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PAPER

Witnessing criticality in non-Hermitian systems via entopic uncertainty relation

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Abstract

Non-Hermitian systems with exceptional points lead to many intriguing phenomena due to the coalescence of both eigenvalues and corresponding eigenvectors, in comparison to Hermitian systems where only eigenvalues degenerate. In this paper, we propose an alternative and accurate proposal based on the entropy uncertainty relation (EUR) to detect the exceptional points and identify different phases of the non-Hermitian systems. In particular, we reveal a general connection between the EUR and the exceptional points of non-Hermitian system. Compared to the unitary Hermitian dynamics, the behaviors of EUR in the non-Hermitian system are well defined into two different ways depending on whether the system is located in unbroken or broken phase regimes. In the unbroken phase regime where EUR undergoes an oscillatory behavior, while in the broken phase regime where the oscillation of EUR breaks down. Moreover, we identify the critical phenomena of non-Hermitian systems in terms of the EUR in the dynamical limit. It is found that the EUR can detect exactly the critical points of non-Hermitian systems beyond (anti-)PT symmetric systems. Finally, we comment on the prospective experimental situation.

1. Introduction

In conventional quantum mechanics, the Hamiltonian of system is required to be Hermitian to ensure real eigenvalues and unitary dynamical evolution of the system. Recently, a class of non-Hermitian Hamiltonians with parity-time (PT) symmetry exhibiting real eigenvalues have triggered numerous attention [1–3]. It has been demonstrated that such a non-Hermitian Hamiltonian in the parameter space usually undergoes the real-to-complex eigenvalue transition. Interestingly, this critical transition occurs at an special point which is known as the exceptional point resulting in the coalescence of both eigenvalues and eigenvectors [4–6]. This point is unique to non-Hermitian systems where many new features inaccessible in Hermitian systems can be observed, such as unidirectional invisibility [7], loss-induced lasing [8, 9], and the optimal brachistochrone problem [10], and so on. What's more, the concept of exceptional point has also been successfully applied to enhance sensing sensitivity [11, 12], as well as fight against decoherence of open systems [13, 14].

Apart from the intriguing phenomena link to the exceptional point, a deeper understanding of critical behavior and quantum phase transition of non-Hermitian systems are also expected. In general, a non-Hermitian system with PT-symmetry exhibits two distinct phases. One is the unbroken phase in which the energy eigenvalues are real, and the other is the broken phase where there exist the complex eigenvalues. A spontaneous symmetry-breaking transition occurs from the unbroken to the broken phases. Consequently, a fundamental and interesting question is how to effectively characterize and capture critical transition in the non-Hermitian systems.

In the present work, we provide the witness based on the entropy uncertainty relation (EUR) [15] to accurately capture signatures of critical transition, compared to the Hilbert–Schmidt speed (HSS) [16]. The phenomena and properties of EUR in Hermitian systems have studied extensively due to potential applications ranging from quantum entanglement witnessing [17–22], to quantum key distribution [23] and quantum cryptography [24]. Here we extend the EUR dynamics to non-Hermitian ones, and demonstrate how the EUR is used to faithfully detect and witness the exceptional points of non-Hermitian systems. The primary motivation for such an extension is at least two-fold: first, there has been a growing interest in non-Hermitian systems based on the fact that many intriguing phenomena inaccessible in Hermitian systems can be observed. Therefore, we would wander whether the EUR can display some new phenomena in the non-Hermitian system. Second, recently several approaches have been proposed to detect and witness criticality in non-Hermitian systems such as the HSS [16], multiple quantum coherence [25], and the corresponding Hermitian factorization surface [26]. These quantities are sensitive to the quantum criticality, and often served as a signature of transition points in Hermitian systems. However, these quantities are state-dependent, especially for some initial states, they have a difficult in witnessing criticality of non-Hermitian systems. EUR is a natural measure of uncertainty in information theory and plays an important role in quantum foundations. Hence, we would wander whether the EUR can be instead employed as an accurate and efficient tool in detecting the critical phenomena of non-Hermitian systems.

To address the above issues, we first demonstrate the behaviors of EUR in non-Hermitian systems and reveal a link between the EUR and the critical points of non-Hermitian system. In contrast to the dynamics of EUR in Hermitian system case, we find that the dynamics of EUR can be well defined from the unbroken to the broken phase regimes. In the unbroken phase regime, the EUR exhibits an oscillatory behavior, while the oscillation of the EUR breaks down in the broken phase regime. Besides, we identify an unique criticality based on the EUR in the long time limit around the critical point, above which the EUR witness increases asymptotically but below which the EUR witness decays asymptotically. Therefore, there exists a sudden change of the EUR witness at critical points which are referred as to the exceptional points of non-Hermitian system. In comparison to the HSS witness, especially for some special initial states, our proposed witness can detect accurately and efficiently the critical behavior of the non-Hermitian systems. Finally, we give a brief discussion of the possible experimental situation.

2. Entropic uncertainty relation

In this section we first briefly review the definition of uncertainty relation. Historically, the uncertainty principle $\Delta x\Delta p \geqslant \hbar/2$, originally proposed by Heisenberg [27], captures that one cannot simultaneously predict the position and momentum with certainty. Subsequently, it has been further formulated by Robertson [28] and generalized to arbitrary pairs of incompatible observables R and Q with $\Delta R\cdot\Delta Q \geqslant \frac{1}{2}|\langle\psi|[R,Q]|\psi\rangle|$, here $\Delta R(\Delta Q)$ denotes the standard deviations and $[R,Q]=RQ-QR$ is the commutator. However, this uncertainty relation in terms of the commutator has a fatal defect because the bound is strongly related to the state $|\psi\rangle$ of system. Specifically, there is a trivial result when $\langle\psi|[R,Q]|\psi\rangle$ is equal to zero for some specific states even though R and Q do not share any common eigenvectors. To overcome this drawback, Deutsch [15] developed the so-called EUR in an information theoretical framework instead of the standard deviation, which later was presented by Maassen and Uffink [29]

$$\mathrm{EUR}\equiv H(R)+H(Q)\geqslant -2\log_2 c, \tag{1}$$

where $H(R(Q))$ denotes the Shannon entropy of the probability distribution of the outcomes when $R(Q)$ is performed on any quantum state ρ of system, respectively. $c=\max_{i,j}|\langle\varphi_i|\psi_j\rangle|$ quantifies the complementarity between R and Q with their corresponding eigenvectors $|\varphi_i\rangle$ and $|\psi_j\rangle$. In fact, the right-hand side of the above inequality can be improved by adding the von-Neumann entropy $S(\rho)=-\mathrm{Tr}(\rho\log_2\rho)$ [30, 31].

Note that equation (1) is a natural measure of uncertainty for a single system A. Extending to bipartite systems with a quantum memory B, the above EUR can be further improved [32]

$$\mathrm{EUR}\equiv S(R|B)+S(Q|B)\geqslant -2\log_2 c+S(A|B), \tag{2}$$

where $S(X|B)$ is the conditional von Neumann entropy of the post measurement states $\rho_{XB}=\sum_X(|\phi_X\rangle\langle\phi_X|\otimes I)\rho_{AB}(|\phi_X\rangle\langle\phi_X|\otimes I)$ and $|\phi_X\rangle$ are the eigenvectors of $X\in(R,Q)$. Particularly, if the system is initially uncorrelated with a quantum memory system, equation (2) reduces to equation (1).

In the following, we adopt the EUR given by equations (1) and (2) to characterize the critical behaviors of the non-Hermitian systems, and reveal a link between the EUR and the exceptional points of non-Hermitian systems.

3. Dynamics of a general two-level non-Hermitian system

To demonstrate the behaviors of EUR in a non-Hermitian system, we take a general two-level non-Hermitian system whose Hamiltonian reads as [2]

$$\mathcal{H}_{\mathrm{NH}} = \begin{pmatrix} r\,\mathrm{e}^{\mathrm{i}\phi} & \sigma \\ s & r\,\mathrm{e}^{-\mathrm{i}\phi} \end{pmatrix}. \tag{3}$$

It is straightforward to check that this non-Hermitian Hamiltonian is not symmetric. The energy eigenvalues of this Hamiltonian $\mathcal{H}_{\mathrm{NH}}$ are $E_{\pm} = r\cos\phi \pm \sqrt{s\sigma - r^2\sin^2\phi}$ and the corresponding eigenvectors are $|E_{\pm}\rangle = \frac{1}{\sqrt{2\cos\Theta}}\begin{pmatrix} \mathrm{e}^{\pm\mathrm{i}\Theta/2} \\ \pm\mathrm{e}^{\mp\mathrm{i}\Theta/2} \end{pmatrix}$, respectively. Defining $\sin\Theta = \sqrt{r^2\sin^2\phi/s\sigma}$, two regions are separated by the point $s\sigma = r^2\sin^2\phi$ where the phase transition takes place from an unbroken phase to a broken phase. For $s\sigma > r^2\sin^2\phi$, the system is denoted in unbroken phase regime and exhibits real spectra, while for $s\sigma < r^2\sin^2\phi$, the system is termed in broken phase regime where complex conjugate eigenvalues emerge.

On the other hand, equation (3) yields a PT symmetric Hamiltonian for $s = \sigma$, i.e., $(PT)H(PT)^{-1} = H$ with parity P and time reversal operations T [33]. Physically, such a non-Hermitian Hamiltonian is often served as phenomenological descriptions of an open quantum system, such as the source-and-sink model consisted of two coupled microwave cavities, one belongs to a source and the other contains a sink [34], or a coupled pair of wave guides, one with loss and the other with gain [35].

The dynamics of non-Hermitian system is determined by the time dependent Schrödinger equation (throughout the rest of the paper, we have set $\hbar = 1$ to simplify expressions)

$$\mathrm{i}\frac{\partial|\Psi(t)\rangle}{\partial t} = \mathcal{H}_{\mathrm{NH}}|\Psi(t)\rangle, \tag{4}$$

with the non-unitary time evolution operator $\mathcal{U}(t) = \exp[-\mathrm{i}\mathcal{H}_{\mathrm{NH}}t]$ which is obtained as,

$$\mathcal{U}(t) = \mathrm{e}^{-itr\cos\phi}\begin{pmatrix} \cos\omega t + \dfrac{r\sin\phi\sin\omega t}{\omega} & -\mathrm{i}\dfrac{\sigma\sin\omega t}{\omega} \\ -\mathrm{i}\dfrac{s\sin\omega t}{\omega} & \cos\omega t - \dfrac{r\sin\phi\sin\omega t}{\omega} \end{pmatrix}, \tag{5}$$

where $\omega = \sqrt{s\sigma - r^2\sin^2\phi}$. However, at the exceptional point, namely, $s\sigma = r^2\sin^2\phi$, the time evolution operator becomes

$$\mathcal{U}(t) \approx \mathrm{e}^{-itr\cos\phi}\begin{pmatrix} 1 + tr\sin\phi & -itr\sin\phi \\ -itr\sin\phi & 1 - tr\sin\phi \end{pmatrix}. \tag{6}$$

It is worthwhile to mention that, the non-Hermitian Hamiltonian given by equation (3) with real spectra can be mapped into a Hermitian one via the Hermitian transformation [36–38], e.g., $\eta\mathcal{H}_{\mathrm{NH}}\eta^{-1} = H$, where the Hermitian matrix is $\eta = \frac{1}{\sqrt{\cos\Theta}}\begin{pmatrix} \cos(\Theta/2) & -\mathrm{i}\sin(\Theta/2) \\ \mathrm{i}\sin(\Theta/2) & \cos(\Theta/2) \end{pmatrix}$. Under this condition, the non-unitary time evolution operator given by equation (5) reduces to an unitary evolution one as a consequence of Hermiticity of H

$$\mathcal{U}(t) = \mathrm{e}^{-itr\cos\phi}\begin{pmatrix} \cos\omega t & -\mathrm{i}\sin\omega t \\ -\mathrm{i}\sin\omega t & \cos\omega t \end{pmatrix}. \tag{7}$$

Given an arbitrary initial state $|\Psi(0)\rangle$, one can express it as a superposition of $|E_{\pm}\rangle$, i.e., $|\Psi(0)\rangle = \sin(\frac{\theta}{2})|E_+\rangle + \cos(\frac{\theta}{2})|E_-\rangle$ which evolves in time according to $|\Psi(t)\rangle = \mathcal{U}(t)|\Psi(0)\rangle = \sin(\frac{\theta}{2})\mathrm{e}^{-\mathrm{i}E_+t}|E_+\rangle + \cos(\frac{\theta}{2})\mathrm{e}^{-\mathrm{i}E_-t}|E_-\rangle$. Unfortunately, due to the nonorthogonality of eigenvectors $|E_{\pm}\rangle$, we have to transform this eigenvector representation into the computational orthonormal vectors $\{|0\rangle, |1\rangle\}$ via the similarity transformation

$$|0\rangle = \frac{1}{\sqrt{2\cos\Theta}}\left(\mathrm{e}^{\mathrm{i}\Theta/2}|E_+\rangle + \mathrm{e}^{-\mathrm{i}\Theta/2}|E_-\rangle\right) \tag{8a}$$

$$|1\rangle = \frac{1}{\sqrt{2\cos\Theta}}\left(\mathrm{e}^{-\mathrm{i}\Theta/2}|E_+\rangle - \mathrm{e}^{\mathrm{i}\Theta/2}|E_-\rangle\right). \tag{8b}$$

In this respect, the time evolution state $|\Psi(t)\rangle$ can be rewritten as

$$|\Psi(t)\rangle = \left(\alpha\,\mathrm{e}^{-\mathrm{i}E_+t} - \mathrm{e}^{-\mathrm{i}\Theta}\,\beta\,\mathrm{e}^{-\mathrm{i}E_-t}\right)|0\rangle + \left(\mathrm{e}^{-\mathrm{i}\Theta}\alpha\,\mathrm{e}^{-\mathrm{i}E_+t} + \beta\,\mathrm{e}^{-\mathrm{i}E_-t}\right)|1\rangle, \tag{9}$$

where $\alpha = \frac{1}{2}\sec\Theta(\cos\frac{\theta}{2} + e^{i\Theta}\sin\frac{\theta}{2})$ and $\beta = \frac{1}{2}\sec\Theta(e^{i\Theta}\cos\frac{\theta}{2} - \sin\frac{\theta}{2})$.

4. Entropic uncertainty relation in non-Hermitian systems

To reveal a relationship between the EUR and the exceptional point of non-Hermitian system, we first calculate the EUR which is only related to the outcomes of probability. Choosing a pair of projective operators as observables $X \in (R, Q)$ which are represented by $P_X = 1/2(I + \vec{n}\cdot\vec{\sigma})$, and $\vec{n} = (n_1, n_2, n_3)$ is an unit vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$, the probability distribution of measurement outcome becomes

$$p_X = \frac{1}{N}\{2\alpha\beta^*[n_3\cos\Theta - i(n_2 - \sin\Theta)] + 2\alpha^*\beta[n_3\cos\Theta + i(n_2 - \sin\Theta)]e^{2i\Delta Et} + [(n_2\sin\Theta - 1)\sec^2\Theta - n_1\sin\theta]e^{i\Delta Et}\}, \tag{10}$$

where $N = 4i\sin\Theta(\alpha^*\beta - \alpha^*\beta\, e^{2i\Delta Et}) - 2\sec^2\Theta\, e^{i\Delta Et}$. From above expression, the outcome probability is depended on the difference of energy eigenvalues $\Delta E \equiv E_+ - E_- = 2\sqrt{s\sigma - r^2\sin^2\phi}$. In particular, for $\Delta E = 0$, namely $s\sigma = r^2\sin^2\phi$ at which a exceptional point takes place, and the probability distribution becomes

$$p_X = \frac{1}{4}\left[2 - 2n_2 + \frac{2(n_2 + n_3\cos\theta + n_1\sin\theta + 2n_3 rt\sin\phi)}{1 + r^2t^2 + rt(2\cos\theta\sin\phi - rt\cos 2\phi)}\right]. \tag{11}$$

Suppose the initial system is prepared in a maximal coherence pure state, $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. According to the equation (9), a normalized density at a later time t is

$$\rho(t) = \begin{pmatrix} \frac{\sigma^2\sin^2\omega t + (\omega\cos\omega t + r\sin\phi\sin\omega t)^2}{2T\omega^2} & \frac{[\omega\cos\omega t + (is - r\sin\phi)\sin\omega t][\omega\cos\omega t - (i\sigma - r\sin\phi)\sin\omega t]}{2T\omega^2} \\ \frac{[\omega\cos\omega t - (is + r\sin\phi)\sin^2\omega t][\omega\cos\omega t + (i\sigma + r\sin\phi)\sin\omega t]}{2T\omega^2} & \frac{s^2\sin^2\omega t + (\omega\cos\omega t - r\sin\phi\sin\omega t)^2}{2T\omega^2} \end{pmatrix}, \tag{12}$$

where $T = \cos^2\omega t + \frac{(s^2 + \sigma^2 + 2r^2\sin^2\phi)\sin^2\omega t}{2\omega^2}$. Taking two complementary observables $Q \equiv \sigma_x$ and $R \equiv \sigma_z$, (namely, choosing $\vec{n} = (\pm 1, 0, 0)$ and $\vec{n} = (0, 0, \pm 1)$ for σ_x and σ_z respectively), the lower bound of the EUR in equation (1) is always 1, while its EUR takes the form as

$$\mathrm{EUR} = -\sum_{i=1}^{2} p_x^i \log_2 p_x^i - \sum_{i=1}^{2} p_z^i \log_2 p_z^i \tag{13}$$

with

$$p_x^1 = \frac{4\omega^2\cos^2\omega t + (s + \sigma)^2\sin^2\omega t}{4T\omega^2},$$

$$p_x^2 = \frac{[4r^2\sin^2\phi + (s - \sigma)^2]\sin^2\omega t}{4T\omega^2},$$

$$p_z^1 = \frac{s^2\sin^2\omega t + (\omega\cos\omega t - r\sin\phi\sin\omega t)^2}{2T\omega^2},$$

$$p_z^2 = \frac{\sigma^2\sin^2\omega t + (\omega\cos\omega t + r\sin\phi\sin\omega t)^2}{2T\omega^2}.$$

For comparison, we have also considered the case where the time evolution is generated by a Hermitian Hamiltonian. Figure 1(a) numerically shows the dynamics of EUR for the initial state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ in both Hermitian and non-Hermitian cases. Obviously, the behavior of EUR in a Hermitian system displays Rabi-type oscillation with the period $2\pi/\omega$ due to intrinsically unitary dynamical evolution. However, things become interesting for the non-Hermitian case. There are three distinct behaviors relating to the exceptional point $s\sigma = r^2\sin^2\phi$. More specifically, for $s\sigma > r^2\sin^2\phi$, the system in unbroken phase has pure real eigenvalues, and the EUR undergoes an periodic oscillatory behavior with $T = \pi/\omega$. This result is not surprised because of the fact that the non-Hermitian system with real eigenvalues is equivalent to Hermitian system by a similarity transformation [39, 40]. While for $s\sigma < r^2\sin^2\phi$ in broken phase regime, the system with complex spectra turns up, the oscillation of EUR breaks down. Particularly, at the exceptional point, a phase transition occurs from an unbroken phase to a broken phase regime, the behavior of EUR increases asymptotically to a stable value. This implies the exceptional point marks the

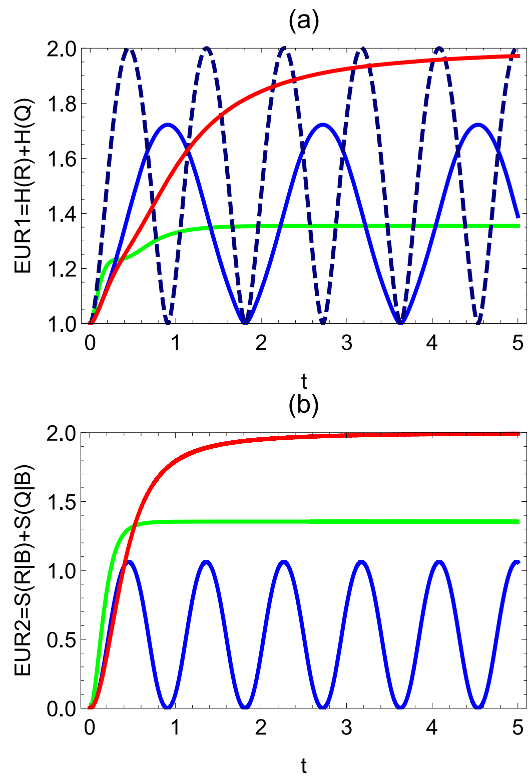

Figure 1. The EUR for different initial states $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ with observables $Q = \sigma_x$ and $R = \sigma_z$ with respect to t in both Hermitian and non-Hermitian systems ((a) $N = 1$ and (b) $N = 2$). Blue dashed curve depicts the behavior of EUR under the Hermitian Hamiltonian. Blue solid, green solid and red solid curves correspond to $s\sigma > r^2 \sin^2\phi$ (e.g., $s = \sigma = 2$ and $r = 1$), $s\sigma < r^2 \sin^2\phi$ (e.g., $s = \sigma = 1$ and $r = 2$) and exceptional points $s\sigma = r^2 \sin^2\phi$, respectively. Other parameters are fixed $\phi = \pi/2$.

oscillatory and non-oscillatory behaviors of the EUR. On the other hand, the similar results are also observed in figure 1(b) where the EUR is described by von-Neumann entropy in the presence of initial quantum correlation $|\Psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Again, the EUR is well defined by the exceptional point changing from oscillations to oscillations broken.

It is worthy mentioning that, the oscillatory or revival dynamical behavior of system has proved to be as an indicator of non-Markovian dynamics leading to information retrieval from the environment to the system [41, 42]. Indeed, a non-Hermitian Hamiltonian is used to characterize and describe an open quantum system. Hence the physical origin behind oscillatory or revival of EUR in the unbroken phase could be attributed to the back-flow of information between the environment and the system, while the unidirectional information flow from the system to the environment leads to a monotone increasing EUR. From the measure of uncertainty view, one can reduce or eliminate the uncertainty of a pairs of incompatible observables if more information is obtained. In other words, one can predict measurement outcome more accurately in the unbroken regime compared to the broken regime due to the fact that the information retrieval from the environment can be retrieved in the unbroken phase, but not in the broken phase.

5. Entropic uncertainty relation as a signature of the exceptional points of non-Hermitian systems

In traditional quantum information approaches, quantum phase transitions can be characterized by the critical behavior of quantities such as entanglement [43–45], quantum Fisher information [46, 47], fidelity [48–50] and so on. These quantities are sensitive to the quantum criticality, and often served as a signature of transition points in Hermitian systems. More recently, Jahromi *et al* [16] have introduced an efficient computable tool, based on the HSS to witness the critical behavior of the non-Hermitian systems. The HSS is defined as the derivative of the Hilbert–Schmidt distance originated from an initial state with respect to changes of the parameter [51, 52]

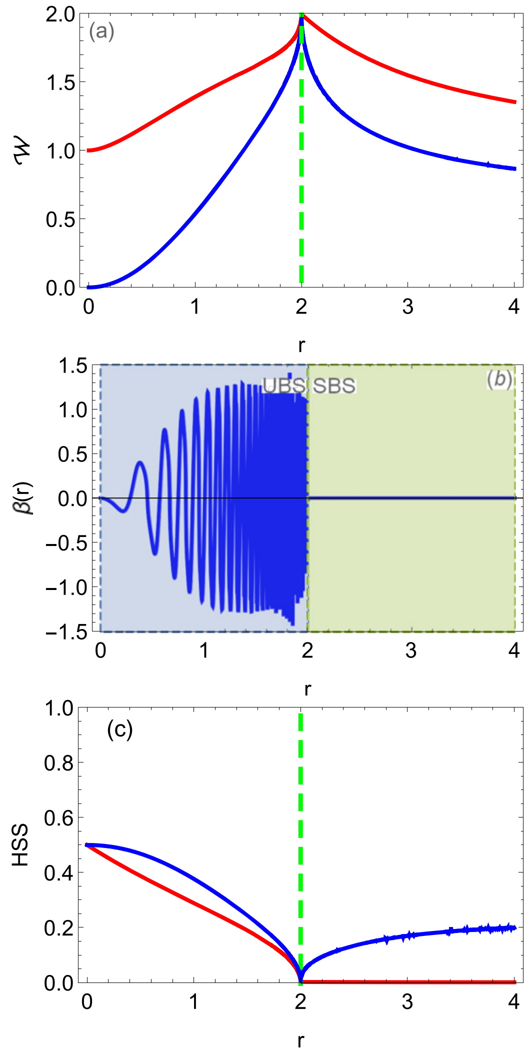

Figure 2. (a) The critical witness $\mathcal{W}$ [red solid curve depicts the critical witness $\mathcal{W}$ determined by equation (1) (Shannon entropy) for $|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\chi}|1\rangle)$ and blue solid curve depicts the critical witness $\mathcal{W}$ determined by equation (2) (von-Neumann entropy) for $|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{i\chi}|11\rangle)$], (b) the rate of change of EUR and (c) HSS witness for $|\Psi_1\rangle$ ($|\Psi_2\rangle$) in a PT symmetric non-Hermitian system with $\sigma = s = 2$ and $\phi = \pi/2$. Therefore, there is a quantum critical point at $r_0 = 2$.

$$S[\rho(\varphi)] = \frac{d}{d\varphi}\sqrt{\frac{1}{2}\,\mathrm{Tr}\,[\rho(\varphi_0+\varphi) - \rho(\varphi_0)]^2} = \sqrt{\frac{1}{2}\,\mathrm{Tr}\left[\left(\frac{d\rho(\varphi)}{d\varphi}\right)^2\right]}, \tag{14}$$

and subsequently the corresponding HSS witness is obtained as follows

$$\mathrm{HSS} = \lim_{T\to\infty}\frac{1}{T}\int_0^\infty S[\rho(\varphi)]dt. \tag{15}$$

Unfortunately, this witness is dependent of the initial state and fails to reveal the critical behavior of the non-Hermitian systems. To overcome this defect, we identify a critical witness based on the long-time average of the EUR

$$\mathcal{W} = \lim_{T\to\infty}\frac{1}{T}\int_0^\infty \mathrm{EUR}(t)dt. \tag{16}$$

In analogy to what has been performed for quantum entanglement as a signature of the quantum criticality [45], a sudden change of $\mathcal{W}$ with respect to varying system parameters occurs at the critical point implying that there exists a quantum phase transition. In the following, we demonstrate how the EUR-based witness $\mathcal{W}$ is as a critical witness to determine the critical point of non-Hermitian systems.

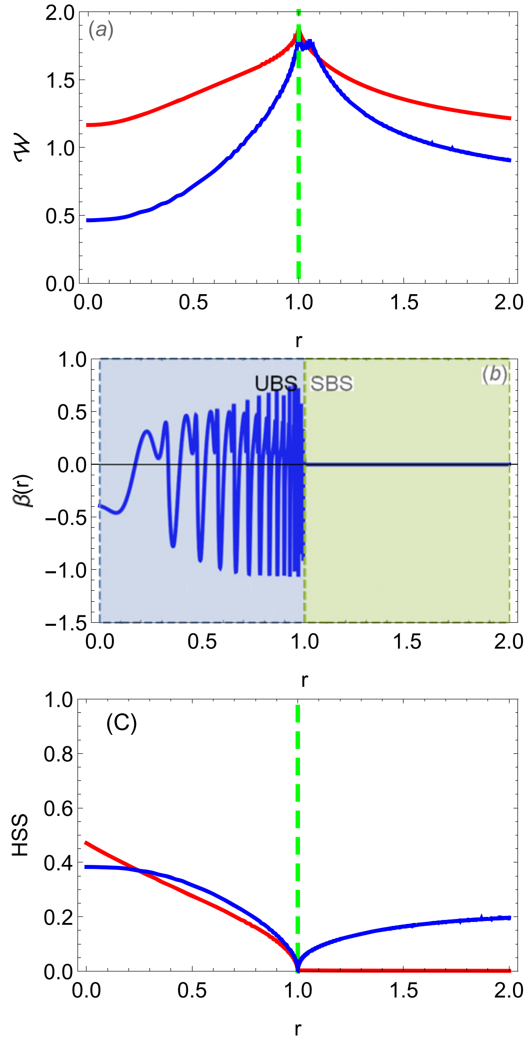

Figure 3. (a) The critical witness $\mathcal{W}$ [red solid curve depicts the critical witness $\mathcal{W}$ determined by equation (1) (Shannon entropy) for $|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\chi}|1\rangle)$ and blue solid curve depicts the critical witness $\mathcal{W}$ determined by equation (2) (von-Neumann entropy) for $|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + e^{i\chi}|11\rangle)$], (b) the rate of change of EUR and (c) HSS witness for $|\Psi_1\rangle$ ($|\Psi_2\rangle$) in a general non-Hermitian system with $\sigma = \sqrt{2}$, $s = \sqrt{2}/2$ and $\phi = \pi/2$. Therefore, there is a quantum critical point at $r_0 = 1$.

5.1. Critical witness in PT-symmetric non-Hermitian systems

In order to check the efficiency of the proposed critical witness $\mathcal{W}$, we first consider the non-Hermitian Hamiltonian given by equation (3). A quantum phase transition takes place at the exceptional point $r_0 = \sqrt{s\sigma/\sin^2\phi}$ ($r_0 = s/\sin\phi$ for a standard PT-symmetric case with $\sigma = s$).

Figure 2(a) numerically displays the behaviors of critical witness $\mathcal{W}$ in the PT-symmetric system with $s = \sigma = 2$ and $\phi = \pi/2$. Clearly, the behavior of $\mathcal{W}$ first increases asymptotically in the unbroken phase $r < 2$, and then decreases asymptotically in the broken phase $r > 2$. A abrupt change of $\mathcal{W}$ happens at the transition point $r_0 = 2$. This result implies the signature of a quantum critical point is confirmed by $\mathcal{W}$. Moreover, by comparing with the HSS-based witness shown in figure 2(c), our proposed witness provides the same sensitivity as the HSS-based witness in detecting the critical behavior of the non-Hermitian systems. On the other hand, the similar behavior is also observed for a general non-Hermitian without PT-symmetric system with $s = \sqrt{2}/2$, $\sigma = \sqrt{2}$ and $\phi = \pi/2$ as shown in figures 3(a) and (c), where a sudden change of $\mathcal{W}$ and HSS witness occur at the critical point $r_0 = 1$. Consequently, we can conclude the EUR-based witness can faithfully identify the critical behavior of the non-Hermitian systems.

To find a deeper insight into the nature of EUR with the critical point in the non-Hermitian system, we have also investigated the rate of change of EUR in dynamics limit $\beta = \lim_{t\to\infty} \frac{d}{dt}\text{EUR}(t)$. The non-vanishing value of β is corresponding to the unbroken phase regime. On the contrary, the vanishing value of β is corresponding to the broken phase regime. As expected, the results are presented in figures 2(b) and 3(b) where the non-vanishing value of $\beta(r)$ is observed in the unbroken phase regime while the zero-value of $\beta(r)$ is always in the broken phase regime.

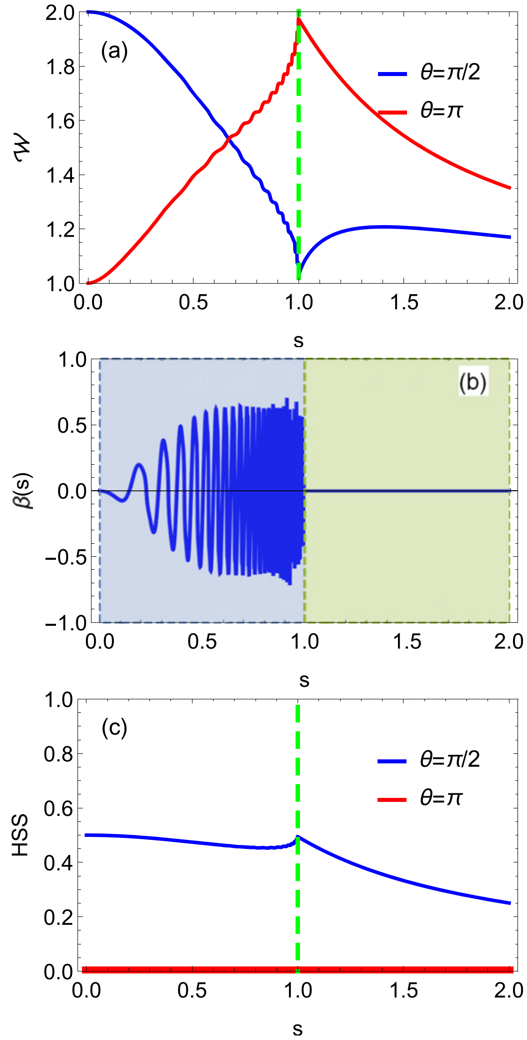

Figure 4. (a) The critical witness $\mathcal{W}$ and (c) HSS witness for different initial states $|\Psi\rangle = \cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})e^{i\chi}|1\rangle$, (b) the rate of change of EUR in an anti-PT-symmetric system with $\lambda = 1$ and $\phi = 2\pi$. Therefore, there is a quantum critical point at $s_0 = 1$.

5.2. Critical witness in anti-PT-symmetric non-Hermitian systems

Following, we go on testing the validity of the critical witness in anti-PT-symmetric non-Hermitian systems. The generalized form of a single-qubit anti-PT-symmetric Hamiltonian can be expressed as [53]

$$\mathcal{H}_{\rm NH} = \begin{pmatrix} \lambda\, e^{i\phi} & is \\ is & -\lambda\, e^{-i\phi} \end{pmatrix}, \qquad (17)$$

where both the parameters λ and s denote real numbers. It is easy to check that this Hamiltonian satisfies the anti-PT invariant Hamiltonian $(PT)H(PT) = -H$ [2, 54], and the corresponding eigenvalue spectra are given by $\varepsilon_{\pm} = i\lambda \sin\phi \pm \sqrt{\lambda^2\cos^2\phi - s^2}$. Obviously, two different phases are defined by the point $|s| = |\lambda\cos\phi|$ [53]. For $|s| > |\lambda\cos\phi|$, the system is termed in the unbroken phase, while $|s| < |\lambda\cos\phi|$ where the system is located in the broken phase.

In figure 4(a), we have plotted the critical witness $\mathcal{W}$ for different initial states $|\Psi\rangle = \cos(\frac{\theta}{2})|0\rangle + \sin(\frac{\theta}{2})e^{i\chi}|1\rangle$ in anti-PT-symmetric system with $\lambda = 1$ and $\phi = 2\pi$. As expected, for any initial states, i.e., $\theta = \pi/2$ or $\theta = \pi$, a abrupt change of $\mathcal{W}$ happens at the transition point $s_0 = 1$, and hence the EUR-based witness can effectively identify the criticality. However, the conclusion is not valid for HSS witness. For some special initial state, i.e., $\theta = \pi$, HSS witness fail to identify the critical behavior of the non-Hermitian systems as shown in figure 4(c). This result indicates our proposed witness is more accurate than HSS witness in detecting the critical behavior of the non-Hermitian systems. At the same time, the rate of change of EUR with non-vanishing to vanishing value in an anti-PT-symmetric non-Hermitian system is also observed in figure 4(b).

5.3. Critical witness in a high dimensional non-Hermitian system

We also check the efficiency of the criticality witness $\mathcal{W}$ in a high dimensional non-Hermitian system described by a 4×4 Hamiltonian [55]

$$H_{\text{NH}} = -JS_x + \text{i}\gamma S_z, \tag{18}$$

where S_x and S_z are spin-3/2 representations of the SU(2) group. The eigenvalues of this Hamiltonian are $\lambda_i = \{\pm\frac{3}{2}, \pm\frac{1}{2}\}\sqrt{J^2 - \gamma^2}$ which give rise to an exceptional point at $J = \gamma$. The real-to-complex eigenvalue transition is accompanied with a symmetry-breaking transition from the unbroken to the broken phase.

In fact, such Hamiltonian given by equation (18) is easily generalized to an arbitrary dimensional system where it still remains analytically solvable. For a initial state $\psi(0)$, the time-evolved state is determined by $\psi(t) = U(t)\psi(0)$ with time evolution operator $U(t) = \exp(-\text{i}H_{\text{NH}}t)$. The probabilities distributions of S_x and S_z over $\Psi(t)$ (a normalized wave function of $\psi(t)$) are

$$p(S_x) = |\langle\Psi(t)|\varphi_x^i\rangle|^2, \quad (i = 1, 2, 3, 4) \tag{19a}$$

$$p(S_z) = |\langle\Psi(t)|\varphi_z^j\rangle|^2, \quad (j = 1, 2, 3, 4), \tag{19b}$$

where $|\varphi_x^i\rangle$ and $|\varphi_z^j\rangle$ are eigenstates of S_x and S_z, respectively.

Figures 5(a) and 5(c) show the critical witness $\mathcal{W}$ and HSS witness for initial state $|\Psi\rangle = (\text{e}^{\text{i}\chi}|0\rangle + |1\rangle + |2\rangle + |3\rangle)/2$ in a high dimensional non-Hermitian system with $J = 2$ and $\chi = \pi/2$, respectively. Similarly, a sudden change of $\mathcal{W}$ at the transition point $\gamma_0 = 2$ is observed in figure 5(a) that is completely consistent with the HSS witness depicted in figure 5(c). On the other hand, the non-vanishing value to the vanishing value of β is also observed in figure 5(b), where a abrupt change of β marks the unbroken phase $\gamma < 2$ and the broken phase $\gamma > 2$.

5.4. Experimental feasibility

Finally, we give a brief discussion of the above predictions which are easy to implement in experiment according to our procedure. To accomplish this, we here restrict our discussions to the single-ion system where the PT-symmetric Hamiltonian with balanced gain and loss has been realized in experiment [56]. The experimental setup refers to the trapped 40Ca^+ ion in a magnetic field and be chosen four Zeeman energy levels labeled by $|0\rangle = |S_{-1/2}\rangle$, $|1\rangle = |D_{5/2}\rangle$, $|2\rangle = |S_{1/2}\rangle$ and $|3\rangle = |P_{3/2}\rangle$. First of all, the ion is initially prepared in the ground state $|0\rangle$, which is driven to the excited state $|1\rangle$ by the first laser. At the same time, another laser is switched on between $|1\rangle$ and $|3\rangle$ which decays quickly to the state $|2\rangle$. Therefore, the loss rate γ between $|1\rangle$ and $|2\rangle$ can be realized by adjusting the intensity of the second laser beam. Under this condition, an effective PT-symmetric Hamiltonian $H_{\text{eff}} = \frac{\Omega}{2}\sigma_x - \text{i}\frac{\gamma}{2}\sigma_z$ is obtained [56], here Ω is the coupling rate between $|0\rangle$ and $|1\rangle$. Compared with the equation (3), one should take $\sigma = s = \Omega/2$, $r = -\gamma/2$ and $\phi = \pi/2$. To verify the EUR dynamical features as well as criticality witness, the probabilities $p_i = \text{Tr}(\sigma_i\rho)$ should be measured from density-matrix of ρ due to the fact that the density-matrix elements of ρ can be directly measured in experiment [56]. Once the probabilities p_i have been experimentally obtained, the EUR can be also presented.

6. Entropic uncertainty relation for multiple two-level non-Hermitian systems

In the preceding part, we have demonstrated the behavior of the EUR in a single non-Hermitian system. Extend the EUR to multiple non-Hermitian systems, we wonder whether or not the relationship between the EUR and the critical points holds for multiple non-Hermitian systems? To address this problem, we consider a composite system consisting of N independent two-level non-Hermitian systems given by equation (3) associated with a pair of observables σ_X and σ_Z,

$$\sigma_X = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{2}\sigma_x^i \tag{20}$$

$$\sigma_Z = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{2}\sigma_z^i. \tag{21}$$

To evaluate the sum of Shannon entropies in equation (1), we assume the initial state of N two-level non-Hermitian systems is $|\Psi_N\rangle = |\Psi\rangle^{\otimes N}$ with $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Therefore, the EUR associated with the measurement in the Z direction can be obtained

$$H(\sigma_Z) = -\sum_{k=0}^{N} C_N^k p^{N-k}(1-p)^k \log_2 C_N^k p^{N-k}(1-p)^k \tag{22}$$

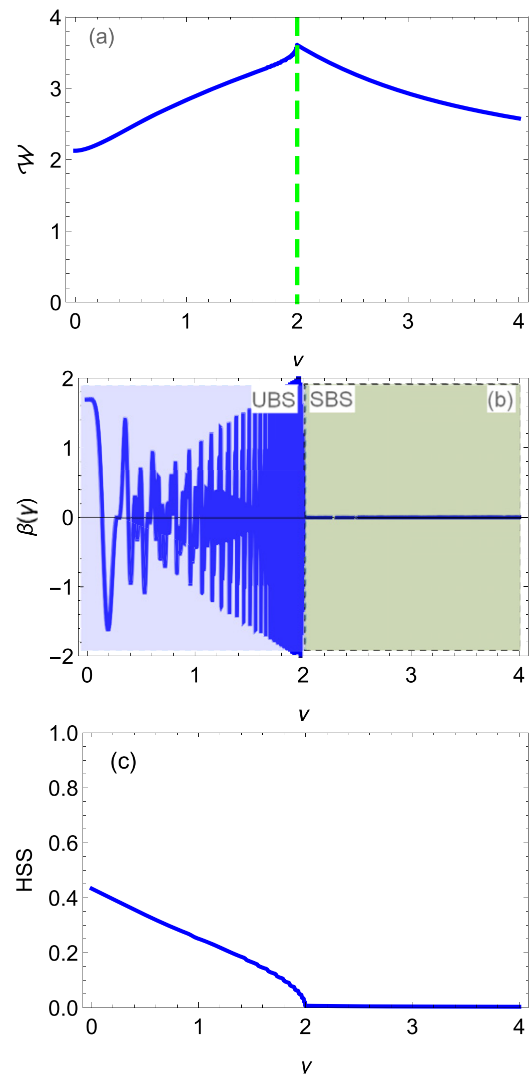

Figure 5. (a) The critical witness $\mathcal{W}$ determined by equation (1) and (c) HSS witness for $|\Psi\rangle = \frac{1}{2}(e^{ix}|0\rangle + |1\rangle + |2\rangle + |3\rangle)$, (b) the rate of change of EUR in a high dimensional non-Hermitian system with $J = 2$. Therefore, there is a quantum critical point at $\gamma_0 = 2$.

with $p = \frac{\sigma^2 \sin^2 \omega t + (\omega \cos \omega t - r \sin \phi \sin \omega t)^2}{2\omega^2 \cos^2 \omega t + (s^2 + \sigma^2 + 2r^2 \sin^2 \phi)\sin^2 \omega t}$. Here $C_N^k = \frac{N!}{(N-k)!k!}$. Similarly, the EUR associated with the measurement in the X direction is obtained by

$$H(\sigma_X) = -\sum_{k=0}^{N} C_N^k q^{N-k}(1-q)^k \log_2 C_N^k q^{N-k}(1-q)^k \tag{23}$$

with $q = \frac{4\omega^2 \cos^2 \omega t + (s+\sigma)^2 \sin^2 \omega t}{4\omega^2 \cos^2 \omega t + 2(s^2 + \sigma^2 + 2r^2 \sin^2 \phi)\sin^2 \omega t}$. Combining equation (22) with equation (23), the sum of Shannon entropies in equation (1) is obtained

$$\begin{aligned} \mathrm{EUR} &= H(\sigma_X) + H(\sigma_Z) \\ &\approx \log_2[2\pi e N\sqrt{pq(1-p)(1-q)}]. \end{aligned} \tag{24}$$

Obviously, the explicit analytic expression of EUR is related to the scale with $\log_2 N$. For $s\sigma > r^2 \sin^2 \phi$, the system is in unbroken phase and the EUR undergoes an periodic oscillatory behavior, while for $s\sigma < r^2 \sin^2 \phi$, the system is in broken phase regime and the oscillation of EUR breaks down. However, at the exceptional point $s\sigma = r^2 \sin^2 \phi$, where a phase transition from the unbroken phase to the broken phase takes place. The expression of EUR in equation (24) with the parameters p and q are replaced by p' and q', respectively,

$$p' = \frac{1}{2} - \frac{tr\sin\phi}{1 + 2t^2r^2\sin^2\phi}$$
$$q' = \frac{1}{2} + \frac{1}{2 + 4t^2r^2\sin^2\phi}, \tag{25}$$

the behavior of EUR increases asymptotically to a stable value. As a result, the EUR in a non-Hermitian system is well defined by the exceptional point changing from oscillations to oscillations broken.

7. Conclusion

In conclusion, we have provided the witness based on the EUR to detect the exceptional points and identified different phases of the non-Hermitian systems. To established a link between the criticality of non-Hermitian systems and the EUR, we have investigated the dynamics of EUR in the non-Hermitian systems. In fact, EUR for non-Hermitian systems has been discussed in [57]. However, compared with the dynamics governed by a Hermitian Hamiltonian, two different types of dynamical behaviors are observed that is closely related to the exceptional point dividing into both the unbroken and the unbroken regimes. In the unbroken regime, the EUR undergoes an oscillatory behavior, while in broken phase regime where the oscillation breaks down. At the exceptional point, the EUR increases asymptotically to a stable value. The exceptional point marks the oscillatory and non-oscillatory behavior of the EUR.

In addition, we have also identified the critical behavior in terms of the EUR in the dynamical limit where different kinds of non-Hermitian models, including a single two-level (anti-)PT-symmetric systems, high-dimensional non-Hermitian system and multiple non-Hermitian system are taken into consideration. In contrast to the HSS-based witness, especially for some initial states of the systems, our findings show that the EUR can be used as an effective and accurate index to identify the exceptional point in these models. Finally, we have also commented on the possible experimental situation. Therefore, our results not only may have potential applications to witness and detect criticality in open physical systems but also could be helpful for understanding the dynamics features of the non-Hermitian systems.

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Data availability statement

The data generated and/or analysed during the current study are not publicly available for legal/ethical reasons but are available from the corresponding author on reasonable request.

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