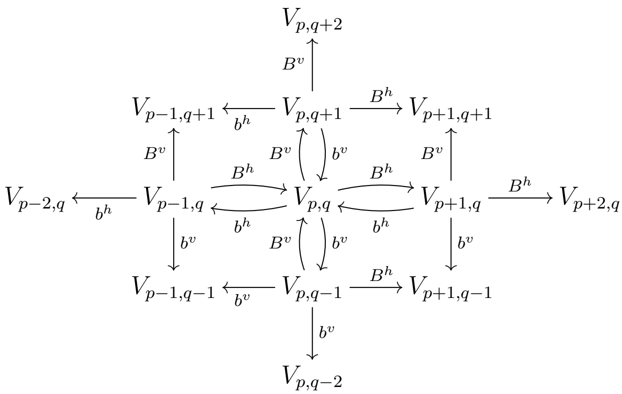
Satisfy

$$(b^h)^2 = 0 \qquad (b^v)^2 = 0 \qquad (B^h)^2 = 0 \qquad (B^v)^2 = 0$$

$$B^v b^h + b^h B^v = 0 \qquad B^v B^h + B^h B^v = 0 \qquad b^v B^h + B^h b^v = 0 \qquad b^h b^v + b^v b^h = 0$$

$$b^h B^h + B^h b^h + B^v b^v + b^v B^v = 1 - (t^h)^{p+1}(t^v)^{q+1}$$

Remark A.4.7 We see that the relations described above can also be represented by the following diagram



in that the sum of all the ways to end up in $V_{p,q}$ equal $1-(t^h)^{p+1}(t^v)^{q+1}$, while the sum of the ways to end up in $V_{p,q+2}$, $V_{p+1,q+1}$, $V_{p+2,q}$, $V_{p+1,q-1}$, $V_{p,q-2}$, $V_{p-1,q-1}$, $V_{p-2,q}$ and $V_{p-1,q-1}$ are zero. In particular, if $(V_{\bullet,\bullet}, d^h, t^h, d^v, t^v)$ is cylindrical, this is essentially the same thing as saying that $(b^h+b^v+B^h+B^v)^2 = 0$. This leads to the following important result.

Corollary A.4.8 If, in the situation of the previous proposition, $V_{\bullet,\bullet}$ is cylindrical, then $(\mathrm{TotC}_{\bullet,\bullet}(V), b^h + b^v, B^h + B^v)$ is a mixed complex.

Remark A.4.9 We can also normalize these double complexes. Indeed, setting

$$D^h_{p,q} = s^h_0 C_{p-1,q} + \cdots + s^h_{p-1} C_{p-1,q}$$

and

$$D_{p,q} = D^h_{p,q} + s^v_0 C_{p,q-1} + \cdots + s^v_{q-1} C_{p,q-1}$$

we can show that $D_{\bullet,\bullet}$ is an acyclic double subcomplex of $(V_{\bullet,\bullet}, b^h, b^v)$. To see this first note that it is obviously true for $D^h_{\bullet,\bullet}$, since it consists of the degeneracies of the horizontal simplicial spaces $V_{\bullet,q}$. In particular $(D^h_{\bullet,\bullet}, b^h)$ is acyclic, and by a spectral sequence argument so is $(D^h_{\bullet,\bullet}, b^h + b^v)$.

Then if we consider the short exact sequence

$$0 \to D^h_{\bullet,\bullet} \to D_{\bullet,\bullet} \to D_{\bullet,\bullet}/D^h_{\bullet,\bullet} \to 0$$

we note that $D_{\bullet,\bullet}/D^h_{\bullet,\bullet}$ is isomorphic to the degeneracies of the verticular simplicial spaces $V_{\bullet,\bullet}/D^h_{\bullet,\bullet}$ and hence similar to the above it is acyclic. In particular we can conclude that $D_{\bullet,\bullet}$ is acyclic with respect to $b^h + b^v$, and hence the quotient map $V_{\bullet,\bullet} \to V_{\bullet,\bullet}/D_{\bullet,\bullet}$ is a quasi-isomorphism.

What rests is seeing how $(\mathrm{C}_\bullet(\mathrm{diag}(V)), b, B)$ and $(\mathrm{Tot}(\mathrm{C}_{\bullet,\bullet}(V)), b^h + b^v, B^h + B^v)$ compare. In the next section we will see that they are quasi-isomorphic.

A.5 The Eilenberg-Zilber Theorem

To compare $(\mathrm{C}_\bullet(\mathrm{diag}(V)), b)$ and $(\mathrm{Tot}(\mathrm{C}_{\bullet,\bullet}(V)), b^h + b^v)$ we discuss the Eilenberg-Zilber Theorem, which shows that they are homotopy equivalent.

So let us fix a bisimplicial vector space $(V_{\bullet,\bullet}, d^h, s^h, d^v, s^v)$. Following [We94, 8.5.4], we write down collections of maps $\mathrm{EZ}_{p,q}\colon V_{p+q,p+q} \to V_{p,q}$ and $\nabla_{p,q}\colon V_{p,q} \to V_{p+q,p+q}$ which will become inverse quasi-isomorphisms.

$$\mathrm{EZ}_{p,q} = d^h_{p+1} \cdots d^h_{p+q} \cdot (d^v_0)^p$$
$$\nabla_{p,q} = \sum_{\sigma \in \mathrm{Sh}(p,q)} (-1)^\sigma s^h_{\sigma(p+q)-1} \cdots s^h_{\sigma(p+1)-1} s^v_{\sigma(p)-1} \cdots s^v_{\sigma(1)-1}$$

here $\mathrm{Sh}(p,q)$ is the set of those permutation σ of the set $\{1, ..., p+q\}$ such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$.

Following [Re00, Thm 3.1], we also write down a map $h\colon V_{n,n} \to V_{n+1,n+1}$ that will induce the homotopy between $\nabla\mathrm{EZ}$ and the identity. We set $h\colon V_{0,0} \to V_{1,1}$ to be zero, and for $n > 0$ define $h\colon V_{n,n} \to V_{n+1,n+1}$ by

$$h = \sum_{q=0}^{n-1} \sum_{p=0}^{n-q-1} \sum_{\sigma \in \mathrm{Sh}(p+1,q)} (-1)^{\overline{n}} (-1)^\sigma s^v_{\sigma(p+q+1)+\overline{n}} \cdots s^v_{\sigma(p+2)+\overline{n}} d^v_{n-q+1} \cdots d^v_n \cdot$$
$$\cdot\, s^h_{\sigma(p+1)+\overline{n}} \cdots s^h_{\sigma(1)+\overline{n}} d^h_{n-p-q} \cdots d^h_{n-q-1}$$

where $\overline{n} = n - p - q - 1$.

The Eilenberg-Zilber Theorem now says that these 3 maps encode enough information to make $\mathrm{diag}V_\bullet$ and $\mathrm{Tot}V_{\bullet,\bullet}$ quasi-isomorphic.

Theorem A.5.1 (Eilenberg-Zilber) [EZ53] The maps EZ, ∇ and h satisfy the following properties:

- $\mathrm{EZ}\colon (\mathrm{diag}V_\bullet, b) \to (\mathrm{Tot}V_{\bullet,\bullet}, b^h + b^v)$ is a chain map
- $\nabla\colon (\mathrm{Tot}V_{\bullet,\bullet}, b^h + b^v) \to (\mathrm{diag}V_\bullet, b)$ is a chain map
- EZ, ∇ and h descend to maps between the normalized complexes induced by $\mathrm{diag}V_\bullet$ by $V_{\bullet,\bullet}$ respectively
- As maps between the normalized complexes, EZ, ∇ and h satisfy
  - $\mathrm{EZ}\nabla = 1$
  - $\nabla\mathrm{EZ} = 1 + bh + hb$

In particular EZ and ∇ are quasi-inverse quasi-isomorphisms in both the unnormalized and normalized settings.

Proof. The bullet-points can all be proven using explicit computations with the bisimplicial identities. As for the last sentence: by the fact that in the normalized setting EZ and ∇ are inverse homotopy equivalences, we now that they are quasi-inverse quasi-isomorphisms in the normalized setting, and by looking at the diagram

$$\begin{array}{ccccc}
\mathrm{diag}V_\bullet & \xrightarrow{\mathrm{EZ}} & \mathrm{Tot}V_{\bullet,\bullet} & \xrightarrow{\nabla} & \mathrm{diag}V_\bullet \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
(\mathrm{diag}V_\bullet)_{\mathrm{norm}} & \xrightarrow[\simeq]{\mathrm{EZ}} & \mathrm{Tot}(V_{\bullet,\bullet})_{\mathrm{norm}} & \xrightarrow[\simeq]{\nabla} & (\mathrm{diag}V_\bullet)_{\mathrm{norm}}
\end{array}$$

we see that the same holds in the unnormalized setting.

Next, we want to jazz this up to the cyclic situation, and for this we refer to work by Khalkali and Rangipour [KR04] which does this for us. They show that the requirements of Example A.2.2 are satisfied, meaning that

$$\mathrm{EZ}B\mathrm{Sh} = B^h + B^v$$

and

$$\mathrm{EZ}(Bh)^i B\mathrm{Sh} = 0$$

for $i \geq 1$. The result that we will use is then as follows.

Theorem A.5.2 (Generalized cyclic Eilenberg-Zilber Theorem) [KR04, Thm 3.1]
For $(V_{\bullet,\bullet}, d^h, s^h, t^h, d^v, s^v, t^v)$ a cylindrical vector space, the map

$$\mathrm{EZ}^{\mathrm{pert}} = \mathrm{EZ} + \mathrm{EZ}\left(\sum_{i\geq 1}(Bh)^i\right)$$

defines a quasi-isomorphism

$$\mathrm{EZ}^{\mathrm{pert}}\colon (\mathrm{CC}_\bullet(\mathrm{diag}(V)), b + B) \xrightarrow{\simeq} ((\mathrm{CTotC})_\bullet(V), b^h + b^v + B^h + B^v)$$

□

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Samenvatting

Equivariante theorie van Lie groepoïden vanuit het perspectief van niet-commutatieve meetkunde

In dit proefschrift beschrijven we verbanden tussen de equivariante theorie van Lie groepoïden en de niet-commutatieve meetkunde van de convolutie algebra.

Lie groepoïden zijn objecten die symmetrieën van een ruimte beschrijven. Ze zijn een generalisatie van Lie groepen in de volgende zin: Lie groepen beschrijven symmetrieën die globaal op de ruimte toe te passen zijn, terwijl Lie groepoïden symmetrieën beschrijven die plaatsafhankelijk zijn.

Deze objecten zijn interessant vanwege hun toepassingen in de natuurkunde. Het bestaan van oplossingen van een natuurkundig systeem kan worden aangetoond of ontkracht via de meetkundige eigenschappen van de ruimte waarop het systeem leeft. In het geval van een symmetrie waaronder het systeem hetzelfde blijft kan de ruimte waarin het systeem opgelost dient te worden verkleind worden. Als voorbeeld kunnen we kijken naar de zwaartekracht die een vast punt uitoefent op een deeltje dat rond het vaste punt draait. Dit systeem is invariant onder rotaties, en in praktijk is de relevante vergelijking die opgelost dient te worden de vergelijking voor de afstand tussen het deeltje en het vaste punt.

In deze gedachtelijn zijn we geïnteresseerd in de 'meetkunde van de ruimte' die 'invariant is onder de symmetrie'. In het geval van een werking van een Lie groep op een ruimte is dit iets wat we redelijk goed snappen, en in dit proefschrift proberen we de ideeën hierover te generaliseren naar Lie groepoïden.

Niet-commutatieve meetkunde is een wiskundige theorie die probeert meetkundige ideeën over ruimtes te herformuleren in termen van algebraïsche eigenschappen van de 'gladde functies' (of 'observabelen') op de ruimte. Voorbeelden van meetkundige informatie die zo verkregen kan worden is het 'aantal gaten', waarbij de lijn geen gaten heeft, de cirkel een '1-dimensionaal gat' en de holle bol een '2-dimensionaal gat'. Het is verbazingwekkend dat dit soort informatie te verkrijgen is uit puur algebraïsche procedures, en als we dit feit omdraaien kunnen we praten over 'meetkundige eigenschappen van een algebra' door dezelfde procedures uit te voeren.

Als we deze ideeën willen toepassen op natuurkundige systemen met symmetrieën zien we dat het oplossen van een symmetrie ook een algebraïsch equivalent heeft. De stap die we maken naar een makkelijkere ruimte via de symmetrie (in het voorbeeld van het draaiende deeltje is dit de stap die we maken van 'plaats' naar 'afstand') is ook te beschrijven via de 'convolutie algebra' van de symmetrie: dit is een algebra die een combinatie is van de gladde functies en de werking die de symmetrie op deze gladde functies heeft. Dit betekent dat we de reductie ook kunnen doen in gevallen waar de symmetrie zo ingewikkeld is, dat de 'makkelijkere ruimte' ingewikkeld is. Belangrijker is dat dergelijke convolutie algebras ook bestaan voor symmetrieën die beschreven worden door Lie groepoïden, niet alleen voor symmetrieën beschreven door groepen.

De conclusie is dat we de niet-commutatieve meetkunde van dergelijke convolutie algebras willen begrijpen, en dat is de wiskundige inhoud van dit proefschrift. We beschrijven verbanden tussen verschillende wiskundige eigenschappen van een Lie groepoïde en de niet-commutatieve meetkunde van de convolutie algebra, met als doel een volledig beeld te scheppen van deze niet-commutatieve meetkunde.

In het eerste deel van dit proefschrift beschrijven we een verband tussen deformaties van de groepoïde -dat zijn 'kleine veranderingen' die je kan doen aan de groepoïde- en de niet-commutatieve meetkunde van de convolutie algebra. In het tweede deel bekijken we het geval van symmetrieën beschreven door een groep, en zien we hoe het verband tussen de convolutie algebra en de 'meetkunde van de ruimte die invariant is onder de symmetrieën' precies werkt. In het laatste stuk behandelen we de theorie van infinitesimale symmetrieën, omdat resultaten voor symmetrieën en infinitesimale symmetrieën veel wisselwerking hebben.

Abstract

Equivariant theory of Lie groupoids from the perspective of non-commutative geometry

In this dissertation we describe connections between the equivariant theory of Lie groupoids and the non-commutative geometry of the convolution algebra.

Lie groupoids are objects that encode symmetries of a space. They are a generalization of Lie groups in the following sense: Lie groups describe symmetries that are globally defined on the space, while Lie groupoids describe symmetries whose application is place-dependent.

These objects are interesting because of their applications in physics. The existence of solutions to a physical system can be proven or disproven by exhibiting geometric properties of the space on which the system is applied. When there is a symmetry under which the system is invariant, you can shrink the space on which one needs to solve the system by factoring out the symmetry. As an example, we can look at a fixed source that exercises a gravitational force on a particle flying around it. This system is invariant under rotation, and in practice the only relevant equation that on needs to solve is that for the distance between the particle and the point source.

In this philosophy we are interested in the 'geometry of the space that is invariant under the symmetry'. In case where there is an action of a Lie group on the space, this is something we understand reasonably well, and in this dissertation we try the generalize these ideas to Lie groupoids.

Non-commutative geometry is a mathematical theory that tries to reformulate geometric ideas of spaces in terms of algebraic properties of the 'smooth functions' (or 'observables') of the space. Examples of geometric information that can be obtained in this way is the 'number of holes', where a line has no holes, a circle has a '1-dimensional hole' and the sphere has a '2-dimensional hole'. It is an astounding fact that this kind of information is obtainable purely using algebraic procedures, and reversing this fact we can talk about 'geometric properties of an algebra' by preforming these procedures.

If we want to apply these ideas to physical systems with symmetries we see that factoring out the symmetry has an algebraic counterpart. The step that we make to an easier space by factoring out (in the example of the orbiting particle this is the step from 'place' to 'distance') can also be described by the 'convolution algebra' of the symmetry:

this is an algebra that combines the smooth function of the space and the action the symmetry has on these smooth functions. This means that we can do the reduction, even when the symmetry is so complicated that the 'easier space' is complicated. More importantly, such convolution algebras also exist for symmetries that are described using Lie groupoids, not just for symmetries defined by groups.

The conclusion is that we want to understand the non-commutative geometry of such convolution algebras, and that is the mathematical content of this dissertation. We described connections between various mathematical properties of a Lie groupoid and the non-commutative geometry of the convolution algebra, with the goal to sketch a complete picture of this non-commutative geometry.

In the first part of this dissertation we describe a connection between deformations of the groupoid -those are 'small changes' one can make to the groupoid- and the non-commutative geometry of the convolution algebra. In the second part we look the case of symmetries described by groups, and look at how the connection between the convolution algebra and the 'invariant geometry of the space' really works. In the last part we treat the theory of infinitesimal symmetries, in part because results for symmetries and infinitesimal symmetries have a lot of interplay.