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A NEW APPROACH TO SOLVING THE RADIATION FIELD PROBLEM OF AN EXTENDED HELICAL UNDULATOR*

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Abstract

An exact solution for the radiation field of a particle in a helical undulator, valid for an arbitrary point in space and an arbitrary particle energy, was obtained by the partial domain method, generalized for the case of a spiral motion of the particle. The interface between the regions is a cylindrical surface containing the spiral trajectory of the particle. A comparison is made with the existing solution, which is valid in the far zone at high particle energies.

INTRODUCTION

The radiation generated by a point charged particle moving along an infinite spiral curve is studied. The spiral period l and its radius a are assumed to be unchanged. The Lorentz factor of the particle γ, its longitudinal velocity V (parallel to the spiral axis) and the frequency of its rotation around the spiral axis ω_0 are also assumed to be constant.

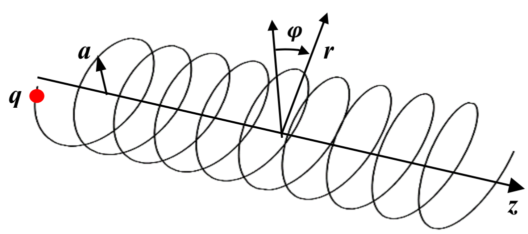

Figure 1: Geometry of the problem.

The usual technique to construct the solution for the radiation field of a helical undulator involves retarded potentials [1]. When describing an observation point in the far zone, a spherical coordinate system is used and, in this case, it is necessary to use some standard simplifications of exponential phase factors, which limits the allowed frequency range of the field distribution and makes the solution approximate. Since the particle trajectory has a cylindrical symmetry, it would be natural to use a cylindrical coordinate system r, φ, z (Fig. 1), associated with the spiral axis. This allows us to obtain a uniform description of the fields in the near and far zones both outside the cylindrical surface $r = a$ containing the particle trajectory, and in the region $r < a$ located inside this surface.

FIELD PRESENTATION

The search form of the field components that are solutions to the inhomogeneous Maxwell equations are initially represented in the form of a multipole expansion in discontinuous functions on a cylindrical surface $r = a$, coaxial with the axis of the spiral containing the particle trajectory. For the case of a particle motion in free space, it has the form [2]:

$$\vec{\mathcal{E}} = \begin{cases} \sum_{m=-\infty}^{\infty} \vec{\mathcal{E}}_m^{(J)} \\ \sum_{m=-\infty}^{\infty} \vec{\mathcal{E}}_m^{(H)} \end{cases}, \quad \vec{\mathcal{H}} = \begin{cases} \sum_{m=-\infty}^{\infty} \vec{\mathcal{H}}_m^{(J)} & r < a \\ \sum_{m=-\infty}^{\infty} \vec{\mathcal{H}}_m^{(H)} & r > a \end{cases} \tag{1}$$

Each term of the multipole expansion is represented as a superposition of TM $\left(\vec{E}_{m,TM}^{(0,Z)}\right)$ and TE $\left(\vec{E}_{m,TE}^{(0,Z)}\right)$ waves with arbitrary weights factors $\left(\mathcal{A}_m^{(Z)}, \mathcal{B}_m^{(Z)}\right)$, composed of fundamental solutions of the homogeneous Maxwell equations in cylindrical coordinates conjugate to the spiral axis:

$$\begin{aligned} \vec{\mathcal{E}}_m^{(Z)} &= \mathcal{A}_m^{(Z)} \vec{E}_{m,TM}^{(0,,Z)} + \mathcal{B}_m^{(Z)} \vec{E}_{m,TE}^{(0,Z)} \\ Z_0 \vec{\mathcal{H}}_m^{(Z)} &= \mathcal{A}_m^{(Z)} \vec{H}_{m,TM}^{(0,Z)} + \mathcal{B}_m^{(Z)} \vec{H}_{m,TE}^{(0,Z)} \end{aligned}, \quad Z = J \text{ or } H, \tag{2}$$

where

$$\begin{aligned} \vec{E}_{m,TM}^{(0,Z)} &= -\nu^{-2} \operatorname{rot} \vec{R}, \quad Z_0 \vec{H}_{m,TM}^{(0,Z)} = jk\nu^{-2} \vec{R}, \\ Z_0 \vec{H}_{m,TE}^{(0,Z)} &= -\nu^{-2} \operatorname{rot} \vec{R}, \quad \vec{E}_{m,TE}^{(0,Z)} = -jk\nu^{-2} \vec{R} \\ \vec{R} &= \vec{e}_z \times \vec{\nabla} P^Z, \quad P^Z = Z_m e^{j(m\varphi + pz - \omega t)}, \quad k = \omega/c \end{aligned} \tag{3}$$

If $Z = J$, $Z_m = J_m(\nu_m r)$ and if $Z = H$, $Z_m = H_m^{(1)}(\nu_m r)$, where J_m and $H_m^{(1)}$ are the Bessel function and the Hankel function of the first kind; $Z_0 = (\varepsilon_0 c)^{-1}$ is the impedance of free space and ε_0 is the dielectric constant of vacuum. In (3) p_m and $\nu_m = \sqrt{\omega^2/c^2 - p_m^2}$ are the longitudinal and transverse eigenvalues of the m^{th} mode. In the case of linear motion of the particle $p_m = \omega/v$ (v is the total velocity of the particle) and $\nu_m = j\omega/v\gamma$ (j is the imaginary unit), while for the helical motion $p_m = (\omega - m\omega_0)/V$ (V is longitudinal component of the particle velocity) and

$$\nu_m = \sqrt{\omega^2/c^2 - (\omega - m\omega_0)^2/V^2} \tag{4}$$

To determine the amplitudes $\mathcal{A}_m^{(J)}, \mathcal{A}_m^{(H)}, \mathcal{B}_m^{(J)}, \mathcal{B}_m^{(H)}$, the boundary conditions [3] are used, that establish a connection between the fields on both sides of the surface $r = a$ containing charges $\rho = q$ and currents $\vec{j} = q\{0, \ \omega_0 a, \ V\}$:

$$\begin{aligned} \left(\vec{\mathcal{E}}_m^{(H)}/_{r\to a+} - \vec{\mathcal{E}}_m^{(J)}/_{r\to a-}\right) \times \vec{e}_r &= 0 \\ \left(\vec{\mathcal{H}}_m^{(H)}/_{r\to a+} - \vec{\mathcal{H}}_m^{(J)}/_{r\to a-}\right) \times \vec{e}_r &= \chi_n \vec{j} \\ \left(\vec{\mathcal{E}}_m^{(H)}/_{r\to a+} - \vec{\mathcal{E}}_m^{(J)}/_{r\to a-}\right) \cdot \vec{e}_r &= -\chi_n q/\varepsilon_0 \\ \left(\vec{\mathcal{H}}_m^{(H)}/_{r\to a+} - \vec{\mathcal{H}}_m^{(J)}/_{r\to a-}\right) \cdot \vec{e}_r &= 0 \end{aligned} \tag{5}$$

Here $\vec{e}_r$ is the unit radial vector (perpendicular to the surface $r = a$), χ_n is the proportionality coefficient to be determined. The system (5) can be solved uniquely:

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$$\begin{matrix}\mathcal{A}_m^{(I)}\\ \mathcal{A}_m^{(H)}\end{matrix} = q\,\frac{\pi a}{2\varepsilon_0 V\omega}\chi_m f_m \begin{cases}H_m^{(1)}(a\nu_m)\\ J_m(a\nu_m)\end{cases},$$
$$\begin{matrix}\mathcal{B}_m^{(I)}\\ \mathcal{B}_m^{(H)}\end{matrix} = jq\,\frac{\pi a^2\omega_0}{2\varepsilon_0 c}\chi_m \nu_m \begin{cases}H_m^{(1)'}(a\nu_m)\\ J_m'(a\nu_m)\end{cases} \quad (6)$$
$$f_m = \omega(\omega_0 m - \omega/\gamma_z^2),\ \ \gamma_z^2 = (1 - V^2/c^2)^{-1}.$$

COUPLING OF TM AND TE MODES

Examining the components of the obtained expressions (3, 6) for the fields for the TM and TE modes separately, we find that the transverse components have a divergence of the order of ν_m^{-2} at $\nu_m \to 0$ (without taking into account the features of the yet unknown function χ_m) for arbitrary integer values $m > 0$, but in their superposition (2) these singularities for $m > 1$ are mutually compensated. For example, the radial magnetic TM and TE components in the vicinity of $\nu_m = 0$ have singularities equal in magnitude and opposite in sign:

$$\left.\begin{matrix}\mathcal{A}_m^{(I)} H_{m,TM_r}^{(0,,I)}\\ \mathcal{A}_m^{(H)} H_{m,TM_r}^{(0,,H)}\end{matrix}\right\}_{\nu_m\to 0} = \left.\begin{matrix}-\mathcal{B}_m^{(I)} H_{m,TE_r}^{(0,,I)}\\ -\mathcal{B}_m^{(I)} H_{m,TE_r}^{(0,,I)}\end{matrix}\right\}_{\nu_m\to 0} = .$$
$$jm\chi_m \frac{q}{2} p_m \omega_0 \left(\frac{r}{a}\right)^{\pm m-1} \nu_m^{-2} \quad (7)$$

Only when $m = 1$ is the logarithmic divergence preserved (at $\nu_1 = 0$). So, for the radial electrical components:

$$\left.\begin{matrix}\mathcal{A}_1^{(I)} E_{1,TM_r}^{(0,,I)}\\ \mathcal{A}_1^{(H)} E_{1,TM_r}^{(0,,H)}\end{matrix}\right\}_{\nu_m\to 0} = \mp\chi_m \frac{q}{2\varepsilon_0} p_1^2 \frac{\omega_0}{\omega}\begin{Bmatrix}1\\ a^2/r^2\end{Bmatrix}\nu_1^{-2} +$$
$$\chi_m \frac{q}{4\varepsilon_0} a^2 p_1^2 \frac{\omega_0}{\omega}\begin{Bmatrix}ln(a\nu_1/2)\\ ln(r\nu_1/2)\end{Bmatrix},$$
$$\left.\begin{matrix}\mathcal{B}_1^{(I)} E_{1,TE_r}^{(0,,I)}\\ \mathcal{B}_1^{(H)} E_{1,TE_r}^{(0,,H)}\end{matrix}\right\}_{\nu_m\to 0} = \pm\chi_m \frac{q\omega\omega_0}{2c^2\varepsilon_0}\begin{Bmatrix}1\\ a^2/r^2\end{Bmatrix}\nu_1^{-2} + \quad .$$
$$\chi_m a^2 \frac{q\omega\omega_0}{4c^2\varepsilon_0}\begin{Bmatrix}ln(a\nu_1/2)\\ ln(r\nu_1/2)\end{Bmatrix} \quad (8)$$

This divergence should be eliminated by an appropriate selection of the factor χ_m. Note that the longitudinal components do not have any singularities. Thus, TM and TE modes are mutually coupled and cannot be generated separately.

DETERMINATION OF χ_m

Based on the results obtained above, it is possible to obtain the new expression for the spectral energy density distribution of the particle radiation:

$$\frac{dJ(\omega)}{d\omega} = \frac{q^2 N}{2\varepsilon_0 c}\sum_{m=1}^{\infty} Q_m X_m \tilde{u}(S_m^2) \quad (9)$$
$$X_m = \tilde{\omega}\left\{\frac{\tilde{D}_m^2}{S_m} J_m^2(\tilde{y}_m) + \beta_\varphi^2 J_m'^2(\tilde{y}_m)\right\}$$

(here $\tilde{u}(x)$ is a unit step function,

$$Q_m = 4\pi^3\chi_m^2\, a^2\omega/c\nu_m,$$
$$\tilde{D}_m = m - \tilde{\omega}/\gamma_z^2\,,\ \tilde{B}_m = m(m - \tilde{\omega}), \quad (10)$$
$$S_m = \tilde{\omega}\tilde{D}_m - \tilde{B}_m,\quad \tilde{y}_m = \frac{\beta_\varphi}{\beta_z}\sqrt{S_m},\quad \tilde{\omega} = \omega/\omega_0,$$
$$\beta_\varphi = \sqrt{\beta_\gamma^2 - \beta_z^2},\qquad \beta_z = V/c,\ \ \beta_\gamma = \sqrt{1-\gamma^{-2}},$$

q particle charge and N number of periods) and compare it with the well-known commonly used formula helical undulator radiation [4]:

$$\frac{dI(\omega)}{d\omega} = \frac{Nq^2K^2\tilde{r}}{\varepsilon_0 c}\sum_{m=1}^{\infty} Y_m \tilde{u}(\tilde{\alpha}_m^2)\,, \quad (11)$$
$$Y_m = J_m'^2(\tilde{x}_m) + (\tilde{\alpha}_m/K - m/\tilde{x}_m)^2 J_m^2(\tilde{x}_m)\,,$$
$$\tilde{\alpha}_m^2 = m/\tilde{r} - 1 - K^2, \tilde{x}_m = 2K\tilde{r}\tilde{\alpha}_m, \tilde{r} = \omega/2\gamma^2\omega_0,$$

with K being the deflecting parameter of the undulator.

To establish a correspondence between (11) and (9), it is necessary, while leaving Q_m unchanged, to express in the remaining fragments of formula (9) the longitudinal velocity V and the spiral radius a through the particle energy γ and the deflecting undulator parameter K [4]:

$$V = c\sqrt{(1-\gamma^{-2})(1-K^2\gamma^{-2})},\ \ a = KV/\gamma\omega_0\ . \quad (12)$$

A coincidence occurs if, after this substitution, we simplify (9) by resorting to approximations $\gamma \gg 1$ and $2\omega \gg m\omega_0$ and setting, in addition, $Q_m = 1$. From the last equality the desired value of the parameter χ_m is determined as:

$$\chi_m = \frac{1}{2a\pi^{3/2}}\left(\frac{c\nu_m}{\omega}\right)^{1/2}. \quad (13)$$

For $\omega \gg m\omega_0$ and (or) $\gamma \gg 1$ it turns into a complex constant independent of m, whereas with

$$m(1+\beta_z)^{-1} < \tilde{\omega} < m(1-\beta_z)^{-1} \quad (14)$$

(when $\nu_m^2 > 0$) it is a positive real function versus ω. The logarithmic divergence of the transverse field components on $\nu_1 = 0$ (when $m = 1$) is eliminated.

Formula (9) with $Q_m = 1$ describes the spectral density of the emitted energy as a function of the normalized frequency $\tilde{\omega} = \omega/\omega_0$ with a parametric dependence on the energy γ and longitudinal velocity of the particle V.

Formula (11) has one remarkable property: it allows the spectra to depend on the generalized parameter $\tilde{r} = \omega/2\gamma^2\omega_0$, leaving only the parameter K free. This is achieved in the approximation of very high energies and far from frequencies that are multiples of the rotation frequency ($2\omega \gg m\omega_0$).

Formula (9), after substitution of the approximate expression for the longitudinal velocity (12), is a refined version of formula (11): In this case, the requirement for γ is relaxed and the constraint $2\omega \gg m\omega_0$ is completely eliminated. This option with the artificial introduction of a functional dependence on $\tilde{r}$, as in formula (11), retains the parametric dependence on K, while acquiring an additional parametric dependence on γ.

The latter makes it possible to superimpose the spectrum, constructed by means of (11) which is independent of γ, with curves constructed by means of formula (9) (taking into account (12)) for different γ and to trace the evolution of their shape as the parameter γ changes (Fig. 2).

The figure 2 demonstrates a general decrease in the distribution levels when moving from Eqn. (11) to Eqn. (9). It is especially noticeable at small values of γ. Displacements of kinks formed by critical frequencies are also observed to decrease with decreasing γ. The local maxima of the distributions of individual modes also shift downward (marked

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by vertical lines in the Figure 2). The reason is that in (9) the Bessel functions use the exact values $\tilde{y}_m$ as arguments instead of their approximate values $\tilde{x}_m$ in (11).

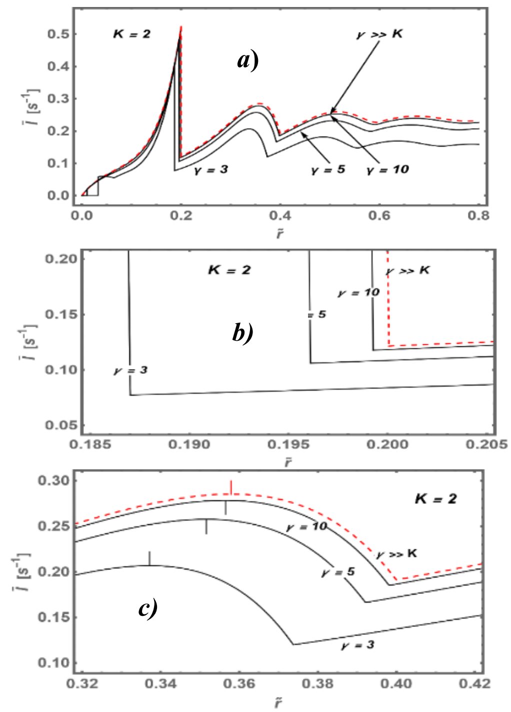

Figure 2: Spectral distributions of the radiation energy density of a particle moving along a spiral trajectory. The red dotted curve is calculated using formula (11); the black solid curves, calculated for three different values of γ, corresponds to formula (9); a) full spectrum; b), c) selected parts of the spectrum; $K = 2$; $\tilde{I}$ – distributions normalized to $Nq^2/\varepsilon_0 c$, calculated by (11) and (9), respectively.

CIRCULAR ROTATION

When $\beta_z \to 0$, the frequency interval (14) of the distribution of the m[th] mode in (9) narrows to the point $\tilde{\omega} = m$. In this case, the step function $\tilde{u}(S_m^2)$ degenerates into a point function $\tilde{d}(\tilde{\omega} - m)$ ($\tilde{d}(x) = 1$ for $x = 0$ and $\tilde{d}(x) = 0$ for $x \neq 0$). Successive substitution of $\tilde{\omega} = m$ and $\beta_z = 0$ into the function X_m (9) gives:

$$\frac{d\tilde{J}(\omega)}{d\omega} = q^2 N \frac{\beta_\gamma^2}{2\varepsilon_0 c} \sum_{m=1}^{\infty} m J_m'^2(m\beta_\gamma) \tilde{d}(\tilde{\omega} - m). \quad (15)$$

The energy radiated in all directions is determined by replacing integration over ω with summation ($\Delta\tilde{\omega} = 1$):

$$\tilde{J} = q^2 N \frac{\omega_0 \beta_\gamma^2}{2\varepsilon_0 c} \Delta\tilde{\omega} \sum_{m=1}^{\infty} m J_m'^2(m\beta_\gamma). \quad (16)$$

To prove the convergence of series (16) for $\beta_\gamma < 1$, the Debye asymptotic is used for the derivative of the Bessel function, the argument of which contains its order [5]. It is valid for a large positive number ν and a fixed positive number α:

$$J_\nu'(\nu \operatorname{sech} \alpha) \approx \sqrt{\frac{sh2\alpha}{4\pi\nu}} exp\{\nu(th\alpha - \alpha)\} \quad (17)$$

$$\alpha = Arcsech\beta_\gamma = ln\left(1 + \sqrt{1 - \beta_\gamma^2}\right) - ln\beta_\gamma . \quad (18)$$

The exponential argument in (17) takes the form:

$$\nu(th\alpha - \alpha) = -\nu u, \;\; u = \alpha - \sqrt{1 - \beta_\gamma^2} \quad (19)$$

and for $\nu > 0$ it is a negative number. Thus, the remainder of the sum (16), starting from $m = n$ ($n \gg 1$), can be written in the form of a convergent geometric progression:

$$\sum_{m=n}^{\infty} m J_m'^2(m\beta_\gamma) \approx \frac{sh2\alpha}{4\pi} \frac{e^{-2un}}{1 - e^{-2u}} . \quad (20)$$

Consequently, the energy emitted by a particle moving in a circle at a constant speed is a finite value at rotational speeds arbitrarily close to the speed of light. When estimating the residual sum in (16) in the ultra-relativistic case ($\beta_\gamma \to 1$), one should take into account the proportionality of the cyclic frequency ω_0 to the factor $\sqrt{1 - \beta_\gamma^2}$ [6]:

$$\omega_0 = \frac{eH}{m_e c} \sqrt{1 - \beta_\gamma^2} . \quad (21)$$

Here m_e and e are the mass and charge of the particle, H is the magnitude of the external longitudinal magnetic field. From (17) - (21), the remainder sum can be written as:

$$\tilde{J}_n = \frac{1}{4\pi} \frac{q^2 N}{\varepsilon_0 c} \frac{eH}{m_e c} \left(1 - \beta_\gamma^2\right) \frac{e^{-2un}}{1 - e^{-2u}} . \quad (22)$$

In the limit $\beta_\gamma \to 1$ (22) takes the form:

$$\tilde{J}_n \to \frac{3}{4\pi\sqrt{2}} \frac{q^2 N}{\varepsilon_0 c} \frac{eH}{m_e c} \frac{1}{\sqrt{1 - \beta_\gamma}} \quad (23)$$

diverging at $\beta_\gamma \to 1$.

The difference between (16) and the corresponding results in [6] is explained by the fact that in our case it is not necessary to average over the rotation period and to integrate over the angles of the spherical coordinate system used in [1, 4] and [6]. Here is the advantage of using a cylindrical coordinate system, organically connected with the spiral-shaped (or, in the limit $\beta_z \to 0$, with a circular) trajectory of the particle, which made it possible to obtain fully integrable and, moreover, accurate (and not averaged, as in [6]) expressions, containing mode contributions not only with even numbers (as in [6]), but also with the odd sequence numbers, i.e., the full set of modes.

CONCLUSION

The use of a cylindrical coordinate system directly related to the particle trajectory made it possible to use the partial domain method for a uniform description of radiation fields both near and far from the particle trajectory and to take into account the discontinuity of the field components on the cylindrical surface containing the particle trajectory, generalizing it to the case of a helical motion of the particle. The main results are: 1) For the radiation of a particle moving along a spiral trajectory with a constant longitudinal velocity and a fixed rotation frequency, expressions for the fields are obtained that are valid at any point in space. 2) The mutual compensation of singularities, present in TM and TE modes with the same order, has been proven. 3) A refined formula for the distribution of the spectral energy density of the radiation of a helical undulator was obtained. 4) A new formula has been derived for the radiated energy of a particle moving along a circle.

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REFERENCES

[1] J. B. Jackson, “*Classical Electrodynamics*”, New York, NY, USA, Willey, 1998.

[2] M. Ivanyan, A. Grigoryan, B. Grigoryan, F. Lemery, K. Floettmann, and L. Aslyan, “Dispersion relations for a cylindrical waveguide with multilayer walls”, in *Proc.* IPAC'23, Venice, Italy, May 2023, pp. 3395-3398. doi:10.18429/JACoW-IPAC2023-WEPL121

[3] J. A. Stratton, “*Electromagnetic Theory*”, New York, NY, USA, McGRAW HILL, 1941.

[4] B. M. Kincaid, “A short-period helical wiggler as an improved source of synchrotron radiation”, *J. Appl. Phys.*, vol. 48, no. 7, pp. 2684-2691, Jul. 1977. doi: 10.1063/1.324138

[5] M. Abramowitz and I. Stegun, “*Handbook of Mathematical Functions*”, Washington, DC, USA, National Bureau of Standards, 1972.

[6] L. D. Landau and E. M. Lifshitz, “*The Classical Theory of Fields*”, New York, NY, USA: Pergamon Press 1971.