SLAC-PUB-231
November 1966

THE BETHE-SALPETER EQUATION IN MOMENTUM SPACE* †

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ABSTRACT

The Wick transformation in momentum space is modified to include all scattering energies by use of a coordinate surface which possesses limited detours into the complex relative energy plane. This device retains the simple form of the equation for the Bethe-Salpeter amplitude $\psi(\underset{\sim}{p}, p_0)$. It is shown that the transformation is valid if $\psi(\underset{\sim}{p}, p_0)$ has the cut structure indicated by field theory, and this structure is shown to be consistent with the Bethe-Salpeter equation provided the interaction $V(x)$ satisfies a simple causality condition. It is further shown that the cut structure of a solution to the transformed equation can be deduced from the causal structure of the interaction alone without reference to field theory. Basic properties of the transformed equation are derived and a numerical treatment for purely elastic scattering is presented.

(To be submitted to Journal of Math. Physics)

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*Work supported in part by the U. S. Atomic Energy Commission.

†All the numerical work and most of the theoretical work in this paper was performed while the author was a National Science Foundation Fellow at Massachussetts Institute of Technology in Cambridge, Massachusetts.

I. INTRODUCTION

This paper will be concerned with the Bethe-Salpeter equation[1] for two non-identical spinless mesons ("a" and "b") of equal mass, m, interacting in the ladder diagram approximation via the field of a third spinless meson of mass μ. Principal attention will be paid to the scattering case in which E, the total center-of-mass frame energy is greater than 2 m. In relative momentum four-space in the center-of-mass frame the equation will be taken as

$$\psi(p) = \psi_o(p) - \frac{1}{D(p)} \int V(p-\eta)\, \psi(\eta)\, d^4\eta \qquad (1.1)$$

where $p = (\underset{\sim}{p}, p_o)$, $\eta = (\underset{\sim}{n}, \eta_o)$, and with $q^2 = \underset{\sim}{q}^2 - q_o^2$ ($\hbar = c = 1$)

The function of $D(p)$ is

$$D(p) = \left[(p + E/2)^2 + m^2\right]\left[(p - E/2)^2 + m^2\right] \qquad (1.2)$$

$$= (p^2 - k^2)^2 - E^2 p_o^2 \quad , \qquad (1.3)$$

$$k^2 \equiv (E/2)^2 - m^2 \quad , \qquad (1.4)$$

$$V(p-\eta) = \frac{1}{\pi^2} \frac{1}{(p-\eta)^2 + \mu^2} \qquad (1.5)$$

and, for the scattering case, $E > 2m$,

$$\psi_o(p) = \delta^3(\underset{\sim}{p} - \underset{\sim}{p}_{in})\, \delta(p_o) \qquad (1.6)$$

with incident center-of-mass momenta

$$\underset{\sim}{p}_{in} = (\underset{\sim}{p}_a)_{in} - (\underset{\sim}{p}_b)_{in} \qquad (1.7)$$

$$|\underset{\sim}{p}_{in}| = k \qquad (1.8)$$

$\psi(x)$, the configuration space amplitude, is to be related to $\psi(p)$ by

$$\psi(x) = \int e^{i(\underset{\sim}{p}\,\underset{\sim}{x} - p_0 x_0)} \psi(p)\, d^4p \qquad (1.9)$$

and the Feynman convention $m \to m - i\epsilon$, $\mu \to \mu - i\epsilon'$ is then understood as the proper way of handling singularities in $1/[D(p)]$ and $V(p - \eta)$. For the case of purely elastic scattering, $2m < E < 2m + \mu$, Charles Schwartz and Charles Zemach[2] have presented an analysis of the Bethe-Salpeter equation in the circumstance considered here and have produced numerical scattering results. Their approach is related to the previous work on the bound state case by G. C. Wick[3] in that, as Wick, they use the device of rotation to imaginary values of the relative time variable to produce a transformed integral equation with the standard Euclidean metric which is more susceptible to solution than the non-rotated equation with its Lorentz metric.[2a]

The treatment of Schwartz and Zemach for elastic scattering energies differs somewhat from the approach of Wick in that the former authors perform the sought-after rotation in configuration space, whereas the rotation of Wick for the bound state case ($E < 2m$, $\psi_0(p) \equiv 0$) is performed primarily in momentum space, i.e., — is primarily concerned with rotation of the variables p_0 and η_0 in Eq. (1.1). For the scattering case the relationship between rotations in these two forms of the equation is not trivial, as is apparent whenever calculations in momentum space become a practical necessity. Thus Schwartz and Zemach are forced to use a distorted contour in momentum space when evaluating certain integrals needed to obtain their numerical results.[4]

We will present below a simple method for extending the Wick rotation in momentum space to scattering energies. In this modified procedure the necessary contour distortion presented by Schwartz and Zemach plays a fündamental role. Many features of the Bethe-Salpeter equation can be easily demonstrated for the resulting momentum space representation. Also, a method of obtaining numerical results in the elastic energy range from the equation in this form will be discussed and some computed values presented which serve as a rough check on the Schwartz-Zemach procedure and other more recent calculations.[2a]

II. TRANSFORMATION IN MOMENTUM SPACE

As an initial step toward transformation of the Bethe-Salpeter equation in momentum space one can refer back to Wick's analysis (based on the operators and states of field theory) of the structure of $\psi(\underset{\sim}{p},p_o)$ as a function of complex p_o. For $E > 2m$, as for $E < 2m$, the Wick analysis states that $\psi(\underset{\sim}{p},p_o)$ is analytic in p_o everywhere except along two cuts. One cut lies just below the real p_o axis extending from $\omega_{min}(\underset{\sim}{p}) - i\epsilon$ to $+\infty - i\epsilon$, while the other lies just above the real axis from $\omega_{max}(\underset{\sim}{p}) + i\epsilon$ to $-\infty + i\epsilon$, where (for the mass equal case),

$$\omega_{min}(\underset{\sim}{p}) = -\omega_{max}(\underset{\sim}{p}) = (\underset{\sim}{p}^2 + m^2)^{1/2} - E/2 \quad . \qquad (2.1)$$

Although this analysis depends on field theory it can be shown that the structure implied by Eq. (2.1) is consistent with the structure of the equation itself, as will be demonstrated later in this section. The location of the cuts in $\psi(\underset{\sim}{p},p_o)$ is shown in Fig. 1.

For $\underset{\sim}{p}^2 > k^2$, $\omega_{min}(\underset{\sim}{p})$ is positive so that there is a gap between the two cuts and one can analytically continue $\psi(\underset{\sim}{p},p_o)$ counter clockwise from the real axis to the imaginary axis in complex p_o, as is indicated in Fig. 1a. For $\underset{\sim}{p}^2 \leq k^2$ the two cuts in $\psi(\underset{\sim}{p},p_o)$ as a function of p_o overlap since then $\omega_{min}(\underset{\sim}{p})$ is negative. Thus for $\underset{\sim}{p}^2 \leq k^2$ one can not analytically continue counter clockwise from the real p_o axis to the imaginary p_o axis without encountering singularities in $\psi(\underset{\sim}{p},p_o)$, but one can continue from the real axis to a contour which follows the imaginary axis except for two detours around those portions of the cuts which protrude below and above $p_o = 0$, continuing to $\omega_{min}(\underset{\sim}{p}) - i\epsilon$ and $-\omega_{min}(\underset{\sim}{p}) + i\epsilon$, respectively. The suggested contour for $\underset{\sim}{p}^2 \leq k^2$ is shown in Fig. 1b, and will be recognized as the contour that was introduced by Schwartz and Zemach. Since the suggested p_o contour for consideration of the analytic continuation of $\psi(\underset{\sim}{p}, p_o)$ depends on $\underset{\sim}{p}^2$, it will be called $C_{\underset{\sim}{p}^2}$. For $\underset{\sim}{p}^2 > k^2$ the contour $C_{\underset{\sim}{p}^2}$ is just the imaginary axis (see Fig. 1a). (The pathology of $\underset{\sim}{p}^2 = k^2$ will be considered in more detail below.)

One now can convert Eq. (1.1) into an integral equation for the analytic continuation of $\psi(\underset{\sim}{p}, p_o)$ from real four-space to a new coordinate surface which consists of all real values of $\underset{\sim}{p}$ but only those complex values of p_o which lie on $C_{\underset{\sim}{p}^2}$. In order that this conversion be accomplished, one must analytically continue in p_o all terms on the right-hand side of Eq. (1.1),

$$\psi_o(\underset{\sim}{p}, p_o) - \frac{1}{D(p)} \iint V(p - \eta)\, \psi(\eta)\, d\eta_o\, d^3\underset{\sim}{n}\ , \tag{2.2}$$

from the real axis to $C_{\underset{\sim}{p}^2}$, and also one must deform the η_o contour in Eq. (2.2) from the real axis to the contour C_{n^2} for every n.

Consider the first term in Eq. (2.2),

$$\psi_o(\underset{\sim}{p}, p_o) = \delta^3(\underset{\sim}{p} - \underset{\sim}{p}_{in})\, \delta(p_o)\ . \tag{2.3}$$

To treat this term consistently one needs a representation of $\delta(p_o)$ which exhibits the singular structure indicated by the Wick analysis. Such a representation is readily achieved by writing

$$\delta(p_o) = \frac{1}{2\pi i}\left(\frac{1}{p_o - i\epsilon} - \frac{1}{p_o + i\epsilon}\right) \tag{2.4}$$

where ϵ is positive real and arbitrarily small. With $\delta(p_o)$ in this form, ψ_o has singularities in p_o at $p_o = \pm i\epsilon$. But the points $\pm i\epsilon$ fall in the cut region allowed by Eq. (2.1) for $\underset{\sim}{p}^2 \leq k^2$, and hence are avoided by counter-clockwise continuation to $C_{\underset{\sim}{p}^2}$ for $\underset{\sim}{p}^2 \leq k^2$. Since the incident particles are strictly on the "mass shell" $\underset{\sim}{p}^2 = k^2$, $\psi_o(\underset{\sim}{p}, p_o) \equiv 0$ if $\underset{\sim}{p}^2 \neq k^2$, and therefore the singularities in $\psi_o(\underset{\sim}{p}, p_o)$ with $\delta(p_o)$ in form Eq. (2.4) conform to the Wick structure indicated by Eq. (2.1).

Further, if one lets ϵ have an infinitesimal positive imaginary part $\epsilon \to \epsilon_1 + i\epsilon_2$, then the points $\pm i\epsilon$ are also avoided by the contour rotation $\underset{\sim}{p}^2 > k^2$ and all ambiguity in defining $C_{\underset{\sim}{p}^2}$ for $\underset{\sim}{p}^2$ near k^2 is eliminated. With this convention for ϵ, the poles in Eq. (2.4) at $\pm i\epsilon$ remain on the same side of the p_o contour throughout and the right-hand side of Eq. (2.4) satisfies

$$\int_C f(p_o) \cdot [\text{R.H.S. of (2.4)}]\, dp_o \to f(0) \tag{2.5}$$

in the limit $|\epsilon| \to 0$ whether C is the real axis or $C_{\underset{\sim}{p}^2}$ for any $\underset{\sim}{p}$ (provided f is analytic at $p_o = 0$). Note (see Fig. 2) the convention $\epsilon \to \epsilon_1 + i\epsilon_2$, $i\epsilon \to i\epsilon_1 - \epsilon_2$ is a natural one for relocating poles relative to $C_{\underset{\sim}{p}^2}$. Thus with Eq. (2.4) and $\epsilon \to \epsilon_1 + i\epsilon_2$, $\psi_o(\underset{\sim}{p}, p_o)$ can be continued to $C_{\underset{\sim}{p}^2}$ for all $\underset{\sim}{p}$ and the formal identification $\psi_o(\underset{\sim}{p}, p_o) = \delta^3(\underset{\sim}{p} - \underset{\sim}{p}_{in})\, \delta(p_o)$ can be retained for the analytically continued ψ_o.

It remains to analytically continue

$$-\frac{i}{D(p)} \int V(p-\eta)\, \psi(\eta)\, d\eta_o\, d^3\underset{\sim}{n} \tag{2.6}$$

in p_o from the real axis to $C_{\underset{\sim}{p}^2}$.

Continuation of the factor

$$1/D(p) = 1 \Big/ \Big[(p_o - \alpha)(p_o + \alpha)(p_o - \beta)(p_o + \beta)\Big] \tag{2.7}$$

where

$$\alpha(\underset{\sim}{p}) = (\underset{\sim}{p}^2 + m^2)^{1/2} - E/2 - i\epsilon \tag{2.8}$$

$$\beta(\underset{\sim}{p}) = (\underset{\sim}{p}^2 + m^2)^{1/2} + E/2 - i\epsilon \qquad (2.9)$$

($-i\epsilon$ is assigned by the Feynman convention and $\epsilon = \epsilon_1 + i\epsilon_2$ is understood to be available if needed) is easy since the poles at $\pm\beta(\underset{\sim}{p})$ never obstruct continuation (note $\mathrm{Im}(\beta) \leq 0$, $\mathrm{Re}(\beta) > 0$) while $\alpha(\underset{\sim}{p})$ and $-\alpha(\underset{\sim}{p})$ lie at $\omega_{min}(\underset{\sim}{p}) - i\epsilon$ and $-\omega_{min}(\underset{\sim}{p}) + i\epsilon$, respectively, and hence are specifically avoided by $C_{\underset{\sim}{p}^2}$ (see Fig. 1b).

Finally, as we shall now show, simultaneous continuation in p_o and deformation of the η_o contour in the term

$$\iint V(p - \eta)\,\psi(n, \eta_o)\, d\eta_o\, d^3\underset{\sim}{n} \equiv T(p) \qquad (2.10)$$

can be accomplished in the manner presented by Wick for the bound state case.[3] In order to simplify the discussion of this process one may consider transformation of the η_o contour for the special value of $p_o = 0$ which lies on both the real axis and $C_{\underset{\sim}{p}^2}$. Subsequently one can consider analytic continuation of this transformed integral to all p_o on $C_{\underset{\sim}{p}^2}$.

It is established by assumption of the Wick structure that singularities in $\psi(\underset{\sim}{n}, \eta_o)$ will not interfere with η_o contour deformation from the real axis (counter clockwise) to $C_{\underset{\sim}{n}^2}$. That the poles in $V(p - \eta)$ at $\pm\left[(\underset{\sim}{p}-\underset{\sim}{n})^2 + \mu^2\right]^{1/2}$ will not interfere with the desired rotation is illustrated in Fig. 3a. Hence the η_o contour change is valid for $p_o = 0$. Now one wants to move p_o from $p_o = 0$ along $C_{\underset{\sim}{p}^2}$, thus analytically continuing the transformed integral

$$\int_{C_{\underset{\sim}{n}^2}} \frac{\psi(\underset{\sim}{n}, \eta_o)\, d\eta_o}{(\underset{\sim}{p} - \underset{\sim}{n})^2 + (p_o - \eta_o)^2 + \mu^2} \qquad (2.11)$$

to all p_o on $C_{\underset{\sim}{p}^2}$. This process of analytic continuation can be taken trivially under the integral sign provided the poles at $r_\pm = p_o \pm \left[(\underset{\sim}{p} - \underset{\sim}{n})^2 + \mu^2\right]^{1/2}$

never cross the contour $C_{\underset{\sim}{n}^2}$. The two paths followed by these poles as p_o moves along $C_{\underset{\sim}{p}^2}$ are shown in Fig. 3a. Defining $M(\underset{\sim}{p}^2)$ to be the maximum real part extension of $C_{\underset{\sim}{p}^2}$,

$$\begin{aligned} M(\underset{\sim}{p}^2) &= 0 \quad \text{if } \underset{\sim}{p}^2 > k^2 \\ M(\underset{\sim}{p}^2) &= E/2 - (\underset{\sim}{p}^2 + m^2)^{1/2} \quad \text{if } \underset{\sim}{p}^2 \leq k^2 \end{aligned} \tag{2.12}$$

it will be seen from Fig. 3a that the poles in question always stay on the same respective sides of $C_{\underset{\sim}{n}^2}$ (although they may come infinitesimally close to it) provided

$$\left[(\underset{\sim}{p} - \underset{\sim}{n})^2 + \mu^2\right]^{1/2} > \left| M(\underset{\sim}{p}^2) - M(\underset{\sim}{n}^2)\right| \tag{2.13a}$$

for all $\underset{\sim}{p}$ and $\underset{\sim}{n}$. If $\underset{\sim}{p}^2$ and $\underset{\sim}{n}^2$ are greater than k^2, Eq. (2.13a) holds trivially and rotation is possible as in the bound state case. Now consider $\underset{\sim}{n}^2 < \underset{\sim}{p}^2, \underset{\sim}{n}^2 < k^2$ (the case $\underset{\sim}{p}^2 < \underset{\sim}{n}^2, \underset{\sim}{p}^2 < k^2$ will be covered by symmetry). Then Eq. (2.13a) becomes

$$\left[(\underset{\sim}{p} - \underset{\sim}{n})^2 + \mu^2\right]^{1/2} > E/2 - (\underset{\sim}{n}^2 + m^2)^{1/2} - M(\underset{\sim}{p}^2) \quad . \tag{2.14}$$

But $M(\underset{\sim}{p}) \geq 0$, so it suffices to show

$$\left[(\underset{\sim}{p} - \underset{\sim}{n})^2 + \mu^2\right]^{1/2} > E/2 - (\underset{\sim}{n}^2 + m^2)^{1/2} . \tag{2.15}$$

Since

$$\left[(\underset{\sim}{p} - \underset{\sim}{n})^2 + \mu^2\right]^{1/2} \geq \mu \tag{2.16}$$

and $(\underset{\sim}{n}^2 + m^2)^{1/2} \geq m$, it suffices that

$$\mu > E/2 - m \quad \text{or} \quad E < 2m + 2\mu \ . \tag{2.17}$$

Thus Eq. (2.13) is easily verifiable if E is less than the energy necessary to produce two real mesons of mass μ by inelastic scattering. To show Eq. (2.14) holds for arbitrary E , note

$$M(\underset{\sim}{p}^2) \geq E/2 - (\underset{\sim}{p}^2 + m^2)^{1/2} \tag{2.18}$$

for all $\underset{\sim}{p}$, so Eq. (2.14) follows if

$$\left[(\underset{\sim}{p}-\underset{\sim}{n})^2+\mu^2\right]^{1/2} > \left(E/2-(\underset{\sim}{n}^2+m^2)^{1/2}\right)-\left(E/2-(\underset{\sim}{p}^2+m^2)^{1/2}\right) \tag{2.19a}$$

i.e., if

$$\left[(\underset{\sim}{p}-\underset{\sim}{n})^2+\mu^2\right]^{1/2} > (\underset{\sim}{p}^2+m^2)^{1/2}-(\underset{\sim}{n}^2+m^2)^{1/2} \tag{2.19b}$$

for all $\underset{\sim}{n}^2 < k^2$ and $\underset{\sim}{p}^2 > \underset{\sim}{n}^2$, and Eq. (2.14) certainly follows if Eq. (2.19b) can be shown for arbitrary $\underset{\sim}{n}$, $\underset{\sim}{p}$. Setting $w = |\underset{\sim}{p}|$, $e = |\underset{\sim}{n}|$, then

$$\left[(\underset{\sim}{p}-\underset{\sim}{n})^2+\mu^2\right]^{1/2} > |e-w| \tag{2.20a}$$

while

$$(\underset{\sim}{p}^2+m^2)^{1/2}-(\underset{\sim}{n}^2+m^2)^{1/2} \leq \left|(w^2+m^2)^{1/2}-(e^2+m^2)^{1/2}\right|$$

$$= \frac{|w^2-e^2|}{(w^2+m^2)^{1/2}+(e^2+m^2)^{1/2}} \leq \frac{|w^2-e^2|}{w+e} = |w-e| \tag{2.20b}$$

verifying Eq. (2.19b). (Inequality Eq. (2.19b) has been fully considered in the general context of Fourier transforms and causality by Dyson.[5] The relationship of causality to the Bethe-Salpeter equation will be emphasized below.)

In a manner analogous to Wick's rotation, one can now continue $T(p)$ in Eq. (2.10) to any p_o which lies in the region swept by counter-clockwise contour deformation from the real p_o axis to $C_{\underset{\sim}{p}2}$: Call the partially deformed contour $C'_{\underset{\sim}{p}}$ (as shown in Fig. 3b). For each n in Eq. (2.10), deform the η_o contour from the real axis to a contour $C'_{\underset{\sim}{pn}}$ horizontally displaced from $C'_{\underset{\sim}{p}}$ by $M(\underset{\sim}{n}) - M(\underset{\sim}{p})$ as shown in Fig. 3b. Finally Eq. (2.10) with this choice of η_o contours can be continued in p_o along $C'_{\underset{\sim}{p}}$ in the manner just considered for continuation Eq. (2.11) along $C_{\underset{\sim}{p}2}$. Thus the extension of the Wick rotation in momentum space to all values of E greater than 2m is complete.

It is significant to note there are circumstances under which $C_{\bar{n}^2}$ will never come infinitesimally close to the poles in $V(p-\eta)$ for p_o on $C_{\bar{p}^2}$. As shown in Fig. 3c, such separation will occur if

$$\left[(\mathbf{p}-\mathbf{n})^2+\mu^2\right]^{1/2} > M(\mathbf{p}^2)+M(\mathbf{n}^2) \qquad (2.13b)$$

for all $\mathbf{p}, \mathbf{n}$, and thus will occur if $\mu > (E/2-m)+(E/2-m) = E-2m$, or $E < 2m+\mu$. This will be recognized as the energy restriction for purely elastic scattering and the momentum space transformation in this case corresponds to the transformation of Schwartz and Zemach in configuration space.

The transformed equation reads

$$\psi(\mathbf{p} > p_o) = \delta^3(\mathbf{p}-\mathbf{p}_{in})\,\delta(p_o) - \frac{1}{D(p)} \iint_{C_{\mathbf{n}^2}} V(p-\eta)\,\psi(\eta)\,d\eta_o\,d^3\mathbf{n} \qquad (2.21)$$

valid for all real $\mathbf{p}$ and all p_o on $C_{\mathbf{p}^2}$ for any value of E greater than $2m$.

A. Singularities in the Transformed Equation

The advantage of transforming to the new coordinate surface is that the singularities in $1\ D(p)$ and $V(p-\eta)$ which occur in Eq. (1.1) over an unbounded region of real four space now are confined to a limited region on the new coordinate surface. Singularities in $1/D(p)$ only occur when the contour $C_{\mathbf{p}^2}$ touches one of the poles in $1/D(\mathbf{p}, p_o)$ at

$$p_o = \pm\left((\mathbf{p}^2+m^2)^{1/2} - E/2 - i\epsilon\right) \quad \text{and} \quad p_o = \pm\left((\mathbf{p}^2+m^2)^{1/2} + E/2 - i\epsilon\right).$$

But the latter set of poles are never touched by $C_{\mathbf{p}^2}$ (see Fig. 1), while the former are only touched for $\mathbf{p}^2 \leq k^2$ by the "detour" segments of $C_{\mathbf{p}^2}$. Thus outside the limited region of detour on the transformed coordinate surface $1/D(p)$ is free of singularity and vanished as $1/p^4$ as $p^2 \to \infty$.

Now consider singularities in η_o in $V(p - \eta)$ for fixed p_o on $C_{\underset{\sim}{p}^2}$. As discussed above poles occur at $r_\pm = p_o \pm \left((\underset{\sim}{p} - n)^2 + \mu^2\right)^{1/2}$. In order for these poles to touch $C_{\underset{\sim}{n}^2}$ at some η_o it must be that $\mathrm{Im}(\eta_o) = \mathrm{Im}(p_o) = 0$, i.e., p_o must be zero ($\underset{\sim}{p}^2 > k^2$) or on a detour section of $C_{\underset{\sim}{p}^2}$ with $\underset{\sim}{p}^2 < k^2$ and η_o must be similarly restricted. Now for $\underset{\sim}{p}^2 > k^2$ inequality Eq. (2.13a) excludes the possibility of $C_{\underset{\sim}{n}^2}$ touching a pole. By symmetry no poles can be touched if $\underset{\sim}{n}^2 > k^2$. Thus singularities can only occur in $V(p - \eta)$, when $\underset{\sim}{p}^2 < k^2$, $\underset{\sim}{n}^2 < k^2$ and p_o and η_o are on the detour portions of $C_{\underset{\sim}{p}^2}$ and $C_{\underset{\sim}{n}^2}$, respectively. In the limit $p^2 \to \infty$ outside the region of detour $V(p - \eta)$ vanishes as $(\lambda/\pi^2) \cdot 1/p^2$ for any fixed η. (It has already been commented that for $E < 2m + \mu$, $V(p - \eta)$ is completely free of singularity on the whole coordinate surface.) The regularity of $1/D(p)$ and $V(p' - \eta')$ away from $\underset{\sim}{p}^2 < k^2$, $\mathrm{Im}(p_o) = \mathrm{Im}(\eta_o) = 0$ allows one to form an intuitive picture of the transformed equation. Asymptotically it is the same regular equation considered by Wick for $E < 2m$.

Since for most of the coordinate surface p_o and η_o are purely imaginary it is useful to write the transformed equation in terms of rotated coordinates

$$p_o' = p_o/i, \quad p' = (\underset{\sim}{p}, p_o'), \quad p'^2 = \underset{\sim}{p}^2 + p_o'^2 \tag{2.22}$$

with analogous definitions for η_o', η', η'^2. Writing $\psi'(p) = i\psi(p)$ and noting $\delta(p_o) = \delta(ip_o') = -i\,\delta(p_o')$, Eq. (2.21) becomes

$$\psi'(p') = \delta^3(\underset{\sim}{p} - \underset{\sim}{p}_{in})\,\delta(p_o') + \frac{1}{D(p')} \iint_{C'_{\underset{\sim}{n}^2}} V(p' - \eta')\,\psi'(\eta')\, d\eta_o'\, d^3\underset{\sim}{n} \tag{2.23}$$

with

$$D(p') = (p'^2 - k^2)^2 + E^2 p_o'^2 \tag{2.24}$$

$$V(p' - \eta') = \lambda/\pi^2 \frac{1}{(p' - \eta')^2 + \mu^2} \tag{2.25}$$

C'_{n^2} is C_{n^2} rotated clockwise by 90^o. The complex p' coordinate surface on which Eq. (2.23) applies can be sketched in terms of $|p|$, Re(p'_o), and Im(p'_o) as shown in Fig. 4.

B. The Wick Frequency Assumption – Causality

It should be noted that the ability to make the contour transformations in complex p_o, η_o demonstrated above only depends on the Wick frequency analysis (based on field theory) through the assumption that $\psi(n, \eta_o)$ in Eq. (1.1) has the structure indicated by Eq. (2.1). But the only assumption really needed is that $\psi(n, \eta_o)$ be analytic in the region swept by counter-clockwise deformation of the η_o contour from the real axis to C_{n^2}. The demonstrated ability to continue the right-hand side of Eq. (1.1) in complex p_o throughout the region of contour distortion between the real axis and C_{p^2} proves that this structure, which was assumed for $\psi(n, \eta_o)$, although suggested by field theory, is fundamentally consistent with the structure of Eq. (1.1).

One may go further and ask whether the full Wick structure implied by Eq. (2.1) can be shown consistent with the structure of Eq. (1.1) in this way. The above analysis has shown that $\psi_o(p, p_o)$ and $1/D(p)$ are consistent with Eq. (2.1). It remains to show that

$$T(p) = \iint V(p - \eta)\, \psi_{(\eta)}\, d^4\eta$$

can be continued throughout the complex p_o plane up to the cuts described by Eq. (2.1), assuming that $\psi(n, \eta_o)$ has the cuts described by Eq. (2.1) in complex η_o.

As illustrated in Fig. 3d, the required continuation, accompanied by the appropriate η_o contour distortion, can be accomplished if

$$\left[(p - n)^2 + \mu^2\right]^{1/2} > \left|(p^2 + m^2)^{1/2} - (n^2 + m^2)^{1/2}\right| \qquad (2.19c)$$

for all $\underset{\sim}{p}, \underset{\sim}{n}$. (One first chooses the path of p_0 continuation $C'_{\underset{\sim}{p}}$, then deforms the η_0 path of integration to $C'_{\underset{\sim}{n}\underset{\sim}{p}}$ for each $\underset{\sim}{n}$, and finally continues p_0 along $C'_{\underset{\sim}{p}}$.) But this inequality is equivalent to inequality Eq. (2.19b). Thus, not surprisingly, ability to perform the p_0, η_0 contour transformations in Eq. (1.1) for arbitrary E follows from the consistence of Eq. (1.1) with the structural requirements of Eq. (2.1) for all $\underset{\sim}{p}$. It is significant to note that in the above proof of this consistency, as indicated by Eq. (2.20a), it suffices that V satisfy a simple condition: $V(\underset{\sim}{q}, q_0)$ can only have singularities in q_0 which are at $(r - i\epsilon)$ and/or $-(r - i\epsilon')$ where r is real and $r \geq |\underset{\sim}{q}|$. In configuration space $r \geq q$ implies that $V(\underset{\sim}{x}, x_0)$ for $x_0 > 0$ and $x_0 < 0$ is a superposition of waves which move with a velocity less than or equal to 1, the velocity of light. This inequality places a very strong restriction on V since it must be satisfied for all $\underset{\sim}{q}$. On the other hand, as indicated by Eq. (2.16), for the restricted range $E < 2m + 2\mu$ the contour transformation to $C_{\underset{\sim}{p}2}$ proposed here only requires $r \geq \mu$, i.e., that $V(\underset{\sim}{x}, x_0)$ have x_0 frequencies greater than μ for $x_0 > 0$, and less than $-\mu$ for $x_0 < 0$ ("causality" need only be obeyed for $|\underset{\sim}{q}| \leq \mu$).

One naturally asks whether the cut structure of ψ can be deduced from the Bethe-Salpeter equation alone without reference to field theory. Toward this end note that if ψ is a solution to the transformed Eq. (2.21), then the known separation of the poles in $V(p-\eta)$ from η_0 on $C_{\underset{\sim}{n}2}$, indicates that $T(\underset{\sim}{p}, p_0) = \iint_{C_{\underset{\sim}{n}2}} V(p-\eta)\,\psi(\eta)\, d^4\eta$ appearing in Eq. (2.21) will be an analytic function of p_0 at least within a strip about $C_{\underset{\sim}{p}2}$ (see Fig. 3e) of half width

$$\Delta_{\underset{\sim}{p}} = \operatorname*{Min}_{\underset{\sim}{n}} \left([(\underset{\sim}{p}-\underset{\sim}{n})^2 + \mu^2]^{1/2} - |M(\underset{\sim}{p}^2) - M(\underset{\sim}{n}^2)| \right)$$

$$\geq \operatorname*{Min}_{\underset{\sim}{n}^2 \leq k^2,\ \underset{\sim}{p}^2 \leq k^2} \left([(\underset{\sim}{p}-\underset{\sim}{n})^2 + \mu^2]^{1/2} - |(\underset{\sim}{p}^2+m^2)^{1/2} - (\underset{\sim}{n}^2+m^2)^{1/2}| \right) \equiv \Delta > 0.$$

(A closer inspection shows that $\Delta = \left(k^2 + (m+\mu)^2\right)^{1/2} - (k^2+m^2)^{1/2}$.)

But noting the analytic structure of $1/D(\underset{\sim}{p}, p_o)$, Eq. (2.21) shows that $\psi(\underset{\sim}{p}, p_o)$ can be continued in p_o (for any $\underset{\sim}{p}$) to the right in the upper half plane (and/or left in the lower half) through a distance Δ from $C_{\underset{\sim}{p}^2}$ to "$C_{\underset{\sim}{p}^2} + \Delta$", where Δ is independent of $\underset{\sim}{p}$. Then putting p_o on $C_{\underset{\sim}{p}^2} + \Delta$, and noting that the transformed equation indicates $\psi(\underset{\sim}{p}, p_o) \sim O(1/p_o^6)$ as $p_o/i \to \pm\infty$ on $C_{\underset{\sim}{p}^2}$, one can deform all the η_o contours in Eq. (2.21) from $C_{\underset{\sim}{n}^2}$ to $C_{\underset{\sim}{n}^2} + \Delta$. Now one can repeat the argument and continue $\psi(\underset{\sim}{p}, p_o)$ to $C_{\underset{\sim}{p}^2} + 2\Delta$, $C_{\underset{\sim}{p}^2} + 3\Delta$, etc.

If, however, one applies this procedure to the left in both upper and lower half planes (see Fig. 3f), one can not continue $\psi(\underset{\sim}{p}, p_o)$ through the poles in $1/D(\underset{\sim}{p}, p_o)$. For any $\underset{\sim}{p}^2$ such that these poles are encountered, the shifted contour must detour around them. One thus "discovers" the Wick cuts in $\psi(\underset{\sim}{p}, p_o)$.

As one shifts successively to the left, adding detours at each step (Fig. 3f), the length Δ_i through which one can move the non-obstructed parts of all the p_o contours in the i^{th} shift decreases as i increases. After $i-1$ shifts the p_o contours in the upper half plane must detour whenever $\underset{\sim}{p}^2 < \underset{\sim}{p}_{i-1}^2$, where

$$\underset{\sim}{p}_{i-1}^2 = \left(E/2 + \sum_{j=1}^{i-1} \Delta_j\right)^2 - m^2 > \underset{\sim}{p}_{i-2}^2 > \cdots > \underset{\sim}{p}_o^2 = k^2. \text{ Thus}$$

$$\Delta_i = \underset{n^2 < \underset{\sim}{p}_{i-1}^2,\ \underset{\sim}{p}^2 < \underset{\sim}{p}_{i-1}^2}{\text{Min}} \left(\left[(\underset{\sim}{p} - \underset{\sim}{n})^2 + \mu^2 \right]^{1/2} - \left| (\underset{\sim}{p}^2 + m^2)^{1/2} - (\underset{\sim}{n}^2 + m^2)^{1/2} \right| \right) < \Delta_{i-1} < \cdots < \Delta_1 = \Delta. \ \Big(\text{More specifically } \Delta_i =$$

$\left(\underset{\sim}{p}_{i-1}^2 + (m+\mu)^2\right)^{1/2} - (\underset{\sim}{p}_{i-1}^2 + m^2)^{1/2}$.) However, since Δ_i is bounded away from zero for any possible bound on $\sum_{j=1}^{i-1} \Delta_j$, the process of continuation illustrated in Fig. 3f can be extended throughout the left half plane (and similarly throughout the right one) despite the decreases in Δ_i. Thus any solution of the transformed equation which has the asymptotic behavior indicated by the equation itself, must have the analytic structure indicated by field theory.

One notes again that the full Wick cut structure only results if V possesses the complete causal structure indicated by Eq. (2.19c).

C. Properties of 1/D (p') — Relationship to Non-relativistic Theory

As in the case of the non-relativistic scattering equation, many important features if Eq. (2.23) are determined by properties of the factor 1/D (p') (see Eqs. (2.7) – (2.9)) which can be written

$$\frac{1}{D(p')} = \frac{1}{4E(\underset{\sim}{p}^2 + m^2)^{1/2}} \left[\frac{1}{\beta(\underset{\sim}{p})} \, i \left(\frac{1}{p_0' - i\beta(\underset{\sim}{p})} - \frac{1}{p_0' + i\beta(\underset{\sim}{p})} \right) - \frac{1}{\alpha(\underset{\sim}{p})} \, i \left(\frac{1}{p_0' - i\alpha(\underset{\sim}{p})} - \frac{1}{p_0' + i\alpha(\underset{\sim}{p})} \right) \right] \tag{2.27}$$

This expression corresponds to the non-relativistic term $1/(\underset{\sim}{p}^2 - k^2)$, and noting that

$$\alpha(\underset{\sim}{p}) \cdot \beta(\underset{\sim}{p}) = \underset{\sim}{p}^2 - k^2 \tag{2.28}$$

one can write Eq. (2.27) in analogy to the non-relativistic factor.

$$\frac{1}{D(p')} = \frac{1}{(\mathbf{p}^2 - k^2)} \cdot d(\mathbf{p}^2, p_o') \quad , \tag{2.29}$$

where, with $\omega(\mathbf{p}) = (\mathbf{p}^2 + m^2)^{1/2}$,

$$d(\mathbf{p}^2, p_o') = \frac{1}{4E}\left[\frac{\alpha(\mathbf{p})}{\omega(\mathbf{p})}\, i\left(\frac{1}{p_o' - i\beta(\mathbf{p})} - \frac{1}{p_o' + i\beta(\mathbf{p})}\right)\right.$$

$$\left. - \frac{\beta(\mathbf{p})}{\omega(\mathbf{p})}\, i\left(\frac{1}{p_o' - i\alpha(\mathbf{p})} - \frac{1}{p_o' + i\alpha(\mathbf{p})}\right)\right]. \tag{2.30}$$

Now note that at $\mathbf{p}^2 = k^2$, the following hold: $\omega = E/2$, $\alpha(\mathbf{p}) = -i\epsilon \rightarrow 0$, $\beta(\mathbf{p}) = E - i\epsilon \rightarrow E$, and thus

$$d(k^2, p_o') = \frac{1}{4E}\left[\frac{-i\epsilon}{E}\, 2i\left(\frac{1}{p_o' - i\epsilon} - \frac{1}{p_o' + i\epsilon}\right) - 2i\left(\frac{1}{p_o' - \epsilon} - \frac{1}{p_o + \epsilon}\right)\right] \tag{2.31a}$$

$$\rightarrow \quad -\frac{i}{2E}\left(\frac{1}{p_o' - \epsilon} - \frac{1}{p_o' + \epsilon}\right) \quad . \tag{2.31b}$$

Setting $\epsilon = \epsilon_1 + i\epsilon_2$, $\epsilon_1 \rightarrow 0$, and taking C_{k2}' as the real axis (see Fig. 2),

$$d(k^2, p_o') \rightarrow -\frac{i}{2E}\left(\frac{1}{p_o' - i\epsilon_2} - \frac{1}{p_o' + i\epsilon_2}\right) \quad , \tag{2.32}$$

so that, with arbitrarily small $|\epsilon|$ understood,

$$d(k^2, p_o) = \frac{\pi}{E}\,\delta(p_o') \quad , \tag{2.33}$$

Now using the familiar formula

$$\frac{1}{\mathbf{p}^2 - k^2} = \mathcal{P}\frac{1}{p^2 - k^2} + i\pi\,\delta(\mathbf{p}^2 - k^2) \tag{2.34}$$

one has

$$\frac{1}{D(p')} = \mathcal{P}\frac{1}{\underset{\sim}{p}^2 - k^2}\ d(\underset{\sim}{p}^2, p_0') + \frac{i\pi^2}{E}\ \delta(\underset{\sim}{p}^2 - k^2)\ \delta(p_0') \tag{2.35}$$

It is easy to check that $d(\underset{\sim}{p}^2, p_0)$ is a real factor in the sense that integrals of the sort $\int C'_{\underset{\sim}{p}^2}\ U(p_0')\, d(\underset{\sim}{p}^2, p_0')\, dp_0'$ will be real if $U^*(p_0') = U(p_0'^*)$ and U is analytic on $C'_{\underset{\sim}{p}^2}$. Thus Eq. (2.35) effectively splits $1/D(p')$ into real and imaginary parts in analogy to Eq. (2.34) for the non-relativistic case.

Equations (2.29) and (2.30) can be used to obtain the non-relativistic limit of $1/D(p')$. Note one can consider the non-relativistic limit to correspond to the limit $m \rightarrow E/2 \rightarrow \infty$ for fixed k, since this implies $mc >> mv$, i.e., $c >> v$ where v is the incident particle velocity. But then for any finite $\underset{\sim}{p}^2$, $\alpha(\underset{\sim}{p}) = [(E/2)^2 + (\underset{\sim}{p}^2 - k^2)]^{1/2} - E/2 \rightarrow (\underset{\sim}{p}^2 - k^2)/E \rightarrow 0$, $\beta(\underset{\sim}{p}) \rightarrow 2m \rightarrow E$, $\omega(p) \rightarrow m \rightarrow E/2$, so that, very much in the manner just considered,

$$d(\underset{\sim}{p}^2, p_0) \rightarrow \frac{i}{2E}\left(\frac{1}{p_0' - i\alpha(\underset{\sim}{p})} - \frac{1}{p_0' + i\alpha(\underset{\sim}{p})}\right) \tag{2.36}$$

in the limit $E \rightarrow \infty$, k fixed. Note that although $\alpha(\underset{\sim}{p})$ becomes a negative infinitesimal for $\underset{\sim}{p}^2 < k^2$, the contour $C'_{\underset{\sim}{p}^2}$ specifically detours for $p^2 < k^2$ so that the poles in Eq. (2.36) at $\pm i\alpha(\underset{\sim}{p})$ stay on the same side of $C'_{\underset{\sim}{p}^2}$ for $\underset{\sim}{p}^2 < k^2$ as for $\underset{\sim}{p}^2 > k^2$ when $\alpha(\underset{\sim}{p}) > 0$. Thus in Eq. (2.36), $\alpha(\underset{\sim}{p})$ can be treated as a positive (real) infinitesimal for all finite $\underset{\sim}{p}^2$, giving

$$d/(\underset{\sim}{p}^2, p_0') \rightarrow \pi/E\ \delta(p_0') \tag{2.37}$$

and

$$1/D(p') \rightarrow \pi/E\ \frac{1}{\underset{\sim}{p}^2 - k^2}\ \delta(p_0') \tag{2.38}$$

in the non-relativistic limit $E \rightarrow \infty$, k fixed.[7]

Having analyzed the location of the poles in $d(\underset{\sim}{p}^2, p_o)$ in detail, note that Eq. (2.35) and Eq. (2.38) can be derived simply by writing

$$1/D(p') = \frac{1}{(p'^2 - k^2)} \rho(p'^2, p_o)$$

where,

$$\rho(p'^2, p'_o) = \frac{1}{E}\left[\frac{(p'^2 - k^2)/E}{[(p'^2 - k^2)/E]^2 + p'^2_o}\right]$$

and $\rho(p'^2, p'_o) \rightarrow \pi/E \;\delta(p'_o)$ for $p'^2 - k^2 = \epsilon_2 \rightarrow 0$, or $E \rightarrow \infty$ by inspection.

Finally, Eqs. (1.1), (2.29) and (2.33) yield a simple derivation of the scattering amplitude. Setting

$$\psi(x) = \int e^{i(\underset{\sim}{p}\cdot\underset{\sim}{x} - p_o x_o)}\, \psi(p)\, d^4p$$

and using Eq. (1.1)

$$\psi(\underset{\sim}{x}, o) = e^{i\underset{\sim}{p}_{in}\cdot\underset{\sim}{x}} - i\int e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} \int \frac{1}{D(p)} \int V(p - \eta)\, d^4\eta\, dp_o d^3\underset{\sim}{p}\,. \qquad (2.39)$$

Rotating contours in the manner introduced above, using $\psi'(\eta') = i\psi(\eta)$ and inserting Eq. (2.29) for $1/D(p')$

$$\psi(\underset{\sim}{x}, o) = e^{i\underset{\sim}{p}_{in}\underset{\sim}{x}} + \int e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} \frac{1}{\underset{\sim}{p}^2 - k^2} \cdot \left(\int_{C'_{\underset{\sim}{p}^2}} d(\underset{\sim}{p}^2, p'_o) \cdot T'(\underset{\sim}{p}, p'_o)\, dp'_o\right) d^3\underset{\sim}{p} \qquad (2.40)$$

with

$$T'(p') = \iint_{C'_{\underset{\sim}{n}^2}} V(p' - \eta')\, \psi'(\eta')\, d\eta'_o\, d^3\eta = T(p) \qquad (2.41)$$

Now it is well known from studies of non-relativistic scattering that in the limit $|\underset{\sim}{x}| \to \infty$

$$\int e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} \frac{1}{\underset{\sim}{p}^2 - k^2} Q(\underset{\sim}{p})\, d^3\underset{\sim}{p} \to f(\Omega_f)\, e^{ikR}/R \tag{2.42}$$

where $R = |\underset{\sim}{x}|$, f refers to the $\underset{\sim}{x}$ direction, and $f(\Omega_f) = 2\pi^2 Q(\underset{\sim}{p}_f)$ with $\underset{\sim}{p}_f = k\hat{x}$. But the second term on the right of Eq. (2.40) is precisely of the form of the term on the left in Eq. (2.42), so that the usual non-relativistic analysis yields[8] for $x_o = 0$, $|\underset{\sim}{x}| \to \infty$

$$\psi(\underset{\sim}{x}, o) \to e^{i\underset{\sim}{p}_{in}\cdot\underset{\sim}{x}} + f(\Omega_f)\, e^{ikR}/R \tag{2.43}$$

where

$$\begin{aligned} f(\Omega_f) &= \int_{C'_{k^2}} d(k^2, p'_o)\, T'(\underset{\sim}{p}_f, p'_o)\, dp'_o \\ &= \int_{C'_{k^2}} \frac{\pi}{E}\, \delta(p'_o)\, T'(\underset{\sim}{p}_f, p'_o)\, dp'_o \\ &= \frac{2\pi^3}{E}\, T'(\underset{\sim}{p}_f, 0) = \frac{2\pi^3}{E}\, T'(p_f)\, . \end{aligned} \tag{2.44}$$

III. A METHOD OF APPROXIMATE SOLUTION

Since the method to be presented here does not produce highly precise results but does serve as a rough check on other methods the following discussion will be brief.[9] Outside the restricted region of detours in the coordinate surface Eq. (2.23) indicates that $|\psi'(p')|$ will vanish rapidly as $p' \to \infty$. Specifically, $1/D(p')$ will vanish as $|1/p'|^4$ while

$$\left|T'(p')\right| = \int_{C'_{\underset{\sim}{\eta}^2}} \frac{\lambda}{(p'-\eta')^2+\mu^2} \psi'(\eta')\, d^4\eta'$$

should vanish $1/|p'|^2$ (assuming the integral over η' is amply convergent). Thus in the limit $|p'| \to \infty$ one expects $|\psi'(p')|$ to vanish as $1/|p'|^6$ (reinforcing the assumption that the integral for T' is amply convergent). This fact suggests that one can obtain good approximations to $\psi'(\underset{\sim}{p}, p'_0)$ by solving a cut-off version of Eq. (2.23), i.e., by considering the restriction of Eq. (2.23) to a large sphere of radius $\Lambda > k$ in p', η' four-space. For $E < 2m + \mu$ it has been commented that $V(p'-\eta')$ is never singular for p' and η' on the coordinate surface. Then for p' and η' within a finite cut-off sphere, $V(p'-\eta')$ can be well approximated by a finite sum of separable potentials. Thus in the elastic scattering range one can take

$$V(p'-\eta') \simeq \sum_{\ell} \sum_{i=1}^{N_\ell} \sum_{j=1}^{N_\ell} V^{\ell}_{ij}\; g^{\ell}_i(|\underset{\sim}{p}|, p'_0)\, g^{\ell}_j(|\underset{\sim}{n}|, \eta'_0)\, P_\ell(\hat{\underset{\sim}{p}})\, P_\ell(\hat{\underset{\sim}{n}}) \tag{3.1}$$

where V^{ℓ}_{ij} are real and g_i are real for real p' and analytic on the whole coordinate surface. But with the right hand side of (3.1) substituted for V in

Eq. (2.23), the new equation is as easy to solve as the non-relativistic equation with a finite sum of (three dim.) separable potentials. Using matrix algebra analogous to the non-relativistic case, setting

$$f(\Omega_f) = \sum f_\ell P_\ell(\hat{p}_f) , \quad f_\ell = \frac{2\ell + 1}{k} e^{i\delta_\ell} \sin\delta_\ell$$

and with Eq. (2.35) for $\mathrm{Im}[1/D(p')]$ one finds

$$\tan\delta_\ell = \frac{2\pi^3 k}{(2\ell+1)E} G_\ell^t V^\ell \left(I - U^\ell V^\ell\right)^{-1} G_\ell \tag{3.2}$$

(with all factors real so elastic unitarity is valid) where matrix notation is used, G is a column matrix

$$(G_\ell)_{j,1} = g_j^\ell(k,0) , \quad I_{ij} = \delta_{ij} , \text{ and } \quad U_{ij}^\ell = \mathcal{P}\int \frac{P_\ell^2(q)\, g_i^\ell(q')\, g_j^\ell(q')}{D(q')} d^4q' . \tag{3.3}$$

The range of q' integration in Eq. (3.3) is limited to the interior of the cut-off sphere of radius Λ.

One method of getting an approximation to V of form Eq. (3.1) inside a sphere of radius Λ_f is to take a least square polynomial fit to $1/[1+x^2]$ on the interval $-2\Lambda_f \leq x \leq 2\Lambda_f$, i.e.,

$$1/[1+x^2] \cong \sum_{i=0}^{M} C_i \cdot (x^2)^i \tag{3.4}$$

Then in units such that $\mu = 1$,

$$V(p' - \eta') \cong \frac{\lambda}{\pi^2} \sum_{i=0}^{M} C_i \left[(p' - \eta')^2\right]^i$$

for p', η' inside a sphere of radius Λ_f .

Now one can expand

$$(p' - \eta')^{2i} = \sum_{j=0}^{i} \underset{\sim}{p}^j \underset{\sim}{n}^j z^j \sum_{\gamma,s,\tau,q} A^j_{\gamma,s,\tau,q} \left(p'^2\right)^\gamma \left(p_o'^2\right)^s \left(\eta_o'^2\right)^q \left(\eta'^2\right)^\tau \tag{3.5}$$

$$s\,;\,q = 0, 1, \ldots \frac{[i-j]}{2}$$

$$\gamma;\ \tau = 0, 1, \ldots, i-j-2s;\ i-j-2s$$

and since

$$z^j = \sum_{\ell=0}^{j} a_{j\ell} P_\ell(z) \rightarrow \sum_{\ell=0}^{j} a_{j\ell} P_\ell(\hat{\underset{\sim}{p}}) P_\ell(\hat{\underset{\sim}{n}}) \;,$$

one has

$$V \cong \sum_{\ell=0}^{M} P_\ell(\hat{\underset{\sim}{p}}) \underset{\sim}{p}^\ell P_\ell(\hat{\underset{\sim}{n}}) \underset{\sim}{n}^\ell \sum_{i=1}^{N_{M-\ell}} \sum_{j=1}^{N_{M-\ell}} V^\ell_{ij}\, g_i\left(p_o'^2, p_o'^2\right) g_i\left(\eta'^2, \eta_o'^2\right) \tag{3.6}$$

where $g_\gamma(q'^2, q_o'^2)$ belong to the set of functions $(q'^2)^s\ (q_o'^2)^t$ for $\tau = 0, 1, \ldots,$ $[(M-\ell)/2]$ and $s = 0, 1, \ldots, M-\ell-2\tau$. The dimension of V^ℓ is $N_{M-\ell}$ where $N_q = \left([q/2] + 1\right)\left(q + 1 - [q/2]\right)$.

The constants V^ℓ_{ij} can be expressed in terms of the constants C_i, $A^j_{\gamma,s,\tau,q}$, and $a_{j\ell}$ by simple arithmetic. The required operations for evaluating V^ℓ_{ij} canbe carried out rapidly by machine. Also, the integrals U^ℓ_{ij} can be done in closed form and expression Eq. (3.2) for $\tan\delta_\ell$ evaluated by machine.[9]

To investigate convergence as a function of the cut-off Λ, the fit to V was kept fixed with $\Lambda_f = 4.0$ and polynomial degree $M = 10$. The resulting $\tan\delta_\ell$ was evaluated for several values of Λ up to and including Λ_f. Convergence as a function of M was investigated by setting $\Lambda = \Lambda_f = \Lambda_{min}$ and varying M up to 10. To standardize this procedure, Λ_{min} was chosen to satisfy

$\left(\Lambda_{min}^2 - k^2\right)^2 = E^2\Lambda_{min}^2$, since then $|p'| = \Lambda_{min}$ roughly marks the beginning of the asymptotic region of $1/D(p')$.

From the results with variable Λ and fixed M an estimate was made of the error incurred by cutting off the equation at $\Lambda = \Lambda_{min}$. The results with $\Lambda = \Lambda_{min}$ and variable M were used to estimate the precise cut-off result at $\Lambda = \Lambda_{min}$. Finally the estimated result at $\Lambda = \Lambda_{min}$ was combined with the estimated error due to cut-off to give an estimate of the non-approximated result. This scheme was only carried out fully for $m = \mu = 1$, $\ell = 0$. The results for $\lambda = 1$, $k^2 = .4$ are listed in Table I. The estimated value of $E/2k \tan \delta_o$ for Eq. (2.21) cut-off at $\Lambda = \Lambda_{min} = 2.52$ is $3.4 \pm .2$. The estimated error due to this cut-off is $-.2$. Thus, with generous allowance for possible error, one can set $E/2k \tan \delta_o = 3.6 \pm .4$. Schwartz and Zemach find for this same case $E/2k \tan \delta_o = 3.5640 \pm .0002$.

The principal difficulty in the approach used here in that polynomial fits to $1/\left[1 + x^2\right]$ are difficult to handle numerically. Thus the degree of convergence was not great, and in some cases it was non-existent. The method worked best at $k^2 = 0$ where for the least positive bound state λ it yielded $.765 \pm .021$, whereas Schwartz and Zemach find a value of .76222.

IV. CONCLUSION

The principal advantage of viewing the Bethe-Salpeter equation in momentum space is that $1/D(p)$ and $V(p - \eta)$ have a simple algebriac form, in sharp contrast to the form of their transforms in configuration space. The only difficulty with these functions in momentum space is the abundance of poles they possess due to the Lorentz metric. By making the modified Wick rotation suggested here one does not eliminate these singularities, but one does restrict their occurrence in the equation to a bounded region. All remaining singularities must be considered carefully in order to arrive at meaningful results. This has only been attempted here for purely elastic scattering $E < 2m + \mu$ where only singularities in $1/D(p)$ must be considered. For higher values of E the task of analysis is more complex. However, the technique of modified rotation, in that it minimizes the added complexity, and emphasizes the simple analytic structure of the equation for all values of E, should provide a useful point of view for further analysis.

ACKNOWLEDGEMENT

The author would like to thank Professor Herman Feshbach for his advice and encouragement, and to thank SLAC for giving him the opportunity to put this paper in final form.

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6. This result corresponds to Schwartz and Zemach's Eq. (3.11).

7. Again the result agrees with Schwartz and Zemach's Eqs. (4.2) and (4.6).

8. This result has been previously obtained via analysis of the Fourier transform of 1/D (see Ref. 2 and: Arthur R. Swift and Benjamin W. Lee, J. Math. Phys. 5, 908 (1964).

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TABLE I

Computed Values of $(E/2k)\ \tan\delta_o$
at $k^2 = .4,\ \lambda = 1,\ m = \mu = 1.$

A. Values for variable Λ with $M = 10$

	Λ	$(E/2k)\ \tan\delta_o$
Λ_{min} =	2.52	3.133
	3.0	3.219
	3.5	3.266
	3.75	3.280
	4.0	3.290
	Limit	3.32 ± .03

B. Values for variable M at $\Lambda = \Lambda_{min}$

M	$(E/2k)\ \tan\delta_o$
7	3.149
8	3.208
9	3.268
10	3.312
Limit	3.4 ± .2

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FIGURE CAPTIONS

1a. Cuts in $\psi(p, p_o)$ for $p^2 > k^2$ (cuts start at $\pm\ \alpha(p) = \pm\left(\omega_{min}(p) - i\epsilon\right)$; arrows indicate continuation in p_o).

1b. Cuts in $\psi(p, p_o)$ for $p^2 \leq k^2$.

2. Location of poles in $1/D(p)$ relative to C_{p^2} (arrows indicate relocation of poles under $\epsilon = \epsilon_1 \rightarrow \epsilon_1 + i\epsilon_2$).

3a. Continuation of $T(p, p_o)$ by moving p_o from $p_o = 0$ along C_{p^2}. All η_o contours have previously been deformed to C_{n^2} after using $\mu \rightarrow \mu - i\epsilon'$ to locate poles in $V(p-\eta)$.

3b. Continuation along C'_p. Arrows are of length $M(n^2) - M(p^2)$. (In the lower half plane the arrows would point in the opposite direction.)

3c. Separation between η_o on C_{n^2} and poles in V for $E < 2m + \mu$. Arrow is of length $\left(\left[(p-n)^2 + \mu^2\right]^{1/2} - M(p^2) - M(n^2)\right)$.

3d. Continuation of $T(p, p_o)$ up to the Wick cuts in $\psi(p, p_o)$. Arrow length is $M(n^2) - M(p^2)$.

3e. Shifting the p_o contour for a specific p, $p^2 < k^2$, in steps of size Δ through the first and third quadrants. Shaded area indicates initial strip of analyticity in $T(p, p_o)$ for this p. (The validity of the procedure shown here is generally independent of small details in contour shape. However, two criteria must be satisfied: 1. The maximum horizontal separation between any two of the original or shifted contours, e.g., C_{p^2} and C_{n^2}, must be given by the difference between their maximum real part extensions, $\left|M(p^2) - M(n^2)\right|$. 2. In shifting horizontally from C_{p^2} to $C_{p^2} + \Delta$, some points on C_{p^2} may be displaced less than Δ but none may be displaced more than Δ. These criteria are easy to meet, as the above choice of linearly segmented contours demonstrates.)

3f. Shifting the p_o contour for a specific $\underset{\sim}{p}$, $\underset{\sim}{p}_1^2 > \underset{\sim}{p}^2 > \underset{\sim}{p}_o^2 = k^2$, in steps of size Δ_i through the left half plane. Shading indicates initial strip of analyticity in $T(\underset{\sim}{p}, p_o)$. (The criteria mentioned under Fig. 3e are again observed.)

4. Transformed coordinate surface.

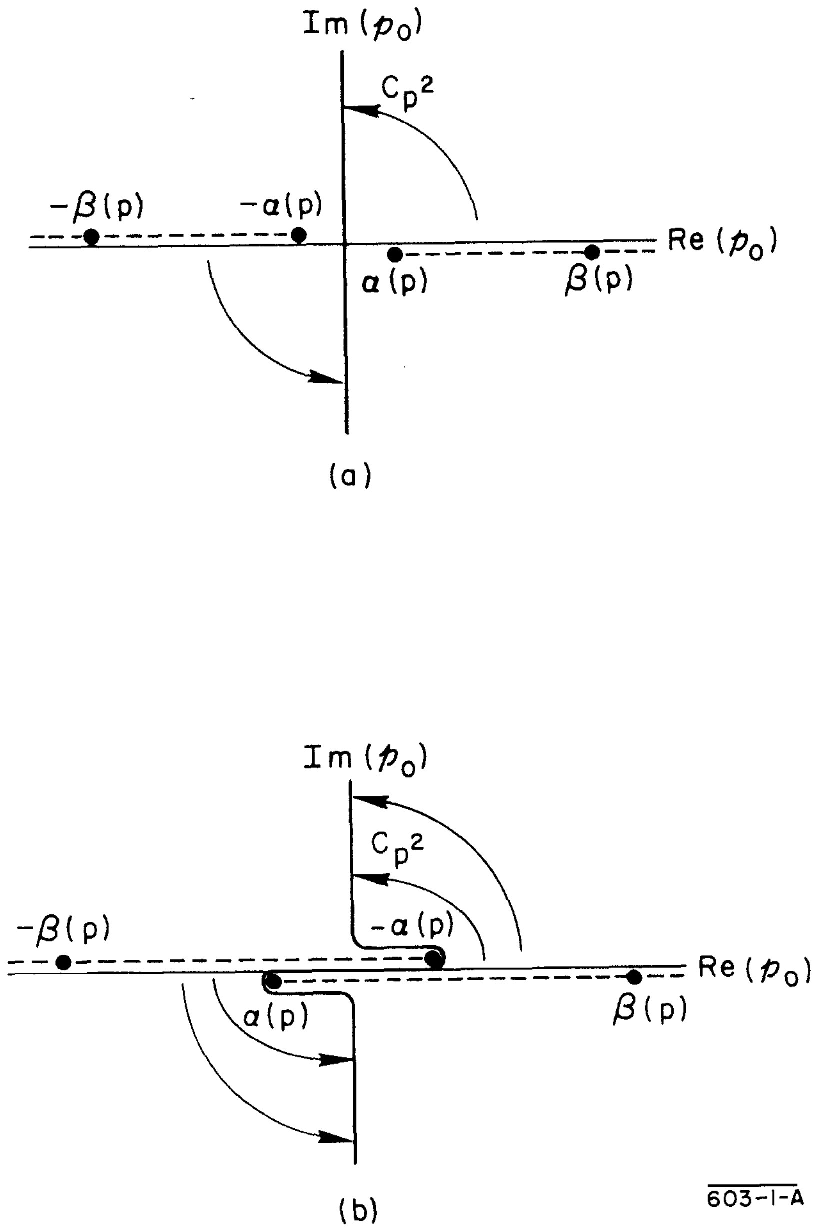
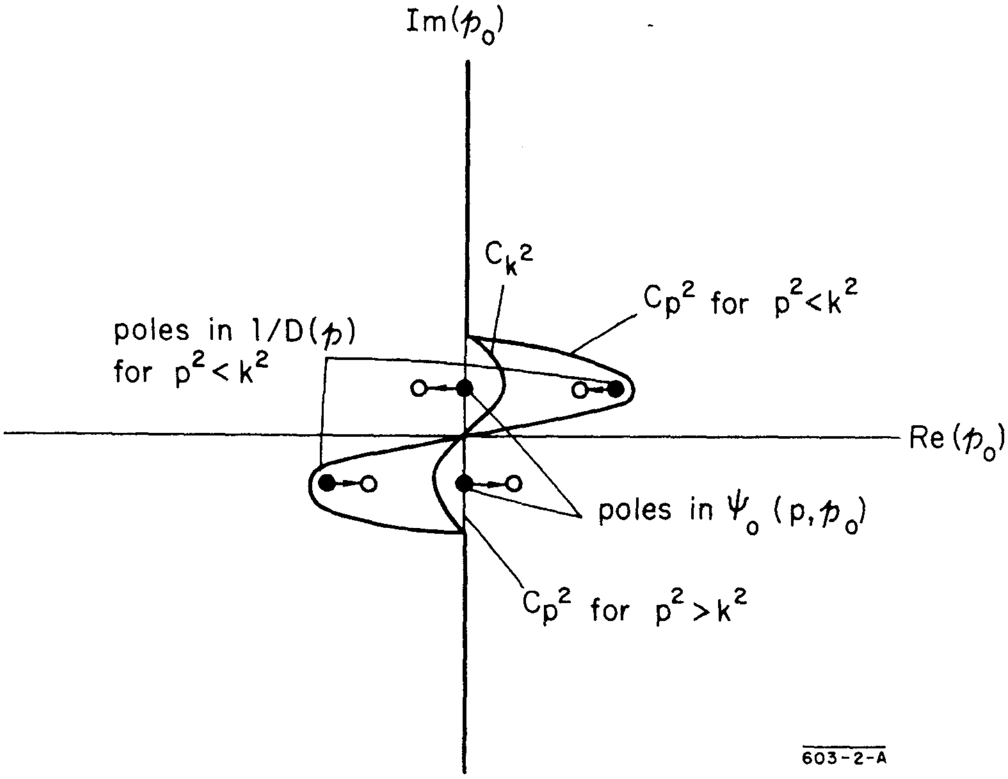
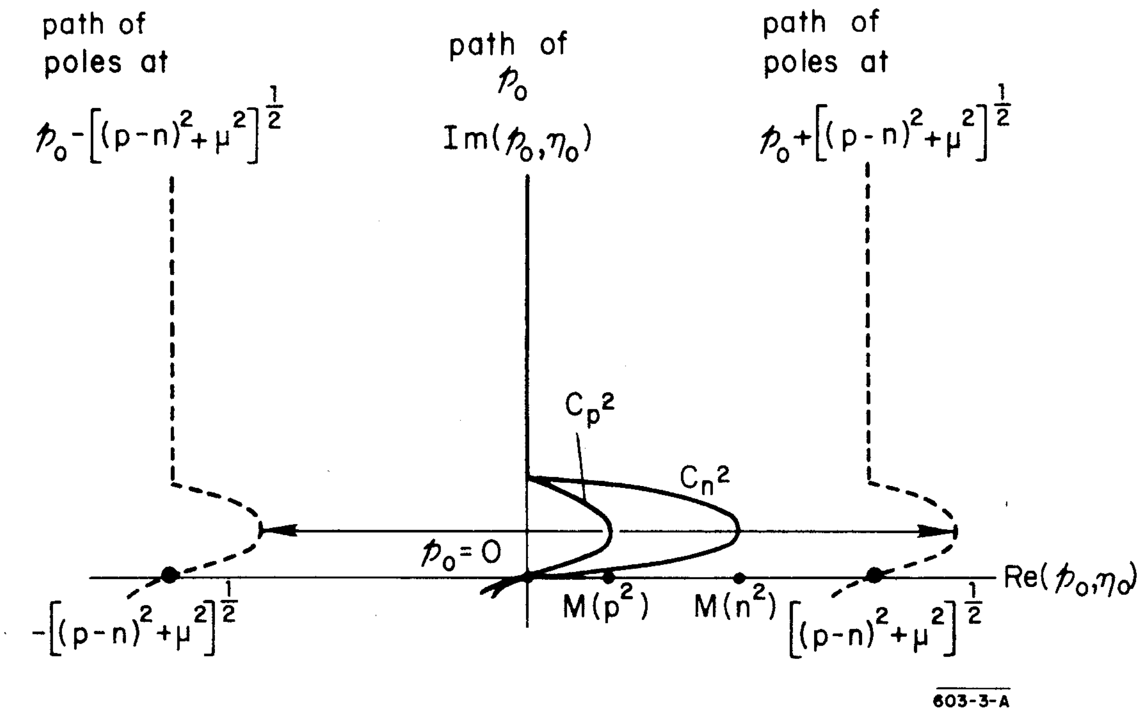


FIG. 1



Im(p_0)
$C_k{}^2$
$C_p{}^2$ for $p^2 < k^2$
poles in $1/D(p)$
for $p^2 < k^2$
Re (p_0)
poles in ψ_0 (p, p_0)
$C_p{}^2$ for $p^2 > k^2$
603-2-A

FIG. 2



path of
poles at
$p_0 - [(p-n)^2 + \mu^2]^{\frac{1}{2}}$
path of
p_0
$Im(p_0, \eta_0)$
path of
poles at
$p_0 + [(p-n)^2 + \mu^2]^{\frac{1}{2}}$
$C_p{}^2$
$C_n{}^2$
$p_0 = 0$
$Re(p_0, \eta_0)$
$-[(p-n)^2 + \mu^2]^{\frac{1}{2}}$
$M(p^2)$
$M(n^2)$
$[(p-n)^2 + \mu^2]^{\frac{1}{2}}$
603-3-A

FIG. 3a

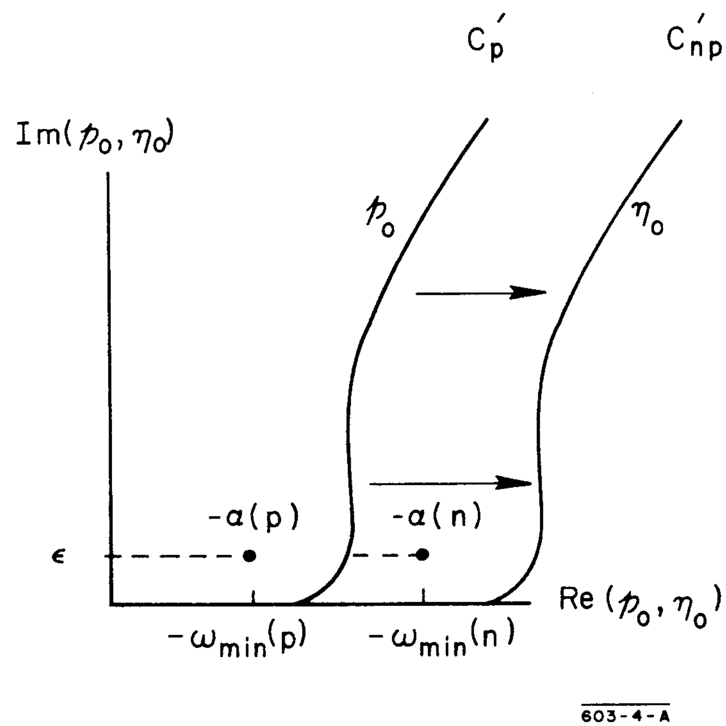


FIG. 3b

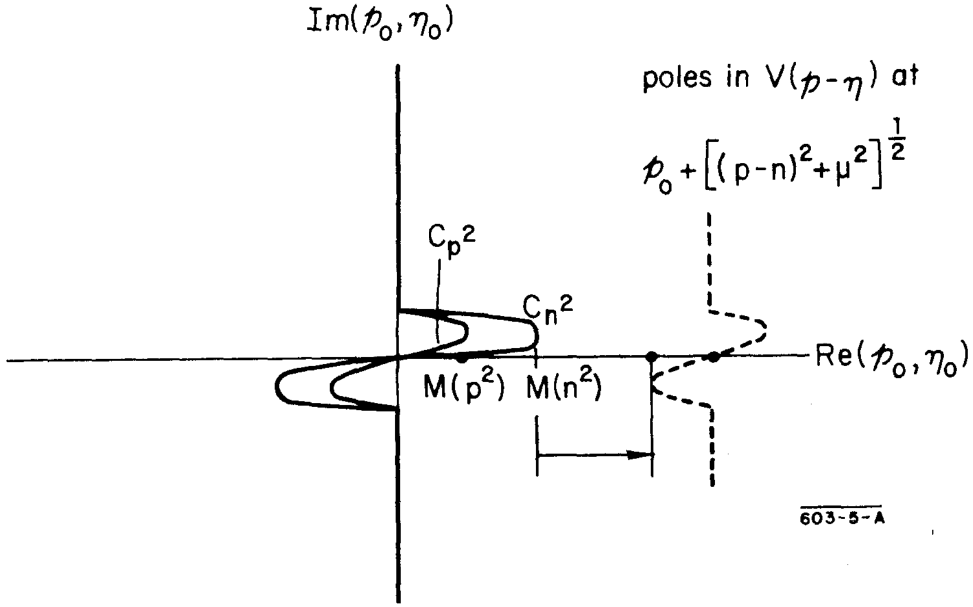

Im(p_0, η_0)
poles in V(p-η) at
$p_0+\left[(p-n)^2+\mu^2\right]^{\frac{1}{2}}$
C_{p^2}
C_{n^2}
Re(p_0, η_0)
M(p^2)
M(n^2)
603-5-A

FIG. 3c

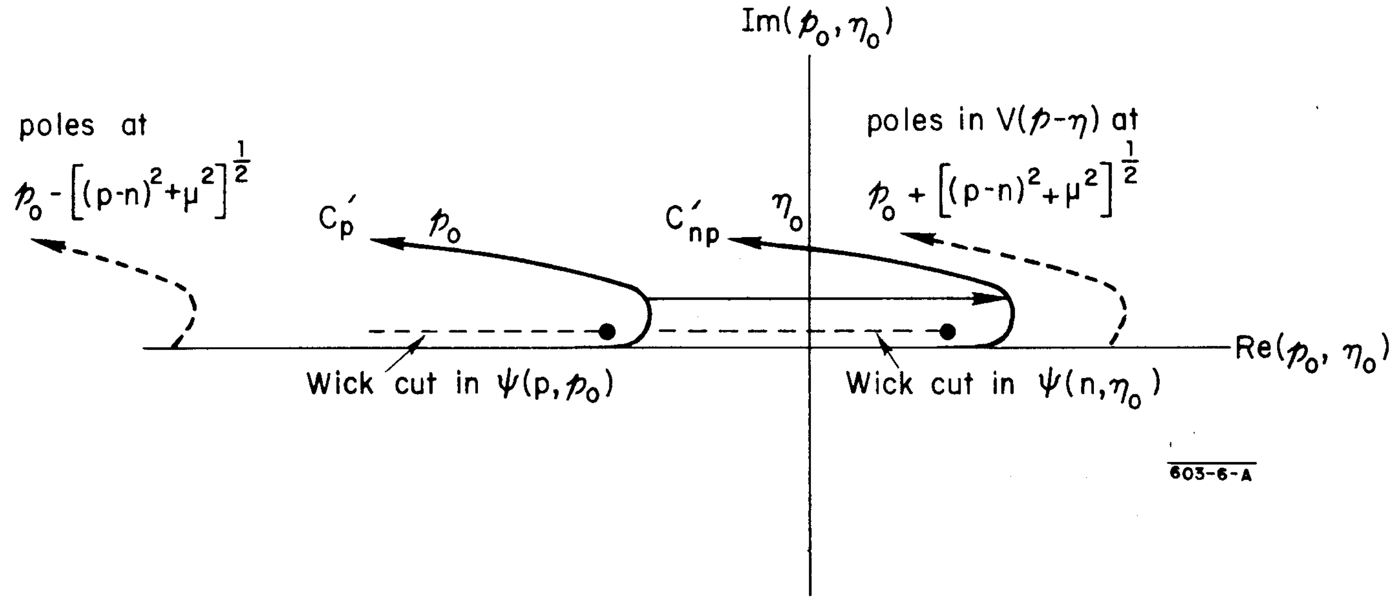
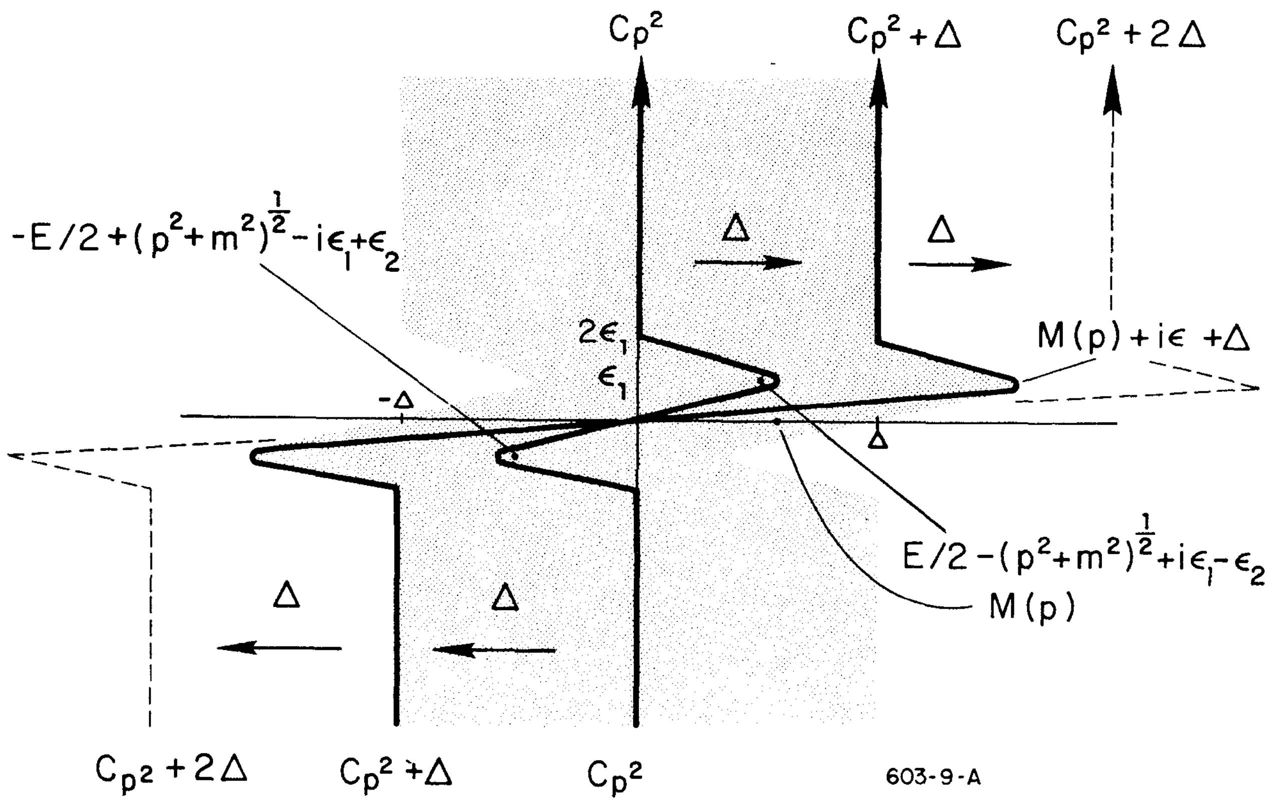
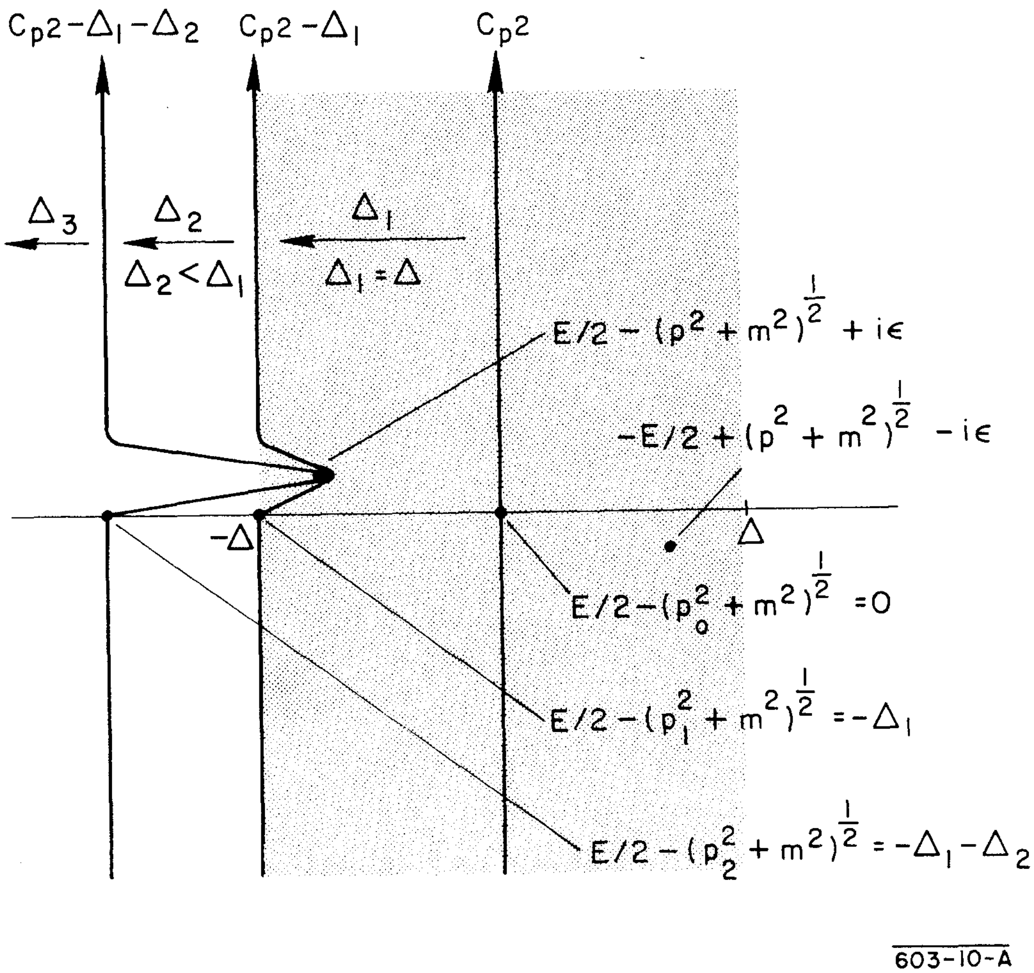

Im(p_0, η_0)
poles at
$p_0 - [(p-n)^2+\mu^2]^{\frac{1}{2}}$
C_p'
p_0
C_{np}'
η_0
poles in V($p-\eta$) at
$p_0 + [(p-n)^2+\mu^2]^{\frac{1}{2}}$
Re(p_0, η_0)
Wick cut in $\psi(p, p_0)$
Wick cut in $\psi(n, \eta_0)$
603-6-A

FIG. 3d



Cp^2
$Cp^2 + \Delta$
$Cp^2 + 2\Delta$
$-E/2+(p^2+m^2)^{\frac{1}{2}}-i\epsilon_1+\epsilon_2$
Δ
Δ
$2\epsilon_1$
ϵ_1
$M(p)+i\epsilon+\Delta$
$-\Delta$
Δ
$E/2-(p^2+m^2)^{\frac{1}{2}}+i\epsilon_1-\epsilon_2$
$M(p)$
Δ
Δ
$C_{p2}+2\Delta$
$Cp^2+\Delta$
Cp^2
603-9-A

FIG. 3e



$C_p2-\Delta_1-\Delta_2$
$C_p2-\Delta_1$
C_p2
Δ_3
Δ_2
$\Delta_2<\Delta_1$
Δ_1
$\Delta_1=\Delta$
$E/2-(p^2+m^2)^{\frac{1}{2}}+i\epsilon$
$-E/2+(p^2+m^2)^{\frac{1}{2}}-i\epsilon$
$-\Delta$
Δ
$E/2-(p_0^2+m^2)^{\frac{1}{2}}=0$
$E/2-(p_1^2+m^2)^{\frac{1}{2}}=-\Delta_1$
$E/2-(p_2^2+m^2)^{\frac{1}{2}}=-\Delta_1-\Delta_2$
603-10-A

FIG. 3 f

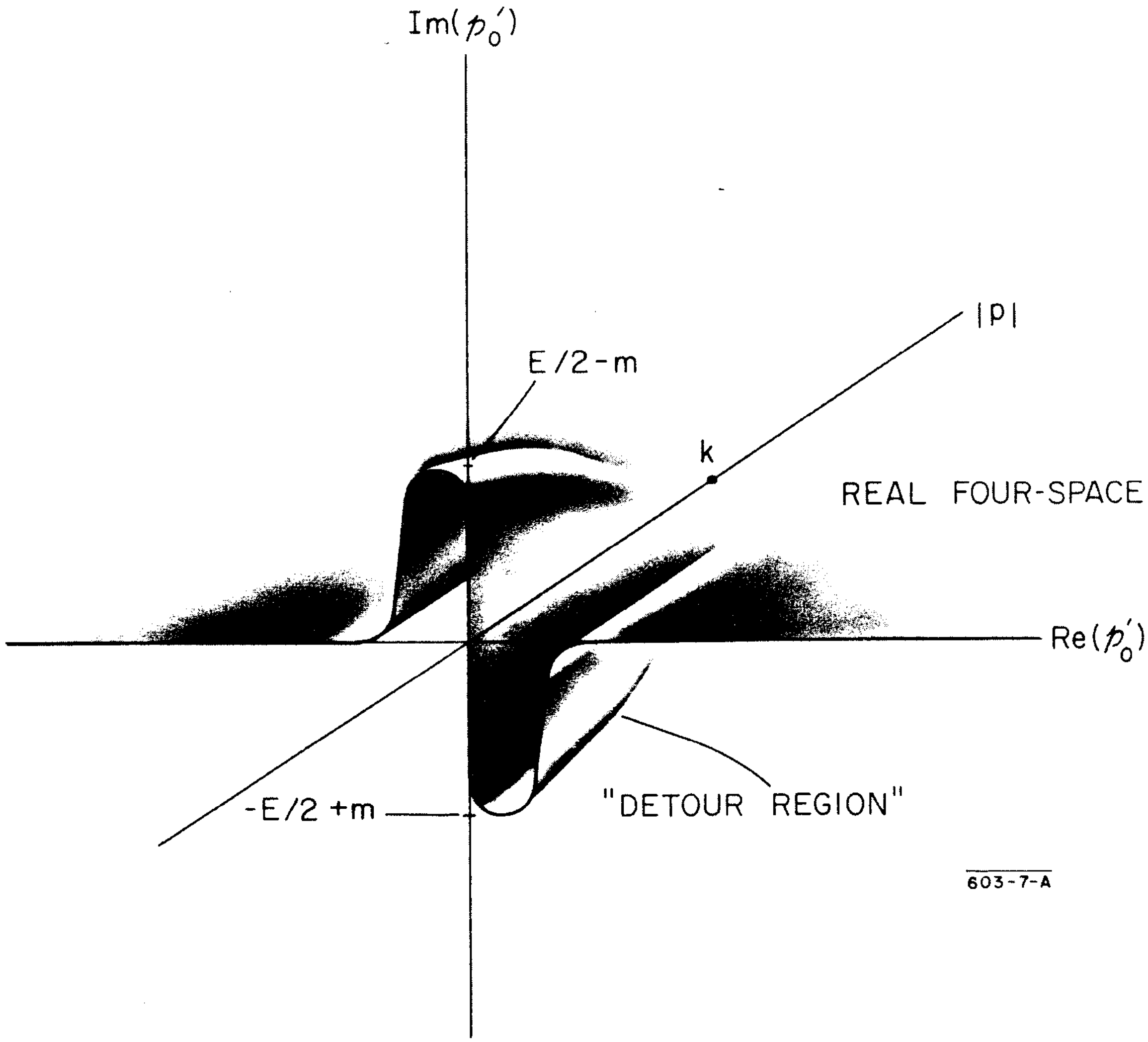


FIG. 4