Linear intersection pairs of negacyclic codes and their applications in construction of entanglement-assisted quantum error-correcting codes

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Abstract. Linear intersection pairs of linear codes have become of interest and continuously studied due to their nice algebraic properties and wide applications. In this article, we focus on linear intersection pairs of negacyclic codes over finite fields and their applications. General characterization and algebraic properties of such pairs are given in terms of their generator polynomials. For $s\in\{0,1\}$, explicit constructions of linear s-intersection pairs and linear s-complementary pairs of negacyclic codes are presented. As applications, constructions of entanglement-assisted quantum error-correcting codes are discussed together with illustrative examples.

1. Introduction

Families of cyclic and negacyclic codes over finite fields have been of interest since they have nice algebraic structures and can be applied in various fields. The algebraic structures of such codes are given though the ideals in $F_q[x]/<x^n-1>$ and $F_q[x]/<x^n+1>$ (see [1]). In applications, such algebraic structured codes can be easily implemented in shift registers [2]. Linear complementary pairs (LCPs) of linear codes over finite fields have been introduced in [3] and shown to have applications in cryptography. In [3] and [4], LCPs have been applied in in counter passive and active side-channel analysis attacks on embedded crypto systems. In [3] and [4], several constructions of LCPs of linear codes were given as well.

In [5], the author formulated the notion of a linear intersection pair of linear codes. Subsequently, their applications in constructions of entanglement-assisted quantum error-correcting codes (EAQECCs) have been presented as well. For an integer $s\geq 0$, a linear s-intersection pair of linear codes is defined to be linear codes C and D of the same length n over F_q whose intersection has dimension s. This can be viewed as a generalization of various concepts in coding theory such as complementary dual codes, self-orthogonal codes, hulls, and LCPs.

Recently, linear s-intersection pairs of cyclic codes were studied in [6] and linear 1-complementary pairs of negacyclic codes were studied in [7]. In this article, we extend some concepts in [6] and [7] to s-intersection pairs and s-complementary pairs of negacyclic codes over finite fields. Such pairs have different parameters compare to [6] and they are optimal in some cases. Moreover, such pairs have more algebraic structures than the pairs of linear codes in [5] and they can be contracted directly using ideals in polynomial rings. Precisely, we focus on linear s-intersection pairs and linear s-complementary pairs of negacyclic codes over finite fields. General characterization of such pairs of negacyclic codes is given in terms of their generator polynomials. For $s\in\{0,1\}$, explicit constructions

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of linear s-intersection pairs and linear s-complementary pairs of negacyclic codes are given as well as applications in construction of EAQECCs. Some illustrative examples of linear intersection pairs derived from negacyclic codes are provided. Some of them have optimal parameters. Examples of EAQECCs and EAQECCs with positive net-rate are presented as well.

After this introductory section, basic concepts and results on codes and linear intersection pairs are recalled in Section 2. Characterizations of linear s-intersection pairs and linear s-complementary pairs of negacyclic codes are presented in Section 3 together with constructions of two negacyclic codes whose intersecting dimension is 0 or 1. Applications in constructions of EAQECCs are provided in Section 4 as well as illustrative examples. Summary and discussion are provided in Section 5.

2. Preliminaries

Basic concepts and properties of linear codes, negacyclic codes, linear intersection pairs, linear complementary pairs are recalled. For more details, the reader may refer to [1], [2], [4], [5], and [7].

2.1. Linear intersection pairs of codes

Let q be a prime power and let n be a positive integer. Let F_q be a finite field of q elements. A linear code of length n is defined to be a subspace of $(F_q)^n$. An $[n, k]_q$ code refers to a linear code C of length n over F_q whose dimension dim(C) is k. An $[n, k]_q$ code C is called an $[n, k, d]_q$ code if the minimum hamming weight of C is d. We denote by $C^{\perp}$ the dual code of C under the Euclidean inner product.

Linear codes C_1 and C_2 of length n over F_q form a linear complementary pair (LCP) if $C_1 \cap C_2=\{\mathbf{0}\}$ and $C_1+C_2=(F_q)^n$ (see [4]). For a non-negative integer s, C_1 and C_2 are said to be a linear s-intersection pair if $\dim(C_1 \cap C_2)=s$ (see, [5]). A linear s-intersection pair C_1 and C_2 is called a linear s-complementary pair if $C_1+C_2=(F_q)^n$ (cf. [7]). Clearly, a linear 0-complementary pair is a LCP.

2.2. Negacyclic codes over finite fields

A linear code C of length n over F_q is called a negacyclic code if $(c_{n-1}, c_0, \ldots, c_{n-2}) \in C$ for all $(c_0, c_1, \ldots, c_{n-1}) \in C$ (see [7]). It is well known that every negacyclic code of length n over F_q can be represented by a principal ideal in $F_q[x]/\langle x^n+1\rangle$ whose generator is a unique monic divisor of x^n+1 of minimal degree. Such a monic polynomial is called the generator polynomial of the negacyclic code C.

Properties of negacyclic codes over finite fields are given in terms of their generator polynomials (see, for example, [1] and [7]).

Theorem 2.1: Let C be a negacyclic code of length n over F_q with generator polynomial g(x) and let k be an integer such that $0 \le k \le n$. Then C has dimension k if and only if deg(g(x))=n-k.

Theorem 2.2: Let C_1 and C_2 be negacyclic codes of length n over F_q with generator polynomials $g_1(x)$ and $g_2(x)$, respectively. Then $C_1 \cap C_2$ is a negacyclic code of length n generated by $\mathrm{lcm}(g_1(x), g_2(x))$ and C_1+C_2 is a negacyclic code of length n generated by $\gcd(g_1(x), g_2(x))$.

3. Linear intersection pairs of negacyclic codes over finite fields

Characterizations of a linear s-intersection pair and a linear s-complementary pair of negacyclic codes are presented. For $s \in \{0,1\}$, explicit constructions of such pairs are given as well.

3.1. Characterization of linear intersection pairs and linear complementary pairs of negacyclic codes

General characterizations of a linear s-intersection pair and a linear s-complementary pair of negacyclic codes are given as follows.

Theorem 3.1: Let F_q be a finite field and let n be a positive integer co-prime to q. Let $s \ge 0$ be an integer. Let C_1 and C_2 be negacyclic codes of length n over F_q with generator polynomials $g_1(x)$ and $g_2(x)$, respectively. Then the following statements hold.

(1) C_1 and C_2 form a linear s-intersection pair if and only if $\deg(\mathrm{lcm}(g_1(x), g_2(x)))=n-s$.
(2) C_1 and C_2 form a linear s-complementary pair if and only if $\deg(\mathrm{lcm}(g_1(x), g_2(x)))=n-s$ and $\gcd(g_1(x),g_2(x))=1$.

Proof. First, we prove 1). Assume that C_1 and C_2 form a linear s-intersection pair. Then $\dim(C_1 \cap C_2)=s$. By Theorem 2.2, $C_1 \cap C_2$ is generated by $\mathrm{lcm}(g_1(x), g_2(x))$. Hence, $\deg(\mathrm{lcm}(g_1(x), g_2(x)))=n-s$ by

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Theorem 2.1. Conversely, assume that $\deg(\mathrm{lcm}(g_1(x), g_2(x)))=n-s$. By Theorem 2.2, $C_1 \cap C_2$ is negacyclic whose generator polynomial is $\mathrm{lcm}(g_1(x), g_2(x))$. Since $\deg(\mathrm{lcm}(g_1(x), g_2(x)))=n-s$, it follows that $\dim(C_1 \cap C_2)=s$ by Theorem 2.1. Hence, C_1 and C_2 form a linear s-intersection pair.

Next, we note that $C_1+C_2=(F_q)^n$ if and only if C_1+C_2 is generated by $1= \gcd(g_1(x), g_2(x))$. Hence, 2) follows from the previous discussion and 1).

Some illustrative examples of a linear s-intersection pair are given.

Example 3.1: Let C_1 and C_1 be $[6, 3]_5$ and $[6, 3]_5$ negacyclic codes of length 6 over F_5 with generator polynomials $g_1(x)=x^3 + 2$ and $g_2(x)=x^3 + x^2 + 3x + 2$, respectively. Since $\mathrm{lcm}(g_1(x), g_2(x))= x^5 + 3x^4 + 4x^3 + 2x^2 + x + 3$, we have $\deg(\mathrm{lcm}(g_1(x), g_2(x)))=5=6-1$ which implies that C_1 and C_1 form a linear 1-intersection pair. Since $\gcd(g_1(x), g_2(x))= x+3 \neq 1$, C_1 and C_1 are not linear 1-complementary pair.

Example 3.2: Let C_1 and C_1 be $[6, 3]_5$ and $[6, 4]_5$ negacyclic codes of length 6 over F_5 with generator polynomials $g_1(x)=x^2 + 2x + 4$ and $g_2(x)=x^3 + x^2 + 3x + 2$, respectively. Since $\mathrm{lcm}(g_1(x), g_2(x))= x^5 + 3x^4 + 4x^3 + 2x^2 + x + 3$, we have $\deg(\mathrm{lcm}(g_1(x), g_2(x)))=5=6-1$ which implies that C_1 and C_1 form a linear 1-intersection pair. Since $\gcd(g_1(x), g_2(x))= 1$, C_1 and C_1 are also a linear 1-complementary pair.

3.2. Constructions of linear intersection pairs negacyclic codes

A general construction for linear s-intersection pairs of negacyclic codes is given.

Theorem 3.2: Let F_q be a finite field and let n be a positive integer such that $\gcd(n,q)=1$. Let $s\geq 0$ be an integer and let e(x) be a monic divisor of x^n+1 of degree s. If a(x) and b(x) are monic polynomials such that $b(x)|a(x)|\ \frac{x^n+1}{e(x)}$ over F_q, then the negacyclic codes with generator polynomials a(x) and $c(x):=\frac{(x^n+1)b(x)}{e(x)a(x)}$ form a linear s-intersection pair.

Proof. From the definition, we have $\mathrm{lcm}(a(x), c(x))=\frac{x^n+1}{e(x)}$ which implies that $\deg(\mathrm{lcm}(a(x), c(x)))=n-s$. It follows that the negacyclic codes with generator polynomials a(x) and c(x) form a linear s-intersection pair by (1) of Theorem 3.1.

By setting e(x)=1 in Theorem 3.2, a linear 0-intersection pair of negacyclic codes can be derived in the following corollary.

Corollary 3.3: Let F_q be a finite field and let n be a positive integer such that $\gcd(n,q)=1$. Let a(x) and b(x) be monic polynomials such that $b(x)|a(x)|(x^n+1)$ over F_q. Then the negacyclic codes with generator polynomials a(x) and $c(x):=\frac{(x^n+1)b(x)}{a(x)}$ form a linear 0-intersection pair.

Clearly, $(x+1)|(x^n+1)$ for all odd positive integer n. By setting e(x)=x+1 in Theorem 3.2, a construction of linear 1-intersection pairs of negacyclic codes is presented as follows.

Corollary 3.4: Let F_q be a finite field and let n be an odd positive integer such that $\gcd(n,q)=1$. Let a(x) and b(x) be monic polynomials such that $b(x)|a(x)|\ \frac{x^n+1}{x+1}$ over F_q. Then the negacyclic codes with generator polynomials a(x) and $c(x):=\frac{(x^n+1)b(x)}{(x+1)a(x)}$ form a linear 1-intersection pair.

Constructions of linear s-complementary pairs of negacyclic codes are now given.

Theorem 3.5: Let F_q be a finite field and let n be a positive integer such that $\gcd(n,q)=1$. Let e(x) be a monic divisor of x^n+1 of degree s. If a(x) is a monic polynomial such that $a(x)|\ \frac{x^n+1}{e(x)}$ over F_q, then the negacyclic codes with generator polynomials a(x) and $c(x):=\frac{(x^n+1)}{e(x)a(x)}$ form a linear s-complementary pair.

Proof. We note that $\mathrm{lcm}(a(x), c(x))=\frac{x^n+1}{e(x)}$ which implies that $\deg(\mathrm{lcm}(a(x), c(x)))=n-s$. Since $\gcd(a(x), c(x))=1$, the negacyclic codes with generator polynomials a(x) and c(x) form a linear s-complementary pair by (2) of Theorem 3.1.

By setting e(x)=1 in Theorem 3.5, the next corollary follows.

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Corollary 3.6: Let F_q be a finite field and let n be a positive integer such that gcd(n,q)=1. Let a(x) be monic polynomial such that $a(x)|(x^n+1)$ over F_q. Then the negacyclic codes with generator polynomials a(x) and $c(x):=\frac{x^n+1}{a(x)}$ are a linear 0-complementary pair (LCP).

By putting e(x)=x+1 in Theorem 3.5, a linear 1-complementary pairs of negacyclic codes is derived in the next corollary.

Corollary 3.7: Let F_q be a finite field and let n be an odd positive integer such that gcd(n,q)=1. Let a(x) be monic polynomial such that $a(x)|\frac{x^n+1}{x+1}$ over F_q. Then the negacyclic codes with generator polynomials a(x) and $c(x):=\frac{x^n+1}{(x+1)a(x)}$ form a linear 1-complementary pair.

In practice, linear intersection pairs of negacyclic codes can be constructed explicitly using Theorem 3.2 in the following steps: 1) Fix an integer s≥0. 2) Fix a monic divisor e(x) of x^n+1 (if exist). 3) Compute all monic divisors of $\frac{(x^n+1)}{e(x)}$. 4) For each monic divisor a(x) of $\frac{(x^n+1)}{e(x)}$, compute all monic divisors of a(x). 5) For each monic divisor b(x) of a(x), compute parameters of negacyclic codes with generator polynomials a(x) and $c(x):=\frac{(x^n+1)b(x)}{e(x)a(x)}$. The output codes form linear s-intersection pairs. In the same fashion, linear s-complementary pairs of negacyclic codes can be computed via Theorem 3.5.

Some illustrative examples of linear s-intersection pairs and linear s-complementary pairs of negacyclic codes over F_3 are given in Table 1. The negacyclic codes in Table 1 are optimal according to the database used in [8]. While the pairs in Table 1 are linear s-intersection pairs, the ones with * are also linear s-complementary pairs. For convenience, denote by $a_0a_1a_2\ldots a_r$ the polynomial $a_0+a_ax+a_2x^2+\cdots+a_rx^r$ over F_3. Using the notation as in Theorem 3.2, the negacyclic codes C_1 and C_2 are generated by a(x) and c(x), respectively.

Table 1. Linear s-intersection pairs and linear s-complementary pairs of negacyclic codes over F_3

e(x)	a(x)	b(x)	c(x)	C_1	C_2	s
1	1121	1	12221201011	$[13, 10, 3]_3$	$[13, 3, 9]_3$	0*
1	1021201	1	10122101	$[13, 7, 5]_3$	$[13, 6, 6]_3$	0*
1	1021201	1121	11010212221	$[13, 7, 5]_3$	$[13, 3, 9]_3$	0
1	1012010221	1	10211	$[13, 4, 7]_3$	$[13, 9, 3]_3$	0*
1	1012010221	1121	11110201	$[13, 4, 7]_3$	$[13, 6, 6]_3$	0
1	1012010221	1111211	11010212221	$[13, 4, 7]_3$	$[13, 3, 9]_3$	0
11	1121	1	1111021001	$[13, 10, 3]_3$	$[13, 4, 7]_3$	1*
11	1021201	1	1220221	$[13, 7, 5]_3$	$[13, 7, 5]_3$	1*
11	1021201	1121	1001201111	$[13, 7, 5]_3$	$[13, 4, 7]_3$	1
11	1012010221	1111211	1001201111	$[13, 4, 7]_3$	$[13, 4, 7]_3$	1
1201	1121	1	10201111	$[13, 10, 3]_3$	$[13, 6, 6]_3$	3*
1201	1121	1121	11102112101	$[13, 10, 3]_3$	$[13, 3, 9]_3$	3
1201	1021201	1	11201	$[13, 7, 5]_3$	$[13, 9, 3]_3$	3*
1201	12220021	1	1211	$[13, 6, 6]_3$	$[13, 10, 3]_3$	3*
1201	1012010221	1	11	$[13, 4, 7]_3$	$[13, 12, 2]_3$	3*
1201	1012010221	1121	12001	$[13, 4, 7]_3$	$[13, 9, 3]_3$	3
1201	1012010221	1111211	12220021	$[13, 4, 7]_3$	$[13, 6, 6]_3$	3

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4. EAQECCs from linear s-intersection pairs of negacyclic codes over finite fields

Entanglement-assisted quantum error correcting codes (EAQECCs) were introduced in [9] and it has been shown that EAQECCs can be constructed from classical codes. Further, the performance of the resulting quantum codes can be determined by the performance of the underlying classical codes. Precisely, an $[[n,k,d;c]]_q$ EAQECC encodes k logical qudits into n physical qudits using c copies of maximally entangled states and its performance is measured by its rate $\frac{k}{n}$ and net rate $\frac{k-c}{n}$. When the net rate of an EAQECC is positive it is possible to obtain catalytic codes as shown in [8]. In [9], good entanglement-assisted quantum codes were constructed. A link between the number of maximally shared qubits required to construct an EAQECC from a classical code and the hull of the classical code were given. For more details on EAQECCs, please refer to [11], [12] and the references therein.

In [11], EAQECCs are constructed from linear intersection pair of classical linear codes.

Theorem 4.1 ([11, Proposition 4.2]): Let $s \geq 0$ be an integer and let C_1 and C_2 be a linear s-intersection pair of linear codes over F_q with parameters $[n, k_1, d_1]_q$ and $[n, k_2, d_2]_q$, respectively. Then there exists an $[[n, k_2-s, \min\{d_1^{\perp}, d_2\}; k_1-s]]_q$ EAQECC with $d_1^{\perp}$ is the minimum Hamming weight of $C_1^{\perp}$.

Based on Theorem 4.1, Theorem 3.2, Corollary 3.3, and Corollary 3.4, constructions of EAQECCs are given in terms of linear intersection pair of negacyclic codes as follows.

Theorem 4.2: Let F_q be a finite field and let n be a positive integer such that gcd(n,q)=1. Let e(x) be a monic divisor of x^n+1 of degree s. If a(x) and b(x) are monic polynomials such that $b(x)|a(x)|\frac{x^n+1}{e(x)}$, then there exists an $[[n, \deg(a(x))-\deg(b(x)), \min\{d_1, d_2\}; n-\deg(a(x))-s]]_q$ EAQECC, where d_1 and d_2 are the minimum Hamming weights of negacyclic codes generated by $\frac{x^n+1}{a(x)}$ and $c(x):=\frac{(x^n+1)b(x)}{e(x)a(x)}$, respectively.

Proof. Based on Theorems 3.2 and 4.1, the desired EAQECC can be constructed directly.

By setting e(x)=1 and e(x)=x+1 in Theorem 4.2, the next corollaries follow immediately.

Corollary 4.3: Let F_q be a finite field and let n be a positive integer such that gcd(n,q)=1. Let a(x) and b(x) be monic polynomials such that $b(x)|a(x)|(x^n+1)$. Then there exists an $[[n, \deg(a(x))-\deg(b(x)), \min\{d_1, d_2\}; n-\deg(a(x))]]_q$ EAQECC, where d is the minimum Hamming weight of the negacyclic code generated by $\frac{x^n+1}{a(x)}$.

Corollary 4.4: Let F_q be a finite field and let n be an odd positive integer such that gcd(n,q)=1. Let a(x) and b(x) be monic polynomials such that $b(x)|a(x)|\frac{x^n+1}{x+1}$. Then there exists an $[[n, \deg(a(x))-\deg(b(x)), \min\{d_1, d_2\}; n-\deg(a(x))-1]]_q$ EAQECC, where d_1 and d_2 are the minimum Hamming weights of negacyclic codes generated by $\frac{x^n+1}{a(x)}$ and $c(x):=\frac{(x^n+1)b(x)}{(x+1)a(x)}$, respectively.

4.1. Examples

In general, EAQECCs can be derived from linear s-intersection pairs of negacyclic codes using Theorem 4.2, Corollary 4.3, and Corollary 4.4. Some illustrative examples of EAQECCs over F_3 are presented in Table 2. The input negacyclic codes C_1 and C_2 with generator polynomials a(x) and c(x) are chosen from Table 1

Table 2. EAQECCs from linear intersection pairs of negacyclic codes over F_3

C_1	C_2	a(x)	c(x)	s	Q
$[13, 10, 3]_3$	$[13, 3, 9]_3$	1121	12221201011	0	$[[13, 3, 9; 10]]_3$
$[13, 7, 5]_3$	$[13, 6, 6]_3$	1021201	10122101	0	$[[13, 6, 6; 7]]_3$
$[13, 7, 5]_3$	$[13, 3, 9]_3$	1021201	11010212221	0	$[[13, 3, 6; 7]]_3$
$[13, 4, 7]_3$	$[13, 9, 3]_3$	1012010221	10211	0	**$[[13, 9, 3; 4]]_3$**
$[13, 4, 7]_3$	$[13, 6, 6]_3$	1012010221	11110201	0	**$[[13, 6, 3; 4]]_3$**
$[13, 10, 3]_3$	$[13, 4, 7]_3$	1121	1111021001	1	$[[13, 3, 7; 9]]_3$
$[13, 7, 5]_3$	$[13, 7, 5]_3$	1021201	1220221	1	$[[13, 6, 5; 6]]_3$

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$[13, 7, 5]_3$	$[13, 4, 7]_3$	1021201	1001201111	1	$[[13, 3, 6; 6]]_3$
$[13, 6, 6]_3$	$[13, 10, 3]_3$	12220021	1211	3	$\mathbf{[[13, 7, 3; 3]]_3}$
$[13, 4, 7]_3$	$[13, 12, 2]_3$	1012010221	11	3	$\mathbf{[[13, 9, 2; 1]]_3}$
$[13, 4, 7]_3$	$[13, 9, 3]_3$	1012010221	12001	3	$\mathbf{[[13, 6, 3; 1]]_3}$

We note that the EAQECCs with boldface parameters in Table 2 have positive net rate which is possible to obtain catalytic codes as shown in [8].

5. Conclusion

Characterization and constructions of linear intersection pairs of negacyclic codes have been presented as well as their applications in constructions of EAQECCs. General characterization and properties of such pairs have been given in terms of their generator polynomials. Explicit constructions of linear s-complementary pairs and linear s-intersection pairs of negacyclic codes have been established for $s \in \{0,1\}$. As applications, constructions of EAQECCs have been given based on these pairs. Some illustrative examples have been presented. It is interesting to study the existence and parameters of linear s-intersection pairs and linear s-complementary pairs of negacyclic codes with arbitrary intersecting dimension as well as the optimality of EAQECCs (see [13]).

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