Expending quantization procedures can help quantum field theories and gravity

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Abstract. Sometimes, canonical quantization has difficulties that can be cured using a new quantization procedure called affine quantization. After briefly introducing this new procedure, we show that its approach to simple models ensures that a valid quantization can be obtained. These valid procedures then are used to help fields and gravity.

1. Introduction

1.1. The highlights of canonical quantization

Canonical quantization (hereafter CQ) can be summarized briefly as follows. Classical variables, $-\infty < p \;\&\; q < \infty$, and a Poisson bracket $\{q, p\} = 1$, may be promoted to the basic quantum operators, $p \to P \;\&\; q \to Q$, for which $[Q, P] = i\hbar 1\!\!1$. However, a valid quantization requires that the classical variables are Cartesian, e.g., $d\sigma^2_{CQ} = \omega^{-1}\, dp^2 + \omega\, dq^2$. In that case, a classical Hamiltonian, $H(p, q) = \mathcal{H}(p, q) \Rightarrow \mathcal{H}(P, Q)$, and the latter term is the quantum Hamiltonian, which usually is a polynomial.

These important features have been developed using coherent states, e.g., $|p, q\rangle = e^{-iqP/\hbar}\, e^{ipQ/\hbar}\, |\omega\rangle$ with $(Q + iP/\omega)\,|\omega\rangle = 0$,

$$H(p, q) = \langle p, q|\mathcal{H}(P, Q)|p, q\rangle = \langle\omega|\mathcal{H}(P + p, Q + q)|\omega\rangle \\ = \mathcal{H}(p, q) + \mathcal{O}(\hbar; p, q)\,, \tag{1}$$

which in the limit $\hbar \to 0$ leads to $H(p, q) = \mathcal{H}(p, q)$. To ensure valid results, CQ requires Cartesian phase space variables. The desired Cartesian variables can be found from the Fubini-Study metric to attain valid coordinates. In particular,

$$d\sigma^2_{CQ} = 2\hbar\,[\,\|\,d|p, q\rangle\|^2 - |\langle p, q|\; d|p, q\rangle|^2 = \omega^{-1}\, dp^2 + \omega\, dq^2\,, \tag{2}$$

an expression designed to cancel any phase factor placed on the coherent states.

A common toy model is a harmonic oscillator, now with simple parameters, in which $H = (p^2 + q^2)/2 \to \mathcal{H} = (P^2 + Q^2)/2$. This models's eigenfunctions and eigenvalues are well known, and we note that, the ground-state eigenvalue is $\hbar/2$, while the full set of eigenvalues are $E_n = \hbar(n + 1/2)$, where $n = 0, 1, 2, 3, ...$ A special fetcher of this model is that it has *equal spacing of the eigenvalues*, and in this case that spacing is $\hbar$.

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It is important to examine different classical variables for the full-harmonic oscillator, such as $q = \bar{q}^3/3$ and $p = \bar{p}/\bar{q}^2$ so that the Poisson bracket is $\{\bar{q}, \bar{p}\} = 1$. So far, such new variables may be promoted to new, and different, quantum operators, $\bar{p} \to \bar{P}$ & $\bar{q} \to \bar{Q}$. The new variables can be used for the classical Hamiltonian, and the quantum Hamiltonian would become $\bar{\mathcal{H}} = [\bar{P}(\bar{Q}^{-4})\bar{P} + \bar{Q}^6/9]/2$. Surely, this new quantum Hamiltonian does not have equally spaced eigenvalues. This example emphasizes that the full set of CQ rules are needed to get valid results. Choosing, and using, proper classical variables to promote to unique quantum operators will fully ensure valid results. Suitable coherent states lead to proper classical variables, which then can help lead to valid results [1].

Now we analyze our other quantization procedure.

1.2. The highlights of an affine quantization

Affine quantization (hereafter AQ) can be summarized briefly as follows. The primary feature of AQ is that we can remove a single coordinate point, say $q = 0$. That automatically implies that $P^\dagger \neq P$, i.e., those two operators are no longer equal. The remaining space includes $q > 0$ and $q < 0$. We discard $q < 0$ and keep $q > 0$. To restore self-adjoint operators, we introduce the dilation classical variable $d = pq \Rightarrow D = [P^\dagger Q + QP]/2 = D^\dagger$. It now follows that $[Q, D] = i\hbar Q$. Instead of Cartesian classical variables, AQ requires that they adopt a different metric, $d\sigma^2_{AQ} = (\beta\hbar)^{-1}\, q^2\, dp^2 + (\beta\hbar)\, q^{-2}\, dq^2$, which is called a constant negative curvature, here with the value $-2/(\beta\hbar)$. In that case, the classical primed-Hamiltonian function, $H'(pq, q) = \mathcal{H}'(pq, q) \Rightarrow \mathcal{H}'(D, Q)$, which is mow the quantum primed-Hamiltonian.

An uncommon toy model is the half-harmonic oscillator, using simple parameters again, for which the classical Hamiltonian is still the same, i.e., $H = (p^2 + q^2)/2$, but now $q > 0$. As a toy, it acts like a particle moving back and forth always bouncing off a wall at $q = 0$ where its momentum instantly reverses direction.

The quantum properties of this models's eigenfunctions and eigenvalues are recently well known, and we note that the eigenvalues are $E_n = 2\hbar(n + 1)$, where, as before, $n = 0, 1, 2, 3, ...$ [2]. Again, we find that a special fetcher of this model is that it also has *equal spacing of the eigenvalues*, and in this case, the spacing is $2\hbar$. In addition the ground state eigenvalue has become $2\hbar$. It is indisputable that the half-harmonic oscillator has a valid solution, and, thanks to the equal spacing and its values, that evidently points to a valid quantization of this model using AQ.

While CQ leads to valid results for the harmonic oscillator, it follows that models like $H = p^2/2 + V(q) \Rightarrow \mathcal{H} = P^2/2 + V(Q)$, provided that $-\infty < V(q) < \infty$, and this example will also lead to valid results. A similar comment holds for AQ, where $q > 0$, and using valid results for the half-harmonic oscillator, it follows that other models, like $H' = (d^2/q^2)/2 + V(q) \Rightarrow \mathcal{H}' = D\,(Q^{-2})\,D/2 + V(Q) = [P^2 + (3/4)\hbar^2/Q^2]/2 + V(Q)$, provided that $-\infty < V(q) < \infty$. Such models will also lead to valid results.[1]

We note that the previous paragraph shows how to create valid quantum Hamiltonians by first ensuring that their kinetic factors are well chosen.

[1] A natural variation of the half-harmonic oscillator uses $0 < q + b < \infty$, in which $0 \leq b < \infty$, only changes the $\hbar$ term from $(3/4)\hbar^2/Q^2$ to $(3/4)\hbar^2/(Q + b)^2$. This model also has a continuously equally spaced eigenvalues for all b, which run from $2\hbar$, for $b = 0$, down to $\hbar$, as $b \to \infty$. An excellent graph of several eigenvalues, expressed as a function of b, may be seen in [3], page 36.

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2. The Quantization of Field Theories

2.1. Examining the territory

A common model, using the momentum field, $\pi(x)$, and its field, $\varphi(x)$, has the classical Hamiltonian

$$H = \textstyle\int\{\frac{1}{2}[\pi(x)^2 + (\overrightarrow{\nabla}\varphi(x))^2 + m^2\varphi(x)^2] + g\,\varphi(x)^p\}\; d^sx\;. \tag{3}$$

Let us suppose that $\varphi(x)$ represents some feature of nature, e.g., a tiny particle named $z1$ to distinguish it from particle $z2$. Note that if $\varphi(x) = 0$, that value could not distinguish the $z1$ particle from that of the $z2$ particle, or any other particle as well. Stated briefly, nothing is lost if we remove points where $\varphi(x) = 0$. In two or more spatial dimensions, paths can pass by any points where $\varphi(x) = 0$. Moreover, the integral in (3) is completely unchanged if we remove points where $\varphi(x) = 0$.

A new topic points to possibilities that $\pi(x) = \pm\infty$, and also possibly $\varphi(x) = \pm\infty$, which can lead to $H(x) = \infty$ at some points, but still find that $H = \int H(x)\, d^sx < \infty$. Such an example could be $H(x) = 1/(x_1^2 + ... + x_s^2)^{s/2}$ integrated over $(x_1^2 + ... + x_s^2) < 1$. Path integration involves that and similar functions. Integrable infinities are acceptable in mathematics, but any field representing nature should never reach infinity.

A way to eliminate infinity expressions is to adopt affine variables, such as $\kappa(x) = \pi(x)\,\varphi(x)$ as the dilation field. Here we remove points where $\varphi(x) = 0$, and also limit all three terms from infinity, specifically such as $0 < |\varphi(x)| < \infty$, while $0 \le |\pi(x)|$ & $|\kappa(x)| < \infty$. Such limitations ensure correct values for all three functions.

This story leads to an infinity-absent, classical Hamiltonian, using affine variables, and which it is given by

$$H' = \textstyle\int\{\frac{1}{2}[\kappa(x)^2/\varphi(x)^2 + (\overrightarrow{\nabla}\varphi(x))^2 + m^2\varphi(x)^2] + g\,\varphi(x)^p\}\; d^sx\;. \tag{4}$$

It is noteworthy that the kinetic term already demands that $0 < |\varphi(x)| < \infty$ and $0 \le |\kappa(x)| < \infty$ so they can represent $\pi(x)$ correctly. Moreover, it follows that $0 < |\varphi(x)|^p < \infty$.

2.2. An affine quantization of field theories

We start with traditional CQ field operators, $\hat{\pi}(x)$ and $\hat{\varphi}(x)$. Removing $\hat{\varphi}(x) = 0$ leads to the fact that now $\hat{\pi}(x)^\dagger \ne \hat{\pi}(x)$. We follow the affine procedures and introduce the dilation quantum field, $\hat{\kappa}(x) = [\hat{\pi}(x)^\dagger\,\hat{\varphi}(x) + \hat{\varphi}(x)\,\hat{\pi}(x)]/2 = \hat{\kappa}(x)^\dagger$. Like the single particles, we find that $[\hat{\varphi}(x), \hat{\kappa}(y)] = i\hbar\delta(x-y)\,\hat{\varphi}(x)$. Just like CQ, the AQ operators need scaling and regularization to get around the facts like $[\hat{\varphi}(x), \hat{\pi}(x)] = i\hbar\,\mathbb{1}\,\infty$ and $[\hat{\varphi}(x), \hat{\kappa}(x)] = i\hbar\,\hat{\varphi}(x)\,\infty$. In this paper we develop expressions that are implicit and accepted as formal expressions. Using formal expressions still requires that their formulation be on the road toward valid results, and not on any other, eventually incorrect, type of road.

Using AQ rules for a formal quantum Hamiltonian of the field models, and using Scrödinger's representation, leads to

$$\mathcal{H}' = \textstyle\int\{\frac{1}{2}[\hat{\kappa}(x)(\varphi(x)^{-2})\hat{\kappa}(x) + (\overrightarrow{\nabla}\varphi(x))^2 + m^2\varphi(x)^2] + g\,\varphi(x)^p\}\; d^sx\;. \tag{5}$$

It is noteworthy that there have been several Monte Carlo (MC) versions of path integration, in which scaling and regularizing have still retained AQ validity possibility, which has to be a useful, and CQ-like, behavior in which the quantum Hamiltonian, again using Schrödinger's representation, becomes[2]

$$\mathcal{H}'' = \textstyle\int\{\frac{1}{2}[\hat{\pi}(x)^2 + 2\hbar^2/\varphi(x)^2 + (\overrightarrow{\nabla}\varphi(x))^2 + m^2\varphi(x)^2] + g\,\varphi(x)^p\}\; d^sx\;. \tag{6}$$

[2] The $2\hbar^2$ numerator term has been recently promoted from $(3/4)\hbar^2$ after further analysis. Clearly, this change boosts the contribution of the $\hbar$-term to the final result.

Observe, that while $\hat{\pi}(x)^\dagger \neq \hat{\pi}(x)$, and thanks to the $\hbar - term$, it ensures that in this equation both $\hat{\pi}(x)^\dagger$ and $(\hat{\pi}(x)^\dagger)^2$ *act* like $\hat{\pi}(x)$ and $\hat{\pi}(x)^2$. That property can aid commutation efforts.

The present MC studies for the model φ^4_4 using CQ, and with no extra $\hbar$-term, have provided 'free results', as if the interaction term was absent when in fact it was present, which is unacceptable. Other MC studies of the same model, now using AQ, which means that the $\hbar$-term is present, has led to 'non-free results', just as they should be, which is acceptable. That may not be the final story yet, but, at least, AQ is clearly closer to valid results than CQ is.

3. The Quantization of Einstein's Gravity

3.1. Examining the territory

Guided by introducing dilation variables and the several examples that appear, we are led to what will be called the gravity dilation field, namely $\pi^a_b(x) = \pi^{ac}(x)\, g_{bc}(x)$, with a new feature in which there is a sum over c. As required by physics, $g_{ab}(x) > 0$, meaning that $ds(x)^2 = g_{ab}(x)\, dx^a\, dx^b > 0$, provided that $\Sigma^3_{a=1}\, (dx^a)^2 > 0$.

Now, the classical ADM Hamiltonian [4], expressed in affine classical variables, uses the metric $g_{ab}(x) > 0$, $g(x) = \det[g_{ab}(x)] > 0$, and $\pi^a_b(x) = \pi^{ac}(x)\, g_{bc}(x)$, which leads to

$$\begin{aligned} H_{AQ-ADM} = \textstyle\int\{ & g(x)^{-1/2}[\pi^a_b(x)\pi^b_a(x) - \tfrac{1}{2}\,\pi^a_a(x)\pi^b_b(x)] \\ & +g(x)^{1/2}\; {}^{(3)}R(x)\}\; d^3x\,, \end{aligned} \tag{7}$$

where ${}^{(3)}R(x)$ is the Ricci scalar for 3 spatial dimensions.

Since the customary classical variables are the momentum field, $\pi^{ab}(x)$ and the metric field, $g_{ab}(x) > 0$, if follows that while the quantum metric is self-adjoint, i.e., $\hat{g}_{ab}(x)^\dagger = \hat{g}_{ab}(x)$, the quantum momentum is not self-adjoint, i.e., $\hat{\pi}^{ab}(x)^\dagger \neq \hat{\pi}^{ab}(x)$. Following previous procedures, we introduce the quantum dilation field, $\hat{\pi}^a_b(x) = [\hat{\pi}^{ac}(x)^\dagger\, \hat{g}_{bc}(x) + \hat{g}_{bc}(x)\, \hat{\pi}^{ac}(x)]/2 = \hat{\pi}^a_b(x)^\dagger$. It follows that the quantum Hamiltonian, in Schrödinger's representation, becomes

$$\begin{aligned} \mathcal{H}_{AQ-ADM} = \textstyle\int\{ & [\hat{\pi}^a_b(x)\, g(x)^{-1/2}\hat{\pi}^b_a(x) - \tfrac{1}{2}\,\hat{\pi}^a_a(x)\, g(x)^{-1/2}\hat{\pi}^b_b(x)] \\ & +g(x)^{1/2}\; {}^{(3)}R(x)\}\; d^3x\,, \end{aligned} \tag{8}$$

A special feature is that $\hat{\pi}^a_b(x)\; g(x)^{-1/2} = 0$. It is as if $\hat{\pi}^a_b(x)\, g(y)^{-1/2} = 0$, where $x \neq y$. That fact simplifies the gravity kinetic factor greatly.

The quantum gravity Hamiltonian is the center piece of a complex analysis, and it is ripe for solution. A valid quantization of the classical Hamiltonian is required to lead the constraints further toward a potentially valid quantization of gravity. A procedure to include constraints and their role in quantum gravity has been presented in [5].

4. Summary

This paper has stressed that a new quantum procedure, i.e., AQ, is prepared to find valid quantizations for problems with incomplete coordinate spaces. The results of the full- and half-harmonic oscillators separately confirmed that they performed valid quantization for CQ and AQ. Applying AQ to field theories and to gravity, exploited the use of appropriate dilation and coordinate variables. It was found that there was no longer any integrable infinities, as nature requires. While both quantum fields and quantum gravity using CQ, have been well examined, they have not as yet been able to claim either one of them as satisfactory valid quantizations. Quite possibly, these problems may be closer to validity using AQ, as this paper has implied. Several articles have already aimed at finding better field theories and gravity results; see [5 - 18].

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