DICE2012 IOP Publishing
Journal of Physics: Conference Series **442** (2013) 012058 doi:10.1088/1742-6596/442/1/012058

Generalized creation and annihilation operators via complex nonlinear Riccati equations

Dieter Schuch

Institut für Theoretische Physik, J.W. Goethe-Universität Frankfurt am Main, Max-von-Laue-Str. 1, D-60438 Frankfurt am Main, Germany

E-mail: Schuch@em.uni-frankfurt.de

Octavio Castaños

Instituto de Ciencias Nucleares, Universidad National Autónoma de México, Apdo. Postal 70-543, 04510 México D.F.

E-mail: ocasta@nucleares.unam.mx

Oscar Rosas-Ortiz

Departamento de Física, Cinvestav, Apdo. Postal 14-740, 07000 México, D.F.

E-mail: orosas@fis.cinvestav.mx

Abstract.
Based on Gaussian wave packet solutions of the time-dependent Schrödinger equation, a generalization of the conventional creation and annihilation operators and the corresponding coherent states can be obtained. This generalization includes systems where also the width of the coherent states is time-dependent as they occur for harmonic oscillators with time-dependent frequency or systems in contact with a dissipative environment. The key point is the replacement of the frequency ω_0 that occurs in the usual definition of the creation/annihilation operator by a complex time-dependent function that fulfils a nonlinear Riccati equation. This equation and its solutions depend on the system under consideration and on the (complex) initial conditions. Formal similarities also exist with supersymmetric quantum mechanics. The generalized creation and annihilation operators also allow to construct exact analytic solutions of the free motion Schrödinger equation in terms of Hermite polynomials with time-dependent variable.

1. Introduction

In classical as well as quantum mechanics, emphasis is placed on Lagrangians, Hamiltonians and potentials, that is, on quantities with the dimension of energy. However, already for the harmonic oscillator (HO) with time-dependent (TD) frequency $\omega = \omega(t)$, the Hamiltonian is no longer a constant of motion. Not to mention dissipative systems with friction forces where from the onset definition of an appropriate Hamiltonian is problematic. However, the quantity that is actually quantized in quantum mechanics is not energy but action since Planck's constant h (or $\hbar = h/2\pi$) is a quantum of action. For example, the fact that the energy of the HO can be quantized is only a consequence of the constant frequency of oscillation, $\omega = \omega_0$.

In the following, it will be shown that a *dynamical* (so-called Ermakov) *invariant* [1] with the dimension of *action* can still exist for systems like the HO with TD frequency or for open systems with a friction force linear proportional to velocity (or momentum), in other words, where the energy or the Hamiltonian is no longer a constant of motion. Factorization of the corresponding operator (in terms of a complex variable fulfilling a nonlinear (NL) Riccati equation) yields generalized creation and annihilation operators for these systems that also allow for construction of the associated coherent states (CS).

In section 2, the standard factorization method, Gaussian wave packets (WP) and CS definitions will be recollected in order to introduce our notation. For treatment of the above-mentioned systems one must go beyond the minimum uncertainty CS with constant width and consider WPs with time-dependent width. The corresponding equations of motion for the width, as well as for the maximum, of the WPs and the resulting dynamical invariant will be discussed in detail in section 3. Here formal similarities with supersymmetric quantum mechanics will also be mentioned. In section 4 generalized creation/annihilation operators are obtained by factorizing the operator corresponding to the Ermakov invariant and generalized CS can be constructed in the usual way.

For some effective models an exact Ermakov invariant also exists for the description of dissipative systems. As an example, an approach based on a modified Lagrange–Hamilton formalism and another using a nonlinear modification of the time-dependent Schrödinger equation (TDSE) will be presented in section 5 along with their interrelation. Section 6 illustrates how generalized creation/annihilation operators (here in the conservative case) can be applied to obtain exact solutions of the TDSE for the free motion in terms of Hermite polynomials with a TD argument. A short summary concludes the paper.

2. Factorization method, wave packets and coherent states

An algebraic method to solve the SE of the HO was proposed by Schrödinger himself [2] although, in principle, this can already be found in Dirac's seminal book [3]. For this purpose, the Hamiltonian operator of the (one-dimensional) HO is expressed in terms of the operators a and a^+ (which is the adjoint of a) as

$$H_{op} \;=\; \frac{1}{2m}\,p_{op}^2 \;+\; \frac{m}{2}\,\omega_0^2\,x^2 \;=\; \hbar\omega_0\,\left(a^+a \;+\; \frac{1}{2}\right) \tag{1}$$

where these operators fulfil the commutator relation $[a, a^+] = 1$ and a^+a is called number operator. This name becomes comprehensible when rewriting Eq. (1) in the conventional way by dividing it by $\hbar\omega_0$ to yield the dimensionless operator

$$\hat{H} \;=\; \frac{H_{op}}{\hbar\omega_0} \;=\; \left(a^+a \;+\; \frac{1}{2}\right)\;. \tag{2}$$

Since $\frac{H}{\omega_0}$ has the dimension of *action* (!), the number operator just counts the quanta of $\hbar$.

The annihilation operator a and the creation operator a^+ (in position representation with $p_{op} = \frac{\hbar}{i}\frac{\partial}{\partial x}$) can then easily be specified as

$$a \;=\; i\,\sqrt{\frac{m}{2\hbar\omega_0}}\,\left(\frac{p_{op}}{m} \;-\; i\,\omega_0\,x\right) \tag{3}$$

$$a^+ \;=\; -\,i\,\sqrt{\frac{m}{2\hbar\omega_0}}\,\left(\frac{p_{op}}{m} \;+\; i\,\omega_0\,x\right)\;. \tag{4}$$

The ground state wave function $\psi_0(x)$ can be obtained from the relation $a\psi_0(x) = 0$, the excited states from the property of the creation operator $a^+\psi_n(x) = \sqrt{n+1}\psi_{n+1}(x)$, yielding $\psi_n(x) = \frac{1}{\sqrt{n!}}(a^+)^n\psi_0(x)$.

Soon after he proposed his equation, Schrödinger himself [4] already found that by superimposing the solutions $\psi_n(x)$ with appropriate weight factors, a stable Gaussian WP solution $\Psi_{WP}(x,t)$ can be obtained. Generalizations of Schrödinger's approach were achieved in the description of coherent light beams emitted by lasers [5, 6, 7] leading to what is now known as CS. There are at least three different formal definitions of these CS in the literature [8] where the minimum uncertainty CS means Gaussian WPs that minimize the Heisenberg uncertainty relation. However, this can only be fulfilled by WPs with constant width. Since we wish to go further and must also consider TD widths, the definition of annihilation operator CS, i.e., eigenstate of a with eigenvalue z, will be applied in our context. In position space, $< x|z >= \Psi_{WP}(x,t)$ is valid. Therefore, in the next section, the dynamics of the two parameters specifying Gaussian WPs, maximum and width, will be examined.

3. Wave packet dynamics, Ermakov invariant and supersymmetric quantum mechanics

In general, Gaussian WP solutions of the TDSE exist for any Hamiltonian that is at most quadratic (or bilinear) in position and momentum operators, e.g.,

$$i\hbar\frac{\partial}{\partial t}\Psi_{WP}(x,t) = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \ + \ V(x,t) \ + \ W(x,p_{op};t,\Psi_{WP})\right\}\Psi_{WP}(x,t) \tag{5}$$

where $W(x,p_{op};t,\Psi_{WP})$ occurs, for instance, in effective models for dissipative systems (see section 5). First we want to consider (one-dimensional) systems without dissipation but with TD WP width or position uncertainty $< \tilde{x}^2 >=< x^2 > - < x >^2$ where $< x >= \int_{-\infty}^{+\infty} dx \Psi^* x \Psi = \eta(t)$ is the mean value of position. Well-known examples of this kind are

a) $V = 0$: free motion with

$$< \tilde{x}^2 > (t) \ = \ < \tilde{x}^2 >_0 \left[1 + (\beta_0 t)^2\right] \tag{6}$$

where the frequency-type quantity β_0 is used to abbreviate $\beta_0 = \frac{\hbar}{2m< \tilde{x}^2 >_0}$;

b) $V = \frac{m}{2}\omega_0^2 x^2$: HO, but for the case $\beta_0 \neq \omega_0$,

$$< \tilde{x}^2 > (t) =< \tilde{x}^2 >_0 \left\{\cos^2\omega_0 t + \left(\frac{\beta_0}{\omega_0}\sin\omega_0 t\right)^2\right\} =< \tilde{x}^2 >_0 \ \left\{1 + \left[\left(\frac{\beta_0^2 - \omega_0^2}{\omega_0^2}\right)\sin^2\omega_0 t\right]\right\}, \tag{7}$$

i.e., WPs with oscillating width;

c) $V = \frac{m}{2}\omega^2(t)x^2$: HO with TD frequency; the explicit form of $< \tilde{x}^2 > (t)$ depends on the specific time-dependence of $\omega(t)$.

The question we wish to pose is: in these cases, do (generalized) creation and annihilation operators also exist that provide the corresponding CSs with TD width?

For this purpose we study the time-evolution of the maximum and width of a Gaussian WP with the general form

$$\Psi_{WP}(x,t) = N(t) \ \exp\left\{i\left[y(t)\tilde{x}^2 + \frac{1}{\hbar} < p > \tilde{x} + K(t)\right]\right\} \tag{8}$$

with $y(t) =$ complex, $\tilde{x} = x - \eta(t)$, i.e., WP maximum at $x = \eta(t)$. The purely TD normalization factor $N(t)$ and phase factor $K(t)$ are not relevant at the moment and will be specified later.

Inserting ansatz (8) into the TDSE for the HO (with arbitrary frequency ω) yields, as equation of motion for the WP maximum, the classical Newton-type equation

$$\ddot{\eta} + \omega^2(t)\eta = 0 \tag{9}$$

and, for the WP width, one obtains the *complex* NL Riccati equation

$$\left(\frac{2\hbar}{m}\dot{y}\right) + \left(\frac{2\hbar}{m}y\right)^2 + \omega^2 = 0 \ . \tag{10}$$

There are different ways of treating this equation. In order to obtain the afore-mentioned action-type invariant, we transform Eq. (10) into a *real* NL differential equation. For this purpose we introduce a new variable $\alpha(t)$ that is related to the imaginary part of the complex Riccati variable via $\frac{2\hbar}{m}y_I = \frac{1}{\alpha^2(t)}$ where $\alpha(t)$ is directly proportional to the position uncertainty or WP width since $\alpha = \sqrt{\frac{2m}{\hbar} < \tilde{x}^2 >}$. The real part of this variable is related to the (relative) change in time of the width what can be expressed as $\frac{2\hbar}{m}y_R = \frac{\frac{d}{dt}<\tilde{x}^2>}{2<\tilde{x}^2>} = \frac{\dot{\alpha}}{\alpha}$ and vanishes for WPs with constant width. Inserting this into the Riccati equation (10), finally turns it into the so-called Ermakov equation for $\alpha(t)$,

$$\ddot{\alpha} + \omega^2(t)\ \alpha = \frac{1}{\alpha^3} \ . \tag{11}$$

This equation was rediscovered (and renamed) several times and studied also in a quantum mechanical context [9].

Eliminating ω^2 from Eq. (11) by using Eq.(9) for the WP maximum, leads to the dynamical invariant

$$I_L \ = \ \frac{1}{2}\left[(\dot{\eta}\alpha - \dot{\alpha}\eta)^2 \ + \ \left(\frac{\eta}{\alpha}\right)^2\right] \ = \ \text{const.} \tag{12}$$

This form, expressed in terms of η, α and their time-derivatives, was used by Hartley and Ray [10] to obtain generalized creation/annihilation operators for the HO with TD frequency but is no longer applicable for open dissipative systems. A further generalization can be achieved to also include these systems if the Ermakov invariant is expressed in terms of η, $\dot{\eta}$ and $\left(\frac{2\hbar}{m}y\right)$ as

$$\begin{aligned} I_L \ &= \ \frac{1}{2}\alpha^2\left[\left(\dot{\eta} - \frac{\dot{\alpha}}{\alpha}\eta\right)^2 \ + \ \left(\frac{\eta}{\alpha^2}\right)^2\right] \ = \ \frac{1}{2}\alpha^2\left[\left(\dot{\eta} - \left(\frac{2\hbar}{m}y_R\right)\eta\right)^2 \ + \ \left(\left(\frac{2\hbar}{m}y_I\right)\eta\right)^2\right] \\ &= \ \frac{1}{2}\alpha^2\left[\left(\dot{\eta} - \left(\frac{2\hbar}{m}y\right)\eta\right)\left(\dot{\eta} - \left(\frac{2\hbar}{m}y^*\right)\eta\right)\right] \end{aligned} \tag{13}$$

where mI_L has the dimension of *action*.

After we have obtained the invariant, now in terms of $\left(\frac{2\hbar}{m}y\right)$ and in a form that can easily be factorized, let us return to the Riccati equation (10) and give some more reasons why a formulation in these terms is advantageous over the one using the real variable $\alpha(t)$. For this, we make use of the fact that the inhomogeneous Riccati equation can be transformed into a homogeneous Bernoulli equation, providing a particular solution $\left(\frac{2\hbar}{m}\tilde{y}\right)$ is known. The general solution of (10) is then given by $\frac{2\hbar}{m}y = \frac{2\hbar}{m}\tilde{y} + \frac{2\hbar}{m}v(t)$, where $\frac{2\hbar}{m}v(t)$ fulfils the Bernoulli equation

$$\left(\frac{2\hbar}{m}\dot{v}\right) + 2\left(\frac{2\hbar}{m}\tilde{y}\right)\left(\frac{2\hbar}{m}v\right) + \left(\frac{2\hbar}{m}v\right)^2 \ = \ 0 \tag{14}$$

and the coefficient of the linear term, $A = 2\left(\frac{2\hbar}{m}\tilde{y}\right)$, depends on the particular solution. This equation can be linearized using $\left(\frac{2\hbar}{m}v\right) = \frac{1}{\kappa(t)}$ to yield $\dot{\kappa} - A\ \kappa \ = \ 1$.

Finally, the general solution of the Riccati equation can then be given in the form

$$\frac{2\hbar}{m}y(t) \ = \ \frac{2\hbar}{m}\tilde{y} \ + \ \frac{d}{dt}\ \ln\left[\kappa_0 + \mathcal{I}(t)\right] \ , \tag{15}$$

with $\kappa_0 = \left(\frac{2\hbar}{m}v_0\right)^{-1}$ and using the abbreviated form $\mathcal{I}(t) = \int^t dt' \ \mathrm{e}^{-\int^{t'} dt" A(t")}$. It is worth mentioning that the solution (15) depends on the (complex) initial condition κ_0 and since $\frac{2\hbar}{m}y(t)$ fulfills a NL differential equation, the dependence on the initial condition can have quite drastic qualitative consequences.

This can be seen by applying a formal comparison with supersymmetric quantum mechanics [11] [12], showing that (15) is formally identical to the most general superpotential

$$W(x) \ = \ \tilde{W}(x) \ + \ \frac{d}{dx} \ln\left[\lambda_1 + \mathcal{I}_1(x)\right] , \tag{16}$$

leading to the one-parameter family of complex isospectral potentials

$$V_1(x) \ = \ W^2 - \frac{d}{dx}W \ = \ \tilde{V}_1 - 2\frac{d^2}{dx^2} \ln\left[\lambda_1 + \mathcal{I}_1(x)\right] \tag{17}$$

that have the same supersymmetric partner potential $\tilde{V}_2(x)$ (see, e.g., [12][13]). The shape of the isospectral potentials strongly depends on the parameter λ_1 and is far different from the original one.

In this case, the integral $\mathcal{I}_1(x)$ is defined as $\mathcal{I}_1(x) \ = \ \int_{-\infty}^{x} dx' \ \Psi_1^2(x')$ where $\Psi_1(x)$ is the normalized ground state wave function of the SE with potential $\tilde{V}_1(x) = \tilde{W}^2(x) - \frac{d}{dx}\tilde{W}(x)$ and λ_1 is a (usually real) parameter.

A major difference between this supersymmetric situation and the one in our case (apart from replacing the spatial variable by a temporal one) is the fact that the variables of the nonlinear Eqs.(10) and (14) are *complex*, whereas $\tilde{W}(x), W(x)$ and $\mathcal{I}_1(x)$ are real quantities; also the parameter κ_0 in our case is generally complex.

This provides a larger variety but, nevertheless, certain methods and results can be transferred from one system to the other (which will be discussed elsewhere).

4. Generalized creation and annihilation operators and corresponding coherent states

As mentioned in section 2, the usual creation and annihilation operators (3) and (4) provide minimum uncertainty CSs with constant width, corresponding to the particular solution of the Riccati equation with $\frac{2\hbar}{m}\tilde{y} = i\frac{2\hbar}{m}\tilde{y}_I = i\frac{2\hbar}{m}y_I = i\omega_0$. Replacing ω_0 by $\frac{1}{\alpha_0^2}$ or $\frac{2\hbar}{m}y_I$, respectively, the operators a and a^+ can be rewritten as

$$a \ = \ i\sqrt{\frac{m}{2\hbar}}\,\alpha_0 \left(\frac{p_{op}}{m} \ - \ i\left(\frac{2\hbar}{m}y_I\right) x\right) \tag{18}$$

$$a^+ \ = \ -\,i\sqrt{\frac{m}{2\hbar}}\,\alpha_0 \left(\frac{p_{op}}{m} \ + \ i\left(\frac{2\hbar}{m}y_I\right) x\right) . \tag{19}$$

As we have shown, already for $\omega = \omega_0 = \text{const.}$, solutions with TD WP width exist, i.e., α_0 turns into $\alpha(t)$ and $\dot{\alpha} \neq 0$, hence $\frac{2\hbar}{m}y_R \ = \ \frac{\dot{\alpha}}{\alpha}$, must be taken into account. Obviously, the same also applies for the HO with TD frequency $\omega = \omega(t)$. Therefore, in Eqs.(18) and (19) α_0 must be replaced by $\alpha(t)$ and $i\left(\frac{2\hbar}{m}y_I\right)$ by $\left(\frac{2\hbar}{m}y\right)$, thus leading to

$$\tilde{a}(t) \ = \ i\sqrt{\frac{m}{2\hbar}}\,\alpha \left(\frac{p_{op}}{m} \ - \ \left(\frac{2\hbar}{m}y\right) x\right) \tag{20}$$

$$\tilde{a}^+(t) \ = \ -\,i\sqrt{\frac{m}{2\hbar}}\,\alpha \left(\frac{p_{op}}{m} \ - \ \left(\frac{2\hbar}{m}y^*\right) x\right) . \tag{21}$$

It is easy to check that they satisfy the standard commutation relations.

Since, at least for TD frequency ω, the corresponding Hamiltonian is no longer a constant of motion, one might ask if $\tilde{a}(t)$ and $\tilde{a}^+(t)$, as defined above, are constants of motion.

It turns out that

$$\frac{\partial}{\partial t}\tilde{a} \ + \ \frac{1}{i\hbar}\,[\tilde{a},H]_- \ = \ -\,i\,\frac{1}{\alpha^2}\,\tilde{a} \ \neq \ 0\;. \tag{22}$$

So, $\tilde{a}(t)$ and $\tilde{a}^+(t)$ are no constants of motion but can be turned into such by simply introducing a phase factor according to

$$a(t) \ = \ \tilde{a}(t)\ \mathrm{e}^{i\int^t dt'\,\frac{1}{\alpha^2}} \tag{23}$$

$$a^+(t) \ = \ \tilde{a}^+(t)\ \mathrm{e}^{-i\int^t dt'\,\frac{1}{\alpha^2}}\quad . \tag{24}$$

Next, the CSs that can be obtained using these generalized operators. For this purpose, the phase factor shall be omitted since it can be absorbed in the purely TD function $K(t)$ in the exponent of the WP/CS or into $N(t)$.

To create a generalized CS $|z>$ we use the property that it is an eigenstate of $a(t)$ with complex eigenvalue z, i.e., $a(t)\ |z> \ = \ z\ |z>$.

The meaning of the complex eigenvalue can be determined using this property and expressing $<x>_z = \eta$ and $<p>_z = m\dot{\eta}$ in terms of z and z^* as

$$<x>_z = \sqrt{\frac{\hbar}{2m}}\,\alpha\,(z^*+z) = \sqrt{\frac{2\hbar}{m}}\,\alpha\,z_R \quad\text{and}\quad <p>_z = \sqrt{\frac{\hbar}{2m}}\,\alpha\,m\,\left[\left(\frac{2\hbar}{m}y\right)z^* \ + \ \left(\frac{2\hbar}{m}y^*\right)z\right] \tag{25}$$

where from it follows that

$$z_R \ = \ \frac{1}{\sqrt{2}}\sqrt{\frac{m}{\hbar}}\,\alpha\,\left(\frac{2\hbar}{m}y_I\right)\eta \quad\text{and}\quad z_I \ = \ \frac{1}{\sqrt{2}}\sqrt{\frac{m}{\hbar}}\,\alpha\,\left[\dot{\eta}-\left(\frac{2\hbar}{m}y_R\right)\eta\right] \tag{26}$$

or

$$z \ = \ \sqrt{\frac{m}{2\hbar}}\,\left[\left(\frac{\eta}{\alpha}\right) \ + \ i\,(\dot{\eta}\alpha-\eta\dot{\alpha})\right] \ = \ \frac{1}{\sqrt{2}}\sqrt{\frac{m}{\hbar}}\,\alpha\,\left[\left(\frac{2\hbar}{m}y_I\right)\eta \ + \ i\left(\dot{\eta}-\left(\frac{2\hbar}{m}y_R\right)\eta\right)\right]\;. \tag{27}$$

Therefore, the complex eigenvalue z is directly related to the Ermakov invariant via

$$I_L \ = \ \frac{\hbar}{m}\left(z_I^2+z_R^2\right) \ = \ \frac{\hbar}{m}zz^* \ = \ \frac{\hbar}{m}|z|^2\;. \tag{28}$$

The operator corresponding to this invariant can be written in terms of the generalized creation/annihilation operators similar to Eq. (2) as $I_{L,op} \ = \ \frac{\hbar}{m}\left[a^+(t)\ a(t)+\frac{1}{2}\right]$ (taking into account the commutation relation between a and a^+), what is in agreement with the result of Hartley and Ray [10].

In position space representation, the generalized CS can be obtained from $<x|a(t)|z> \ = \ z<x|z> \ = \ z\ \Psi_z(x,t)$, and finally be written as

$$\Psi_z(x,t) \ = \ \left(\frac{m}{\pi\hbar}\right)^{1/4}\left(\frac{1}{\lambda}\right)^{1/2}\exp\left\{i\left[y\tilde{x}^2+\frac{1}{\hbar}<p>\tilde{x}+\frac{1}{2\hbar}<p><x>\right]\right\}\;, \tag{29}$$

corresponding to our above Gaussian WP $\Psi_{WP}(x,t)$ with $N(t) = (\frac{m}{\pi\hbar})^{1/4}\left(\frac{1}{\lambda}\right)^{1/2}$ and $K(t) = \frac{1}{2\hbar}<p><x>$. For $\alpha^2 = \alpha_0^2 = \omega_0^{-1}$, $N(t)$ turns into $N(t) = (\frac{m\omega_0}{\pi\hbar})^{1/4}\ \mathrm{e}^{-i\omega_0 t/2}$, where the exponential contributes the ground state energy to the energy of the WP.

5. Dissipative systems with effective Hamiltonians

There exist several approaches for describing dissipative quantum systems using modified effective one-body SEs where the effect of the environment on the observable system is taken into account by dissipative friction terms without considering the individual degrees of freedom of the environment. Several of these approaches are discussed in the literature [14][15][16]. In this context it was also investigated [17] if Ermakov invariants also exist for these approaches. An explicitly TD Hamiltonian [14] and a NLSE [16] for which this is the case will be discussed subsequently.

The explicitly TD approach of Caldirola and Kanai [14] starts on the classical level with a *non-canonical* transformation between the *physical* variables x and $p = m\dot{x}$ and the *canonical* variables $\hat{x} = x$, $\hat{p} = p\ \mathrm{e}^{\gamma}t$.

Expressed in the canonical variables, the Hamiltonian can be formulated as

$$\hat{H}_{CK} \ = \ \frac{1}{2m}\mathrm{e}^{-\gamma t}\ \hat{p}^2 + \mathrm{e}^{\gamma t}V\left(x\right) \tag{30}$$

which yields the proper equation of motion including a linear momentum (or velocity) dependent friction force.

The transition to quantum mechanics is achieved by replacing, as usual, the *canonical* momentum with a differential operator according to $\hat{p} \rightarrow \hat{p}_{op} = \frac{\hbar}{i}\frac{\partial}{\partial x}$, leading to the corresponding explicitly TD Hamiltonian operator and, thus, to a modified SE.

For the systems considered in this work, this equation also possesses exact Gaussian WP solutions like (8). The equation of motion for the WP maximum is just the one for the classical trajectory $x(t)$ including the friction force (in our notation, $x(t)$ corresponds to $\eta(t)$),

$$\ddot{\eta} + \gamma\dot{\eta} + \omega^2\eta = 0\ . \tag{31}$$

The modified Riccati equation for the complex variable $\left(\frac{2\hbar}{m}\hat{y}\right)_{CK}$ reads

$$\left(\frac{2\hbar}{m}\dot{\hat{y}}\right)_{CK} + \ \mathrm{e}^{-\gamma t}\left(\frac{2\hbar}{m}\hat{y}\right)^2_{CK} + \ \omega^2\ \mathrm{e}^{\gamma t} \ = \ 0\ . \tag{32}$$

The imaginary part of this variable is again connected with the WP width and the real variable α_{CK} in the same way as in the conservative case. This again allows for the transformation of the Riccati equation into a (real) Ermakov-type equation,

$$\ddot{\alpha}_{CK} + \gamma\dot{\alpha}_{CK} + \omega^2\alpha_{CK} = \frac{\mathrm{e}^{-2\gamma t}}{\alpha^3_{CK}}\ . \tag{33}$$

This equation, together with Eq.(31) for η, forms the required system of equations that possesses an exact Ermakov-type invariant, here given in the form

$$\hat{I}_{CK} = \frac{1}{2}\left[e^{2\gamma t}\left(\dot{\eta}\ \alpha_{CK} - \eta\ \dot{\alpha}_{CK}\right)^2 + \left(\frac{\eta}{\alpha_{CK}}\right)^2\right] = \mathrm{const.} \tag{34}$$

The NL approach [16] was motivated by the attempt to break the time-reversal symmetry on all levels to include irreversibility, a phenomenon usually associated also with dissipative systems (but not necessarily occurring only in those systems).

This can be achieved by introducing a diffusion term into the continuity equation thus turning it into a Fokker–Planck-type equation (in particular a Smoluchowski equation). Following a method described in [16], the (real) Smoluchowski equation can be separated into two complex equations, namely a (modified) SE for the wave function $\Psi(x,t)$ and its complex conjugate, $\Psi^*(x,t)$, if a separation condition concerning the diffusion term is fulfilled.

This leads to the NLSE

$$i\hbar\frac{\partial}{\partial t}\Psi_{NL}(x,t) \;=\; \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \;V(x)\;+\;\gamma\frac{\hbar}{i}\left(\ln\Psi_{NL} - <\ln\Psi_{NL}>\right)\right\}\Psi_{NL}(x,t) \tag{35}$$

with a complex logarithmic nonlinearity (for details, see e.g., [16]).

The equation of motion for the WP maximum is again identical to Eq.(31). The equation for the time-dependence of the width can now be obtained from the modified complex Riccati equation

$$\left(\frac{2\hbar}{m}\dot{y}\right)_{NL} + \gamma\left(\frac{2\hbar}{m}y\right)_{NL} + \left(\frac{2\hbar}{m}y\right)^2_{NL} + \omega^2(t) \;=\; 0\;. \tag{36}$$

Also in this case, the imaginary part has the same relation to the WP width and the real variable α_{NL} as in the conservative case. The real part of the complex Riccati-variable now takes the modified form $\frac{2\hbar}{m}y_{R,NL} = \frac{\dot{\alpha}_{NL}}{\alpha_{NL}} - \frac{\gamma}{2}$.

With the help of these relations, the Riccati equation now turns into the Ermakov-type equation

$$\ddot{\alpha}_{NL} + \left(\omega^2 - \frac{\gamma^2}{4}\right)\alpha_{NL} = \frac{1}{\alpha^3_{NL}}\quad. \tag{37}$$

Together with the dissipative equation for the WP maximum, Eq.(31), this leads to the exact Ermakov invariant

$$I_{NL} \;== \frac{1}{2}e^{\gamma t}\;\alpha^2_{NL}\left[\left(\dot{\eta}\; - \left(\frac{\dot{\alpha}_{NL}}{\alpha_{NL}} - \frac{\gamma}{2}\right)\eta\right)^2 \;+\; \left(\frac{1}{\alpha^2_{NL}}\;\eta\right)^2\right] \;=\; \text{const.} \tag{38}$$

To establish the connection between the explicitly TD Caldirola–Kanai approach and the logarithmic NLSE (35) we refer to Schrödinger's first communication on wave mechanics [18] where he starts from the Hamilton–Jacobi equation with the action function S. He introduced the wave function $\Psi(x,t)$ via $S_c = \frac{\hbar}{i}\ln\Psi$, where the subscript c (added by us) indicates that this action is a complex quantity, since Ψ is, in general, a complex function. Via a variational ansatz, Schrödinger arrived at the Hamiltonian operator $H_L = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$.

We now reverse Schrödinger's procedure, starting with Eq. (35) (divided by Ψ, which causes no problems for Gaussian WPs) and, using the definition of S_c, arrive at $\left(\frac{\partial}{\partial t} + \gamma\right)S_c + H \;=\; -\gamma < S_c >$. The purely TD term $-\gamma < S_c >$ is necessary mainly for normalization purposes (can therefore be absorbed by the normalization coefficient) and shall be neglected in the following.

Multiplying the remaining equation by $\mathrm{e}^{\gamma t}$ and using the definitions $\hat{S}_c \;=\; \mathrm{e}^{\gamma t}S_c$ and $\hat{H} \;=\; \mathrm{e}^{\gamma t}H$, it can be rewritten as Hamilton–Jacobi equation in terms of $\hat{S}_c$ and $\hat{H}$.

From the definition of the action function, it follows that the wave function $\hat{\Psi}(x,t)$ in the transformed (canonical) system is connected with the wave function $\Psi(x,t)$ in the physical system via the *non-unitary* relation

$$\ln\hat{\Psi} \;=\; e^{\gamma t}\,\ln\Psi\quad. \tag{39}$$

Consequently, the (complex) momenta in the two systems are connected via

$$\hat{p}_c \;=\; \frac{\hbar}{i}\,\frac{\partial}{\partial x}\,\ln\hat{\Psi} \;=\; e^{\gamma t}\,\frac{\hbar}{i}\,\frac{\partial}{\partial x}\,\ln\Psi_{NL} \;=\; e^{\gamma t}p_c\quad, \tag{40}$$

which is equivalent to the connection between the canonical and the kinetic momentum in the Caldirola–Kanai approach. The $non-canonical$ connection between the classical variables (x,p) and $(\hat{x},\hat{p})$ corresponds to the $non-unitary$ transformation between Ψ and $\hat{\Psi}$ (see also [19]).

Expressing $\hat{H}$ in terms of the canonical momentum $\hat{p}_c$ and following Schrödinger's quantization procedure, finally yields the modified SE of the Caldirola–Kanai approach. Because the WP solution of this equation, $\hat{\Psi}_{CK}(x,t)$, is related to the WP solution $\Psi_{NL}(x,t)$ of the NLSE (35) via (39), this means particularly for the corresponding Riccati and Ermakov equations that

$$\left(\frac{2\hbar}{m}\hat{y}\right)_{CK} = e^{\gamma t}\left(\frac{2\hbar}{m}y\right)_{NL} \quad \text{and} \quad \alpha_{CK} = e^{-\gamma t/2}\alpha_{NL} \; . \tag{41}$$

Taking this into account, Eqs.(32) and (33) turn into Eqs.(36) and (37) and the Ermakov invariant $\hat{I}_{CK}$ turns exactly into I_{NL}. Factorization of the operator corresponding to (34) would provide the creation/annihilation operators on the canonical level, therefore, factorization of the operator corresponding to (38) without the exponential factor (because of relation (39)) but now in terms of $\left(\frac{2\hbar}{m}y\right)$, provides the generalized creation/annihilation operators for the dissipative case on the physical level in the same form as in the conservative case.

6. Application of $a(t)$ and $a^+(t)$ to obtain new exact solutions of the TDSE

The generalized creation and annihilation operators will now be used to obtain exact solutions of the TDSE that are different from the standard textbook solutions. As an example the free motion, i.e. $V = 0$, without dissipation shall be considered, generalizations to other quadratic Hamiltonians and the inclusion of dissipation are, according to the results obtained above, straightfotward.

The exact solutions of the free motion TDSE are usually given by plane waves. A different complete set of solutions in terms of Hermite polynomials with TD variable can be obtained via the generalized creation and annihilation operators (what is in agreement with results of [20] obtained in a different context). The ground state is obtained via $a(t)|0>= 0$ and (in position representation) given by

$$< x|0 > \; = \; \Psi_0(x,t) \; = \; \left(\frac{m}{\pi\hbar}\right)^{1/4}\left(\frac{1}{\alpha}\right)^{1/2} e^{iy(t)x^2} \; \exp\left\{-\frac{i}{2}\int^t dt' \frac{1}{\alpha^2}\right\} \; . \tag{42}$$

The excited states are obtained by applying $a^+(t)$ as

$$\Psi_n(x,t) = \left[\left(\frac{m}{\pi\hbar}\right)^{1/2}\frac{1}{\alpha}\,\frac{1}{n!\;2^n}\right]^{1/2} \mathrm{H}_n\left(\frac{x}{x_0}\right)\, e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2} \; \exp\left\{-i\left(n+\frac{1}{2}\right)\int^t dt'\frac{1}{\alpha^2}\right\} \tag{43}$$

with the TD variable $\xi = \frac{x}{x_0} = \xi(t)$ where $x_0 = \sqrt{\frac{\hbar}{m}}\alpha(t)$ and $\mathrm{H}_n(\xi)$ are Hermite polynomials. The functions $\Psi_n(x,t)$ are *exact solutions* of the TDSE for $V = 0$, but *no eigenfunctions* of $H_{L,0}$, the free motion Hamiltonian (but of $I_{L,0}$), since

$$\begin{aligned} H_{L,0}\;\Psi_n \;&=\; -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\;\Psi_n \;=\; \left\{\hbar\,\frac{1}{\alpha^2}\left(n+\frac{1}{2}\right) + \frac{m}{2}\left(\frac{2\hbar}{m}y\right)^2 x^2 + \frac{\hbar}{2i}\left(\frac{2\hbar}{m}y_R\right)\right\}\Psi_n \\ &+ \left[\left(\frac{m}{\pi\hbar}\right)^{1/2}\frac{1}{\alpha}\,\frac{1}{n!\;2^n}\right]^{1/2} e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}\left\{-i\hbar\left(\frac{2\hbar}{m}y_R\right)\left(\frac{x}{x_0}\right)\mathrm{H}_n'\left(\frac{x}{x_0}\right)\right\} \end{aligned} \tag{44}$$

with $\mathrm{H}_n'\left(\frac{x}{x_0}\right) = \frac{d}{d\xi}\mathrm{H}_n(\xi)$.

However, using the relations $\xi\mathrm{H}_n' = 2\xi^2\mathrm{H}_n - \mathrm{H}_{n+1} = \left[2\xi^2 - (n+1)\right]\mathrm{H}_n - \frac{1}{2}\mathrm{H}_{n+2}$ and $< n|n+2 >= 0$, it can be shown that for $\left(\frac{2\hbar}{m}y\right)$ obeying Eq. (13) with $\omega = 0$ the mean values fulfill

$$\left\langle -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right\rangle \;=\; \frac{\hbar}{2}\left(n+\frac{1}{2}\right)\left[\dot{\alpha}^2 + \frac{1}{\alpha^2}\right] \;=\; \left\langle i\hbar\frac{\partial}{\partial t}\right\rangle \; . \tag{45}$$

7. Conclusions

In this paper a generalization of the creation/annihilation operators is investigated for cases, where the corresponding CSs or WPs have TD width and the energy (generally) is no longer a constant of motion. Nevertheless, for these systems still a dynamical invariant can exist. This can be obtained from the Newtonian equation of motion for the WP maximum and a complex NL Riccati equation for the WP width. Factorization of the corresponding operator supplies the generalized creation/annihilation operators in terms of position and momentum operators and the complex variable $\left(\frac{2\hbar}{m}y\right)$ obeing the Riccati equation. These results are also in agreement with the work of Malkin et al. [21] where these operators were expressed in terms of a complex variable that can be obtained by linearizing the Riccati equation. In our case, however, new insight is gained from the information contained in the nonlinearity of the Riccati equation and the relation to supersymmetric quantum mechanics.

Using the definition of the CS as eigenstate of the annihilation operator, it is possible to determine the complex eigenvalue $z(t)$ in terms of the classical quantities η and $\dot{\eta}$ and of real and imaginary parts of $\left(\frac{2\hbar}{m}y\right)$. The CS in position space obtained in this way is identical to the Gaussian WP that is a solution of the corresponding TDSE.

In the second part, the formalism is extended to include dissipative environmental effects. For this purpose effective descriptions in terms of an explicitly TD Hamiltonian based on a modified canonical approach and a logarithmic NLSE are compared. In both cases, the equations of motion for the corresponding WP maximum and width again supply an Ermakov invariant. Furthermore, both approaches are related via the non-unitary transformation (39). Factorization of the corresponding operator again provides the modified creation/annihilation operators that have, expressed ind terms of $\left(\frac{2\hbar}{m}y\right)$, the same form as in the conservative case.

Finally, these generalized operators are used to construct exact analytic solutions of the free motion Schrödinger equation in terms of Hermite polynomials with time-dependent variable.

References

[1] Ermakov V P 1880 *Univ. Izv. Kiev* **20** 1
[2] Schrödinger E 1940 *Proc. Roy. Irish Acad.* **46** A 9
[3] Dirac P A M 1935 *The Principles of Quantum Mechanics* 34 (Oxford: Oxford Univ. Press)
[4] Schrödinger E 1926 *Naturwissenschaften* **14** 664
[5] Glauber R J 1963 *Phys. Rev.* **130** 2529
[6] Sudarshan E C G 1963 *Phys. Rev. Lett.* **10** 227
[7] Klauder J R 1963 *J. Math. Phys.* **4** 1055
[8] Nieto M M and Simmons Jr L M 1979 *Phys. Rev.* D **20** 1321
[9] Lewis H R and Riesenfeld W B 1969 *J. Math. Phys.* **10** 1458
[10] Hartley J G and Ray J R 1982 *Phys. Rev.* D **25** 382
[11] Mielnik B and Rosas-Ortiz O 2004 *J. Phys. A: Math. Gen.* **37** 10007
[12] Cooper F, Khare A and Sukhatme U 2001 *Supersymmetry in Quantum Mechanics* (Singapore: World Scientific) Chap. 6.1
[13] Rosas-Ortiz O and Muñoz R 2003 *J. Phys. A: Math. Gen.* **36** 8497; Fernández D J, Muñoz R and Ramos A 2003 *Phys. Lett.* A **308** 11
[14] Caldirola P 1941 *Nuovo Cimento* **18** 393; Kanai E 1948 *Progr. Theor. Phys.* **3** 440
[15] Hasse R W 1975 *J. Math. Phys.* **16** 2005
[16] Schuch D, Chung K-M and Hartmann H 1983 *J. Math. Phys.* **24** 1652; ibid 1984 *J. Math. Phys.* **25** 3086; Schuch D and Chung K-M 1986 *Int. J. Quantum Chem.* **29** 1561
[17] Nassar A B 1986 *J. Math. Phys.* **27** 755; ibid 1986 *J. Math. Phys.* **27** 2949
[18] Schrödinger E 1926 *Ann. d. Phys.* **79** 361
[19] Schuch D 1997 *Phys. Rev.* A **55** 935
[20] Guerrero J, López-Ruiz F F, Aldaya V and Cossío F 2011 *J. Phys. A: Math. Theor.* **44** 445307
[21] Malkin I A, Man'ko V I and Trifonov D A 1970 *Phys. Rev.* D **2** 1371