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RESEARCH ARTICLE

Conformal wave equations for the Einstein-tracefree matter system

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Abstract

Inspired by a similar analysis for the vacuum conformal Einstein field equations by Paetz (Ann Henri Poincaré 16:2059, 2015), in this article we show how to construct a system of quasilinear wave equations for the geometric fields associated to the conformal Einstein field equations coupled to matter models whose energy-momentum tensor has vanishing trace. In this case, the equation of conservation for the energy-momentum tensor is conformally invariant. Our analysis includes the construction of a subsidiary evolution which allows to prove the propagation of the constraints. We discuss how the underlying structure behind these systems of equations is the set of integrability conditions satisfied by the conformal field equations. The main result of our analysis is that both the evolution and subsidiary equations for the geometric part of the conformal Einstein-tracefree matter field equations close without the need of any further assumption on the matter models other that the vanishing of the trace of the energy-momentum tensor. Our work is supplemented by an analysis of the evolution and subsidiary equations associated to three basic tracefree matter models: the conformally invariant scalar field, the Maxwell field and the Yang–Mills field. As an application we provide a global existence and stability result for de Sitter-like spacetimes. In particular, the result for the conformally invariant scalar field is new in the literature.

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1 Introduction

The *conformal Einstein field equations* are a conformal representation of the Einstein field equations which permit us to study the global properties of the solutions to equations of General Relativity by means of Penrose's procedure of conformal compactification—see e.g. [11,15] for an entry point to the literature on the subject. Crucially, a solution to the conformal Einstein field equations implies a solution to the Einstein field equations away from the conformal boundary.

A key step in the analysis involving the conformal Einstein field equations is the so-called *procedure of hyperbolic reduction*, in which a subset of the field equations is cast in the form of a hyperbolic evolution system (the *evolution system*) for which known techniques of the theory of partial differential equations allow us to establish well-posedness. An important ingredient in the hyperbolic reduction is the choice of a gauge, which in the case of the conformal Einstein field equations involves not only fixing coordinates (the *coordinate gauge*) but also the representative of the conformal class of the spacetime metric (the so-called *unphysical metric*) to be considered (the *conformal gauge*). Naturally, gauge choices should bring to the fore the physical and geometric features of the setting under consideration. In order to make contact with the Einstein field equations, the procedure of hyperbolic reduction has to be supplemented by an argument concerning the *propagation of the constraints*, by means of which one identifies the conditions under which one can guarantee that a solution to the evolution system implies a solution to the full system of conformal equations, independently of the gauge choice. The propagation of the constraints involves the construction of a *subsidiary evolution system* describing the evolution of the conformal field equations and of the conditions representing the gauge. The construction of the subsidiary system requires lengthy manipulations of the equations which are underpinned by integrability conditions inherent to the field equations.

Most of the results concerning the conformal Einstein field equations available in the literature make use of hyperbolic reductions leading to first order *symmetric hyperbolic evolution systems*. This approach works best for the frame and spinorial versions of the conformal equations. Arguably, the simplest variant of the conformal Einstein field equations is given by the so-called *metric conformal Einstein field equations* in which the field equations are presented in tensorial form and the unphysical metric is determined by means of an *unphysical Einstein field equation* relating the Ricci tensor of the unphysical metric to the various geometric fields entering in the conformal equations—these can be thought of as corresponding to some fictitious unphysical matter. Remarkably, until recently, there was no suitable hyperbolic reduction procedure available for this version of the conformal field equations. In [17] Paetz has obtained a satisfactory hyperbolic procedure for the metric vacuum Einstein field equations which is based on the construction of second order wave equations. To round up his analysis, Paetz then proceeds to construct a system of subsidiary wave equations for tensorial fields encoding the conformal Einstein field equations (the so-called *geometric zero-quantities*) showing, in this way, the propagation of the constraints. The motivation behind Paetz's approach is that the use of second order hyperbolic equations gives access to a different part of the theory of partial differential equations which complements the results available for first order symmetric hyperbolic systems—see e.g. [3,6]. Paetz's construction of an evolution system consisting of wave equations has been adapted to the case of the spinorial conformal Einstein field equations in [12]. In addition to its interest in analytic considerations, the construction of wave equations for the metric conformal Einstein field equations is also of relevance in numerical studies, as the gauge fixing procedure and the particular form of the equations is more amenable to implementation in current mainstream numerical codes than other formulations of the conformal equations.

The purpose of the present article is twofold: first, it generalises Paetz's construction of a system of wave equations for the conformal Einstein field equations to the case of matter models whose energy-momentum tensor has a vanishing trace—i.e. so-called *tracefree matter*. The case of tracefree matter is of particular interest since the equation of conservation satisfied by the energy-momentum is conformally invariant; moreover, the associated equations of motion for the matter fields can, usually, be shown to possess good conformal properties—see [15], Chapter 9. Second, it clarifies the inner structure of Paetz's original construction by identifying the integrability conditions underlying the mechanism of the propagation of the constraints. The motivation behind this analysis is to extend the recent analysis of the construction of vacuum anti-de Sitter-like spacetimes in [3] to the case of tracefree matter. However, we believe that the analysis we present has an interest on its own right as it brings to the fore the subtle structure of the metric conformal Einstein field equations.

The main results of this article can be summarised as follows:

Theorem *The geometric fields in the metric conformal Einstein field equations coupled to a tracefree matter field satisfy a system of wave equations which is regular up to and beyond the conformal boundary of a spacetime admitting a conformal extension. Moreover, the associated geometric zero-quantities satisfy a (subsidiary) system of homogeneous wave equations independently of the matter model. The subsidiary system is also regular on the conformal boundary.*

The precise statements concerning the above main result are contents of Lemmas 1 and 3.

Remark 1 A remarkable property of our analysis is that it renders suitable evolution equations for the conformal fields and the zero-quantities without having to make any assumptions on the matter model except that it satisfies *good* evolution equations in the conformally rescaled spacetime. Thus, our discussion can be regarded as a *once-for-all* analysis of the evolution equations associated to the geometric part of the metric conformal field equations valid for a wide class of coordinate gauges prescribed in terms of the coordinate gauge source function appearing in the *generalised wave coordinate condition*.

Remark 2 The homogeneity of the subsidiary system on the geometric zero-quantities is the key structural property required to ensure the propagation of the constraints by exploiting the uniqueness of solutions to a system of wave equations.

The approach followed to obtain our main result is based on the identification of a family of integrability conditions associated to the metric conformal Einstein field equations. To the best of our knowledge, these integrability conditions have not appeared elsewhere in the literature. In our opinion this approach brings better to the fore the structural properties of the conformal Einstein field equations and, in particular, it makes the construction of the subsidiary evolution system more transparent than the *brute force* approach adopted in [17]. A similar strategy is also adopted to study the propagation of the gauge. In particular, by setting the matter fields to zero, our analysis provides an alternative version of the main results of [17]—the initial conditions on the gauge required in the present analysis differ from those in [17] though. Despite offering a more sleek approach to the construction of an evolution system for the conformal Einstein field equations, our analysis still requires heavy computations which are best carried out in a computer algebra system. In the present case we have made systematic use of the suite `xAct` for the manipulation of tensorial expressions in `Mathematica`—see [16].

We supplement our general analysis of the metric conformal Einstein field equations with an analysis of the evolution and subsidiary evolution equations of some of the tracefree matter models more commonly used in the literature: the Maxwell field, the Yang–Mills field and the conformally invariant scalar field. For each of these fields we construct suitably second order wave equations for the matter fields and the associated *matter zero-quantities*. For the case of the Yang–Mills field, our analysis makes no assumptions on the gauge group.

As an application of our analysis, in the final section of this article we present stability results for the de Sitter spacetime for perturbations which include the Maxwell, Yang–Mills or conformally invariant scalar field. Proofs of this result for the Maxwell and Yang–Mills fields have been obtained in [9] using the spinorial version of the conformal equations and a first order hyperbolic reduction. The stability result for the conformally invariant scalar field is, to the best of our knowledge, new.

Overview of the article

In Sect. 2 we briefly summarise the key properties of the metric conformal Einstein field equations coupled to tracefree matter and their relation to the Einstein field equations. Section 3 provides the derivation of the geometric wave equations for the geometric fields appearing in the conformal Einstein field equations. Section 4 introduces the key notion of geometric zero-quantity and discusses the identities and integrability conditions associated to objects of this type. Section 5 provides the construction of the subsidiary evolution system for the geometric zero-quantities used in the argument of the propagation of the constraints. This is, in principle, the most calculationally intensive part of our analysis. However, using the integrability conditions of Sect. 4 we provide a streamlined presentation thereof. In Sect. 6 we discuss the gauge freedom inherent in the geometric evolution systems obtained in Sects. 3 and 5 and how this freedom can be used to complete the hyperbolic reduction of the equations. Section 7 establishes the consistency of the gauge introduced in the previous section, independently of the particular tracefree matter model. Section 8 provides a case-by-case analysis of three prototypical tracefree matter models—the conformally invariant scalar field (Sect. 8.1), the Maxwell field (Sect. 8.2) and the Yang–Mills field (Sect. 8.3). The discussion for each of these matter models includes the construction of suitable wave evolution equations and subsidiary evolution equations. Finally, Sect. 9 provides an application of the analysis developed in this article to the global existence and stability of de Sitter-like spacetimes.

Conventions

In what follows, $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ will denote a spacetime satisfying the Einstein equations with matter—later we will make the further assumption that the energy-momentum tensor is tracefree. The signature of the spacetime metric is $(-,+,+,+)$. The lowercase Latin letters $a, b, c, \ldots$ are used as abstract spacetime indices, while Greek letters $\mu, \nu, \lambda, \ldots$ will be used as spacetime coordinate indices. Our conventions for the curvature are

$$\nabla_c \nabla_d u^a - \nabla_d \nabla_c u^a = R^a{}_{bcd} u^b.$$

2 The metric conformal Einstein field equations with tracefree matter

The purpose of this section is to provide a brief overview of the basic properties of the conformal Einstein field equations with tracefree matter. A more extended discussion of the properties of these equations, as well as their derivation, can be found in Chapter 8 of [15].

2.1 Basic relations

In what follows let $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ denote a spacetime satisfying the *Einstein field equations with matter*

$$\tilde{R}_{ab} - \frac{1}{2}\tilde{R}\tilde{g}_{ab} + \lambda \tilde{g}_{ab} = \tilde{T}_{ab}, \tag{1}$$

where $\tilde{R}_{ab}$ and $\tilde{R}$ denote, respectively, the Ricci tensor and Ricci scalar of the metric $\tilde{g}_{ab}$, λ is the Cosmological constant and $\tilde{T}_{ab}$ is the energy-momentum tensor. As a consequence of the contracted Bianchi identity one obtains the conservation law

$$\tilde{\nabla}^a \tilde{T}_{ab} = 0. \tag{2}$$

Here $\tilde{\nabla}_a$ denotes the Levi-Civita covariant derivative of the metric $\tilde{g}_{ab}$. Now, let $(\mathcal{M}, g_{ab})$ denote a spacetime related to $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ via a conformal embedding

$$\tilde{\mathcal{M}} \overset{\varphi}{\hookrightarrow} \mathcal{M}, \qquad \tilde{g}_{ab} \overset{\varphi}{\mapsto} g_{ab} \equiv \Xi^2 (\varphi^{-1})^* \tilde{g}_{ab}, \qquad \Xi|_{\varphi(\tilde{\mathcal{M}})} > 0,$$

where is Ξ a smooth scalar field—the so-called *conformal factor*. With a slight abuse of notation we write

$$g_{ab} = \Xi^2 \tilde{g}_{ab}. \tag{3}$$

Remark 3 Following the standard usage, we refer to $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ as the *physical spacetime* while $(\mathcal{M}, g_{ab})$ will be called the *unphysical spacetime*.

2.1.1 The unphysical energy-momentum tensor

Since Eq. (3) does not determine the way $\tilde{T}_{ab}$ transforms, it will be convenient to define the *unphysical energy-momentum tensor* as

$$T_{ab} \equiv \Xi^{-2}\tilde{T}_{ab}.$$

Using the transformation rules between the Levi-Civita covariant derivatives of conformally related metrics, Eq. (2) takes the form

$$\nabla^a T_{ab} = \Xi^{-1} T \nabla_b \Xi,$$

with ∇_a the Levi-Civita covariant derivative of g_{ab} and $T \equiv g^{ab}T_{ab}$. It then follows that

$$\nabla^a T_{ab} = 0 \qquad \text{if and only if} \qquad T = 0.$$

Assumption 1 In the remainder of this article we restrict our attention to matter models for which $T = 0$, so that the corresponding unphysical energy-momentum tensor T_{ab} is divergence-free, that is,

$$\nabla^a T_{ab} = 0. \tag{4}$$

2.2 Basic properties of the conformal Einstein field equations

The *metric tracefree conformal Einstein field equations* have been first discussed in [9]. In terms of the notation and conventions used in this article they are given by

$$\nabla_a \nabla_b \Xi = -\Xi L_{ab} + s g_{ab} + \tfrac{1}{2}\Xi^3 T_{ab}, \tag{5a}$$

$$\nabla_a s = -L_{ab}\nabla^b \Xi + \tfrac{1}{2}\Xi^2 \nabla^b \Xi T_{ab}, \tag{5b}$$

$$\nabla_a L_{bc} - \nabla_b L_{ac} = \nabla_e \Xi d^e{}_{cab} + \Xi T_{abc}, \tag{5c}$$

$$\nabla_e d^e{}_{abc} = T_{bca}, \tag{5d}$$

$$6\Xi s - 3\nabla_c \Xi \nabla^c \Xi = \lambda, \tag{5e}$$

$$R^c{}_{dab} = \Xi d^c{}_{dab} + 2(\delta^c{}_{[a} L_{b]d} - g_{d[a} L_{b]}{}^c). \tag{5f}$$

A detailed derivation of these equations can be found in [15]. In the above expressions L_{ab}, s, $d^a{}_{bcd}$ and T_{abc} denote, respectively, the Schouten tensor, the Friedrich scalar, the rescaled Weyl tensor and the rescaled Cotton tensor. These objects are defined as

$$L_{ab} \equiv \tfrac{1}{2}R_{ab} - \tfrac{1}{12}g_{ab}R, \tag{6a}$$
$$s \equiv \tfrac{1}{4}\nabla^c\nabla_c\Xi + \tfrac{1}{24}R\Xi, \tag{6b}$$
$$d^a{}_{bcd} \equiv \Xi^{-1}C^a{}_{bcd}, \tag{6c}$$
$$T_{abc} \equiv \Xi\nabla_{[a}T_{b]c} + 3\nabla_{[a}\Xi T_{b]c} - g_{c[a}T_{b]e}\nabla^e\Xi, \tag{6d}$$

where $C^a{}_{bcd}$ is the conformally invariant Weyl tensor. Observe that T_{abc} has the following symmetries:

$$T_{abc} = T_{[ab]c}, \quad T_{[abc]} = 0. \tag{7}$$

Relevant for the subsequent discussion is the well-known fact that the rescaled Weyl tensor has two associated Hodge dual tensors, namely

$$^*d_{abcd} \equiv \tfrac{1}{2}\epsilon_{ab}{}^{ef}d_{efcd}, \quad d^*_{abcd} \equiv \tfrac{1}{2}\epsilon_{cd}{}^{ef}d_{abef},$$

where ϵ_{abcd} is the 4-volume form of the metric g_{ab}. One can check that $^*d_{abcd} = d^*_{abcd}$. Similarly, we also define the Hodge dual of T_{abc} as

$$^*T_{abc} \equiv \tfrac{1}{2}\epsilon_{ab}{}^{de}T_{dec}. \tag{8}$$

Moreover, if Assumption 1 and Eq. (5a) are taken into account, one obtains some additional relations, namely

$$\nabla_c T_{ab}{}^c = 0, \tag{9a}$$
$$\nabla_c {}^*T_{ab}{}^c = 0, \tag{9b}$$
$$\nabla_c T_a{}^c{}_b = \nabla_c T_{(a}{}^c{}_{b)}. \tag{9c}$$

Remark 4 Equations (5a)–(5d) will be regarded as a set of differential conditions for the fields Ξ, s, L_{ab} and $d^a{}_{bcd}$. Equation (5e) can be shown to play the role of a constraint which only needs to be verified at a single point—see e.g. [15], Lemma 8.1. Eq. (5f), providing the link between the conformal fields $d^c{}_{dab}$, L_{ab} and the irreducible decomposition of the Riemann tensor, allows us to deduce a differential condition for the components of the unphysical metric g_{ab}—see Sect. 6.2.

Remark 5 By a solution to the metric tracefree conformal Einstein field equations it will be understood a collection of fields $(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd}, T_{ab})$ satisfying Eqs. (4) and (5a)–(5f).

The relation between the metric tracefree conformal Einstein field equations and the Einstein field equations (1) is given in the following proposition—see [15], Proposition 8.1.

Proposition 1 *Let $(g_{ab}, \Xi, s, L_{ab}, d^a{}_{bcd}, T_{ab})$ denote a solution to the metric tracefree conformal Einstein field equations such that $\Xi \neq 0$ on an open set $\mathcal{U} \subset \mathcal{M}$. Then* *the metric* $\tilde{g}_{ab} = \Xi^{-2} g_{ab}$ *is a solution to the Einstein field equations* (1) *with energy momentum tensor given by* $\tilde{T}_{ab} = \Xi^2 T_{ab}$ *on* $\mathcal{U}$.

Proof The proof given in [15] omits Eq. (5f) and implicitly assumes that the field L_{ab} can be identified with the Schouten tensor of the metric g_{ab}. With Eq. (5f) at hand, one is allowed to make this identification. From here onwards one can apply the argument in Proposition 8.1 in [15]. □

2.2.1 An alternative equation for $d^a{}_{bcd}$

For our purposes, it will be convenient to consider an alternative version of the conformal field equation for the rescaled Weyl tensor. This can be obtained as follows: multiplying (5d) by $\epsilon_{fg}{}^{bc}$ and exploiting the identity ${}^*d_{abcd} = d^*_{abcd}$ results in

$$2\nabla_a {}^*d_{fgc}{}^a = 2\nabla_a d^*_{fgc}{}^a = -2{}^*T_{fgc}.$$

From here it follows that

$$3\nabla_{[e} d_{ab]cd} + \epsilon_{eabf} {}^*T_{cd}{}^f = 0. \tag{10}$$

Remark 6 This last equation is equivalent to (5d) and will be essential in Sects. 3 and 4 where a system of wave equations for the geometric fields and the zero-quantities associated to the Eqs. (5a)–(5f) is discussed.

2.3 An equation for the components of the metric g_{ab}

Taking the natural trace in Eq. (5f) leads to the relation

$$R_{ab} = 2L_{ab} + \tfrac{1}{6} R g_{ab}. \tag{11}$$

Here, R_{ab} and L_{ab} are considered as independent variables. In particular, the Ricci tensor R_{ab} is assumed to be expressed in terms of first and second derivatives of the components of the metric whilst L_{ab} is a field satisfying equations (5a)–(5e). This will be further discussed in Sect. 6 where a suitable wave equation for the components of the metric is constructed.

Remark 7 As pointed out in [10], Eq. (11) can be regarded as an Einstein field equation for the unphysical metric g_{ab}. From this point of view, the geometric fields Ξ, s, L_{ab} and d_{abcd} can be regarded as unphysical matter fields. Accordingly, in the following we refer to Eq. (11) as the *unphysical Einstein equation*. This approach should allow to adapt well-tested numerical methods for the Einstein field equations to the case of the conformal field equations.

3 The evolution system for the geometric fields

In this section we show how to construct an evolution system for the geometric fields appearing in the conformal Einstein field equations, Eqs. (5a)–(5f). These evolution equations take the form of *geometric wave equations*—that is, their principal part involves the D'Alambertian $\square \equiv \nabla_a \nabla^a$ associated to the conformal metric g_{ab}.

In [17], Paetz has obtained a system of geometric wave equations for the set of conformal fields $(\Xi, s, L_{ab}, d^a{}_{bcd})$ in the vacuum case. This can be generalised to include a tracefree matter component. The next statement summarises this result:

Lemma 1 *The metric tracefree conformal Einstein field equations* (5a)–(5f) *imply the following system of geometric wave equations for the conformal fields:*

$$
\begin{aligned}
\square \Xi &= 4s - \tfrac{1}{6}\Xi R, && \text{(12a)}\\
\square s &= -\tfrac{1}{6} s R + \Xi L_{ab} L^{ab} - \tfrac{1}{6}\nabla_a R \nabla^a \Xi + \tfrac{1}{4}\Xi^5 T_{ab}T^{ab} - \Xi^3 L_{ab}T^{ab}\\
&\quad + \Xi \nabla^a \Xi \nabla^b \Xi T_{ab}, && \text{(12b)}\\
\square L_{ab} &= -2\Xi d_{acbd} L^{cd} + 4 L_a{}^c L_{bc} - L_{cd}L^{cd} g_{ab} + \tfrac{1}{6}\nabla_a \nabla_b R + \tfrac{1}{2}\Xi^3 d_{acbd} T^{cd}\\
&\quad - \Xi \nabla_c T_a{}^c{}_b - 2 T_{(a|c|b)} \nabla^c \Xi, && \text{(12c)}\\
\square d_{abcd} &= -4\Xi d_a{}^f{}_{[c}{}^e d_{d]ebf} - 2\Xi d_a{}^f{}_b{}^e d_{cdfe}\\
&\quad + \tfrac{1}{2} d_{abcd} R - T_{[a}{}^f \Xi^2 d_{b]fcd} - \Xi^2 T_{[c}{}^f d_{d]fab}\\
&\quad - \Xi^2 g_{a[c} d_{d]gbf} T^{fg} + \Xi^2 g_{b[c} d_{d]gaf} T^{fg}\\
&\quad + 2\nabla_{[a} T_{|cd|b]} + \epsilon_{abef} \nabla^f {}^*T_{cd}{}^e. && \text{(12d)}
\end{aligned}
$$

Proof Equation (12a) is a direct consequence of (5a). Equations (12b) and (12c) result, respectively, from applying a covariant derivative to (5b) and (5c), and using the second Bianchi identity. The wave equation for $d^a{}_{bcd}$, on the other hand, requires to consider the alternative conformal field equation (10). Applying ∇^e to the latter and using Eq. (5d) along with the first Bianchi identity, a long but straightforward calculation yields the wave equation

$$
\begin{aligned}
\square d_{abcd} =& -4\Xi d_a{}^f{}_{[c}{}^e d_{d]ebf} - 2\Xi d_a{}^f{}_b{}^e d_{cdfe} + \tfrac{1}{3} d_{abcd} R\\
& - 2 d_{cdf[a} L_{b]}{}^f - 2 d_{abf[c} L_{d]}{}^f\\
& - 2 g_{a[c} d_{d]ebf} L^{fe} + 2 g_{b[c} d_{d]fae} L^{ef}\\
& + 2\nabla_{[a} T_{|cd|b]} + \epsilon_{abef} \nabla^f {}^*T_{cd}{}^e. \qquad (13)
\end{aligned}
$$

It is possible to eliminate terms containing L_{ab} from the wave equation (13) through the generalisation of an identity obtained in [17] to the case of tracefree matter. Multiplying Eq. (10) by Ξ, using the definitions of $d^a{}_{bcd}$ and ${}^*T_{abc}$, Eq. (5c) and the second Bianchi identity to simplify it, one finds that

$$
d_{cd[ag}\nabla_{b]}\Xi + d_{de[ag} g_{b]c} \nabla^e \Xi - d_{ce[ag} g_{b]d} \nabla^e \Xi = 0. \qquad (14)
$$

Applying a further covariant derivative ∇^g to the last expression and making use of Eqs. (5a), (5d) and (10) as well as the properties of the rescaled Cotton tensor, the following identity is obtained:

$$\begin{aligned} & 2\Xi d_{cdf[a}L_{b]}{}^f + 2\Xi d_{abf[c}L_{d]}{}^f + 2g_{a[c}\Xi d_{d]gbf}L^{fg} \\ & \quad - 2\Xi g_{b[c}d_{d]gaf}L^{fg} + \tfrac{1}{6}\Xi d_{abcd}R \\ & \quad - \Xi^3 d_{cdf[a}T_{b]}{}^f - \Xi^3 d_{abf[c}T_{d]}{}^f \\ & \quad - \Xi^3 g_{a[c}d_{d]gbf}T^{fg} + \Xi^3 g_{b[c}d_{d]gaf}T^{fg} = 0. \end{aligned} \tag{15}$$

By substituting this into expression (13) we get Eq. (12d), which does not involve the Schouten tensor. □

Remark 8 In concrete applications it may prove useful to express the Schouten tensor in terms of the tracefree Ricci tensor and the Ricci scalar through the formula

$$L_{ab} = \Phi_{ab} + \tfrac{1}{24}Rg_{ab}. \tag{16}$$

As will be discussed in Sect. 6.1, the Ricci scalar R is associated to the particular choice of conformal gauge. Thus, the decomposition (16) allows us to split the field L_{ab} into a gauge part and a part which is determined through the field equations. Keeping the simplicity of the presentation in mind, we do not pursue this approach further as it leads to lengthier expressions.

4 Zero-quantities and integrability conditions

In this section we consider a convenient setting for the discussion and book-keeping of the evolution equations implied by the conformal Einstein field equations with tracefree matter. Our approach is based on the observation that the metric conformal Einstein field equations constitute an overdetermined system of differential conditions for the various conformal fields. Thus, the equations are related to each other through *integrability conditions* — i.e. necessary conditions for the existence of solutions to the equations.

4.1 Definitions and basic properties

First we proceed to introduce the set of *geometric zero-quantities* (also called *subsidiary variables*) associated to the system of metric tracefree conformal Einstein field equations (5a)–(5e). These fields are defined as:

$$\Upsilon_{ab} \equiv \nabla_a\nabla_b\Xi + \Xi L_{ab} + sg_{ab} - \tfrac{1}{2}\Xi^3 T_{ab}, \tag{17a}$$

$$\Theta_a \equiv \nabla_a s + L_{ac}\nabla^c\Xi - \tfrac{1}{2}\Xi^2\nabla^c\Xi T_{ac}, \tag{17b}$$

$$\Delta_{abc} \equiv \nabla_a L_{bc} - \nabla_b L_{ac} - \nabla_e\Xi d^e{}_{cab} - \Xi T_{abc}, \tag{17c}$$

$$\Lambda_{abc} \equiv T_{bca} - \nabla_e d^e{}_{abc}, \tag{17d}$$

$$Z \equiv \lambda - 6\Xi s + 3\nabla_c \Xi \nabla^c \Xi, \tag{17e}$$

$$P^c{}_{dab} \equiv R^c{}_{dab} - \Xi d^c{}_{dab} - 2(\delta^c{}_{[a} L_{b]d} - g_{d[a} L_{b]}{}^c). \tag{17f}$$

In terms of the above, the conformal Einstein field equations (5a)–(5f) can be expressed as the conditions

$$\Upsilon_{ab} = 0, \qquad \Theta_a = 0, \qquad \Delta_{abc} = 0, \qquad \Lambda_{abc} = 0, \qquad Z = 0, \qquad P^c{}_{dab} = 0,$$

from where these fields take their name.

4.1.1 Properties of the zero-quantities

By definition, the zero-quantities possess the following symmetries:

$$\Upsilon_{ab} = \Upsilon_{(ab)}, \quad \Delta_{abc} = \Delta_{[ab]c}, \quad \Delta_{[abc]} = 0, \quad \Lambda_{abc} = \Lambda_{a[bc]}, \quad \Lambda_{[abc]} = 0,$$
$$\Delta_a{}^b{}_b = 0, \quad \Lambda^b{}_{ab} = 0. \tag{18}$$

Moreover, one can check that Δ_{abc} and Λ_{abc} satisfy the identities

$$\Delta_{abc} = \tfrac{2}{3}\Delta_{abc} + \tfrac{1}{3}\Delta_{acb} - \tfrac{1}{3}\Delta_{bca}, \qquad \Lambda_{abc} = \tfrac{2}{3}\Lambda_{abc} + \tfrac{1}{3}\Lambda_{bac} - \tfrac{1}{3}\Lambda_{cab}, \tag{19}$$

which are useful for simplifying certain combinations of zero-quantities. Regarding $P^a{}_{bcd}$, it inherits the symmetries of the Riemann tensor; in particular, we can define its Hodge dual tensors

$$^*P_{abcd} \equiv \tfrac{1}{2}\epsilon_{ab}{}^{ef} P_{efcd}, \quad P^*_{abcd} \equiv \tfrac{1}{2}\epsilon_{cd}{}^{ef} P_{abef}. \tag{20}$$

In addition, it will result useful to introduce a further auxiliary zero-quantity associated to Eq. (10)—see Remark 6:

$$\Lambda_{abcde} \equiv 3\nabla_{[a} d_{bc]de} + \epsilon_{abcf}{}^* T_{de}{}^f = 3\Lambda_{d[ab} g_{c]e} - 3\Lambda_{e[ab} g_{c]d}. \tag{21}$$

Here, the second equality has been obtained through a calculation similar to the one yielding (10). From the above definition it follows that $\Lambda_{ab}{}^d{}_{cd} = \Lambda_{abc}$, as well as

$$\Lambda_{abcde} = \Lambda_{[abc]de}, \qquad \Lambda_{abcde} = \Lambda_{abc[de]}. \tag{22}$$

4.1.2 Some consequences of the wave equations

Key for our subsequent analysis is the observation that assuming the validity of the geometric wave equations for the conformal fields implies a further set of relations satisfied by the zero-quantities. These are summarised in the following lemma:

Lemma 2 *Assume that the wave equations* (11)*,* (12a)–(12d)*, and Assumption* 1 *hold. Then the geometric zero-quantities satisfy the identities*

$$\Upsilon_a{}^a = 0, \tag{23a}$$
$$P^c{}_{acb} = 0, \tag{23b}$$
$$\nabla_b \Upsilon_a{}^b = 3\Theta_a, \tag{23c}$$
$$\nabla_a \Theta^a = \Upsilon^{ab} L_{ab} - \tfrac{1}{2}\Xi^2 \Upsilon^{ab} T_{ab}, \tag{23d}$$
$$\nabla_c \Delta_a{}^c{}_b = \Upsilon^{cd} d_{acbd} + \Lambda_{abc} \nabla^c \Xi - L^{cd} P_{acbd}, \tag{23e}$$
$$\nabla_c \Delta_{ab}{}^c = 2\Xi T_{c[a} \Upsilon_{b]}{}^c - \Lambda_{cab} \nabla^c \Xi, \tag{23f}$$
$$\nabla_c \Lambda^c{}_{ab} = d_{[a}{}^{cde} P_{b]cde} - 2T_{c[a} \Upsilon_{b]}{}^c, \tag{23g}$$
$$\nabla_c \Lambda_{[ab]}{}^c = 2d_{[a}{}^{cde} P_{b]dec}, \tag{23h}$$
$$\nabla_d P_{abc}{}^d = -\Delta_{abc} - \Xi \Lambda_{cab}, \tag{23i}$$
$$\nabla_c \Lambda_{eg}{}^c{}_{mn} = 2\nabla_{[e} \Lambda_{g]mn} + 2d_{[e}{}^c{}_{|m|}{}^h P_{g]cnh} - 2d_{[e}{}^c{}_{|n|}{}^h P_{g]cmh} + 2d_{mn}{}^{ch} P_{ecgh}. \tag{23j}$$

Proof The result follows directly from the definitions of the zero-quantities with the aid of the wave equations for the conformal fields (11) and (12a)–(12d), the second Bianchi identity and the properties of the rescaled Cotton tensor. It is worth mentioning that (23j) is obtained by using (13) instead of (12d) as it considerably simplifies the calculation. □

4.2 Integrability conditions

The zero-quantities are not independent of each other but they are related via a set of identities, the so-called *integrability conditions*. These relations are key for the computation of a suitable (subsidiary) system of wave equations for the zero-quantities. The procedure to obtain these relations is to compute suitable antisymmetrised covariant derivatives of the zero-quantities which, in turn, are expressed in terms of lower order objects. Following this general strategy we obtain the following:

Proposition 2 *The geometric zero-quantities defined in* (17a)–(17c) *and* (17e)–(17f) *satisfy the identities*

$$2\nabla_{[a} \Upsilon_{c]b} = 2g_{b[a} \Theta_{c]} + \Xi \Delta_{acb} + P_{acbd} \nabla^d \Xi, \tag{24a}$$
$$2\nabla_{[a} \Theta_{b]} = -2L_{[a}{}^c \Upsilon_{b]c} + \Delta_{abc} \nabla^c \Xi + \Xi^2 T_{c[a} \Upsilon_{b]}{}^c, \tag{24b}$$
$$\begin{aligned} 3\nabla_{[d} \Delta_{ab]c} = {} & \Lambda_{abdce} \nabla^e \Xi + 3\Upsilon_{[a}{}^e d_{bd]ce} \\ & + 3L_{[a}{}^e P_{bd]ce} - \tfrac{3}{2}\Xi^2 P_{[ab|c|}{}^e T_{d]e} + 2\Xi \Upsilon_{[a}{}^e g_{b|c|} T_{d]e} \\ & + \Xi \Upsilon_{[a}{}^e g_{|c|b} T_{d]e}, \end{aligned} \tag{24c}$$
$$\nabla_a Z = -6\Xi \Theta_a + 6\Upsilon_{ab} \nabla^b \Xi, \tag{24d}$$
$$3\nabla_{[e} P_{gh]mn} = \Xi \Lambda_{eghnm} - 3\Delta_{[eg|m|} g_{h]n} + 3\Delta_{[eg|n|} g_{h]m}. \tag{24e}$$

Proof Equations (24a)–(24d) follow from direct calculations employing the definitions of the zero-quantities, the rescaled Cotton tensor and the first Bianchi identity. Equation (24e), on the other hand, can be obtained in a similar manner as (10): multiplying (23i) by $\epsilon_{mn}{}^{cd}$ and exploiting the fact that ${}^*P_{abcd} = P^*_{abcd}$—which is a consequence of (23b)—yields

$$2\nabla_a {}^*P_{mnb}{}^a = 2\nabla_a P^*_{mnb}{}^a = -\epsilon_{mnac}(\Xi\Lambda_b{}^{ac} + \Delta^{ac}{}_b). \tag{25}$$

By substituting back the definition of P^*_{mnab}, (24e) is found after some simplifications. □

Remark 9 Observe that these relations have right-hand sides consisting of lower order expressions which are homogeneous in the zero-quantities. This property will be key when suitable wave equations for these fields are derived in the next section. Equations (24a)–(24e) together with (23j) constitute the set of integrability conditions for the geometric zero-quantities associated to the tracefree conformal Einstein field equations.

Remark 10 The expressions in Lemma (2) and Proposition (2) allow us to show, in particular, that the wave equations (12d) and (13) differ from each other by a homogeneous combination of zero-quantities. Thus, in arguments involving the propagation of the constraints, both forms of the evolution equation can be used interchangeably.

5 The subsidiary evolution system for the zero-quantities

An important aspect of any *hyperbolic reduction procedure* for the (conformal) Einstein field equations is the identification of the conditions upon which a solution to the (reduced) evolution equations implies a solution to the full set of field equations — this type of analysis is generically known as the *propagation of the constraints*. In practice, the propagation of the constraints requires the construction of a suitable system of evolution equations for the zero-quantities associated to the field equations.

5.1 Construction of the subsidiary system

In this section it is shown how the set of integrability conditions provides a systematic and direct way to obtain wave equations for the zero-quantities—a so-called *subsidiary evolution system*. The propagation of the constraints then follows from the structural properties of the subsidiary system as a consequence of the uniqueness of solutions to systems of wave equations.

5.1.1 Equations for Υ_{ab}, Θ_a, Δ_{abc}, Z and P_{abcd}

Equation (24a) serves as the starting point to obtain a wave equation for Υ_{ab}. After applying ∇^c and commuting derivatives, Eq. (23c) renders it as a suitable wave equation. Remaining first order derivatives can be rewritten and simplified via Eqs. (19), (23a), (23d), (23e) and (23i) resulting in:

$$\begin{aligned}\Box\Upsilon_{ab} &= \tfrac{1}{6}\Upsilon_{ab}R - 2\Upsilon^{cd}L_{cd}g_{ab} + \tfrac{1}{2}\Xi^2\Upsilon^{cd}g_{ab}T_{cd} + 4\nabla_{(a}\Upsilon_{b)} \\ &\quad - 2\Xi\Upsilon^{cd}d_{acbd} + 4\Upsilon_{(a}{}^{c}L_{b)c} \\ &\quad - 2\Upsilon^{cd}P_{acbd} + 2\Xi L^{cd}P_{acbd} - \tfrac{1}{2}\Xi^3 P_{abcd}T^{cd}. \end{aligned} \tag{26}$$

Regarding Θ_a, an analogous calculation using expression (24b) in conjunction with the same equations as in the previous case leads directly to a wave equation for this field. Exploiting (5c), (6d) and (24a) to simplify it one obtains

$$\begin{aligned}\Box\Theta_c &= 6L_{ca}\Theta^a - 2\Upsilon^{ab}\Delta_{cab} + 2\Xi L^{ab}\Delta_{cab} - \Xi^3\Delta_c{}^{ab}T_{ab} \\ &\quad - 2\Xi^2\Theta^a T_{ca} - 2\Upsilon^{bd}d_{cbad}\nabla^a\Xi \\ &\quad + \tfrac{3}{2}\Xi\Upsilon_c{}^{b}T_{ab}\nabla^a\Xi + \tfrac{1}{2}\Xi^2 P_{cbad}T^{bd}\nabla^a\Xi + \tfrac{1}{2}\Xi\Upsilon_a{}^{b}T_{cb}\nabla^a\Xi \\ &\quad - \tfrac{1}{6}\Upsilon_{ca}\nabla^a R - \tfrac{5}{2}\Xi\Upsilon^{ab}T_{ab}\nabla_c\Xi \\ &\quad + 2\Upsilon^{ab}\nabla_c L_{ab} - \Xi^2\Upsilon^{ab}\nabla_c T_{ab}. \end{aligned} \tag{27}$$

A wave equation for Λ_{abc} can be obtained by applying ∇^d to integrability condition (24c), commuting derivatives and using (23e) to eliminate the second order derivatives. A direct but long calculation exploiting the same relations used in the previous two cases, along with (5d) and (21), yields

$$\begin{aligned}\Box\Delta_{abc} &= 2\Lambda_{cab}s - \Upsilon_c{}^{d}T_{abd} - \Xi\Lambda_{abdce}L^{de} + 3d_{abcd}\Theta^d \\ &\quad + \tfrac{1}{3}R\Delta_{abc} + L_c{}^{d}\Delta_{abd} + \tfrac{1}{2}\Xi^3\Lambda_{abdce}T^{de} \\ &\quad - \Xi P_{abce}T_d{}^{e}\nabla^d\Xi + \tfrac{1}{6}P_{abcd}\nabla^d R + \nabla^d\Xi\nabla_e\Lambda_{ab}{}^{e}{}_{cd} \\ &\quad + 2\Upsilon^{de}\nabla_e d_{abcd} + L^{de}\nabla_e P_{abcd} \\ &\quad - \tfrac{1}{2}\Xi^2 T^{de}\nabla_e P_{abcd} + 2\Upsilon_{[a}{}^{d}T_{b]cd} - \Xi\Upsilon_{[a}{}^{d}\nabla_{|c|}T_{b]d} \\ &\quad - 2\Xi d_{[a}{}^{d}{}_{b]}{}^{e}\Delta_{dec} + 2\Xi d_{[a}{}^{d}{}_{|c|}{}^{e}\Delta_{b]de} \\ &\quad + 2d_{[a}{}^{d}{}_{|c|}{}^{e}\nabla_{b]}\Upsilon_{de} - 2d_{[a}{}^{d}{}_{|c}{}^{e}\nabla_{d|}\Upsilon_{b]e} \\ &\quad - 2L_{[a}{}^{d}\Delta_{b]dc} + 2L^{de}\nabla_{[a}P_{b]dce} - 2P_{[a}{}^{d}{}_{b]}{}^{e}\Delta_{dec} \\ &\quad + 2P_{[a}{}^{d}{}_{|c|}{}^{e}\Delta_{b]de} - 2P_{[a}{}^{d}{}_{|c|}{}^{e}\nabla_{b]}L_{de} - 2P_{[a}{}^{d}{}_{|c}{}^{e}\nabla_{d|}L_{b]e} \\ &\quad + \Xi^2 P_{[a}{}^{d}{}_{|c}{}^{e}\nabla_{d|}T_{b]e} - \Xi^2\Delta_c{}^{d}{}_{[a}T_{b]d} \\ &\quad + \Xi T_{[a}{}^{d}\nabla_{|c|}\Upsilon_{b]d} - 2\nabla^d\Xi\nabla_{[a}\Lambda_{b]cd} + 2\Upsilon^{de}T_{[a|de|}g_{b]c} \\ &\quad + \Xi\Upsilon^{de}g_{[a|c}\nabla_{d|}T_{b]e} - \Upsilon_{[a}{}^{d}T_{b]d}\nabla_c\Xi \\ &\quad - 2L^{de}\Delta_{[a|de|}g_{b]c} + 3\Xi\Upsilon^{d}g_{[a|c|}T_{b]d} + 2\Xi P_{[a}{}^{d}{}_{|c|}{}^{e}T_{b]e}\nabla_d\Xi \\ &\quad - \Xi g_{[a|c}T^{de}\nabla_{d|}\Upsilon_{b]e} \\ &\quad + \Upsilon_{[a}{}^{d}g_{b]c}T_d{}^{e}\nabla_e\Xi + \Upsilon^{de}g_{[a|c|}T_{b]d}\nabla_e\Xi. \end{aligned} \tag{28}$$

A wave equation for Z is readily found by simply applying ∇^a to Eq. (24d):

$$\Box Z = 6\Upsilon_{ab}\Upsilon^{ab} - 12\Xi\Upsilon^{ab}L_{ab} + 6\Xi^3\Upsilon^{ab}T_{ab} + 12\Theta^a\nabla_a\Xi. \tag{29}$$

In the case of P_{abcd}, application of ∇^h together with Eqs. (23b), (23e), (23i), as well as the various symmetries of Λ_{abc} and $P^a{}_{bcd}$ results, after a rather direct calculation in:

$$
\begin{aligned}
\Box P_{egmn} = {} & \tfrac{1}{3} R P_{egmn} - 2L_{[m}{}^h P_{n]heg} + 2\Lambda_{[n|eg|}\nabla_{m]}\Xi + 2\Xi\nabla_{[m}\Lambda_{n]eg} \\
& + 2\nabla_{[m}\Delta_{|eg|n]} + 2\Xi\nabla_{[e}\Lambda_{g]mn} \\
& + 2\nabla_{[e}\Delta_{|mn|g]} - 2\Lambda_{[e|mn|}\nabla_{g]}\Xi - 2\Xi d_{[e}{}^h{}_{g]}{}^a P_{mnha} \\
& - 2\Xi d_{[e}{}^h{}_{|m|}{}^a P_{g]hna} + 2\Xi d_{[e}{}^h{}_{|n|}{}^a P_{g]hma} \\
& - 2L_{[e}{}^h P_{g]hmn} - 2P_{[e}{}^h{}_{g]}{}^a P_{mnha} - 4P_{[e}{}^h{}_{|m|}{}^a P_{g]hna} \\
& + 2\Xi g_{[e|m}\nabla^h\Lambda_{n|g]h} - 2\Xi g_{[e|n}\nabla^h\Lambda_{m|g]h} \\
& + 2\Upsilon^{ha} d_{[e|hma|}g_{g]n} - 2\Upsilon^{ha} d_{[e|hna|}g_{g]m} + 2\Lambda_{[g|nh|}g_{e]m} \\
& + 2\Lambda_{n[g|h|}g_{e]m} + 2\Lambda_{m[e|h|}g_{g]n} \\
& + 2\Lambda_{[e|mh|}g_{g]n} - 4L^{ha}P_{[e|hma|}g_{g]n} + 4L^{ha}P_{[e|hna|}g_{g]m}.
\end{aligned} \tag{30}
$$

5.1.2 Equation for Λ_{abc}

Notice that the integrability condition for Λ_{abc}, Eq. (23j), contains derivatives of zero-quantities on both sides of the equation. This feature seems to hinder our standard approach for the construction of a subsidiary equation. Then, in order to construct a suitable wave equation it will be necessary to exploit the symmetries of Λ_{abcde}. Applying ∇^e to the integrability condition (23j) and commuting derivatives leads to

$$
\begin{aligned}
\Box\Lambda_{gmn} = {} & \Lambda^c{}_{mn}R_{gc} + \nabla_g\nabla_c\Lambda^c{}_{mn} - 2P_g{}^{ceh}\nabla_h d_{mnce} \\
& - 2d_{mn}{}^{ce}\nabla_h P_{gce}{}^h - \nabla^c\nabla^e\Lambda_{gce[mn]} \\
& - 2\Lambda^c{}_{[m}{}^e R_{|gc|n]e} - 2d_{[m}{}^{ceh}\nabla_{|e}P_{gh|n]c} \\
& - 2d_g{}^c{}_{[m}{}^e\nabla^h P_{n]ech} - 2P_{[m}{}^{ceh}\nabla_{|e}d_{gh|n]c} \\
& - 2P_g{}^c{}_{[m}{}^e\nabla^h d_{n]ech}.
\end{aligned}
$$

Here, the double-derivative terms put at risk the hyperbolicity of the system. For the second derivative of Λ_{abc} one can use (23g), while the one involving Λ_{abcde} can be eliminated by recalling that this field is antisymmetric under any permutation of the first three indices—see (22). Using this property and commuting derivatives gives

$$
\begin{aligned}
\Box\Lambda_{gmn} = {} & -\Xi\Lambda^c{}_g{}^e d_{mnce} + 4\Lambda^c{}_{mn}L_{gc} + 2d_{mnce}\Delta_g{}^{ce} \\
& - 2P_g{}^{ceh}\nabla_h d_{mnce} + 2\Upsilon_{[m}{}^c\nabla_{|g|}T_{n]c} \\
& - 2\Xi\Lambda^c{}_{[m}{}^e d_{|g|n]ce} - 4\Xi\Lambda^c{}_{[m}{}^e d_{|ge|n]c} - 4\Lambda^c{}_{g[m}L_{n]c} \\
& + 2\Lambda_{[m}{}^{ce}P_{|gc|n]e} + 2\Lambda^c{}_g{}^e P_{[m|c|n]e} \\
& - 2T_{[m}{}^{ce}P_{|ge|n]c} + 2d_g{}^c{}_{[m}{}^e\Delta_{n]ec} - 2d_{[m}{}^{ceh}\nabla_{|e}P_{gh|n]c}
\end{aligned}
$$

$$
\begin{aligned}
& - 2P_{[m}{}^{ceh}\nabla_{|e}d_{gh|n]c} - 2T_{[m}{}^{c}\nabla_{|g|}\Upsilon_{n]c} \\
& - \Xi\Lambda^{ceh}d_{[m|ceh}g_{g|n]} - 4\Lambda^{c}{}_{[m}{}^{e}L_{|ce}g_{g|n]} - \Lambda^{ceh}P_{[m|ceh}g_{g|n]}.
\end{aligned} \tag{31}
$$

The results of this section can be summarised in the following lemma:

Lemma 3 *Assume that the conformal fields satisfy equations* (11) *and* (12a)–(12d). *Then, the geometric zero-quantities* (17a)–(17f) *satisfy the homogeneous system of geometric wave equations* (26)–(31).

5.2 Propagation of the constraints

As it will be discussed in detail in Sect. 6, the system of geometric wave equations (26)–(31) implies, in turn, a system of proper (hyperbolic) wave equations for which a theory of the existence and uniqueness of solutions is readily available—see e.g. [13]. From the latter one directly obtains the following result:

Proposition 3 *Assume that the geometric zero-quantities and their first derivatives vanish on a fiduciary spacelike hypersurface $\mathcal{S}_\star$ of an unphysical spacetime $(\mathcal{M}, g_{ab})$. Then, the geometric zero-quantities vanish on the domain of dependence $D(\mathcal{S}_\star)$ of $\mathcal{S}_\star$.*

Remark 11 Working, for example, with coordinates adapted to the hypersurface $\mathcal{S}_\star$, it can be readily checked that the completely spatial parts of the zero-quantities Υ_{ab}, Θ_a, Λ_{abc}, Δ_{abc}, Z and $P^a{}_{bcd}$ encode the same information as the conformal Einstein constraint equations — see e.g. [15], Chapter 11. Similarly, projections with a transversal (i.e. timelike) component can be read as a first order evolution system for the geometric conformal fields—we ignore null components as these can be obtained as linear combinations of transversal and intrinsic components. Thus, in order to ensure the vanishing of the zero-quantities on the initial hypersurface $\mathcal{S}_\star$, one needs, firstly, to produce a solution to the conformal constraint equations; this ensures the vanishing of the spatial part of the zero-quantities. Secondly, one reads the transversal components of the zero-quantities as definitions for the normal derivatives of the conformal fields which can be readily computed from the solution to the conformal constraints. In this way, the transversal components of the zero-quantities vanish *a fortiori*.

6 Gauge considerations

This section provides a brief overview of the gauge freedom inherent to the conformal Einstein field equations and the associated evolution equations. This gauge freedom is of two types: *conformal* and *coordinate*. The discussion in this section follows closely Section 2.3 in [3] and is provided for completeness and to ease the reading of the article.

6.1 Conformal gauge source functions

An important feature of the conformal Einstein field equations is that the Ricci scalar R of the metric g_{ab} can be regarded as a *conformal gauge source* specifying the representative in the conformal class $[\tilde{\boldsymbol{g}}]$ of the (conformal) unphysical metric. Accordingly, one can always find (locally) a conformal rescaling such that the metric g'_{ab} has a prescribed Ricci scalar R'.

Remark 12 Based on the previous discussion, in what follows the Ricci scalar of the metric g_{ab} is regarded as a prescribed function $\mathcal{R}(x)$ of the coordinates, so one writes

$$R = \mathcal{R}(x).$$

6.2 Generalised harmonic coordinates and the reduced Ricci operator

The components of the Ricci tensor R_{ab} can be explicitly written in terms of the components of the metric tensor g_{ab} in general coordinates $x = (x^\mu)$ as

$$\begin{aligned} R_{\mu\nu} = & -\frac{1}{2} g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} + g_{\sigma(\mu}\nabla_{\nu)}\Gamma^\sigma + g_{\lambda\rho}g^{\sigma\tau}\Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\tau\nu} \\ & + 2\Gamma^\sigma{}_{\lambda\rho}g^{\lambda\tau}g_{\sigma(\mu}\Gamma^\rho{}_{\nu)\tau}, \end{aligned}$$

with

$$\Gamma^\nu{}_{\mu\lambda} \equiv \frac{1}{2} g^{\nu\rho}(\partial_\mu g_{\rho\lambda} + \partial_\lambda g_{\mu\rho} - \partial_\rho g_{\mu\lambda}),$$

where we have defined the *contracted Christoffel symbols* as $\Gamma^\nu \equiv g^{\mu\lambda}\Gamma^\nu{}_{\mu\lambda}$. A direct computation then gives $\Box x^\mu = -\Gamma^\mu$. Following the well-known procedure for the hyperbolic reduction of the Einstein field equations, we introduce *coordinate gauge source functions* $\mathcal{F}^\mu(x)$ to prescribe the value of the contracted Christoffel symbols via the condition $\Gamma^\mu = \mathcal{F}^\mu(x)$. This means that the coordinates $x = (x^\mu)$ satisfy the *generalised wave coordinate condition*

$$\Box x^\mu = -\mathcal{F}^\mu(x) \tag{32}$$

—see e.g. [5,15,18]. Associated to the latter, it is convenient to define the *reduced Ricci operator* $\mathscr{R}_{\mu\nu}[\boldsymbol{g}]$ as

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] \equiv R_{\mu\nu} - g_{\sigma(\mu}\nabla_{\nu)}\Gamma^\sigma + g_{\sigma(\mu}\nabla_{\nu)}\mathcal{F}^\sigma(x). \tag{33}$$

More explicitly, one has that

$$\begin{aligned} \mathscr{R}_{\mu\nu}[\boldsymbol{g}] = & -\frac{1}{2} g^{\lambda\rho}\partial_\lambda\partial_\rho g_{\mu\nu} - g_{\sigma(\mu}\nabla_{\nu)}\mathcal{F}^\sigma(x) + g_{\lambda\rho}g^{\sigma\tau}\Gamma^\lambda{}_{\sigma\mu}\Gamma^\rho{}_{\tau\nu} \\ & + 2\Gamma^\sigma{}_{\lambda\rho}g^{\lambda\tau}g_{\sigma(\mu}\Gamma^\rho{}_{\nu)\tau}. \end{aligned}$$

Thus, by choosing coordinates satisfying the generalised wave coordinates condition (32), the unphysical Einstein equation (11) takes the form

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] = 2L_{\mu\nu} + \frac{1}{6}\mathcal{R}(x)g_{\mu\nu}.$$

Assuming that the components $L_{\mu\nu}$ are known, the latter is a quasilinear wave equation for the components of the metric tensor.

6.2.1 The reduced wave operator

The geometric wave operator $\square$ acting on tensorial fields contains derivatives of the Christoffel symbols which, in turn, contain second order derivatives of the components of the metric tensor. The presence of these second order derivative terms is problematic as they destroy, in principle, the hyperbolicity of the evolution Eqs. (12c) and (12d) since they enter in the principal part of the system. However, as discussed in e.g. [3,17], the generalised wave coordinate condition (32) can be used to reduce the geometric wave operator $\square$ to a proper second order hyperbolic operator.

Definition 1 The reduced wave operator $\blacksquare$ acting on a covariant tensor field $T_{\lambda\cdots\rho}$ is defined as

$$\blacksquare T_{\lambda\cdots\rho} \equiv \square T_{\lambda\cdots\rho} + \Big((2L_{\tau\lambda} + \frac{1}{6}\mathcal{R}(x)g_{\tau\lambda} - R_{\tau\lambda}) - g_{\sigma\tau}\nabla_\lambda(\mathcal{F}^\sigma(x) - \Gamma^\sigma) \Big) T^\tau{}_{\cdots\rho} + \cdots$$
$$\cdots + \Big((2L_{\tau\rho} + \frac{1}{6}\mathcal{R}(x)g_{\tau\rho} - R_{\tau\rho}) - g_{\sigma\tau}\nabla_\rho(\mathcal{F}^\sigma(x) - \Gamma^\sigma) \Big) T_{\lambda\cdots}{}^\tau,$$

where $\square \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. The action of $\blacksquare$ on a scalar ϕ is simply given by

$$\blacksquare\phi \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu\phi.$$

Remark 13 The operator $\blacksquare$ provides a proper second order hyperbolic operator for systems which involve the metric as an unknown, in contrast to $\square$. Accordingly, when working in generalised harmonic coordinates, all the second order derivatives of the metric tensor can be removed from the principal part of geometric wave equations. A system of evolutions equations expressed in terms of the reduced wave operator $\blacksquare$ (rather than in terms of the geometric wave operator $\square$) will be said to be *proper*.

6.3 Summary: gauge reduced evolution equations

The discussion of the previous sections leads us to consider the following *gauge reduced* system of evolution equations for the components of the conformal fields Ξ, s, L_{ab}, d_{abcd} and g_{ab} with respect to coordinates $x = (x^\mu)$ satisfying the generalised wave coordinate condition (32):

$$\blacksquare\Xi = 4s - \frac{1}{6}\Xi\mathcal{R}(x), \tag{34a}$$

$$\begin{aligned}\blacksquare s = {} & -\tfrac{1}{6}s\mathcal{R}(x) + \Xi L_{\mu\nu}L^{\mu\nu} - \tfrac{1}{6}\nabla_\mu\mathcal{R}(x)\nabla^\mu\Xi + \tfrac{1}{4}\Xi^5 T_{\mu\nu}T^{\mu\nu} \\ & -\Xi^3 L_{\mu\nu}T^{\mu\nu} + \Xi\nabla^\mu\Xi\nabla^\nu\Xi T_{\mu\nu},\end{aligned} \tag{34b}$$

$$\begin{aligned}\blacksquare L_{\mu\nu} = {} & -2\Xi d_{\mu\rho\nu\lambda}L^{\rho\lambda} + 4L_\mu{}^\lambda L_{\nu\lambda} - L_{\lambda\rho}L^{\lambda\rho}g_{\mu\nu} \\ & +\tfrac{1}{6}\nabla_\mu\nabla_\nu\mathcal{R}(x) + \tfrac{1}{2}\Xi^3 d_{\mu\lambda\nu\rho}T^{\lambda\rho} \\ & -\Xi\nabla_\lambda T_\mu{}^\lambda{}_\nu - 2T_{(\mu|\lambda|\nu)}\nabla^\lambda\Xi,\end{aligned} \tag{34c}$$

$$
\begin{aligned}
\blacksquare d_{\mu\nu\lambda\rho} = & -4\Xi d_{\mu}{}^{\tau}{}_{[\lambda}{}^{\sigma} d_{\rho]\sigma\nu\tau} - 2\Xi d_{\mu}{}^{\tau}{}_{\nu}{}^{\sigma} d_{\lambda\rho\tau\sigma} + \tfrac{1}{2} d_{\mu\nu\lambda\rho}\mathcal{R}(x) \\
& - T_{[\mu}{}^{\sigma}\,\Xi^2 d_{\nu]\sigma\lambda\rho} - \Xi^2 T_{[\lambda}{}^{\sigma} d_{\rho]\sigma\mu\nu} \\
& - \Xi^2 g_{\mu[\lambda} d_{\rho]\sigma\nu\tau} T^{\tau\sigma} + \Xi^2 g_{\nu[\lambda} d_{\rho]\sigma\mu\tau} T^{\tau\sigma} \\
& + 2\nabla_{[\mu} T_{|\lambda\rho|\nu]} + \epsilon_{\mu\nu\sigma\tau}\nabla^{\tau}\, {}^{*}T_{\lambda\rho}{}^{\sigma},
\end{aligned} \tag{34d}
$$

$$
\mathscr{R}_{\mu\nu}[\boldsymbol{g}] = 2L_{\mu\nu} + \frac{1}{6}\mathcal{R}(x) g_{\mu\nu}. \tag{34e}
$$

Remark 14 The reduced system of evolution Eqs. (34a)–(34e) is a system of quasilinear wave equations for the fields Ξ, s, $L_{\mu\nu}$, $d_{\mu\nu\lambda\rho}$ and $g_{\mu\nu}$. More explicitly, one has that

$$
\begin{aligned}
g^{\sigma\tau}\partial_\sigma\partial_\tau \Xi &= X\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, s, \mathcal{R}(x)\big), \\
g^{\sigma\tau}\partial_\sigma\partial_\tau s &= S\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, \partial\Xi, s, \boldsymbol{L}, \mathcal{R}(x), \partial\mathcal{R}(x), \boldsymbol{T}\big), \\
g^{\sigma\tau}\partial_\sigma\partial_\tau L_{\mu\nu} &= F_{\mu\nu}\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, \boldsymbol{L}, \boldsymbol{d}, \mathcal{R}(x), \partial^2\mathcal{R}(x), \boldsymbol{T}, \partial\boldsymbol{T}\big), \\
g^{\sigma\tau}\partial_\sigma\partial_\tau d_{\mu\nu\lambda\rho} &= D_{\mu\nu\lambda\rho}\big(\boldsymbol{g}, \partial\boldsymbol{g}, \Xi, \boldsymbol{d}, \mathcal{R}(x), \partial\boldsymbol{T}\big), \\
g^{\sigma\tau}\partial_\sigma\partial_\tau g_{\mu\nu} &= G_{\mu\nu}\big(\boldsymbol{g}, \partial\boldsymbol{g}, \boldsymbol{L}, \mathcal{R}(x)\big),
\end{aligned}
$$

where X, S, $F_{\mu\nu}$, $D_{\mu\nu\lambda\rho}$ and $G_{\mu\nu}$ are polynomial expressions of their arguments. Strictly speaking, the system is a system of wave equations only if $g_{\mu\nu}$ is known to be Lorentzian. The basic existence, uniqueness and stability results of systems of the above type have been given in [13]—these results are the second order analogues of the theory developed in [14] for symmetric hyperbolic systems. The basic theory for initial-boundary value problems can be found in [4,7].

7 Propagation of the gauge

This section is devoted to studying the consistency of the conformal and coordinate gauge introduced in Sect. 6 by constructing a system of homogeneous wave equations for a set of subsidiary fields. The coming discussion extends the analysis in [3], Sect. 5, for the vacuum case which is closely followed—accordingly, we mainly focus on the new features arising from the presence of matter.

7.1 Basic relations

Consider a set of coordinates $x = (x^\mu)$. Let $g_{\mu\nu}$ denote the components of a metric g_{ab} in these coordinates. Similarly, $R_{\mu\nu}$ denotes the components of the associated Ricci tensor R_{ab}, while R is the corresponding Ricci scalar. We now investigate the requirements for R and $R_{\mu\nu}$ to coincide, respectively, with $\mathcal{R}(x)$ and $\mathscr{R}_{\mu\nu}$. In addition, we also need to investigate the conditions under which $L_{\mu\nu}$ corresponds to the components of the Schouten tensor. This can be expressed as the vanishing of the following fields:

$$Q \equiv R - \mathcal{R}(x), \tag{35a}$$
$$Q^{\mu} \equiv \Gamma^{\mu} - \mathcal{F}^{\mu}(x), \tag{35b}$$
$$Q_{\mu\nu} \equiv R_{\mu\nu} - \mathscr{R}_{\mu\nu}. \tag{35c}$$

We make the following assumption:

Assumption 2 Let $T_{\mu\nu}$ and $T_{\mu\nu\lambda}$ be, respectively, the components of a tracefree energy momentum tensor with vanishing divergence and its associated rescaled Cotton tensor. Let $g_{\mu\nu}$ and $L_{\mu\nu}$ be solutions to the equations:

$$\mathscr{R}_{\mu\nu} = 2L_{\mu\nu} + \tfrac{1}{6}\mathcal{R}(x)g_{\mu\nu}, \tag{36a}$$
$$\begin{aligned}\blacksquare L_{\mu\nu} =& -2\Xi d_{\mu\rho\nu\lambda}L^{\rho\lambda} + 4L_{\mu}{}^{\lambda}L_{\nu\lambda} - L_{\lambda\rho}L^{\lambda\rho}g_{\mu\nu} \\ &+ \tfrac{1}{6}\nabla_{\mu}\nabla_{\nu}\mathcal{R}(x) + \tfrac{1}{2}\Xi^{3}d_{\mu\lambda\nu\rho}T^{\lambda\rho} \\ &- \Xi\nabla_{\lambda}T_{\mu}{}^{\lambda}{}_{\nu} - 2T_{(\mu|\lambda|\nu)}\nabla^{\lambda}\Xi.\end{aligned} \tag{36b}$$

As a direct consequence of Eq. (36a), one can find that the gauge zero-quantities (35a)–(35c) are not independent of each other. Simple calculations yield

$$Q_{\mu\nu} = \nabla_{(\mu}Q_{\nu)}, \tag{37a}$$
$$Q = Q_{\mu}{}^{\mu} = \nabla_{\mu}Q^{\mu}. \tag{37b}$$

Furthermore, Eq. (33) and Definition 1 lead to

$$\mathscr{R}_{\mu\nu}[\boldsymbol{g}] = R_{\mu\nu} - \nabla_{(\mu}Q_{\nu)}, \tag{38a}$$
$$\blacksquare L_{\mu\nu} = \Box L_{\mu\nu} - (Q_{\mu\sigma} - \nabla_{\mu}Q_{\sigma})L^{\sigma}{}_{\nu} - (Q_{\nu\sigma} - \nabla_{\nu}Q_{\sigma})L^{\sigma}{}_{\mu}. \tag{38b}$$

Remark 15 Equations (37a)–(37b) show that if $Q^{\mu} = 0$ then Q and $Q_{\mu\nu}$ automatically vanish. In this sense, we will consider Q^{μ} as the basic gauge zero-quantity of the system.

7.2 The gauge subsidiary evolution system

In this subsection we obtain a system of homogeneous wave equations for the gauge subsidiary variables. This will be achieved by exploiting the properties of the so-called *Bach tensor* which will play the role of an integrability condition for the system.

7.2.1 The Bach tensor

Let g_{ab} be a 4-dimensional metric. The Bach tensor is defined as:

$$B_{ab} \equiv \nabla^{c}\nabla_{a}L_{bc} - \nabla^{c}\nabla_{c}L_{ab} - C_{acdb}L^{cd}. \tag{39}$$

From this definition it is easy to verify that B_{ab} is symmetric and tracefree. Additionally, it satisfies the following identity, independently of the validity of the Einstein field equations:

$$\nabla^a B_{ab} = 0. \tag{40}$$

Remark 16 A straightforward calculation shows that the Bach tensor can be expressed in terms of the geometric zero-quantities as

$$B_{ab} = -L^{cd} P_{acbd} - \tfrac{1}{2}\Xi^3 d_{acbd} T^{cd} + \Xi \nabla_c T_a{}^c{}_b + 2T_{(a|c|b)} \nabla^c \Xi.$$

Consequently, if g_{ab} is a solution to the tracefree metric conformal Einstein field equations then the Bach tensor vanishes if $T_{ab} = 0$.

Remark 17 In view of the fact that trivial initial conditions for the zero-quantities imply the vanishing of $P^a{}_{bcd}$—see Proposition 3—throughout the remainder of the article, and for the sake of simplicity, our calculations will assume that $P^a{}_{bcd} = 0$.

7.2.2 Wave equations for the gauge subsidiary variables

The Bach tensor can be conveniently expressed in terms of the gauge zero-quantities. Terms containing $R_{\mu\nu}$ and R can be rewritten according to definitions (35a) and (35c) along with (37a) and (38a). A procedure similar to that of Section 5.2 in [3] allows us to show that the Bach tensor can be expressed in the form

$$B_{\mu\nu} = B'_{\mu\nu} + N_{\mu\nu}, \tag{41}$$

where $B'_{\mu\nu}$ is an expression homogeneous on Q, Q_μ, $Q_{\mu\nu}$ and its derivatives up to fourth order and which is identical to the one found in [3]. Here, the contributions from $T_{\mu\nu}$ have been grouped in the symmetric tensor

$$N_{\mu\nu} \equiv -\tfrac{1}{2}\Xi^3 d_{\mu\lambda\nu\rho} T^{\lambda\rho} + 2T_{(\mu|\lambda|\nu)} \nabla^\lambda \Xi + \Xi \nabla_\lambda T_\mu{}^\lambda{}_\nu.$$

Next, we introduce the auxiliary field

$$M_\mu \equiv \square Q_\mu. \tag{42}$$

Taking the divergence of equation (41), and after some direct manipulations, Eqs. (37a)–(37b) and (40) imply that

$$\square M_\mu = H_\mu(\nabla \boldsymbol{M}, \nabla \boldsymbol{Q}, \nabla Q, \boldsymbol{Q}, Q) + 4\nabla^\nu N_{\nu\mu},$$

where $\boldsymbol{Q}$ stands for Q_μ and, for simplicity, H_ν represents a homogeneous function of its arguments. On the other hand, we can rewrite the term $\nabla^\nu N_{\nu\mu}$ in a suitable way by using the symmetries of T_{abc} along with the help of Eqs. (35c), (38a) and the geometric zero-quantities. A direct calculation shows that

$$\nabla^\nu N_{\nu\mu} = -T_{\mu\nu\lambda} \Upsilon^{\nu\lambda} - \tfrac{1}{2}\Xi^3 T_{\nu\lambda} \Lambda^\nu{}_\mu{}^\lambda,$$

so the wave equation for M_μ takes the schematic form

$$\Box M_\mu = H_\mu(\nabla \boldsymbol{M}, \nabla \boldsymbol{Q}, \nabla Q, \boldsymbol{Q}, Q, \boldsymbol{\Upsilon}, \boldsymbol{\Lambda}). \tag{43}$$

Lastly, a wave equation for Q is required to close the system. This can be obtained by direct application of the $\Box$ operator on the definition of Q along with the aid of Eqs. (35a), (37b) and (38a), resulting in

$$\begin{aligned}\Box Q = &-2L_{\mu\nu}\nabla^\mu Q^\nu - \nabla^\mu Q^\nu \nabla_{(\mu} Q_{\nu)} - \tfrac{1}{2} Q^\mu \nabla_\mu Q \\ &- \tfrac{1}{2} Q^\mu \nabla_\mu \mathcal{R}(x) - \tfrac{1}{6}\mathcal{R}(x) Q + \nabla^\mu M_\mu.\end{aligned} \tag{44}$$

Remark 18 The gauge subsidiary evolution system, Eqs. (42)–(44), is homogeneous in $M_\mu,\ Q_\mu,\ Q,\ \Upsilon_{\mu\nu},\ \Lambda_{\mu\nu\lambda}$ and their first derivatives.

The previous discussion leads to the following result:

Lemma 4 *Assume that the hypotheses of Lemma 3 hold. Moreover, let the quantities M_μ, Q_μ, Q, $\Upsilon_{\mu\nu}$ and $\Lambda_{\mu\nu\lambda}$ along with their first covariant derivatives vanish on a fiduciary hypersurface $\mathcal{S}_\perp$. Then the unique solution to the system* (42)–(44) *on a small enough slab of $\mathcal{S}_\star$ corresponds to $Q = 0$, $Q_\mu = 0$ and $M_\mu = 0$, which in turn implies that $Q_{\mu\nu} = 0$.*

Remark 19 As discussed in Section 5.2.3 of [3] the initial gauge conditions in Lemma 4 can be rephrased in terms of conditions on the lapse and shift (and their derivatives) associated to the coordinate gauge source function $\mathcal{F}^\mu(x)$. It must be pointed out that these initial gauge conditions are not equivalent, in the vacuum case, to those considered in [17] which only require the vanishing of the gauge zero-quantities and their first derivatives on the initial hypersurface. In the present case, the conditions require the vanishing of third order derivatives via the definition of M_μ.

8 Evolution equations for the matter fields

Having settled the analysis of the *geometric* part of the metric tracefree conformal Einstein equations, we now proceed to investigate the evolution of the subsidiary equations associated to a number of matter models of interest: the conformally invariant scalar field, the Maxwell field and the Yang–Mills field.

8.1 The conformally invariant scalar field

It is well-known that the equation

$$\tilde{\nabla}^a \tilde{\nabla}_a \tilde{\phi} = 0,$$

where $\tilde{\phi}$ is a scalar field, is not conformally invariant. This deficiency can be healed by the addition of a term involving the coupling with the Ricci scalar, namely

$$\tilde{\nabla}^a \tilde{\nabla}_a \tilde{\phi} - \tfrac{1}{6}\tilde{R}\tilde{\phi} = 0. \tag{45}$$

Defining the unphysical scalar field

$$\phi \equiv \Xi^{-1}\tilde{\phi},$$

a direct computation shows that Eq. (45) implies

$$\Box\phi - \tfrac{1}{6}R\phi = 0. \tag{46}$$

In what follows, for convenience, Eq. (45) will known as the *conformally invariant wave equation*—or the *conformally coupled wave equation*. The energy-momentum tensor associated to this field takes the form

$$T_{ab} = \nabla_a\phi\nabla_b\phi - \tfrac{1}{4}g_{ab}\nabla_c\phi\nabla^c\phi - \tfrac{1}{2}\nabla_a\nabla_b\phi + \tfrac{1}{2}\phi^2 L_{ab}, \tag{47}$$

so that $\nabla^a T_{ab} = 0$ holds if Eq. (46) is satisfied. It can be readily verified that T_{ab}, as given by the expression above, is also tracefree.

Remark 20 The second derivatives of ϕ in Eq. (47) will lead to the appearance of second and third order derivatives of the matter field in the expression of the rescaled Cotton tensor—see Eq. (6d)—which may affect the hyperbolicity of the system (34a)–(34e). Moreover, T_{ab} is also coupled to the geometric sector via the Schouten tensor. These difficulties will be addressed in the sequel.

Remark 21 The conformally invariant scalar field is related to the standard scalar field satisfying the wave equation through a transformation originally due to Bekenstein [1]. Thus, in principle, the theory for the conformally invariant scalar field developed in this section can be rephrased in terms of the standard scalar field.

8.1.1 Auxiliary fields and the evolution equations

We start the analysis by observing that the third order derivative terms in the expression of the rescaled Cotton tensor for the conformally invariant scalar field are of the form $\nabla_{[a}\nabla_{b]}\nabla_c\phi$. Using the commutator of covariant derivatives, these terms can be transformed into first order derivative terms according to the formula

$$\nabla_{[a}\nabla_{b]}\nabla_c\phi = -\tfrac{1}{2}R_{abc}{}^{d}\nabla_d\phi.$$

Thus, one is left with an expression for the Cotton tensor containing, at most, second order derivatives. In order to eliminate these derivatives which, potentially, could destroy the hyperbolic nature of the wave equations (34a)–(34d), one needs to promote the first and second derivatives of ϕ as further (independent) unknowns. Accordingly, we define

$$\phi_a \equiv \nabla_a\phi, \qquad \phi_{ab} \equiv \nabla_a\nabla_b\phi. \tag{48}$$

Following the previous discussion, and exploiting equation (5c), one can write the rescaled Cotton tensor for the conformally invariant scalar field as

$$
\begin{aligned}
T_{abc} = \left(1 - \tfrac{1}{4}\Xi^2\phi^2\right)^{-1} \Big(& \tfrac{3}{2}\Xi\phi L_{c[b}\phi_{a]} + \tfrac{3}{2}\Xi\phi_{[b}\phi_{a]c} \\
& - \tfrac{1}{4}\Xi\phi^2 d_{abcd}\nabla^d\Xi - \tfrac{1}{4}\Xi^2\phi d_{abcd}\phi^d \\
& + \tfrac{1}{2}\Xi\phi g_{c[b}L_{a]d} + \tfrac{1}{2}\Xi g_{c[a}\phi_{b]d}\phi^d + g_{c[b}T_{a]d}\nabla^d\Xi + 3T_{c[b}\nabla_{a]}\Xi \Big).
\end{aligned} \tag{49}
$$

We now proceed to construct suitable evolution equations for ϕ_a and ϕ_{ab} by means of a set of integrability conditions for these fields. Firstly, the identity $\nabla_a\phi_b = \nabla_b\phi_a$ represents an integrability condition for ϕ_a. A wave equation then readily follows after applying ∇^b and using Eq. (46):

$$
\square\phi_a = 2\phi^b L_{ab} + \tfrac{1}{3}R\phi_a + \tfrac{1}{6}\phi\nabla_a R. \tag{50}
$$

On the other hand, an integrability condition for ϕ_{ab} can be obtained directly from its definition:

$$
2\nabla_{[c}\phi_{a]b} = \phi_d R_{cab}{}^d = -\Xi\phi^d d_{acbd} - 2\phi_{[c}L_{a]b} + 2\phi^d g_{b[c}L_{a]d}.
$$

Applying ∇^c to this relation and using Eqs. (5c), (5d), (46) and (50), a straightforward calculation leads to:

$$
\begin{aligned}
\square\phi_{ab} =& \tfrac{1}{2}\phi_{ab}R - \tfrac{1}{3}R\phi L_{ab} - 2\phi^{cd}L_{cd}g_{ab} - \tfrac{1}{6}\phi^c g_{ab}\nabla_c R \\
& + \tfrac{1}{6}\phi\nabla_{(a}\nabla_{b)}R - 2\Xi\phi^{cd}d_{(a|c|b)d} \\
& + 8\phi_{(a}{}^c L_{b)c} + 2\Xi\phi^c T_{(a|c|b)} + \tfrac{2}{3}\phi_{(a}\nabla_{b)}R \\
& + 2\phi^c\nabla_{(a}L_{b)c} - 2\phi^c d_{(acb)}{}^d\nabla_d\Xi.
\end{aligned} \tag{51}
$$

Remark 22 In Eq. (51) it is understood that the rescaled Cotton tensor T_{bca} is expressed in terms of the auxiliary fields ϕ_a and ϕ_{ab} according to (49) so does not contain second or higher derivatives of the fields.

Remark 23 When coupling the wave equations (46), (50) and (51) to the system (34a)–(34e) satisfied by the geometric conformal fields, it is understood that the geometric wave operator $\square$ is replaced by its reduced counterpart $\blacksquare$ as discussed in Sect. 6.2.1.

8.1.2 Subsidiary equations

To verify the consistency of our approach in dealing with the higher order derivative terms in the rescaled Cotton tensor for the conformally invariant scalar field we introduce the following subsidiary fields:

$$
Q_a \equiv \phi_a - \nabla_a\phi, \tag{52a}
$$

$$Q_{ab} \equiv \phi_{ab} - \nabla_a \nabla_b \phi. \tag{52b}$$

A wave equation for Q_a can be obtained in a straightforward way: applying $\square$ to definition (52a) and with the help of relations (46) and (50), a short calculation yields

$$\square Q_a = \square \phi_a - \nabla_a \square \phi - R_{ab} \nabla^b \phi = \tfrac{1}{3} R Q_a + 2 L_a{}^b Q_b. \tag{53}$$

Similarly, applying $\square$ to Eq. (52b), commuting covariant derivatives and using the definitions of the geometric zero-quantities one obtains

$$\begin{aligned}\square Q_{ab} = & \tfrac{1}{2} Q_{ab} R - 2 Q^{cd} L_{cd} g_{ab} - \tfrac{1}{6} Q^c g_{ab} \nabla_c R + 2 Q^c \nabla_c L_{ab} \\ & - 2 \Xi Q^{cd} d_{acbd} + 8 Q_{(a}{}^c L_{b)c} \\ & - 2 \phi^c \Delta_{(a|c|b)} + 4 \Xi Q^c T_{(a|c|b)} + 4 Q^c \Delta_{(a|c|b)} \\ & + \tfrac{2}{3} Q_{(a} \nabla_{b)} R - 4 Q^c d_{(a|c|b)}{}^d \nabla_d \Xi.\end{aligned} \tag{54}$$

Remark 24 The system of wave equations (53) and (54) is homogeneous in Q_a, Q_{ab} and Δ_{abc} Thus, it follows from general uniqueness results for solutions to wave equations that if these quantities and their derivatives vanish on an initial hypersurface $\mathcal{S}_\star$, then necessarily $Q_a = 0$ and $Q_{ab} = 0$ at least on a small enough slab around $\mathcal{S}_\star$.

8.1.3 Summary

The analysis of the conformally invariant scalar field can be summarised in the following manner:

Proposition 4 *The system of equations* (34a)–(34e) *with rescaled Cotton tensor given by* (49)*, together with the conformally invariant wave equation* (46) *and the auxiliary system* (50)–(51) *written in terms of the reduced wave operator* $\blacksquare$*, constitute a* proper *system of quasilinear wave equations—see Remark* 13.

8.2 The Maxwell field

The next example under consideration is the electromagnetic field. The physical Maxwell equations expressed in terms of the antisymmetric Faraday tensor $\tilde{F}_{ab}$ are given by

$$\tilde{\nabla}^a \tilde{F}_{ab} = 0,$$
$$\tilde{\nabla}_{[a} \tilde{F}_{bc]} = 0.$$

It is well-known that the Maxwell equations are conformally invariant by defining the *unphysical Faraday tensor* F_{ab} as

$$F_{ab} \equiv \tilde{F}_{ab}.$$

From here it follows that the physical Maxwell equations imply

$$\nabla^a F_{ab} = 0, \tag{55a}$$

$$\nabla_{[a} F_{bc]} = 0, \tag{55b}$$

with the associated *unphysical Maxwell energy-momentum tensor* given by

$$T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd}. \tag{56}$$

Alternatively, defining the Hodge dual F^*_{ab} of the Faraday tensor as

$$F^*_{ab} \equiv -\frac{1}{2} \epsilon_{ab}{}^{cd} F_{cd}, \tag{57}$$

the second unphysical Maxwell equation (55b) can be written as:

$$\nabla^a F^*_{ab} = 0. \tag{58}$$

8.2.1 Auxiliary field and the evolution equations

A geometric wave equation for the Faraday tensor can be obtained from differentiation of the Maxwell equation (55b), which represents a natural integrability condition for this field. Commuting covariant derivatives and applying Eq. (55a), a calculation yields

$$\square F_{bc} = \tfrac{1}{3} F_{bc} R - 2\Xi F^{ad} d_{bacd}. \tag{59}$$

From Eq. (6d) it follows that the rescaled Cotton tensor contains first derivatives of F_{ab}. This puts at risk the hyperbolicity of the system (34a)–(34d). In order to deal with this problem we introduce the auxiliary variable

$$F_{abc} \equiv \nabla_a F_{bc}, \tag{60}$$

satisfying $F_{abc} = F_{a[bc]}$. By virtue of Eq. (55b) it also follows that $F_{[abc]} = 0$. In terms of this quantity, it can be readily checked that the rescaled Cotton tensor for the Maxwell field takes the form

$$\begin{aligned} T_{abc} = {} & \Xi F_{[b}{}^d F_{a]cd} - \tfrac{1}{2} \Xi F_c{}^d F_{dab} + \tfrac{1}{2} \Xi g_{c[a} F^{de} F_{b]de} \\ & - 3 F_{cd} F_{[a}{}^d \nabla_{b]} \Xi + F_{de} F^{de} g_{c[a} \nabla_{b]} \Xi \\ & - g_{c[a} F_{b]}{}^e F_{de} \nabla^d \Xi. \end{aligned} \tag{61}$$

From definition (60) it follows that F_{abc} possesses two independent divergences: $\nabla^a F_{abc}$ is simply the right-hand side of wave equation (59) whilst the other is given by

$$\nabla_c F_{ab}{}^c = \Xi F^{cd} d_{acbd} - \frac{1}{6} R F_{ab} + 2 F_{[a}{}^c L_{b]c}, \tag{62}$$

as a direct calculation confirms. In order to obtain an integrability condition for F_{abc}, consider the expression $3\nabla_{[d}F_{|a|bc]}$. Commuting covariant derivatives and using the first Bianchi identity for the Weyl tensor, a straightforward calculation results in:

$$3\nabla_{[d}F_{|a|bc]} = -3\Xi F_{[d}{}^{e}d_{|ae|bc]} + 6F_{[db}L_{c]a} + 6g_{a[d}F_{b}{}^{e}L_{c]e}. \tag{63}$$

A geometric wave equation can be obtained by applying ∇^{d} to the last expression and commuting derivatives. Using Eqs. (5c), (5d), (8), (10), (62) as well as the symmetries of d_{abcd} and T_{abc} to simplify it, a long but direct calculation yields

$$\begin{aligned}\square F_{abc} = &-2\Xi F_{a}{}^{d}T_{bcd} + 4\Xi F_{[b}{}^{d}T_{|ad|c]} - 2\Xi F_{a}{}^{de}d_{bdce}\\ &- 4\Xi F^{d}{}_{[b}{}^{e}d_{c]ead} + \tfrac{1}{2}F_{abc}R + 4F^{d}{}_{bc}L_{ad}\\ &- 4F^{d}{}_{a[b}L_{c]d} - 4F^{d}{}_{[b}{}^{e}g_{c]a}L_{de} + \tfrac{1}{3}F_{bc}\nabla_{a}R\\ &- 2F^{de}d_{ade[b}\nabla_{c]}\Xi - 4\Xi F^{de}\nabla_{[b}d_{c]ead}\\ &- \tfrac{1}{3}F_{a[b}\nabla_{c]}R - 2F_{[b}{}^{e}d_{c]ead}\nabla^{d}\Xi - F_{d}{}^{e}d_{aebc}\nabla^{d}\Xi\\ &- 4F_{[b}{}^{e}d_{c]dae}\nabla^{d}\Xi - F_{a}{}^{e}d_{bcde}\nabla^{d}\Xi\\ &+ 2F^{ef}g_{a[b}d_{c]edf}\nabla^{d}\Xi + \tfrac{1}{3}g_{a[b}F_{c]d}\nabla^{d}R.\end{aligned} \tag{64}$$

This equation can be further simplified via a pair of observations. Firstly, by multiplying Eq. (14) by F^{dg} the following auxiliary identity is found:

$$2F_{[a}{}^{e}d_{b]ecd}\nabla^{d}\Xi - 2F_{[c}{}^{e}d_{d]eab}\nabla^{d}\Xi + 2F^{de}d_{ced[a}\nabla_{b]}\Xi - 2F^{eg}g_{c[a}d_{b]edg}\nabla^{d}\Xi = 0. \tag{65}$$

Secondly, from Eq. (10) we have the following relations:

$$\begin{aligned}4\Xi F^{de}\nabla_{[b}d_{c]ead} &= -2\Xi\epsilon_{bcef}F^{de}{}^{*}T_{ad}{}^{f} + 2\Xi F^{de}\nabla_{e}d_{adbc},\\ \Xi F^{de}\nabla_{e}d_{adbc} &= -\tfrac{1}{2}\Xi\epsilon_{adef}F^{de}{}^{*}T_{bc}{}^{f} - \tfrac{1}{2}\Xi F^{de}\nabla_{a}d_{bcde}.\end{aligned}$$

Combining them we readily obtain the identity

$$4\Xi F^{de}\nabla_{[b}d_{c]ead} = 4\Xi F_{[b}{}^{d}T_{|a|c]d} - 2\Xi F_{a}{}^{d}T_{bcd} + \Xi F^{de}\nabla_{a}d_{bcde}. \tag{66}$$

Making use of (65) and (66), the wave equation for F_{abc} takes a simpler form:

$$\begin{aligned}\square F_{abc} = &\,4\Xi F_{[b}{}^{d}T_{c]da} - 2\Xi F_{a}{}^{de}d_{bdce} - 4F^{d}{}_{[b}{}^{e}d_{c]ead}\\ &+ \tfrac{1}{2}F_{abc}R + 4F^{d}{}_{bc}L_{ad} - 4F^{d}{}_{a[b}L_{c]d}\\ &- 4F^{d}{}_{[b}{}^{e}g_{c]a}L_{de} + \tfrac{1}{3}F_{bc}\nabla_{a}R - \tfrac{1}{3}F_{a[b}\nabla_{c]}R\\ &+ \tfrac{1}{3}g_{a[b}F_{c]d}\nabla^{d}R - 4F^{de}d_{ade[b}\nabla_{c]}\Xi\\ &- 4F_{[b}{}^{e}d_{c]dae}\nabla^{d}\Xi - 2F_{a}{}^{e}d_{bcde}\nabla^{d}\Xi - \Xi F^{de}\nabla_{a}d_{bcde}.\end{aligned} \tag{67}$$

As remarked in the case of the conformally invariant scalar field, the geometric operator $\square$ is to be replaced by $\blacksquare$ when Eqs. (59) and (67) are coupled to the system (34a)–(34e).

8.2.2 Subsidiary equations

In order to complete the discussion of the Maxwell field it is necessary to construct suitable evolution equations for the zero-quantities

$$M_b \equiv \nabla^a F_{ab}, \tag{68a}$$

$$M_{abc} \equiv \nabla_{[a} F_{bc]}, \tag{68b}$$

$$Q_{abc} \equiv F_{abc} - \nabla_a F_{bc}. \tag{68c}$$

Here, M_{abc} possesses the symmetries

$$M_{abc} = M_{a[bc]} = M_{[ab]c} = M_{[abc]}. \tag{69a}$$

Also, one can verify the following identities:

$$\nabla^a M_a = 0, \tag{70a}$$

$$\nabla^c M_{abc} = -\tfrac{2}{3}\nabla_{[a} M_{b]}. \tag{70b}$$

Remark 25 Following the spirit of the discussion in the previous section, the zero-quantities M_a and M_{abc} encode Maxwell equations (55a) and (55b), respectively, while Q_{abc} does so for the auxiliary field F_{abc}.

Equation for M_a. Observe that Eq. (70b) works as an integrability condition for M_a. Applying ∇^b, using (70a) and exploiting the various symmetries of M_{abc}, one obtains

$$\square M_a = \tfrac{1}{6} M_a R + 2M^b L_{ab}. \tag{71}$$

Equation for M_{abc}. In order to avoid lengthy expressions it is simpler to consider the Hodge dual of M_{abc} defined as

$$M^*_a \equiv \nabla^b F^*_{ba} = \frac{1}{2}\epsilon_a{}^{bcd} M_{bcd}. \tag{72}$$

Here, the second equality is a consequence of Eqs. (60) and (68b). From this definition it can be easily checked that M^*_a is divergencefree which, in turn, implies an integrability condition. More explicitly:

$$\nabla^a M^*_a = 0 \iff \nabla_{[d} M_{abc]} = 0. \tag{73}$$

Applying ∇^d to (73) and commuting derivatives, a straightforward calculation leads to

$$\square M_{abc} = \tfrac{1}{2} R M_{abc} - 6\Xi d_{[a}{}^{d}{}_{b}{}^{e} M_{c]de} - 6L_{[a}{}^{d} M_{bc]d}, \tag{74}$$

where it has been used that $\nabla_{[a}\nabla_{|d|}M_{bc]}{}^{d}$ vanishes by virtue of Eq. (70b).
Equation for Q_{abc}. A wave equation for the field Q_{abc} can be obtained by direct application of the $\square$ operator. Employing definitions (68a) and (68c), along with Eqs. (17c), (17d), (59) and (67), one obtains the expression

$$\begin{aligned}\square Q_{abc} =& 4\Xi F_{[b}{}^{d}\Lambda_{|a|c]d} - 2\Xi Q_{a}{}^{de}d_{bdce} - 2\Xi Q^{d}{}_{[b}{}^{e}d_{c]ead}\\ &+ \tfrac{1}{2}Q_{abc}R - 4M_{[b}L_{c]a} + 4Q^{d}{}_{bc}L_{ad}\\ &- 4Q^{d}{}_{a[b}L_{c]d} + 6L_{a}{}^{d}M_{bcd} - 4Q^{d}{}_{[b}{}^{e}g_{c]a}L_{de}\\ &+ 2F^{de}d_{bdce}\nabla_{a}\Xi - 4F^{de}d_{ade[b}\nabla_{c]}\Xi\\ &- 6F_{[a}{}^{e}d_{bc]de}\nabla^{d}\Xi. \end{aligned} \tag{75}$$

In order to show that the terms not containing zero-quantities vanish, observe that the first Bianchi identity implies that

$$2F^{de}d_{bdce}\nabla_{a}\Xi - 4F^{de}d_{ade[b}\nabla_{c]}\Xi = 3F^{de}d_{de[ab}\nabla_{c]}\Xi.$$

On the other hand, multiplying definition (21) by F^{de}, a short calculation yields the auxiliary identity

$$3F^{de}d_{de[ab}\nabla_{c]}\Xi - 6F_{[a}{}^{e}d_{bc]de}\nabla^{d}\Xi = 0.$$

From the last two expressions it follows then that

$$\begin{aligned}\square Q_{abc} =& 4\Xi F_{[b}{}^{d}\Lambda_{|a|c]d} - 2\Xi Q_{a}{}^{de}d_{bdce} - 2\Xi Q^{d}{}_{[b}{}^{e}d_{c]ead}\\ &+ \tfrac{1}{2}Q_{abc}R - 4M_{[b}L_{c]a} + 4Q^{d}{}_{bc}L_{ad}\\ &- 4Q^{d}{}_{a[b}L_{c]d} + 6L_{a}{}^{d}M_{bcd} - 4Q^{d}{}_{[b}{}^{e}g_{c]a}L_{de}. \end{aligned} \tag{76}$$

Remark 26 Geometric wave equations (71), (74) and (76) are crucially homogeneous in M_{a}, M_{abc}, Q_{abc} and Λ_{abc}. Thus, if these quantities and their first covariant derivatives vanish on an initial hypersurface $\mathcal{S}_{\star}$, it can be guaranteed that there exists a unique solution on a small enough slab of $\mathcal{S}_{\star}$, and it corresponds to $M_{a}=0$, $M_{abc}=0$ and $Q_{abc}=0$.

8.2.3 Summary

The previous discussion about the coupling of the Maxwell field to the metric tracefree conformal Einstein field equations can be summarised as follows:

Proposition 5 *The system of wave equations* (34a)–(34e) *with rescaled Cotton tensor given by* (61) *together with the wave equations* (59) *and* (67) *written in terms of the wave operator* $\blacksquare$ *is a proper quasilinear system of wave equations for the Einstein-Maxwell system.*

8.3 The Yang–Mills field

The Yang–Mills field is the last example of a tracefree matter model we study in this paper. Due to its similarities with the Faraday field, some of the calculations will result analogous to the ones performed in the previous subsection. However, one of the distinctive features of the Yang–Mills field is the fact that, in order to obtain a hyperbolic reduction of the equations, one needs to introduce a set of gauge source functions fixing the divergence of the gauge potential. The consistency of this gauge choice will be analysed towards the end of the section.

8.3.1 Basic equations

The *Yang–Mills field* consists of a set of *gauge potentials* $\tilde{A}^{\mathfrak{a}}{}_{\mu}$ and *gauge fields* $\tilde{F}^{\mathfrak{a}}{}_{\mu\nu}$ where the indices $\mathfrak{a},\ \mathfrak{b},\ldots$ take values in a Lie algebra $\mathfrak{g}$ of a Lie group $\mathfrak{G}$. The equations satisfied by the fields $\tilde{A}^{\mathfrak{a}}{}_{\mu}$ and $\tilde{F}^{\mathfrak{a}}{}_{\mu\nu}$ are:

$$\begin{aligned}
&\tilde{\nabla}_a\tilde{A}^{\mathfrak{a}}{}_b-\tilde{\nabla}_a\tilde{A}^{\mathfrak{a}}{}_b+C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\tilde{A}^{\mathfrak{b}}{}_a\tilde{A}^{\mathfrak{c}}{}_b-\tilde{F}^{\mathfrak{a}}{}_{ab}=0,\\
&\tilde{\nabla}^a\tilde{F}^{\mathfrak{a}}{}_{ab}+C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\tilde{A}^{\mathfrak{b}a}\tilde{F}^{\mathfrak{c}}{}_{ab}=0,\\
&\tilde{\nabla}_{[a}\tilde{F}^{\mathfrak{a}}{}_{bc]}+C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\tilde{A}^{\mathfrak{b}}{}_{[a}\tilde{F}^{\mathfrak{c}}{}_{bc]}=0.
\end{aligned}$$

Here $C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}=C^{\mathfrak{a}}{}_{[\mathfrak{b}\mathfrak{c}]}$ denote the structure constants of the Lie algebra $\mathfrak{g}$ which satisfy the *Jacobi identity*

$$C^{\mathfrak{a}}{}_{\mathfrak{d}\mathfrak{e}}C^{\mathfrak{b}}{}_{\mathfrak{a}\mathfrak{c}}+C^{\mathfrak{a}}{}_{\mathfrak{e}\mathfrak{c}}C^{\mathfrak{b}}{}_{\mathfrak{a}\mathfrak{d}}+C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{d}}C^{\mathfrak{b}}{}_{\mathfrak{a}\mathfrak{e}}=0. \tag{77}$$

Also, the energy-momentum tensor associated to the Yang–Mills field is given by

$$\tilde{T}_{ab}=\tfrac{1}{4}\delta_{\mathfrak{a}\mathfrak{b}}\tilde{F}^{\mathfrak{a}}{}_{cd}\tilde{F}^{\mathfrak{b}cd}\tilde{g}_{ab}-\delta_{\mathfrak{a}\mathfrak{b}}\tilde{F}^{\mathfrak{a}}{}_{ac}\tilde{F}^{\mathfrak{b}}{}_b{}^c.$$

Conformal invariance. The Yang–Mills equations are well-known to be conformally invariant. More precisely, defining the *unphysical fields*:

$$F^{\mathfrak{a}}{}_{ab}\equiv\tilde{F}^{\mathfrak{a}}{}_{ab},\qquad A^{\mathfrak{a}}{}_a\equiv\tilde{A}^{\mathfrak{a}}{}_a,$$

a direct computation under the conformal rescaling (3) renders the *unphysical Yang–Mills equations*

$$\nabla_aA^{\mathfrak{a}}{}_b-\nabla_bA^{\mathfrak{a}}{}_a+C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}A^{\mathfrak{b}}{}_aA^{\mathfrak{c}}{}_b-F^{\mathfrak{a}}{}_{ab}=0, \tag{78a}$$

$$\nabla^aF^{\mathfrak{a}}{}_{ab}+C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}A^{\mathfrak{b}a}F^{\mathfrak{c}}{}_{ab}=0, \tag{78b}$$

$$\nabla_{[a}F^{\mathfrak{a}}{}_{bc]}+C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}A^{\mathfrak{b}}{}_{[a}F^{\mathfrak{c}}{}_{bc]}=0. \tag{78c}$$

In addition, the unphysical energy-momentum tensor is

$$T_{ab}=\tfrac{1}{4}\delta_{\mathfrak{a}\mathfrak{b}}F^{\mathfrak{a}}{}_{cd}F^{\mathfrak{b}cd}g_{ab}-\delta_{\mathfrak{a}\mathfrak{b}}F^{\mathfrak{a}}{}_{ac}F^{\mathfrak{b}}{}_b{}^c. \tag{79}$$

Finally, it will result useful to introduce the dual of $F^{\mathfrak{a}}{}_{ab}$ defined as

$$F^{*\mathfrak{a}}{}_{ab} \equiv -\tfrac{1}{2}\epsilon_{ab}{}^{cd}F^{\mathfrak{a}}{}_{cd}. \tag{80}$$

Remark 27 Due to the form of the energy-momentum tensor given in (79), first derivatives of $F^{\mathfrak{a}}{}_{ab}$ will appear in the rescaled Cotton tensor, putting at risk the hyperbolicity of the system (34a)–(34d). As in the case of the Maxwell field, this will make necessary the introduction of an auxiliary quantity.

8.3.2 Evolution equations for the Yang–Mills fields

Suitable wave equations for the Yang–Mills fields can be obtained by a procedure analogous to the one used for the Maxwell field. Accordingly, we introduce the auxiliary field

$$F^{\mathfrak{a}}{}_{abc} \equiv \nabla_a F^{\mathfrak{a}}{}_{bc} + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}A^{\mathfrak{b}}{}_a F^{\mathfrak{c}}{}_{bc}. \tag{81}$$

Moreover, the construction of a geometric wave equation for the Yang–Mills gauge potentials requires the introduction of *gauge source functions* $f^{\mathfrak{a}}(x)$ depending in a smooth way on the coordinates and fixing the value of the divergence of the potential. More precisely, in the following we set

$$\nabla^a A^{\mathfrak{a}}{}_a \equiv f^{\mathfrak{a}}(x). \tag{82}$$

Equation for the field strength. The Yang–Mills Bianchi identity, Eq. (78c), represents an integrability condition for the field strength tensors $F^{\mathfrak{a}}{}_{ab}$. Differentiating it and making use of Eqs. (77) and (78a)–(78c), a straightforward calculation results in

$$\begin{aligned}\square F^{\mathfrak{a}}{}_{ab} =& -2\Xi F^{\mathfrak{a}cd}d_{acbd} + \tfrac{1}{3}F^{\mathfrak{a}}{}_{ab}R + 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}F^{\mathfrak{b}}{}_a{}^c F^{\mathfrak{c}}{}_{bc}\\ &- 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}F^{\mathfrak{c}}{}_{cab}A^{\mathfrak{b}c} - C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}F^{\mathfrak{d}}{}_{ab}A^{\mathfrak{b}c}A^{\mathfrak{c}}{}_c\\ &+ C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}f^{\mathfrak{b}}(x)F^{\mathfrak{c}}{}_{ab}.\end{aligned} \tag{83}$$

Equation for the gauge potential. Equation (78a) provides a natural integrability condition for the gauge potential field. After applying ∇^b, commuting derivatives and using Eq. (78b), one arrives to:

$$\begin{aligned}\square A^{\mathfrak{a}}{}_a =& \tfrac{1}{6}A^{\mathfrak{a}}{}_a R + 2A^{\mathfrak{a}b}L_{ab} + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}F^{\mathfrak{c}}{}_{ab}A^{\mathfrak{b}b} + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}f^{\mathfrak{c}}(x)A^{\mathfrak{b}}{}_a\\ &- C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}A^{\mathfrak{b}b}\nabla_b A^{\mathfrak{c}}{}_a + \nabla_a f^{\mathfrak{a}}(x).\end{aligned} \tag{84}$$

Equation for the auxiliary field. A suitable integrability condition for the field $F^{\mathfrak{a}}{}_{abc}$ can be obtained from its definition. Using this and Eq. (78c), some manipulations yield

$$\begin{aligned}3\nabla_{[d}F^{\mathfrak{a}}{}_{|a|bc]} =& -3\Xi F^{\mathfrak{a}}{}_{[b}{}^e d_{cd]ae} + 6F^{\mathfrak{a}}{}_{[bc}L_{d]a} + 6g_{a[b}F^{\mathfrak{a}}{}_c{}^e L_{d]e}\\ &- 3C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}F^{\mathfrak{c}}{}_{a[bc}A^{\mathfrak{b}}{}_{d]} - 3C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_a A^{\mathfrak{b}}{}_{[b}F^{\mathfrak{c}}{}_{cd]}.\end{aligned}$$

Proceeding as in the case of the wave equation for $F^{\mathfrak{a}}{}_{abc}$, as well as using the Jacobi identity and definitions (86a)–(86d), a lengthy calculation results in

$$\begin{aligned}
\square F^{\mathfrak{a}}{}_{abc} = {} & \tfrac{1}{2} F^{\mathfrak{a}}{}_{abc} R + 4F^{\mathfrak{a}d}{}_{bc} L_{ad} + 2F^{\mathfrak{b}d}{}_{bc} F^{\mathfrak{c}}{}_{ad} C^{\mathfrak{a}}{}_{\mathfrak{bc}} - F^{\mathfrak{c}}{}_{abc} f^{\mathfrak{b}}(x) C^{\mathfrak{a}}{}_{\mathfrak{bc}} \\
& - F^{\mathfrak{d}}{}_{abc} A^{\mathfrak{b}d} A^{\mathfrak{c}}{}_{d} C^{\mathfrak{a}}{}_{\mathfrak{be}} C^{\mathfrak{e}}{}_{\mathfrak{cd}} \\
& + \tfrac{1}{3} F^{\mathfrak{a}}{}_{bc} \nabla_a R - 2A^{\mathfrak{b}d} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \nabla_d F^{\mathfrak{c}}{}_{abc} \\
& - F^{\mathfrak{a}}{}_{d}{}^{e} d_{aebc} \nabla^d \Xi - F^{\mathfrak{a}}{}_{a}{}^{e} d_{bcde} \nabla^d \Xi + 2\Xi F^{\mathfrak{a}de} \nabla_e d_{adbc} \\
& - 4\Xi F^{\mathfrak{a}d}{}_{[b}{}^{e} d_{|ad|c]e} - 2\Xi F^{\mathfrak{a}}{}_{a}{}^{de} d_{[b|d|c]e} \\
& - 4F^{\mathfrak{a}d}{}_{a[b} L_{c]d} + 4\Xi F^{\mathfrak{a}}{}_{[b}{}^{d} T_{c]da} + 4\Xi F^{\mathfrak{a}}{}_{[b}{}^{d} T_{|ad|c]} \\
& - \tfrac{1}{3} F^{\mathfrak{a}}{}_{a[b} \nabla_{c]} R + 4F^{\mathfrak{b}}{}_{a[b}{}^{d} F^{\mathfrak{c}}{}_{c]d} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \\
& - 4F^{\mathfrak{a}d}{}_{[b}{}^{e} L_{|de} g_{a|c]} - 2\Xi F^{\mathfrak{a}de} T_{[b|de} g_{a|c]} \\
& - 4F^{\mathfrak{a}}{}_{[b}{}^{d} d_{|ad|c]}{}^{e} \nabla_e \Xi - 2F^{\mathfrak{a}}{}_{[b}{}^{d} d_{|a|}{}^{e}{}_{c]d} \nabla_e \Xi \\
& + 2F^{\mathfrak{a}de} d_{ad[b|e|} \nabla_{c]} \Xi - \tfrac{1}{3} F^{\mathfrak{a}}{}_{[b}{}^{d} g_{|a|c]} \nabla_d R \\
& - 2F^{\mathfrak{a}de} g_{a[b} \nabla_{|d|} L_{c]e}.
\end{aligned} \tag{85}$$

In a similar manner to the two previous matter models, when Eqs. (83), (84) and (85) are coupled to the system of wave equations for the conformal fields, the $\square$ operator is to be replaced by its counterpart $\blacksquare$.

8.3.3 Subsidiary equations

The next step in the analysis of the Yang–Mills field is the introduction of the corresponding subsidiary quantities and the consequent construction of suitable geometric wave equations for them. For this purpose define the following set of zero-quantities:

$$M^{\mathfrak{a}}{}_{a} \equiv \nabla^b F^{\mathfrak{a}}{}_{ba} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}b} F^{\mathfrak{c}}{}_{ba}, \tag{86a}$$

$$M^{\mathfrak{a}}{}_{ab} \equiv \nabla_a A^{\mathfrak{a}}{}_{b} - \nabla_b A^{\mathfrak{a}}{}_{a} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}}{}_{a} A^{\mathfrak{c}}{}_{b} - F^{\mathfrak{a}}{}_{ab}, \tag{86b}$$

$$M^{\mathfrak{a}}{}_{abc} \equiv \nabla_{[a} F^{\mathfrak{a}}{}_{bc]} + C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}}{}_{[a} F^{\mathfrak{c}}{}_{bc]}, \tag{86c}$$

$$Q^{\mathfrak{a}}{}_{abc} \equiv F^{\mathfrak{a}}{}_{abc} - \nabla_a F^{\mathfrak{a}}{}_{bc} - C^{\mathfrak{a}}{}_{\mathfrak{bc}} A^{\mathfrak{b}}{}_{a} F^{\mathfrak{c}}{}_{bc}. \tag{86d}$$

Notice that, unlike the Maxwell field analysis, an additional field $M^{\mathfrak{a}}{}_{ab}$ must be considered due to the introduction of the gauge potential $A^{\mathfrak{a}}{}_{a}$. Combining (86c) and (86d) an auxiliary relation is directly obtained, namely

$$3M^{\mathfrak{a}}{}_{abc} + 3Q^{\mathfrak{a}}{}_{[abc]} - 3F^{\mathfrak{a}}{}_{[abc]} = 0. \tag{87}$$

From these definitions, it follows that $M^{\mathfrak{a}}{}_{abc}$ and $M^{\mathfrak{a}}{}_{ab}$ possess the symmetries

$$M^{\mathfrak{a}}{}_{abc} = M^{\mathfrak{a}}{}_{a[bc]} = M^{\mathfrak{a}}{}_{[ab]c} = M^{\mathfrak{a}}{}_{[abc]}, \quad M^{\mathfrak{a}}{}_{ab} = -M^{\mathfrak{a}}{}_{ba}. \tag{88}$$

Furthermore, direct calculations show that the Yang–Mills zero-quantities satisfy the relations

$$\nabla_a M^{\mathfrak{a}a} = -C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} A^{\mathfrak{b}a} M^{\mathfrak{c}}{}_a + \tfrac{1}{2} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} F^{\mathfrak{b}ab} M^{\mathfrak{c}}{}_{ab}, \tag{89a}$$

$$\nabla^b M^{\mathfrak{a}}{}_{ab} = M^{\mathfrak{a}}{}_a, \tag{89b}$$

$$\begin{aligned}\nabla_a M^{\mathfrak{a}}{}_{bc}{}^a = &-\tfrac{2}{3}\nabla_{[b} M^{\mathfrak{a}}{}_{c]} - \tfrac{2}{3} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} A^{\mathfrak{b}}{}_{[b} M^{\mathfrak{c}}{}_{c]} - C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} A^{\mathfrak{b}a} M^{\mathfrak{c}}{}_{abc} \\ &-\tfrac{2}{3} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} A^{\mathfrak{b}a} Q^{\mathfrak{c}}{}_{abc} - \tfrac{2}{3} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} F^{\mathfrak{b}}{}_{[b}{}^a M^{\mathfrak{c}}{}_{c]a}.\end{aligned} \tag{89c}$$

Equation for $M^{\mathfrak{a}}{}_{ab}$. Consider the expression $3\nabla_{[c} M^{\mathfrak{a}}{}_{ab]}$. Commuting covariant derivatives, substituting expressions (86c), (86d) and exploiting the Jacobi identity for the structure constants, the integrability condition is obtained:

$$3\nabla_{[c} M^{\mathfrak{a}}{}_{ab]} = -M^{\mathfrak{a}}{}_{abc} - 3C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} A^{\mathfrak{b}}{}_{[a} M^{\mathfrak{c}}{}_{bc]}. \tag{90}$$

Applying ∇^c to the last equation, a short calculation using Eqs. (89a) and (89c) yields

$$\begin{aligned}\Box M^{\mathfrak{a}}{}_{ab} = &\, 3C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} A^{\mathfrak{b}} M^{\mathfrak{c}}{}_{abc} + 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} A^{\mathfrak{b}c} Q^{\mathfrak{c}}{}_{cab} + \tfrac{1}{3} R M^{\mathfrak{a}}{}_{ab} \\ &- 2d_{acbd} M^{\mathfrak{a}cd} - C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} f^{\mathfrak{b}} M^{\mathfrak{c}}{}_{ab} \\ &- 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} F_{[a}{}^c M^{\mathfrak{c}}{}_{b]c} + 2\nabla_c M^{\mathfrak{a}}{}_{ab}{}^c - C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} A^{\mathfrak{b}c} \nabla_c M^{\mathfrak{c}}{}_{ab} \\ &- 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} M^{\mathfrak{c}}{}_{c[a} \nabla^c A^{\mathfrak{b}}{}_{b]}.\end{aligned} \tag{91}$$

Equation for $M^{\mathfrak{a}}{}_a$. Equation (89c) constitutes an integrability condition for the field $M^{\mathfrak{a}}{}_a$. A suitable wave equation can be obtained by first applying ∇^c, commuting derivatives and observing that $\nabla_c \nabla_a M^{\mathfrak{a}}{}_b{}^{ac} = \nabla_{[c}\nabla_{a]} M^{\mathfrak{a}}{}_b{}^{ac}$. Then, using definitions (86a)–(86d) along with (89a), (89b), (90), the Jacobi identity, and an appropriate substitution of (87), a long but straightforward computation results in:

$$\begin{aligned}\Box M^{\mathfrak{a}}{}_b = &\, 2L_{ba} M^{\mathfrak{a}a} + \tfrac{1}{6} R M^{\mathfrak{a}}{}_b + 2F^{\mathfrak{c}}{}_{ba} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} M^{\mathfrak{b}a} - f^{\mathfrak{b}}(x) C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} M^{\mathfrak{c}}{}_b \\ &- A^{\mathfrak{b}a} A^{\mathfrak{c}}{}_a C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}} C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}} M^{\mathfrak{d}}{}_b \\ &- \tfrac{3}{2} F^{\mathfrak{b}ac} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} M^{\mathfrak{c}}{}_{bac} + 3A^{\mathfrak{b}a} A^{\mathfrak{c}c} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}} C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}} M^{\mathfrak{e}}{}_{bac} \\ &+ 2A^{\mathfrak{b}a} A^{\mathfrak{c}c} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}} C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}} Q^{\mathfrak{e}}{}_{cba} \\ &- \tfrac{3}{2} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} M^{\mathfrak{c}}{}_{bac} M^{\mathfrak{b}ac} + 2F^{\mathfrak{b}a}{}_b{}^c C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} M^{\mathfrak{c}}{}_{ac} - 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} Q^{\mathfrak{b}a}{}_b{}^c M^{\mathfrak{c}}{}_{ac} \\ &+ F^{\mathfrak{c}}{}_b{}^c A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{d}} C^{\mathfrak{d}}{}_{\mathfrak{b}\mathfrak{e}} M^{\mathfrak{e}}{}_{ac} - 2A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} \nabla_a M^{\mathfrak{c}}{}_b + 2A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} \nabla_c Q^{\mathfrak{c}}{}_{ab}{}^c \\ &- 3C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} M^{\mathfrak{c}}{}_{bac} \nabla^c A^{\mathfrak{b}a} + 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} Q^{\mathfrak{c}}{}_{abc} \nabla^c A^{\mathfrak{b}a}.\end{aligned} \tag{92}$$

Equation for $M^{\mathfrak{a}}{}_{abc}$. In a similar fashion to the approach adopted for the electromagnetic zero-quantity M_{abc}, and in order to simplify the calculations, we introduce the Hodge dual of $M^{\mathfrak{a}}{}_{abc}$:

$$M^{*\mathfrak{a}}{}_a \equiv C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}} F^{*\mathfrak{c}}{}_{ba} A^{\mathfrak{b}b} + \nabla^b F^{*\mathfrak{a}}{}_{ba} = \tfrac{1}{2}\epsilon_a{}^{bcd} M^{\mathfrak{a}}{}_{bcd}. \tag{93}$$

Here, the second equality has been obtained with help of (80) and (86c). With this expression we compute the divergence of $M^{*\mathfrak{a}}{}_a$. Making use of (86b) and the Jacobi identity, a calculation yields

$$\begin{aligned}\nabla_a M^{*\mathfrak{a}a} &= -C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}C^{\mathfrak{e}}{}_{\mathfrak{c}\mathfrak{d}}F^{*\mathfrak{d}}{}_{ab}A^{\mathfrak{b}a}A^{\mathfrak{c}b} - C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}A^{\mathfrak{b}a}M^{*\mathfrak{c}}{}_a + C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}F^{*\mathfrak{c}}{}_{ab}\nabla^b A^{\mathfrak{b}a}\\ &= -C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}A^{\mathfrak{b}a}M^{*\mathfrak{c}}{}_a - \tfrac{1}{4}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{e}}\epsilon_{ab}{}^{cd}F^{\mathfrak{b}ab}M^{\mathfrak{c}}{}_{cd}.\end{aligned}$$

In terms of non-dual objects this takes the form of an integrability condition:

$$\begin{aligned}&\epsilon^{abcd}\nabla_d M^{\mathfrak{a}}{}_{abc} = C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\epsilon^{abcd}A^{\mathfrak{b}}{}_a M^{\mathfrak{c}}{}_{bcd} + \tfrac{1}{2}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\epsilon^{abcd}F^{\mathfrak{b}}{}_{ab}M^{\mathfrak{c}}{}_{cd}\\ \Longleftrightarrow\ &4\nabla_{[a}M^{\mathfrak{a}}{}_{bcd]} = 4C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}A_{[a}M^{\mathfrak{a}}{}_{bcd]} + 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}F^{\mathfrak{b}}{}_{[ab}M^{\mathfrak{c}}{}_{cd]}.\end{aligned} \tag{94}$$

Then, a suitable wave equation can be obtained applying ∇^d and commuting derivatives. After a long calculation in which definitions (86a)–(86d), Eqs. (87)–(90) and the Jacobi identity are employed, one finds that

$$\begin{aligned}\Box M^{\mathfrak{a}}{}_{abc} = &\tfrac{1}{2}RM^{\mathfrak{a}}{}_{abc} - A^{\mathfrak{b}d}A^{\mathfrak{c}}{}_d C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}M^{\mathfrak{e}}{}_{abc}\\ &- C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}f^{\mathfrak{b}}(x)M^{\mathfrak{c}}{}_{abc} - 2A^{\mathfrak{b}d}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_d M^{\mathfrak{c}}{}_{abc}\\ &- 6\Xi d_{[a}{}^d{}_b{}^e M^{\mathfrak{a}}{}_{c]de} - 6L_{[a}{}^d M^{\mathfrak{a}}{}_{bc]d}\\ &+ 2F^{\mathfrak{b}d}{}_{[ab}C^{\mathfrak{a}}{}_{|\mathfrak{b}\mathfrak{c}|}M^{\mathfrak{c}}{}_{c]d} - 6F^{\mathfrak{b}}{}_{[a}{}^d C^{\mathfrak{a}}{}_{|\mathfrak{b}\mathfrak{c}|}M^{\mathfrak{c}}{}_{bc]d}\\ &- 2A^{\mathfrak{b}d}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{[a}Q^{\mathfrak{c}}{}_{|d|bc]} + 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}Q^{\mathfrak{b}d}{}_{[ab}\nabla_{c]}A^{\mathfrak{c}}{}_d\\ &- 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}Q^{\mathfrak{b}d}{}_{[ab}M^{\mathfrak{c}}{}_{c]d}\\ &+ F^{\mathfrak{b}}{}_{[ab}A^{\mathfrak{c}d}C^{\mathfrak{a}}{}_{|\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}|}M^{\mathfrak{e}}{}_{c]d} - 2A^{\mathfrak{b}}{}_{[a}A^{\mathfrak{c}d}C^{\mathfrak{a}}{}_{|\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}Q^{\mathfrak{e}}{}_{d|bc]}.\end{aligned} \tag{95}$$

Equation for $Q^{\mathfrak{a}}{}_{abc}$. Similar to the case for the Maxwell field, a wave equation for $Q^{\mathfrak{a}}{}_{abc}$ can be obtained by directly applying the $\Box$ operator to its definition. Since the identity used in the deduction of Eq. (76) has the same form for the Yang–Mills strength field, an analogous procedure can be followed. A long computation gives:

$$\begin{aligned}\Box Q^{\mathfrak{a}}{}_{abc} = &6L_a{}^d M^{\mathfrak{a}}{}_{bcd} + \tfrac{1}{2}RQ^{\mathfrak{a}}{}_{abc} + 4L_a{}^d Q^{\mathfrak{a}}{}_{dbc} - f^{\mathfrak{b}}(x)C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}Q^{\mathfrak{c}}{}_{abc}\\ &- 2F^{\mathfrak{b}}{}_a{}^d C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}Q^{\mathfrak{c}}{}_{dbc}\\ &- A^{\mathfrak{b}d}A^{\mathfrak{c}}{}_d C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}Q^{\mathfrak{e}}{}_{abc} + 2A^{\mathfrak{b}}{}_a A^{\mathfrak{c}d}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}Q^{\mathfrak{e}}{}_{dbc}\\ &+ F^{\mathfrak{c}}{}_{bc}A^{\mathfrak{b}d}C^{\mathfrak{a}}{}_{\mathfrak{c}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{b}\mathfrak{e}}M^{\mathfrak{e}}{}_{ad}\\ &- 2F^{\mathfrak{c}}{}_{bc}A^{\mathfrak{b}d}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{d}}C^{\mathfrak{d}}{}_{\mathfrak{c}\mathfrak{e}}M^{\mathfrak{e}}{}_{ad} + 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}Q^{\mathfrak{c}}{}_{dbc}\nabla_a A^{\mathfrak{b}d}\\ &+ 4A^{\mathfrak{b}d}C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}\nabla_{[a}Q^{\mathfrak{c}}{}_{d]bc}\\ &+ 2C^{\mathfrak{a}}{}_{\mathfrak{b}\mathfrak{c}}M^{\mathfrak{b}}{}_a{}^d\nabla_d F^{\mathfrak{c}}{}_{bc} + 4\Xi F^{\mathfrak{a}}{}_{[b}{}^d\Lambda_{c]ad}\\ &+ 4\Xi F^{\mathfrak{a}}{}_{[b}{}^d\Lambda_{|a|c]d} - 2\Xi d_{[b}{}^d{}_{c]}{}^e Q^{\mathfrak{a}}{}_{ade}\\ &+ 4\Xi d_a{}^d{}_{[b}{}^e Q^{\mathfrak{a}}{}_{|d|c]e} + 4L_{a[b}M^{\mathfrak{a}}{}_{c]} + 4L_{[b}{}^d Q^{\mathfrak{a}}{}_{|da|c]} + \Xi F^{\mathfrak{a}de}\Lambda_{[b|de}g_{a|c]}\\ &+ 4F^{\mathfrak{b}}{}_{[b}{}^d C^{\mathfrak{a}}{}_{|\mathfrak{b}\mathfrak{c}}Q^{\mathfrak{c}}{}_{a|c]d} + 4L^{de}g_{a[b}Q^{\mathfrak{a}}{}_{|d|c]e}.\end{aligned} \tag{96}$$

8.3.4 Propagation of the gauge

In this subsection we show the consistency of the introduction of the gauge source functions $f^{\mathfrak{a}}(x)$ into the analysis of the propagation of the constraints for the Yang–Mills potential. For this purpose we introduce the zero-quantity $P^{\mathfrak{a}}$ defined as:

$$P^{\mathfrak{a}} \equiv \nabla^{a} A^{\mathfrak{a}}{}_{a} - f^{\mathfrak{a}}(x). \tag{97}$$

The computation of a wave equation for this field is straightforward: first, a short calculation employing Eqs. (84), (86a), (86b) and (89b) gives

$$\nabla_{a} P^{\mathfrak{a}} = -A^{\mathfrak{b}}{}_{b} C^{\mathfrak{a}}{}_{\mathfrak{bc}} P^{\mathfrak{c}} - M^{\mathfrak{a}}{}_{b} + \nabla_{a} M^{\mathfrak{a}}{}_{b}{}^{a}.$$

From here, application of a further covariant derivative results directly in

$$\square P^{\mathfrak{a}} = -f^{\mathfrak{b}} C^{\mathfrak{a}}{}_{\mathfrak{bc}} P^{\mathfrak{c}} + A^{\mathfrak{b}a} C^{\mathfrak{a}}{}_{\mathfrak{bc}} M^{\mathfrak{c}}{}_{a} - \tfrac{1}{2} F^{\mathfrak{b}ab} C^{\mathfrak{a}}{}_{\mathfrak{bc}} M^{\mathfrak{c}}{}_{ab} - A^{\mathfrak{b}b} C^{\mathfrak{a}}{}_{\mathfrak{bc}} \nabla_{b} P^{\mathfrak{c}}. \tag{98}$$

Remark 28 Geometric wave equations (91), (92), (95), (96) and (98) are homogeneous in $M^{\mathfrak{a}}{}_{a}$, $M^{\mathfrak{a}}{}_{ab}$, $M^{\mathfrak{a}}{}_{abc}$, $Q^{\mathfrak{a}}{}_{abc}$, $P^{\mathfrak{a}}$, Λ_{abc} and their first covariant derivatives. Thus, if these fields vanish on an initial hypersurface $\mathcal{S}_{\star}$, it can be guaranteed that there exists a unique solution on a small enough slab of $\mathcal{S}_{\star}$ and this solution is the trivial one.

8.3.5 Summary

The previous discussion about the Yang–Mills field coupled to the conformal Einstein field equations leads to the following statement:

Proposition 6 *The system of wave equations* (34a)–(34e) *with energy-momentum tensor given by* (79) *coupled to wave equations* (83)*,* (84) *and* (85) *written in terms of the operator* $\blacksquare$ *is a proper quasilinear system of wave equations for the Einstein–Yang–Mills system.*

9 Applications

The purpose of this section is to provide a direct application of the analysis of the evolution systems and subsidiary equations associated to the conformal Einstein field equations coupled to tracefree matter. Arguably, the simplest applications of our analysis to a problem of global nature is that of the existence and stability of de-Sitter like spacetimes. The original stability result of this type, for vacuum perturbations, was carried in [8]. For the sake of conciseness of the presentation and given that the key technical details have been discussed in the literature—see e.g. [15], Chapter 15—here we pursue a *high-level* presentation in the spirit of [11].

In order to present the result, it is recalled that one of the key features of the conformal Einstein field equations is that they are regular up to the conformal boundary.

This property is also satisfied by the conformally invariant scalar field equation, the Maxwell equations and the Yang–Mills equations. Thus, they admit initial data prescribed on spacelike hypersurfaces describing the conformal boundary of spacetime. In an analogous way to the Einstein field equations, the metric conformal Einstein field equations admit a 3+1 decomposition with respect to a foliation of spacelike hypersurfaces. The equations in this decomposition which are intrinsic to the spacelike hypersurfaces are known as the *conformal Einstein constraint equations*—see e.g. [15], Chapter 11. When evaluated on a spacelike hypersurface representing the conformal boundary of a de Sitter-like spacetime, these equations simplify considerably and a procedure to construct the solutions to these equations is available—see [15], Proposition 11.1 for the vacuum case; this result can be generalised to include tracefree matter models. From the geometric side, the freely specifiable data in this construction are given by the intrinsic metric of the conformal boundary and a TT-tensor prescribing the electric part of the rescaled Weyl tensor. The initial data obtained by this type of construction will be known as *asymptotic de Sitter-like initial data*. The component of the conformal boundary where the asymptotic de Sitter-like data are prescribed can be either the future or the past one. In the following, for convenience, we restrict the discussion to the case of the past component of the conformal boundary.

For asymptotic initial data sets of the type described in the previous paragraph one has the following result:

Theorem 1 *Consider (past) asymptotic de-Sitter initial data for the Einstein field equations with positive Cosmological constant coupled to any of the following matter models:*

(i) the conformally invariant scalar field,
(ii) the Maxwell field,
(iii) the Yang–Mills field.

Then one has that:

(a) The initial data determine a unique, maximal, globally hyperbolic solution to the Einstein field equations which admits a smooth de Sitter-like conformal future extension.
(b) The set of initial data sets leading to developments which admit smooth conformal extensions to both the future and past is an open set (in the appropriate Sobolev norm) of the set of asymptotic initial data.

Proof We only provide a sketch of the proof as the strategy is similar to the one followed in the proof of the stability of the Milne spacetime in [12]. A version of the proof which uses first order symmetric hyperbolic systems can be found in [15], Chapter 15.

The first main observation is that the conformal representation of the (vacuum) de Sitter spacetime in terms of the embedding into the Einstein cylinder gives rise to a solution to the conformal Einstein field equations. Coordinates $(x) = (t, \underline{x})$ can be chosen so that the two components of the conformal boundary are located at $t = \pm\frac{1}{2}\pi$. For this representation the Ricci scalar takes the value -6 and the conformal factor is given by $\mathring{\Xi} = \cos t$. In the following we denote by $\mathring{\mathbf{u}}$ this solution to the conformal equations and by $\mathring{\mathbf{u}}_\star$ its restriction to the hypersurface $t = -\frac{1}{2}\pi$ which corresponds to the past conformal boundary $\mathscr{I}^-$. We will look for solutions to the conformal evolution equations of the form $\mathbf{u} = \mathring{\mathbf{u}} + \breve{\mathbf{u}}$ with initial data given by $\mathbf{u}_\star = \mathring{\mathbf{u}}_\star + \breve{\mathbf{u}}_\star$. The fields $\breve{\mathbf{u}}$ and $\breve{\mathbf{u}}_\star$ describe the (non-linear) perturbations. Substituting this form of the solution into the evolution equations one obtains a system of quasilinear equations for the components of $\breve{\mathbf{u}}$ which can be schematically written as

$$\big(\mathring{g}^{\mu\nu}(x) + \breve{g}^{\mu\nu}(x;\breve{\mathbf{u}})\big)\partial_\mu\partial_\nu\breve{\mathbf{u}} = \boldsymbol{F}(x;\breve{\mathbf{u}},\partial\breve{\mathbf{u}}). \tag{99}$$

In the above expression $\mathring{g}^{\mu\nu}$ denote the components of the contravariant metric on the Einstein cylinder. The above equation is in the form for which the local existence and Cauchy stability theory of quasilinear wave equations as given in, say, [13] applies. Initial data for the system (99) are of the form $(\breve{\mathbf{u}}_\star, \partial_t\breve{\mathbf{u}}_\star)$. The size of the initial data is encoded in the expression

$$\| (\breve{\mathbf{u}}_\star, \partial_t\breve{\mathbf{u}}_\star) \|_{\mathbb{S}^3,m} \equiv \| \breve{\mathbf{u}}_\star \|_{\mathbb{S}^3,m} + \| \partial_t\breve{\mathbf{u}}_\star \|_{\mathbb{S}^3,m}$$

where $\| \quad \|_{\mathbb{S}^3,m}$ denotes the standard Sobolev norm of order $m \geq 4$ on a manifold which is topologically $\mathbb{S}^3$. If the initial data $(\breve{\mathbf{u}}_\star, \partial_t\breve{\mathbf{u}}_\star)$ are sufficiently small then the contravariant metric on $\mathscr{I}^-$ given by $\mathring{g}^{\mu\nu}(x_\star) + \breve{g}^{\mu\nu}(x_\star;\breve{\mathbf{u}}_\star)$ is Lorentzian—this property is preserved in the evolution. Now, the background solution $\mathring{\mathbf{u}}$ is well-defined and smooth on the whole of the Einstein cylinder; in particular, up to $t = \pi$ for which one has that $\mathring{\Xi}|_{t=\pi} = -1$. It follows from the Cauchy stability statements in [13] that if $\| (\breve{\mathbf{u}}_\star, \partial_t\breve{\mathbf{u}}_\star) \|_{\mathbb{S}^3,m}$ is sufficiently small then the solution will exists up to $t = \pi$. By restricting, if necessary, the size of the data one has that

$$\Xi|_{t=\pi} = -1 + \breve{\Xi}_{t=\pi} < 0.$$

From the above observation it can be argued that the function $\Xi = \mathring{\Xi} + \breve{\Xi}$ over the Einstein cylinder becomes zero on a spacelike hypersurface which lies between the times $t = 0$ and $t = \pi$. This hypersurface corresponds to the future conformal boundary ($\mathscr{I}^+$) arising from the data $(\breve{\mathbf{u}}_\star, \partial_t\breve{\mathbf{u}}_\star)$ on $\mathscr{I}^-$.

Once the existence of a global solution to the evolution system has been established, one makes use of the uniqueness of solutions to systems of quasilinear wave equations to prove the propagation of the constraints. To this end one observes that if the initial data satisfies the conformal constraints at the past conformal boundary $\mathscr{I}^-$, then a calculation shows that the zero-quantities and their normal derivatives also vanish on $\mathscr{I}^-$. As the subsidiary evolution system is homogeneous in the zero-quantities, it follows that its unique solution must be the trivial (i.e. vanishing) one. Thus, one has obtained a global solution to the conformal Einstein field equations. From the general theory of the conformal Einstein field equations—see e.g. Proposition 8.1 in Chapter 8 of [15]—this solution implies, in turn, a solution to the Einstein field equations with positive Cosmological constant having de Sitter-like asymptotics. □

Remark 29 The above theorem is a global stability result for the de Sitter spacetime under perturbations involving a conformally invariant scalar field, a Maxwell field or a Yang–Mills field as (trivially) the de Sitter spacetime can be constructed from asymptotic initial data. Thus, for a suitably small neighbourhood of asymptotic de Sitter data, all data in the neighbourhood give rise to global solutions.

Remark 30 The cases (ii) and (iii)—the Maxwell and Yang–Mills fields, have been studied using first order symmetric hyperbolic systems in [9]. However, the case (i)—the conformally invariant scalar field—has, hitherto, not been considered in the literature.

Remark 31 The theory in [13] is the analogue for systems wave equations of the theory for symmetric hyperbolic systems developed in [14]. A version of the key existence and Cauchy stability result in [13] given in the form used in Theorem 1 can be found in the Appendix of [12].

10 Concluding remarks

The global existence and stability result presented in Theorem 1 is the simplest application of the analysis of the second order conformal evolution equations developed in this article. A further application is to the construction of anti-de Sitter-like spacetimes with tracefree matter models following the strategy implemented in [3]—this construction will be presented elsewhere [2]. The theory developed in this article should also allow to obtain matter generalisation of the existence results for characteristic initial value problems considered in [6].

More crucially, the analysis in this article should also pave the road for numerical simulations of spacetimes with tracefree matter in the conformal setting. The use of the metric conformal Einstein equations in conjunction with a coordinate gauge prescribed in terms of generalised wave condition provides a formulation of the evolution equations for the conformal fields which can be regarded as a (unphysical) *reduced Einstein equation* with (unphysical) matter described by the conformal factor, Friedrich scalar, Schouten tensor and the rescaled Weyl tensor. Viewed in this way, one can readily adapt the plethora of numerical know-how that has been developed in the numerical simulations of the Einstein field equations. A further discussion of this idea can be found in [10].

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