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Quantization of the Rank Two Heisenberg–Virasoro Algebra

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Abstract: Quantum groups occupy a significant position in both mathematics and physics, contributing to progress in these fields. It is interesting to obtain new quantum groups by the quantization of Lie bialgebras. In this paper, the quantization of the rank two Heisenberg–Virasoro algebra by Drinfel'd twists is presented, Lie bialgebra structures of which have been investigated by the authors recently.

Keywords: quantization; Lie bialgebras; Drinfel'd twists; the rank two Heisenberg–Virasoro algebra; Hopf algebras

MSC: 17B62; 17B05; 17B37

1. Introduction

Quantum groups were first independently introduced by Drinfel'd [1,2] and Jimbo [3] around 1985 with the aim of constructing solutions to the quantum Yang–Baxter equations. They have been identified by Drinfel'd and Jimbo with a certain class of Hopf algebras. In Hopf algebra or quantum group theory, there exist two conventional approaches for generating new bialgebras from existing ones. One approach involves twisting the product by a 2-cocycle while maintaining the coproduct unchanged. Alternatively, one can twist the coproduct utilizing a Drinfel'd twist element while preserving the product. The process of quantizing Lie bialgebras serves as a crucial approach in generating new quantum groups (cf. [2,4], etc.). Since quantum groups have been discovered to possess numerous applications across diverse fields, encompassing statistical physics, symplectic geometry, knot theory, and even modular representations of reductive algebraic groups, quantizations of Lie bialgebras have received considerable attention in many studies (e.g., [5–22]). In [5], the infinite dimensional Witt algebra with characteristic 0 was explicitly quantized through the utilization of the twist initially discovered by Giaquinto and Zhang in [6]. Afterwards, quantizations of the generalized Witt algebra with characteristic 0 were provided in [7], whereas its Lie bialgebra structures were determined in [8]. The quantizations of generalized Kac–Moody algebras were obtained by Etingof and Kazhdan (see [9,10]). The quantizations of generalized Virasoro-like-type, Block-type, W-algebra $W(2, 2)$ and Schroding–Virasoro algebra were given in [11–14], while Lie bialgebra structures of these algebras were considered in [15–18], respectively. Recently, the authors proved in [23] that every Lie bialgebra structure on the rank two Heisenberg–Virasoro algebra is triangular coboundary. However its quantum group structure is not known, which is what our paper shall focus on.

The rank two Heisenberg–Virasoro algebra L is an infinite-dimensional Lie algebra with a $\mathbb{C}$-basis $\{t_\alpha,\ E_\alpha \mid \alpha \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}\}$ and the following Lie brackets:

$$[t_\alpha, E_\beta] = \det\binom{\beta}{\alpha} t_{\alpha+\beta},\ [E_\alpha, E_\beta] = \det\binom{\beta}{\alpha} E_{\alpha+\beta},\ [t_\alpha,\ t_\beta] = 0, \quad (1)$$

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where $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, $\det\begin{pmatrix}\beta \\ \alpha\end{pmatrix} = \beta_1\alpha_2 - \alpha_1\beta_2$, and $\mathbf{0} = (0, 0)$. In [24], the derivations, automorphism group, and central extension of L were thoroughly investigated. Furthermore, the irreducibility of universal Whittaker modules related to L was conclusively determined in [25]. Lastly, authors in [26] delved into the Verma module structure associated with L, offering a comprehensive characterization.

We present two degree derivations D_1 and D_2 on L, i.e.,

$$[D_i, t_\alpha] = \alpha_i t_\alpha,\ [D_i, E_\alpha] = \alpha_i E_\alpha,\ [D_1, D_2] = 0,\ \text{for } i = 1,\ 2. \tag{2}$$

And subsequently, we arrive at our Lie algebra $\widetilde{L} = L \oplus \mathbb{C}D_1 \oplus \mathbb{C}D_2$. For convenience, we still refer to it as the rank two Heisenberg–Virasoro algebra. In the present paper, we shall consider the quantization of the rank two Heisenberg–Virasoro algebra $\widetilde{L}$. We use the general quantization method by Drinfel'd twists (cf. [6,27]) to quantize explicitly the Lie algebra $\widetilde{L}$. Actually, the entirety of this process relies solely on the construction of Drinfel'd twists. The main results of this article are Theorems 1 and 2, which provide the quantizations of the rank two Heisenberg–Virasoro algebra $\widetilde{L}$. Our findings have broadened the category of illustrative instances related to non-commutative and non-cocommutative Hopf algebras.

In this paper, we use the notations $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{Z}$ and $\mathbb{C}$ to represent the sets of nonnegative integers, positive integers, integers, and complex numbers, respectively.

2. Preliminaries

In this section, we first revisit several fundamental definitions and outcomes pertaining to quantization techniques, which will be used in subsequent discussions.

Let A denote a unitary algebra over $\mathbb{C}$. For an arbitrary element x of A, $\lambda \in \mathbb{C}$, $n \in \mathbb{N}$, define

$$x_\lambda^{<n>} := (x+\lambda)(x+\lambda+1)\cdots(x+\lambda+n-1) \tag{3}$$

$$x_\lambda^{[n]} := (x+\lambda)(x+\lambda-1)\cdots(x+\lambda-n+1) \tag{4}$$

where $x_\lambda^{<0>} = x_\lambda^{[0]} = 1$. For convenience, we use $x^{<n>}$ and $x^{[n]}$ to represent $x_0^{<n>}$ and $x_0^{[n]}$, respectively.

The following result and definition belongs to [2,5,6].

Lemma 1 ([5,6]). *Let x be an arbitrary element of the unitary algebra A over $\mathbb{C}$. For given $\lambda,\ \rho \in \mathbb{C}$ and $m,\ n,\ l \in \mathbb{Z}_+$, the following equations hold.*

$$x_\lambda^{<m+n>} = x_\lambda^{<m>} x_{\lambda+m}^{<n>},\ x_\lambda^{[m+n]} = x_\lambda^{[m]} x_{\lambda-m}^{[n]},\ x_\lambda^{[m]} = x_{\lambda-m+1}^{<m>}, \tag{5}$$

$$\sum_{m+n=l} \frac{(-1)^n}{m!n!} x_\lambda^{[m]} x_\rho^{<n>} = \binom{\lambda-\rho}{l} = \frac{(\lambda-\rho)(\lambda-\rho-1)\cdots(\lambda-\rho-l+1)}{l!} \tag{6}$$

$$\sum_{m+n=l} \frac{(-1)^n}{m!n!} x_\lambda^{[m]} x_{\rho-m}^{[n]} = \binom{\lambda-\rho+l-1}{l} = \frac{(\lambda-\rho)(\lambda-\rho+1)\cdots(\lambda-\rho+l-1)}{l!} \tag{7}$$

Definition 1 ([2]). *Let $(W, \sigma, \tau, \Delta_0, \varepsilon_0, S_0)$ be a Hopf algebra over a commutative ring. $\mathfrak{F}$ is called a Drinfel'd twist on W, if it is an invertible element of $W \otimes W$ such that*

$$(\mathfrak{F} \otimes 1)(\Delta_0 \otimes \mathrm{Id})(\mathfrak{F}) = (1 \otimes \mathfrak{F})(\mathrm{Id} \otimes \Delta_0)(\mathfrak{F}) \tag{8}$$

$$(\varepsilon_0 \otimes \mathrm{Id})(\mathfrak{F}) = 1 \otimes 1 = (\mathrm{Id} \otimes \varepsilon_0)(\mathfrak{F}) \tag{9}$$

The well-known results mentioned below come from [2,4,27].

Lemma 2 ([2,4,27])**.** *Let $(W, \sigma, \tau, \Delta_0, \varepsilon_0, S_0)$ be a Hopf algebra over a commutative ring, $\mathfrak{F}$ a Drinfel'd twist of W. Then*

(1) *$f = \sigma(\mathrm{Id} \otimes S_0)(\mathfrak{F})$ is an invertible element of $W \otimes W$ with $f^{-1} = \sigma(S_0 \otimes \mathrm{Id})(\mathfrak{F}^{-1})$.*

(2) *the algebra $(W, \sigma, \tau, \Delta, \varepsilon, S)$ is a new Hopf algebra, that is referred to as the twisting of W by the Drinfel'd twist $\mathfrak{F}$, if we remains the counit unchanged (i.e., $\varepsilon = \varepsilon_0$) and define $\Delta : W \to W \otimes W$, $S : W \to W$ by*

$$\Delta(h) = \mathfrak{F}\Delta_0(h)\mathfrak{F}^{-1}, S(h) = f\, S_0(h) f^{-1}, \forall\, h \in W. \tag{10}$$

3. The Main Results

Let $(\mathfrak{A}(\widetilde{L}), \sigma, \tau, \Delta_0, \varepsilon_0, S_0)$ denote the standard Hopf algebra, which is characterized by the specified definitions of the coproduct, the antipode, and the counit, as outlined below.

$$\Delta_0(t_\alpha) = t_\alpha \otimes 1 + 1 \otimes t_\alpha,\ \Delta_0(E_\alpha) = E_\alpha \otimes 1 + 1 \otimes E_\alpha,\ \Delta_0(D_i) = D_i \otimes 1 + 1 \otimes D_i, \tag{11}$$

$$S_0(t_\alpha) = -t_\alpha,\ S_0(E_\alpha) = -E_\alpha,\ S_0(D_i) = -D_i,\ \varepsilon_0(t_\alpha) = \varepsilon_0(E_\alpha) = \varepsilon_0(D_i) = 0, \tag{12}$$

for $\alpha \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, $i = 1,\ 2$.

Let $\mathfrak{A}(\widetilde{L})[\![x]\!]$ denote an associative algebra over $\mathbb{C}$, where $\mathfrak{A}(\widetilde{L})[\![x]\!]$ consists of formal power series with coefficients belonging to $\mathfrak{A}(\widetilde{L})$. Then, the Hopf algebra structure of $\mathfrak{A}(\widetilde{L})[\![x]\!]$ is naturally induced from $(\mathfrak{A}(\widetilde{L}), \sigma, \tau, \Delta_0, \varepsilon_0, S_0)$. For convenience, we also denote it by $(\mathfrak{A}(\widetilde{L})[\![x]\!], \sigma, \tau, \Delta_0, \varepsilon_0, S_0)$.

The key findings of this paper are summarized in the following two theorems, which give the quantizations of $\mathfrak{A}(\widetilde{L})$ by the Drinfel'd twist $\mathfrak{F}$ defined in (21).

Theorem 1. *Let $\widetilde{L}$ be the rank two Heisenberg–Virasoro algebra. For any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, $E_\alpha \in \widetilde{L}$, we choose $H = \frac{1}{\mu}(\eta_1 D_1 + \eta_2 D_2)$ with $\mu := \eta_1\alpha_1 + \eta_2\alpha_2 \neq 0$ and $\eta_1, \eta_2 \in \mathbb{C}$ to satisfy $[H, E_\alpha] = E_\alpha$. Then there exists a non-commutative and non-cocommutative Hopf algebra structure $(\mathfrak{A}(\widetilde{L})[\![x]\!], \sigma, \tau, \Delta, \varepsilon, S)$ on $\mathfrak{A}(\widetilde{L})[\![x]\!]$ over $\mathbb{C}[\![x]\!]$ with $\mathfrak{A}(\widetilde{L})[\![x]\!]/x\mathfrak{A}(\widetilde{L})[\![x]\!] \cong \mathfrak{A}(\widetilde{L})$, which preserves the product and counit of $\mathfrak{A}(\widetilde{L})[\![x]\!]$, while the deformed coproduct and antipode are defined as follows.*

$$\begin{aligned}
\Delta(E_\beta) &= E_\beta \otimes (1 - E_\alpha x)^{\frac{\eta}{\mu}} + \sum_{s=0}^{\infty} (-1)^s b_s H^{<s>} \otimes (1 - E_\alpha x)^{-s} E_{\beta+s\alpha} x^s,\\
\Delta(t_\beta) &= t_\beta \otimes (1 - E_\alpha x)^{\frac{\eta}{\mu}} + \sum_{s=0}^{\infty} (-1)^s b_s H^{<s>} \otimes (1 - E_\alpha x)^{-s} t_{\beta+s\alpha} x^s,\\
\Delta(D_j) &= D_j \otimes 1 + 1 \otimes D_j + \alpha_j H^{<1>} \otimes (1 - E_\alpha x)^{-1} E_\alpha x,\\
S(E_\beta) &= -(1 - E_\alpha x)^{-\frac{\eta}{\mu}} \sum_{s=0}^{\infty} b_s E_{\beta+s\alpha} H_1^{<s>} x^s,\\
S(t_\beta) &= -(1 - E_\alpha x)^{-\frac{\eta}{\mu}} \sum_{s=0}^{\infty} b_s t_{\beta+s\alpha} H_1^{<s>} x^s,\\
S(D_j) &= \alpha_j H (1 - E_\alpha x)^{-1} (E_\alpha x - E_\alpha^2 x^2) - D_j,
\end{aligned}$$

where $j = 1,\ 2$, for any $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, we denote $\eta = \eta_1\beta_1 + \eta_2\beta_2$, $b_s = \frac{1}{s!}(\beta_1\alpha_2 - \beta_2\alpha_1)^s$, $b_0 = 1$.

For the sake of simplicity, we adopt the same notations as those utilized in Theorem 1 for the subsequent theorem.

Theorem 2. *Let $\widetilde{L}$ be the rank two Heisenberg–Virasoro algebra. For any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, $t_\alpha \in \widetilde{L}$, we choose $H = \frac{1}{\mu}(\eta_1 D_1 + \eta_2 D_2)$ with $\mu := \eta_1\alpha_1 + \eta_2\alpha_2 \neq 0$ and $\eta_1, \eta_2 \in \mathbb{C}$ to* *satisfy* $[H,\ t_\alpha] = t_\alpha$. *Then there exists another non-commutative and non-cocommutative Hopf algebra structure* $(\mathfrak{A}(\widetilde{L})\,[\![x]\!],\ \sigma,\ \tau,\ \Delta,\ \varepsilon,\ S)$ *on* $\mathfrak{A}(\widetilde{L})\,[\![x]\!]$ *over* $\mathbb{C}\,[\![x]\!]$ *with* $\mathfrak{A}(\widetilde{L})\,[\![x]\!]/x\mathfrak{A}(\widetilde{L})\,[\![x]\!] \cong \mathfrak{A}(\widetilde{L})$, *which preserves the product and counit of* $\mathfrak{A}(\widetilde{L})\,[\![x]\!]$, *while the deformed coproduct and antipode are defined as follows.*

$$\begin{aligned}
&\Delta(E_\beta) = E_\beta \otimes (1 - t_\alpha x)^{\frac{\eta}{\mu}} + 1 \otimes E_\beta - b_1 H^{<1>} \otimes (1 - t_\alpha x)^{-1} t_{\beta+\alpha} x,\\
&\Delta(t_\beta) = t_\beta \otimes (1 - t_\alpha x)^{\frac{\eta}{\mu}} + 1 \otimes t_\beta,\\
&\Delta(D_j) = D_j \otimes 1 + 1 \otimes D_j + \alpha_j H^{<1>} \otimes (1 - t_\alpha x)^{-1} t_\alpha x,\\
&S(E_\beta) = -(1 - t_\alpha x)^{-\frac{\eta}{\mu}} (E_\beta + b_1 t_{\beta+\alpha} H_1^{<1>} x),\\
&S(t_\beta) = -(1 - t_\alpha x)^{-\frac{\eta}{\mu}} t_\beta,\\
&S(D_j) = \alpha_j H (1 - t_\alpha x)^{-1} (t_\alpha x - t_\alpha^2 x^2) - D_j,
\end{aligned}$$

where $j = 1,\ 2$, $b_1 = \beta_1\alpha_2 - \beta_2\alpha_1$.

4. Proof of the Main Results

The proof of Theorems 1 and 2 shall be divided into a series of lemmas. The formulas in the following lemma will be used later in the quantizations of the rank two Heisenberg–Virasoro algebra.

Lemma 3. *For any* $\alpha = (\alpha_1,\ \alpha_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, *we choose* $H = \frac{1}{\mu}(\eta_1 D_1 + \eta_2 D_2)$ *and* $G = E_\alpha$ *or* $G = t_\alpha$ *with* $\mu := \eta_1\alpha_1 + \eta_2\alpha_2 \neq 0$ *and* $\eta_1,\ \eta_2 \in \mathbb{C}$ *such that* $[H,\ G] = G$. *For any* $\beta = (\beta_1,\ \beta_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, *denote* $\eta = \eta_1\beta_1 + \eta_2\beta_2$. *Then the following equations hold in* $\mathfrak{A}(\widetilde{L})$ *for* $\lambda \in \mathbb{C}$, $n,\ l \in \mathbb{Z}_+$, $\gamma = (\gamma_1,\ \gamma_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$.

$$E_\beta H_\lambda^{[n]} = H_{\lambda-\frac{\eta}{\mu}}^{[n]} E_\beta,\ t_\beta H_\lambda^{[n]} = H_{\lambda-\frac{\eta}{\mu}}^{[n]} t_\beta,\ E_\beta H_\lambda^{<n>} = H_{\lambda-\frac{\eta}{\mu}}^{<n>} E_\beta,\ t_\beta H_\lambda^{<n>} = H_{\lambda-\frac{\eta}{\mu}}^{<n>} t_\beta, \tag{13}$$

$$G^l H_\lambda^{[n]} = H_{\lambda-l}^{[n]} G^l,\ G^l H_\lambda^{<n>} = H_{\lambda-l}^{<n>} G^l, \tag{14}$$

$$D_j^l H_\lambda^{[n]} = H_\lambda^{[n]} D_j^l,\ D_j^l H_\lambda^{<n>} = H_\lambda^{<n>} D_j^l, \tag{15}$$

$$E_\beta E_\gamma^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\beta_1\gamma_2 - \beta_2\gamma_1)^k E_\gamma^{n-k} E_{\beta+k\gamma}, \tag{16}$$

$$E_\beta t_\gamma^n = t_\gamma^n E_\beta - n(\beta_1\gamma_2 - \beta_2\gamma_1) t_\gamma^{n-1} t_{\beta+\gamma}, \tag{17}$$

$$t_\beta E_\gamma^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (\beta_1\gamma_2 - \beta_2\gamma_1)^k E_\gamma^{n-k} t_{\beta+k\gamma} \tag{18}$$

$$D_j E_\gamma^n = n\gamma_j E_\gamma^n + E_\gamma^n D_j,\ D_j t_\gamma^n = n\gamma_j t_\gamma^n + t_\gamma^n D_j,\ j = 1,\ 2. \tag{19}$$

Proof. For any $\alpha = (\alpha_1,\ \alpha_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, we choose $\mu := \eta_1\alpha_1 + \eta_2\alpha_2 \neq 0$ with $\eta_1,\ \eta_2 \in \mathbb{C}$. Denote $H = \frac{1}{\mu}(\eta_1 D_1 + \eta_2 D_2)$ and $G = E_\alpha$ or $G = t_\alpha$. Then by (2), it is obvious that

$$[H,\ G] = G.$$

For any $\beta = (\beta_1,\ \beta_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, we denote $\eta = \eta_1\beta_1 + \eta_2\beta_2$. Using (2), we have

$$[H,\ E_\beta] = \frac{\eta}{\mu} E_\beta,\ [H,\ t_\beta] = \frac{\eta}{\mu} t_\beta,$$

then

$$E_\beta H = H E_\beta - \frac{\eta}{\mu} E_\beta,\ \ t_\beta H = H t_\beta - \frac{\eta}{\mu} t_\beta.$$

Applying (3) and (4), we obtain

$$E_\beta H_\lambda^{[1]} = E_\beta(H+\lambda) = (H - \tfrac{\eta}{\mu} + \lambda)E_\beta = H^{[1]}_{\lambda-\frac{\eta}{\mu}} E_\beta,$$
$$t_\beta H_\lambda^{[1]} = t_\beta(H+\lambda) = (H - \tfrac{\eta}{\mu} + \lambda)t_\beta = H^{[1]}_{\lambda-\frac{\eta}{\mu}} t_\beta,$$
$$E_\beta H_\lambda^{<1>} = E_\beta(H+\lambda) = (H - \tfrac{\eta}{\mu} + \lambda)E_\beta = H^{<1>}_{\lambda-\frac{\eta}{\mu}} E_\beta,$$
$$t_\beta H_\lambda^{<1>} = t_\beta(H+\lambda) = (H - \tfrac{\eta}{\mu} + \lambda)t_\beta = H^{<1>}_{\lambda-\frac{\eta}{\mu}} t_\beta,$$

which shows that the case $n = 1$ of (13) is true. Suppose that (13) is true for n. Using (5), we can derive

$$E_\beta H_\lambda^{[n+1]} = E_\beta H_\lambda^{[n]} H_{\lambda-n}^{[1]} = H^{[n]}_{\lambda-\frac{\eta}{\mu}} E_\beta H^{[1]}_{\lambda-n} = H^{[n]}_{\lambda-\frac{\eta}{\mu}} H^{[1]}_{\lambda-n-\frac{\eta}{\mu}} E_\beta = H^{[n+1]}_{\lambda-\frac{\eta}{\mu}} E_\beta,$$
$$t_\beta H_\lambda^{[n+1]} = t_\beta H_\lambda^{[n]} H_{\lambda-n}^{[1]} = H^{[n]}_{\lambda-\frac{\eta}{\mu}} t_\beta H^{[1]}_{\lambda-n} = H^{[n]}_{\lambda-\frac{\eta}{\mu}} H^{[1]}_{\lambda-n-\frac{\eta}{\mu}} t_\beta = H^{[n+1]}_{\lambda-\frac{\eta}{\mu}} t_\beta,$$
$$E_\beta H_\lambda^{<n+1>} = E_\beta H_\lambda^{<n>} H_{\lambda+n}^{<1>} = H^{<n>}_{\lambda-\frac{\eta}{\mu}} E_\beta H^{<1>}_{\lambda+n} = H^{<n>}_{\lambda-\frac{\eta}{\mu}} H^{<1>}_{\lambda+n-\frac{\eta}{\mu}} E_\beta = H^{<n+1>}_{\lambda-\frac{\eta}{\mu}} E_\beta,$$
$$t_\beta H_\lambda^{<n+1>} = t_\beta H_\lambda^{<n>} H_{\lambda+n}^{<1>} = H^{<n>}_{\lambda-\frac{\eta}{\mu}} t_\beta H^{<1>}_{\lambda+n} = H^{<n>}_{\lambda-\frac{\eta}{\mu}} H^{<1>}_{\lambda+n-\frac{\eta}{\mu}} t_\beta = H^{<n+1>}_{\lambda-\frac{\eta}{\mu}} t_\beta.$$

Thus (13) follows.

From (7), one has

$$E_\alpha H_\lambda^{[n]} = H_{\lambda-1}^{[n]} E_\alpha,\ t_\alpha H_\lambda^{[n]} = H_{\lambda-1}^{[n]} t_\alpha,$$
$$E_\alpha H_\lambda^{<n>} = H_{\lambda-1}^{<n>} E_\alpha,\ t_\alpha H_\lambda^{<n>} = H_{\lambda-1}^{<n>} t_\alpha.$$

So (14) holds for $l = 1$. Suppose (14) holds for l. Then we obtain

$$E_\alpha^{l+1} H_\lambda^{[n]} = E_\alpha E_\alpha^l H_\lambda^{[n]} = E_\alpha H_{\lambda-l}^{[n]} E_\alpha^l = H_{\lambda-(l+1)}^{[n]} E_\alpha^{l+1},$$
$$t_\alpha^{l+1} H_\lambda^{[n]} = t_\alpha t_\alpha^l H_\lambda^{[n]} = E_\alpha H_{\lambda-l}^{[n]} t_\alpha^l = H_{\lambda-(l+1)}^{[n]} t_\alpha^{l+1},$$
$$E_\alpha^{l+1} H_\lambda^{<n>} = E_\alpha E_\alpha^l H_\lambda^{<n>} = E_\alpha H_{\lambda-l}^{<n>} E_\alpha^l = H_{\lambda-(l+1)}^{<n>} E_\alpha^{l+1},$$
$$t_\alpha^{l+1} H_\lambda^{<n>} = t_\alpha t_\alpha^l H_\lambda^{<n>} = t_\alpha H_{\lambda-l}^{<n>} t_\alpha^l = H_{\lambda-(l+1)}^{<n>} t_\alpha^{l+1}.$$

Hence, (14) holds for all l.

(15) follows from $[D_j,\ H] = 0$ for $j = 1,\ 2$.

For (16), we first prove the following equation by induction on n.

$$E_\beta E_\gamma^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} E_\gamma^{n-k} (ad\, E_\gamma)^k (E_\beta) \tag{20}$$

Because $E_\beta E_\gamma = E_\gamma E_\beta - [E_\gamma, E_\beta]$, it is clear that (20) is true for $n = 1$. Suppose that (20) is true for n, then

$$\begin{aligned}
E_\beta E_\gamma^{n+1} &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} E_\gamma^{n-k} (ad\, E_\gamma)^k (E_\beta) E_\gamma \\
&= \sum_{k=0}^{n} (-1)^k \binom{n}{k} E_\gamma^{n-k} \left[-(ad\, E_\gamma)^{k+1}(E_\beta) + E_\gamma (ad\, E_\gamma)^k (E_\beta) \right] \\
&= \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} E_\gamma^{n-k} (ad\, E_\gamma)^{k+1}(E_\beta) + \sum_{k=0}^{n} (-1)^k \binom{n}{k} E_\gamma^{n+1-k} (ad\, E_\gamma)^k (E_\beta) \\
&= \sum_{k=1}^{n+1} (-1)^k \binom{n}{k-1} E_\gamma^{n+1-k} (ad\, E_\gamma)^k (E_\beta) + \sum_{k=0}^{n} (-1)^k \binom{n}{k} E_\gamma^{n+1-k} (ad\, E_\gamma)^k (E_\beta) \\
&= \sum_{k=1}^{n} (-1)^k \left[\binom{n}{k-1} + \binom{n}{k} \right] E_\gamma^{n+1-k} (ad\, E_\gamma)^k (E_\beta) + (-1)^{n+1} (ad\, E_\gamma)^{n+1} (E_\beta) \\
&\quad + E_\gamma^{n+1} E_\beta \\
&= \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} E_\gamma^{n+1-k} (ad\, E_\gamma)^k (E_\beta).
\end{aligned}$$

Thus, (20) holds for all n. Furthermore, the following equation holds.

$$(ad\, E_\gamma)^k(E_\beta) = (\beta_1\gamma_2 - \beta_2\gamma_1)^k E_{\beta+k\gamma}.$$

Hence, (16) follows. We can similarly obtain (18) by induction. For (17), we have

$$E_\beta t_\gamma = t_\gamma E_\beta - [t_\gamma,\ E_\beta] = t_\gamma E_\beta - (\beta_1\gamma_2 - \beta_2\gamma_1)t_{\beta+\gamma}.$$

So (17) is true for $n = 1$. Suppose that (17) is true for n, then

$$\begin{aligned} E_\beta t_\gamma^{n+1} &= t_\gamma^n E_\beta t_\gamma - n(\beta_1\gamma_2 - \beta_2\gamma_1)t_\gamma^n t_{\beta+\gamma} \\ &= t_\gamma^{n+1} E_\beta - (n+1)(\beta_1\gamma_2 - \beta_2\gamma_1)t_\gamma^n t_{\beta+\gamma}. \end{aligned}$$

Hence, (17) holds for all $n = 1$. Noting that

$$D_j X = \gamma_j X + X D_j \text{ for } X = E_\gamma \text{ or } X = t_\gamma,\ j = 1,\ 2,$$

which imply the case $n = 1$ of (19). Suppose all equations of (19) hold for n. Then

$$\begin{aligned} D_j X^{n+1} &= n\gamma_j X^{n+1} + X^n D_j X \\ &= (n+1)\gamma_j X^{n+1} + X^{n+1} D_j. \end{aligned}$$

Thus, (19) follows. □

For the rank two Heisenberg–Virasoro algebra $\widetilde{L}$, in order to describe a quantization of $\mathfrak{A}(\widetilde{L})$ by a Drinfel'd twist $\mathfrak{F}$ over $\mathfrak{A}(\widetilde{L})[\![x]\!]$, we must explicitly construct such a Drinfel'd twist.

For any $\alpha = (\alpha_1,\ \alpha_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, we choose $H = \frac{1}{\mu}(\eta_1 D_1 + \eta_2 D_2)$ and $G = E_\alpha$ or $G = t_\alpha$ with $\mu := \eta_1\alpha_1 + \eta_2\alpha_2 \neq 0$ and $\eta_1,\ \eta_2 \in \mathbb{C}$ such that $[H,\ G] = G$. For any $\lambda \in \mathbb{C}$, we set

$$\mathfrak{F}_\lambda = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H_\lambda^{[k]} \otimes G^k x^k,\ F_\lambda = \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{<k>} \otimes G^k x^k, \tag{21}$$

$$U_\lambda = \sigma \cdot (S_0 \otimes Id)(F_\lambda),\ V_\lambda = \sigma \cdot (Id \otimes S_0)(\mathfrak{F}_\lambda). \tag{22}$$

For convenience, write $H^{<k>} = H_0^{<k>}$, $H^{[k]} = H_0^{[k]}$, $\mathfrak{F} = \mathfrak{F}_0$, $F = F_0$, $U = U_0$,$V = V_0$. Since $S_0(H_\lambda^{<k>}) = (-1)^k H_{-\lambda}^{[k]}$ and $S_0(G^k) = (-1)^k G^k$, we obtain

$$U_\lambda = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H_{-\lambda}^{[k]} G^k x^k,\ V_\lambda = \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{[k]} G^k x^k. \tag{23}$$

Lemma 4. *Whether $G = E_\alpha$ or $G = t_\alpha$, the following equations hold for any $\lambda,\ \rho \in \mathbb{C}$.*

$$\mathfrak{F}_\lambda F_\rho = 1 \otimes (1 - Gx)^{\lambda-\rho},\ V_\lambda U_\rho = (1 - Gx)^{-(\lambda+\rho)}.$$

Therefore the elements $\mathfrak{F}_\lambda,\ F_\lambda,\ U_\lambda,\ V_\lambda$ are invertible and $\mathfrak{F}_\lambda^{-1} = F_\lambda$, $U_\lambda^{-1} = V_{-\lambda}$. In particular, $\mathfrak{F}^{-1} = F$, $U^{-1} = V$.

Proof. Using (6) and (21), we deduce

$$\begin{aligned} \mathfrak{F}_\lambda F_\rho &= \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H_\lambda^{[k]} \otimes G^k x^k\right) \cdot \left(\sum_{s=0}^{\infty} \frac{1}{s!} H_\rho^{<s>} \otimes G^s x^s\right) \\ &= \sum_{k,s=0}^{\infty} \frac{(-1)^k}{k!s!} H_\lambda^{[k]} H_\rho^{<s>} \otimes G^{k+s} x^{k+s} \\ &= \sum_{t=0}^{\infty} (-1)^t \left(\sum_{k+s=t} \frac{(-1)^s}{k!s!} H_\lambda^{[k]} H_\rho^{<s>}\right) \otimes G^t x^t \\ &= \sum_{t=0}^{\infty} (-1)^t \begin{pmatrix} \lambda-\rho \\ t \end{pmatrix} \otimes G^t x^t \\ &= 1 \otimes (1 - Gx)^{\lambda-\rho}. \end{aligned}$$

Using (7), (14) and (23), one has

$$\begin{aligned} V_\lambda U_\rho &= \left(\sum_{k=0}^{\infty}\frac{1}{k!}H_\lambda^{[k]}G^k x^k\right)\cdot\left(\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}H_{-\rho}^{[s]}G^s x^s\right) \\ &= \sum_{k,s=0}^{\infty}\frac{(-1)^s}{k!s!}H_\lambda^{[k]}H_{-\rho-k}^{[s]}G^{k+s}x^{k+s} \\ &= \sum_{t=0}^{\infty}\left(\sum_{k+s=t}\frac{(-1)^s}{k!s!}H_\lambda^{[k]}H_{-\rho-k}^{[s]}\right)G^{k+s}x^{k+s} \\ &= \sum_{t=0}^{\infty}\binom{\lambda+\rho+t-1}{t}G^t x^t \\ &= (1-Gx)^{-(\lambda+\rho)}. \end{aligned}$$

Hence, Lemma 4 follows. □

The formula in Lemma 5 will be used to prove that $\mathfrak{F}$ defined in (21) is a Drinfel'd twist in Lemma 6.

Lemma 5. *For any positive integer n and $\lambda \in \mathbb{C}$, one can write*

$$\Delta_0(H^{[n]}) = \sum_{k=0}^{n}\binom{n}{k}H_{-\lambda}^{[k]}\otimes H_\lambda^{[n-k]}.$$

In particular, one has $\Delta_0(H^{[n]}) = \sum_{k=0}^{n}\binom{n}{k}H^{[k]}\otimes H^{[n-k]}$.

Proof. We will use induction on n. Obviously, it is true for $n = 1$, since $\Delta_0(H) = H\otimes 1 + 1\otimes H$. Suppose it holds for n. Then we can deduce

$$\begin{aligned} &\Delta_0(H^{[n+1]}) = \Delta_0(H^{[n]})\Delta_0(H-n) \\ &= \left[\sum_{k=0}^{n}\binom{n}{k}H_{-\lambda}^{[k]}\otimes H_\lambda^{[n-k]}\right][(H-\lambda-n)\otimes 1 + 1\otimes(H+\lambda-n) + n(1\otimes 1)] \\ &= \left[\sum_{k=1}^{n-1}\binom{n}{k}H_{-\lambda}^{[k]}\otimes H_\lambda^{[n-k]}\right][(H-\lambda-n)\otimes 1 + 1\otimes(H+\lambda-n)] + X_{-\lambda}^{[n]}\otimes(H+\lambda-n) \\ &+n\left[\sum_{k=0}^{n}\binom{n}{k}H_{-\lambda}^{[k]}\otimes H_\lambda^{[n-k]}\right] + \left(1\otimes H_\lambda^{[n+1]} + H_{-\lambda}^{[n+1]}\otimes 1\right) + (H-\lambda-n)\otimes H_\lambda^{[n]} \\ &= 1\otimes H_\lambda^{[n+1]} + H_{-\lambda}^{[n+1]}\otimes 1 + n\sum_{k=1}^{n-1}\binom{n}{k}H_{-\lambda}^{[k]}\otimes H_\lambda^{[n-k]} + (H-\lambda)\otimes H_\lambda^{[n]} \\ &+X_{-\lambda}^{[n]}\otimes(H+\lambda) + \sum_{k=1}^{n-1}\binom{n}{k}H_{-\lambda}^{[k+1]}\otimes H_\lambda^{[n-k]} + \sum_{k=1}^{n-1}(k-n)\binom{n}{k}H_{-\lambda}^{[k]}\otimes H_\lambda^{[n-k]} \\ &+\sum_{k=1}^{n-1}\binom{n}{k}H_{-\lambda}^{[k]}\otimes H_\lambda^{[n-k+1]} - \sum_{k=1}^{n-1}k\binom{n}{k}H_{-\lambda}^{[k]}\otimes H_\lambda^{[n-k]} \\ &= 1\otimes X_\lambda^{[n+1]} + X_{-\lambda}^{[n+1]}\otimes 1 + \left[\sum_{k=1}^{n-1}\binom{n}{k}H_{-\lambda}^{[k+1]}\otimes H_\lambda^{[n-k]} + (H-\lambda)\otimes H_\lambda^{[n]}\right] \\ &+\left[\sum_{k=1}^{n-1}\binom{n}{k}H_{-\lambda}^{[k]}\otimes H_\lambda^{[n-k+1]} + H_{-\lambda}^{[n]}\otimes(H+\lambda)\right] \\ &= 1\otimes H_\lambda^{[n+1]} + H_{-\lambda}^{[n+1]}\otimes 1 + \sum_{k=1}^{n}\left[\binom{n}{k-1} + \binom{n}{k}\right]H_{-\lambda}^{[k]}\otimes H_\lambda^{[n-k+1]} \\ &= \sum_{k=0}^{n+1}\binom{n+1}{k}H_{-\lambda}^{[k]}\otimes H_\lambda^{[n+1-k]}. \end{aligned}$$

Thus, the lemma follows. □

The following lemma shows that $\mathfrak{F}$ defined in (21) is a Drinfel'd twist.

Lemma 6. *For* $H = \frac{1}{\mu}(\eta_1 D_1 + \eta_2 D_2)$ *with* $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ *and* $\mu = \eta_1\alpha_1 + \eta_2\alpha_2 \neq 0\,(\eta_i \in \mathbb{C})$, $\mathfrak{F} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H^{[k]} \otimes G^k x^k$ *is a Drinfel'd twist on* $\mathfrak{A}(\widetilde{L})[\![x]\!]$, *i.e.,* $\mathfrak{F}$ *satisfies the following equalities, no matter whether* $G = E_\alpha$ *or* $G = t_\alpha$.

$$(\mathfrak{F} \otimes 1)(\Delta_0 \otimes \mathrm{Id})(\mathfrak{F}) = (1 \otimes \mathfrak{F})(\mathrm{Id} \otimes \Delta_0)(\mathfrak{F}), (\varepsilon_0 \otimes \mathrm{Id})(\mathfrak{F}) = 1 \otimes 1 = (\mathrm{Id} \otimes \varepsilon_0)(\mathfrak{F}).$$

Proof. Using Lemma 5, (5) and (14), one obtains

$$\begin{aligned}
&(\mathfrak{F} \otimes 1)(\Delta_0 \otimes \mathrm{Id})(\mathfrak{F}) \\
&= \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H^{[k]} \otimes G^k x^k \otimes 1\right] (\Delta_0 \otimes \mathrm{Id}) \left[\sum_{s=0}^{\infty} \frac{(-1)^s}{s!} H^{[s]} \otimes G^s x^s\right] \\
&= \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H^{[k]} \otimes G^k x^k \otimes 1\right] \cdot \left[\sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \sum_{t=0}^{s} \binom{s}{t} H_{-k}^{[t]} \otimes H_k^{[s-t]} \otimes G^s x^s\right] \\
&= \sum_{k,s=0}^{\infty} \frac{(-1)^{k+s}}{k!s!} x^{k+s} \left[\sum_{t=0}^{s} \binom{s}{t} H^{[k]} H_{-k}^{[t]} \otimes G^k H_k^{[s-t]} \otimes G^s\right] \\
&= \sum_{k,s=0}^{\infty} \frac{(-1)^{k+s}}{k!s!} x^{k+s} \left[\sum_{t=0}^{s} \binom{s}{t} H^{[k+t]} \otimes H^{[s-t]} G^k \otimes G^s\right],
\end{aligned}$$

and,

$$\begin{aligned}
&(1 \otimes \mathfrak{F})(\mathrm{Id} \otimes \Delta_0)(\mathfrak{F}) \\
&= \left[\sum_{m=0}^{\infty} \frac{(-1)^m}{m!} x^m \otimes H^{[m]} \otimes G^m\right] \cdot \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n H^{[n]} \otimes \sum_{l=0}^{n} \binom{n}{l} G^l \otimes G^{n-l}\right] \\
&= \sum_{m,n=0}^{\infty} \frac{(-1)^{m+n}}{m!n!} x^{m+n} \left[\sum_{l=0}^{n} \binom{n}{l} H^{[n]} \otimes H^{[m]} G^l \otimes G^{m+n-l}\right].
\end{aligned}$$

It suffices to establish the validity of the following equality for any fixed $p \in \mathbb{Z}$.

$$\begin{aligned}
&\sum_{k+s=p}^{\infty} \frac{1}{k!s!} x^{k+s} \left[\sum_{t=0}^{s} \binom{s}{t} H^{[k+t]} \otimes H^{[s-t]} G^k \otimes G^s\right] \\
&= \sum_{m+n=p}^{\infty} \frac{1}{m!n!} x^{m+n} \left[\sum_{l=0}^{n} \binom{n}{l} H^{[n]} \otimes H^{[m]} G^l \otimes G^{m+n-l}\right]
\end{aligned}$$

Fixing m, n, l such that $m + n = p$, $0 \leq l \leq n$. Set $k = l$, $k + t = n$. Then $s = p - l$, $s - t = m$. It is obvious that the coefficients of $H^{[n]} \otimes H^{[m]} G^l \otimes G^{p-l}$ in both sides are equal.

The second equality follows from that

$$\begin{aligned}
(\varepsilon_0 \otimes \mathrm{Id})(\mathfrak{F}) &= (\varepsilon_0 \otimes \mathrm{Id}) \left[1 \otimes 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} H^{[k]} \otimes G^k x^k\right] \\
&= (\varepsilon_0 \otimes \mathrm{Id})(1 \otimes 1) = 1 \otimes 1,
\end{aligned}$$

and

$$\begin{aligned}
(\mathrm{Id} \otimes \varepsilon_0)(\mathfrak{F}) &= (\mathrm{Id} \otimes \varepsilon_0) \left[1 \otimes 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} H^{[k]} \otimes G^k x^k\right] \\
&= (\mathrm{Id} \otimes \varepsilon_0)(1 \otimes 1) = 1 \otimes 1.
\end{aligned}$$

Hence, the lemma is proved. □

By Lemma 6, we can carry out the process of twisting the standard Hopf structure $(\mathfrak{A}(\widetilde{L})[\![x]\!], \sigma, \tau, \Delta_0, \varepsilon_0, S_0)$ by the Drinfel'd twist $\mathfrak{F}$.

Lemma 7. *If $G = E_\alpha$ with $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ and $\mu = \eta_1\alpha_1 + \eta_2\alpha_2 \neq 0 (\eta_i \in \mathbb{C})$, then for any $\lambda \in \mathbb{C}$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$, denote $\eta = \eta_1\beta_1 + \eta_2\beta_2$, we have the following identities:*

$$(E_\beta \otimes 1)F_\lambda = F_{\lambda-\frac{\eta}{\mu}}(E_\beta \otimes 1),\ (t_\beta \otimes 1)F_\lambda = F_{\lambda-\frac{\eta}{\mu}}(t_\beta \otimes 1), \tag{24}$$

$$(1 \otimes E_\beta)F_\lambda = \sum_{s=0}^{\infty} (-1)^s b_s F_{\lambda+s}\big(H_\lambda^{<s>} \otimes E_{\beta+s\alpha} x^s\big), \tag{25}$$

$$(1 \otimes t_\beta)F_\lambda = \sum_{s=0}^{\infty} (-1)^s b_s F_{\lambda+s}\big(H_\lambda^{<s>} \otimes t_{\beta+s\alpha} x^s\big), \tag{26}$$

$$E_\beta U_\lambda = U_{\lambda+\frac{\eta}{\mu}} \sum_{s=0}^{\infty} b_s E_{\beta+s\alpha} H_{1-\lambda}^{<s>} x^s,\ t_\beta U_\lambda = U_{\lambda+\frac{\eta}{\mu}} \sum_{s=0}^{\infty} b_s t_{\beta+s\alpha} H_{1-\lambda}^{<s>} x^s, \tag{27}$$

$$D_j U_\lambda = -\alpha_j H_{-\lambda}^{[1]} U_{\lambda+1} E_\alpha x + U_\lambda D_j,\ (D_j \otimes 1)F_\lambda = F_\lambda(D_j \otimes 1), \tag{28}$$

$$(1 \otimes D_j)F_\lambda = F_{\lambda+1}(H_\lambda^{<1>} \otimes \alpha_j E_\alpha x) + F_\lambda(1 \otimes D_j), \tag{29}$$

$$E_\alpha U_\lambda = U_{\lambda+1} E_\alpha,\ t_\alpha U_\lambda = U_{\lambda+1} t_\alpha,\ V_\lambda H_{-\lambda}^{[1]} = H_{-\lambda}^{[1]} V_\lambda - H_\lambda^{[1]} V_{\lambda-1} E_\alpha x, \tag{30}$$

where $j = 1, 2$, and $b_s = \frac{1}{s!}(\beta_1\alpha_2 - \beta_2\alpha_1)^s$, $s \in \mathbb{N}$.

Proof. For (24), using (13), we can deduce

$$\begin{aligned}
(E_\beta \otimes 1)F_\lambda &= \sum_{k=0}^{\infty} \frac{1}{k!} E_\beta H_\lambda^{<k>} \otimes G^k x^k = \sum_{k=0}^{\infty} \frac{1}{k!} H_{\lambda-\frac{\eta}{\mu}}^{<k>} E_\beta \otimes G^k x^k \\
&= \left(\sum_{k=0}^{\infty} \frac{1}{k!} H_{\lambda-\frac{\eta}{\mu}}^{<k>} \otimes G^k x^k \right) (E_\beta \otimes 1) = F_{\lambda-\frac{\eta}{\mu}}(E_\beta \otimes 1), \\
(t_\beta \otimes 1)F_\lambda &= \sum_{k=0}^{\infty} \frac{1}{k!} t_\beta H_\lambda^{<k>} \otimes G^k x^k = \sum_{k=0}^{\infty} \frac{1}{k!} H_{\lambda-\frac{\eta}{\mu}}^{<k>} t_\beta \otimes G^k x^k \\
&= \left(\sum_{k=0}^{\infty} \frac{1}{k!} H_{\lambda-\frac{\eta}{\mu}}^{<k>} \otimes G^k x^k \right) (t_\beta \otimes 1) = F_{\lambda-\frac{\eta}{\mu}}(t_\beta \otimes 1).
\end{aligned}$$

For (25), using (5) and (16), we obtain

$$\begin{aligned}
(1 \otimes E_\beta)F_\lambda &= \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{<k>} \otimes E_\beta G^k x^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{<k>} \otimes \left[\sum_{s=0}^{k} (-1)^s \binom{k}{s} (\beta_1\alpha_2 - \beta_2\alpha_1)^s E_\alpha^{k-s} E_{\beta+s\alpha} \right] x^k \\
&= \sum_{k=0}^{\infty} \left[\sum_{s=0}^{k} \frac{(-1)^s}{(k-s)!s!} H_\lambda^{<k>} \otimes (\beta_1\alpha_2 - \beta_2\alpha_1)^s E_\alpha^{k-s} E_{\beta+s\alpha} \right] x^k \\
&= \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s}{k!s!} H_\lambda^{<k+s>} \otimes (\beta_1\alpha_2 - \beta_2\alpha_1)^s E_\alpha^k E_{\beta+s\alpha} x^{k+s} \\
&= \sum_{s=0}^{\infty} (-1)^s \left(\sum_{k=0}^{\infty} \frac{1}{k!} H_{\lambda+s}^{<k>} \otimes E_\alpha^k x^k \right) \big(b_s H_\lambda^{<s>} \otimes E_{\beta+s\alpha} x^s \big) \\
&= \sum_{s=0}^{\infty} (-1)^s b_s F_{\lambda+s}\big(H_\lambda^{<s>} \otimes E_{\beta+s\alpha} x^s\big),
\end{aligned}$$

where $b_s = \frac{1}{s!}(\beta_1\alpha_2 - \beta_2\alpha_1)^s$, $s \in \mathbb{N}$.

For (26), using (5) and (18), we can deduce

$$\begin{aligned}(1\otimes t_\beta)F_\lambda &= \sum_{k=0}^{\infty}\frac{1}{k!}H_\lambda^{<k>}\otimes t_\beta G^k x^k\\ &= \sum_{k=0}^{\infty}\frac{1}{k!}H_\lambda^{<k>}\otimes\left[\sum_{s=0}^{k}(-1)^s\binom{k}{s}(\beta_1\alpha_2-\beta_2\alpha_1)^s E_\alpha^{k-s}t_{\beta+s\alpha}\right]x^k\\ &= \sum_{k=0}^{\infty}\left[\sum_{s=0}^{k}\frac{(-1)^s}{(k-s)!s!}H_\lambda^{<k>}\otimes(\beta_1\alpha_2-\beta_2\alpha_1)^s E_\alpha^{k-s}t_{\beta+s\alpha}\right]x^k\\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s}{k!s!}H_\lambda^{<k+s>}\otimes(\beta_1\alpha_2-\beta_2\alpha_1)^s E_\alpha^{k}t_{\beta+s\alpha}x^{k+s}\\ &= \sum_{s=0}^{\infty}(-1)^s\left(\sum_{k=0}^{\infty}\frac{1}{k!}H_{\lambda+s}^{<k>}\otimes E_\alpha^k x^k\right)\left(b_s H_\lambda^{<s>}\otimes t_{\beta+s\alpha}x^s\right)\\ &= \sum_{s=0}^{\infty}(-1)^s b_s F_{\lambda+s}\left(H_\lambda^{<s>}\otimes t_{\beta+s\alpha}x^s\right).\end{aligned}$$

For (27), using (5), (13), (14), (16), (18), we obtain

$$\begin{aligned}E_\beta U_\lambda &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}E_\beta H_{-\lambda}^{[k]}G^k x^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}E_\beta G^k x^k\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}\left[\sum_{s=0}^{k}(-1)^s\binom{k}{s}(\beta_1\alpha_2-\beta_2\alpha_1)^s E_\alpha^{k-s}E_{\beta+s\alpha}\right]x^k\\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{(-1)^{k+s}}{(k-s)!s!}(\beta_1\alpha_2-\beta_2\alpha_1)^s H_{-\lambda-\frac{\eta}{\mu}}^{[k]}E_\alpha^{k-s}E_{\beta+s\alpha}x^k\\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s}{k!}b_s H_{-\lambda-\frac{\eta}{\mu}}^{[k+s]}E_\alpha^k E_{\beta+s\alpha}x^{k+s}\\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k}{k!}b_s H_{-\lambda-\frac{\eta}{\mu}}^{[k]}H_{-\lambda-\frac{\eta}{\mu}-k}^{[s]}E_\alpha^k E_{\beta+s\alpha}x^{k+s}\\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k}{k!}b_s H_{-\lambda-\frac{\eta}{\mu}}^{[k]}E_\alpha^k H_{-\lambda-\frac{\eta}{\mu}}^{[s]}E_{\beta+s\alpha}x^{k+s}\\ &= \sum_{s=0}^{\infty}b_s\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}E_\alpha^k x^k\right)H_{-\lambda-\frac{\eta}{\mu}}^{[s]}E_{\beta+s\alpha}x^s\\ &= U_{\lambda+\frac{\eta}{\mu}}\sum_{s=0}^{\infty}b_s H_{-\lambda-\frac{\eta}{\mu}}^{[s]}E_{\beta+s\alpha}x^s = U_{\lambda+\frac{\eta}{\mu}}\sum_{s=0}^{\infty}b_s E_{\beta+s\alpha}H_{-\lambda+1}^{<s>}x^s,\end{aligned}$$

and

$$\begin{aligned}t_\beta U_\lambda &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t_\beta H_{-\lambda}^{[k]}G^k x^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}t_\beta G^k x^k\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}\left[\sum_{s=0}^{k}(-1)^s\binom{k}{s}(\beta_1\alpha_2-\beta_2\alpha_1)^s E_\alpha^{k-s}t_{\beta+s\alpha}\right]x^k\\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{k}\frac{(-1)^{k+s}}{(k-s)!s!}(\beta_1\alpha_2-\beta_2\alpha_1)^s H_{-\lambda-\frac{\eta}{\mu}}^{[k]}E_\alpha^{k-s}t_{\beta+s\alpha}x^k\\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^s}{k!}b_s H_{-\lambda-\frac{\eta}{\mu}}^{[k+s]}E_\alpha^k t_{\beta+s\alpha}x^{k+s}\\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k}{k!}b_s H_{-\lambda-\frac{\eta}{\mu}}^{[k]}H_{-\lambda-\frac{\eta}{\mu}-k}^{[s]}E_\alpha^k t_{\beta+s\alpha}x^{k+s}\\ &= \sum_{k=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^k}{k!}b_s H_{-\lambda-\frac{\eta}{\mu}}^{[k]}E_\alpha^k H_{-\lambda-\frac{\eta}{\mu}}^{[s]}t_{\beta+s\alpha}x^{k+s}\\ &= \sum_{s=0}^{\infty}b_s\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}E_\alpha^k x^k\right)t_{\beta+s\alpha}H_{-\lambda+s}^{[s]}x^s\\ &= U_{\lambda+\frac{\eta}{\mu}}\sum_{s=0}^{\infty}b_s t_{\beta+s\alpha}H_{-\lambda+s}^{[s]}x^s = U_{\lambda+\frac{\eta}{\mu}}\sum_{s=0}^{\infty}b_s t_{\beta+s\alpha}H_{1-\lambda}^{<s>}x^s.\end{aligned}$$

For (28) and (29), using (5), (15), (19), we obtain

$$\begin{aligned}
D_jU_\lambda &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H_{-\lambda}^{[k]} D_j E_\alpha^k x^k \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!} \alpha_j H_{-\lambda}^{[k]} E_\alpha^k x^k + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H_{-\lambda}^{[k]} E_\alpha^k D_j x^k \\
&= -\alpha_j H_{-\lambda}^{[1]} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} H_{-\lambda-1}^{[k-1]} E_\alpha^{k-1} x^{k-1} E_\alpha x + \left[\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H_{-\lambda}^{[k]} E_\alpha^k x^k\right] D_j \\
&= -\alpha_j H_{-\lambda}^{[1]} U_{\lambda+1} E_\alpha x + U_\lambda D_j, \\
(D_j \otimes 1)F_\lambda &= \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{<k>} D_j \otimes G^k x^k \\
&= \left(\sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{<k>} \otimes G^k x^k\right)(D_j \otimes 1) \\
&= F_\lambda (D_j \otimes 1), \\
(1 \otimes D_j)F_\lambda &= \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{<k>} \otimes D_j E_\alpha^k x^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{<k>} \otimes (k\alpha_j E_\alpha^k + E_\alpha^k D_j) x^k \\
&= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} H_\lambda^{<k>} \otimes \alpha_j E_\alpha^k x^k + \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{<k>} \otimes E_\alpha^k D_j x^k \\
&= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} H_\lambda^{<1>} H_{\lambda+1}^{<k-1>} \otimes \alpha_j E_\alpha^k x^k + \left(\sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{<k>} \otimes E_\alpha^k x^k\right)(1 \otimes D_j) \\
&= \left(\sum_{k=1}^{\infty} \frac{1}{(k-1)!} H_{\lambda+1}^{<k-1>} \otimes E_\alpha^{k-1} x^{k-1}\right)(H_\lambda^{<1>} \otimes \alpha_j E_\alpha x) + F_\lambda (1 \otimes D_j) \\
&= F_{\lambda+1}(H_\lambda^{<1>} \otimes \alpha_j E_\alpha x) + F_\lambda (1 \otimes D_j).
\end{aligned}$$

For (30), using (5) and (13), we deduce

$$E_\alpha U_\lambda = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} E_\alpha H_{-\lambda}^{[k]} E_\alpha^k x^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H_{-\lambda-1}^{[k]} E_\alpha^{k+1} x^k = U_{\lambda+1} E_\alpha,$$

$$t_\alpha U_\lambda = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} t_\alpha H_{-\lambda}^{[k]} E_\alpha^k x^k = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} H_{-\lambda-1}^{[k]} t_\alpha E_\alpha^k x^k = U_{\lambda+1} t_\alpha,$$

$$\begin{aligned}
V_\lambda H_{-\lambda}^{[1]} &= \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{[k]} E_\alpha^k x^k H_{-\lambda}^{[1]} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{[k]} H_{-\lambda-k}^{[1]} E_\alpha^k x^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{[k]} (H-\lambda) E_\alpha^k x^k - \sum_{k=1}^{\infty} \frac{1}{(k-1)!} H_\lambda^{[k]} E_\alpha^k x^k \\
&= H_{-\lambda}^{[1]} \sum_{k=0}^{\infty} \frac{1}{k!} H_\lambda^{[k]} E_\alpha^k x^k - \sum_{k=1}^{\infty} \frac{1}{(k-1)!} H_\lambda^{[1]} H_{\lambda-1}^{[k-1]} E_\alpha^k x^k \\
&= H_{-\lambda}^{[1]} V_\lambda - H_\lambda^{[1]} V_{\lambda-1} E_\alpha x.
\end{aligned}$$

Therefore, Lemma 7 is proved. □

The proof of the first principal result in this paper is as follows.

Proof of Theorem 1. For any E_β, t_β with $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2 \setminus \{0\}$ and $D_j \in \widetilde{L}$, $j = 1, 2$, by Lemmas 2 and 4 and (24)–(26), (28) and (29), we obtain

$$\begin{aligned}
&\Delta(E_\beta) = \mathfrak{F}\Delta_0(E_\beta)\mathfrak{F}^{-1} = \mathfrak{F}(E_\beta \otimes 1)\mathfrak{F}^{-1} + \mathfrak{F}(1 \otimes E_\beta)\mathfrak{F}^{-1}\\
&= \mathfrak{F}(E_\beta \otimes 1)F + \mathfrak{F}(1 \otimes E_\beta)F\\
&= \mathfrak{F}F_{-\frac{\eta}{\mu}}(E_\beta \otimes 1) + \mathfrak{F}\sum_{s=0}^{\infty}(-1)^s b_s F_s\left(H^{<s>} \otimes E_{\beta+s\alpha}x^s\right)\\
&= \left(1 \otimes (1-E_\alpha x)^{\frac{\eta}{\mu}}\right)(E_\beta \otimes 1) + \sum_{s=0}^{\infty}(-1)^s b_s\left(1 \otimes (1-E_\alpha x)^{-s}\right)\left(H^{<s>} \otimes E_{\beta+s\alpha}x^s\right)\\
&= E_\beta \otimes (1-E_\alpha x)^{\frac{\eta}{\mu}} + \sum_{s=0}^{\infty}(-1)^s b_s\left(H^{<s>} \otimes (1-E_\alpha x)^{-s}E_{\beta+s\alpha}x^s\right),\\
&\Delta(t_\beta) = \mathfrak{F}\Delta_0(t_\beta)\mathfrak{F}^{-1} = \mathfrak{F}(t_\beta \otimes 1)\mathfrak{F}^{-1} + \mathfrak{F}(1 \otimes t_\beta)\mathfrak{F}^{-1}\\
&= \mathfrak{F}(t_\beta \otimes 1)F + \mathfrak{F}(1 \otimes t_\beta)F\\
&= \mathfrak{F}F_{-\frac{\eta}{\mu}}(t_\beta \otimes 1) + \mathfrak{F}\sum_{s=0}^{\infty}(-1)^s b_s F_s\left(H^{<s>} \otimes t_{\beta+s\alpha}x^s\right)\\
&= \left(1 \otimes (1-E_\alpha x)^{\frac{\eta}{\mu}}\right)(t_\beta \otimes 1) + \sum_{s=0}^{\infty}(-1)^s b_s\left(1 \otimes (1-E_\alpha x)^{-s}\right)\left(H^{<s>} \otimes t_{\beta+s\alpha}x^s\right)\\
&= t_\beta \otimes (1-E_\alpha x)^{\frac{\eta}{\mu}} + \sum_{s=0}^{\infty}(-1)^s b_s\left(H^{<s>} \otimes (1-E_\alpha x)^{-s}t_{\beta+s\alpha}x^s\right),
\end{aligned}$$

$$\begin{aligned}
\Delta(D_j) &= \mathfrak{F}\Delta_0(D_j)\mathfrak{F}^{-1} = \mathfrak{F}(D_j \otimes 1)\mathfrak{F}^{-1} + \mathfrak{F}(1 \otimes D_j)\mathfrak{F}^{-1}\\
&= \mathfrak{F}(D_j \otimes 1)F + \mathfrak{F}(1 \otimes D_j)F\\
&= \mathfrak{F}F(D_j \otimes 1) + \mathfrak{F}\left[F_1\left(H^{<1>} \otimes \alpha_j E_\alpha x\right) + F(1 \otimes D_j)\right]\\
&= D_j \otimes 1 + 1 \otimes D_j + \alpha_j H^{<1>} \otimes (1-E_\alpha x)^{-1}E_\alpha x,
\end{aligned}$$

where $\eta = \eta_1\beta_1 + \eta_2\beta_2$, $\mu = \eta_1\alpha_1 + \eta_2\alpha_2 \neq 0\ (\eta_i \in \mathbb{C})$, $b_s = \frac{1}{s!}(\beta_1\alpha_2 - \beta_2\alpha_1)^s$, $s \in \mathbb{N}$, $j = 1, 2$.

By Lemma 4, (27) and (30), we deduce

$$\begin{aligned}
S(E_\beta) &= f\,S_0(E_\beta)f^{-1} = -VE_\beta U = -VU_{\frac{\eta}{\mu}}\sum_{s=0}^{\infty} b_s E_{\beta+s\alpha}H_1^{<s>}x^s\\
&= -(1-E_\alpha x)^{-\frac{\eta}{\mu}}\sum_{s=0}^{\infty} b_s E_{\beta+s\alpha}H_1^{<s>}x^s,\\
S(t_\beta) &= f\,S_0(t_\beta)f^{-1} = -Vt_\beta U = -VU_{\frac{\eta}{\mu}}\sum_{s=0}^{\infty} b_s t_{\beta+s\alpha}H_1^{<s>}x^s\\
&= -(1-E_\alpha x)^{-\frac{\eta}{\mu}}\sum_{s=0}^{\infty} b_s t_{\beta+s\alpha}H_1^{<s>}x^s,\\
S(D_j) &= f\,S_0(D_j)f^{-1} = -VD_j U = -V(-\alpha_j H^{[1]}U_1 E_\alpha x + UD_j)\\
&= \alpha_j(HV - HV_{-1}E_\alpha x)U_1 E_\alpha x - D_j,\\
&= \alpha_j H(1-E_\alpha x)^{-1}E_\alpha x - \alpha_j HV_{-1}U_2 E_\alpha^2 x^2 - D_j\\
&= \alpha_j H(1-E_\alpha x)^{-1}(E_\alpha x - E_\alpha^2 x^2) - D_j.
\end{aligned}$$

Therefore we complete the proof of Theorem 1. □

Lemma 8. *If $G = t_\alpha$ with $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2 \setminus \{0\}$ and $\mu = \eta_1\alpha_1 + \eta_2\alpha_2 \neq 0\,(\eta_i \in \mathbb{C})$, then for any $\lambda \in \mathbb{C}$, $\beta = (\beta_1, \beta_2) \in \mathbb{Z}^2 \setminus \{0\}$, denote $\eta = \eta_1\beta_1 + \eta_2\beta_2$, one has*

$$(E_\beta \otimes 1)F_\lambda = F_{\lambda-\frac{\eta}{\mu}}(E_\beta \otimes 1),\ (t_\beta \otimes 1)F_\lambda = F_{\lambda-\frac{\eta}{\mu}}(t_\beta \otimes 1), \tag{31}$$

$$(1 \otimes E_\beta)F_\lambda = F_\lambda(1 \otimes E_\beta) - b_1F_{\lambda+1}\left(H_\lambda^{<1>} \otimes t_{\beta+\alpha}x\right),\ (1 \otimes t_\beta)F_\lambda = F_\lambda(1 \otimes t_\beta), \tag{32}$$

$$E_\beta U_\lambda = U_{\lambda+\frac{\eta}{\mu}}E_\beta + b_1U_{\lambda+\frac{\eta}{\mu}}t_{\beta+\alpha}H_{1-\lambda}^{<1>}x,\ t_\beta U_\lambda = U_{\lambda+\frac{\eta}{\mu}}t_\beta, \tag{33}$$

$$D_jU_\lambda = -\alpha_jH_{-\lambda}^{[1]}U_{\lambda+1}t_\alpha x + U_\lambda D_j,\ (D_j \otimes 1)F_\lambda = F_\lambda(D_j \otimes 1), \tag{34}$$

$$(1 \otimes D_j)F_\lambda = F_{\lambda+1}(H_\lambda^{<1>} \otimes \alpha_jt_\alpha x) + F_\lambda(1 \otimes D_j), \tag{35}$$

$$E_\alpha U_\lambda = U_{\lambda+1}E_\alpha,\ t_\alpha U_\lambda = U_{\lambda+1}t_\alpha,\ V_\lambda H_{-\lambda}^{[1]} = H_{-\lambda}^{[1]}V_\lambda - H_\lambda^{[1]}V_{\lambda-1}t_\alpha x, \tag{36}$$

where $j = 1,\ 2$, and $b_1 = \beta_1\alpha_2 - \beta_2\alpha_1$.

Proof. (31), (34)–(36) can be derived in a manner analogous to those presented in Lemma 7. The first equation of (32) is a consequence of the following equation.

$$\begin{aligned}
(1 \otimes E_\beta)F_\lambda &= \sum_{k=0}^{\infty}\frac{1}{k!}H_\lambda^{<k>} \otimes E_\beta t_\alpha^k x^k \\
&= \sum_{k=0}^{\infty}\frac{1}{k!}H_\lambda^{<k>} \otimes \left[t_\alpha^kE_\beta - k(\beta_1\alpha_2 - \beta_2\alpha_1)t_\alpha^{k-1}t_{\beta+\alpha}\right]x^k \\
&= \left(\sum_{k=0}^{\infty}\frac{1}{k!}H_\lambda^{<k>} \otimes t_\alpha^k x^k\right)(1 \otimes E_\beta) - \sum_{k=1}^{\infty}\frac{1}{(k-1)!}H_\lambda^{<k>} \otimes (\beta_1\alpha_2 - \beta_2\alpha_1)t_\alpha^{k-1}t_{\beta+\alpha}x^k \\
&= F_\lambda(1 \otimes E_\beta) - \sum_{k=0}^{\infty}\frac{1}{k!}H_\lambda^{<k+1>} \otimes (\beta_1\alpha_2 - \beta_2\alpha_1)t_\alpha^k t_{\beta+\alpha}x^{k+1} \\
&= F_\lambda(1 \otimes E_\beta) - \left(\sum_{k=0}^{\infty}\frac{1}{k!}H_{\lambda+1}^{<k>} \otimes t_\alpha^k x^k\right)\left(H_\lambda^{<1>} \otimes b_1t_{\beta+\alpha}x\right) \\
&= F_\lambda(1 \otimes E_\beta) - b_1F_{\lambda+1}\left(H_\lambda^{<1>} \otimes t_{\beta+\alpha}x\right).
\end{aligned}$$

where $b_1 = \beta_1\alpha_2 - \beta_2\alpha_1$. For the latter part of (32), one has

$$\begin{aligned}
(1 \otimes t_\beta)F_\lambda = \sum_{k=0}^{\infty}\frac{1}{k!}H_\lambda^{<k>} \otimes t_\beta t_\alpha^k x^k &= \left(\sum_{k=0}^{\infty}\frac{1}{k!}H_\lambda^{<k>} \otimes t_\alpha^k x^k\right)(1 \otimes t_\beta) \\
&= F_\lambda(1 \otimes t_\beta).
\end{aligned}$$

For (33), using (5), (13), (14) and (17), one obtains

$$\begin{aligned}
E_\beta U_\lambda &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}E_\beta H_{-\lambda}^{[k]}t_\alpha^k x^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}E_\beta t_\alpha^k x^k \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}\left[t_\alpha^kE_\beta - k(\beta_1\alpha_2 - \beta_2\alpha_1)t_\alpha^{k-1}t_{\beta+\alpha}\right]x^k \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}t_\alpha^k x^kE_\beta - \sum_{k=1}^{\infty}\frac{(-1)^k}{(k-1)!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}(\beta_1\alpha_2 - \beta_2\alpha_1)t_\alpha^{k-1}t_{\beta+\alpha}x^k \\
&= U_{\lambda+\frac{\eta}{\mu}}E_\beta - \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k+1]}(\beta_1\alpha_2 - \beta_2\alpha_1)t_\alpha^k t_{\beta+\alpha}x^{k+1} \\
&= U_{\lambda+\frac{\eta}{\mu}}E_\beta - b_1\left[\sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}t_\alpha^k x^k\right]H_{-\lambda-\frac{\eta}{\mu}}^{[1]}t_{\beta+\alpha}x \\
&= U_{\lambda+\frac{\eta}{\mu}}E_\beta + b_1U_{\lambda+\frac{\eta}{\mu}}H_{-\lambda-\frac{\eta}{\mu}}^{[1]}t_{\beta+\alpha}x \\
&= U_{\lambda+\frac{\eta}{\mu}}E_\beta + b_1U_{\lambda+\frac{\eta}{\mu}}t_{\beta+\alpha}H_{1-\lambda}^{<1>}x,
\end{aligned}$$

$$\begin{aligned}
t_\beta U_\lambda &= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t_\beta H_{-\lambda}^{[k]}t_\alpha^k x^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}t_\beta t_\alpha^k x^k \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}H_{-\lambda-\frac{\eta}{\mu}}^{[k]}t_\alpha^k x^kt_\beta = U_{\lambda+\frac{\eta}{\mu}}t_\beta
\end{aligned}$$

Thus, Lemma 8 is proved. □

The proof of the second principal result in this paper is presented as follows.

Proof of Theorem 2. For any E_β, t_β with $\beta \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$ and $D_j \in \widetilde{L}$, $j = 1,\ 2$, by Lemmas 2, 4 and 8, we deduce

$$\begin{aligned}
\Delta(E_\beta) &= \mathfrak{F}\Delta_0(E_\beta)\mathfrak{F}^{-1} = \mathfrak{F}(E_\beta \otimes 1)F + \mathfrak{F}(1 \otimes E_\beta)F \\
&= \mathfrak{F}F_{-\frac{\eta}{\mu}}(E_\beta \otimes 1) + \mathfrak{F}\big[F(1 \otimes E_\beta) - b_1 F_1\big(H^{<1>} \otimes t_{\beta+\alpha}x\big)\big] \\
&= \Big[1 \otimes (1 - t_\alpha x)^{\frac{\eta}{\mu}}\Big](E_\beta \otimes 1) + (1 \otimes 1)(1 \otimes E_\beta) \\
&\quad -b_1\Big[1 \otimes (1 - t_\alpha x)^{-1}\Big]\big(H^{<1>} \otimes t_{\beta+\alpha}x\big) \\
&= E_\beta \otimes (1 - t_\alpha x)^{\frac{\eta}{\mu}} + 1 \otimes E_\beta - b_1 H^{<1>} \otimes (1 - t_\alpha x)^{-1} t_{\beta+\alpha}x, \\
\Delta(t_\beta) &= \mathfrak{F}\Delta_0(t_\beta)\mathfrak{F}^{-1} = \mathfrak{F}(t_\beta \otimes 1)F + \mathfrak{F}(1 \otimes t_\beta)F \\
&= \mathfrak{F}F_{-\frac{\eta}{\mu}}(t_\beta \otimes 1) + \mathfrak{F}F(1 \otimes t_\beta) = t_\beta \otimes (1 - t_\alpha x)^{\frac{\eta}{\mu}} + 1 \otimes t_\beta, \\
\Delta(D_j) &= \mathfrak{F}\Delta_0(D_j)\mathfrak{F}^{-1} = \mathfrak{F}(D_j \otimes 1)F + \mathfrak{F}(1 \otimes D_j)F \\
&= \mathfrak{F}F(D_j \otimes 1) + \mathfrak{F}\big[F_1\big(H^{<1>} \otimes \alpha_j t_\alpha x\big) + F(1 \otimes D_j)\big] \\
&= D_j \otimes 1 + 1 \otimes D_j + \alpha_j H^{<1>} \otimes (1 - t_\alpha x)^{-1} t_\alpha x, \\
S(E_\beta) &= f\, S_0(E_\beta) f^{-1} = -V E_\beta U = -V\Big(U_{\frac{\eta}{\mu}} E_\beta + b_1 U_{\frac{\eta}{\mu}} t_{\beta+\alpha} H_1^{<1>} x\Big) \\
&= -(1 - t_\alpha x)^{-\frac{\eta}{\mu}}\big(E_\beta + b_1 t_{\beta+\alpha} H_1^{<1>} x\big), \\
S(t_\beta) &= f\, S_0(t_\beta) f^{-1} = -V t_\beta U = -V U_{\frac{\eta}{\mu}} t_\beta = -(1 - t_\alpha x)^{-\frac{\eta}{\mu}} t_\beta,
\end{aligned}$$

$$\begin{aligned}
S(D_j) &= f\, S_0(D_j) f^{-1} = -V D_j U = -V(-\alpha_j H^{[1]} U_1 t_\alpha x + U D_j) \\
&= \alpha_j (HV - HV_{-1} t_\alpha x) U_1 t_\alpha x - D_j \\
&= \alpha_j H (1 - t_\alpha x)^{-1} t_\alpha x - \alpha_j H (1 - t_\alpha x)^{-1} E_\alpha^2 x^2 - D_j \\
&= \alpha_j H (1 - t_\alpha x)^{-1} (t_\alpha x - t_\alpha^2 x^2) - D_j,
\end{aligned}$$

where $j = 1,\ 2$, $b_1 = \beta_1\alpha_2 - \beta_2\alpha_1$. Therefore, Theorem 2 is proved. □

5. Conclusions

Heisenberg–Virasoro and in general Virasoro algebras are useful in Conformal Field Theory (CFT). In [28], the authors shows this clearly by applying these algebras to CFT. The papers by O. B. Fournier and P. Mathieu on and around the subject are useful to see the consequences, notably the Virasoro characters (cf. [29–31] and other similar papers). Heisenberg–Virasoro algebras and in general Virasoro algebras also have many applications in Quantum Mechanics and Quantum Field Theory (QFT) (cf. [32–35], etc.). Furthermore, Heisenberg–Virasoro algebras and in general Virasoro algebras hold a profound connection with Vertex operator algebras (VOA) (cf. [36,37], etc.). The exploration of vertex operator algebras in relation to Virasoro algebras serves as the algebraic cornerstone for investigating minimal models in CFT.

Quantum groups play important roles in many fields such as mathematics and physics. It is an important and interesting approach to construct new quantum groups through the quantization of Lie bialgebras. In this paper, the explicit formulas of the quantization of the rank two Heisenberg–Virasoro algebra $\widetilde{L}$ by Drinfel'd twists (see Theorems 1 and 2) are presented, Lie bialgebra structures of which were considered by the authors in a recent paper [23]. It is found that the quantization of $\widetilde{L}$ is not unique since there are two types of Drinfel'd twists (see Lemma 6). Our results broaden the scope of examples encompassed by non-commutative and non-cocommutative Hopf algebras.

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References

1. Drinfel'd, V.G. Hopf algebras and the quantum Yang–Baxter equation. *Soviet Math. Dokl.* **1985**, *32*, 254–258.
2. Drinfel'd, V.G. Quantum Groups. In *Proceeding of the International Congress of Mathematicians, Berkeley, CA, USA, 3–11 August 1986*; American Mathematical Society: Providence, RI, USA, 1987; Volume 1–2, pp. 798–820.
3. Jimbo, M. Aq-difference analogue of U(g) and the Yang–Baxter equation. *Lett. Math. Phys.* **1985**, *10*, 63–69. [CrossRef]
4. Etingof, P.; Schiffmann, O. *Lectures on Quantum Groups*, 2nd ed.; International Press: Somerville, MA, USA, 2002.
5. Grunspan, C. Quantizations of the Witt algebra and of simple Lie algebras in characteristic p. *J. Algebra* **2004**, *280*, 145–161. [CrossRef]
6. Giaquinto, A.; Zhang, J. Bialgebra action, twists and universal deformation formulas. *J. Pure Appl. Algebra* **1998**, *128*, 133–151. [CrossRef]
7. Hu, N.; Wang, X. Quantizations of generalized-Witt algebra and of Jacobson-Witt algebra in the modular case. *J. Algebra* **2007**, *312*, 902–929. [CrossRef]
8. Song, G.; Su, Y. Lie bialgebras of generalized Witt type. *Sci. China Ser. A* **2006**, *49*, 533–544. [CrossRef]
9. Etingof, P.; Kazhdan, D. Quantization of Lie bialgebras I. *Selecta Math. New Ser.* **1996**, *2*, 1–41. [CrossRef]
10. Etingof, P.; Kazhdan, D. Quantization of Lie bialgebras, part VI: Quantization of generalized Kac–Moody Algebras. *Transform. Groups* **2008**, *13*, 527–539. [CrossRef]
11. Song, G.; Su, Y.; Wu, Y. Quantization of generalized Virasoro-like algebras. *Linear Algebra Appl.* **2008**, *428*, 2888–2899. [CrossRef]
12. Cheng, Y.; Su, Y. Quantization of Lie algebras of Block type. *Acta Math. Sci.* **2010**, *30B*, 1134–1142.
13. Li, J.; Su, Y. Quantizations of the W-Algebra W (2, 2). *Acta Math. Sin. Engl. Ser.* **2011**, *27*, 647–656. [CrossRef]
14. Su, Y.; Yuan, L. Quantization of Schrödinger-Virasoro Lie algebra. *Front. Math. China* **2010**, *5*, 701–715. [CrossRef]
15. Wu, Y.; Song, G.; Su, Y. Lie bialgebras of generalized Virasoro-like type. *Acta Math. Sin. Engl. Ser.* **2006**, *22*, 1915–1922. [CrossRef]
16. Cheng, Y.; Song, G.; Xin, B. Lie bialgebra structures on Lie algebras of Block type. *Algebra Colloq.* **2009**, *16*, 677–690. [CrossRef]
17. Li, J.; Su, Y. Lie bialgebra structures on the W-algebra W (2, 2). *Algebra Colloq.* **2010**, *17*, 181–190. [CrossRef]
18. Han, J.; Li, J.; Su, Y. Lie bialgebra structures on the Schrödinger-Virasoro Lie algebra. *J. Math. Phys.* **2009**, *50*, 083504. [CrossRef]
19. Yue, X.; Jiang, Q.; Xin, B. Quantization of Lie algebras of generalized Weyl type. *Algebra Colloq.* **2009**, *16*, 437–448. [CrossRef]
20. Chen, H.; Shen, R.; Zhang, J. Quantization on generalized Heisenberg-Virasoro algebra. *Acta Math. Sin. Chin. Ser.* **2014**, *57*, 109–116.
21. Yuan, L.; He, C. Lie super-bialgebra and quantization of the super Virasoro algebra. *J. Math. Phys.* **2016**, *57*, 053508. [CrossRef]
22. Su, Y.; Yue, X.; Zhu, X. Quantization of the Lie superalgebra of Witt type. *J. Geom. Phys.* **2021**, *160*, 103987. [CrossRef]
23. Su, Y.; Chen, X. Lie Bialgebras on the Rank Two Heisenberg–Virasoro Algebra. *Mathematics* **2023**, *11*, 1030. [CrossRef]
24. Xue, M.; Lin, W.; Tan, S. Central extension, derivations and automorphism group for Lie algebras arising from 2-dimensional torus. *J. Lie Theory* **2006**, *16*, 139–153.
25. Tan, S.; Wang, Q.; Xu, C. On Whittaker modules for a Lie algebra arising from the 2-dimensional torus. *Pac. J. Math.* **2015**, *273*, 147–167. [CrossRef]
26. Li, Z.; Tan, S. Verma module for rank two Heisenberg-Virasoro algebra. *Sci. China Math.* **2020**, *63*, 1259–1270. [CrossRef]
27. Chari, V.; Pressley, A. *A Guide to Quantum Groups*; Cambridge University Press: Cambridge, UK, 1995.
28. Di Francesco, P.; Mathieu, P.; Sénéchal, D. Quantum Field Theory. In *Conformal Field Theory. Graduate Texts in Contemporary Physics*; Springer: New York, NY, USA, 1997.
29. Blondeau-Fournier, O.; Mathieu, P.; Welsh, T.A. A quartet of fermionic expressions for M(k, 2k±1) Virasoro characters via half-lattice paths. *Nucl. Phys. B* **2017**, *924*, 643–683. [CrossRef]
30. Mathieu, P.; Blondeau-Fournier, O.; Wood, S.; Ridout, D. The super-Virasoro singular vectors and jack superpolynomials relationship revisited. *Nucl. Phys. B* **2016**, *913*, 34–63.
31. Blondeau-Fournier, O.; Mathieu, P.; Welsh, T.A. A Bijection Between Paths for the $M(p, 2p + 1)$ Minimal Model Virasoro Characters. *Ann. Henri Poincaré* **2010**, *11*, 101–125. [CrossRef]
32. Oh, C.H.; Singh, K. Realizations of the q-Heisenberg and q-Virasoro algebras. *J. Phys. A* **1994**, *27*, 3439–3444. [CrossRef]
33. Adamović, D.; Radobolja, G. Free field realization of the twisted Heisenberg-Virasoro algebra at level zero and its applications. *J. Pure Appl. Algebra* **2015**, *219*, 4322–4342. [CrossRef]
34. Aizawa, N.; Sato, H. q-Deformation of the Virasoro algebra with central extension. *Phys. Lett. B* **1991**, *256*, 185–190. [CrossRef]
35. Frenkel, E.; Reshetikhin, N. Quantum affine algebras and deformations of the Virasoro and W-algebras. *Commun. Math. Phys.* **1996**, *178*, 237–264. [CrossRef]

36. Guo, H.; Wang, Q. Twisted Heisenberg-Virasoro vertex operator algebra. *Glas. Mat.* **2019**, *54*, 369–407. [CrossRef]
37. Wang, W. Rationality of Virasoro vertex operator algebras. *Int. Math. Res. Not.* **1993**, *7*, 197–211. [CrossRef]