

COMPACTIFIED IMAGINARY LIOUVILLE THEORY

COLIN GUILLARMOU, ANTTI KUPIAINEN, AND RÉMI RHODES

ABSTRACT. On a given Riemann surface, we construct a path integral based on the Liouville action functional with imaginary parameters. The construction relies on the compactified Gaussian Free Field (GFF), which we perturb with a curvature term and an exponential potential. In physics this path integral is conjectured to describe the scaling limit of critical loop models such as Potts and $O(n)$ models. The potential term is defined by means of imaginary Gaussian Multiplicative Chaos theory. The curvature term involves integrated 1-forms, which are multivalued on the manifold, and requires a delicate regularisation in order to preserve diffeomorphism invariance. We prove that the probabilistic path integral satisfies the axioms of Conformal Field Theory (CFT) including Segal’s gluing axioms and we construct the correlation functions for this CFT, involving electro-magnetic operators. This CFT has several conjectural exotic features: most importantly, it should be non-unitary with the structure of a logarithmic CFT. Our motivation is thus to provide a concrete setup for the mathematical study of logarithmic CFTs.

CONTENTS

1. Introduction	572
2. Outline and main results	577
3. Geometric preliminaries	586
4. Curvature term	605
5. Imaginary Gaussian multiplicative chaos	617
6. The path integral and correlation functions	622
7. Three point correlation functions on the Riemann sphere	643
8. Segal’s axioms	649
Appendix A. Proof of Lemma 4.5	685
Appendix B. Proof of Lemma 6.14	689
Acknowledgments	691
References	691

Received by the editors February 2, 2024, and, in revised form, May 26, 2025, and July 15, 2025.

2020 *Mathematics Subject Classification.* Primary 60D05, 81T40.

Key words and phrases. Liouville quantum gravity, quantum field theory, Gaussian multiplicative chaos, Ward identities, BPZ equations, DOZZ formula.

The second author was supported by the ERC Advanced Grant 741487 and of Academy of Finland. The third author was partially supported by the Institut Universitaire de France (IUF). The third author was supported by the ANR-21-CE40-0003.

1. INTRODUCTION

1.1. Background: QFT and CFT. Quantum field theory (QFT) is one of the success stories of 20th century physics. Providing the basic theoretical framework for both high-energy and condensed matter physics, it has met overwhelming experimental success in both domains. QFT has also had a profound impact on nearly all areas of mathematics, ranging from analysis and geometry to topology and algebra. QFT was originally introduced for the purpose of extending the quantum formalism to the description of the electro-magnetic field and its interaction with charged matter, and was later extended to nuclear forces. Mathematically, quantum fields are defined as maps from the physical Minkowski space-time $\mathbb{R}^{1,3}$ to operators acting on a Hilbert space. When setting up axiomatic schemes to describe QFTs, it was observed that the formalism could be analytically continued in the time variable to imaginary time, making the quantum fields random fields. These random fields are defined as maps from the Euclidean space time \mathbb{R}^4 to random variables defined on some underlying probability space. Such random fields have been used in statistical physics to model thermal fluctuations of quantities such as the density of a fluid or the microscopic magnetisation of a metal in three or fewer dimensions. Thus, QFT provided a common formalism for both subjects.

High-energy physics QFTs have the property of becoming asymptotically scale-invariant at small spatial scales. In turn, statistical physics systems have scale invariance at large scales when the temperature parameter takes the critical value at which a second-order phase transition occurs, as in the case of magnetism. These limits are described by scale-invariant QFTs, which, in most known cases, possess an even stronger symmetry: conformal invariance. Such QFTs are called conformal field theories (CFTs). Renormalisation group theory, developed by Ken Wilson and others, provides a dynamical systems picture of the ‘space of all QFTs’, where dynamics are generated by a one-parameter group of scale transformations, with CFTs given by its fixed points. This naturally led to the quest to find all CFTs. Such a project is believed to be feasible because CFTs are thought to have a rigid structure constrained by the concept of conformal bootstrap [25, 61]. In an axiomatic formulation, a CFT is characterised by a set of random fields $V_\alpha(x)$, where x belongs to the space (-time) manifold Σ and α is an index labelling the different fields. The basic objects of a CFT are the n -point correlation functions $\langle \prod_{i=1}^n V_{\alpha_i}(x_i) \rangle$, where $\langle - \rangle$ denotes the probabilistic expectation. In short, the conformal bootstrap hypothesis states that an n -point function for $n > 3$ is explicitly given in terms of $(n - 1)$ -point functions and 3-point functions, and therefore, by induction, in terms of 3-point functions. Conformal symmetry fixes the spatial dependence of the latter, meaning they are determined up to constants $C(\alpha_1, \alpha_2, \alpha_3)$, called the **structure constants** of the CFT. Furthermore, consistency of the inductive scheme imposes strong constraints on these structure constants, opening the door to finding solutions either analytically or numerically.

The two-dimensional case is particularly suitable for mathematical study, as complex analysis comes into play here. Furthermore, the conformal symmetry extends to an infinite-dimensional symmetry algebra known as the Virasoro algebra, which is a central extension of the Witt algebra of vector fields on the circle. In their pioneering work, Belavin, Polyakov and Zamolodchikov (BPZ84) used the bootstrap method coupled with Virasoro algebra symmetry to solve significant two-dimensional CFTs by

expressing their correlation functions in terms of special functions arising from representation theory. Since then, CFT has also had a deep influence on mathematics, with applications in modular forms, representation theory of infinite-dimensional Lie algebras and vertex algebras, Monstrous Moonshine, geometric Langlands theory, and knot theory, to name a few.

The rigorous foundation of CFT also triggered interest among mathematicians. Several axiomatic setups have been proposed, each of them relying mainly on one of the various aspects of CFT. Kac’s treatment of CFTs [39] as vertex algebras grew out of their representation-theoretical structure, whereas Frenkel and Ben-Zvi [27] formalised CFT within the context of algebraic geometry. This was further expanded by Beilinson and Drinfeld [8]. The lecture notes [28] emphasised the probabilistic approach based on the Feynman path integral, which is closer in spirit to statistical physics. Segal [68] proposed an axiomatic definition of CFT inspired by the path integral approach, designed to capture geometrically the conformal bootstrap for CFT. Despite the various axiomatic definitions of CFTs, finding concrete examples that obey any given set of axioms has proven challenging and remains a rare occurrence in mathematics. For a long time, the preferential approach has mostly been algebraic but, even though advances have been made, even the most basic examples of CFT, the minimal models, are still not fully constructed mathematically (see [30] for recent advances in the context of Vertex Operator Algebras, in particular concerning their chiral part).

1.2. Liouville CFT. More recently, the probabilistic formulation has resulted in the first proof of the conformal bootstrap in a non-trivial CFT, specifically the Liouville CFT [31, 32]. The approach taken in that work is a probabilistic formulation of the favoured approach to QFT among physicists: Feynman’s path integral. In this approach one starts from an action functional $S(\phi, g)$ defined on some space of functions $\phi : \Sigma \rightarrow M$ where, in the 2d CFT case, Σ is a compact Riemann surface equipped with a Riemannian metric g and M is a manifold. The probabilistic expectation is formally given by the path integral

$$(1.1) \quad \langle F \rangle = \int_{\phi: \Sigma \rightarrow M} F(\phi) e^{-S(\phi, g)} D\phi,$$

where the integration is over some space of maps ϕ and $D\phi$ denotes a formal Lebesgue measure on that space. In order to define such path integrals, one first needs to introduce a regularised expression, e.g. by replacing Σ with a finite grid and the integral with a standard finite-dimensional integral, and then study the limit as the grid’s mesh size ϵ tends to zero. In this process, it is usually found that the parameters of S need to be made dependent on the mesh size ϵ , i.e. renormalised. In the case of the Liouville CFT, S is the Liouville functional that appears in uniformisation theory of Riemann surfaces. Here $M = \mathbb{R}$ and

$$S(\phi, g) := \frac{1}{4\pi} \int_{\Sigma} (|\mathrm{d}\phi|_g^2 + QK_g\phi + \mu e^{\gamma\phi}) \mathrm{d}v_g,$$

where $\gamma > 0$, $\mu \in \mathbb{R}_+$ and $Q = \frac{2}{\gamma}$ are parameters, and v_g and K_g are respectively the volume form and the scalar curvature in the metric g . The minimiser ϕ_m of this action functional is such that the metric $e^{\gamma\phi_m} g$ has constant negative curvature (for the genus of Σ greater than one). The path integral corresponding to this action functional

was constructed in [17, 34]. The renormalisation needed here is the replacement of $e^{\gamma\phi} dv_g$ by the Gaussian Multiplicative Chaos measure $\lim_{\epsilon \rightarrow 0} \epsilon^{\frac{\gamma^2}{2}} e^{\gamma\phi_\epsilon} dv_g$ where ϕ_ϵ is a mollification at scale $\epsilon > 0$ of ϕ . Furthermore, the Q parameter is renormalised to $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$. The path integral construction was later shown to satisfy Segal’s axioms for CFT and obey the conformal bootstrap [31, 32]. The Liouville CFT is an essential building block in string theory and its connection to statistical physics is via the KPZ conjecture [43], which relates a statistical physics model on a random lattice to the same model on a non-random lattice coupled with Liouville CFT.

1.3. Compactified imaginary Liouville CFT. Whereas the connection of Liouville CFT to statistical physics is indirect, an imaginary version of Liouville CFT is expected in physics to describe a major class of two dimensional statistical physics models, the so-called loop models. This CFT is formulated in terms of an action functional

$$(1.2) \quad S_L(\phi, g) := \frac{1}{4\pi} \int_{\Sigma} (|d\phi|_g^2 + iQK_g\phi + \mu e^{i\beta\phi}) dv_g,$$

where $\mu \in \mathbb{C}$ and $Q, \beta \in \mathbb{R}$. The manifold M where ϕ takes values is taken to be the circle $\mathbb{T}_R := \mathbb{R}/(2\pi R\mathbb{Z})$ of radius R . If the parameters μ and Q are set to 0 then the path integral corresponding to the action (1.2) gives rise to the compactified Gaussian Free Field (or compactified GFF for short, see [28, Lecture 1.4] or [21, subsections 6.3.5 and 10.4.1] for physics references, or [20, subsection 2.1.3] in mathematics), a CFT with central charge 1, where the central charge is the number multiplying the central element in the Virasoro algebra. The case $\mu = 0$ goes in physics under the name of Coulomb gas, in which case the parameters Q, R are constrained by the relation $QR \in \mathbb{Z}$ and the so-called background charge (the curvature term in the action) shifts the central charge of the compactified GFF to the value $c_{IL} = 1 - 6Q^2$.

When $\mu \neq 0$, the exponential term gives rise to a non-trivial interacting CFT provided the value of Q is fixed to

$$(1.3) \quad Q = \frac{\beta}{2} - \frac{2}{\beta}.$$

The radius R is a free parameter of our model up to the fact that it obeys the constraint $R \in \frac{1}{\beta}\mathbb{Z}$. Yet the algebraic structure of this CFT depends strongly on the following alternative:

$$(1.4) \quad \text{either (rational) } \frac{1}{\beta}\mathbb{Z} \cap \frac{1}{Q}\mathbb{Z} \neq \{0\} \quad \text{or (irrational) } \frac{1}{\beta}\mathbb{Z} \cap \frac{1}{Q}\mathbb{Z} = \{0\}.$$

The first situation is called rational case (or non-generic in physics) and in that case we impose $R \in \frac{1}{\beta}\mathbb{Z} \cap \frac{1}{Q}\mathbb{Z}$ with $R > 0$. This forces the central charge to be of the form $c_{IL} = 1 - 6\frac{(p-q)^2}{pq}$ for some (relatively prime) integers p, q .¹ The second situation is called non-rational case (or generic in physics) in which case only the constraint $R \in \frac{1}{\beta}\mathbb{Z}$ remains.

In this paper we give a probabilistic construction of the path integral (1.1) corresponding to the action functional (1.2). We prove that the resulting theory satisfies Segal’s axioms of conformal field theory in the rational case. The irrational case will

¹This matches the possible set of central charges for minimal models.

appear in a follow up paper. We also give an explicit integral representation for the structure constants (i.e. 3 point correlation functions) of this CFT in terms of twisted generalization of the famous Dotsenko-Fateev integrals [19]. These results provide the foundation for proving conformal bootstrap for the theory to be discussed in later works. Let us now briefly discuss the mathematical and physical motivations for studying this theory.

1.4. Logarithmic CFT. There are two major differences between the action functional (1.2) and that of Liouville CFT. In the latter, one takes $i\beta \in \mathbb{R}$ and the field ϕ takes values in \mathbb{R} in contrast to the circle in (1.2). These two changes have a drastic effect on the algebraic, geometric and probabilistic aspect of this CFT. First it is a **non-unitary** (i.e. non-reflection positive) CFT. This means its Hamiltonian, obtained as a generator of dilations in a canonical Hilbert space of the theory which we construct in this paper, is **not self-adjoint**. Second and most importantly, it gives rise to a **logarithmic CFT**, topic which has been under active research in physics [24, 26, 35, 48, 52, 57, 60, 62, 64, 66] and in mathematics [6, 47, 51]. Concretely, this means that the Hamiltonian should be diagonalizable in Jordan blocks. Important questions in this context are the classification of the reducible but indecomposable representations of the Virasoro algebra and the structure of the conformal bootstrap, for which the holomorphic factorization is no more expected, at least under its standard form. Third, we prove that this CFT possesses **non-scalar primary fields**, namely they possess a spin. Fourth it is **non-rational and non-diagonal**: its spectrum is countable and different representations of the Virasoro algebra are involved in the chiral and anti-chiral parts entering the correlation functions. Hence our path integral provides the foundations to the mathematical study of all these concepts. The aspects related to the spectrum of this model and the associated representation of the Virasoro algebra involved here are not discussed in our paper but are the subject of forthcoming works.

Beyond the CFT aspects, we also expect this CFT will have a strong interplay with Schramm Loewner Evolution and Conformal Loop Ensembles. Indeed, this was advocated first in physics (see for instance [37] or the review [65]) and a similar interplay occurred recently in the case of the Liouville CFT, in which case these two objects were bridged by the mating-of-trees formalism² [21]. This has led to mutual better understanding of these objects [1, 2, 4] as well as unexpected results for various statistical physics models [3, 58] and the same is expected in case of the CFT studied in this paper.

1.5. Physics: From loop models to the Coulomb gas and imaginary Liouville theory. In the approach of [9], the critical exponents of a statistical physics model were related to the representation theory of the Virasoro algebra of the CFT conjectured to describe its scaling limit. Progress in the physical understanding of two-dimensional critical phenomena has therefore been obtained on the one hand by algebraically classifying CFTs and on the other hand by associating them with putative scaling limits of concrete lattice models [9, 13]. It has been known since the 70's that the critical points of many models of statistical physics can be described in terms of the GFF, dubbed the Coulomb gas representation (see [13, 18, 56] for instance). This picture turned out to be particularly suited to describing the scaling limit of so called loop models. These

²A framework to study the coupling between SLE or CLE and Liouville CFT in terms of more classical probabilistic objects such as Brownian motion, Levy process, and Bessel process.

are models for a random ensemble of loops (closed paths) on a two dimensional lattice and they could be mapped to models of fluctuating surfaces described by a height function h mapping the lattice points to \mathbb{Z} or \mathbb{R} , the basic idea being to think of loops as contour lines of the random surface. The scaling limit of h was then argued to be related to GFF. The variance of the GFF (here rescaled to the parameter β in (1.2)) was then fixed by a priori knowledge of an exact value of some critical exponent, see [13] for instance.

However, for the full description of the loop models, the naive GFF theory needs to be modified in several ways: first the Gaussian height function has to be taken periodic, i.e. to take values on a circle instead of the real line, giving rise this way to the compactified GFF described before. Secondly, to obtain the expected central charge c of the CFT one needs to include the curvature term in (1.2) which shifts c from the GFF value $c = 1$ to $c = 1 - 6Q^2$. Finally, later on, it was argued in [38, 44, 46]³ (see also the review paper [65] or some criticism in [29]) that the action should also include the Liouville potential $e^{i\beta\phi}$ and requiring it to be a marginal perturbation (and thus potentially giving rise to a CFT with the same c) then fixes the value of β in terms of Q as prescribed by (1.3). This is how physicists ended up with the path integral (1.1) with action functional (1.2) to describe the long-distance properties of self-avoiding loop models. For instance, the critical q -state Potts model on a connected planar graph $G = (V, E)$ where V is a set of vertices and E the set of edges has the following loop gas representation of its partition function [7]

$$Z = q^{|V|/2} \sum x^{|E'|} q^{\ell(E')/2},$$

where the sum runs over subgraphs $(V, E') \subset (V, E)$ and $\ell(E')$ being the number of loops in the loop configurations on the medial graph. Here $x > 0$, and $q^{1/2}$ appears as a formal parameter and can thus be assigned complex values. At the critical point $x = x_c$, the conjectural relation with the path integral is

$$q^{1/2} = -2 \cos(\pi\beta^2/4)$$

for $0 < \beta < 2$, with corresponding central charge $c = 1 - 6(\frac{\beta}{2} - \frac{2}{\beta})^2$. A similar correspondence exists with the $O(n)$ -model, see [14, 29, 65].⁴ These conjectures are important but still far from being understood in mathematics. Let us stress that in the different context of the dimer model, yet of the same flavour, there has been a long series of works studying the convergence towards the compactified GFF, see [10–12, 20, 41, 42]. We will not develop any further this aspect in the paper. Our main goal is here to construct mathematically the path integral (1.1), (1.2) and establish the basic CFT axioms.

Let us close this introduction by stressing that one important byproduct of our work is a mathematical construction of the Coulomb gas and we feel that some foundational aspects of this fundamental model have been a bit overlooked in the physics literature, which has some impact on the conclusions drawn for the path integral (1.1). Our concerns come from two modifications needed to define the curvature term: one is

³See also [45] for multicomponent compactified bosons with Kac-Moody symmetry.

⁴A further orbifold of our path integral might be involved: this question is under investigation in physics but the mathematical construction of the path integral is quite similar.

topological and the other one comes from the windings of the electro-magnetic operators. The topological modification for the curvature was already considered in [69] but to our knowledge the other problem has not been addressed before. However, even in [69], the main properties of the modification are not addressed in detail and it turns out that the resulting path integral is not diffeomorphism invariant unless $R \in \frac{1}{Q}\mathbb{Z}$. In the case of the Coulomb gas (hence with no potential), this condition is always assumed in physics (like in [69]). In the case of the imaginary Liouville path integral, the presence of the potential forces $R \in \frac{1}{\beta}\mathbb{Z}$. If we further impose $R \in \frac{1}{Q}\mathbb{Z}$, we end up considering the rational case. Yet, in physics, the path integral is also considered in the irrational case, i.e. $R \notin \frac{1}{Q}\mathbb{Z}$, and then the path integral violates the most basic requirement (diffeomorphism invariance) of a Quantum Field Theory. We haven't found any reference to this fact in the literature except for a remark by A.B. Zamolodchikov in [70] for a somewhat related model (Generalized Minimal Models), quoting: "It remains a question if such infinite algebra is consistent with general requirements of quantum field theory. In particular, the construction of a modular invariant partition function of Generalized Minimal Models obviously encounters severe problems."

2. OUTLINE AND MAIN RESULTS

Because the construction of the path integral requires some geometric background, we sketch here the construction, as a guideline for the paper. First we discuss informally some aspects of the construction at a general level but we will then give two concrete examples with simple topologies to introduce pedagogically two important notions of the construction: the defect graph and the topological instantons.

To make sense of the measure (1.1) with action (1.2), it is first necessary to understand the path integral for the compactified GFF (i.e. with $Q = \mu = 0$ in (1.2)). For this, one observes that the differential $d\phi$ of a (say smooth) map $\phi : \Sigma \rightarrow \mathbb{T}_R$ defines a real valued closed 1-form ω on Σ , with periods $\int_\gamma \omega \in 2\pi R\mathbb{Z}$ if γ is any closed curve on Σ . The Hodge decomposition then allows to uniquely decompose $\omega = \omega_h + df$, where $f : \Sigma \rightarrow \mathbb{R}$ is a function and ω_h is a harmonic 1-form (the De Rham cohomology group is represented by harmonic 1-forms). In other words the \mathbb{T}_R valued map ϕ can be viewed as a sum of f and a multivalued harmonic function $\int_{\gamma_{x_0,x}} \omega := I_{x_0}(\omega_h)$ on Σ where $\gamma_{x_0,x}$ is a path from x_0 to x . This decomposition is orthogonal in the space $\Omega^1(\Sigma)$ of real valued 1-forms on Σ equipped with the inner product

$$(2.1) \quad \langle \omega, \omega' \rangle_2 := \int_\Sigma \omega \wedge * \omega',$$

where $*$ is the Hodge operator, in such a way that

$$\int_\Sigma |d\phi|_g^2 dv_g = \int_\Sigma |df|_g^2 dv_g + \int_\Sigma \omega_h \wedge * \omega_h.$$

Therefore the formal measure in (1.1) for the compactified GFF can be understood as the measure

$$(2.2) \quad e^{-\frac{1}{4\pi} \int_\Sigma |d\phi|_g^2 dv_g} D\phi = e^{-\frac{1}{4\pi} \int_\Sigma |df|_g^2 dv_g} Df \times e^{-\frac{1}{4\pi} \int_\Sigma \omega_h \wedge * \omega_h} d\mu(\omega_h) \times dc,$$

where $d\mu(\omega_h)$ is the counting measure on the De Rham cohomology group (isomorphic to \mathbb{Z}^{2g} where g is the genus of Σ), dc is the Lebesgue measure on $\mathbb{R}/2\pi R\mathbb{Z}$ (c plays

the role of the zero mode) and the formal measure $e^{-\frac{1}{4\pi} \int_{\Sigma} |df|_g^2 dv_g} Df$ is interpreted in our work as the distribution law of the usual Gaussian Free Field on Σ . The functional $F(\phi)$ in (1.1) then has to be well chosen so that the result does not depend on the ambiguity related to the fact that $I_{x_0}(\omega_h)$ is multivalued on Σ (see Section 6). This aspect is quite subtle and it manifests, for instance, as soon as one wishes to make sense of the curvature term in (1.2). We expand this point a bit further now.

There are actually two main difficulties with the curvature term, which we explain in the rational case. First, since the zero mode c lives in $\mathbb{R}/2\pi R\mathbb{Z}$, in order to have $e^{i\frac{Q}{4\pi} \int_{\Sigma} \phi K_g dv_g}$ univalued, we need to have $\frac{Q}{4\pi} \int_{\Sigma} 2\pi R K_g dv_g \in 2\pi\mathbb{Z}$. By Gauss-Bonnet $\frac{1}{4\pi} \int_{\Sigma} K_g dv_g = 2 - 2g$ so this forces $2Q$ to be an integer multiple of $1/R$.⁵ The same argument holds for the Liouville potential and this forces β to be an integer multiple of $1/R$. Second, and more subtly, the curvature term contains $\phi = c + f + I_{x_0}(\omega_h)$ which is a multivalued function on Σ . Therefore the integral $\int_{\Sigma} \phi K_g dv_g$ does not make sense unambiguously; even worse, if we define the primitive on the universal cover, and then descend it on a fundamental domain, the quantity $\int_{\Sigma} \phi K_g dv_g$ will depend on the choice of the fundamental domain. Concretely, this means that a naive definition of the curvature term produces a theory that is not diffeomorphism invariant. A considerable part of our work consists in regularising this integral via a family of branch cuts σ on Σ and in proving that the result does not depend on the choice of the branch cuts. These branch cuts are chosen to be a symplectic basis of the first homology group. The result is that a change of σ modifies the regularised curvature integral by an integer multiple of $2\pi R$ and thus one has to impose $QR \in \mathbb{Z}$. Together with $\beta R \in \mathbb{Z}$ and the expression for Q given by (1.3) we end up with the condition for the rational case written in (1.4).

Concerning the Liouville potential term, the construction involves some technology related to the renormalisation of imaginary Gaussian Multiplicative Chaos (GMC for short), in particular some tools developed in [49, 50], and is restricted to $0 < \beta^2 < 2$. The full picture of this path integral is expected to cover the values $0 < \beta^2 < 4$; this is an important future direction of development. The construction for $0 < \beta^2 < 2$ in the rational case of the path integral and the correlation functions is carried out in Section 6. We give now more details in two simple cases.

2.1. Construction on the Riemann sphere. Now we explain the construction on the simplest possible topology, the Riemann sphere, because the compactified GFF then coincides with the standard GFF. We first give the construction of the path integral without electro-magnetic operators. Consider a smooth Riemannian metric g on the Riemann sphere \hat{C} , with volume form v_g and scalar curvature K_g . The GFF in the metric g is denoted by X_g (see Section 5 for details). We consider β such that $\beta^2 < 2$ and $Q = \frac{\beta}{2} - \frac{2}{\beta}$, and then we pick $R > 0$ with $R \in \frac{1}{\beta}\mathbb{Z} \cap \frac{1}{Q}\mathbb{Z}$.

The construction involves imaginary GMC, namely the limit

$$M_{\beta}(X_g, dx) := \lim_{\epsilon \rightarrow 0} \epsilon^{-\frac{\beta^2}{2}} e^{i\beta X_{g,\epsilon}(x)} dx,$$

where $X_{g,\epsilon}$ is a reasonable regularisation of the GFF at scale ϵ in the metric g . The convergence is non-trivial for $\beta^2 \in (0, 2)$ and the limit is a distribution of order 2 with

⁵Actually, this point could be circumvented by adding artificial (electric) operators as is usually done in the physics literature, leading to the possibility to consider the irrational case.

exponential moments. More generally, we will consider imaginary GMC $M_\beta(\phi_g, dx)$ where we replace the GFF X_g by a more general functional of the GFF, denoted by ϕ_g and called the *Liouville field*, which will be made precise depending on the context. The definition of the path integral is then (up to some metric dependent terms encoded in the constant $C(g)$, the precise meaning of which we skip for now)

$$(2.3) \quad \langle F \rangle_{\hat{C},g} := C(g) \int_{\mathbb{R}/2\pi R\mathbb{Z}} \mathbb{E} \left[F(\phi_g) e^{-\frac{iQ}{4\pi} \int_{\hat{C}} K_g \phi_g \, dv_g - \mu M_\beta^g(\phi_g, \hat{C})} \right] dc$$

for all reasonable test functions F , periodic with period $2\pi R\mathbb{Z}$ in the variable c , and here the Liouville field is $\phi_g = c + X_g$. Note the integration dc of the constant mode c of the field ϕ_g over the circle of radius R . Also, the curvature term perfectly makes sense here because, at this stage, the Liouville field is a well-defined distribution on \hat{C} and can be integrated over \hat{C} .

2.1.1. *Electric operators.* Next we introduce electric operators: given a point $x \in \hat{C}$ on the sphere and a weight $\alpha \in R^{-1}\mathbb{Z}$, they are formally defined by

$$V_{(\alpha,0)}(x) = \lim_{\epsilon \rightarrow 0} \epsilon^{-\frac{\alpha^2}{2}} e^{i\alpha \phi_{g,\epsilon}(x)}.$$

The condition $\alpha \in R^{-1}\mathbb{Z}$ makes sure that the electric operator, seen as a function of c , is well-defined on the circle. Products of such operators for distinct points $\mathbf{x} = (x_1, \dots, x_n) \in \hat{C}^n$ with respective weights $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (R^{-1}\mathbb{Z})^n$ will be denoted by $V_{(\boldsymbol{\alpha},0)}(\mathbf{x}) := \prod_{j=1}^n V_{(\alpha_j,0)}(x_j)$. Correlation functions for electric operators are formally given by evaluating the path integral above with $F(\phi_g) = V_{(\boldsymbol{\alpha},0)}(\mathbf{x})$, namely $\langle V_{(\boldsymbol{\alpha},0)}(\mathbf{x}) \rangle_{\hat{C},g}$. Of course, the limit $\epsilon \rightarrow 0$ is ill-defined because, making sense only as a distribution, the GFF cannot be evaluated pointwise. To remedy this, the usual trick is to apply the Girsanov transform; special care has to be taken here because of the imaginary weights so that rigorously implementing the Girsanov transform has to go through analytic continuation arguments (see Section 6.2). The outcome is that the path integral with electric operators is (again, up to explicit trivial factors $C_{g,\mathbf{x},\boldsymbol{\alpha}}$)

$$(2.4) \quad \langle FV_{(\boldsymbol{\alpha},0)}(\mathbf{x}) \rangle_{\hat{C},g} := C_{g,\mathbf{x},\boldsymbol{\alpha}} \int_{\mathbb{R}/2\pi R\mathbb{Z}} e^{ic \sum_{j=1}^n \alpha_j} \mathbb{E} \left[F(\phi_g) e^{-\frac{iQ}{4\pi} \int_{\hat{C}} K_g \phi_g \, dv_g - \mu M_\beta^g(\phi_g, \hat{C})} \right] dc,$$

where the Liouville field is now $\phi_g = c + X_g + u_{\mathbf{x}}$, and the function

$$u_{\mathbf{x}}(x) := \sum_{j=1}^n i\alpha_j G_g(x, x_j)$$

(with G_g the Green function on \hat{C} in the metric g) stands for the shift resulting from the Girsanov transform. Observe that this shift creates singularities in the potential $M_\beta^g(\phi_g, \Sigma)$, which can be formally written as $e^{i\beta c} \int_{\hat{C}} e^{-\beta \sum_j \alpha_j G_g(x, x_j)} M_\beta(X_g, dx)$. Well-posedness and existence of exponential moments of this random variable are part of our work.

2.1.2. *Magnetic operators.* The next step is to introduce the magnetic operators. A magnetic operator creates a magnetic charge $m \in \mathbb{Z}$ at its insertion point z . Its effect in the loop model is to create a discontinuity $2\pi Rm$ in the height field ϕ along a line emanating from z . Here, this has the effect to make the Liouville field ϕ_g acquire a winding $2\pi Rm$ around the point z . The discontinuity curve has to end at the location

of another magnetic charge and if we consider distinct points $\mathbf{z} = (z_1, \dots, z_n) \in \hat{\mathbb{C}}^n$ with respective magnetic charges $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ we must impose the neutrality condition $\sum_i m_i = 0$. To construct these operators, we consider a closed 1-form $\nu_{\mathbf{z}, \mathbf{m}}$ on $\hat{\mathbb{C}} \setminus \{\mathbf{z}\}$ (with $\{\mathbf{z}\} := \cup_{j=1}^n \{z_j\}$) such that $\nu_{\mathbf{z}, \mathbf{m}}$ is of the form $m_j R d\theta$ in local radial coordinates $z - z_j = re^{i\theta}$ near the point z_j , for $j = 1, \dots, n$. Such a 1-form exists if and only if the charges satisfy $\sum_{j=1}^n m_j = 0$. Also, note that the choice of this closed 1-form is not unique but the path integral we will construct will not depend on this choice. Then we want to offset the GFF with a primitive of this 1-form, namely we want to consider the path integral (2.4) where the Liouville field is now

$$\phi_g = c + X_g + u_x + I_{x_0}(\nu_{\mathbf{z}, \mathbf{m}}),$$

and $I_{x_0}(\nu_{\mathbf{z}, \mathbf{m}}) := \int_{x_0}^x \nu_{\mathbf{z}, \mathbf{m}}$ is a primitive of the 1-form $\nu_{\mathbf{z}, \mathbf{m}}$ with base point x_0 . The point is that $\nu_{\mathbf{z}, \mathbf{m}}$ is not exact (and $\hat{\mathbb{C}} \setminus \{\mathbf{z}\}$ is of course not simply connected if $n > 1$) so that the primitive is multivalued on $\hat{\mathbb{C}} \setminus \{\mathbf{z}\}$: it has a monodromy $2\pi m_j R$ when turning once around a given point z_j . The first important remark is that any function of the form $e^{i\frac{k}{R} I_{x_0}(\nu_{\mathbf{z}, \mathbf{m}})}$, for $k \in \mathbb{Z}$, is unambiguously defined as a smooth function on $\hat{\mathbb{C}} \setminus \{\mathbf{z}\}$. In particular, since β is an integer multiple of $1/R$, the potential term in (2.4) is well-defined as it reads

$$e^{i\beta c} \int_{\hat{\mathbb{C}}} e^{-\beta \sum_j \alpha_j G_g(x, x_j)} e^{i\beta I_{x_0}(\nu_{\mathbf{z}, \mathbf{m}})(x)} M_\beta(X_g, dx).$$

Also, and at the level of electric operators, offsetting the field by $I_{x_0}(\nu_{\mathbf{z}, \mathbf{m}})$ produces a further factor of the form $\prod_{j=1}^n e^{i\alpha_j I_{x_0}(\nu_{\mathbf{z}, \mathbf{m}})(x_j)}$; again, the conditions $\alpha_j \in R^{-1}\mathbb{Z}$ make this product unambiguously defined. Actually the main problem comes from the curvature term in (2.4): the monodromy of the field ϕ_g makes this term being ill-defined and depending on the choice of the primitive. It must be regularised. For this, we first introduce a system of branch cuts for $I_{x_0}(\nu_{\mathbf{z}, \mathbf{m}})$, which we call *defect graph*. It consists in a family of smooth simple curves $(\xi_p)_{p=1, \dots, n-1}$, which do not intersect except eventually at their endpoints. Each arc $\xi_p : [0, 1] \rightarrow \hat{\mathbb{C}}$ is a smooth oriented curve parametrised by arclength with endpoints $\xi_p(0) = z_j$ and $\xi_p(1) = z_{j'}$ for $j \neq j'$. We must further impose a direction along which the arcs reach the endpoints. So we fix a family of unit tangent vectors $v_j \in T_{z_j} \hat{\mathbb{C}}$ ($j = 1, \dots, n$) and we further require these arcs to obey: if $\xi_p(a) = z_j$ with $a \in \{0, 1\}$, then $\xi'_p(a)$ is positively colinear to v_j . Finally, consider the oriented graph with vertices $\{\mathbf{z}\}$ and edges $(z_j, z_{j'})$ when there is an arc connecting z_j to $z_{j'}$. This graph must be connected and without cycle. What we call defect graph is the set $\mathcal{D} := \bigcup_{p=1}^{n-1} \xi_p([0, 1])$, see Figure 1.

The defect graph is a proper branch cut for $\nu_{\mathbf{z}, \mathbf{m}}$ in the sense that this 1-form is exact on the complement $\hat{\mathbb{C}} \setminus \mathcal{D}$. Therefore it admits an unambiguously defined primitive denoted by $I_{x_0}^\xi(\nu_{\mathbf{z}, \mathbf{m}})$. Then we introduce the regularised curvature term as

$$\int_{\hat{\mathbb{C}}}^{\text{reg}} K_g \phi_g dv_g := \int_{\hat{\mathbb{C}}} K_g (c + X_g + u_x) dv_g + \int_{\hat{\mathbb{C}}}^{\text{reg}} I_{x_0}^\xi(\nu_{\mathbf{z}, \mathbf{m}}) K_g dv_g,$$

where

$$(2.5) \quad \int_{\hat{\mathbb{C}}}^{\text{reg}} I_{x_0}^\xi(\nu_{\mathbf{z}, \mathbf{m}}) K_g dv_g := \int_{\hat{\mathbb{C}}} I_{x_0}^\xi(\nu_{\mathbf{z}, \mathbf{m}}) K_g dv_g - 2 \sum_{p=1}^n \kappa(\xi_p) \int_{\xi_p} k_g d\ell_g,$$

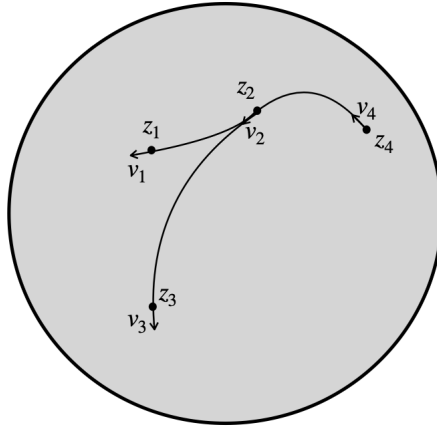


FIGURE 1. A defect graph with 4 points

where k_g denotes the geodesic curvature of an oriented curve, $d\ell_g$ the Riemannian measure on ξ_p induced by g and $\kappa(\xi_p)$ are coefficients that have explicit expressions in terms of the magnetic charges; their value is actually imposed by an argument relying on the Gauss-Bonnet theorem (see Subsection 4.2). The definition of the path integral with both electric and magnetic operators is then

$$(2.6) \quad \langle FV_{(\alpha,0)}(\mathbf{x})V_{(0,\mathbf{m})}(\mathbf{v}) \rangle_{\hat{c},g} := C_{g,\mathbf{x},\mathbf{z},\alpha,\mathbf{m}} \times \int_{\mathbb{R}/2\pi RZ} e^{i\mathbf{c} \cdot \sum_{j=1}^n \alpha_j} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \nu_{\mathbf{z},\mathbf{m}} \rangle} F(\phi_g) e^{-\frac{iQ}{4\pi} \int_{\hat{c}}^{\text{reg}} K_g \phi_g \, dv_g - \mu M_{\beta}^g(\phi_g, \hat{c})} \right] d\mathbf{c}$$

with $\phi_g = c + X_g + u_x + I_{x_0}(\nu_{\mathbf{z},\mathbf{m}})$ and $\mathbf{v} = ((z_1, v_1), \dots, (z_n, v_n)) \in (T\hat{c})^n$, and $C_{g,\mathbf{x},\mathbf{z},\alpha,\mathbf{m}}$ is a constant encoding trivial factors (in particular it contains the product $\prod_{j=1}^n e^{i\alpha_j I_{x_0}(\nu_{\mathbf{z},\mathbf{m}})(x_j)}$). What is non-trivial is to show that this regularised curvature does not depend on the choice of the defect graph: this is proved in Subsection 4.2 using Gauss-Bonnet.

Electro-magnetic operators, denoted by $V_{(\alpha,\mathbf{m})}(\mathbf{v})$, are then defined by merging both electric and magnetic operators, namely by taking the limit $\mathbf{x} \rightarrow \mathbf{z}$ in (2.6). Yet, this limit is ill-defined because of the windings around the points \mathbf{z} . This is why we must further impose the direction along which each x_j reaches z_j , and we require this limit to be taken in the direction v_j , in which case we prove that the limit makes sense and defines this way correlation functions of electro-magnetic operators $\langle V_{(\alpha,\mathbf{m})}(\mathbf{v}) \rangle_{\Sigma,g}$. Furthermore, we obtain an explicit description how these correlation functions are affected by a change of the choice of the unit vectors v_j , which translates the notion of spin for primary fields in the physics literature.

2.2. Other topologies. Now we sketch the construction for more complex topologies, and we focus here on the case of complex tori. As soon as the genus of the surface is non-zero, the compactified GFF integration measure (2.2) has a so called instanton component corresponding to the summation over the De Rham cohomology. Concretely, the torus $\mathbb{T}_{\tau}^2 = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with modulus $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$ (with $\tau_2 > 0$) has a basis of homology given by the cycles $a(t) = t$ and $b(t) = t\tau$ for $t \in [0, 1]$. A dual basis

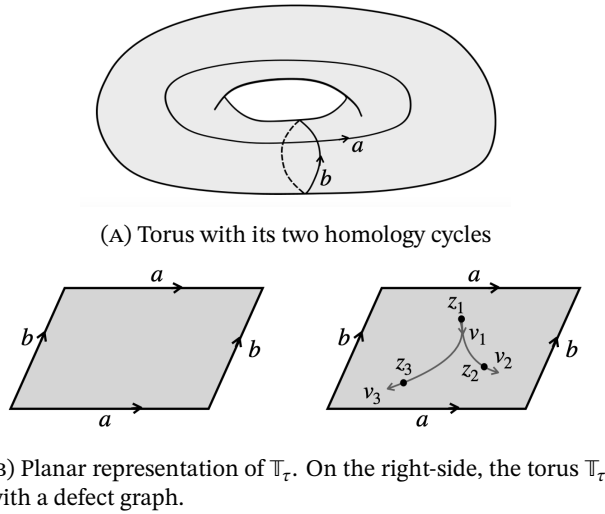


FIGURE 2. Homology cycles and defect graph on the torus

of cohomology is given by $\omega_1 = dx - \frac{\tau_1}{\tau_2} dy$ and $\omega_2 = \frac{1}{\tau_2} dy$. For $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$ we set $\omega_{\mathbf{k}} = k_1 \omega_1 + k_2 \omega_2$. Then, the path integral basically corresponds to (2.3) with the Liouville field given (for $I_{x_0}(\omega_{\mathbf{k}})(x) = \int_{x_0}^x \omega_{\mathbf{k}}$ multivalued) by

$$\phi_g = c + X_g + I_{x_0}(\omega_{\mathbf{k}}),$$

and a further summation over $\mathbf{k} \in \mathbb{Z}^2$. Again, the problem here is that the curvature term is ill-defined⁶ as there is an arbitrary choice to be made for the primitive $I_{x_0}(\omega_{\mathbf{k}})$, which is multivalued on Σ , and the resulting integral $\int_{\Sigma} K_g I_{x_0}(\omega_{\mathbf{k}}) dv_g$ does depend on this choice. So the curvature term has to be regularised: the cycles a, b are chosen as branch cuts and then the 1-form $\omega_{\mathbf{k}}$ is exact on $\Sigma \setminus (a \cup b)$. One then needs to introduce counterterms in a way similar to (2.5) to finally get the path integral

$$(2.7) \quad \langle F \rangle_{\mathbb{T}_{\tau}, g} := C(g) \sum_{\mathbf{k} \in \mathbb{Z}^2} e^{-\frac{1}{4\pi} \|\omega_{\mathbf{k}}\|_g^2} \times \int_{\mathbb{R}/2\pi R\mathbb{Z}} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle} F(\phi_g) e^{-\frac{iQ}{4\pi} \int_{\mathbb{T}_{\tau}}^{\text{reg}} K_g \phi_g dv_g - \mu M_{\beta}^g(\phi_g, \mathbb{T}_{\tau})} \right] dc.$$

On general Riemann surfaces, a considerable part of our work consists in proving that the path integral does not depend on these branch cuts, with the invariance under diffeomorphisms as a consequence. One can then define electro-magnetic operators on \mathbb{T}_{τ} as was done for the Riemann sphere, introducing further branch cuts for the magnetic operators (see Figure 2). This concludes our short overview of the construction.

2.3. Main results. Let us expand now in further details the properties of this CFT. First we stress that the correlation functions of interest are expectation values of the electro-magnetic fields, the construction of which is summarized above. In the Coulomb gas picture, an electric operator with weight $\alpha = \frac{e}{R}$ creates an electric charge

⁶Note that we do not assume the curvature is uniformised with $K_g = 0$.

$e \in \mathbb{Z}$ at the insertion point $x \in \Sigma$. A magnetic operator in turn creates a magnetic charge $m \in \mathbb{Z}$ at its insertion point z . Electro-magnetic operators $V_{\alpha,m}(z, v)$ mix the two effects. They are the primary fields of the CILT; they are not scalar when $m \neq 0$ as they possess a spin, details can be found in Section 6.2.

In CFTs, the 3 point correlation functions on the Riemann sphere, or structure constants, play a special role as building blocks of the conformal bootstrap formulae. In Section 7 we compute the three-point functions of the electro-magnetic operators and show them to be given by a generalization of the well known Dotsenko-Fateev integrals. In the case when the magnetic charges are set to 0, these integrals are given by the imaginary DOZZ formula [70], whereas the presence of magnetic charges make these structure constants appear as a square root of products of imaginary DOZZ coefficients, see Section 7.

Finally we complete the CFT axioms by establishing Segal’s gluing axioms [68] for the path integral. Segal’s axioms were designed to capture the conformal bootstrap approach to CFTs using a geometrical perspective (for a nice introduction to mathematicians see [28]). They are related to the Operator Product Expansion and are fundamental to obtain a consistent CFT on all closed surfaces. To prove these axioms, we define the path integral on surfaces with boundaries and construct associated operators on a Hilbert (called amplitudes), and we show that these operators compose under gluing of surfaces. This part uses our previous work [31] but we stress that there are many non-trivial new aspects due to the topological terms in the path integral definition due to the fact that we work with the compactified boson. We believe that even for the compactified free field, our gluing results are new.

Our results depend on the alternative (1.4). Theorem 2.1 summarizes the main results of this paper for the correlation functions of the electro-magnetic operators in the **rational case** (see (1.4)); we refer to Theorem 6.13 and to Propositions 8.13 and 8.14 for more detailed statements. It is also possible, with the techniques of this paper, to define correlation functions in the case where the central charge is irrational. The invariance by diffeomorphisms is not as general as in the rational case, so it is not completely clear how that fits in the full CFT picture. On the other hand the spectral analysis and representation theory for this model is possible and seems very interesting. This will be studied elsewhere.

In the rational case, some elementary algebra shows that there exist p, q coprime integers such that $\beta^2/4 = \frac{p}{q}$. Furthermore, $\frac{1}{\beta}\mathbb{Z} \cap \frac{1}{Q}\mathbb{Z} = \frac{\sqrt{pq}}{2}\mathbb{Z}$ if p, q are odd, or $\frac{1}{\beta}\mathbb{Z} \cap \frac{1}{Q}\mathbb{Z} = \sqrt{pq}\mathbb{Z}$ if p or q is even. The central charge becomes

$$(2.8) \quad c_{\text{IL}} = 1 - 6 \frac{(p - q)^2}{pq}.$$

Theorem 2.1. *Let $\beta^2/4 = \frac{p}{q}$ where $p, q \in \mathbb{N}$ are coprime and let the compactification radius $R = \frac{k}{2}\sqrt{pq}$ with $k \in \mathbb{N}^*$ if p, q are odd, or $k \in 2\mathbb{N}^*$ if p or q is even. Let (Σ, g) be a closed Riemannian surface and $\mathbf{v} = ((z_1, v_1), \dots, (z_n, v_n)) \in (T\Sigma)^n$. Assume the electric charges satisfy $\alpha_j := e_j/R > Q$, with $e_j \in \mathbb{Z}$, and that the magnetic charges $m_j \in \mathbb{Z}$ satisfy $\sum_{j=1}^n m_j = 0$. Then the following hold true:*

(i) The correlation functions $\langle V_{(\alpha, \mathbf{m})}(\mathbf{v}) \rangle_{\Sigma, g}$ exist as limits of regularised objects and do not depend on the choice of the cohomology basis nor on the magnetic discontinuity curves.

(ii) $\langle V_{(\alpha, \mathbf{m})}(\mathbf{v}) \rangle_{\Sigma, g}$ satisfy the axioms of a Conformal Field Theory with central charge (2.8) and conformal weights $\Delta_{\alpha, m}$ and spin $S_{\alpha, m}$ given by

$$\Delta_{\alpha, m} = \frac{e}{2R} \left(\frac{e}{2R} - Q \right) + \frac{m^2 R^2}{4}, \quad S_{\alpha, m} = em - QRm,$$

in the sense that they transform in a covariant way under the action of diffeomorphisms $g \rightarrow \psi^* g$, see (6.21), Weyl scaling $g \rightarrow e^\rho g$, $\rho \in C^\infty(\Sigma)$, see (6.19), and rotations of the vectors $v_j \rightarrow O_j v_j$, $O_j \in \text{SO}(2)$, see (6.22).

(iii) $\langle V_{(\alpha, \mathbf{m})}(\mathbf{v}) \rangle_{\Sigma, g}$ satisfy the gluing axioms of Segal for conformal field theory under cutting of the surface along analytically parametrised simple curves, see Proposition 8.13 and 8.14.

(iv) If (e_1, e_2) is the canonical basis of $\mathbb{R}^2 = T\hat{C}$ and $g_0 = \frac{|dz|^2}{\max(|z|, 1)^4}$ the choice of metric on the Riemann sphere, then the 3-point function on the Riemann sphere is given by

$$\begin{aligned} & \langle V_{(\alpha_1, m_1)}(0, e_1) V_{(\alpha_2, m_2)}(1, e_1) V_{(\alpha_3, m_3)}(\infty, e_1) \rangle_{\hat{C}, g_0} = \\ & 2\pi R \frac{(-\mu)^\ell}{\ell!} \int_{\mathbb{C}^\ell} \prod_{j=1}^{\ell} x_j^{\Delta_1} \bar{x}_j^{\bar{\Delta}_1} (1-x_j)^{\Delta_2} (1-\bar{x}_j)^{\bar{\Delta}_2} \prod_{j < j'} |x_j - x_{j'}|^{\beta^2} dx_1 \dots dx_\ell, \end{aligned}$$

where

$$\ell := \frac{2Q - \sum_j \alpha_j}{\beta} \in \mathbb{N}, \quad \Delta_j = \frac{\beta \alpha_j + k p m_j}{2}, \quad \bar{\Delta}_j = \frac{\beta \alpha_j - k p m_j}{2}.$$

When $m_i = 0$, this can also be written as

$$\begin{aligned} & \langle V_{(\alpha_1, 0)}(0, e_1) V_{(\alpha_2, 0)}(1, e_1) V_{(\alpha_3, 0)}(\infty, e_1) \rangle_{\hat{C}, g_0} \\ & = 2\pi R (-\mu)^{\frac{2Q - \sum_i \alpha_i}{\beta}} C_\beta^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3), \end{aligned}$$

and in general

$$\begin{aligned} & \langle V_{(\alpha_1, m_1)}(0, e_1) V_{(\alpha_2, m_2)}(1, e_1) V_{(\alpha_3, m_3)}(\infty, e_1) \rangle_{\hat{C}, g_0}^2 \\ & = 4\pi^2 R^2 \mu^{\frac{2}{\beta}(2Q - \sum \alpha_i)} C_\beta^{\text{DOZZ}}(\alpha_1 + m_1 R, \alpha_2 + m_2 R, \alpha_3 + m_3 R) \\ & \quad \times C_\beta^{\text{DOZZ}}(\alpha_1 - m_1 R, \alpha_2 - m_2 R, \alpha_3 - m_3 R), \end{aligned}$$

where $C_\beta^{\text{DOZZ}}(\cdot, \cdot, \cdot)$ is the imaginary DOZZ constant of (7.10).

Remarks.

(1) Segal’s axioms give access to the spectrum of the CFT by considering the generator of the semigroup of annuli (see [68] or [31] in the case of Liouville theory) i.e. the Hamiltonian operator of the CFT. This operator is not self-adjoint but it has discrete spectrum and non-trivial Jordan blocks. This will be studied in a forthcoming work.

(2) It may be instructive to write our conformal weights in terms of the usual Kac table parametrisation. The conformal weight and spin are then characterised by a pair

(h, \bar{h}) such that $h + \bar{h} = 2\Delta_{\alpha, m}$ and $S_{\alpha, m} = h - \bar{h}$. Here we get

$$h = \frac{1}{4} \left(\frac{e}{R} + mR - Q \right)^2 - \frac{Q^2}{4} \quad \text{and} \quad \bar{h} = \frac{1}{4} \left(\frac{e}{R} - mR - Q \right)^2 - \frac{Q^2}{4}.$$

In the Kac notations, one introduces the number $h_{r,s} := \frac{1}{4} \left(r \frac{2}{\beta} - s \frac{\beta}{2} \right)^2 - \frac{Q^2}{4}$ for $r, s \in \mathbb{R}$ and one has to determine for which pairs (r, s) and (r', s') the relation $(h, \bar{h}) = (h_{r,s}, h_{r',s'})$ holds. Here, this is equivalent to finding all the pairs (r, s) and (r', s') that can be written as

$$\frac{e}{R} + mR - Q = r \frac{2}{\beta} - s \frac{\beta}{2} \quad \text{and} \quad \frac{e}{R} - mR - Q = r' \frac{2}{\beta} - s' \frac{\beta}{2},$$

for $e, m \in \mathbb{Z}$. Some algebra then gives the condition that any r, r', s, s' satisfying

$$\frac{r+r'}{2}, \frac{s+s'}{2} \in \frac{2}{k}\mathbb{Z} \quad \text{and} \quad r-r' \in kp\mathbb{Z}, \quad s-s' \in kq\mathbb{Z}$$

can be written as desired.

One can then use this information to compare with the set of conformal weights of other CFTs. If we impose the field to be diagonal (or spinless), meaning $h_{r,s} = h_{r',s'}$, we find that either $r = r', s = s' \in \frac{2}{k}\mathbb{Z}$ or $r = -r' \in \frac{kq}{2}\mathbb{Z}$ and $s = -s' \in \frac{kp}{2}\mathbb{Z}$. In particular, taking $k = 2$, we find that all the diagonal fields have conformal weights corresponding to the degenerate representations of the Virasoro algebra.

If $p = 2$ then the central charge of our construction coincides with that of the $(2, q)$ -minimal models [9] and the set of its spectral weights thus contains the weights of the degenerate Virasoro representations for $k = 2$. It has been argued in physics by [40] that the A-series minimal model CFT with the same central charge can be recovered from this compactified imaginary Liouville theory by a so-called BRST reduction, providing this way a path integral construction of the minimal models. This question, raised to us by N. Seiberg, was the original motivation for this work, and we plan to understand this aspect in future work.

(3) Our construction works also in the irrational case under the condition that $\sum_j \alpha_j \in \chi(\Sigma)Q + \frac{1}{R}\mathbb{Z}$. When we work on the torus or the sphere without magnetic field, the correlation functions satisfy the same invariance properties as in the rational case. On the other hand, for higher genus or in the case with magnetic fields, the diffeomorphism invariance does not seem to hold in general. This will be studied in a separate work and it is not clear it produces a CFT with all the required axioms in the general geometric case.

(4) There is an important open question in physics to construct a CFT dubbed Time-like Liouville Theory or Imaginary Liouville Theory [36, 63, 67, 70]. This is a non-compact CFT with continuous spectrum and structure constants given by the imaginary DOZZ formula. Even though there may be some links with such a CFT, our path integral does not coincide with it (among other reasons, it has discrete spectrum). We choose to call our path integral CILT because our construction is based on the Liouville action with imaginary parameters, yet on the compactified boson so that it is not mistaken with the putative Timelike Liouville Theory.

3. GEOMETRIC PRELIMINARIES

Notations. We shall denote $\dot{u}(t) = \partial_t u(t)$ the derivative of C^1 curves $u : [0, 1] \rightarrow \Sigma$ with values in a surface. If (Σ, g) is a closed, or compact with boundary, oriented Riemannian surface, we denote $\langle f, f' \rangle_2 = \int f \bar{f}' dv_g$ the L^2 pairing with respect to the Riemannian measure v_g , we denote $\langle u, f \rangle$ the distributional pairing if $u \in \mathcal{D}'(\Sigma)$ is a distribution and $f \in C_c^\infty(\Sigma^\circ, \mathbb{R})$ compactly supported in the interior Σ° of Σ , with the convention that if $u \in C^\infty(\Sigma)$ then $\langle u, f \rangle = \langle u, f \rangle_2$.

3.1. Closed surfaces. Let Σ be an oriented compact surface of genus g and let g be a smooth Riemannian metric. The metric g induces a conformal class $[g] := \{e^\varphi g \mid \varphi \in C^\infty(\Sigma)\}$, which is equivalent to a complex structure, i.e. a field $J \in C^\infty(\Sigma; \text{End}(T\Sigma))$ of endomorphisms of the tangent bundle such that $J^2 = -\text{Id}$. There are local charts $\omega_j : U_j \rightarrow \mathbb{D}$ such that $\omega_j \circ \omega_k^{-1}$ are holomorphic functions and $(\omega_j^{-1})^* g = e^{\rho_j} |dz|^2$ on \mathbb{D} , where $z = x+iy$ is the usual complex coordinate on \mathbb{C} and $\rho_j \in C^\infty(\mathbb{D})$. The complex structure in the holomorphic charts ω_j is given by $J\partial_x = \partial_y$ and $J\partial_y = -\partial_x$. The Hodge operator $*$: $T^*\Sigma \rightarrow T^*\Sigma$ is the dual to J , it satisfies $*dx = dy$ and $*dy = -dx$ in local holomorphic charts. The pair (Σ, J) (or equivalently $(\Sigma, [g])$) is called a *closed Riemann surface*. The orientation is given by any non-vanishing 2-form $w_\Sigma \in C^\infty(\Sigma; \Lambda^2 T^*\Sigma)$ so that $(\omega_j^{-1})^* w_\Sigma = e^{f_j} dx \wedge dy$ in \mathbb{D} for some function f_j .

The Gauss-Bonnet formula reads

$$(3.1) \quad \int_\Sigma K_g dv_g = 4\pi\chi(\Sigma),$$

where $\chi(\Sigma) = (2 - 2g)$ is the Euler characteristic, K_g the scalar curvature of g and dv_g the Riemannian measure. The uniformisation theorem says that in the conformal class of g , there exists a metric $g_0 = e^{\rho_0} g$ of scalar curvature $K_{g_0} = -2$ if $g \geq 2$, $K_{g_0} = 0$ if $g = 1$ or $K_{g_0} = 2$ if $g = 0$. It is unique if $g \geq 2$. For a metric $\hat{g} = e^\rho g$, one has the relation

$$K_{\hat{g}} = e^{-\rho}(\Delta_g \rho + K_g),$$

where $\Delta_g = d^*d$ is the non-negative Laplacian (here d is exterior derivative and d^* its $L^2(\Sigma, v_g)$ adjoint). Let us recall a result of Aubin [5] on prescribing the curvature.

Lemma 3.1 (Aubin). *Let (Σ, g) be a closed Riemannian surface with genus $g \geq 2$. Let $f \in C^\infty(\Sigma)$ be a non-positive function such that $\int_\Sigma f dv_g < 0$. Then there exists a conformal metric $\hat{g} := e^\rho g$ for some $\rho \in C^\infty(\Sigma)$ such that $K_{\hat{g}} = f$.*

3.2. Surfaces with analytic parametrised boundary. Let $\mathbb{T} = \{e^{i\theta} \in \mathbb{C} \mid \theta \in [0, 2\pi]\}$ be the unit circle. A compact Riemann surface Σ with parametrised analytic boundary $\partial\Sigma = \sqcup_{j=1}^{\mathfrak{b}} \partial_j \Sigma$ is a compact oriented surface with smooth boundary with a family of charts $\omega_j : U_j \rightarrow \omega_j(U_j) \subset \mathbb{C}$ for $j = 1, \dots, j_0$ and smooth diffeomorphisms $\zeta_j : \mathbb{T} \rightarrow \partial_j \Sigma$ where

- $\cup_j U_j$ is an open covering of Σ with $U_j \cap \partial_j \Sigma \neq \emptyset$ if and only if $j \in [1, \mathfrak{b}]$
- there exists $\delta < 1$ such that for all $j > \mathfrak{b}$ we have $\omega_j(U_j) = \mathbb{D}$, and for all $j \leq \mathfrak{b}$ we have $\omega_j(U_j) = \mathbb{A}_\delta := \{z \in \mathbb{C} \mid |z| \in (\delta, 1]\}$ and $\omega_j(\partial_j \Sigma) = \mathbb{T}$,
- $\omega_k \circ \omega_j^{-1}$ are holomorphic maps in the interior of the domain $\omega_j(U_j \cap U_k)$,
- $\omega_j \circ \zeta_j : \mathbb{T} \rightarrow \mathbb{T}$ is real analytic, i.e. it extends holomorphically in a neighborhood of \mathbb{T} .

The charts induce a complex structure J as for the closed case. A Riemannian metric g is *compatible* with J if $(\omega_j^{-1})^*g = e^{\rho_j} |dz|^2$ for some smooth function ρ_j on $\omega_j(U_j)$. The boundary circles $\partial_j\Sigma$ inherit an orientation from the orientation of Σ , simply by taking the 1-form $-i_{\partial_j\Sigma}^*(i_\nu\omega_\Sigma)$ where i_ν is the interior product with non-vanishing interior pointing vector field ν to Σ and $i_{\partial_j\Sigma} : \partial_j\Sigma \rightarrow \Sigma$ is the natural inclusion. In the chart given by the annulus \mathbb{A}_δ , the orientation is then given by $d\theta$ (i.e. the counterclockwise orientation) on the unit circle parametrised by $(e^{i\theta})_{\theta \in [0, 2\pi]}$. We say that the boundary $\partial_j\Sigma$ is *outgoing* if the orientation $(\zeta_j)_*(d\theta)$ is the orientation of $\partial_j\Sigma$ induced by that of Σ (we also say that the orientation of $\partial_j\Sigma$ is positive) otherwise the parametrised boundary $\partial_j\Sigma$ is called *incoming*. We define $\varsigma_j \in \{\pm 1\}$ with $\varsigma_j = -1$ if $\partial_j\Sigma$ outgoing and $\varsigma_j = 1$ if $\partial_j\Sigma$ is incoming. Composing ζ_j by the inversion $o(z) := 1/z$ reverses orientation and transforms an outgoing boundary into an incoming one and conversely. Notice that there is $\delta < 1$ such that ζ_j is holomorphic from an annular neighborhood \mathbb{A}_δ of \mathbb{T} (resp. $\mathbb{A}_\delta^{-1} := \delta^{-1}\mathbb{A}_\delta$) to a neighborhood U'_j of $\partial_j\Sigma$ if $\omega_j \circ \zeta_j|_{\mathbb{T}}$ preserves orientation, i.e. $\partial_j\Sigma$ is outgoing (resp. reverses orientation, i.e. $\partial_j\Sigma$ is incoming). Up to adding U'_j to the set of charts and replacing U_j by $U_j \cap \Sigma^\circ$ for $j = 1, \dots, \mathfrak{b}$, the new set of charts $((U'_j, \omega'_j), (U_j, \omega_j))$ with $\omega'_j := o^{(1+\varsigma_j)/2} \circ \zeta_j^{-1}$ produces the same complex structure on Σ as the original one, and the parametrisation of the boundary component $\partial_j\Sigma$ is given by $(\omega'_j)^{-1}|_{\mathbb{T}}$. Without loss of generality, we can and will thus assume from now that, when choosing our set of charts (U_j, ω_j) above, the boundary parametrisations for $j = 1, \dots, \mathfrak{b}$ are given by

$$\zeta_j : \mathbb{T} \rightarrow \partial_j\Sigma, \quad \zeta_j(e^{i\theta}) := \begin{cases} \omega_j^{-1}(e^{i\theta}) & \text{if } \varsigma_j = -1 \\ \omega_j^{-1}(e^{-i\theta}) & \text{if } \varsigma_j = 1 \end{cases}.$$

The metric g is said *admissible* if it is compatible with the complex structure and for (U_j, ω_j) with $j = 1, \dots, \mathfrak{b}$, we have

$$(\omega_j^{-1})^*g = |dz|^2/|z|^2,$$

on $\omega_j(U_j)$. In that case the boundary is geodesic for g , with length 2π , and the metric g has curvature $K_g = 0$ near $\partial\Sigma$.

3.3. Gluing and cutting surfaces. Let Σ be a Riemann surface, not necessarily connected, with \mathfrak{b} parametrised analytic boundary connected components $\zeta_j : \mathbb{T} \rightarrow \partial_j\Sigma$. Let $\omega_j : U_j \rightarrow \mathbb{A}_\delta$ be the charts near $\partial_j\Sigma$ with $\omega_j^{-1}|_{\mathbb{T}} = \zeta_j$ as above. If $\partial_j\Sigma$ is outgoing and $\partial_k\Sigma$ is incoming, we can glue $\partial_j\Sigma$ to $\partial_k\Sigma$ to obtain a new Riemann surface $\Sigma^\#$ with $(\mathfrak{b} - 2)$ -boundary components: this is done by identifying $\zeta_j(e^{i\theta}) \simeq \zeta_k(e^{i\theta})$. A neighborhood in $\Sigma^\#$ of the identified circle $\partial_j\Sigma \sim \partial_k\Sigma$ is given by $(U_j \cup U_k)/\sim$ where \sim means the identification $\partial_j\Sigma \sim \partial_k\Sigma$, and

$$\omega_{jk} : z \in U_j \mapsto \omega_j(z) \in \mathbb{A}_\delta, \quad z \in U_k \mapsto \frac{1}{\omega_k(z)} \in \mathbb{A}_\delta^{-1}$$

produces a chart. If $\partial_j\Sigma$ and $\partial_k\Sigma$ belong to different connected components of Σ , $b_0(\Sigma^\#) = b_0(\Sigma) - 1$ if b_0 denotes the 0-th Betti number; otherwise $b_0(\Sigma^\#) = b_0(\Sigma)$. If Σ is equipped with an admissible metric, then g induces a smooth metric on the glued Riemann surface $\Sigma^\#$ compatible with the complex structure.

Conversely, starting from a Riemann surface (Σ, J) with $\mathfrak{b} \geq 0$ boundary circles $\partial_1\Sigma, \dots, \partial_\mathfrak{b}\Sigma$ and choosing an analytic embedded circle \mathcal{C} in the interior Σ° of Σ and a

chart $\omega : V \rightarrow \{|z| \in [\delta, \delta^{-1}]\} \subset \mathbb{C}$ for some $\delta < 1$ with $\omega(\mathcal{C}) = \mathbb{T}$, there is a natural Riemann surface $\bar{\Sigma}_e$ obtained by compactifying $\Sigma_e := \Sigma \setminus \mathcal{C}$ into a Riemann surface with $b + 2$ boundary circles by adding two copies $\partial_{b+1}\bar{\Sigma}_e = \mathcal{C}$, $\partial_{b+2}\bar{\Sigma}_e = \mathcal{C}$ of \mathcal{C} to Σ_e using the chart ω on respectively $\omega^{-1}(\{|z| \leq 1\})$ and $\omega^{-1}(\{|z| \geq 1\})$ (one outgoing, one incoming). Using the gluing procedure of $\partial_{b+1}\bar{\Sigma}_e$ with $\partial_{b+2}\bar{\Sigma}_e$ in $\bar{\Sigma}_e$ just described above using the parametrisations $\zeta_1 := \omega^{-1}|_{\mathbb{T}}$ and $\zeta_2 := \omega^{-1}|_{\mathbb{T}}$, we recover (Σ, J) .

3.4. Determinant of Laplacians. For a Riemannian metric g on a connected oriented compact surface Σ , the non-negative Laplacian $\Delta_g = d^*d$ has discrete spectrum $\text{Sp}(\Delta_g) = (\lambda_j)_{j \in \mathbb{N}_0}$ with $\lambda_0 = 0$ and $\lambda_j \rightarrow +\infty$ and we shall denote e_j the associated orthonormalised eigenfunctions. We can define the determinant of Δ_g by

$$\det'(\Delta_g) = \exp(-\partial_s \zeta(s)|_{s=0}),$$

where $\zeta(s) := \sum_{j=1}^{\infty} \lambda_j^{-s}$ is the spectral zeta function of Δ_g , which admits a meromorphic continuation from $\text{Re}(s) \gg 1$ to $s \in \mathbb{C}$ and is holomorphic at $s = 0$ (the series converges for $\text{Re}(s) \gg 1$ due to Weyl’s law). We recall that if $\hat{g} = e^\varphi g$ for some $\varphi \in C^\infty(\Sigma)$, one has the so-called Polyakov formula (see [59, eq. (1.13)])

$$(3.2) \quad \log \frac{\det'(\Delta_{\hat{g}})}{v_{\hat{g}}(\Sigma)} = \log \frac{\det'(\Delta_g)}{v_g(\Sigma)} - \frac{1}{48\pi} \int_{\Sigma} (|d\varphi|_g^2 + 2K_g \varphi) dv_g,$$

where K_g is the scalar curvature of g as above.

When (Σ, g) is a connected oriented compact surface Σ with boundary, the Laplacian Δ_g with Dirichlet boundary conditions has discrete spectrum $(\lambda_{j,D})_{j \geq 1}$ and the determinant is defined in a similar way as $\det(\Delta_{g,D}) := e^{-\zeta'_{g,D}(0)}$ where $\zeta_{g,D}$ is the spectral zeta function of Δ_g with Dirichlet boundary conditions defined for $\text{Re}(s) \gg 1$ by $\zeta_{g,D}(s) := \sum_{j=1}^{\infty} \lambda_{j,D}^{-s}$. The function $\zeta_{g,D}(s)$ admits a meromorphic extension to $s \in \mathbb{C}$ and is holomorphic at $s = 0$.

3.5. Green’s function and resolvent of Laplacian. Each compact Riemannian surface (Σ, g) has a (non-negative) Laplace operator Δ_g . The Green function G_g on a surface Σ without boundary is defined to be the integral kernel of the resolvent operator $R_g : L^2(\Sigma) \rightarrow L^2(\Sigma)$ satisfying $\Delta_g R_g = 2\pi(\text{Id} - \Pi_0)$, $R_g^* = R_g$ and $R_g 1 = 0$, where Π_0 is the orthogonal projection in $L^2(\Sigma, dv_g)$ on $\ker \Delta_g$ (the constants). By integral kernel, we mean that for each $f \in L^2(\Sigma, dv_g)$

$$R_g f(x) = \int_{\Sigma} G_g(x, x') f(x') dv_g(x').$$

The Laplacian Δ_g has an orthonormal basis of real valued eigenfunctions $(e_j)_{j \in \mathbb{N}_0}$ in $L^2(\Sigma, dv_g)$ with associated eigenvalues $\lambda_j \geq 0$; we set $\lambda_0 = 0$ and $e_0 = (v_g(\Sigma))^{-1/2}$. The Green function then admits the following Mercer’s representation in $L^2(\Sigma \times \Sigma, dv_g \otimes dv_g)$

$$(3.3) \quad G_g(x, x') = 2\pi \sum_{j \geq 1} \frac{1}{\lambda_j} e_j(x) e_j(x').$$

Similarly, on a surface with smooth boundary Σ , we will consider the Green function with Dirichlet boundary conditions $G_{g,D}$ associated to the Laplacian Δ_g with Dirichlet

condition on $\partial\Sigma$. In this case, the associated resolvent operator

$$R_{g,D}f(x) = \int_{\Sigma} G_{g,D}(x, x')f(x')dv_g(x')$$

solves $\Delta_g R_{g,D} = 2\pi\text{Id}$.

3.6. First homology group and symplectic basis on closed surfaces. Let Σ be a closed oriented surface of genus g . We denote by $\mathcal{H}_1(\Sigma)$ the first homology group of Σ with value in \mathbb{Z} . It is an abelian group isomorphic to \mathbb{Z}^{2g} , which can be obtained as the abelianization of the fundamental group $\pi_1(\Sigma, x_0)$ for some fixed $x_0 \in \Sigma$. Recall that elements in $\pi_1(\Sigma, x_0)$ are equivalence classes of closed C^1 curves on the surfaces (equivalence been given by homotopy fixing x_0), and for a closed curve c on Σ , we denote by $[c]$ its class in $\mathcal{H}_1(\Sigma)$.

For two transverse oriented C^1 curves $a : \mathbb{T} \rightarrow \Sigma, b : \mathbb{T} \rightarrow \Sigma$ on Σ , the algebraic intersection number $\iota(a, b) \in \mathbb{Z}$ is the number of intersection points $p = a(t_1) = b(t_2)$ weighted by $+1$ (resp. -1) if the orientation of the basis $(\dot{a}(t_1), \dot{b}(t_2))$ of $T_p\Sigma$ at p is positive (resp. negative) with respect to the orientation of Σ . This number only depends on the homology class $[a], [b]$ of a, b , it defines a bilinear skew-symmetric map

$$\iota : \mathcal{H}_1(\Sigma) \wedge \mathcal{H}_1(\Sigma) \rightarrow \mathbb{Z}.$$

This map is a symplectic form and there exists a symplectic basis $([a_i], [b_i])_{i=1, \dots, g}$ of $\mathcal{H}_1(\Sigma)$ represented by closed simple curves $(a_i, b_i)_i$ such that

$$(3.4) \quad \iota(a_i, b_j) = \delta_{ij}, \quad \iota(a_i, a_j) = 0, \quad \iota(b_i, b_j) = 0.$$

In fact, we can and will choose a_i, b_i so that the intersection number $\iota(a_i, b_j)$ is also equal to the geometric intersection number: for example a_1 intersects b_1 in a unique point, etc (see Figure 3). A symplectic basis $([a_i], [b_i])_{i=1, \dots, g}$ of $\mathcal{H}_1(\Sigma)$ represented by simple closed curves $(a_i, b_i)_i$ with the property just described will be called a *geometric symplectic basis*. See Figure 3. If $c = \sum_{j=1}^g \alpha_j a_j + \beta_j b_j$ and $c' = \sum_{j=1}^g \alpha'_j a_j + \beta'_j b_j$, we have

$$\iota(c, c') = \sum_{j=1}^g \alpha_j \beta'_j - \alpha'_j \beta_j = (\alpha, \beta)J(\alpha', \beta')^\top,$$

where J is the canonical symplectic matrix on \mathbb{R}^{2g} . We denote by $\text{Sp}(2g, \mathbb{R})$ the symplectic group, defined by $A \in \text{Sp}(2g, \mathbb{R})$ if $AJA^\top = J$, and let $\text{Sp}(2g, \mathbb{Z}) := \text{Sp}(2g, \mathbb{R}) \cap \text{GL}(2g, \mathbb{Z})$ the subgroup whose coefficients are integers. Any other basis $(a', b') = (a'_1, \dots, a'_g, b'_1, \dots, b'_g)$ (not necessarily symplectic) of $\mathcal{H}_1(\Sigma)$ is related to $(a, b) = (a_1, \dots, a_g, b_1, \dots, b_g)$ by a matrix $A \in \text{SL}(2g, \mathbb{Z})$ with integer coefficients and of determinant 1

$$(a', b') = A(a, b).$$

The basis (a', b') is symplectic if and only if $A \in \text{Sp}(2g, \mathbb{Z})$. We shall typically use the notation $\sigma = (\sigma_1, \dots, \sigma_{2g})$ for a basis of $\mathcal{H}_1(\Sigma)$ and $(a_1, b_1, \dots, a_g, b_g)$ when the basis is a geometric symplectic basis. We refer to [23, Chapter 6.1] for a detailed discussion on the symplectic structure of $\mathcal{H}_1(\Sigma)$. It follows from the proof of [23, Theorem 6.4] that each symplectic basis of $\mathcal{H}_1(\Sigma)$ can be realized by a geometric symplectic basis. Finally, as explained in [23, Section 6.3], if $\psi : \Sigma \rightarrow \Sigma$ is an orientation preserving diffeomorphism, it induces an automorphism $\psi_* : \mathcal{H}_1(\Sigma) \rightarrow \mathcal{H}_1(\Sigma)$ which belongs to $\text{Sp}(2g, \mathbb{Z})$ (as it preserves the intersection form).

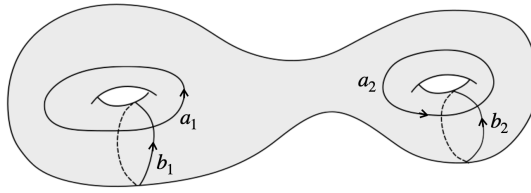


FIGURE 3. A geometric symplectic basis of $\mathcal{H}_1(\Sigma)$

3.7. De Rham first cohomology group on closed Riemann surfaces. We equip Σ with a complex structure $J \in \text{End}(T\Sigma)$, i.e. satisfying $J^2 = -\text{Id}$. Denote by $\Lambda^p \Sigma := \Lambda^p T^* \Sigma$ the bundle of differential p forms. Let $d : C^\infty(\Sigma, \Lambda^p \Sigma) \rightarrow C^\infty(\Sigma, \Lambda^{p+1} \Sigma)$ be the exterior derivative and $*$: $T^* \Sigma \rightarrow T^* \Sigma$ be the Hodge operator, dual to $-J$. The Hodge operator induces a scalar product on the space $C^\infty(\Sigma, \Lambda^1 \Sigma)$ of real valued 1-forms

$$\langle \omega, \omega' \rangle_2 := \int_{\Sigma} \omega \wedge * \omega'.$$

The formal adjoint of $d : C^\infty(\Sigma) \rightarrow C^\infty(\Sigma, \Lambda^1 \Sigma)$ with respect to this scalar product on 1-forms and the Riemannian L^2 scalar product on functions is given by $d^* = - * d *$. The first de Rham cohomology space is defined by

$$\mathcal{H}^1(\Sigma) := \ker d|_{C^\infty(\Sigma, \Lambda^1 \Sigma)} / \text{Im } d|_{C^\infty(\Sigma)}.$$

It is isomorphic to the real vector space of real-valued closed and co-closed forms, or equivalently real harmonic 1-forms,

$$\text{Harm}^1(\Sigma) = \{ \omega \in C^\infty(\Sigma, \Lambda^1 \Sigma) \mid d\omega = 0, d * \omega = 0 \}.$$

The space $\mathcal{H}^1(\Sigma)$ is dual to $\mathcal{H}_1(\Sigma)$ with the duality map given by

$$\mathcal{H}^1(\Sigma) \times \mathcal{H}_1(\Sigma) \rightarrow \mathbb{R}, \quad \langle \omega, \sigma \rangle := \int_{\sigma} \omega$$

and called the period of ω over σ . If we fix a basis $(\sigma_1, \dots, \sigma_{2g})$ of $\mathcal{H}_1(\Sigma)$, one can then find a unique basis $(\omega_1, \dots, \omega_{2g})$ dual to $(\sigma_1, \dots, \sigma_{2g})$. For $R > 0$, we define the \mathbb{Z} -module consisting of cohomology classes with periods in $2\pi R\mathbb{Z}$:

$$(3.5) \quad \mathcal{H}_R^1(\Sigma) := \{ \omega \in \mathcal{H}^1(\Sigma) \mid \forall \sigma \in \mathcal{H}_1(\Sigma), \langle \omega, \sigma \rangle \in 2\pi R\mathbb{Z} \}.$$

The discussion above implies the following:

Lemma 3.2. *Let $R > 0$ and let $\sigma = (\sigma_1, \dots, \sigma_{2g})$ be a basis of $\mathcal{H}_1(\Sigma)$. Then there exist $2g$ independent closed smooth 1-forms $\omega_1, \dots, \omega_{2g}$ forming a basis of $\mathcal{H}_R^1(\Sigma)$ dual to $(\sigma_1, \dots, \sigma_{2g})$ in the sense that*

$$\forall i, j, \quad \frac{1}{2\pi R} \int_{\sigma_i} \omega_j = \delta_{ij}.$$

*If $\omega'_1, \dots, \omega'_{2g}$ is another such family, then for each $j = 1, \dots, 2g$ there exists $\varphi_j \in C^\infty(\Sigma)$ such that $\omega'_j = \omega_j - d\varphi_j$. There is a unique such basis of \mathcal{H}_R^1 so that in addition $d * \omega_j = 0$ for all $j = 1, \dots, 2g$.*

We deduce that if ω_j are some closed forms as in Lemma 3.2, for each $\mathbf{k} = (k_1, \dots, k_{2g}) \in \mathbb{Z}^{2g}$, the form

$$(3.6) \quad \omega_{\mathbf{k}} := \sum_{j=1}^{2g} k_j \omega_j$$

satisfies $\int_{\sigma_i} \omega_{\mathbf{k}} = k_i 2\pi R$ and the map $\mathbf{k} \mapsto \omega_{\mathbf{k}}$ identifies \mathbb{Z}^{2g} with $\mathcal{H}_R^1(\Sigma)$.

We pursue with a technical lemma that will be useful later:

Lemma 3.3. *Let Σ be a closed Riemann surface and let $\omega \in C^\infty(\Sigma, \Lambda^1 \Sigma)$ be a closed 1-form. Then*

$$\langle dR_g d^* \omega, \omega \rangle_2 = 2\pi \|(\text{Id} - \Pi_1)\omega\|_2^2,$$

where $\Pi_1 : L^2(\Sigma, \Lambda^1 \Sigma) \rightarrow \text{Harm}^1(\Sigma)$ is the orthogonal projection.

Proof. We observe the following about the Green's function: G_g is the integral kernel of the operator $R_g = 2\pi \Delta_g^{-1}$ satisfying

$$\Delta_g \Delta_g^{-1} = \text{Id} - \Pi_0,$$

where Π_0 is the projector on constants with respect to the volume form v_g . Let $\Delta_{g,1} = d^*d + dd^*$ be the Laplacian on 1-forms and $R_{g,1}$ be the operator so that

$$\Delta_{g,1} R_{g,1} = R_{g,1} \Delta_{g,1} = 2\pi(\text{Id} - \Pi_1),$$

where Π_1 is the orthogonal projector on $\text{Harm}_1(\Sigma) = \ker d \cap \ker d^*|_{C^\infty(\Sigma, \Lambda^1 \Sigma)}$. Then

$$\Delta_{g,1} dR_g = d\Delta_g R_g = 2\pi d(\text{Id} - \Pi_0) = 2\pi d,$$

thus applying $R_{g,1}$ on the left,

$$(3.7) \quad 2\pi(\text{Id} - \Pi_1) dR_g = 2\pi dR_g = 2\pi R_{g,1} d.$$

This proves that if $d\omega = 0$, then

$$\langle dR_g d^* \omega, \omega \rangle_2 = \langle R_{g,1} dd^* \omega, \omega \rangle_2 = \langle R_{g,1} \Delta_{g,1} \omega, \omega \rangle = 2\pi \|(1 - \Pi_1)\omega\|_2^2. \quad \square$$

3.8. Homology and cohomology on surfaces with boundary. Now, let us consider the case of a Riemann surface (Σ, J) with boundary (note that J induces an orientation). Assume that there are \mathfrak{b} boundary connected components $\partial_1 \Sigma, \dots, \partial_{\mathfrak{b}} \Sigma$, oriented positively with respect to the orientation of Σ , and the genus is denoted by $g \geq 0$.

As for closed surfaces, the first homology group $\mathcal{H}_1(\Sigma)$ is represented by oriented closed curves in Σ . It is isomorphic to $\mathbb{Z}^{2g+\mathfrak{b}-1}$. The positively oriented boundary circles $c_j = \partial_j \Sigma$ for $j = 1, \dots, \mathfrak{b}$ are elements in $\mathcal{H}_1(\Sigma)$ and $\sum_{j=1}^{\mathfrak{b}} [c_j] = [\partial \Sigma] = 0$ is the class of the boundary, which is trivial. If $\Sigma^{\#2} = \Sigma \# \Sigma$ is the double of Σ ((Σ, J) glued with $(\Sigma, -J)$ along the boundary), the inclusion map of the right copy of Σ into $\Sigma^{\#2}$ induces a linear injection

$$i_\Sigma : \mathcal{H}_1(\Sigma) \rightarrow \mathcal{H}_1(\Sigma^{\#2})$$

and the intersection pairing ι defined above for closed surfaces also makes sense on Σ using this injection. The involution τ_Σ exchanging the right copy of Σ with the left copy induces a linear map on $\mathcal{H}_1(\Sigma)$ fixing the boundary curves $[c_j]$. The group $\mathcal{H}_1(\Sigma)$ is generated by $([\sigma_1], \dots, [\sigma_{2g}], [c_1], \dots, [c_{\mathfrak{b}-1}])$ where σ_j are simple curves that do not intersect $\partial \Sigma$ and $i_1, \dots, i_{\mathfrak{b}-1} \in \{1, \dots, \mathfrak{b}\}$ are distinct. Notice that these cycles σ can

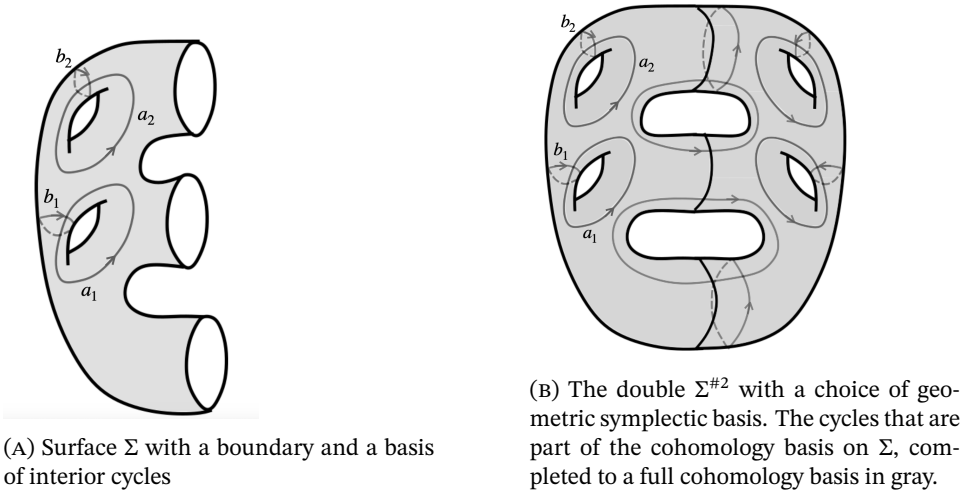


FIGURE 4. Homology on the doubled surface

also be viewed as element in the relative homology $\mathcal{H}_1(\Sigma, \partial\Sigma)$, and we can assume that $[\sigma_j] \neq 0$ in $\mathcal{H}_1(\Sigma, \partial\Sigma)$. In fact, one can choose $(\sigma_j)_j$ to be $(a_1, b_1, \dots, a_g, b_g)$ with

$$\iota(a_j, a_i) = 0 = \iota(b_j, b_i), \quad \iota(a_j, b_i) = \delta_{ij},$$

and that the geometric intersection of these curves is at most one point. In particular, $(i_\Sigma(a_j), i_\Sigma(b_j))_{j=1, \dots, g}$ is a subset of a geometric symplectic basis in $\mathcal{H}_1(\Sigma^{\#2})$; see Figure 4. We call such a basis

$$(3.8) \quad (a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_{b-1})$$

a *canonical geometric basis* of $\mathcal{H}_1(\Sigma)$. Up to renumbering the boundary components, we will assume that $i_j = j$.

Elements in the relative homology $\mathcal{H}_1(\Sigma, \partial\Sigma)$ are represented by either oriented closed curves or oriented curves with endpoints on the boundary $\partial\Sigma$. The relative homology $\mathcal{H}_1(\Sigma, \partial\Sigma)$ also has dimension $2g + b - 1$. A basis of cycles of $\mathcal{H}_1(\Sigma, \partial\Sigma)$ can be obtained by taking

$$(3.9) \quad (a_1, b_1, \dots, a_g, b_g, d_1, \dots, d_{b-1}),$$

where (a_j, b_j) are chosen inside Σ° such that the intersection numbers are equal to their geometric intersection number and are given as in (3.4), and the $(d_j)_j$ are disjoint non-intersecting oriented simple curves, not intersecting $\cup_{i=1}^g (a_i \cup b_i)$, with endpoints on $\partial\Sigma$ and satisfying the following properties:

(1) each oriented arc d_j has its initial endpoint on a connected component $\partial_{i_j}\Sigma$ and the final endpoint on a different connected component $\partial_{f_j}\Sigma$ of $\partial\Sigma$, and it is tangent at $\partial_{i_j}\Sigma$ to the inward vector field normal to $\partial\Sigma$ and at $\partial_{f_j}\Sigma$ to the outward vector field normal to $\partial\Sigma$.

(2) The oriented graph \mathcal{G} whose b vertices are $\partial_1\Sigma, \dots, \partial_b\Sigma$ and oriented edges are d_1, \dots, d_{b-1} must be connected and without cycle, i.e. there is no sequence of edges $(\partial_{j_1}\Sigma, \partial_{j_2}\Sigma), \dots, (\partial_{j_k}\Sigma, \partial_{j_{k+1}}\Sigma)$ with $j_1 = j_{k+1}$. See Figure 5 for an example.

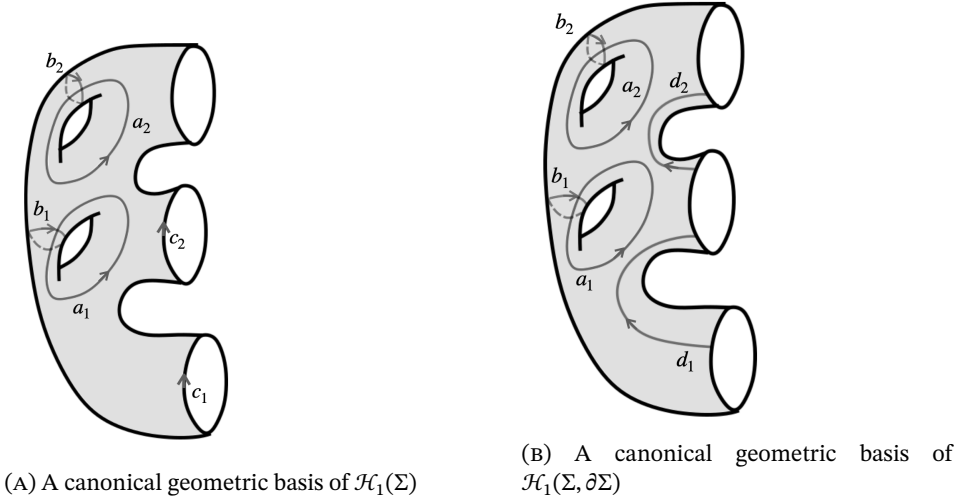


FIGURE 5. Absolute and relative homology on Σ

Such a basis (3.9) will be called a *canonical geometric basis* of $\mathcal{H}_1(\Sigma, \partial\Sigma)$.

The *absolute cohomology* $\mathcal{H}^k(\Sigma)$ of degree $k \in \{0, 1, 2\}$ is defined by

$$\mathcal{H}^k(\Sigma) := \ker d|_{C_{\text{abs}}^\infty(\Sigma, \Lambda^k \Sigma)} / \text{Im } d|_{C_{\text{abs}}^\infty(\Sigma, \Lambda^{k-1} \Sigma)},$$

where, if ν is the interior pointing unit normal vector to the boundary and $\iota_{\partial\Sigma} : \partial\Sigma \rightarrow \Sigma$ denotes the inclusion map, we define the set of real-valued absolute forms (i_ν denotes interior product)

$$C_{\text{abs}}^\infty(\Sigma, \Lambda^k \Sigma) := \{u \in C^\infty(\Sigma, \Lambda^k \Sigma) \mid \iota_{\partial\Sigma}^*(i_\nu u) = 0, \iota_{\partial\Sigma}^*(i_\nu du) = 0\}.$$

The space $\mathcal{H}^k(\Sigma)$ is isomorphic to the real vector space of real-valued absolute closed and co-closed forms

$$\text{Harm}^k(\Sigma) := \{u \in C^\infty(\Sigma, \Lambda^k \Sigma) \mid du = 0, d * u = 0, \iota_{\partial\Sigma}^*(i_\nu u) = 0\}.$$

The *relative cohomology* $\mathcal{H}^k(\Sigma, \partial\Sigma)$ of degree $k \in \{0, 1, 2\}$ is defined by

$$\mathcal{H}^k(\Sigma, \partial\Sigma) := \ker d|_{C_{\text{rel}}^\infty(\Sigma, \Lambda^k \Sigma)} / \text{Im } d|_{C_{\text{rel}}^\infty(\Sigma, \Lambda^{k-1} \Sigma)},$$

where

$$C_{\text{rel}}^\infty(\Sigma, \Lambda^k \Sigma) := \{u \in C^\infty(\Sigma, \Lambda^k \Sigma) \mid \iota_{\partial\Sigma}^* u = 0\}.$$

The space $\mathcal{H}^k(\Sigma, \partial\Sigma)$ is isomorphic to the real vector space of real-valued closed and co-closed relative forms

$$\text{Harm}^k(\Sigma, \partial\Sigma) := \{u \in C^\infty(\Sigma, \Lambda^k \Sigma) \mid du = 0, d * u = 0, \iota_{\partial\Sigma}^* u = 0\}.$$

The Poincaré duality says that the Hodge star operator

$$* : \text{Harm}^1(\Sigma) \rightarrow \text{Harm}^1(\Sigma, \partial\Sigma)$$

is an isomorphism.

Using the double $\Sigma^{\#2}$ of Σ obtained by gluing Σ with itself at the boundary and the natural involution τ_Σ on $\Sigma^{\#2}$, one has that $\mathcal{H}^1(\Sigma, \partial\Sigma) \simeq \{u \in \mathcal{H}^1(\Sigma^{\#2}) \mid \tau_\Sigma^* u = -u\}$ and $\mathcal{H}^1(\Sigma) \simeq \{u \in \mathcal{H}^1(\Sigma^{\#2}) \mid \tau_\Sigma^* u = u\}$. We recover that

$$4g + 2b - 2 = \dim \mathcal{H}^1(\Sigma^{\#2}) = \dim \mathcal{H}^1(\Sigma) + \dim \mathcal{H}^1(\Sigma, \partial\Sigma) = 2 \dim \mathcal{H}^1(\Sigma)$$

and there are $2g + b - 1$ independent forms in $\mathcal{H}^1(\Sigma)$ (resp. $\mathcal{H}^1(\Sigma, \partial\Sigma)$). The duality between $\mathcal{H}^1(\Sigma)$ and $\mathcal{H}_1(\Sigma)$ is given by the pairing

$$(3.10) \quad \mathcal{H}^1(\Sigma) \times \mathcal{H}_1(\Sigma) \rightarrow \mathbb{R}, \quad \langle u, \sigma \rangle := \int_\sigma u.$$

The fact that $\sum_{j=1}^b [c_j] = 0$ in $\mathcal{H}_1(\Sigma)$ becomes simply Stokes formula: for any closed 1-form ω on Σ

$$\sum_{j=1}^b \int_{c_j} \omega = \int_{\partial\Sigma} \omega = \int_\Sigma d\omega = 0.$$

As in (3.5) we define for $R > 0$

$$\mathcal{H}_R^1(\Sigma) := \{\omega \in \mathcal{H}^1(\Sigma) \mid \forall \sigma \in \mathcal{H}_1(\Sigma), \langle \omega, \sigma \rangle \in 2\pi R\mathbb{Z}\}.$$

For the relative homology $\mathcal{H}_1(\Sigma, \partial\Sigma)$, the relative condition ensures that the integral of a closed 1-form on a curve σ representing an element $[\sigma] \in \mathcal{H}_1(\Sigma, \partial\Sigma)$ only depends on the homotopy class of σ if the homotopy is such that the endpoints of the family of curves stay on $\partial\Sigma$. The relative homology $\mathcal{H}_1(\Sigma, \partial\Sigma)$ is dual to $\mathcal{H}^1(\Sigma, \partial\Sigma)$ by the pairing

$$(3.11) \quad \mathcal{H}^1(\Sigma, \partial\Sigma) \times \mathcal{H}_1(\Sigma, \partial\Sigma) \rightarrow \mathbb{R}, \quad \langle u, \sigma \rangle := \int_\sigma u$$

and we define

$$\mathcal{H}_R^1(\Sigma, \partial\Sigma) := \{\omega \in \mathcal{H}^1(\Sigma, \partial\Sigma) \mid \forall \sigma \in \mathcal{H}_1(\Sigma, \partial\Sigma), \langle \omega, \sigma \rangle \in 2\pi R\mathbb{Z}\}.$$

The duality isomorphisms (3.10) and (3.11) imply the following:

Lemma 3.4.

(1) Fix a basis of $\mathcal{H}_1(\Sigma)$. Let $\omega_1, \omega_2 \in C_{\text{abs}}^\infty(\Sigma, \Lambda^1\Sigma)$ such that $\int_\sigma \omega_1 = \int_\sigma \omega_2$ for all σ in the basis of $\mathcal{H}_1(\Sigma)$. Then there is $f \in C^\infty(\Sigma)$ with $\partial_\nu f|_{\partial\Sigma} = 0$ such that

$$\omega_1 = \omega_2 + df.$$

(2) Fix a basis of $\mathcal{H}_1(\Sigma, \partial\Sigma)$. Let $\omega_1, \omega_2 \in C_{\text{rel}}^\infty(\Sigma, \Lambda^1\Sigma)$ such that $\int_\sigma \omega_1 = \int_\sigma \omega_2$ for all σ in the basis of $\mathcal{H}_1(\Sigma, \partial\Sigma)$. Then there is $f \in C^\infty(\Sigma)$ with $f|_{\partial\Sigma} = 0$ such that

$$\omega_1 = \omega_2 + df.$$

Now, we will construct particular bases of $\mathcal{H}_R^1(\Sigma, \partial\Sigma)$, which have the nice property of being compactly supported. This will be particularly useful for gluing Segal’s amplitudes later.

Lemma 3.5. Let $R > 0$ and let Σ be an oriented compact surface with genus g and b boundary connected components $\partial_1\Sigma, \dots, \partial_b\Sigma$ and $\sigma_1, \dots, \sigma_{2g+b-1}$ be a basis of $\mathcal{H}_1(\Sigma, \partial\Sigma)$. Then there is a basis $\omega_1^c, \dots, \omega_{2g+b-1}^c$ of $\mathcal{H}_R^1(\Sigma, \partial\Sigma)$ made of closed forms that are compactly supported inside the interior Σ° of Σ and dual to $\sigma_1, \dots, \sigma_{2g+b-1}$ in the sense that

$$\frac{1}{2\pi R} \int_{\sigma_k} \omega_j^c = \delta_{jk}.$$

Moreover, $\omega_{2g+j}^c = df_j$ for some smooth function f_j on Σ that is locally constant in a neighborhood of $\partial\Sigma$.

Proof. Let us start with a basis $\omega_1^r, \dots, \omega_{2g+b-1}^r$ of $\mathcal{H}^1(\Sigma, \partial\Sigma)$, dual to the basis $\sigma_1, \dots, \sigma_{2g+b-1}$ of $\mathcal{H}_1(\Sigma, \partial\Sigma)$ in the sense above. Since $t_{\partial\Sigma}^* \omega_j^r = 0$, we see that $\int_{\partial_i \Sigma} \omega_j^r = 0$ for all i, j . In particular, in a collar neighborhood U_i of $\partial_i \Sigma$, we can define for $z_i \in \partial_i \Sigma$

$$f_{ij}(z) := \int_{\alpha_{z_i, z}} \omega_j^r$$

which is a well-defined smooth function in U_i if $\alpha_{z_i, z} \subset U_i$ is a smooth curve with initial endpoint z_i and final endpoint z . Notice also that $f_{ij}|_{\partial_i \Sigma} = 0$. Let $\chi_i \in C^\infty(\Sigma)$ with support in U_i and equal to 1 near $\partial_i \Sigma$. Then $\omega_j^c := \omega_j^r - \sum_{i=1}^b d(\chi_i f_{ij})$ belongs to $\mathcal{H}^1(\Sigma, \partial\Sigma)$ and is compactly supported inside Σ° , and it provides a basis of $\mathcal{H}^1(\Sigma, \partial\Sigma)$ since the integrals of ω_j^c on the cycles forming a basis of $\mathcal{H}_1(\Sigma, \partial\Sigma)$ are equal to the integrals of ω_j^r on these cycles. The fact that $\omega_{2g+j}^c = df_j$ for some f_j can be checked by writing $f_j(x) = \int_{\alpha_{x_0, x}} \omega_{2g+j}^c$ for some $x_0 \in \partial\Sigma$ where $\alpha_{x_0, x}$ is a smooth curve with endpoint at x_0 and x (depending smoothly on x), and noticing that the result does not depend on the curve by our assumptions on ω_{2g+j}^c . It is also locally constant near $\partial\Sigma$ since ω_{2g+j}^c is compactly supported in Σ° . \square

Lemma 3.6. *Let $R > 0$ and let Σ be an oriented compact surface with genus g and b boundary connected components $\partial_1 \Sigma, \dots, \partial_b \Sigma$, oriented positively with respect to Σ , let c_i be the cycle corresponding to $\partial_i \Sigma$. Let $(\sigma_j)_{j=1, \dots, 2g}$ be independent cycles of $\mathcal{H}_1(\Sigma)$ so that $((\sigma_j)_{j \leq 2g}, (c_i)_{i \leq b-1})$ form a basis of $\mathcal{H}_1(\Sigma)$. Then there are $b - 1$ independent closed forms $\omega_2^a, \dots, \omega_b^a$ in $\mathcal{H}_R^1(\Sigma)$ such that*

$$\begin{aligned} \forall \ell = 2, \dots, b, \quad \frac{1}{2\pi R} \int_{\partial_1 \Sigma} \omega_\ell^a &= -1, \quad \forall i \in [2, b], \quad \frac{1}{2\pi R} \int_{\partial_i \Sigma} \omega_\ell^a = \delta_{\ell i}, \\ \forall j = 1, \dots, 2g, \forall \ell = 2, \dots, b, \quad \int_{\sigma_j} \omega_\ell^a &= 0. \end{aligned}$$

Moreover, ω_ℓ^a can be chosen so that $\omega_\ell^a|_{U_i} = 0$ for some open neighborhood U_i of $\partial_i \Sigma$ if $i \notin \{\ell, 1\}$, while in U_ℓ and U_1 , there are biholomorphisms $\psi_\ell : \{z \in \mathbb{C} \mid |z| \in (\delta, 1]\} \rightarrow U_\ell$ and $\psi_1 : \{z \in \mathbb{C} \mid |z| \in (\delta, 1]\} \rightarrow U_1$ for some $\delta < 1$ such that $\psi_\ell(\mathbb{T}) = \partial_\ell \Sigma$, $\psi_1(\mathbb{T}) = \partial_1 \Sigma$ and if $z = re^{i\theta}$ are radial coordinates

$$\psi_\ell^* \omega_\ell^a = R d\theta, \quad \psi_1^* \omega_\ell^a = -R d\theta.$$

Proof. Let us take $\omega_2, \dots, \omega_b \in \mathcal{H}^1(\Sigma)$ such that $(2\pi R)^{-1} \int_{c_i} \omega_\ell = \delta_{i\ell}$ for $i \in [2, b]$ and $\int_{\sigma_j} \omega_\ell = 0$ for all $j \leq 2g$. We necessarily have $(2\pi R)^{-1} \int_{c_1} \omega_\ell = -1$ since $\sum_{i=1}^b c_i = 0$ in $\mathcal{H}_1(\Sigma)$. With the arguments of Lemma 3.5, we can replace ω_ℓ by $\omega'_\ell := \omega_\ell - \sum_{i \notin \{1, \ell\}} df_{\ell i}$ for some $f_{\ell i} \in C^\infty(\Sigma)$ supported in U_i so that $\text{supp}(\omega'_\ell) \cap \partial_i \Sigma = \emptyset$ for $i \notin \{\ell, 1\}$. Now, in U_ℓ , we see that $\omega'_\ell - R(\psi_\ell)_* d\theta$ integrates to 0 on $c_\ell = \partial_\ell \Sigma$. Therefore there is $f_{\ell \ell} \in C^\infty(U_\ell)$ with compact support in U_ℓ such that $\omega'_\ell = R(\psi_\ell)_* d\theta + df_{\ell \ell}$ in U_ℓ . The same argument shows that there is $f_{\ell b} \in C^\infty(U_b)$ with compact support in U_b such that $\omega'_\ell = -R(\psi_b)_* d\theta + df_{\ell b}$ in U_b , and we can just choose

$$\omega_\ell^a := \omega_\ell - \sum_{i=1}^b df_{\ell i}$$

which satisfies the desired properties. □

3.9. Gluing of surfaces and homology/cohomology. Consider two surfaces Σ_1, Σ_2 with parametrised boundary $\zeta_i = (\zeta_{i1}, \dots, \zeta_{ib_i})$ for $i = 1, 2$ where $\zeta_{ij} : \mathbb{T} \rightarrow \partial_j \Sigma_i$ by identifying $\partial_1 \Sigma_1 \sim \partial_1 \Sigma_2$ using the parametrisations of the boundary. We can construct a basis of homology on the glued surface $\Sigma := \Sigma_1 \# \Sigma_2$ from bases of relative/absolute homology/cohomology on Σ_j .

Lemma 3.7. *For $i = 1, 2$, let Σ_i be two oriented surfaces with genus g_i and b_i boundary connected components parametrised by $\zeta_i = (\zeta_{i1}, \dots, \zeta_{ib_i})$ with $\zeta_{ij} : \mathbb{T} \rightarrow \partial_j \Sigma_i$ all oriented positively. Let Σ be the oriented surface obtained by gluing Σ_1 to Σ_2 using the identifications $\partial_1 \Sigma_1 \sim -\partial_1 \Sigma_2$ given by $\zeta_{11}(e^{i\theta}) = \zeta_{21}(e^{-i\theta})$. The surface Σ has genus $g_1 + g_2$ and $b_1 + b_2 - 2$ boundary connected components.*

(1) Let $\sigma_i := S_i \cup D_i$ be a canonical geometric basis of the relative homology $\mathcal{H}_1(\Sigma_i, \partial \Sigma_i)$ as described in (3.9) for $i = 1, 2$, with

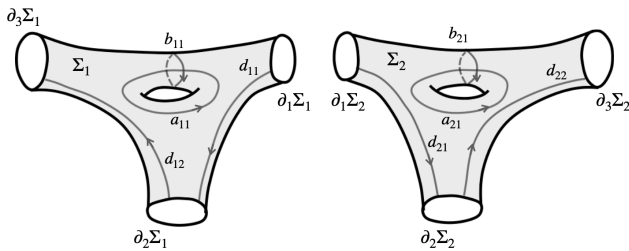
$$S_i = (a_{i1}, b_{i1}, \dots, a_{ig_i}, b_{ig_i}), \quad D_i := (d_{i1}, \dots, d_{ib_i-1})$$

and where d_{i1} are chosen so that the endpoint of d_{11} on $\partial_1 \Sigma_1$ coincides with the endpoint of d_{21} on $\partial_1 \Sigma_2$. Let

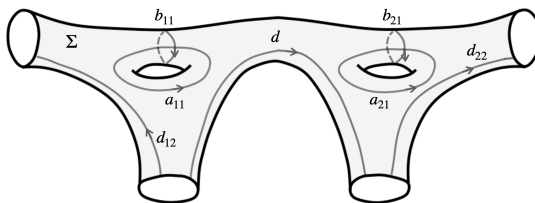
$$D := d_{21} - d_{11} \subset \mathcal{H}_1(\Sigma, \partial \Sigma), \quad \hat{D}_i = (d_{i2}, \dots, d_{ib_i-1}) \subset \mathcal{H}_1(\Sigma, \partial \Sigma),$$

where we identify cycles in Σ_i with their image in Σ induced by the inclusion $\Sigma_i \rightarrow \Sigma$. Then the set

$$(3.12) \quad \sigma := D \cup S_1 \cup S_2 \cup \hat{D}_1 \cup \hat{D}_2 \subset \mathcal{H}_1(\Sigma, \partial \Sigma)$$



(A) Relative homology on Σ_1 and Σ_2



(B) A canonical geometric basis of $\mathcal{H}_1(\Sigma, \partial \Sigma)$

FIGURE 6. Gluing relative homology bases on $\Sigma = \Sigma_1 \# \Sigma_2$

forms a canonical geometric basis of $\mathcal{H}_1(\Sigma, \partial\Sigma)$. See Figure 6. We say that σ is the gluing of the relative homology bases σ_1 and σ_2 and we use the notation

$$\sigma = \sigma_1 \# \sigma_2.$$

(2) Let $\sigma_i = S_i \cup C_i$ be a canonical geometric basis of the absolute homology $\mathcal{H}_1(\Sigma_i)$ as described in (3.8) for $i = 1, 2$, with

$$S_i = (a_{i1}, b_{i1}, \dots, a_{i\mathfrak{b}_i}, b_{i\mathfrak{b}_i}), \quad C_i = (c_{i2}, \dots, c_{i\mathfrak{b}_i}) \subset \mathcal{H}_1(\Sigma_i)$$

and $c_{ij} = \partial_j \Sigma_i$ the positively oriented boundary cycles associated to $\partial_1 \Sigma_i, \dots, \partial_{\mathfrak{b}_i-1} \Sigma_i$. Let

$$\hat{C}_i = (c_{i2}, \dots, c_{i(\mathfrak{b}_i-1)}) \subset \mathcal{H}_1(\Sigma),$$

where we identify cycles in Σ_i with their image in Σ induced by the inclusion $\Sigma_i \rightarrow \Sigma$. Then the set

$$(3.13) \quad \sigma := S_1 \cup S_2 \cup C_1 \cup \hat{C}_2 \subset \mathcal{H}_1(\Sigma)$$

is a canonical geometric basis of $\mathcal{H}_1(\Sigma)$. See Figure 7. We say that σ is the gluing of the absolute homology bases σ_1 and σ_2 and we use the notation

$$\sigma = \sigma_1 \# \sigma_2.$$

(3) For $i = 1, 2$, let $\omega_1^{i,c}, \dots, \omega_{2\mathfrak{q}_i+\mathfrak{b}_i-1}^{i,c} \in \mathcal{H}_R^1(\Sigma_i, \partial\Sigma_i)$ be the compactly supported basis from Lemma 3.5 associated to $S_i \cup D_i$, with $\omega_{2\mathfrak{q}_i+j}^{i,c}$ being the form dual to d_{ij} for $j = 1, \dots, \mathfrak{b}_i - 1$. Let $\omega_2^{1,a}, \dots, \omega_{\mathfrak{b}_1}^{1,a} \in \mathcal{H}_R^1(\Sigma_1)$ and $\omega_1^{2,a}, \dots, \omega_{\mathfrak{b}_2-1}^{2,a} \in \mathcal{H}_R^1(\Sigma_2)$ be the closed forms from Lemma 3.6 chosen such that,

$$\begin{aligned} \forall j, \ell \in [2, \mathfrak{b}_1], \quad \frac{1}{2\pi R} \int_{c_{11}} \omega_j^{1,a} &= -1, & \frac{1}{2\pi R} \int_{c_{1\ell}} \omega_j^{1,a} &= \delta_{j\ell}, \\ \forall j, \ell \in [1, \mathfrak{b}_2 - 1], \quad \frac{1}{2\pi R} \int_{c_{2\mathfrak{b}_2}} \omega_j^{2,a} &= -1, & \frac{1}{2\pi R} \int_{c_{2\ell}} \omega_j^{2,a} &= \delta_{j\ell}. \end{aligned}$$

Then $\omega_1^{i,c}, \dots, \omega_{2\mathfrak{q}_i+\mathfrak{b}_i-1}^{i,c} \subset \mathcal{H}_R^1(\Sigma_i, \partial\Sigma_i)$ for $i = 1, 2$, and $\omega_2^{1,a}, \dots, \omega_{\mathfrak{b}_2-1}^{2,a} \subset \mathcal{H}_R^1(\Sigma_2)$, can all be considered as smooth closed forms on Σ , extending them by 0 outside Σ_j and Σ_2 respectively — recall that $\omega_j^{2,a} = 0$ near $\partial_1 \Sigma_2$ for $j > 1$. The forms

$$\omega_j^a := \mathbf{1}_{\Sigma_1} \omega_j^{1,a} + \mathbf{1}_{\Sigma \setminus \Sigma_1} \omega_1^{2,a} \text{ for } j \in [2, \mathfrak{b}_1]$$

are also smooth and closed on Σ . The set

$$(3.14) \quad \omega_1^{1,c}, \dots, \omega_{2\mathfrak{q}_1}^{1,c}, \omega_1^{2,c}, \dots, \omega_{2\mathfrak{q}_2}^{2,c}, \omega_2^{1,a}, \dots, \omega_{\mathfrak{b}_1}^{1,a}, \omega_2^{2,a}, \dots, \omega_{\mathfrak{b}_2-1}^{2,a}$$

is a basis of $\mathcal{H}_R^1(\Sigma)$, dual to (3.13). The set

$$(3.15) \quad \omega_1^{1,c}, \dots, \omega_{2\mathfrak{q}_1+\mathfrak{b}_1-1}^{1,c}, \omega_1^{2,c}, \dots, \omega_{2\mathfrak{q}_2}^{2,c}, \omega_{2\mathfrak{q}_2+2}^{2,c}, \dots, \omega_{2\mathfrak{q}_2+\mathfrak{b}_2-1}^{2,c}$$

is a basis of $\mathcal{H}_R^1(\Sigma, \partial\Sigma)$, dual to (3.12).

Proof. Part (1) is a straightforward exercise of topology using Mayer-Vietoris. For (2), first consider the relative cohomology case. It is readily checked that the forms $\omega_j^{i,c}$

have vanishing integrals on all cycles of (3.12) except

$$\frac{1}{2\pi R} \int_{d_{11}-d_{21}} \omega_{2g_1+1}^{1,c} = 1, \quad \frac{1}{2\pi R} \int_{d_{1j}} \omega_{2g_1+j}^{1,c} = 1 \text{ if } j \in [2, \mathfrak{b}_1 - 1],$$

$$\frac{1}{2\pi R} \int_{d_{2j}} \omega_{2g_2+j}^{2,c} = 1, \text{ if } j \in [2, \mathfrak{b}_2 - 1], \quad \frac{1}{2\pi R} \int_{\sigma_{ij}} \omega_j^{i,c} = 1 \text{ if } j \in [1, 2g_i],$$

where $(\sigma_{i1}, \dots, \sigma_{i2g_i}) = (a_{i1}, b_{i1}, \dots, a_{ig_i}, b_{ig_i})$. This shows that (3.15) is a dual basis to (3.12).

Similarly, the forms $\omega_j^{2,a}$ and ω_j^a have vanishing integrals on all cycles of (3.12) except for the following

$$\frac{1}{2\pi R} \int_{c_{1j}} \omega_j^a = 1 \text{ if } j \in [2, \mathfrak{b}_1], \quad \frac{1}{2\pi R} \int_{c_{2j}} \omega_j^{2,a} = 1 \text{ if } j \in [2, \mathfrak{b}_2 - 1]$$

which implies, using the pairing we already did in the relative case, that (3.14) is a dual basis to (3.13). □

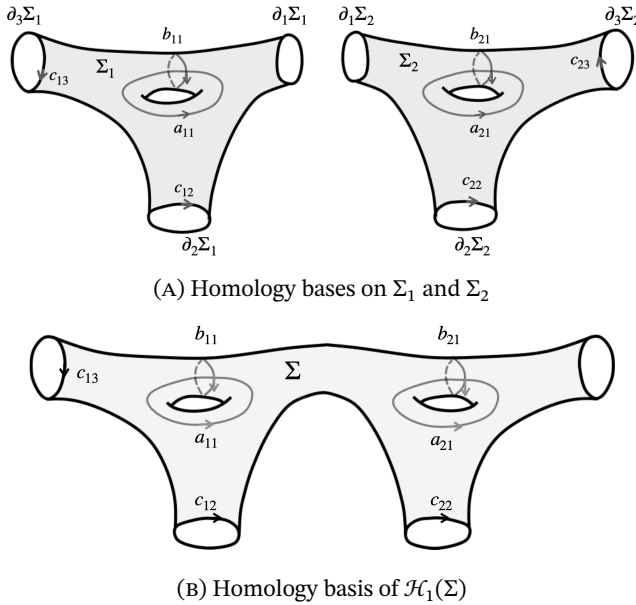


FIGURE 7. Gluing homology bases on $\Sigma = \Sigma_1 \# \Sigma_2$

Similarly, with the same argument, we can glue two boundary components on a connected oriented surface.

Lemma 3.8. *Let Σ be an oriented surface with genus g and \mathfrak{b} boundary connected components $\zeta_\ell : \mathbb{T} \rightarrow \partial_\ell \Sigma$ for $\ell \in [1, \mathfrak{b}]$, all oriented positively. Let $\Sigma^\#$ be the oriented surface obtained by identifying $\partial_1 \Sigma \sim -\partial_2 \Sigma$ in Σ via $\zeta_1(e^{i\theta}) = \zeta_2(e^{-i\theta})$, where the minus sign $-\partial_2 \Sigma$ denotes $\partial_2 \Sigma$ with the reverse orientation. The surface $\Sigma^\#$ has genus $g + 1$ and $\mathfrak{b} - 2$ boundary connected components.*

(1) Let $\sigma = S \cup D$ be a canonical geometric basis of the relative homology $\mathcal{H}_1(\Sigma, \partial\Sigma)$ as described in (3.9), with

$$S = (a_1, \dots, a_g, b_1, \dots, b_b), \quad D := (d_1, \dots, d_{b-1})$$

and where d_1 is chosen so that the endpoint of d_1 on $\partial_1\Sigma$ coincides with the endpoint of d_1 on $\partial_2\Sigma$ after gluing $\partial_1\Sigma$ with $\partial_2\Sigma$. Let

$$C = (c_1, \dots, c_b) \subset \mathcal{H}_1(\Sigma)$$

be the boundary cycles associated to $\partial_1\Sigma, \dots, \partial_b\Sigma$ and let $c_\ell^\#$ and $d_\ell^\#$ the images of c_ℓ and d_ℓ in $\Sigma^\#$ after gluing; in particular $c_1^\# = -c_2^\#$ and $d_1^\#$ is a closed curve. The set

$$(3.16) \quad \sigma^\# := S \cup d_1^\# \cup c_1^\# \cup \cup_{\ell=3}^{b-1} d_\ell^\# \subset \mathcal{H}_1(\Sigma, \partial\Sigma)$$

is a basis of $\mathcal{H}_1(\Sigma^\#, \partial\Sigma^\#)$.

(2) With the same notation as in (1), the set

$$(3.17) \quad \sigma^\# := S \cup d_1^\# \cup c_1^\# \cup \cup_{\ell=3}^{b-1} c_\ell^\# \subset \mathcal{H}_1(\Sigma^\#)$$

is a canonical geometric basis of $\mathcal{H}_1(\Sigma^\#)$.

(3) Let $\omega_1^c, \dots, \omega_{2g+b-1}^c \in \mathcal{H}^1(\Sigma, \partial\Sigma)$ be the compactly supported basis from Lemma 3.5 associated to $S \cup D$, with $\omega_{2g+\ell}^c$ being the form dual to d_ℓ for $\ell = 1, \dots, b-1$. Let $\omega_1^a, \dots, \omega_{b-1}^a \in \mathcal{H}^1(\Sigma)$ be the closed forms from Lemma 3.6. Then the forms $\omega_1^a - \omega_2^a$ and ω_{2g+1}^c both induce smooth closed 1-forms $\omega_{c_1^\#}$ and $\omega_{d_1^\#}$ on the glued surface $\Sigma^\#$. The set

$$(3.18) \quad \omega_1^c, \dots, \omega_{2g}^c, \omega_{d_1^\#}, \omega_{c_1^\#}, \omega_3^a, \dots, \omega_{b-1}^a$$

is a basis of $\mathcal{H}^1(\Sigma^\#)$, dual to (3.17). The set

$$(3.19) \quad \omega_1^c, \dots, \omega_{2g}^c, \omega_{d_1^\#}, \omega_{c_1^\#}, \omega_{2g+3}^c, \dots, \omega_{2g+b-1}^c$$

is a basis of $\mathcal{H}^1(\Sigma^\#, \partial\Sigma^\#)$, dual to (3.16).

3.10. Equivariant functions and distributions. Let Σ be a compact Riemann surface, with or without boundary. We let $b \geq 0$ the number of boundary connected components and g the genus, and $\beta_1 = \dim \mathcal{H}^1(\Sigma)$ the first Betti number ($\beta_1 = 2g$ if $\partial\Sigma = \emptyset$ and $\beta_1 = 2g + b - 1$ if $\partial\Sigma \neq \emptyset$). The universal cover $\pi : \tilde{\Sigma}_{x_0} \rightarrow \Sigma$ of Σ can be constructed from a fixed base point $x_0 \in \Sigma$ as the set of continuous paths $c : [0, 1] \rightarrow \Sigma$ with $c(0) = x_0$ up to homotopy. It has a distinguished point \tilde{x}_0 projecting to x_0 (given by the curve $c(t) = c(0) = x_0$), and we can view the fundamental group $\pi_1(\Sigma, x_0)$ as a group of deck transformations on $\tilde{\Sigma}_{x_0}$.

For each closed 1-form ω on Σ (possibly with a boundary), we can construct a function (a primitive) of ω on $\tilde{\Sigma}_{x_0}$ by

$$(3.20) \quad I_{x_0}(\omega)(\tilde{x}) = \int_{\alpha_{x_0, x}} \omega,$$

where the integral is along any C^1 path $\alpha_{x_0, x} : [0, 1] \rightarrow \Sigma$ such that $\alpha_{x_0, x}(0) = x_0$, $\alpha_{x_0, x}(1) = x := \pi(\tilde{x})$ and $\alpha_{x_0, x}$ lifts to $\tilde{\Sigma} = \tilde{\Sigma}_{x_0}$ as a curve with initial point \tilde{x}_0 and endpoint \tilde{x} . Notice that $dI_{x_0}(\omega) = \pi^*\omega$, and if ω is harmonic on Σ then $I_{x_0}(\omega)$ is a harmonic function on $\tilde{\Sigma}$.

Lemma 3.9. *Let $\omega \in \mathcal{H}_R^1(\Sigma)$ be a closed 1-form on Σ with integrals in $2\pi R\mathbb{Z}$ along the homology curves. Then $e^{\frac{i}{R}I_{x_0}(\omega)}$ descends to a well-defined smooth function on Σ .*

Proof. For $x \in \widetilde{\Sigma}_{x_0}$ and $\gamma \in \pi_1(\Sigma, x_0)$, one has

$$I_{x_0}(\omega)(\gamma.x) - I_{x_0}(\omega)(x) = \int_{\alpha_{x,\gamma.x}} \pi^* \omega \in 2\pi R\mathbb{Z},$$

where $\alpha_{x,y}$ is any $C^1(\widetilde{\Sigma}_{x_0})$ curve with $\alpha(0) = x$ and $\alpha(1) = y$ and $\alpha_{x,\gamma.x}$ descends to a closed curve on Σ , thus an element in $\mathcal{H}_1(\Sigma)$. This shows that $e^{\frac{i}{R}I_{x_0}(\omega)}$ is invariant by $\pi_1(\Sigma, x_0)$, and thus descends on Σ as a smooth function. \square

We can now define the space of equivariant functions on $\widetilde{\Sigma} = \widetilde{\Sigma}_{x_0}$: let $\Gamma := \pi_1(\Sigma, x_0)$ be the fundamental group of Σ , then we define the \mathbb{Z} -module

$$C_\Gamma^\infty(\widetilde{\Sigma}) := \{u \in C^\infty(\widetilde{\Sigma}) \mid \forall \gamma \in \Gamma, \gamma^* u - u \in 2\pi R\mathbb{Z}\}.$$

This space of functions can also be defined by the property that $e^{\frac{i}{R}u}$ is a smooth function that descends on Σ . Now, we observe that each $u \in C_\Gamma^\infty(\widetilde{\Sigma})$ induces a map

$$\chi_u : \Gamma \rightarrow 2\pi R\mathbb{Z}, \quad \chi_u(\gamma) := \gamma^* u - u.$$

One easily checks that $\chi_u(\text{Id}) = 0$ and $\chi_u(\gamma_1\gamma_2) = \chi_u(\gamma_1) + \chi_u(\gamma_2)$, so that χ_u is a group morphism. Now, each group morphism $\chi : \Gamma \rightarrow 2\pi R\mathbb{Z}$ is equivalent to an element in $\mathcal{H}_R^1(\Sigma)$. Fixing a basis $\omega_1, \dots, \omega_{\beta_1}$ of $\mathcal{H}_R^1(\Sigma)$ there is $\mathbf{k} \in \mathbb{Z}^{\beta_1}$ such that $\chi(\gamma) = \int_\gamma \omega_{\mathbf{k}}$ for all $\gamma \in \Gamma$ if $\omega_{\mathbf{k}} = \sum_{j=1}^{\beta_1} k_j \omega_j$. We denote $\chi_{\mathbf{k}}$ this morphism associated to $\omega_{\mathbf{k}}$. Next we define

$$C_\chi^\infty(\widetilde{\Sigma}) := \{u \in C^\infty(\widetilde{\Sigma}) \mid \forall \gamma \in \Gamma, \gamma^* u - u = \chi(\gamma)\} \subset C_\Gamma^\infty(\widetilde{\Sigma}).$$

For $u \in C_{\chi_{\mathbf{k}}}^\infty(\widetilde{\Sigma})$, we see that $u - I_{x_0}(\omega_{\mathbf{k}})$ is Γ -invariant, thus descends to a smooth function on Σ . We can thus rewrite the affine space $C_{\chi_{\mathbf{k}}}^\infty(\widetilde{\Sigma})$ under the form

$$C_{\chi_{\mathbf{k}}}^\infty(\widetilde{\Sigma}) = \{\pi^* f + I_{x_0}(\omega_{\mathbf{k}}) \mid f \in C^\infty(\Sigma)\}.$$

The discussion above shows that

$$C_\Gamma^\infty(\widetilde{\Sigma}) = \cup_{\mathbf{k} \in \mathbb{Z}^{\beta_1}} C_{\chi_{\mathbf{k}}}^\infty(\widetilde{\Sigma}).$$

In particular, if the representatives $\omega_{\mathbf{k}}$ of the cohomology space $\mathcal{H}_R^1(\Sigma)$ are fixed, each $u \in C_\Gamma^\infty(\widetilde{\Sigma})$ can be written in a unique way as $u = \pi^* f + I_{x_0}(\omega_{\mathbf{k}})$ for some $f \in C^\infty(\Sigma)$ and some $\mathbf{k} \in \mathbb{Z}^{\beta_1}$. We can also consider, for $s \in \mathbb{R}$, the Sobolev \mathbb{Z} -module

$$(3.21) \quad H_\Gamma^s(\widetilde{\Sigma}) := \{f \in H_{\text{loc}}^s(\widetilde{\Sigma}) \mid \forall \gamma \in \Gamma, \gamma^* f - f \in 2\pi R\mathbb{Z}\},$$

where $H_{\text{loc}}^s(\widetilde{\Sigma})$ is the space of distributions that are in the Sobolev space $H^s(U)$ on each relatively compact open set $U \subset \widetilde{\Sigma}$. As for smooth functions, each $u \in H_\Gamma^s(\widetilde{\Sigma})$ can be written as $u = \pi^* f + I_{x_0}(\omega_{\mathbf{k}})$ for some $f \in H^s(\Sigma)$ and some $\mathbf{k} \in \mathbb{Z}^{\beta_1}$. In other words, we get an identification

$$\mathbb{Z}^{\beta_1} \times H^s(\Sigma) \rightarrow H_\Gamma^s(\widetilde{\Sigma}), \quad (\mathbf{k}, f) \mapsto \pi^* f + I_{x_0}(\omega_{\mathbf{k}}).$$

3.11. Harmonic 1-forms with poles at prescribed marked points. Let Σ be a Riemann surface of genus g , with or without boundary. In case there is a boundary, we denote $\partial_j \Sigma$ for $j = 1, \dots, b$ the oriented boundary connected components, where $\zeta_j = -1$ if the boundary $\partial_j \Sigma$ is outgoing (i.e. positive) and $\zeta_j = 1$ if it is incoming. We write $\beta_1 = \dim \mathcal{H}_1(\Sigma)$ the first Betti number. Consider some distinct marked points $\mathbf{z} = (z_1, \dots, z_n) \in \Sigma^n$ in the interior of Σ , and we denote by $\{\mathbf{z}\} = \cup_j z_j$ the union of these points as a set in Σ . We attach some winding numbers $\mathbf{m} := (m_1, \dots, m_n) \in \mathbb{Z}^n$ to these points, these will be called *magnetic charges*. Denote by U_1, \dots, U_n neighborhoods of z_1, \dots, z_n and biholomorphic maps $\psi_j : \mathbb{D} \rightarrow U_j$ such that $\psi_j(0) = z_j$ and which extends holomorphically in a larger disc $\delta^{-1}\mathbb{D}$ for some $\delta < 1$. In the disc \mathbb{D} , we can use the variable $z = re^{i\theta}$ and we denote by $d\theta$ the 1-form in $\mathbb{D} \setminus \{0\}$ that is closed and coclosed in that pointed disk.

Proposition 3.10. *Let $\sigma = (\sigma_1, \dots, \sigma_{\beta_1}) \subset \mathcal{H}_1(\Sigma)$ be a basis realized by closed curves not intersecting the disks U_j around z_j , and such that $\sigma_{2g+\ell} = \partial_\ell \Sigma$ for $\ell = 1, \dots, b-1$ and $\sigma_\ell \cap \partial \Sigma = \emptyset$ if $\ell \leq 2g$. Let $\mathbf{k} = (k_1, \dots, k_b) \in \mathbb{Z}^b$ and $\mathbf{m} := (m_1, \dots, m_n) \in \mathbb{Z}^n$ and assume that $\sum_{\ell=1}^b \zeta_\ell k_\ell + \sum_{j=1}^n m_j = 0$. Then there exists a smooth real valued closed 1-form on $\Sigma \setminus \{z_1, \dots, z_n\}$, denoted by $\nu_{\mathbf{z}, \mathbf{m}}$ if $\partial \Sigma = \emptyset$, resp. $\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}$ if $\partial \Sigma \neq \emptyset$, such that $\iota_{\partial \Sigma}(i_\nu \nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}) = 0$ (with ν the interior pointing vector at $\partial \Sigma$) and*

$$\begin{cases} \psi_j^*(\nu_{\mathbf{z}, \mathbf{m}}|_{U_j}) = m_j \text{Rd}\theta, & \partial \Sigma = \emptyset, \\ \psi_j^*(\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}|_{U_j}) = m_j \text{Rd}\theta, & \partial \Sigma \neq \emptyset \end{cases}$$

and such that for all $j = 1, \dots, 2g$ and $\ell = 1, \dots, b$,

$$\begin{cases} \int_{\sigma_j} \nu_{\mathbf{z}, \mathbf{m}} = 0, & \partial \Sigma = \emptyset, \\ \int_{\sigma_j} \nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}} = 0 \quad \text{and} \quad \frac{1}{2\pi R} \int_{\partial_\ell \Sigma} \nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}} = \zeta_\ell k_\ell, & \partial \Sigma \neq \emptyset. \end{cases}$$

If $\psi_{n+\ell} : \{z \in \mathbb{C} \mid |z| \in (\delta, 1]\} \rightarrow V_\ell$ is a biholomorphism for some $\delta < 1$ and V_ℓ a collar neighborhood of $\partial_\ell \Sigma$, the form $\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}$ can be chosen so as to satisfy in addition $\psi_{n+\ell}^* \nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}} = -k_\ell \text{Rd}\theta$ near \mathbb{T} . The form $\nu_{\mathbf{z}, \mathbf{m}}$ satisfies $d^* \nu_{\mathbf{z}, \mathbf{m}} \in C^\infty(\Sigma) \cap C_c^\infty(\Sigma \setminus \{\mathbf{z}\})$ when $\partial \Sigma = \emptyset$ and $d^* \nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}} \in C^\infty(\Sigma) \cap C_c^\infty(\Sigma \setminus \{\mathbf{z}\})$ when $\partial \Sigma \neq \emptyset$. Moreover we have, in the distribution sense,

$$\begin{cases} d\nu_{\mathbf{z}, \mathbf{m}} = -2\pi R \sum_{j=1}^n m_j \delta_{z_j} & \partial \Sigma = \emptyset, \\ d\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}} = -2\pi R \sum_{j=1}^n m_j \delta_{z_j} & \partial \Sigma \neq \emptyset, \end{cases}$$

where δ_{z_j} is the Dirac measure at z_j . There is a unique real-valued closed and coclosed 1-form $\nu_{\mathbf{z}, \mathbf{m}}^h$ if $\partial \Sigma = \emptyset$, resp. $\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}^h$ if $\partial \Sigma \neq \emptyset$, on $\Sigma \setminus \{\mathbf{z}\}$ such that

$$(3.22) \quad \begin{cases} \nu_{\mathbf{z}, \mathbf{m}}^h - \nu_{\mathbf{z}, \mathbf{m}} = df_{\mathbf{m}}, & \partial \Sigma = \emptyset, \\ \nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}^h - \nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}} = df_{\mathbf{m}, \mathbf{k}}, & \partial \Sigma \neq \emptyset \end{cases}$$

for some $f_{\mathbf{m}} \in C^\infty(\Sigma)$ if $\partial \Sigma = \emptyset$, resp. $f_{\mathbf{m}, \mathbf{k}} \in C^\infty(\Sigma)$ with $f_{\mathbf{m}, \mathbf{k}}|_{\partial \Sigma} = 0$ if $\partial \Sigma \neq \emptyset$.

Proof. Let $\hat{\Sigma} := \Sigma \cup \cup_{j=1}^n \psi_j(\mathbb{D}^\circ)$. For the first claim, when $\partial \Sigma = \emptyset$ it suffices to take a linear combination $\nu_{\mathbf{z}, \mathbf{m}} = -\sum_{j=2}^n m_j \omega_j^a$ of the forms ω_j^a of Lemma 3.6 applied to the surface with boundary $\hat{\Sigma}$, and extend smoothly this form by setting $\nu_{\mathbf{z}, \mathbf{m}}|_{U_j} = (\psi_j)_*(m_j \text{Rd}\theta)$ for all $j = 1, \dots, n$. Here, we notice that we apply Lemma 3.6 with the biholomorphisms $\hat{\psi}_j : z \mapsto \psi_j(1/z)$ to get the right orientation of $\partial_j \hat{\Sigma}$ needed for that Lemma,

which is the reason of the minus sign in front of m_j in the choice of $\nu_{z,m}$. Next, assume $\partial\Sigma \neq \emptyset$ and, for simplicity of exposition, that the boundary components are all outgoing. We define $\partial_j \hat{\Sigma} := \partial U_j$ for $j = 1, \dots, n$ (oriented negatively in Σ) and $\partial_{n+j} \hat{\Sigma} := \partial_j \Sigma$ when $j = 1, \dots, b$ (oriented positively in $\hat{\Sigma}$) and for $\ell > n$ we choose biholomorphisms $\psi_\ell : \{z \in \mathbb{C} \mid |z| \in (\delta, 1]\} \rightarrow V_\ell$ for some $\delta < 1$ and V_ℓ a collar neighborhood of $\partial_\ell \hat{\Sigma}$. We then set $\nu_{z,m,k} = -\sum_{j=2}^n m_j \omega_j^a + \sum_{j=1}^b k_j \omega_{n+j}^a$ where the forms ω_j^a are the forms of Lemma 3.6 applied to the surface with boundary $\hat{\Sigma}$, chosen so that near \mathbb{T} we have

$$\begin{aligned} \forall j > n, \forall i > 1, \quad \psi_i^* \omega_j^a &= \delta_{ij} R d\theta, \text{ and } \psi_1^* \omega_j^a = R d\theta, \\ \forall j \in [2, n], \forall i > 1, \quad \psi_i^* \omega_j^a &= -\delta_{ij} R d\theta \text{ and } \psi_1^* \omega_j^a = R d\theta. \end{aligned}$$

Notice that near \mathbb{T} , we have

$$\begin{aligned} \psi_1^* \nu_{z,m,k} &= -\sum_{j=2}^n m_j \psi_1^* \omega_j^a + \sum_{j=1}^b k_j \psi_1^* \omega_{n+j}^a = \left(-\sum_{j=2}^n m_j + \sum_{j=1}^b k_j\right) R d\theta = m_1 R d\theta \\ \forall j = 2, \dots, n, \quad \psi_j^* \nu_{z,m,k} &= m_j R d\theta \end{aligned}$$

and we extend smoothly $\nu_{z,m,k}$ on U_j by setting $\nu_{z,m,k}|_{U_j} = (\psi_j)_*(m_j R d\theta)$ for all $j = 1, \dots, n$. In both cases, $\partial\Sigma = \emptyset$ or $\partial\Sigma \neq \emptyset$, the forms $\nu_{z,m}$ and $\nu_{z,m,k}$ satisfy the desired properties. Next, we compute $d\nu_{z,m}$ and $d^*\nu_{z,m}$ in the distribution sense. Using that $d * d\theta = 0$ in $\mathbb{D} \cap \{|z| > \epsilon\}$ and that $*d\theta = -dr/r$, if $f \in C^\infty(\mathbb{D}^o)$ is real valued we get

$$\langle d^* \nu_{z,m}, f \rangle = \int_\Sigma \nu_{z,m} \wedge *df = \lim_{\epsilon \rightarrow 0} \int_{|z| > \epsilon} \nu_{z,m} \wedge *df = -\lim_{\epsilon \rightarrow 0} \int_{|z| = \epsilon} f * \nu_{z,m} = 0$$

which shows that $d^* \nu_{z,m} = 0$ in the distribution sense near z_j , and $d^* \nu_{z,m}$ is thus smooth on Σ since $\nu_{z,m}$ is smooth outside $\{z\}$. Now for $d\nu_{z,m}$, we already know this is 0 outside $\{z\}$. We check that near z_j

$$\int_\Sigma \nu_{z,m} \wedge *d^*(f\nu_g) = \lim_{\epsilon \rightarrow 0} \int_{|z| > \epsilon} \nu_{z,m} \wedge d^*(f\nu_g) = -\lim_{\epsilon \rightarrow 0} \int_{|z| = \epsilon} f \nu_{z,m} = -m_j 2\pi R f(0).$$

The same argument applies for $\nu_{z,m,k}$ in the case $\partial\Sigma \neq \emptyset$.

To find the harmonic form, let $\omega := \nu_{z,m}$ if $\partial\Sigma = \emptyset$ and $\omega := \nu_{z,m,k}$ if $\partial\Sigma \neq \emptyset$.

By Stokes formula as above, we have $\langle d^* \omega, 1 \rangle_2 = 0$ if $\partial\Sigma = \emptyset$ and thus there is a unique smooth solution f_0 of

$$\Delta_g f_0 + d^* \omega = 0$$

with $\langle f_0, 1 \rangle_2 = 0$ if $\partial\Sigma = \emptyset$ and with $f_0|_{\partial\Sigma} = 0$ if $\partial\Sigma \neq \emptyset$. We then define $\omega^h := \omega + df_0$: it satisfies $d\omega^h = 0$ outside $\{z\}$ and $d^* \omega^h = \Delta_g f_0 + d^* \omega = 0$. \square

The form $\nu_{z,m} \in L^1(\Sigma)$ (resp. $\nu_{z,m,k} \in L^1(\Sigma)$) does not belong to $L^2(\Sigma)$, but we can define a renormalised L^2 -norm: first, for $\omega \in C^\infty(\Sigma \setminus \{z\})$, let

$$\|\omega\|_{g,\epsilon}^2 := \int_{\Sigma_{z,\epsilon,g}} \omega \wedge *\bar{\omega},$$

where $\Sigma_{z,\epsilon,g} := \Sigma \setminus \bigcup_{j=1}^n B_g(z_j, \epsilon)$ with $B_g(z_j, \epsilon)$ the geodesic ball centered at z_j with radius ϵ with respect to g .

Lemma 3.11. *Let $\omega = \nu_{z,m}$ if $\partial\Sigma = \emptyset$ or $\omega = \nu_{z,m,k}$ if $\partial\Sigma \neq \emptyset$. As $\epsilon \rightarrow 0$, the following limit exists*

$$(3.23) \quad \|\omega\|_{g,0}^2 := \lim_{\epsilon \rightarrow 0} \left(\|\omega\|_{g,\epsilon}^2 + 2\pi R^2 (\log \epsilon) \sum_{j=1}^n m_j^2 \right).$$

Furthermore, if $g' = e^\rho g$ is conformal to g then

$$\|\omega\|_{g',0}^2 = \|\omega\|_{g,0}^2 + \pi R^2 \sum_{j=1}^n m_j^2 \rho(z_j).$$

The same holds with $\omega = \nu_{z,m}^h$ or $\omega = \nu_{z,m,k}^h$.

Proof. We consider the case $\partial\Sigma = \emptyset$, the proof in the case with boundary being exactly the same. Let us write the metric in a small geodesic ball $B_g(z_j, \epsilon)$ in complex coordinates (using $\psi_j : \mathbb{D} \rightarrow U_j$) under the form $g_j := e^{\rho_j} |dz|^2$. One has for $\epsilon > 0$ small that

$$\int_{U_j \setminus B_g(z_j, \epsilon)} \nu_{z,m} \wedge \overline{\nu_{z,m}} = R^2 m_j^2 \int_{\mathbb{D} \setminus B_{g_j}(0, \epsilon)} \frac{1}{r^2} (dr \wedge rd\theta),$$

where $z = re^{i\theta}$. Note that $dr \wedge rd\theta = v_{|dz|^2}$ is the Euclidean volume form, which is smooth. Let us introduce geodesic polar coordinates for g_j around z_j

$$(r_j, \theta_j) \in (0, \delta) \times \mathbb{R}/2\pi\mathbb{Z} \mapsto \exp_{z_j}^{g_j}(r_j e^{-\rho_j/2} (\cos(\theta_j)\partial_x + \sin(\theta_j)\partial_y)),$$

where $\exp_{z_j}^{g_j}$ is the exponential map at z_j for the metric g_j . Notice that $r_j(z) = d_{g_j}(z, z_j)$ and that $r_j = re^{\rho_j(0)/2}(1 + F_j(r_j, \theta_j))$ with F_j smooth on $[0, \delta) \times \mathbb{R}/2\pi\mathbb{Z}$ and $F_j(0, \theta_j) = 0$. Moreover, using $e^{-\rho_j} dv_{g_j} = dv_{|dz|^2}$ and $dv_{g_j} = J_j(r_j, \theta_j) r_j dr_j d\theta_j$ with J_j smooth on $[0, \delta) \times \mathbb{R}/2\pi\mathbb{Z}$ satisfying $J_j(0, \theta_j) = 1$ one checks that for $\delta > 0$ fixed small, as $\epsilon \rightarrow 0$

$$(3.24) \quad \begin{aligned} \int_{r_j \in (\epsilon, \delta)} \frac{1}{r^2} v_{|dz|^2} &= \int_\epsilon^\delta \int_0^{2\pi} r_j^{-1} (1 + F_j(r_j, \theta_j))^2 J_j(r_j, \theta_j) dr_j d\theta_j \\ &= -2\pi \log(\epsilon) + 2\pi \log \delta + \int_0^\delta \int_0^{2\pi} L_j(r_j, \theta_j) dr_j d\theta_j + \mathcal{O}(\epsilon) \end{aligned}$$

for $L_j(r_j, \theta_j) := ((1 + F_j(r_j, \theta_j))^2 J_j(r_j, \theta_j) - 1)/r_j$ smooth on $[0, \delta) \times \mathbb{R}/2\pi\mathbb{Z}$. The function, for $\text{Re}(s) > 0$,

$$K_j : s \mapsto \int_{r_j \in (\epsilon, \delta)} r_j(z)^s \frac{1}{r^2} v_{|dz|^2} = \int_0^\delta \int_0^{2\pi} r_j^{s-1} (1 + F_j(r_j, \theta_j))^2 J_j(r_j, \theta_j) dr_j d\theta_j$$

extends meromorphically in $\text{Re}(s) > -1$ with a single pole of order 1 at $s = 0$, since

$$\begin{aligned} K_j(s) &= 2\pi \frac{\delta^s}{s} + \int_0^\delta \int_0^{2\pi} r_j^s L_j(r_j, \theta_j) dr_j d\theta_j \\ &= \frac{2\pi}{s} + \int_0^\delta \int_0^{2\pi} L_j(r_j, \theta_j) dr_j d\theta_j + 2\pi \log \delta + \mathcal{O}(s), \end{aligned}$$

where the last equality is a Laurent expansion at $s = 0$. We observe, by comparing to (3.24), that (here FP denotes finite part)

$$\text{FP}_{\epsilon=0} \int_{r_j \in (\epsilon, \delta)} \frac{1}{r^2} \mathbb{V}_{|dz|^2} = \text{FP}_{s=0} \int_{r_j \in (\epsilon, \delta)} r_j(z)^s \frac{1}{r^2} \mathbb{V}_{|dz|^2}.$$

This implies that, if r_g is any smooth positive function on $\Sigma \setminus \{\mathbf{z}\}$ equal to r_j near z_j for each j , then

$$\|\omega\|_{g,0}^2 = \text{FP}_{s=0} \int_{\Sigma} r_g(z)^s \nu_{z,\mathbf{m}} \wedge \overline{* \nu_{z,\mathbf{m}}}.$$

Now, if $g' = e^\rho g$ and $r_{g'}$ defined as r_g but using g' , we have

$$\|\omega\|_{g',0}^2 - \|\omega\|_{g,0}^2 = \text{FP}_{s=0} \int_{\Sigma} (r_{g'}^s - r_g^s) \nu_{z,\mathbf{m}} \wedge \overline{* \nu_{z,\mathbf{m}}}$$

and using that $r_{g'}^s(z) = r_g(z)^s q(z)^s$ for some smooth q with $q(z_j) = e^{\rho(z_j)/2}$, we obtain, writing $q_j := q|_{U_j} \circ \psi_j$,

$$\begin{aligned} \|\omega\|_{g',0}^2 - \|\omega\|_{g,0}^2 &= \lim_{s \rightarrow 0} \int_{\Sigma} r_g^s (q^s - 1) \nu_{z,\mathbf{m}} \wedge \overline{* \nu_{z,\mathbf{m}}} \\ &= \lim_{s \rightarrow 0} \sum_j R^2 m_j^2 \int_0^\delta \int_0^{2\pi} r^{s-1} (e^{s \frac{\rho(z_j)}{2}} - 1) \text{drd}\theta \\ &= \sum_j \pi R^2 m_j^2 \rho(z_j) \end{aligned}$$

and this ends the proof. The same proof works with $\nu_{\mathbf{x},\mathbf{m}}^h$ by using (3.22). □

3.12. Equivariant functions and Sobolev distributions. Case with marked points. Let (Σ, g) be a closed Riemannian surface and $\mathbf{z} = (z_1, \dots, z_n)$ disjoint marked points on Σ . As in Section 3.10, let $\Gamma := \pi_1(\Sigma \setminus \{\mathbf{z}\}, x_0)$ be the fundamental group of the punctured surface $\Sigma_{\mathbf{z}} := \Sigma \setminus \{\mathbf{z}\}$, $\tilde{\Sigma}_{\mathbf{z}}$ be the universal cover of $\Sigma_{\mathbf{z}}$ with $\pi : \tilde{\Sigma}_{\mathbf{z}} \rightarrow \Sigma_{\mathbf{z}}$ the projection, and $\tilde{x}_0 \in \tilde{\Sigma}_{\mathbf{z}}$ is a fixed preimage of x_0 used to define $\tilde{\Sigma}_{\mathbf{z}}$. The metric g lifts to $\tilde{\Sigma}_{\mathbf{z}}$ and provides a Riemannian measure. We say that $u \in L^2_{\text{loc}}(\tilde{\Sigma}_{\mathbf{z}})$ if on each fundamental domain \mathcal{F} of Γ , $u \in L^2(\mathcal{F}, \text{dv}_g)$. Then we can consider the space

$$L^2_{\Gamma}(\tilde{\Sigma}_{\mathbf{z}}) := \{u \in L^2_{\text{loc}}(\tilde{\Sigma}_{\mathbf{z}}) \mid \forall \gamma \in \Gamma, \gamma^* u - u \in 2\pi R\mathbb{Z}\}.$$

We have $L^2_{\Gamma}(\tilde{\Sigma}_{\mathbf{z}}) = \bigcup_{\chi} L^2_{\chi}(\tilde{\Sigma}_{\mathbf{z}})$ where the union is over the set of group morphisms $\chi : \Gamma_{\mathbf{z}} \rightarrow 2\pi R\mathbb{Z}$ and

$$L^2_{\chi}(\tilde{\Sigma}_{\mathbf{z}}) = \{u \in L^2_{\text{loc}}(\tilde{\Sigma}_{\mathbf{z}}) \mid \forall \gamma \in \Gamma, \gamma^* u - u = \chi(\gamma)\}.$$

Each such group morphism is represented by an element $\omega_{\mathbf{k}} + \nu_{z,\mathbf{m}}$ for $(\mathbf{k}, \mathbf{m}) \in \mathbb{Z}^{2g} \times \mathbb{Z}^n$ via

$$\chi_{\mathbf{k},\mathbf{m}}(\gamma) = \int_{\gamma} (\nu_{z,\mathbf{m}} + \omega_{\mathbf{k}}),$$

where $\omega_{\mathbf{k}} \in \mathcal{H}^1_R(\Sigma)$ for $\mathbf{k} \in \mathbb{Z}^{2g}$ as in (3.6) using a basis of $\mathcal{H}^1_R(\Sigma)$. This means that the affine space $L^2_{\chi_{\mathbf{k},\mathbf{m}}}(\tilde{\Sigma}_{\mathbf{z}})$ can be represented by

$$L^2_{\chi_{\mathbf{k},\mathbf{m}}}(\tilde{\Sigma}_{\mathbf{z}}) = \{\pi^* f + I_{x_0}(\omega_{\mathbf{k}}) + I_{x_0}(\nu_{z,\mathbf{m}}) \mid f \in L^2(\Sigma)\}$$

and each element $u \in L^2_{\chi_{\mathbf{k},\mathbf{m}}}(\tilde{\Sigma}_{\mathbf{z}})$ has a unique decomposition under the form $u = \pi^*f + I_{x_0}(\omega_{\mathbf{k}}) + I_{x_0}(\nu_{\mathbf{z},\mathbf{m}})$. Here we have set $I_{x_0}(\nu_{\mathbf{z},\mathbf{m}})(\tilde{x}) = \int_{\alpha_{x_0,x}} \nu_{\mathbf{z},\mathbf{m}}$ if $\alpha_{x_0,x}$ lifts to a curve with initial point \tilde{x}_0 and endpoint \tilde{x} . We also consider, for $s \in (-1/2, 0)$,

$$H^s_{\Gamma}(\tilde{\Sigma}_{\mathbf{z}}) := L^2_{\Gamma}(\tilde{\Sigma}_{\mathbf{z}}) + \pi^*(H^s(\Sigma)),$$

where $H^s(\Sigma)$ is the Sobolev space of order s on Σ . The forms $(\omega_{\mathbf{k}})_{\mathbf{k} \in \mathbb{Z}^{2g}} \cup (\nu_{\mathbf{z},\mathbf{m}})_{\mathbf{m} \in \mathbb{Z}^{2n}}$ are representatives of all the cohomology classes in the cohomology space $\mathcal{H}^1_R(\Sigma_{\mathbf{z}})$ of the surface $\Sigma \setminus \{\mathbf{z}\}$. With these representatives fixed, there is a one-to-one correspondence

$$\mathbb{Z}^{2g+\mathbf{m}} \times H^s(\Sigma) \rightarrow H^s_{\Gamma}(\tilde{\Sigma}_{\mathbf{z}}), \quad (\mathbf{k}, \mathbf{m}, f) \mapsto \pi^*f + I_{x_0}(\omega_{\mathbf{k}}) + I_{x_0}(\nu_{\mathbf{z},\mathbf{m}}).$$

4. CURVATURE TERM

Let (Σ, g) be a closed oriented Riemannian surface. For $\omega_{\mathbf{k}} \in \mathcal{H}^1_R(\Sigma)$ a closed 1-form, the construction of the path integral will require to make sense of the integrals

$$\int_{\Sigma} K_g I_{x_0}(\omega_{\mathbf{k}}) dv_g, \quad \int_{\Sigma} K_g I_{x_0}(\nu_{\mathbf{z},\mathbf{m}}) dv_g,$$

the problem being that $I_{x_0}(\omega_{\mathbf{k}})$ and $I_{x_0}(\nu_{\mathbf{z},\mathbf{m}})$ are multivalued on Σ , i.e. they live on the universal cover $\tilde{\Sigma}$ of Σ and $\tilde{\Sigma}_{\mathbf{z}}$ of $\Sigma \setminus \{\mathbf{z}\}$. We will thus consider $I_{x_0}(\omega_{\mathbf{k}})$ and $I_{x_0}(\nu_{\mathbf{z},\mathbf{m}})$ as well-defined functions on a dense open set of Σ and $\Sigma \setminus \{\mathbf{z}\}$ by removing curves. To obtain an invariant definition, it will be required to remove a curvature term coming from these curves.

4.1. Curvature term associated to $\mathcal{H}^1_R(\Sigma)$. Before giving the definition of the regularised curvature term, we let $\sigma = (a_j, b_j)_{j=1,\dots,g}$ be a geometric symplectic basis of $\mathcal{H}_1(\Sigma)$ and let

$$(4.1) \quad \Sigma_{\sigma} := \Sigma \setminus \sigma = \Sigma \setminus \cup_{j=1}^g (a_j \cup b_j).$$

We observe that any closed form ω on Σ is exact when restricted on Σ_{σ} (see below) and we denote, for $x_0 \in \Sigma_{\sigma}$ a fixed base point,

$$I^{\sigma}_{x_0}(\omega)(x) := \int_{\alpha_{x_0,x}} \omega$$

defined using the integral of ω along an oriented path $\alpha_{x_0,x} \subset \Sigma_{\sigma}$ with x_0, x as initial and final points, depending smoothly on x . This is a well-defined smooth function on Σ_{σ} , not depending on the choice of path $\alpha_{x_0,x}$ in Σ_{σ} , and $dI^{\sigma}_{x_0}(\omega) = \omega$.

Definition 4.1. For $\sigma = (a_j, b_j)_{j=1,\dots,g}$ a geometric symplectic basis of $\mathcal{H}_1(\Sigma)$ and $\omega \in \mathcal{H}^1_R(\Sigma)$ with associated morphism $\chi_{\omega} : \mathcal{H}_1(\Sigma) \rightarrow 2\pi R\mathbb{Z}$ given by $\chi_{\omega}(\gamma) := \int_{\gamma} \omega$, we define the regularised integral

$$(4.2) \quad \int_{\Sigma_{\sigma}}^{\text{reg}} K_g I^{\sigma}_{x_0}(\omega) dv_g := \int_{\Sigma_{\sigma}} I^{\sigma}_{x_0}(\omega) K_g dv_g + 2 \sum_{j=1}^g (\chi_{\omega}(a_j) \int_{b_j} k_g d\ell_g - \chi_{\omega}(b_j) \int_{a_j} k_g d\ell_g),$$

where we use the convention that the geodesic curvature of an oriented curve $c(t) \subset \Sigma$ parametrised by arclength⁷ is defined by

$$(4.3) \quad k_g(c(t)) = \langle \nabla_{\dot{c}(t)} \dot{c}(t), \nu(t) \rangle_g,$$

⁷If the curve is not parametrised by arclength, the integral $\int_c k_g d\ell_g$ is defined by changing parametrisation to make it arclength, while keeping the orientation, and using (4.3).

where $\nu(t)$ is the unit normal to the curve c at $c(t)$ such that $(\dot{c}(t), \nu)$ is positively oriented in Σ , ∇ the Levi-Civita connection of g .

We notice that in the physics literature, a related renormalisation was proposed by Verlinde-Verlinde [69, Section 6.3] in the context of bozonization of fermions. Our renormalisation only uses a symplectic homology basis and not a fundamental domain for Σ in the universal cover $\tilde{\Sigma}$. This seems more adapted to study the invariance of the curvature integral under change of diffeomorphisms.

We state the important invariance properties of the regularised integral in 4 Lemmas: local invariance with respect to the geometric symplectic basis, invariance by conformal change of metric, diffeomorphism invariance and global invariance by change of symplectic basis of $\mathcal{H}_1(\Sigma)$. The proofs will be done below.

Lemma 4.2 (Local invariance). *Let $\sigma' = (a'_j, b'_j)_{j=1, \dots, g}$ be another geometric symplectic basis of $\mathcal{H}_1(\Sigma)$ representing the symplectic homology basis $[\sigma] = ([a_j], [b_j])_j$. Then*

$$\int_{\Sigma_\sigma}^{\text{reg}} K_g I_{x_0}^\sigma(\omega) \, dv_g - \int_{\Sigma_{\sigma'}}^{\text{reg}} K_g I_{x_0}^{\sigma'}(\omega) \, dv_g \in 8\pi^2 RZ.$$

Lemma 4.3 (Conformal change of metrics). *Let $x_0 \in \Sigma$ and $\sigma = (a_j, b_j)_{j=1, \dots, g}$ be a geometric symplectic basis of $\mathcal{H}_1(\Sigma)$. Let $\rho \in C^\infty(\Sigma)$ and $\hat{g} = e^\rho g$ be two conformally related metrics on Σ and $\omega \in \mathcal{H}_R^1(\Sigma)$ a closed 1-form. Then the following identity holds true*

$$\int_{\Sigma_\sigma}^{\text{reg}} I_{x_0}^\sigma(\omega) K_g \, dv_{\hat{g}} = \int_{\Sigma_\sigma}^{\text{reg}} I_{x_0}^\sigma(\omega) K_g \, dv_g + \langle d\rho, \omega \rangle_2.$$

Lemma 4.4 (Invariance by diffeomorphism). *For $\psi : \Sigma \rightarrow \Sigma$ an orientation preserving diffeomorphism and $\sigma = (a_j, b_j)_{j=1, \dots, g}$ a geometric symplectic basis of $\mathcal{H}_1(\Sigma)$, let $\psi(\sigma) = (\psi(a_j), \psi(b_j))_{j=1, \dots, g}$ be the image geometric symplectic basis representing the basis $(\psi_*[a_j], \psi_*[b_j])_{j=1, \dots, g}$ of $\mathcal{H}_1(\Sigma)$. Let $x_0 \in \Sigma$, then, the following identity holds true*

$$\int_{\Sigma_\sigma}^{\text{reg}} K_g I_{x_0}^\sigma(\omega) \, dv_g = \int_{\Sigma_{\psi(\sigma)}}^{\text{reg}} K_{\psi_*g} I_{\psi(x_0)}^{\psi(\sigma)}(\psi_*\omega) \, dv_{\psi_*g}.$$

Lemma 4.5 (Invariance by change of symplectic basis of $\mathcal{H}_1(\Sigma)$). *Let $\sigma = (a_j, b_j)_{j=1, \dots, g}$ and $\sigma' = (a'_j, b'_j)_{j=1, \dots, g}$ be two geometric symplectic bases of $\mathcal{H}_1(\Sigma)$. Let $x_0 \in \Sigma$, then the following identity holds true*

$$\int_{\Sigma_\sigma}^{\text{reg}} K_g I_{x_0}^\sigma(\omega) \, dv_g - \int_{\Sigma_{\sigma'}}^{\text{reg}} K_g I_{x_0}^{\sigma'}(\omega) \, dv_g \in 8\pi^2 RZ.$$

In order to write the proofs of Lemmas 4.2–4.5, we first need to introduce a few notations and geometric decomposition of Σ_σ . Choose a geometric symplectic basis $\sigma := (a_j, b_j)_{j=1, \dots, g}$ of $\mathcal{H}_1(\Sigma)$ with intersection points $x_j := a_j \cap b_j$. The surface Σ can be decomposed under the form

$$\Sigma = S_\sigma \cup \bigcup_{j=1}^g \mathcal{T}_j,$$

where S_σ is a sphere with g disks $\mathcal{D}_1, \dots, \mathcal{D}_g$ removed, thus having g boundary circles $(c_j)_j$, and each \mathcal{T}_j is a torus with a disk \mathcal{D}'_j removed and whose non-trivial homology cycles are (a_j, b_j) . The boundary of \mathcal{D}'_j is glued to c_j . Thus

$$\Sigma_\sigma := \Sigma \setminus \bigcup_{i=1}^g (a_i \cup b_i)$$

is an open surface which decomposes as

$$\Sigma_\sigma = S_\sigma \cup \cup_{i=1}^g K_j,$$

where

$$K_j = \{z \in \mathbb{C} \mid \operatorname{Re}(z) \in (0, 1), \operatorname{Im}(z) \in (0, 1)\} \setminus D(e, \epsilon), \quad e = \frac{1}{2}(1 + i),$$

is a square with a small disk $\mathcal{D}'_j = D(e, \epsilon)$ centered at e and of radius $\epsilon < 1/4$ removed (see Figure 20). The circle c_j is glued to the circle $\partial\mathcal{D}'_j \subset K_j$, and the closure \bar{K}_j is a surface with a boundary circle $\partial D(e, \epsilon)$ and with 4 oriented boundary curves

$$\begin{aligned} \sigma_{a_j} &= \{t \in \mathbb{C} \mid t \in [0, 1]\}, & \bar{\sigma}_{a_j} &= \{i + t \in \mathbb{C} \mid t \in [0, 1]\}, \\ \sigma_{b_j} &= \{it \in \mathbb{C} \mid t \in [0, 1]\}, & \bar{\sigma}_{b_j} &= \{1 + it \in \mathbb{C} \mid t \in [0, 1]\} \end{aligned}$$

forming 4 corners. The torus \mathcal{T}_j is realized as a quotient $\mathcal{T}_j = \tilde{\mathcal{T}}_j / (\mathbb{Z} + i\mathbb{Z})$ where $\tilde{\mathcal{T}}_j = \mathbb{C} \setminus \cup_{k \in \mathbb{Z} + i\mathbb{Z}} D(e + k, \epsilon)$ of \mathcal{T}_j , with the action of the abelian group $\mathbb{Z} + i\mathbb{Z}$ being by translation. The generators $1, i$ of $\mathbb{Z} + i\mathbb{Z} \simeq \mathbb{Z}^2$ are identified to the cycles a_j, b_j and we write $\gamma_{a_j}(z) = z + 1$ and $\gamma_{b_j}(z) = z + i$. The set K_j is a fundamental domain for the quotient map $\pi_j : \tilde{\mathcal{T}}_j \rightarrow \tilde{\mathcal{T}}_j / (\mathbb{Z} + i\mathbb{Z})$, and $\gamma_{a_j}(\sigma_{b_j}) = \bar{\sigma}_{b_j}$ and $\gamma_{b_j}(\sigma_{a_j}) = \bar{\sigma}_{a_j}$. The curves $\sigma_{a_j}, \bar{\sigma}_{a_j}$ (resp. $\sigma_{b_j}, \bar{\sigma}_{b_j}$) are lifts of $a_j \cap \mathcal{T}_j$ (resp. $b_j \cap \mathcal{T}_j$) to \bar{K}_j .

Since c_j is the boundary of \mathcal{T}_j (viewed as embedded in Σ), we observe that $\int_{c_j} \omega = 0$ for all closed forms ω . Since the only non-contractible closed curves in Σ_σ are generated by the $(c_j)_j$, this shows that all closed forms ω on Σ_σ are exact, as was claimed above, and $I_{x_0}^\sigma(\omega)$ is well-defined on Σ_σ . This function, restricted to $\mathcal{T}_j \setminus \{a_j, b_j\}$, pulls-back by π_j to a smooth function on K_j that extends smoothly to \bar{K}_j satisfying

$$\begin{aligned} \forall z \in \sigma_{a_j}, \quad I_{x_0}^\sigma(\omega)(\gamma_{b_j}(z)) &= I_{x_0}^\sigma(\omega)(z) + \int_{b_j} \omega, \\ \forall z \in \sigma_{b_j}, \quad I_{x_0}^\sigma(\omega)(\gamma_{a_j}(z)) &= I_{x_0}^\sigma(\omega)(z) + \int_{a_j} \omega. \end{aligned}$$

If we glue $\tilde{\mathcal{T}}_j$ to S_σ by identifying $\partial\mathcal{D}'_j \subset K_j$ with $c_j \subset \partial S_\sigma$, then $I_{x_0}^\sigma(\omega)|_{K_j}$ extends smoothly from K_j to $\tilde{\mathcal{T}}_j$ satisfying

$$I_{x_0}^\sigma(\omega)(\gamma_{b_j}(z)) = I_{x_0}^\sigma(\omega)(z) + \int_{b_j} \omega, \quad I_{x_0}^\sigma(\omega)(\gamma_{a_j}(z)) = I_{x_0}^\sigma(\omega)(z) + \int_{a_j} \omega.$$

We will call it the *equivariant extension* of $I_{x_0}^\sigma(\omega)$ to $\tilde{\mathcal{T}}_j$ and denote it in the same way $I_{x_0}^\sigma(\omega)$.

We also make the following observation: let $U_{x_0}^\sigma \subset \tilde{\Sigma}$ be the connected component, in the universal cover of Σ , of the open set $\pi^{-1}(\Sigma_\sigma)$ that contains the point \tilde{x}_0 . Then $\pi^* I_{x_0}^\sigma(\omega) = I_{x_0}^\sigma(\omega)$ on $U_{x_0}^\sigma$ and therefore $\pi^* I_{x_0}^\sigma(\omega)$ extends on $\tilde{\Sigma}$ as a smooth function in $C_\Gamma^\infty(\tilde{\Sigma})$.

Proof of Lemma 4.2. We let $(a_j, b_j)_{j=1, \dots, g}$ be a geometric symplectic basis of $\mathcal{H}^1(\Sigma)$ and $(a'_j, b'_j)_{j=1, \dots, g}$ another geometric symplectic basis of $\mathcal{H}^1(\Sigma)$ such that $[a_j] = [a'_j]$ for all j . Recall that this means that $(a_j, b_j)_j$ is a basis of $\mathcal{H}_1(\Sigma)$ made of simple curves, with $a_j \cap b_j$ being a point, $a_i \cap a_j = b_i \cap b_j = a_i \cap b_j = \emptyset$ if $i \neq j$, and same for $(a'_j, b'_j)_j$. First, consider the case where a'_j is homotopic to a_j and b'_j homotopic to b_j for all j .

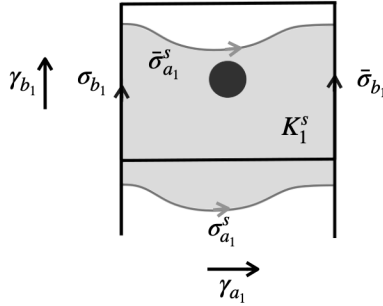


FIGURE 8. Moving the curve a_1

It suffices to prove that, if $(a_j^s, b_j^s)_{j=1, \dots, g}$ is a smooth 1-parameter family of geometric symplectic basis (thus homotopically equivalent to $(a_j, b_j)_j$) with $a_j^0 = a_j, b_j^0 = b_j$ and disjoint from x_0 for all s , then for $\sigma^s = (a_j^s, b_j^s)_j$ the derivative

$$\partial_s \left(\int_{\Sigma_{\sigma^s}}^{\text{reg}} K_g I_{x_0}^{\sigma^s}(\omega) \, dv_g \right) \Big|_{s=0} = 0.$$

Without loss of generality, we prove this when only one a_j^s is depending on s , and for small s the curve $a_j^s \subset \mathcal{T}_j^\circ$. Let $a_j^s(t) = a_j(t) + sv_j(t) + \mathcal{O}(s^2)$ and we consider its lifts to the cover $\tilde{\mathcal{T}}_j$: this defines two families of curves $\sigma_{a_j^s}$ and $\tilde{\sigma}_{a_j^s} = \gamma_{b_j}(\sigma_{a_j^s})$ joining the vertical lines $i\mathbb{R}$ and $i\mathbb{R} + 1$ (the lifts of σ_{b_j} and $\tilde{\sigma}_{b_j}$) and the domain $K_j^s \subset \{\text{Re}(z) \in [0, 1]\}$ enclosed between these two curves produces a new fundamental domain for \mathcal{T}_j (see Figure 8). If $I_{x_0}^\sigma(\omega)$ is the smooth equivariant extension of $I^\sigma(\omega)|_{K_j}$ to $\tilde{\mathcal{T}}_j$, one has

$$\int_{K_j^s} I_{x_0}^{\sigma^s}(\omega) K_g \, dv_g = \int_{K_j^s} I_{x_0}^\sigma(\omega) K_g \, dv_g$$

and it thus suffices to compute the variation

$$\begin{aligned} \partial_s \left(\int_{K_j^s} I_{x_0}^\sigma(\omega) K_g \, dv_g \right) \Big|_{s=0} &= - \int_{\sigma_{a_j}} g(v_j, \nu) I_{x_0}^\sigma(\omega) K_g \, d\ell_g \\ &\quad + \int_{\tilde{\sigma}_{a_j}} g(d\gamma_{b_j} \cdot \nu_j, d\gamma_{b_j} \cdot \nu) I_{x_0}^\sigma(\omega) K_g \, d\ell_g \\ &= \chi(\gamma_{b_j}) \int_{\sigma_{a_j}} g(v_j, \nu) K_g \, d\ell_g, \end{aligned}$$

where ν is the incoming unit normal vector field to σ_{a_j} in K_j . We can now differentiate the Gauss-Bonnet formula in the polygonal domain \hat{K}_j^s bounded by $\sigma_{a_j^s} \cup \tilde{\sigma}_{a_j^s} \cup i\mathbb{R} \cup (1 + i\mathbb{R})$: since the sum of the interior angles is constant (equal to 2π), we get

$$\int_{\sigma_{a_j}} g(v_j, \nu) K_g \, d\ell_g = -\partial_s \left(\int_{\hat{K}_j^s} K_g \, dv_g \right) \Big|_{s=0} = 2\partial_s \left(\int_{\sigma_{a_j^s}} k_g \, d\ell_g \right) \Big|_{s=0}.$$

This implies that

$$\partial_s \left(\int_{K_j^s} I_{x_0}^\sigma(\omega) K_g \, dv_g \right) |_{s=0} = 2\chi(\gamma_{b_j}) \partial_s \left(\int_{\sigma_{a_j^s}} k_g \, d\ell_g \right) |_{s=0}$$

and thus

$$\partial_s \left(\int_{\Sigma_{\sigma^s}}^{\text{reg}} I_{x_0}^{\sigma^s}(\omega) K_g \, dv_g \right) |_{s=0} = 0.$$

The other case we have to deal with is when one of the curves, say a'_j , is not homotopic to a_j (but homologous). Up to deforming a'_j via a homotopy, we can assume that $a'_j \cap a_j = \emptyset$ (here a'_j does not intersect the other curves a_i, b_i), since by the previous proof the regularised curvature integral is invariant by homotopy. Since $[a_j] - [a'_j] = 0$ in homology, this means that $a_j \cup a'_j$ is the boundary of a surface with boundary, which has two connected components Σ_1, Σ_2 . We can assume that $x_0 \in \Sigma_2$ (up to exchanging Σ_1 with Σ_2). Then we observe that in Σ_1

$$I_{x_0}^{\sigma'}(\omega) - I_{x_0}^\sigma(\omega) = \int_{b_j} \omega = \chi_\omega(b_j) \in 2\pi R\mathbb{Z}$$

since the difference is an integral of ω along a loop in $(\Sigma \setminus \sigma) \cup a_j$ that intersects a_j once. Then we can apply Gauss-Bonnet on $\Sigma_1 \setminus (\sigma \cup \sigma')$

$$\begin{aligned} & \int_{\Sigma_\sigma}^{\text{reg}} K_g I_{x_0}^\sigma(\omega) \, dv_g - \int_{\Sigma_{\sigma'}}^{\text{reg}} K_g I_{x_0}^{\sigma'}(\omega) \, dv_g \\ &= \chi_\omega(b_j) \left(\int_{\Sigma_1} K_g \, dv_g - 2 \int_{a_j} k_g \, d\ell_g + 2 \int_{a'_j} k_g \, d\ell_g \right) \\ &= 4\pi\chi(\Sigma_1)\chi_\omega(b_j) \in 8\pi^2 R\mathbb{Z}. \end{aligned} \quad \square$$

Next, we check the conformal covariance of the regularised integral.

Proof of Lemma 4.3. We use the relation $K_{\hat{g}} = e^{-\rho}(K_g + \Delta_g \rho)$ and $dv_{\hat{g}} = e^\rho dv_g$, then by integration by parts

$$\begin{aligned} \int_{\Sigma_\sigma} I_{x_0}^\sigma(\omega) K_{\hat{g}} \, dv_{\hat{g}} &= \int_{\Sigma_\sigma} I_{x_0}^\sigma(\omega) K_g \, dv_g + \langle d\rho, \omega \rangle_2 + \sum_{j=1}^g \int_{\sigma_{a_j}} \partial_\nu \rho I_{x_0}^\sigma(\omega) \, d\ell_g \\ &+ \sum_{j=1}^g \int_{\bar{\sigma}_{a_j}} \partial_\nu \rho I_{x_0}^\sigma(\omega) \, d\ell_g + \int_{\sigma_{b_j}} \partial_\nu \rho I_{x_0}^\sigma(\omega) \, d\ell_g \\ &+ \int_{\bar{\sigma}_{b_j}} \partial_\nu \rho I_{x_0}^\sigma(\omega) \, d\ell_g, \end{aligned}$$

where ∂_ν is the interior boundary unit normal pointing vector in K_j . We can use that $\bar{\sigma}_{a_j} = \gamma_{b_j}(\sigma_{a_j})$ and $\bar{\sigma}_{b_j} = \gamma_{a_j}(\sigma_{b_j})$ that ρ is a well-defined smooth function on Σ : since $I_{x_0}^\sigma(\omega)(\gamma_\sigma x) = I_{x_0}^\sigma(\omega)(x) + \chi(\gamma_\sigma)$ for $\sigma \in \{a, b\}$ and $(\gamma_{b_j})_* \partial_\nu = -\partial_\nu$ on σ_{a_j} ,

$$\int_{\sigma_{a_j}} \partial_\nu \rho I_{x_0}^\sigma(\omega) \, d\ell_g + \int_{\sigma_{a_j}} \partial_\nu \rho I_{x_0}^\sigma(\omega) \, d\ell_g = -\chi(\gamma_{b_j}) \int_{\sigma_{a_j}} \partial_\nu \rho \, d\ell_g,$$

$$\int_{\sigma_{b_j}} \partial_{\nu} \rho I_{x_0}^{\sigma}(\omega) d\ell_g + \int_{\bar{\sigma}_{b_j}} \partial_{\nu} \rho I_{x_0}^{\sigma}(\omega) d\ell_g = \chi(\gamma_{a_j}) \int_{\sigma_{b_j}} \partial_{\nu} \rho d\ell_g,$$

and we compute that $k_{\hat{g}} d\ell_{\hat{g}} = k_g d\ell_g - \frac{1}{2} \partial_{\nu} \rho d\ell_g$ on σ_j . Combining these facts, we obtain the desired formula for the conformal change of $\int_{\Sigma_{\sigma}}^{\text{reg}} K_g I_{x_0}^{\sigma}(\omega) d\nu_g$. \square

The next step is to check the invariance of the regularised integral with respect to diffeomorphism.

Proof of Lemma 4.4. First we observe that for $\omega \in \mathcal{H}_R^1(\Sigma)$ and $y \in \Sigma_{\psi(\sigma)}$

$$I_{x_0}(\omega)(\psi^{-1}(y)) = \int_{x_0}^{\psi^{-1}(y)} \omega = \int_{\psi(x_0)}^y \psi_* \omega = I_{\psi(x_0)}^{\psi(\sigma)}(\psi_* \omega)(y),$$

thus we get

$$\int_{\Sigma_{\sigma}} I_{x_0}^{\sigma}(\omega) K_g d\nu_g = \int_{\Sigma_{\psi(\sigma)}} I_{\psi(x_0)}^{\psi(\sigma)}(\psi_* \omega) K_{\psi_* g} d\nu_{\psi_* g}.$$

On the other hand, we also have for $\sigma \in \{a, b\}$

$$\chi(\gamma_{\sigma_j}) = \int_{\sigma_j} \omega = \int_{\psi(\sigma_j)} \psi_* \omega,$$

and thus we obtain

$$\sum_{j=1}^g \int_{b_j} \omega \int_{a_j} k_g d\nu_g = \sum_{j=1}^g \int_{\psi(b_j)} \psi_* \omega \int_{a_j} k_{\psi_* g} d\nu_{\psi_* g}$$

and similarly when we exchange a_j and b_j . This ends the proof. \square

The final step consists in proving Lemma 4.5 that the regularised curvature integral of $I_{x_0}^{\sigma}(\omega)$ on Σ_{σ} does not depend on the choice of canonical basis of $\mathcal{H}_1(\Sigma)$. Since the proof is slightly more technical and longer, we defer it to Appendix A for readability.

4.2. Curvature terms associated to magnetic points. Let $\mathbf{z} = (z_1, \dots, z_n)$ be disjoint marked points and $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{Z}^n$ some associated magnetic charges, and let $v_j \in T_{z_j} \Sigma$ some unit tangent vectors (with respect to the metric g on Σ), and denote $\mathbf{v} = ((z_1, v_1), \dots, (z_n, v_n)) \in (T\Sigma)^n$. Consider the closed 1 form $\nu_{\mathbf{z}, \mathbf{m}}$ of Proposition 3.10. As above, we will need to define $\int_{\Sigma} K_g I_{x_0}(\nu_{\mathbf{z}, \mathbf{m}}) d\nu_g$, but $I_{x_0}(\nu_{\mathbf{z}, \mathbf{m}})$ being multivalued, we have to remove some curves and curvature terms along these curves to obtain a natural quantity.

We need a family of arcs, which we call *defect lines* and form a defect graph, constructed as follows:

Definition 4.6 (Defect graph). We consider a family of $n - 1$ arcs as follows:

- these arcs are indexed by $p \in \{1, \dots, n - 1\}$, are simple and do not intersect except possibly at their endpoints.
- Each arc is a smooth oriented curve $\xi_p : [0, 1] \rightarrow \Sigma$ with endpoints $\xi_p(0) = z_j$ and $\xi_p(1) = z_{j'}$ for some $j \neq j'$.
- These arcs reach the endpoints in the directions prescribed by \mathbf{v} , meaning $\dot{\xi}_p(0) = \lambda_{p,j} v_j$ and $\dot{\xi}_p(1) = \lambda_{p,j'} v_{j'}$ for some $\lambda_{p,j}, \lambda_{p,j'} > 0$.

- Consider the oriented graph with vertices \mathbf{z} and edges $(z_j, z_{j'})$ when there is an arc connecting z_j to $z_{j'}$. This graph must be connected and without cycle, i.e. there is no sequence of edges $(z_{j_1}, z_{j_2}), \dots, (z_{j_k}, z_{j_{k+1}})$ with $j_1 = j_{k+1}$.

In what follows, the union $\mathcal{D}_{\mathbf{v},\xi} := \bigcup_{p \in \{1, \dots, n-1\}} \xi_p([0, 1])$ will be called the defect graph associated to \mathbf{v} and the collection of arcs $\xi := (\xi_1, \dots, \xi_{n-1})$.

We notice that the graph $\mathcal{D}_{\mathbf{v},\xi}$, viewed as a subset of Σ , is homotopic to a point. Let us first state Lemma 4.7, the proof of which will be done below.

Lemma 4.7. *On $\Sigma \setminus \mathcal{D}_{\mathbf{v},\xi}$, the 1-form $\nu_{\mathbf{z},\mathbf{m}}$ is exact.*

If ξ is a defect graph for \mathbf{z} , we denote by

$$I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}})(x) = \int_{\alpha_{x_0,x}} \nu_{\mathbf{z},\mathbf{m}}$$

the primitive of $\nu_{\mathbf{z},\mathbf{m}}$ on $\Sigma \setminus \mathcal{D}_{\mathbf{v},\xi}$ vanishing at $x_0 \in \Sigma$, where $\alpha_{x_0,x}$ is any smooth curve in $\Sigma \setminus \mathcal{D}_{\mathbf{v},\xi}$, the result being independent of the curve by Lemma 4.7. Note that the mapping

$$z \mapsto e^{\frac{i}{R} I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}})}$$

is single valued on $\Sigma \setminus \{\mathbf{z}\}$.

Definition 4.8 (Regularised curvature). We assign to each arc ξ_p in the defect graph a value $\kappa(\xi_p) \in 2\pi R\mathbb{Z}$, corresponding to the difference of the values of $I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h)$ on both sides of the arc, as follows: take a small neighborhood $\mathcal{D}_{\mathbf{v},\xi}(\epsilon)$ of $\mathcal{D}_{\mathbf{v},\xi}$ homeomorphic to a disk for some small $\epsilon > 0$, and for $x \in \xi_p(]0, 1[)$, consider a C^1 simple curve $\alpha_x : [0, 1] \rightarrow \mathcal{D}_{\mathbf{v},\xi}(\epsilon)$ with endpoints $\alpha_x(0) = \alpha_x(1) = x$, with $\alpha_x(]0, 1[) \cap \mathcal{D}_{\mathbf{v},\xi} = \emptyset$, such that the disk bounded by $\alpha_x([0, 1])$ contains at least one point of \mathbf{z} , $(\dot{\alpha}_x(t), \nu(t))$ is a positive basis of the tangent space of Σ at $\alpha_x(t)$ if $\nu(t)$ is the unit inward normal to the disk enclosed by α_x and the angle between the curve ξ_p at x and $\dot{\alpha}(0)$ is $\pi/2$ (i.e. we start from the left face of the arc ξ_p , see Figure 9). Then we set

$$(4.4) \quad \kappa(\xi_p) := \int_{\alpha_x} \nu_{\mathbf{z},\mathbf{m}}.$$

We define the regularised curvature term similarly to Definition 4.1 by

$$(4.5) \quad \int_{\Sigma}^{\text{reg}} I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}) K_g \, dv_g := \int_{\Sigma} I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}) K_g \, dv_g - 2 \sum_{p=1}^{n-1} \kappa(\xi_p) \int_{\xi_p} k_g \, d\ell_g.$$

This quantity can also be written as $\kappa(\xi_p) := 2\pi R \sum_{j \in J_p} m_j$ where J_p is set of points z_j enclosed by the curve α_x . Here we assume that the number of turns of α_x around $\mathcal{D}_{\mathbf{v},\xi}$ is 1, but taking curves which turn $k \geq 1$ times (with positive orientation) would lead to the same result by using that $\sum_{j=1}^n m_j = 0$. As before, let $\pi : \tilde{\Sigma}_{\mathbf{z}} \rightarrow \Sigma_{\mathbf{z}}$ be the covering map on the universal cover of $\Sigma_{\mathbf{z}} = \Sigma \setminus \{\mathbf{z}\}$, $\Gamma = \pi_1(\Sigma_{\mathbf{z}}, x_0)$ the fundamental group of $\Sigma_{\mathbf{z}}$ with $x_0 \in \Sigma_{\mathbf{z}}$, $\tilde{x}_0 \in \tilde{\Sigma}_{\mathbf{z}}$ a point so that $\pi(\tilde{x}_0) = x_0$ and $U_{x_0} \subset \tilde{\Sigma}_{\mathbf{z}}$ the connected component of $\pi^{-1}(\Sigma_{\mathbf{z}})$ containing \tilde{x}_0 . Then the function $\pi^* I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}})|_{U_{x_0}}$ is equal to

$$I_{x_0}(\nu_{\mathbf{z},\mathbf{m}})(\tilde{x}) = \int_{\alpha_{x_0,x}} \nu_{\mathbf{z},\mathbf{m}},$$

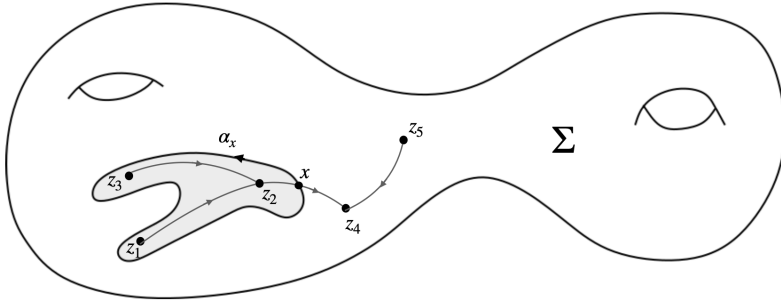


FIGURE 9. Curve α_x : In gray the defect graph. The curve α_x starting at the point $x \in \xi_p$ is the boundary of the shaded area corresponding to a topological disk. On this example $\kappa(\xi_p) = 2\pi R(m_1 + m_2 + m_3)$.

where $\alpha_{x_0,x} \subset \Sigma_z$ is any smooth curve with initial point x_0 and endpoint x so that its lift to $\tilde{\Sigma}_z$ is a curve with initial point \tilde{x}_0 and endpoint \tilde{x} . The function $\pi^* I_{x_0}^\xi(\nu_{z,m}^h)|_{U_{x_0}}$ thus extends smoothly as an equivariant function in $C_1^\infty(\tilde{\Sigma}_z)$.

We state now the main properties of the regularised curvature and prove all the lemmas in this subsection after this.

Lemma 4.9 (Dependence with respect to defect graph). *If ξ and ξ' are two defect graphs, we have*

$$\int_\Sigma^{\text{reg}} I_{x_0}^\xi(\nu_{z,m})K_g \, dv_g - \int_\Sigma^{\text{reg}} I_{x_0}^{\xi'}(\nu_{z,m})K_g \, dv_g \in 8\pi^2 RZ.$$

Lemma 4.10 (Conformal change of metrics). *Consider two conformal metrics $g' = e^\rho g$. The regularised magnetic curvature term defined by (4.5) satisfies*

$$\int_\Sigma^{\text{reg}} I_{x_0}^\xi(\nu_{z,m}^h)K_{g'} \, dv_{g'} = \int_\Sigma^{\text{reg}} I_{x_0}^\xi(\nu_{z,m}^h)K_g \, dv_g.$$

Proof of Lemma 4.7. By construction $\int_{\sigma_j} \nu_{z,m} = 0$ for all cycles σ_j in $\mathcal{H}_1(\Sigma)$. Let $u(x) := \int_{\alpha_{x_0,x}} \nu_{z,m}$ where $\alpha_{x_0,x}$ is a C^1 curve with image in $\Sigma \setminus \mathcal{D}_{v,\xi}$ and endpoints at x_0 and x . The function u is a priori multivalued and $du = \nu_{z,m}$. To prove it is singled valued, it suffices to check that the value of u does not depend on the choice of curve $\alpha_{x_0,x}$. Taking two such curves, we get two (a priori multivalued) primitives u and u' of $\nu_{z,m}$ with $u(x) - u'(x) = \int_{\beta_{x_0}} \nu_{z,m}$ for some closed curve $\beta_{x_0} \in \pi_1(\Sigma \setminus \mathcal{D}_{v,\xi}, x_0)$ in the fundamental group of $\Sigma \setminus \mathcal{D}_{v,\xi}$. By assumption, the graph $\mathcal{D}_{v,\xi}$, viewed as a subset of Σ , is homotopic to a point, thus the first absolute homology group of $\Sigma \setminus \mathcal{D}_{v,\xi}$ is that of Σ with a small disk removed, thus isomorphic to $\mathcal{H}_1(\Sigma)$. This means that the homology class $[\beta_{x_0}]$ of β_{x_0} in $\Sigma \setminus \mathcal{D}_{v,\xi}$ is a linear combination of the basis elements $[\sigma_j] \in \mathcal{H}_1(\Sigma) \simeq \mathcal{H}_1(\Sigma \setminus \mathcal{D}_{v,\xi})$, and therefore $\int_{\beta_{x_0}} \nu_{z,m} = 0$. \square

Proof of Lemma 4.9. We will describe elementary deformations (S for smooth, A for arrival, D for departure) of the defect graph (for fixed v) that produce the same correlation functions. In that aim, we need to first introduce the basic moves:

Definition 4.11 (Basic moves).

- An S-move on the defect graph $\mathcal{D}_{\mathbf{v},\xi}$ consists in picking $p \in \{1, \dots, n-1\}$ and in replacing ξ_p by another smooth simple oriented curve $\tilde{\xi}_p : [0, 1] \rightarrow \Sigma$ with the same endpoints $\tilde{\xi}_p(0) = z_j$ and $\tilde{\xi}_p(1) = z_{j'}$ for $j \neq j'$, such that $(\mathcal{D}_{\mathbf{v},\xi} \setminus \xi_p) \cap \tilde{\xi}_p = \emptyset$, and still reaching the endpoints in the directions prescribed by \mathbf{v} , i.e. $\dot{\tilde{\xi}}_p(0) = \lambda_{p,j}v_j$ and $\dot{\tilde{\xi}}_p(1) = \lambda_{p,j'}v_{j'}$ for some $\lambda_{p,j}, \lambda_{p,j'} > 0$. An S-move thus consists in changing the shape of the curve between the endpoints of an edge. See Figure 10.
- An R-move on the defect graph $\mathcal{D}_{\mathbf{v},\xi}$ consists in picking $p \in \{1, \dots, n-1\}$ and in replacing ξ_p by another smooth simple oriented curve $\tilde{\xi}_p : [0, 1] \rightarrow \Sigma$ with the reverse endpoints $\tilde{\xi}_p(0) = z_{j'}$ and $\tilde{\xi}_p(1) = z_j$ for $j \neq j'$, such that $(\mathcal{D}_{\mathbf{v},\xi} \setminus \xi_p) \cap \tilde{\xi}_p = \emptyset$ and still reaching the endpoints in the directions prescribed by \mathbf{v} , i.e. $\dot{\tilde{\xi}}_p(0) = \lambda_{p,j'}v_{j'}$ and $\dot{\tilde{\xi}}_p(1) = \lambda_{p,j}v_j$ for some $\lambda_{p,j}, \lambda_{p,j'} > 0$. See Figure 10.
- An A-move changes the structure of the graph. Assume we are given distinct $p, p' \in \{1, \dots, n-1\}$ such that $\xi_p(1) = \xi_{p'}(1) = z_j$ and $\xi_p(0) = z_{j'}$, $\xi_{p'}(0) = z_{j''}$ with $z_{j'} \neq z_{j''}$. Choose a smooth oriented curve $\tilde{\xi} : [0, 1] \rightarrow \Sigma$ with endpoints $\tilde{\xi}(0) = z_{j'}$ and $\tilde{\xi}(1) = z_{j''}$, and $\dot{\tilde{\xi}}(0) = \lambda_{j'}v_{j'}$ and $\dot{\tilde{\xi}}(1) = \lambda_{j''}v_{j''}$ for some $\lambda_{j'}, \lambda_{j''} > 0$ and such that $(\mathcal{D}_{\mathbf{v},\xi} \setminus \xi_p) \cap \tilde{\xi} = \emptyset$. The A-move then consists in removing the edge ξ_p and in replacing it by $\tilde{\xi}$. See Figure 12.
- A D-move changes the structure of the graph too. Assume we are given distinct $p, p' \in \{1, \dots, n-1\}$ such that $\xi_p(0) = \xi_{p'}(0) = z_j$ and $\xi_p(1) = z_{j'}$, $\xi_{p'}(1) = z_{j''}$ with $z_{j'} \neq z_{j''}$. Choose a smooth oriented curve $\tilde{\xi} : [0, 1] \rightarrow \Sigma$ with endpoints $\tilde{\xi}(0) = z_{j'}$ and $\tilde{\xi}(1) = z_{j''}$, $\dot{\tilde{\xi}}(0) = \lambda_{j'}v_{j'}$ and $\dot{\tilde{\xi}}(1) = \lambda_{j''}v_{j''}$ for some $\lambda_{j'}, \lambda_{j''} > 0$ and such that $(\mathcal{D}_{\mathbf{v},\xi} \setminus \xi_p) \cap \tilde{\xi} = \emptyset$. The D-move then consists in removing the edge $\xi_{p'}$ and in replacing it by $\tilde{\xi}$. See Figure 11.

We denote $\tilde{\xi}$ the new family of curves after such changes, and $\mathcal{D}_{\mathbf{v},\tilde{\xi}}$ the new defect graph. Now we claim that the value of $\int_{\Sigma}^{\text{reg}} I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m}})K_g \, dv_g$ remains unchanged if we perform S-moves, R-moves, A-moves and D-moves to the defect graph. Let us focus first on the case of S-move. We consider first the case when the curves ξ_p and $\tilde{\xi}_p$ do not intersect (except at their endpoints). We can always reduce to this case since, if two curves intersect, by choosing a third one that does not intersect these two curves and comparing the integrals for the two curves with this third one, we get the desired result. Let us call $\tilde{\mathcal{D}}_{\mathbf{v},\xi}$ the defect graph after the S-move. Consider the domain D bounded by the union of the curves ξ_p and $\tilde{\xi}_p$ and not containing x_0 (see Figure 13); notice that the interior of D may possibly contain $\mathcal{D}_{\mathbf{v},\xi} \setminus \xi_p$, depending on where x_0 is located. The boundary of D inherits an orientation from Σ (with the positive orientation given by $-J\nu$ if ν is the inward unit normal to ∂D and J the rotation of $+\pi/2$), which coincides with the orientation of either ξ_p or $\tilde{\xi}_p$. Without loss of generality, we may assume this coincides with the orientation of $\tilde{\xi}_p$. The two defect graphs give rise to two different primitives $I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m}})$ and $I_{x_0}^{\tilde{\xi}}(\nu_{\mathbf{z},\mathbf{m}})$ which agree outside D and, on D , we have $I_{x_0}^{\tilde{\xi}}(\nu_{\mathbf{z},\mathbf{m}}) + \kappa(\xi_p) = I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m}})$. Moreover $\kappa(\tilde{\xi}_p) = \kappa(\xi_p)$. The difference of the two regularised

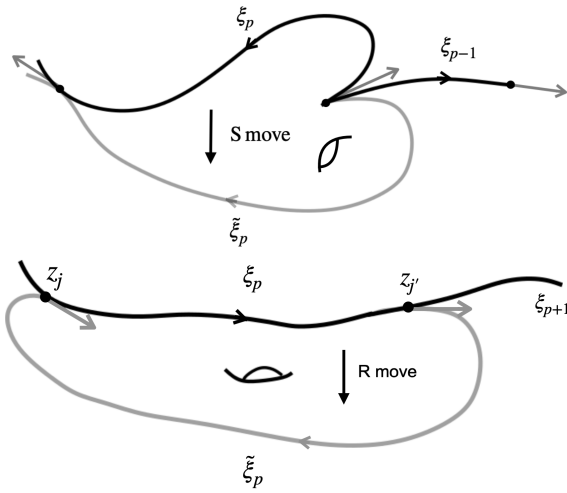


FIGURE 10. S-move and R-move

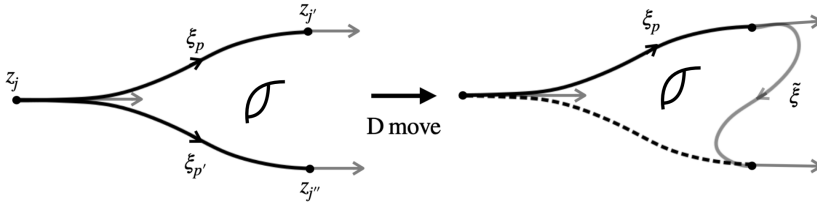


FIGURE 11. D-Move

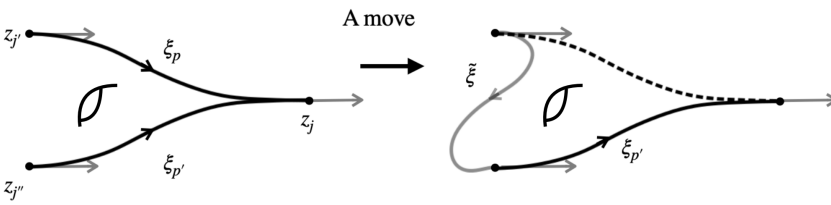


FIGURE 12. A-Move

integrals is then

$$\int_{\Sigma}^{\text{reg}} I_{x_0}^{\xi}(\nu_{z,m})K_g \, dv_g - \int_{\Sigma}^{\text{reg}} I_{x_0}^{\tilde{\xi}}(\nu_{z,m})K_g \, dv_g$$

$$= \int_D (I_{x_0}^{\xi}(\nu_{z,m}) - I_{x_0}^{\tilde{\xi}}(\nu_{z,m}))K_g \, dv_g - 2\kappa(\xi_p) \int_{\xi_p} k_g \, d\ell_g + 2\kappa(\tilde{\xi}_p) \int_{\tilde{\xi}_p} k_g \, d\ell_g.$$

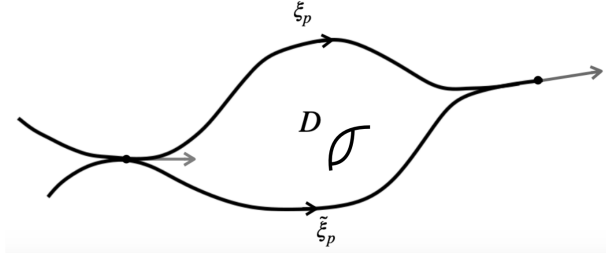


FIGURE 13. Gauss-Bonnet in D for S-move

Now we apply the Gauss-Bonnet theorem on D to get

$$\begin{aligned} & \int_D (I_{x_0}^{\xi_p}(\nu_{z,m}) - I_{x_0}^{\tilde{\xi}_p}(\nu_{z,m})) K_g dv_g \\ &= \int_D \kappa(\xi_p) K_g dv_g \\ &= 4\pi\kappa(\xi_p)(\chi(D) - 1) - 2\kappa(\xi_p) \left(\int_{\tilde{\xi}_p} k_g d\ell_g - \int_{\xi_p} k_g d\ell_g \right) \end{aligned}$$

and we deduce that the difference of the two regularised integrals is equal to $4\pi(\chi(D) - 1)\kappa(\xi_p) \in 8\pi^2R\mathbb{Z}$. This means that the regularised curvature term only changes by $8\pi^2R\mathbb{Z}$ if we perform an S-move to the defect graph. For the case of R-move, we can apply a similar argument: consider the connected domain D not containing x_0 and bounded by $\xi_p \cup \tilde{\xi}_p$, then assuming $\xi_p \cup \tilde{\xi}_p$ has negative orientation we still have $I_{x_0}^{\tilde{\xi}_p}(\nu_{z,m}) + \kappa(\xi_p) = I_{x_0}^{\xi_p}(\nu_{z,m})$ but $\kappa(\xi_p) = -\kappa(\tilde{\xi}_p)$. Applying Gauss-Bonnet in D , we get

$$\begin{aligned} & \int_{\Sigma}^{\text{reg}} I_{x_0}^{\xi_p}(\nu_{z,m}) K_g dv_g - \int_{\Sigma}^{\text{reg}} I_{x_0}^{\tilde{\xi}_p}(\nu_{z,m}) K_g dv_g \\ &= \kappa(\xi_p) \left(\int_D K_g dv_g - 2 \int_{\xi_p} k_g d\ell_g - 2 \int_{\tilde{\xi}_p} k_g d\ell_g \right) \\ &= 4\pi\chi(D)\kappa(\xi_p) \in 8\pi^2R\mathbb{Z}. \end{aligned}$$

The cases of A-moves and D-moves can be treated similarly by applying the Gauss-Bonnet theorem in the domain T not containing x_0 bounded by $\xi_p, \xi_{p'}, \tilde{\xi}$, with vertices $z_j, z_{j'}, z_{j''}$. For the A-Move, assume for example that $\tilde{\xi}, \xi_{p'}$ are oriented positively and ξ_p negatively (see Figure 12), then we have $I_{x_0}^{\tilde{\xi}}(\nu_{z,m}) + \kappa(\xi_p) = I_{x_0}^{\xi_{p'}}(\nu_{z,m})$ in T and $\kappa(\xi_p) = \kappa(\tilde{\xi})$. We then get by Gauss-Bonnet in D

$$\begin{aligned} & \int_{\Sigma}^{\text{reg}} I_{x_0}^{\xi_{p'}}(\nu_{z,m}) K_g dv_g - \int_{\Sigma}^{\text{reg}} I_{x_0}^{\tilde{\xi}}(\nu_{z,m}) K_g dv_g \\ &= \kappa(\xi_p) \left(\int_T K_g dv_g - 2 \int_{\xi_p} k_g d\ell_g + 2 \int_{\tilde{\xi}} k_g d\ell_g \right) \\ &= 4\pi\chi(T)\kappa(\xi_p) \in 8\pi^2R\mathbb{Z}. \end{aligned}$$

For the D-Move, assume for example that $\xi_{p'}$ is oriented positively and $\tilde{\xi}, \xi_p$ negatively (see Figure 11), then we have $I_{x_0}^{\tilde{\xi}}(\nu_{z,m}) - \kappa(\tilde{\xi}) = I_{x_0}^{\xi}(\nu_{z,m})$ in T and $\kappa(\xi_{p'}) = \kappa(\tilde{\xi})$. We then get by Gauss-Bonnet in D

$$\begin{aligned} & \int_{\Sigma}^{\text{reg}} I_{x_0}^{\tilde{\xi}}(\nu_{z,m}) K_g \, dv_g - \int_{\Sigma}^{\text{reg}} I_{x_0}^{\xi}(\nu_{z,m}) K_g \, dv_g \\ &= \kappa(\xi_{p'}) \left(- \int_T K_g \, dv_g - 2 \int_{\xi_{p'}} k_g \, d\ell_g + 2 \int_{\tilde{\xi}} k_g \, d\ell_g \right) \\ &= -4\pi\chi(T)\kappa(\xi_{p'}) \in 8\pi^2 R\mathbb{Z}. \end{aligned}$$

Now we claim that every defect graph can be deformed, via S-R-A-D-moves, to the defect graph, called path, where each ξ_p is a fixed curve joining z_p to z_{p+1} (here we have fixed an order on the points z_1, \dots, z_{n_m}). By a sequence of R-moves, we can revert the orientations of the edges in such a way that each edge is oriented from z_j to $z_{j'}$ if $j < j'$. We say that z_j is a *peak* if z_j is attached to at least two edges $[z_{j'}, z_j]$ and $[z_{j''}, z_j]$ with $j'' < j' < j$. Notice that if z_j is a peak, attached to m edges $[z_{i_1}, z_j], \dots, [z_{i_m}, z_j]$ with $i_1 < \dots < i_m < j$, then we can change the graph by $(m - 1)$ A-moves by replacing iteratively these edges to the sequence of edges $[z_{i_1}, z_{i_2}], \dots, [z_{i_m}, z_j]$, in such a way that there is only one edge of the form $[z_{j'}, z_j]$ with $j' < j$. We shall now proceed by induction: let $\mathcal{D}(j)$ be the set of defect graphs satisfying the property that each point $z_{j'}$ with $j' \geq j$ is attached to the only edges $[z_{j'-1}, z_{j'}]$ and $[z_{j'}, z_{j'+1}]$, or to the only edge $[z_{j'-1}, z_{j'}]$ if $j' = n_m$. We start by showing that from the initial defect graph, we can make a finite number of A and D moves to transform it into a graph contained in the class $\mathcal{D}(n_m)$. First, if z_{n_m} is a peak, we make finitely many A-moves as explained above to remove the peak. Now, there are two cases: either z_{n_m} is attached to the single edge $[z_{n_{m-1}}, z_{n_m}]$ and the new graph \mathcal{D}_0 is then in $\mathcal{D}(n_m)$, or it is attached to a single edge $[z_j, z_{n_m}]$ for $j < n_m - 1$. In that last case, since our graph \mathcal{D}_0 is a tree, by adding the edge $[z_{n_{m-1}}, z_{n_m}]$ to \mathcal{D}_0 , we create a graph \mathcal{D}'_0 with one cycle made of $k + 1$ edges relating the points $z_{n_m}, z_{i_1}, \dots, z_{i_k}, z_{n_{m-1}}, z_{n_m}$ for some i_1, \dots, i_k , ordered in the sense of the cycle (not that \mathcal{D}'_0 is not a defect graph anymore). We then apply a D-move to \mathcal{D}_0 (in the triplet $z_{n_m}, z_{i_1}, z_{i_2}$) to replace the edge $[z_{i_1}, z_{n_m}]$ by $[z_{i_2}, z_{n_m}]$. The new graph is a defect graph and we next apply a D-move replacing the edge $[z_{i_2}, z_{n_m}]$ by $[z_{i_3}, z_{n_m}]$. Continuing this process, making in total $(k - 1)$ D-moves with the last move

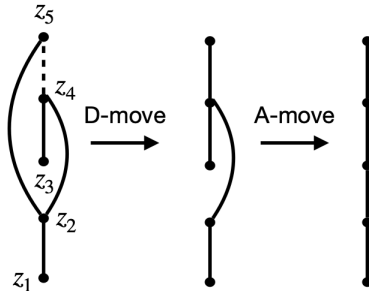


FIGURE 14. Sequence of moves to reduce the graph to a path

replacing the edge $[z_{i_k}, z_{n_m}]$ by $[z_{n_m-1}, z_{n_m}]$, we end up with a defect graph in $\mathcal{D}(n_m)$. Now we apply the same procedure to the subgraph obtained by removing z_{n_m} and the edge $[z_{n_m-1}, z_{n_m}]$, we obtain a graph in $\mathcal{D}(n_m - 1)$. Iterating this, we get down to a graph in $\mathcal{D}(0)$, which is a path with the desired shape. This proves that the magnetic correlation functions do not depend on the defect graph. This is illustrated in Figure 14. \square

Proof of Lemma 4.10. We use the relation $K_{g'} = e^{-\rho}(K_g + \Delta_g \rho)$ and $dv_{g'} = e^{\rho} dv_g$, then by integration by parts

$$\int_{\Sigma} I_{x_0}^{\xi}(\nu_{z,m}^h) K_{g'} dv_{g'} = \int_{\Sigma} I_{x_0}^{\xi}(\nu_{z,m}^h) K_g dv_g + \langle d\rho, \nu_{z,m}^h \rangle_2 - \sum_{p=1}^{n_m-1} \int_{\xi_p} \partial_{\nu} \rho (I_{x_0}^{\xi}(\nu_{z,m}^h)^+ - I_{x_0}^{\xi}(\nu_{z,m}^h)^-) d\ell_g,$$

where $\nu = J\xi_p$ is the left normal pointing vector with respect to the orientation of ξ_p (J being the rotation of $+\pi/2$ in the tangent space), $I_{x_0}^{\xi}(\nu_{z,m}^h)^+$ is the limit of $I_{x_0}^{\xi}(\nu_{z,m}^h)$ on ξ_p from the side given by $-\nu$ (the right side) and $I_{x_0}^{\xi}(\nu_{z,m}^h)^-$ the value from the side given by ν (the left side). Also $I_{x_0}^{\xi}(\nu_{z,m}^h)^+ - I_{x_0}^{\xi}(\nu_{z,m}^h)^- = \kappa(\xi_p)$. Next we use that $k_{g'} d\ell_{g'} = k_g d\ell_g - \frac{1}{2} \partial_{\nu} \rho d\ell_g$ on ξ_p . Finally $\langle d\rho, \nu_{z,m}^h \rangle_2 = 0$ because $\nu_{z,m}^h$ is co-closed. Combining these facts, we obtain our claim. \square

5. IMAGINARY GAUSSIAN MULTIPLICATIVE CHAOS

In this section, we first recall some facts about the Gaussian Free Field X_g (resp. $X_{g,D}$) on closed surfaces (resp. surfaces with boundary). This is a Gaussian random distribution on the surface, living in a negative Sobolev space $H^s(\Sigma)$ for $s < 0$. In order to make sense of certain functionals, we need to regularise it at a small scale $\epsilon > 0$. This will be done either in a geometric fashion or using white noise, as we explain below. Finally, we recall the construction and properties of the Imaginary Gaussian multiplicative chaos $e^{i\beta X_g} dv_g$, which is a random distribution on Σ . We shall need estimates on its exponential moments.

5.1. Gaussian free fields. On a Riemann surface without boundary, the Gaussian Free Field (GFF in short) is defined as follows. Let $(a_j)_j$ be a sequence of i.i.d. real Gaussians $\mathcal{N}(0, 1)$ with mean 0 and variance 1, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and define the Gaussian Free Field with vanishing mean in the metric g by the random distribution (recall that $(e_j)_j$ is an orthonormal basis of eigenfunctions for Δ_g with eigenvalues λ_j)

$$(5.1) \quad X_g := \sqrt{2\pi} \sum_{j \geq 1} a_j \frac{e_j}{\sqrt{\lambda_j}},$$

where the sum converges almost surely in the Sobolev space $H^s(\Sigma)$ for $s < 0$ defined by

$$(5.2) \quad H^s(\Sigma) := \{f = \sum_{j \geq 0} f_j e_j \mid \|f\|_s^2 := |f_0|^2 + \sum_{j \geq 1} \lambda_j^s |f_j|^2 < +\infty\}.$$

This Hilbert space is independent of g , only its norm depends on a choice of g . The covariance X_g is then the Green function when viewed as a distribution, which we will write with a slight abuse of notation

$$\mathbb{E}[X_g(x)X_g(x')] = G_g(x, x').$$

In the case of a surface with boundary Σ , the Dirichlet Gaussian free field (with covariance $G_{g,D}$) will be denoted $X_{g,D}$. It is defined similarly to the sum (5.1) with the $(e_j)_j$ and $(\lambda_j)_j$ replaced by the normalised eigenfunctions $(e_{j,D})_j$ and ordered eigenvalues $(\lambda_{j,D})_j$ of the Laplacian with Dirichlet boundary conditions, the sum being convergent almost surely in the Sobolev space $H^s(\Sigma)$ (for all $s \in (-1/2, 0)$) defined by

$$(5.3) \quad H^s(\Sigma) := \{f = \sum_{j \geq 0} f_j e_{j,D} \mid \|f\|_s^2 := \sum_{j \geq 0} \lambda_{j,D}^s |f_j|^2 < +\infty\}.$$

In both cases (closed or open surfaces), we will denote by $(\cdot, \cdot)_s$ the inner product in $H^s(\Sigma)$, and by extension also the duality bracket on $H^s(\Sigma) \times H^{-s}(\Sigma)$.

5.2. Metric regularisations and white noise regularisation. As Gaussian Free Fields are rather badly behaved (they are distributions), we will need to consider regularisations, and we will mainly consider two of them. First we introduce a regularisation procedure, which we will call *g-regularisation*. Let Σ be a surface with or without boundary equipped with a Riemannian metric g and associated distance d_g . For a random distribution h on Σ and for $\epsilon > 0$ small, we define a regularisation h_ϵ of h by averaging on geodesic circles of radius $\epsilon > 0$: let $x \in \Sigma$ and let $\mathcal{C}_g(x, \epsilon)$ be the geodesic circle of center x and radius $\epsilon > 0$, and let $(f_{x,\epsilon}^n)_{n \in \mathbb{N}} \in C^\infty(\Sigma)$ be a sequence with $\|f_{x,\epsilon}^n\|_{L^1} = 1$ which is given by $f_{x,\epsilon}^n = \theta^n(d_g(x, \cdot)/\epsilon)$ where $\theta^n \in C_c^\infty((0, 2))$ is non-negative, supported near $r = 1$ and such that $f_{x,\epsilon}^n dv_g$ converges in $\mathcal{D}'(\Sigma)$ to the uniform probability measure $\mu_{x,\epsilon}$ on $\mathcal{C}_g(x, \epsilon)$ as $n \rightarrow \infty$ (defined using the metric g). If the pairing $\langle h, f_{x,\epsilon}^n \rangle$ converges almost surely towards a random variable $h_\epsilon(x)$ that has a modification which is continuous in the parameters (x, ϵ) , we will say that h admits a *g-regularisation* $(h_\epsilon)_\epsilon$. This is the case for the GFF $X_g, X_{g,D}$ and we denote by $X_{g,\epsilon}, X_{g,D,\epsilon}$ their *g-regularisation*.

In the case of the Dirichlet GFF over a surface Σ with boundary, we introduce another regularisation, called *white-noise regularisation*. The Green function $G_{g,D}$ can be written as

$$G_{g,D}(x, x') = 2\pi \int_0^\infty p_t(x, x') dt,$$

where $p_t(x, x')$ denotes the transition densities of the Brownian motion on Σ killed upon touching the boundary $\partial\Sigma$ (i.e. the heat kernel of the Laplacian with Dirichlet condition). Let W be a white noise on $\mathbb{R}_+ \times \Sigma$ and define for $\delta > 0$ with intensity $dt \otimes v_g(dx)$

$$X_{g,D,\delta}(x) := (2\pi)^{1/2} \int_{\delta^2}^\infty \int_\Sigma p_{t/2}(x, y) W(dt, dy).$$

Then the covariance kernel of these processes is given by

$$\mathbb{E}[X_{g,D,\delta}(x)X_{g,D,\delta'}(x')] = 2\pi \int_{\delta^2 \vee \delta'^2}^\infty p_t(x, x') dt.$$

5.3. Markov property of GFF. We recall here the domain Markov property of the GFF on Riemann surfaces, whose proof can be found in the appendix of [31]. As mentioned in the previous section, the GFF can be restricted to a smooth simple curve \mathcal{C} , as a random $H^{-s}(\mathcal{C})$ -valued variable for $s > 0$.

Proposition 5.1. *Let (Σ, g) be a Riemannian manifold with smooth boundary $\partial\Sigma$. Let \mathcal{C} be a union of smooth non-overlapping closed simple curves separating Σ into two connected components Σ_1 and Σ_2 .*

(1) *if $\partial\Sigma \neq \emptyset$ then the Dirichlet GFF $X_{g,D}$ on Σ admits the following decomposition in law as a sum of independent processes*

$$X_{g,D} \stackrel{\text{law}}{=} Y_1 + Y_2 + P$$

with Y_q a Dirichlet GFF on Σ_q for $q = 1, 2$ and P the harmonic extension on $\Sigma \setminus \mathcal{C}$ of the restriction of $X_{g,D}$ to \mathcal{C} with boundary value 0 on $\partial\Sigma$.

(2) *if $\partial\Sigma = \emptyset$ then the GFF X_g on Σ admits the following decomposition in law*

$$X_g \stackrel{\text{law}}{=} Y_1 + Y_2 + P - c_g,$$

where Y_1, Y_2, P are independent, Y_q is a Dirichlet GFF on Σ_q for $q = 1, 2$, P is the harmonic extension on $\Sigma \setminus \mathcal{C}$ of the restriction of X_g to \mathcal{C} and $c_g := \frac{1}{v_g(\Sigma)} \int_{\Sigma} (Y_1 + Y_2 + P) dv_g$.

5.4. Gaussian multiplicative chaos. For $\beta \in \mathbb{R}$ and h a random distribution admitting a g -regularisation $(h_\epsilon)_\epsilon$, we define the complex measure

$$(5.4) \quad M_\beta^{g,\epsilon}(h, dx) := \epsilon^{-\frac{\beta^2}{2}} e^{i\beta h_\epsilon(x)} dv_g(x).$$

Of particular interest for us is the case when $h = X_g$ or $h = X_{g,D}$. In that case, for $\beta^2 < 2$, the random measures above converge as $\epsilon \rightarrow 0$ in $L^2(\Omega)$ and weakly in the space of distributions [49, Theorem 3.1] to a non-trivial distribution of order 2 denoted by $M_\beta^g(X_g, dx)$ or $M_\beta^g(X_{g,D}, dx)$; this means that there exists a random variable $D_\Sigma \in L^2(\Omega)$, such that

$$(5.5) \quad \forall \varphi \in C^\infty(\Sigma), \quad \left| \int_{\Sigma} \varphi(x) M_\beta^g(X_g, dx) \right| \leq D_\Sigma \|\varphi\|_{C^2}.$$

For notational readability and if there is no risk of confusion, we will use the notation $M_\beta^{g,\epsilon}(dx)$ for $M_\beta^{g,\epsilon}(X_g, dx)$ (i.e. we skip the field dependence). Also we stress that the condition $\beta^2 < 2$ will always be assumed throughout the paper.

Also, from [34, Lemma 3.2], we recall that there exist W_g (resp. $W_{g,D}$) such that uniformly on the compact subsets of the interior Σ° of Σ

$$(5.6) \quad \lim_{\epsilon \rightarrow 0} \mathbb{E}[X_{g,\epsilon}^2(x)] + \log \epsilon = W_g(x) \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \mathbb{E}[X_{g,D,\epsilon}^2(x)] + \log \epsilon = W_{g,D}(x),$$

where the function W_g , called the Robin mass, is smooth on Σ . In particular, considering a metric $g' = e^\omega g$ conformal to the metric g , we obtain the relation

$$(5.7) \quad M_\beta^{g'}(X_{g'}, dx) = e^{(1-\frac{\beta^2}{4})\omega(x)} M_\beta^g(X_g, dx).$$

The Robin mass of the Dirichlet GFF can also be rewritten in terms of the heat kernel

$$(5.8) \quad W_{g,D}(x) = 2\pi \int_0^\infty (p_t(x, x) - \frac{1}{4\pi t} \mathbf{1}_{[0,1]}(t)) dt$$

from which we deduce, pointwise,

$$(5.9) \quad \lim_{\delta \rightarrow 0} \mathbb{E}[X_{g,D,\delta}^2(x)] + \log \delta = W_{g,D}(x).$$

For later use, we mention here the following estimate [53], valid for some $C > 0$ and $\forall t \in (0, 1)$,

$$(5.10) \quad |4\pi t p_t(x, x) - 1| \leq C(t + e^{-\frac{\text{dist}(x, \partial\Sigma)^2}{Ct}}).$$

We state the following elementary, though important, equivalence between the GMC construction via white-noise or g-regularisation.

Proposition 5.2. *Assume Σ is a surface without boundary and \mathcal{C} is a family of smooth simple curves splitting Σ into two connected components Σ_1 and Σ_2 . Consider the equality in law stated in Proposition 5.1*

$$X_g = Y_1 + Y_2 + P - c_g,$$

where Y_1, Y_2, P are three independent Gaussian fields with Y_i the Dirichlet GFF on Σ_i for $i = 1, 2$, P the harmonic extension of the boundary values $X_{g|\mathcal{C}}$ and $c_g = \frac{1}{v_g(\Sigma)} \int_{\Sigma} (Y_1 + Y_2 + P) dv_g$. Then, for $\beta^2 < 2$, the following limits agree in law

$$M_{\beta}^g(X_g, dx) \stackrel{\text{law}}{=} M_{\beta}^g(Y_1 + Y_2 + P - c_g, dx),$$

where the limit in the left hand side is taken via g-regularisation and the limit in the right hand side can be either white noise or g-regularisation.

Proof. This result is standard and left to the reader as an exercise: it results from straightforward L^2 -computations. Recall that $\beta^2 < 2$ and GMC is therefore in L^2 . \square

5.5. Exponential moments. In this section, we assume that $\beta^2 < 2$ and we recall the following result, originally proved in [50] but here adapted to our context, mainly to deal with boundary terms, which will be fundamental for the existence of the path integral and correlation functions:

Proposition 5.3. *Assume Σ is a Riemann surface with or without boundary. If Σ has a non-empty boundary, we set $\mathcal{D}' = \Sigma$. If Σ has empty boundary, we consider an open subset \mathcal{D}' of Σ , with closure $\overline{\mathcal{D}'} \neq \Sigma$. We consider the Dirichlet GFF $X_{g,D}$ on \mathcal{D}' and an open subset \mathcal{D} of \mathcal{D}' .*

Let Z be a real valued random variable (not necessarily assumed to be independent of $X_{g,D}$). Finally we consider a measurable function $f : \mathcal{D} \rightarrow \mathbb{C}$. Then for $\mu \in \mathbb{R}$

$$\mathbb{E} \left[\exp \left(\mu \left| \int_{\mathcal{D}} f(x) M_{\beta}^g(Z + X_{g,D}, dx) \right| \right) \right] \leq e^{C\mu v} (1 + C\mu u e^{C\mu^2 u^2})$$

with

$$u^2 := \iint_{\mathcal{D}^2} |f(x)| |f(y)| d_g(x, y)^{-\beta^2} dv_g(x) dv_g(y),$$

$$v := \int_{\mathcal{D}} |f(x)| \text{dist}(x, \partial\Sigma)^{-\frac{\beta^2}{2}} dv_g(x),$$

for some constant C only depending on β .

Proof. Before proceeding to the proof, let us stress that the crucial input for the proof is an integral representation of the Green function in terms of a positive heat kernel, in order to have a white noise representation of the GFF and to use stochastic analysis. This is valid for the Dirichlet GFF and that is the reason why we focus on this case. From this result, we will deduce later exponential moment estimates for the GFF on closed Riemann surfaces, based on the Dirichlet case via the Markov property. Of course, on closed Riemann surfaces there is no globally defined Dirichlet GFF and this is the reason why we restrict to a strict subset in this case.

Note that adding Z to the GFF is harmless because it multiplies the GMC by a number with modulus 1, so we may as well assume that $Z = 0$. Let us first prove exponential moments for the real part of $\int_{\mathcal{D}} f(x)M_{\beta}^g(X_{g,D}, dx)$; the argument for the imaginary part is the same. The real part can be obtained as the limit as $\delta \rightarrow 0$ of the process

$$\int_{\mathcal{D}} \delta^{-\frac{\beta^2}{2}} \operatorname{Re}(f(x)) \cos(\beta X_{g,D,\delta}(x)) dv_g(x) - \int_{\mathcal{D}} \delta^{-\frac{\beta^2}{2}} \operatorname{Im}(f(x)) \sin(\beta X_{g,D,\delta}(x)) dv_g(x).$$

For readability we will write the proof with details in the case when $\operatorname{Im}(f(x)) = 0$ but the argument is similar for the second term in the relation above. The limit above can be obtained as the limit $t \rightarrow \infty$ of the following semimartingale

$$\mathcal{M}_t = \int_{\mathcal{D}} f(x) \cos(\beta X_{g,D,e^{-t}}(x)) e^{\frac{\beta^2}{2}t} dv_g(x).$$

The argument will follow from the control of the terms appearing in its Itô decomposition. Indeed, the Itô formula gives the decomposition

$$\begin{aligned} d\mathcal{M}_t &= \frac{\beta^2}{2} \int_{\mathcal{D}} f(x) \cos(\beta X_{g,D,e^{-t}}(x)) e^{\frac{\beta^2}{2}t} dv_g(x) (dt - d\langle X_{g,D,e^{-t}} \rangle_t) \\ &\quad - \beta \int_{\mathcal{D}} f(x) \sin(\beta X_{g,D,e^{-t}}(x)) e^{\frac{\beta^2}{2}t} dv_g(x) dX_{g,D,e^{-t}}(x) \\ &=: dA_t + dL_t. \end{aligned}$$

Concerning the local martingale part, its quadratic variations are

$$\begin{aligned} \langle L \rangle_t - \langle L \rangle_0 &= \beta^2 \int_0^t \iint_{\mathcal{D}^2} f(x)f(y) \sin(\beta X_{g,D,e^{-u}}(x)) \sin(\beta X_{g,D,e^{-u}}(y)) \\ &\quad e^{\beta^2 u} dv_g(x) dv_g(y) d\langle X_{g,D,e^{-u}}(x), X_{g,D,e^{-u}}(y) \rangle_u. \end{aligned}$$

The bracket is given by (using the Markov property of the transition kernels)

$$(5.11) \quad d\langle X_{g,D,e^{-u}}(x), X_{g,D,e^{-u}}(y) \rangle_u = 4\pi e^{-2u} p_{e^{-2u}}(x, y) du.$$

Next we bound the quadratic variation, by estimating both sines by 1, by using the standard inequality for the Dirichlet heat kernel (it follows from [16] and the monotonicity of the Dirichlet heat kernel with respect to the domain)

$$e^{-2u} p_{e^{-2u}}(x, y) \leq C e^{-d_g(x,y)^2 e^{2u}/C}$$

for some constant C which does not depend on \mathcal{D} (and which may change along lines in the following), to get that

$$\begin{aligned}
 \sup_{t \geq 0} \langle L \rangle_t - \langle L \rangle_0 &\leq C \beta^2 \int_0^\infty \iint_{\mathcal{D}^2} |f(x)| |f(y)| e^{\beta^2 t} e^{-d_g(x,y)^2 e^{2t}/C} dv_g(x) dv_g(y) dt \\
 (5.12) \qquad \qquad \qquad &\leq C \iint_{\mathcal{D}^2} |f(x)| |f(y)| d_g(x,y)^{-\beta^2} dv_g(x) dv_g(y),
 \end{aligned}$$

where we have performed the change of variables $s = d_g(x,y)e^t$ to get the last line.

Concerning the bounded variation part, we use again (5.11) to get that it converges towards A_∞ when $t \rightarrow \infty$ and that it is bounded by

$$|A_\infty - A_0| \leq \frac{\beta^2}{2} \int_0^\infty \int_{\mathcal{D}} |f(x)| e^{\frac{\beta^2}{2} u} |1 - 4\pi e^{-2u} p_{e^{-2u}}(x,x)| du dv_g(x).$$

Using next the estimate (5.10), we deduce that

$$\begin{aligned}
 |A_\infty - A_0| &\leq \frac{\beta^2}{2} \int_0^\infty \int_{\mathcal{D}} |f(x)| e^{\frac{\beta^2}{2} u} C(e^{-2u} + e^{-\text{dist}(x,\partial\Sigma)^2 e^{2u}/C}) du dv_g(x) \\
 (5.13) \qquad \qquad \qquad &\leq C \int_{\mathcal{D}} |f(x)| \text{dist}(x,\partial\Sigma)^{-\frac{\beta^2}{2}} dv_g(x).
 \end{aligned}$$

This shows that the semimartingale $\mathcal{M}_t - \mathcal{M}_0$ has exponential moments provided that the two quantities (5.12) and (5.13) are finite. Indeed, recall that a bounded continuous martingale \mathcal{M}_t starting from 0 satisfies

$$\mathbb{E}[e^{\mu \mathcal{M}_t - \frac{\mu^2}{2} \langle \mathcal{M} \rangle_t}] = 1.$$

If the bracket is bounded (by a deterministic bound), we deduce for $\mu \in \mathbb{R}$

$$\mathbb{E}[e^{\mu \mathcal{M}_\infty}] \leq e^{\frac{\mu^2}{2} \sup_t \langle \mathcal{M} \rangle_t}.$$

Next, if we set $u = \sup_t \langle \mathcal{M} \rangle_t$, we deduce $\mathbb{P}(\mathcal{M}_\infty > x) \leq \exp(-\frac{x^2}{2u^2})$, and the same for $\mathbb{P}(\mathcal{M}_\infty < -x)$, in such a way that $\mathbb{P}(|\mathcal{M}_\infty| > x) \leq 2 \exp(-\frac{x^2}{2u^2})$. Finally, we can use the standard trick

$$\mathbb{E}[e^{\mu |\mathcal{M}_\infty|}] = 1 + \int_0^\infty \mathbb{P}(|\mathcal{M}_\infty| > \frac{x}{\mu}) e^x dx$$

to get that

$$\mathbb{E}[e^{\mu |\mathcal{M}_\infty|}] \leq 1 + C \mu e^{C u^2 \mu^2},$$

for some irrelevant constant C . We can use this estimate to the local martingale part of our semimartingale $\mathcal{M}_t - \mathcal{M}_0$. Furthermore, \mathcal{M}_0 obviously has bounded exponential moments and $\mathbb{E}[e^{\mu |\mathcal{M}_0|}] < e^{\mu \int_{\mathcal{D}} |f(x)| dv_g(x)}$ for $\mu > 0$. All in all, we obtain the claimed estimate. □

6. THE PATH INTEGRAL AND CORRELATION FUNCTIONS

Throughout this section we will work under the constraint $\beta^2 < 2$, in order to ensure convergence of the imaginary GMC. We also assume that $\beta > 0$ since the case $\beta < 0$

would be symmetric to $\beta > 0$. We further impose the compactification radius $R > 0$ to obey

$$(6.1) \quad R \in \frac{1}{\beta}\mathbb{Z}.$$

Let us introduce a further parameter $\mu \in \mathbb{C} \setminus \{0\}$ and set

$$(6.2) \quad Q = \frac{\beta}{2} - \frac{2}{\beta}.$$

Finally we introduce the central charge $c_{\text{IL}} = 1 - 6Q^2$. We focus in this section on the rational case

$$(6.3) \quad Q \in \frac{1}{R}\mathbb{Z}.$$

6.1. Path integral. Consider now a closed Riemann surface Σ equipped with a metric g . To construct the path integral, we make Assumptions 6.1:

Assumption 6.1. We fix:

- a geometric symplectic basis $\sigma = (\sigma_j)_{j=1, \dots, 2g}$ of $\mathcal{H}_1(\Sigma)$ and $2g$ independent closed smooth 1-forms $\omega_1, \dots, \omega_{2g}$ forming a basis of the cohomology $\mathcal{H}_R^1(\Sigma)$ dual to σ (see Lemma 3.2). Let $\omega_{\mathbf{k}} = \sum_{j=1}^{2g} k_j \omega_j$ if $\mathbf{k} = (k_1, \dots, k_{2g})$, as defined in (3.6).
- a base point $x_0 \in \Sigma_\sigma = \Sigma \setminus \cup_{j=1}^{2g} \sigma_j$, and we define $I_{x_0}^\sigma(\omega_{\mathbf{k}})$ the function (3.20) on Σ_σ obtained from the closed form $\omega_{\mathbf{k}}$.

The first step is to introduce a space of reasonable test functions for our path integral. Recall that the family $(e_j)_{j \geq 0}$ stands for an orthonormal basis of $L^2(\Sigma, \nu_g)$ of eigenfunctions of the Laplacian. Write any function $f \in H^s(\Sigma)$ (for $s < 0$) as

$$(6.4) \quad f = f_0 + \sqrt{2\pi} \sum_{j \geq 1} f_j e_j$$

and notice that the zero mode f_0 is unnormalised in the sense that it is not multiplied by $\nu_g(\Sigma)^{-1/2}$, which corresponds to the constant eigenfunction e_0 .⁸ We equip $H^s(\Sigma)$ with the pushforward of the measure $dc \otimes \mathbb{P}$ on $(\mathbb{R}/2\pi R\mathbb{Z}) \times \Omega$ through the map $(c, \omega) \mapsto c + X_g(\omega)$.

Recall from Section 3.10 that equivariant distributions $u \in H_\Gamma^s(\tilde{\Sigma})$ can be uniquely decomposed as

$$(6.5) \quad u = \pi^*(f_0 + \sqrt{2\pi} \sum_{j \geq 1} f_j e_j) + I_{x_0}(\omega_{\mathbf{k}})$$

for some $\mathbf{k} \in \mathbb{Z}^{2g} \simeq \mathcal{H}_R^1(\Sigma)$ and $f \in H^s(\Sigma)$ of the form (6.4), where $\pi : \tilde{\Sigma} \rightarrow \Sigma$ is the projection on the base. Each connected component of $H_\Gamma^s(\tilde{\Sigma})$ has the structure of an affine space with associated vector space $H^s(\Sigma)$, parametrised by one element $[\omega_{\mathbf{k}}]$ in the cohomology class $\mathcal{H}_R^1(\Sigma)$. The decomposition (6.5) in the component associated to $[\omega_{\mathbf{k}}]$ depends on a choice of representative $\omega_{\mathbf{k}}$ in the cohomology class $[\omega_{\mathbf{k}}]$, which can be viewed as choosing a base point in the connected component. We consider the following space $\mathcal{E}_R(\Sigma)$ of functionals F defined on $H_\Gamma^s(\tilde{\Sigma})$: in the connected component

⁸This term can be absorbed in the c -integral up to changing the compactification radius and this is why we choose this normalisation.

associated to $[\omega_{\mathbf{k}}]$ (for $\mathbf{k} \in \mathbb{Z}^{2g}$), writing distributions u in this component under the form (6.5), we say that $F \in \mathcal{E}_R(\Sigma)$ if it has the form

$$(6.6) \quad F(u) = \sum_{n=-N}^N e^{\frac{i}{R}nf_0} P_n(f - f_0) G(e^{\frac{i}{R}I_{x_0}(\omega_{\mathbf{k}})})$$

for arbitrary $N \in \mathbb{N}$, where P_n are polynomials, depending on \mathbf{k} , of the form $P_n(\langle f - f_0, g_1 \rangle, \dots, \langle f - f_0, g_{m_n} \rangle)$ where g_1, \dots, g_{m_n} belong to $H^{-s}(\Sigma)$, and G is continuous and bounded on $C^0(\Sigma, \mathbb{S}^1)$, also depending on \mathbf{k} . Notice in particular that these functionals are $2\pi R$ -periodic in the zero mode f_0 . Let us check that the space $\mathcal{E}_R(\Sigma)$ is not depending on the choice of representative $\omega_{\mathbf{k}} \in [\omega_{\mathbf{k}}] \in \mathcal{H}_R^1(\Sigma)$: indeed, if $\omega'_{\mathbf{k}}$ is another representative, we have $\omega'_{\mathbf{k}} = \omega_{\mathbf{k}} + dh_{\mathbf{k}}$ for some smooth function $h_{\mathbf{k}}$ and u can be rewritten as

$$u = \pi^* f' + I_{x_0}(\omega'_{\mathbf{k}}), \quad f' := f - h_{\mathbf{k}} + h_{\mathbf{k}}(x_0)$$

and we see that there is a polynomial P'_n (depending on $h_{\mathbf{k}}, x_0$ and P_n) and a continuous function G' on $C^0(\Sigma, \mathbb{S}^1)$ given by $G'(w) := G(e^{\frac{i}{R}(h_{\mathbf{k}} - h_{\mathbf{k}}(x_0))} w)$ for $w \in C^0(\Sigma, \mathbb{S}^1)$ such that

$$F(u) = \sum_{n=-N}^N e^{\frac{i}{R}nf'_0} P'_n(\langle f' - f'_0, g_1 \rangle, \dots, \langle f' - f'_0, g_{m_n} \rangle) G'(e^{\frac{i}{R}I_{x_0}(\omega'_{\mathbf{k}})}).$$

The reason for this choice of space, in particular the polynomial part in the variable $f - f_0$, is that it is convenient to perform an analytic continuation argument later to define the correlation functions, see the proof of Theorem 6.11. Next we define the space $\mathcal{L}^{\infty,p}(H^s(\Sigma))$ as the closure of $\mathcal{E}_R(\Sigma)$ with respect to the norm

$$(6.7) \quad \|F\|_{\mathcal{L}^{\infty,p}} := \sup_{\mathbf{k}} \left(\int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle - \frac{1}{4\pi} \|df_{\mathbf{k}}\|_2^2} |F(\pi^*(c + X_g) + I_{x_0}(\omega_{\mathbf{k}}))|^p \right] dc \right)^{\frac{1}{p}},$$

where $(1 - \Pi_1)\omega_{\mathbf{k}} = df_{\mathbf{k}}$ with Π_1 is the projection on harmonic forms (recall Lemma 3.3).

Lemma 6.2. *The norm $\|F\|_{\mathcal{L}^{\infty,p}}$ does not depend on the choice of representatives of the cohomology $\mathcal{H}_R^1(\Sigma)$.*

Proof. For $\mathbf{k} \in \mathbb{Z}^{2g} \simeq \mathcal{H}_R^1(\Sigma)$, let $\omega_{\mathbf{k}}^h := \Pi_1 \omega_{\mathbf{k}}$ be the harmonic 1 form in the same cohomology class as $\omega_{\mathbf{k}}$, so that $\omega_{\mathbf{k}} = \omega_{\mathbf{k}}^h + df_{\mathbf{k}}$. Using the Girsanov transform and a shift in the c -variable we have

$$\begin{aligned} & \int_{\mathbb{R}/2\pi R\mathbb{Z}} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle - \frac{1}{4\pi} \|df_{\mathbf{k}}\|_2^2} |F(\pi^*(c + X_g) + I_{x_0}(\omega_{\mathbf{k}}))|^p \right] dc \\ &= \int_{\mathbb{R}/2\pi R\mathbb{Z}} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}}^h \rangle} |F(\pi^*(c + X_g) + I_{x_0}(\omega_{\mathbf{k}}^h))|^p \right] dc \\ &= \int_{\mathbb{R}/2\pi R\mathbb{Z}} \mathbb{E} \left[|F(\pi^*(c + X_g) + I_{x_0}(\omega_{\mathbf{k}}^h))|^p \right] dc, \end{aligned}$$

where we have used that $\langle dX_g, \omega_{\mathbf{k}}^h \rangle_2 = \langle X_g, d^* \omega_{\mathbf{k}}^h \rangle_2 = 0$ because $\omega_{\mathbf{k}}^h$ is harmonic. This proves the claim. \square

On closed surfaces Σ , we will denote the Liouville field by

$$\phi_g := c + X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}).$$

This field belongs to $H^s(\Sigma_\sigma)$ but can also be considered as an element in $H^s_\Gamma(\tilde{\Sigma})$. Indeed, recall from Section 4 that $I_{x_0}^\sigma(\omega_{\mathbf{k}})$ is a smooth function on Σ_σ such that there is an open set $U_{x_0}^\sigma \subset \tilde{\Sigma}$ containing \tilde{x}_0 for which $I_{x_0}(\omega_{\mathbf{k}})|_{U_{x_0}^\sigma} = \pi^* I_{x_0}^\sigma(\omega_{\mathbf{k}})$ and $\pi : U_{x_0}^\sigma \rightarrow \Sigma_\sigma$ a surjective local diffeomorphism. This means that the lift $\pi^* \phi_g|_{U_{x_0}^\sigma}$ has a unique extension to $\tilde{\Sigma}$ as an equivariant element in $H^s_\Gamma(\tilde{\Sigma})$, and we shall therefore freely identify ϕ_g with this extension when considering $F(\phi_g)$ with F defined on $H^s_\Gamma(\tilde{\Sigma})$.

The Liouville field is thus a function of the zero mode $c \in \mathbb{T}_R = \mathbb{R}/2\pi R\mathbb{Z}$, the free field X_g , $\mathbf{k} \in \mathbb{Z}^{2g}$, the base point x_0 , the canonical basis σ and the choice of cohomology basis $\omega_1, \dots, \omega_{2g}$.

Definition 6.3 (Path integral). We consider the path integral, defined for all $F \in \mathcal{E}_R(\Sigma)$,

$$(6.8) \quad \begin{aligned} \langle F \rangle_{\Sigma, g} &:= \left(\frac{V_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} \|\omega_{\mathbf{k}}\|_2^2} \\ &\times \int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle_2} F(\phi_g) e^{-\frac{iQ}{4\pi} \int_{\Sigma_\sigma}^{\text{reg}} K_g \phi_g \, dv_g - \mu M_\beta^g(\phi_g, \Sigma)} \right] dc, \end{aligned}$$

where the curvature term is defined following (4.2), namely

$$(6.9) \quad \int_{\Sigma}^{\text{reg}} K_g \phi_g \, dv_g := \int_{\Sigma} (c + X_g) K_g \, dv_g + \int_{\Sigma_\sigma}^{\text{reg}} I_{x_0}^\sigma(\omega_{\mathbf{k}}) K_g \, dv_g.$$

Note that Lemma 3.9 entails that the potential term $M_\beta^g(\phi_g, \Sigma)$ is well-defined since the (regularised) integrand descends to a function on Σ . Note also that Definition 6.3 a priori depends on the marking σ (i.e. the basis of $\mathcal{H}_1(\Sigma)$), the choice of closed forms representing a basis of cohomology (used to define the $(\omega_{\mathbf{k}})_{\mathbf{k}}$) as well as the base point x_0 . We will show that actually it does not and this is why we dropped all of these dependences from the notations.

Proposition 6.4. *The path integral (6.8) satisfies the following basic properties:*

- (1) *the quantity $\langle F \rangle_{\Sigma, g}$ is well-defined and finite for $F \in \mathcal{E}_R(\Sigma)$, and extends to $F \in \mathcal{L}^{\infty, p}(H^s(\Sigma))$ for $p > 1$.*
- (2) *the quantity $\langle F \rangle_{\Sigma, g}$ depends neither on the base point $x_0 \in \Sigma$, nor on the choice of the homology basis σ , nor on the closed forms representing the cohomology basis dual to σ .*

Proof. We first prove (1). For this we observe that

$$\mathbb{E} \left[\left| \exp \left(-\mu M_\beta^g(c + X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}), \Sigma) \right) \right| \right] \leq C$$

for some constant C depending only on Σ and β (and thus not on \mathbf{k}). To see this, we want to use Proposition 5.3. Let us consider an analytic closed simple curve \mathcal{C} disconnecting the surface Σ into two connected components Σ_1 and Σ_2 (for example \mathcal{C} bound a small disk). Now we use the Markov decomposition in Proposition 5.1 (item (2)) to write the GFF as the sum

$$X_g = X_1 + X_2 + P - c_g,$$

where X_1, X_2, P are independent Gaussian processes, X_1, X_2 are Dirichlet GFF respectively on Σ_1, Σ_2 , P is the harmonic extension of the boundary values $X_g|_{\mathcal{C}}$ (which we

also write X_c) and c_g is a Gaussian random variable. Conditioning on the values X_c , we can then bound our expectation as

$$\begin{aligned} & \mathbb{E} \left[\left| \exp \left(-\mu M_\beta^g(c + X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}), \Sigma) \right) \right| \right] \\ & \leq \mathbb{E} \left[\mathbb{E} \left[\prod_{j=1,2} \left| \exp \left(-\mu M_\beta^g(c - c_g + X_j + P + I_{x_0}^\sigma(\omega_{\mathbf{k}}), \Sigma_j) \right) \right| \mid X_c \right] \right] \\ & \leq \mathbb{E} \left[\prod_{j=1,2} \mathbb{E} \left[\exp \left(\left| \mu M_\beta^g(c + X_j + P + I_{x_0}^\sigma(\omega_{\mathbf{k}}), \Sigma_j) \right| \right) \mid X_c \right] \right]. \end{aligned}$$

Proposition 5.3 (applied with $Z = c$ and $f = e^{i\beta(P + I_{x_0}^\sigma(\omega_{\mathbf{k}}))}$) then ensures that the following estimate holds true for the conditional expectation given X_c

$$\mathbb{E} \left[\exp \left(\left| \mu M_\beta^g(c + X_j + P + I_{x_0}^\sigma(\omega_{\mathbf{k}}), \Sigma_j) \right| \right) \mid X_c \right] \leq e^{C|\mu|v} (1 + C|\mu|ue^{C\mu^2 u^2})$$

with

$$\begin{aligned} u^2 &:= \iint_{\Sigma_j^2} |f(x)| |f(y)| d_g(x, y)^{-\beta^2} dv_g(x) dv_g(y) \\ v &:= \int_{\Sigma_j} |f(x)| \text{dist}(x, \partial \Sigma_j)^{-\frac{\beta^2}{2}} dv_g(x), \end{aligned}$$

for some constant C only depending on β . Since $|f(x)| = 1$ the above conditional expectation is uniformly bounded by a deterministic constant (independent of c, \mathbf{k}). We deduce

$$(6.10) \quad \mathbb{E} \left[\left| \exp \left(-\mu M_\beta^g(c + X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}), \Sigma) \right) \right| \right] < +\infty.$$

As a consequence, our claim (1) follows easily: indeed, the term $e^{-\frac{iQ}{4\pi} \int_{\Sigma_\sigma}^{\text{reg}} K_g \phi_g dv_g}$ has absolute value bounded by 1. Using Hölder inequality, the integrand in (6.8) is thus bounded by

$$\begin{aligned} & \left(\frac{V_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2q}} e^{-\frac{1}{4\pi} \|\omega_{\mathbf{k}}\|_2^2} \left(\int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle_2} |F(\phi_g)|^{p_1} \right] d\mathbf{c} \right)^{1/p_1} \\ & \quad \left(\int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle_2} |e^{-\mu M_\beta^g(\phi_g, \Sigma)}|^{p_2} \right] d\mathbf{c} \right)^{1/p_2} \end{aligned}$$

for some $p_1, p_2 > 1$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$. Using (6.7) this is bounded by

$$\begin{aligned} & \left(\frac{V_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2q}} e^{-\frac{1}{4\pi} \|\omega_{\mathbf{k}}\|_2^2} \|F\|_{\mathcal{L}^\infty, p_1} e^{\frac{1}{4\pi p_1} \|(1 - \Pi_1)\omega_{\mathbf{k}}\|_2^2} \\ & \quad \times \left(\int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle_2} |e^{-\mu M_\beta^g(\phi_g, \Sigma)}|^{p_2} \right] d\mathbf{c} \right)^{1/p_2}. \end{aligned}$$

We can apply the Girsanov transform to the last expectation above to get (by Lemma 3.3)

$$\begin{aligned} & \left(\int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_k \rangle_2} |e^{-\mu M_\beta^g(\phi_g, \Sigma)}|_{p_2} \right] dc \right)^{1/p_2} \\ & \leq e^{\frac{1}{4\pi p_2} \|(1-\Pi_1)\omega_k\|_2^2} \left(\int_{\mathbb{T}_R} \mathbb{E} \left[|e^{-\mu M_\beta^g(c+X_g+I_{x_0}^\sigma(\omega_k^h, \Sigma))}|_{p_2} \right] dc \right)^{1/p_2}. \end{aligned}$$

The last expectation is bounded by some constant C independent of \mathbf{k} as shown in (6.10). Summarizing,

$$\begin{aligned} \langle F \rangle_{\Sigma, g} & \leq C \left(\frac{V_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} \|\omega_k\|_2^2} \|F\|_{\mathcal{L}^{\infty, p_1}} e^{\frac{1}{4\pi} \|(1-\Pi_1)\omega_k\|_2^2} \\ & = C \left(\frac{V_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \|F\|_{\mathcal{L}^{\infty, p_1}} \sum_{\mathbf{k} \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} \|\omega_k^h\|_2^2}. \end{aligned}$$

We conclude about integrability and well-posedness of the path integral, as well as to its extension to $\mathcal{L}^{\infty, p}(H^s(\Sigma))$.

For (2), observe that changing the base point x_0 amounts to shifting the zero mode c by some constant, and this is absorbed by a change of variables in the dc -integral, since the integrand is periodic in c (the assumption $F \in \mathcal{L}^{\infty, p}(H^s(\Sigma))$, hence periodic in c , is crucial). Next we show that the path integral is invariant under change of cohomology basis (even if not dual to σ). Let $\hat{\omega}_j$, for $j = 1, \dots, 2g$, be another basis of cohomology. For $\mathbf{k} \in \mathbb{Z}^{2g}$, we set $\hat{\omega}_k := \sum_{j=1}^{2g} k_j \hat{\omega}_j$. Then there is $A \in \text{GL}_{2g}(\mathbb{Z})$ such that $\omega_k = \hat{\omega}_{A\mathbf{k}} + df_{A\mathbf{k}}$ for all \mathbf{k} and for some exact form $df_{A\mathbf{k}}$ (see Lemma 3.2). We can then replace ω_k by $\hat{\omega}_{A\mathbf{k}} + df_{A\mathbf{k}}$ in the expression for the path integral. By making a change of variables in the summation over \mathbf{k} , we can get rid of the change of basis matrix A , i.e. we get

$$\begin{aligned} (6.11) \quad \langle F \rangle_{\Sigma, g} & = \left(\frac{V_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} \|\hat{\omega}_k + df_k\|_2^2} \\ & \quad \times \int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \hat{\omega}_k + df_k \rangle_2} F(\phi_g) e^{-\frac{iQ}{4\pi} \int_{\Sigma} f_k^{\text{reg}} K_g \phi_g dv_g - \mu M_\beta^g(\phi_g, \Sigma)} \right] dc, \end{aligned}$$

where the Liouville field is now $\phi_g = c + X_g + I_{x_0}^\sigma(\hat{\omega}_k) + f_k(x) - f_k(x_0)$. Next we apply the Girsanov transform to the term $e^{-\frac{1}{2\pi} \langle dX_g, df_k \rangle_2}$. It produces a variance type term $e^{\frac{1}{4\pi} \|df_k\|_2^2}$ and it shifts the law of the GFF as $X_g \rightarrow X_g - (f_k - m_g(f_k))$ where $m_g(f_k) := \frac{1}{V_g(\Sigma)} \int_{\Sigma} f_k dv_g$. We can then shift the c -integral to absorb the constant $m_g(f_k) - f_k(x_0)$. Combining with the norm term in front of the expectation, we get the result.

Finally if we consider another basis of homology σ' . Let $\hat{\omega}_j$, for $j = 1, \dots, 2g$, be another basis of cohomology, dual to σ' . For $\mathbf{k} \in \mathbb{Z}^{2g}$, we set $\hat{\omega}_k := \sum_{j=1}^{2g} k_j \hat{\omega}_j$. Then there is $A \in \text{GL}_{2g}(\mathbb{Z})$ such that $\omega_k = \hat{\omega}_{A\mathbf{k}} + df_{A\mathbf{k}}$ for all \mathbf{k} and for some exact form $df_{A\mathbf{k}}$ (see Lemma 3.2). Then, the previous result tells us that we can replace, in the path integral associated to σ and the ω_k 's, the closed 1-forms ω_k 's by the $\hat{\omega}_k$'s. Next we want to change the homology basis. Only three terms depend now on σ : $F(\phi_g)$, the curvature term and the potential term. Note that $e^{\frac{i}{R} I_{x_0}^\sigma(\hat{\omega}_k)} = e^{\frac{i}{R} I_{x_0}^{\sigma'}(\hat{\omega}_k)}$. Also Lemma

4.5 shows that $e^{-\frac{iQ}{4\pi} \int_{\Sigma_\sigma}^{\text{reg}} K_g \phi_g \, dv_g} = e^{-\frac{iQ}{4\pi} \int_{\Sigma_{\sigma'}}^{\text{reg}} K_g \phi_g \, dv_g}$ because $Q \in \frac{1}{R}\mathbb{Z}$. Hence we are done. \square

Remark 6.5. The invariance under change of cohomology basis in Proposition 6.4 is quite intuitive from the path integral perspective. Indeed, note that formally, $\|\hat{\omega}_{\mathbf{k}}\|_2^2 + 2\langle dX_g, \hat{\omega}_{\mathbf{k}} \rangle_2 = \|\hat{\omega}_{\mathbf{k}} + dX_g\|_2^2 - \|dX_g\|_2^2$. Next, the GFF expectation can be understood as a path integral $e^{-\frac{1}{4\pi} \|dX_g\|_2^2} DX_g$ so that, all in all, the path integral can be understood as $e^{-\frac{1}{4\pi} \|d\phi_g\|_2^2} D\phi_g$, namely a ‘‘Gaussian’’ measure over closed 1-forms.

6.2. Correlation functions: Electric and magnetic operators. In this section, we introduce the correlation functions for all the operators we need in this theory. They are of two types, electric or magnetic, and we will construct each of them in the next two subsections. Finally we will construct mixed electric-magnetic operators by combining the two constructions.

6.2.1. Magnetic operators. Let z_1, \dots, z_{n_m} be distinct points on a closed Riemann surface Σ . For each such a point z_j we assign a unit tangent vector $v_j \in T_{z_j}\Sigma$ and a magnetic charge $m_j \in \mathbb{Z}$. We collect those data in $\mathbf{z} = (z_1, \dots, z_{n_m}) \in \Sigma^{n_m}$, $\mathbf{v} = ((z_1, v_1), \dots, (z_{n_m}, v_{n_m})) \in (T\Sigma)^{n_m}$ and $\mathbf{m} \in \mathbb{Z}^{n_m}$. We assume that $\sum_{j=1}^{n_m} m_j = 0$ and will denote by $\{\mathbf{z}\} = \cup_{j=1}^{n_m} \{z_j\} \subset \Sigma$. We wish to insert on Σ magnetic defects at the z_j 's so that the field $\phi_g(z)$ is multivalued and gains a factor $2\pi m_j R$ when z turns once around a small circle around z_j (and not the other $z_{j'}$'s). As before, we choose a set of data given by Assumption 6.1. We assume that the geometric symplectic basis σ as well as the base point x_0 is distinct from $\{\mathbf{z}\}$, in particular $\{\mathbf{z}\} \subset \Sigma_\sigma$ (recall (4.1)).

The structure of the magnetic operators relies on the construction of the harmonic 1-forms of Proposition 3.10. Consider the harmonic 1-form $\nu_{z, \mathbf{m}}^h$ with windings $2\pi R m_j$ around the point z_j given by this Proposition.⁹

We define the Liouville field

$$(6.12) \quad \phi_g := c + X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{z, \mathbf{m}}^h).$$

As explained in Section 4 and Section 4.2, this field belongs to $H^s(\Sigma \setminus (\sigma \cup \xi))$ but can alternatively be viewed as an element in $H_1^s(\tilde{\Sigma}_z)$ as $I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{z, \mathbf{m}}^h)$ has a lift to a fundamental domain of $\pi_1(\Sigma_z, x_0)$ in $\tilde{\Sigma}_z$ given by $I_{x_0}(\omega_{\mathbf{k}}) + I_{x_0}(\nu_{z, \mathbf{m}}^h)$. Recall that, if the basis of $\mathcal{H}^1(\Sigma \setminus \{\mathbf{z}\})$ is fixed, each $u \in H_1^s(\tilde{\Sigma}_z)$ decomposes uniquely as

$$(6.13) \quad u = \pi^* f + I_{x_0}(\omega_{\mathbf{k}}) + I_{x_0}(\nu_{z, \mathbf{m}}^h)$$

for some $f \in H^s(\Sigma)$, $(\mathbf{k}, \mathbf{m}) \in \mathbb{Z}^{2g+n_m}$. We also write $f_0 = v_g(\Sigma)^{-1} \int_\Sigma f \, dv_g$ as in (6.4). Each connected component of $H_1^s(\tilde{\Sigma}_z)$ has the structure of an affine space with associated vector space $H^s(\Sigma)$, parametrised by one element $[\omega_{\mathbf{k}} + \nu_{z, \mathbf{m}}^h]$ in the cohomology class $\mathcal{H}_R^1(\Sigma_z)$. The decomposition (6.13) in the component associated to $[\omega_{\mathbf{k}}]$ depends on a choice of representative $\omega_{\mathbf{k}} + \nu_{z, \mathbf{m}}^h$ in the cohomology class $[\omega_{\mathbf{k}} + \nu_{z, \mathbf{m}}^h]$, which can be viewed as choosing a base point in the connected component. For each $\mathbf{m} \in \mathbb{Z}^{n_m}$ fixed, we consider the following space $\mathcal{E}_R^{\mathbf{m}}(\Sigma)$ of functionals F defined on $H_1^s(\tilde{\Sigma}_z)$ as follows: in the connected component associated to $[\omega_{\mathbf{k}} + \nu_{z, \mathbf{m}}^h]$ (for $\mathbf{k} \in \mathbb{Z}^{2g}$), writing

⁹Being harmonic is not necessary, we could choose closed 1-forms instead, according to the same proposition.

distributions u in this component under the form (6.13), we say that $F \in \mathcal{E}_R^{\mathbf{m}}(\Sigma)$ if it has the form

$$(6.14) \quad F(u) = \sum_{n=-N}^N e^{\frac{i}{R}nf_0} P_n(f - f_0) G(e^{\frac{i}{R}(I_{x_0}(\omega_{\mathbf{k}}) + I_{x_0}(\nu_{z,\mathbf{m}}^h))})$$

for arbitrary $N \in \mathbb{N}$, where P_n are polynomials (depending on (\mathbf{k}, \mathbf{m})) of the form $P_n(\langle f - f_0, g_1 \rangle, \dots, \langle f - f_0, g_{m_n} \rangle)$ where g_1, \dots, g_{m_n} belong to $H^{-s}(\Sigma)$, and G is continuous and bounded on $C^0(\Sigma, \mathbb{S}^1)$ (depending on (\mathbf{k}, \mathbf{m})). Notice in particular that these functionals are $2\pi R$ -periodic in the zero mode f_0 . The same argument as in the case with no magnetic field shows that the space $\mathcal{E}_R^{\mathbf{m}}(\Sigma)$ is well-defined, in the sense that it does not depend on the choice of representative $\omega_{\mathbf{k}} + \nu_{z,\mathbf{m}}^h$ of the cohomology class.

Next, for \mathbf{m} fixed and a fixed family of representatives $\omega_{\mathbf{k}}$ of the cohomology classes in $\mathcal{H}_R^1(\Sigma)$, we define the space $\mathcal{L}_{\mathbf{m}}^{\infty,p}(H^s(\Sigma))$ as the closure of $\mathcal{E}_R^{\mathbf{m}}(\Sigma, g)$ with respect to the norm defined by

$$\begin{aligned} & \|F\|_{\mathcal{L}_{\mathbf{m}}^{\infty,p}} \\ &= \sup_{\mathbf{k}} \left(\int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{\langle dX_g, \omega_{\mathbf{k}} \rangle}{2\pi} - \frac{\|df_{\mathbf{k}}\|_2^2}{4\pi}} |F(\pi^*(c + X_g) + I_{x_0}(\omega_{\mathbf{k}}) + I_{x_0}(\nu_{z,\mathbf{m}}^h))|^p \right] dc \right)^{\frac{1}{p}}, \end{aligned}$$

where $(1 - \Pi_1)\omega_{\mathbf{k}} = df_{\mathbf{k}}$ with Π_1 the projection on harmonic forms (recall Lemma 3.3).

Lemma 6.6. *The norm $\|F\|_{\mathcal{L}_{\mathbf{m}}^{\infty,p}}$ does not depend on the choice of representatives $\omega_{\mathbf{k}}$ in the cohomology $\mathcal{H}_R^1(\Sigma)$.*

Proof. The proof is the same as that of Lemma 6.2. □

Definition 6.7. The definition of the path integral with magnetic operators at locations $\mathbf{z} = (z_1, \dots, z_{n_m})$ with magnetic charges $\mathbf{m} = (m_1, \dots, m_{n_m})$ and tangent vectors $\mathbf{v} = ((z_1, v_1), \dots, (z_{n_m}, v_{n_m})) \in (T\Sigma)^{n_m}$ reads for $F \in \mathcal{E}_R^{\mathbf{m}}(\Sigma)$

$$(6.15) \quad \begin{aligned} \langle FV_{(0,\mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma,g} &:= \left(\frac{V_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi}\|\omega_{\mathbf{k}}\|_2^2 - \frac{1}{4\pi}\|\nu_{z,\mathbf{m}}^h\|_{g,0}^2 - \frac{1}{2\pi}\langle \omega_{\mathbf{k}}, \nu_{z,\mathbf{m}}^h \rangle_2} \\ &\quad \times \int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi}\langle dX_g, \omega_{\mathbf{k}} \rangle_2} F(\phi_g) e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g \phi_g \, dv_g - \mu M_{\beta}^g(\phi_g, \Sigma)} \right] dc, \end{aligned}$$

where $V_{(0,\mathbf{m})}^g(\mathbf{v})$ is a formal notation to indicate no electric charge (the 0 index) but the presence of magnetic charges $\mathbf{m} = (m_1, \dots, m_{n_m})$, and where the regularised curvature term has now a further magnetic term

$$(6.16) \quad \int_{\Sigma}^{\text{reg}} K_g \phi_g \, dv_g := \int_{\Sigma} (c + X_g) K_g \, dv_g + \int_{\Sigma_g}^{\text{reg}} I_{x_0}^{\sigma}(\omega_{\mathbf{k}}) K_g \, dv_g + \int_{\Sigma}^{\text{reg}} I_{x_0}^{\xi}(\nu_{z,\mathbf{m}}^h) K_g \, dv_g$$

with $\int_{\Sigma}^{\text{reg}} I_{x_0}^{\xi}(\nu_{z,\mathbf{m}}^h) K_g \, dv_g$ defined by (4.5) and $\int_{\Sigma}^{\text{reg}} I_{x_0}^{\sigma}(\omega_{\mathbf{k}}) K_g \, dv_g$ by (4.2).

This definition is similar to (6.8); notice in particular that we have not put the term $e^{-\frac{1}{2\pi}\langle dX_g, \nu_{z,\mathbf{m}}^h \rangle_2}$ since by Proposition 3.10, we know that $d^* \nu_{z,\mathbf{m}}^h \in C^{\infty}(\Sigma)$ ($d^* \nu_{z,\mathbf{m}}^h$ is understood in the distributional sense), and in turn equal to 0, thus

$$\lim_{\epsilon \rightarrow 0} \langle dX_{g,\epsilon}, \nu_{z,\mathbf{m}}^h \rangle_2 = \lim_{\epsilon \rightarrow 0} \langle X_{g,\epsilon}, d^* \nu_{z,\mathbf{m}}^h \rangle_2 = 0.$$

This term would however appear if we were using closed 1-form $\nu_{z,m}$ (with prescribed windings) instead of the harmonic 1-form $\nu_{z,m}^h$.¹⁰

Also this path integral possesses the same properties as (6.8): this can be shown in the same way up to some caveats that we explain in the proof of Proposition 6.8. One further important property is that the path integral does not depend on the choice of the defect graph.

Proposition 6.8. *The path integral (6.15) satisfies the following basic properties:*

- (1) *the quantity $\langle FV_{(0,m)}^g(\mathbf{v}) \rangle_{\Sigma,g}$ is well-defined and finite for $F \in \mathcal{E}_R^m(\Sigma)$, and extends to $F \in \mathcal{L}_m^{\infty,p}(H^s(\Sigma))$ for $p > 1$.*
- (2) *the quantity $\langle FV_{(0,m)}^g(\mathbf{v}) \rangle_{\Sigma,g}$ depends neither on the base point $x_0 \in \Sigma$, nor on the choice of geometric symplectic basis σ of $\mathcal{H}_1(\Sigma)$, nor on the choice of the cohomology basis dual to σ ,*
- (3) *the quantity $\langle FV_{(0,m)}^g(\mathbf{v}) \rangle_{\Sigma,g}$ only depends on the location of the points $\mathbf{v} = (z_j, v_j)_{j=1, \dots, n_m}$ in $T\Sigma$ and the charges \mathbf{m} , but not on the defect graph.*

Proof. The proof of items (1) and (2) is similar to Proposition 6.4, but there is only one point to be careful with. We have to check the summability over \mathbf{k} as this expression features now a further term $e^{-\frac{1}{2\pi}\langle \omega_{\mathbf{k}}, \nu_{z,m}^h \rangle}$. This term is bounded by $e^{C_m|\mathbf{k}|}$ for some $C_m > 0$ and thus does not affect the summability over \mathbf{k} in the proof of Proposition 6.4. To prove (3), it suffices to use Lemma 4.9. □

Below, we denote by $S\Sigma := \{(x, v) \in T\Sigma \mid |v|_{g_x} = 1\}$ the unit sphere bundle.

Corollary 6.9. *For r_θ being the rotation of angle θ in the tangent bundle, set*

$$r_\theta \mathbf{v} := ((z_1, r_{\theta_1} v_1), \dots, (z_{n_m}, r_{\theta_{n_m}} v_{n_m})) \in (T\Sigma)^{n_m}.$$

Then

$$\langle FV_{(0,m)}^g(r_\theta \mathbf{v}) \rangle_{\Sigma,g} = e^{-iQR(\mathbf{m}, \theta)} \langle FV_{(0,m)}^g(\mathbf{v}) \rangle_{\Sigma,g}.$$

Denoting $RQ = -\ell \in -\mathbb{N}$, the correlation functions, viewed as functions

$$\mathbf{v} \in (S\Sigma)^{n_m} \mapsto \langle FV_{(0,m)}^g(\mathbf{v}) \rangle_{\Sigma,g}$$

are sections of $\mathcal{K}^{\ell m_1} \otimes \dots \otimes \mathcal{K}^{\ell m_{n_m}}$ where $\mathcal{K} = (T^{1,0}\Sigma)^*$ is the canonical line bundle and $\mathcal{K}^{-1} = (T^{0,1}\Sigma)^*$ the anti-canonical bundle; by convention, if $k \geq 1$, we write $\mathcal{K}^k := \otimes_{j=1}^k \mathcal{K}$ and $\mathcal{K}^{-k} := \otimes_{j=1}^k \mathcal{K}^{-1}$.

Proof. It suffices to consider the case where only one vector is rotated, as we can apply recursively the result to each angle. Consider the defect graph $\mathcal{D}_{\mathbf{v}, \xi}$. Since the correlation functions do not depend on the graph, we may choose the canonical defect graph $z_1 \rightarrow z_2 \rightarrow \dots \rightarrow z_{n_m}$. Let us first investigate the case when the 1st vector is rotated. We proceed as in the proof of Lemma 4.9 using Gauss-Bonnet. Denote by $(\xi_p)_p$ the paths of the defect graph $\mathcal{D}_{\mathbf{v}, \xi}$. Let us consider another path $\tilde{\xi}_1$ such that $\tilde{\xi}(0) = z_1, \tilde{\xi}(1) = z_2$ with $\partial_t \tilde{\xi}(0) = \tilde{\lambda}_1 r_{\theta_1} v_1, \partial_t \tilde{\xi}(1) = \tilde{\lambda}_2 v_2$ for $\tilde{\lambda}_1, \tilde{\lambda}_2 > 0$. We compute the change in the correlation functions when replacing ξ_1 by $\tilde{\xi}$. Let us call $\tilde{\mathcal{D}}_{r_\theta \mathbf{v}, \xi}$ the defect graph after this replacement. We can assume the curves ξ_1 and $\tilde{\xi}$ don't overlap and bound a domain D homeomorphic to a disk, and the boundary of D inherits an

¹⁰Adding an exact form to $\nu_{z,m}^h$ in the path integral expression amounts to adding this exact form to $\omega_{\mathbf{k}}$ so that our statement is already completely equivalent to considering closed 1-forms instead of $\nu_{z,m}^h$.

orientation from Σ . Without loss of generality, we may assume that $\tilde{\xi}$ is positively oriented, and ξ_1 negatively oriented with respect to the orientation of ∂D . The two defect graphs give rise to two different primitives $I_{x_0}^{\tilde{\xi}}(\nu_{z,m}^h)$ and $I_{x_0}^{\xi_1}(\nu_{z,m}^h)$ and, on D , we have $I_{x_0}^{\tilde{\xi}}(\nu_{z,m}^h) = I_{x_0}^{\xi_1}(\nu_{z,m}^h) - 2\pi R m_1$, where we noted that $\kappa(\xi_1) = \kappa(\tilde{\xi}) = m_1$. The difference of the two regularised integrals is then

$$\begin{aligned} & \int_{\Sigma}^{\text{reg}} I_{x_0}^{\tilde{\xi}}(\nu_{z,m}^h) K_g \, dv_g - \int_{\Sigma}^{\text{reg}} I_{x_0}^{\xi_1}(\nu_{z,m}^h) K_g \, dv_g \\ &= \int_D (I_{x_0}^{\tilde{\xi}}(\nu_{z,m}^h) - I_{x_0}^{\xi_1}(\nu_{z,m}^h)) K_g \, dv_g + 4\pi m_1 R \left(\int_{\xi_1} k_g \, d\ell_g - \int_{\tilde{\xi}} k_g \, d\ell_g \right). \end{aligned}$$

Now we apply again the Gauss-Bonnet theorem on D to get

$$\begin{aligned} \int_D (I_{x_0}^{\tilde{\xi}}(\nu_{z,m}^h) - I_{x_0}^{\xi_1}(\nu_{z,m}^h)) K_g \, dv_g &= -2\pi R m_1 \int_D K_g \, dv_g \\ &= 4\pi R m_1 \left(\int_{\tilde{\xi}} k_g \, d\ell_g - \int_{\xi_1} k_g \, d\ell_g \right) + 4\pi R m_1 \theta_1 \end{aligned}$$

and we deduce that the difference of the two regularised integrals is $4\pi m_1 \theta_1 R$. Hence $\langle FV_{(0,m)}^g(r_{\theta} \mathbf{v}) \rangle_{\Sigma,g} = e^{-iQRm_1 \theta_1} \langle FV_{(0,m)}^g(\mathbf{v}) \rangle_{\Sigma,g}$. The same argument works when we rotate the last vector. The proof has a little twist when rotating an intermediate point because turning an angle affects then two domains, each of which has to be applied the Gauss-Bonnet theorem. The main change is that when adding the contributions of these two domains, the difference $\kappa(\xi_p) - \kappa(\xi_{p-1}) = m_p$ appears now instead of m_1 above, and this yields similarly $\langle FV_{(0,m)}^g(r_{\theta} \mathbf{v}) \rangle_{\Sigma,g} = e^{-iQRm_p \theta_p} \langle FV_{(0,m)}^g(\mathbf{v}) \rangle_{\Sigma,g}$ in case we rotate the p -th vector only. Hence our claim. Any C^0 function f on $S\Sigma$ can be decomposed in Fourier modes in the fibers (which are circles), the fact that for $k \in \mathbb{Z}$ one has $f(z, r_{\theta} v) = e^{ik\theta} f(z, v)$ for all z means exactly that f has only Fourier modes in the fibers of order k , which means that f is the restriction of a C^0 section of \mathcal{K}^k to the unit sphere bundle (see [33, Chapter 5.4.] for instance). \square

6.2.2. Electric operators. We construct now the electric operators in the presence of magnetic operators. Pure electric correlations can be obtained as a particular case of the following by taking the magnetic field to be 0. Such fields need to be regularised. Recall that each $u \in H_{\Gamma}^s(\tilde{\Sigma}_z)$ decomposes uniquely as $u = \pi^* f + I_{x_0}(\omega_{\mathbf{k}}) + I_{x_0}(\nu_{z,\mathbf{k}}^h)$ for some $f \in H^s(\Sigma)$, $(\mathbf{k}, \mathbf{m}) \in \mathbb{Z}^{2g+n_m}$. We introduce the regularised electric operators, for fixed electric charge $\alpha \in \mathbb{R}$ and $x \in \Sigma$,

$$V_{\alpha,g,\epsilon}(u, x) = \epsilon^{-\alpha^2/2} e^{i\alpha u_{g,\epsilon}(x)},$$

where $u_{g,\epsilon}$ is a g -regularisation of the field u . When $u = \phi_g$ is the Liouville field (as below), we will shortcut this expression as $V_{\alpha,g,\epsilon}(x)$.

Next, we choose distinct points x_1, \dots, x_{n_e} on Σ (and distinct from the locations \mathbf{z} of the magnetic defects), which we collect in the vector $\mathbf{x} \in \Sigma^{n_e}$, with associated electric charges $\alpha := (\alpha_1, \dots, \alpha_{n_e}) \in \mathbb{R}^{n_e}$. We denote $V_{(\alpha,0)}^{g,\epsilon}(u, \mathbf{x}) := \prod_{j=1}^{n_e} V_{\alpha_j,g,\epsilon}(u, x_j)$ (which we shortcut as $V_{(\alpha,0)}^{g,\epsilon}(\mathbf{x})$ if u is the Liouville field). Note that this functional belongs to $\mathcal{L}_{\mathbf{m}^{\infty,p}}^{\infty,p}(H^s(\Sigma))$ iff the charges satisfy $\alpha_j \in \frac{1}{R}\mathbb{Z}$, which we will assume from now on. Let us introduce $u_{\mathbf{x}}(x) = \sum_{j=1}^{n_e} i\alpha_j G_g(x, x_j)$ and note that $u_{\mathbf{x}} \in H^s(\Sigma)$ for $s < 1$. We consider

the space $\mathcal{E}_R^{\mathbf{m}}(\Sigma)$ as before. Next, we define the space $\mathcal{L}_{\alpha, \mathbf{m}}^{\infty, p}(H^s(\Sigma))$ as the closure of $\mathcal{E}_R^{\mathbf{m}}(\Sigma)$ with respect to the seminorm

$$\begin{aligned} & \|F\|_{\mathcal{L}_{\alpha, \mathbf{m}}^{\infty, p}} \\ &= \sup_{\mathbf{k}} \left(\int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{\langle dX_g, \omega_{\mathbf{k}} \rangle}{2\pi} - \frac{\|df_{\mathbf{k}}\|_2^2}{4\pi}} |F(c + \pi^* X_g + u_{\mathbf{x}} + I_{x_0}(\omega_{\mathbf{k}}) + I_{x_0}(v_{z, \mathbf{m}}^h))|^p \right] dc \right)^{\frac{1}{p}}, \end{aligned}$$

where $(1 - \Pi_1)\omega_{\mathbf{k}} = df_{\mathbf{k}}$ with Π_1 is the projection on harmonic forms (recall Lemma 3.3).

Similarly to Lemma 6.6, we claim the following:

Lemma 6.10. *The norm $\|F\|_{\mathcal{L}_{\alpha, \mathbf{m}}^{\infty, p}}$ does not depend on the choice of representatives $\omega_{\mathbf{k}}$ in the cohomology $\mathcal{H}_R^1(\Sigma)$.*

The path integral with both electric and magnetic operators is defined by the limit

$$(6.17) \quad \langle FV_{(\alpha, 0)}^g(\mathbf{x})V_{(0, \mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma, g} := \lim_{\epsilon \rightarrow 0} \langle FV_{(\alpha, 0)}^{g, \epsilon}(\mathbf{x})V_{(0, \mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma, g}$$

for $F \in \mathcal{E}_R^{\mathbf{m}}(\Sigma)$. The existence of the limit is non-trivial and only holds under some constraints that we summarize below:

Theorem 6.11. *Assume that*

$$(6.18) \quad \forall j, \quad \alpha_j > Q \quad \text{and} \quad \alpha_j \in \frac{1}{R}\mathbb{Z}, \quad \sum_{j=1}^{n_m} m_j = 0.$$

The mapping $F \in \mathcal{E}_R^{\mathbf{m}}(\Sigma) \mapsto \langle FV_{(\alpha, 0)}^g(\mathbf{x})V_{(0, \mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma, g}$ satisfies the following properties:

(1) **Existence:** *it is well-defined and extends to $F \in \mathcal{L}_{\alpha, \mathbf{m}}^{\infty, p}(H^s(\Sigma))$ for $s < 0$. For $F = 1$, it defines the correlation functions $\langle V_{(\alpha, 0)}^g(\mathbf{x})V_{(0, \mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma, g}$.*

(2) **Conformal anomaly:** *let $g' = e^\rho g$ be two conformal metrics on the closed Riemann surface Σ for some $\rho \in C^\infty(\Sigma)$, and let $\mathbf{x} = (x_1, \dots, x_{n_e}) \in \Sigma^{n_e}$, $\mathbf{v} = ((z_1, v_1), \dots, (z_{n_m}, v_{n_m})) \in (T\Sigma)^{n_m}$ with z_j and x_i distinct for all i, j , and $\alpha = (\alpha_1, \dots, \alpha_{n_e}) \in \mathbb{R}^{n_e}$ obeying the constraint (6.18). Then for $\mathbf{m} = (m_1, \dots, m_{n_m}) \in \mathbb{Z}^{n_m}$, we have*

$$(6.19) \quad \begin{aligned} \langle FV_{(\alpha, 0)}^{g'}(\mathbf{x})V_{(0, \mathbf{m})}^{g'}(\mathbf{v}) \rangle_{\Sigma, g'} &= \langle F(\cdot - \frac{iQ}{2}\rho)V_{(\alpha, 0)}^g(\mathbf{x})V_{(0, \mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma, g} \\ &\times e^{-\sum_{j=1}^{n_e} \Delta_{(\alpha_j, 0)}\rho(x_j) - \sum_{j=1}^{n_m} \Delta_{(0, m_j)}\rho(z_j)} \\ &\times e^{\frac{c_{\text{IL}}}{96\pi} \int_{\Sigma} (|d\rho|_g^2 + 2K_g \rho) dv_g}, \end{aligned}$$

where the real numbers $\Delta_{\alpha, m}$, called conformal weights, are defined by the relation for $\alpha \in \mathbb{R}$

$$(6.20) \quad \Delta_{(\alpha, m)} = \frac{\alpha}{2} \left(\frac{\alpha}{2} - Q \right) + \frac{m^2 R^2}{4}$$

and the central charge is $c_{\text{IL}} := 1 - 6Q^2$.

(3) **Diffomorphism invariance:** *let $\psi : \Sigma' \rightarrow \Sigma$ be an orientation preserving diffeomorphism. Then*

$$(6.21) \quad \langle F(\phi_{\psi^* g})V_{(\alpha, 0)}^{\psi^* g}(\mathbf{x})V_{(0, \mathbf{m})}^{\psi^* g}(\mathbf{v}) \rangle_{\Sigma', \psi^* g} = \langle F(\phi_g \circ \psi)V_{(\alpha, 0)}^g(\psi(\mathbf{x}))V_{(0, \mathbf{m})}^g(\psi_* \mathbf{v}) \rangle_{\Sigma, g},$$

where we used the collective notations

$$\psi_* \mathbf{v} := ((\psi(z_1), d\psi_{z_1} \cdot v_1), \dots, (\psi(z_{n_m}), d\psi_{z_{n_m}} \cdot v_{n_m})), \quad \psi(\mathbf{x}) = (\psi(x_1), \dots, \psi(x_{n_e})).$$

(4) *Spins*: with $r_\theta \mathbf{v} := ((z_1, r_{\theta_1} v_1), \dots, (z_{n_m}, r_{\theta_{n_m}} v_{n_m}))$, then

$$(6.22) \quad \langle FV_{(\alpha,0)}^g(\mathbf{x})V_{(0,\mathbf{m})}^g(r_\theta \mathbf{v}) \rangle_{\Sigma,g} = e^{-iQR(\mathbf{m},\theta)} \langle FV_{(\alpha,0)}^g(\mathbf{x})V_{(0,\mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma,g}.$$

Proof. We split the proof in 4 parts: the existence and convergence of the path integral, the conformal anomaly and the diffeomorphism invariance.

(1) **Existence.** The condition $\alpha_j \in \frac{1}{R}\mathbb{Z}$ ensures that the product $\prod_j V_{\alpha_j,g,\epsilon}(x_j)$ is in $\mathcal{L}_{\alpha,\mathbf{m}}^{\infty,p}(H^s(\Sigma))$. We will use the Cameron-Martin theorem to transform the electric insertions into singularities in the potential. There is some caveat here: this theorem applies only for real valued Gaussians whereas we face here imaginary Gaussians. We thus need to use an analytic continuation argument. The fact that $F \in \mathcal{E}_R^{\mathbf{m}}(\Sigma)$ is crucial for this: indeed F is polynomial, hence analytic, in linear observables of the GFF. This argument only needs to be applied to the GFF expectation and that is why we only average over \mathbb{E} below (and irrelevant factors are removed from computations). So, consider the map

$$\mathbf{w} := (w_1, \dots, w_{n_\epsilon}) \in \mathbb{C}^{n_\epsilon} \mapsto A(\mathbf{w})$$

with A defined by (the variables c and \mathbf{k} are fixed)

$$A(\mathbf{w}) := \mathbb{E} \left[e^{-\frac{\langle dX_g, \omega_{\mathbf{k}} \rangle_2}{2\pi}} F(\phi_g) \prod_{j=1}^{n_\epsilon} \epsilon^{-\frac{w_j^2}{2}} e^{iw_j(c+X_{g,\epsilon}(x_j))} e^{-\frac{iQ}{4\pi} \int_\Sigma^{\text{reg}} K_g \phi_g \, dv_g - \mu M_\beta^g(\phi_g, \Sigma)} \right],$$

where $\phi_g = c + X_g + I_{x_0}^\xi(\omega_{\mathbf{k}}) + I_{z,\mathbf{m}}^\xi(\nu_{z,\mathbf{m}}^h)$ is the Liouville field. For fixed $\epsilon > 0$ this quantity is obviously holomorphic on \mathbb{C}^{n_ϵ} . For $w_1, \dots, w_{n_\epsilon} \in i\mathbb{R}$, we can use the Cameron-Martin theorem to shift the GFF X_g by the term $u_{0,\epsilon}$, and this shift has Radon-Nikodym derivative

$$e^{i \sum_j w_j X_{g,\epsilon}(x_j) + \frac{1}{2} \mathbb{E}[(\sum_{j=1}^{n_\epsilon} w_j X_{g,\epsilon}(x_j))^2]},$$

where we have set $G_{\epsilon,\epsilon'}(x, x') := \mathbb{E}[X_{g,\epsilon}(x)X_{g,\epsilon'}(x')]$ (with the convention that $X_{g,0} = X_g$) and $u_{\epsilon,\epsilon'}(x) := \sum_{j=1}^{n_\epsilon} iw_j G_{\epsilon,\epsilon'}(x, x_j)$, which is a continuous function of $x \in \Sigma$, holomorphic in \mathbf{w} . We get that

(6.23)

$$A(\mathbf{w}) = e^{-\frac{1}{2} \mathbb{E}[(\sum_{j=1}^{n_\epsilon} w_j X_{g,\epsilon}(x_j))^2]} \prod_{j=1}^{n_\epsilon} \epsilon^{-\frac{w_j^2}{2}} e^{i \sum_j w_j c} \\ \times \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g + du_{0,\epsilon}, \omega_{\mathbf{k}} \rangle_2} F(\phi_g + u_{0,\epsilon}) e^{-\frac{iQ}{4\pi} \int_\Sigma^{\text{reg}} K_g(\phi_g + u_{0,\epsilon}) \, dv_g - \mu M_\beta^g(\phi_g + u_{0,\epsilon}, \Sigma)} \right].$$

Now we would like to argue that the right hand side is a holomorphic function of \mathbf{w} . As already explained, the fact that F is polynomial in the GFF is crucial but there is a further subtlety here in the potential: we stress that M_β^g is not a.s. a measure, but a distribution of order 2. Therefore, to apply the theorem of complex differentiation for parametrised integrals, we need to control the quantities $\partial_x^2 u_{0,\epsilon}$ uniformly over x and the compact subsets in \mathbf{w} . The point is that, because of our regularisation along geodesic circles, the partial derivatives $\partial_x^2 u_{0,\epsilon}$ do not exist as functions, hence are not bounded. Therefore, it is not clear that a.s. the mapping $\mathbf{w} \mapsto M_\beta^g(\phi_g + u_{0,\epsilon}, \Sigma)$ is holomorphic (recall that the dependence on \mathbf{w} is encoded in the function $u_{0,\epsilon}$). Furthermore, the term $I_{x_0}^\xi(\nu_{z,\mathbf{m}}^h)$ appearing in the potential is not of class C^2 . Yet, this statement is true at the level of expectation values and this is all what we need. To prove this,

we will approximate $u_{0,\epsilon}$ by a family of \mathbf{w} -holomorphic and two times x -differentiable functions. Let us thus consider a family $(u_{0,\epsilon,\delta})_\delta$ obtained by convolution in the x -variable of the function $u_{0,\epsilon}$ with a mollifying family indexed by δ , which stands for the regularisation scale, and such that $\sup_\Sigma |u_{0,\epsilon} - u_{0,\epsilon,\delta}| \rightarrow 0$ as $\delta \rightarrow 0$. Such a family is holomorphic in \mathbf{w} and two times differentiable in x for each fixed $\delta > 0$. The fact that $I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h)$ is not C^2 is not really problematic: indeed, since it is a deterministic smooth function outside of a set of zero Lebesgue measure, the singularities are not seen by the imaginary GMC. To see this, observe that $\int_\Sigma f(x)M_\beta^g(X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h), dx)$ can be obtained as L^2 limit of regularised approximations for each smooth f . Now we claim:

Lemma 6.12. *The random variable $M_\beta^g(X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h), dx)$ is a random distribution (in the sense of Schwartz) of order 2 on Σ and there exists some L^2 random variable D_Σ such that for all $f \in C^\infty(\Sigma)$*

$$\left| \int_\Sigma f(x)M_\beta^g(X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h), dx) \right| \leq D_\Sigma(\mathbf{k})(\|f\|_\infty + \|\Delta_g f\|_\infty).$$

Proof. Notice that all $f \in C^\infty(\Sigma)$ can be written as

$$f(x) = m_g(f) + \int G_g(x, y)\Delta_g f(y) dv_g(y),$$

where $m_g(f) = \frac{1}{v_g(\Sigma)} \int f(y) dv_g(y)$. Then for all fixed f we have

$$\begin{aligned} & \left| \int_\Sigma f(x)M_\beta^g(X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h), dx) \right| \\ &= \left| \int_\Sigma (m_g(f) + \int_\Sigma G_g(x, y)\Delta_g f(y) dv_g(y))M_\beta^g(X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h), dx) \right| \\ &\leq |m_g(f)| |M_\beta^g(X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h), \Sigma)| \\ &\quad + \int |\Delta_g f(y)| \left(\int G_g(x, y)M_\beta^g(X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h), dx) \right) |dv_g(y) \\ &\leq (\|f\|_\infty + \|\Delta_g f\|_\infty) \left(|M_\beta^g(X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h), \Sigma)| \right. \\ &\quad \left. + \int \left| \int G_g(x, y)M_\beta^g(X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h), dx) \right| dv_g(y) \right). \end{aligned}$$

Since this bound is valid almost surely for a countable dense family of $C^\infty(\Sigma)$ equipped with the norm $\|f\|_\infty + \|\Delta_g f\|_\infty$, we deduce that $M_\beta^g(X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h), dx)$ is a distribution of order 2 almost surely. The random variable in the right-hand side above is our $D_\Sigma(\mathbf{k})$. One can easily check that it is an L^2 random variable: this amounts to computing the following integral (in local coordinates, and using that $e^{i\beta(I_{x_0}^\sigma(\omega_{\mathbf{k}}) + I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{m}}^h))}$ are bounded)

$$u^2(y) := \iint_{D \times D} \log \frac{1}{|x - y|} \log \frac{1}{|x' - y|} |x - x'|^{-\beta^2} dx dx' < +\infty,$$

where D is a disk. This is not only obviously true since $\beta^2 < 2$, but also this quantity is bounded uniformly in y over compact subsets. \square

Next we claim that, for $\delta > 0$ fixed, the expectation

$$(6.24) \quad e^{i \sum_j w_j c} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g + du_{0,\epsilon}, \omega_k \rangle_2} F(\phi_g + u_{0,\epsilon}) e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g(\phi_g + u_{0,\epsilon}) dv_g - \mu M_{\beta}^g(\phi_g + u_{0,\epsilon}, \Sigma)} \right]$$

is holomorphic in $\mathbf{w} \in \mathbb{C}^{n_t}$. We have to check that, for any compact subset $K \subset \mathbb{C}^{n_t}$,

$$\sup_{\mathbf{w} \in K} \mathbb{E} \left[\left| e^{-\frac{\langle dX_g + du_{0,\epsilon}, \omega_k \rangle_2}{2\pi}} F(\phi_g + u_{0,\epsilon}) e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g(\phi_g + u_{0,\epsilon}) dv_g - \mu M_{\beta}^g(\phi_g + u_{0,\epsilon}, \Sigma)} \right| \right] < \infty.$$

Up to using Hölder inequality, this amounts to proving that

$$\sup_{\mathbf{w} \in K} \mathbb{E} \left[\left| e^{-\mu M_{\beta}^g(\phi_g + u_{0,\epsilon}, \Sigma)} \right| \right] < \infty.$$

This follows from Proposition 5.3.

Hence our claim for the holomorphicity of (6.24). Now we claim that the integral (6.24) converges locally uniformly with respect to \mathbf{w} towards the same expression with $\delta = 0$. To see this, it is enough to observe that, locally uniformly in \mathbf{w} ,

$$(6.25) \quad \mathbb{E} \left[\left| e^{\mu(M_{\beta}^g(\phi_g + u_{0,\epsilon,\delta}, \Sigma) - M_{\beta}^g(\phi_g + u_{0,\epsilon}, \Sigma))} - 1 \right|^2 \right] \rightarrow 0, \quad \text{as } \delta \rightarrow 0.$$

Indeed, from Proposition 5.3, we have for each $\alpha \in \mathbb{R}$

$$\begin{aligned} & \mathbb{E} \left[e^{\alpha |M_{\beta}^g(\phi_g + u_{0,\epsilon,\delta}, \Sigma) - M_{\beta}^g(\phi_g + u_{0,\epsilon}, \Sigma)|} \right] \\ & \leq e^{C\alpha} \|e^{i\beta u_{0,\epsilon,\delta}} - e^{i\beta u_{0,\epsilon}}\|_{\infty} \left(1 + C\alpha \|e^{i\beta u_{0,\epsilon,\delta}} - e^{i\beta u_{0,\epsilon}}\|_{\infty} e^{C\alpha^2 \|e^{i\beta u_{0,\epsilon,\delta}} - e^{i\beta u_{0,\epsilon}}\|_{\infty}^2} \right) \end{aligned}$$

and the latter estimate goes to 1 as $\delta \rightarrow 0$ locally uniformly in \mathbf{w} . The claim (6.25) follows. In conclusion the right hand side of (6.23) defines a holomorphic quantity of $\mathbf{w} \in \mathbb{C}^{n_t}$. So does the left hand side, and both sides coincide on $(i\mathbb{R})^{n_t}$, therefore on \mathbb{R}^{n_t} .

Next, we want to take $\mathbf{w} = \alpha$ in (6.23), integrate over c and \mathbf{k} , and then pass to the limit $\epsilon \rightarrow 0$ in the right hand side to give sense to the limit of the left hand side. The limit candidate is the same expression with $\epsilon = 0$

$$(6.26) \quad e^{-\frac{1}{2} \sum_j \alpha_j^2 W_g(x_j) - \sum_{j < j'} \alpha_j \alpha_{j'} G_g(x_j, x_{j'}) + i \sum_j \alpha_j c} \times \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g + du_{0,0}, \omega_k \rangle_2} F(\phi_g + u_{0,0}) e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g(\phi_g + u_{0,0}) dv_g - \mu M_{\beta}^g(\phi_g + u_{0,0}, \Sigma)} \right].$$

The main issue is to make sense of the limit of the potential $M_{\beta}^g(\phi_g + u_{0,\epsilon}, \Sigma)$. In the limit, the contribution from $u_{0,\epsilon}$ will create a singularity in the surface Σ and we have to show that we can integrate M_{β}^g against those singularities. Actually, it is not clear that we can make sense of $\int_{\Sigma} e^{-\beta \sum_j \alpha_j G_g(x, x_j)} M_{\beta}^g(\phi_g, dx)$ almost surely for all possible values of the x_j 's, as we have an understanding of M_{β}^g only as a distribution of order 2. Yet, since we fix x_1, \dots, x_{n_t} , we can still make sense of this quantity on average. Indeed, under the condition (6.18), it is plain to see that the family $(M_{\beta}^g(\phi_g + u_{0,\epsilon}, \Sigma))_{\epsilon}$ is Cauchy

in L^2 and converges towards a random variable denoted by $M_\beta^g(\phi_g + u_{0,0}, \Sigma)$ satisfying

$$\begin{aligned} & \mathbb{E}[|M_\beta^g(\phi_g + u_{0,0}, \Sigma)|^2] \\ &= \iint_{\Sigma^2} e^{i\beta(u_{0,0}(x)-u_{0,0}(y))+\beta^2 G_g(x,y)-\frac{\beta^2}{2}(W_g(x)+W_g(y))} \\ & \quad \times e^{i\beta(I_{x_0}^g(\omega_k)+I_{x_0}^h(\nu_{z,m}^h))(x)-i\beta(I_{x_0}^g(\omega_k)+I_{x_0}^h(\nu_{z,m}^h))(y)} dv_g(x) dv_g(y). \end{aligned}$$

The control of this integral uses the elementary computation that the integral

$$(6.27) \quad \int_{|x|,|y|\leq 1} \frac{|x|^{\beta\alpha}|y|^{\beta\alpha} dx dy}{|x-y|^{\beta^2}}$$

is finite provided that $\beta\alpha > -2 + \frac{\beta^2}{2}$, i.e. $\alpha > Q$. Furthermore, using Fatou’s lemma in Proposition 5.3 gives the estimate for $\alpha \in \mathbb{R}$

$$(6.28) \quad \mathbb{E}\left[\exp\left(\alpha|M_\beta^g(\phi_g + u_{0,0}, \Sigma)|\right) \right] \leq e^{C\alpha v}(1 + C\alpha u \exp(C\alpha^2 u^2))$$

with

$$\begin{aligned} u^2 &:= \iint_{\Sigma^2} e^{i\beta(u_{0,0}(x)+u_{0,0}(y))} d_g(x,y)^{-\beta^2} dv_g(x) dv_g(y), \\ v &:= \int_{\mathcal{D}} e^{i\beta u_{0,0}(x)} \text{dist}(x, \partial\mathcal{D})^{-\frac{\beta^2}{2}} dv_g(x) \end{aligned}$$

for any domain \mathcal{D} with smooth boundary $\partial\mathcal{D}$ that does not cross the x_j ’s. Indeed, to apply Proposition 5.3, we choose a smooth simple loop \mathcal{C} in Σ and condition the above expectation on the values of the restriction of X_g to \mathcal{C} . Using the Markov property in Proposition 5.1, we can then apply Proposition 5.3 to the Dirichlet GFF on the connected components of $\Sigma \setminus \mathcal{C}$, producing quantities that are bounded by the above quantities. Details are left to the reader.

Let us write $\mathbb{T}_R := \mathbb{R}/2\pi R\mathbb{Z}$. Using Hölder inequality, we can then bound the difference between regularised amplitudes and their candidate for the limit, call Δ_ϵ this difference,

$$|\Delta_\epsilon| \leq C \sum_{\mathbf{k} \in \mathbb{Z}^{2d}} e^{-\frac{1}{4\pi}\|\omega_{\mathbf{k}}\|_2^2 - \frac{1}{4\pi}\|\nu_{z,m}^h\|_{g,0}^2 - \frac{1}{2\pi}\langle \omega_{\mathbf{k}}, \nu_{z,m}^h \rangle_2} (R_\epsilon^1 + R_\epsilon^2 + R_\epsilon^3 + R_\epsilon^4)$$

with (for $1/p + 1/q = 1$)

$$\begin{aligned} R_\epsilon^1 &:= |e^{-\frac{1}{2\pi}\langle du_{0,\epsilon}, \omega_{\mathbf{k}} \rangle_2} \left(\int_{\mathbb{T}_R} \mathbb{E}\left[e^{-\frac{1}{2\pi}\langle dX_g, \omega_{\mathbf{k}} \rangle_2} |F(\phi_g + u_{0,\epsilon}) - F(\phi_g + u_{0,0})|^p \right] d\mathbf{c} \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_{\mathbb{T}_R} \mathbb{E}\left[e^{-\frac{1}{2\pi}\langle dX_g, \omega_{\mathbf{k}} \rangle_2} |e^{-\mu M_\beta^g(\phi_g + u_{0,\epsilon}, \Sigma)}|^q \right] d\mathbf{c} \right)^{\frac{1}{q}}, \\ R_\epsilon^2 &:= |e^{-\frac{1}{2\pi}\langle du_{0,\epsilon}, \omega_{\mathbf{k}} \rangle_2} \left(\int_{\mathbb{T}_R} \mathbb{E}\left[e^{-\frac{1}{2\pi}\langle dX_g, \omega_{\mathbf{k}} \rangle_2} |F(\phi_g + u_{0,0})|^p \right] d\mathbf{c} \right)^{\frac{1}{p}} \\ & \quad \times \left(\int_{\mathbb{T}_R} \mathbb{E}\left[e^{-\frac{1}{2\pi}\langle dX_g, \omega_{\mathbf{k}} \rangle_2} |e^{-\mu M_\beta^g(\phi_g + u_{0,\epsilon}, \Sigma)} - e^{-\mu M_\beta^g(\phi_g + u_{0,0}, \Sigma)}|^q \right] d\mathbf{c} \right)^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned}
 R_\epsilon^3 &:= |e^{-\frac{\langle du_{0,\epsilon}, \omega_{\mathbf{k}} \rangle_2}{2\pi}} - e^{-\frac{\langle du_{0,0}, \omega_{\mathbf{k}} \rangle_2}{2\pi}}| \left(\int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle_2} |F(\phi_g + u_{0,0})|^p \right] dc \right)^{\frac{1}{p}} \\
 &\quad \times \left(\int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle_2} |e^{-\mu M_\beta^g(\phi_g + u_{0,\epsilon}, \Sigma)} |q| \right] dc \right)^{\frac{1}{q}}, \\
 R_\epsilon^4 &:= |e^{-\frac{1}{2} \mathbb{E}[(\sum_{j=1}^{n_\epsilon} \alpha_j X_{g,\epsilon}(x_j))^2]} \prod_{j=1}^{n_\epsilon} \epsilon^{-\frac{\alpha_j^2}{2}} - e^{-\frac{1}{2} \sum_j \alpha_j^2 W_g(x_j) - \sum_{j < j'} \alpha_j \alpha_{j'} G_g(x_j, x_{j'})}| \\
 &\quad \times |e^{-\frac{1}{2\pi} \langle du_{0,0}, \omega_{\mathbf{k}} \rangle_2}| \left(\int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle_2} |F(\phi_g + u_{0,0})|^p \right] dc \right)^{\frac{1}{p}} \\
 &\quad \times \left(\int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle_2} |e^{-\mu M_\beta^g(\phi_g + u_{0,0}, \Sigma)} |q| \right] dc \right)^{\frac{1}{q}}.
 \end{aligned}$$

In the terms R_ϵ^1 , R_ϵ^2 and R_ϵ^4 above, there are two trivial terms $|e^{-\frac{1}{2\pi} \langle du_{0,\epsilon}, \omega_{\mathbf{k}} \rangle_2}|$ and $|e^{-\frac{1}{2\pi} \langle du_{0,0}, \omega_{\mathbf{k}} \rangle_2}|$, which we bound by $Ce^{C|\mathbf{k}|}$ for some constant $C > 0$ uniformly in ϵ . In R_ϵ^3 , the difference $|e^{-\frac{1}{2\pi} \langle du_{0,\epsilon}, \omega_{\mathbf{k}} \rangle_2} - e^{-\frac{1}{2\pi} \langle du_{0,0}, \omega_{\mathbf{k}} \rangle_2}|$ is bounded by $Ce^{C|\mathbf{k}|}(e^{C|\mathbf{k}|o(1)} - 1)$ (using Landau notation as $\epsilon \rightarrow 0$).

Next, after using the Girsanov transform in R_ϵ^1 , the first integral term is bounded by $e^{\frac{1}{4\pi p} \|df_{\mathbf{k}}\|_2^2} \|F(\cdot + u_{0,\epsilon} - u_{0,0}) - F(\cdot)\|_{\mathcal{L}_{\alpha,\mathbf{m}}^{\infty,p}}$ (recall that $df_{\mathbf{k}} = (1 - \Pi_1)\omega_{\mathbf{k}}$) by definition of $\|F(\cdot)\|_{\mathcal{L}_{\alpha,\mathbf{m}}^{\infty,p}}$. It is straightforward to check that $\|F(\cdot + u_{0,\epsilon} - u_{0,0}) - F(\cdot)\|_{\mathcal{L}_{\alpha,\mathbf{m}}^{\infty,p}} \rightarrow 0$ as $\epsilon \rightarrow 0$ for $F \in \mathcal{E}_R^{\mathbf{m}}(\Sigma)$: indeed, this follows from the fact that $u_{0,\epsilon} \rightarrow u_{0,0}$ in $H^s(\Sigma)$ for $s < 1$ and from the fact that $F(f)$ depends on f in terms of a polynomial in the variables $\langle f, g_1 \rangle, \dots, \langle f, g_n \rangle$ for some functions $g_1, \dots, g_n \in H^{-s}(\Sigma)$. The second integral in R_ϵ^1 is bounded by $Ce^{\frac{1}{4\pi q} \|df_{\mathbf{k}}\|_2^2}$ for some universal constant C as a result of the Girsanov transform and Proposition 5.3.

Concerning R_ϵ^2 , the first integral is bounded by $e^{\frac{1}{4\pi p} \|df_{\mathbf{k}}\|_2^2} \|F(\cdot)\|_{\mathcal{L}_{\alpha,\mathbf{m}}^{\infty,p}}$, similarly to the first integral in R_ϵ^1 . The main problem lies in evaluating the last integral. First, using the Girsanov transform in the first line and then Hölder inequality for conjugate exponents p_1, p_2 , we bound

$$\begin{aligned}
 &\sup_c \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} \rangle_2} |e^{-\mu M_\beta^g(\phi_g + u_{0,\epsilon}, \Sigma)} - e^{-\mu M_\beta^g(\phi_g + u_{0,0}, \Sigma)} |q| \right] \\
 &\leq \sup_c e^{\frac{1}{4\pi} \|df_{\mathbf{k}}\|_2^2} \mathbb{E} \left[|e^{-\mu M_\beta^g(\phi_g + u_{0,\epsilon} + f_{\mathbf{k}}, \Sigma)} - e^{-\mu M_\beta^g(\phi_g + u_{0,0} + f_{\mathbf{k}}, \Sigma)} |q| \right] \\
 &\leq \sup_c e^{\frac{1}{4\pi} \|df_{\mathbf{k}}\|_2^2} \mathbb{E} \left[|e^{-\mu M_\beta^g(\phi_g + u_{0,0} + f_{\mathbf{k}}, \Sigma)} |qp_1| \right]^{\frac{1}{p_1}} \\
 &\quad \times \mathbb{E} \left[|e^{-\mu(M_\beta^g(\phi_g + u_{0,\epsilon} + f_{\mathbf{k}}, \Sigma) - M_\beta^g(\phi_g + u_{0,0} + f_{\mathbf{k}}, \Sigma))} - 1 |qp_2| \right]^{\frac{1}{p_2}}.
 \end{aligned}$$

The first expectation above is bounded by constant (independent of c, \mathbf{k}) by Proposition 5.3, and as explained above. Now we focus on the second. Recall first the trivial inequality $|e^z - 1| \leq e^{|z|} - 1$ for $z \in \mathbb{C}$, and then $(e^u - 1)^q \leq C(e^{qu} - 1)$ for $u \in \mathbb{R}_+$ and

$q > 1$. Therefore the second expectation is bounded by

$$C\mathbb{E}\left[e^{|\mu|qp_2} |M_\beta^g(\phi_g + u_{0,\epsilon} + f_{\mathbf{k},\Sigma}) - M_\beta^g(\phi_g + u_{0,0} + f_{\mathbf{k},\Sigma})| - 1\right]^{\frac{1}{p_2}},$$

which is bounded by (using Proposition 5.3)

$$\left(e^{C\alpha v}(1 + C\alpha u e^{C\alpha^2 u^2}) - 1\right)^{1/p_2}$$

with $\alpha = |\mu|qp_2$, $v := \int_{\Sigma \setminus \mathcal{C}} |e^{i\beta u_{0,\epsilon}(x)} - e^{i\beta u_{0,0}(x)}| \text{dist}(x, \mathcal{C})^{-\frac{\beta^2}{2}} dv_g(x)$, \mathcal{C} is a smooth simple curve in Σ which does not cross the x_j 's as before, and

$$u^2 := \iint_{\Sigma^2} |e^{i\beta u_{0,\epsilon}(x)} - e^{i\beta u_{0,0}(x)}| |e^{i\beta u_{0,\epsilon}(y)} - e^{i\beta u_{0,0}(y)}| e^{\beta^2 G_g(x,y)} dv_g(x) dv_g(y).$$

This quantity goes to 0 as $\epsilon \rightarrow 0$ as a simple consequence of Lebesgue dominated convergence (recall $\beta^2 < 2$).

For R_ϵ^3 , the two integral terms are bounded by $Ce^{\frac{1}{4\pi} \|df_{\mathbf{k}}\|_2^2} \|F(\cdot)\|_{\mathcal{L}_{\alpha,m}^{\infty,p}}$, as above. Overall, we have the bound $R_\epsilon^3 \leq Ce^{\frac{1}{4\pi} \|df_{\mathbf{k}}\|_2^2} \|F(\cdot)\|_{\mathcal{L}_{\alpha,m}^{\infty,p}} e^{C|\mathbf{k}|} (e^{C|\mathbf{k}|o(1)} - 1)$.

Finally, the analysis of R_ϵ^4 easily follows the previous arguments. We bound the product of the two expectations as above by $Ce^{C|\mathbf{k}|} e^{\frac{1}{4\pi} \|df_{\mathbf{k}}\|_2^2} \|F(\cdot)\|_{\mathcal{L}_{\alpha,m}^{\infty,p}}$. The first term in the expression for R_ϵ^4 tends to 0 as $\epsilon \rightarrow 0$ by (5.6).

Gathering these estimates, we deduce (using the estimate $e^{-\frac{1}{2\pi} \langle \omega_{\mathbf{k}}, \nu_{z,m}^h \rangle_2} \leq e^{C|\mathbf{k}|}$ for some $C > 0$)

$$|\Delta_\epsilon| \leq C \sum_{\mathbf{k} \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} \|\Pi_1 \omega_{\mathbf{k}}\|_2^2 + C|\mathbf{k}|} \times (\|F(\cdot + u_{0,\epsilon} - u_{0,0}) - F(\cdot)\|_{\mathcal{L}_{\alpha,m}^{\infty,p}} + (C_\epsilon + e^{C_\epsilon|\mathbf{k}|} - 1) \|F(\cdot)\|_{\mathcal{L}_{\alpha,m}^{\infty,p}})$$

for some constant C_ϵ such that $\lim_{\epsilon \rightarrow 0} C_\epsilon = 0$. Therefore, up to the multiplicative factor $(\frac{v_g(\Sigma)}{\det(\Delta_g)})^{\frac{1}{2}}$ that is harmless, the regularised correlation functions in the right hand side of (6.17) converge as $\epsilon \rightarrow 0$ towards

$$(6.29) \quad e^{-\frac{1}{2} \sum_j \alpha_j^2 W_g(x_j) - \sum_{j < j'} \alpha_j \alpha_{j'} G_g(x_j, x_{j'})} \sum_{\mathbf{k} \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} (\|\omega_{\mathbf{k}}\|_2^2 + \|\nu_{z,m}^h\|_{g,0}^2 + 2\langle \omega_{\mathbf{k}}, \nu_{z,m}^h \rangle_2)} \times \prod_j e^{i\alpha_j I_{x_0}(\omega_{\mathbf{k}} + \nu_{z,m}^h)(x_j)} \times \int_{\mathbb{T}^R} e^{i \sum_j \alpha_j c} G(c) dc,$$

where

$$G(c) = \mathbb{E}\left[e^{-\frac{1}{2\pi} \langle dX_g + du_{0,0}, \omega_{\mathbf{k}} \rangle_2} F(\phi_g + u_{0,0}) e^{(-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g(\phi_g + u_{0,0}) dv_g - \mu M_\beta^g(\phi_g + u_{0,0}, \Sigma))}\right]$$

expression that we take as a definition of $\langle FV_{(\alpha,0)}^g(\mathbf{x})V_{(0,m)}^g(\mathbf{v}) \rangle_{\Sigma,g}$. This expression extends to functionals $F \in \mathcal{L}_{\alpha,m}^{\infty,p}(H^s(\Sigma))$ for $s < -1$.

(2) Conformal anomaly. Next, we prove the conformal anomaly. The argument is similar to [34, Prop. 4.4], so we sketch the proof up to the crucial argument, following [34, Prop. 4.4]. But first of all, and in order to simplify the proof, let us recall that the path integral is invariant under change of cohomology basis. It will then be convenient to choose a basis of harmonic 1-forms, hence the $\omega_{\mathbf{k}}$'s are harmonic in the following.

Let $g' = e^\rho g$ conformal to g . It suffices to consider the case $F \in \mathcal{E}_R^m(\Sigma)$. We recall the equality in law $X_{g'} = X_g - m_{g'}(X_g)$ with $m_{g'}(f) := \frac{1}{v_{g'}(\Sigma)} \int_\Sigma f dv_{g'}$ (see [34, Lemma 3.1]). Using invariance under translations of the Lebesgue measure on the circle, we deduce

$$\begin{aligned} \langle FV_{(\alpha,0)}^{g'}(\mathbf{x})V_{(0,\mathbf{m})}^{g'}(\mathbf{v}) \rangle_{\Sigma,g'} &= \left(\frac{V_{g'}(\Sigma)}{\det'(\Delta_{g'})} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} \|\omega_{\mathbf{k}}\|_2^2 - \frac{1}{4\pi} \|v_{z,\mathbf{m}}^h\|_{g',0}^2 - \frac{1}{2\pi} \langle \omega_{\mathbf{k}}, v_{z,\mathbf{m}}^h \rangle_2} \\ &\times \int_{\mathbb{T}_R} \mathbb{E}[V_{(\alpha,0)}^{g'}(\phi_g, \mathbf{x})F(\phi_g)e^{-\frac{iQ}{4\pi} \int_\Sigma^{\text{reg}} K_{g'} \phi_g dv_{g'} - \mu M_\beta^{g'}(\phi_g, \Sigma)}] dc. \end{aligned}$$

The point, in the expression above, is that we are integrating the Liouville field $\phi_g = c + X_g + I_{x_0}(\omega_{\mathbf{k}} + v_{z,\mathbf{m}}^h)$ in the path integral regularised in the metric g' . So we have to remove every g' -dependency. We treat first the curvature term. For this we need to use Lemma 4.10. Using this Lemma, the relations (5.7) and $K_{g'} = e^{-\rho}(K_g + \Delta_g \rho)$ and Lemma 4.3 (and note that $\langle d\rho, \omega_{\mathbf{k}} \rangle_2 = 0$ because $\omega_{\mathbf{k}}$ is harmonic), we deduce

$$\begin{aligned} \langle FV_{(\alpha,0)}^{g'}(\mathbf{x})V_{(0,\mathbf{m})}^{g'}(\mathbf{v}) \rangle_{\Sigma,g'} &= \left(\frac{V_{g'}(\Sigma)}{\det'(\Delta_{g'})} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} \|\omega_{\mathbf{k}}\|_2^2 - \frac{1}{4\pi} \|v_{z,\mathbf{m}}^h\|_{g',0}^2 - \frac{1}{2\pi} \langle \omega_{\mathbf{k}}, v_{z,\mathbf{m}}^h \rangle_2} \\ &\times \int_{\mathbb{T}_R} \mathbb{E}[e^{-\frac{iQ}{4\pi} \int_\Sigma \Delta_g \rho X_g dv_g} V_{(\alpha,0)}^{g'}(\phi_g, \mathbf{x})F(\phi_g)e^{-\frac{iQ}{4\pi} \int_\Sigma^{\text{reg}} K_g \phi_g dv_g - \mu M_\beta^g(\phi_g + i\frac{Q}{2}\rho, \Sigma)}] dc. \end{aligned}$$

The same argument of analytic continuation as before allows us to use the (imaginary) Cameron-Martin theorem with the term $e^{-\frac{iQ}{4\pi} \int_\Sigma \Delta_g \rho X_g dv_g}$: the field X_g in the above expression is then replaced by $X_g - \frac{iQ}{2}(\rho - m_g(\rho))$ and the variance of this transform is

$$\frac{Q^2}{16\pi^2} \iint_{\Sigma^2} \Delta_g \rho(x) G_g(x, x') \Delta_g \rho(x') dv_g(x) dv_g(x') = \frac{Q^2}{8\pi} \int_\Sigma |d\rho|_g^2 dv_g.$$

Therefore, using (3.2) and Lemma 3.11 to transform both the det term and the regularised norm, we deduce

$$\begin{aligned} \langle FV_{(\alpha,0)}^{g'}(\mathbf{x})V_{(0,\mathbf{m})}^{g'}(\mathbf{v}) \rangle_{\Sigma,g'} &= e^{\frac{1-6Q^2}{96\pi} \int_\Sigma (|d\rho|_g^2 + 2K_g \rho) dv_g + \sum_j \frac{Q\alpha_j}{2} \rho(x_j) - \sum_j \frac{R^2 m_j^2}{4} \rho(z_j)} \\ &\times \left(\frac{V_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2g}} e^{-\frac{\|\omega_{\mathbf{k}}\|_2^2}{4\pi} - \frac{\|v_{z,\mathbf{m}}^h\|_{g,0}^2}{4\pi} - \frac{\langle \omega_{\mathbf{k}}, v_{z,\mathbf{m}}^h \rangle_2}{2\pi}} \\ &\times \int_{\mathbb{T}_R} H_0(c, \mathbf{k}) dc \end{aligned}$$

with

$$\begin{aligned} H_0(c, \mathbf{k}) &= \mathbb{E}[V_{(\alpha,0)}^{g'}(\phi_g + i\frac{Q}{2}m_g(\rho), \mathbf{x})F(\phi_g - \frac{iQ}{2}(\rho - m_g(\rho))) \\ &\times e^{-\frac{iQ}{4\pi} \int_\Sigma^{\text{reg}} K_g(\phi_g + i\frac{Q}{2}m_g(\rho)) dv_g - \mu M_\beta^g(\phi_g + i\frac{Q}{2}m_g(\rho), \Sigma)}]. \end{aligned}$$

Note now that the vertex operator $V_{\alpha_j, g'}(\phi_g, x_j)$ is not regularised in the metric g , but g' instead. Repeating the argument for the construction of the correlation functions before, we see that this only affects the variance in the Cameron-Martin theorem. Otherwise stated, a straightforward consequence of (5.6) is the relation

$$(6.30) \quad V_{\alpha, g'}(\phi_g, x) = e^{-\frac{\alpha^2}{4} \rho(x)} V_{\alpha, g}(\phi_g, x)$$

when plugging this relation into the expectation. In conclusion, and using Gauss-Bonnet for the constant in the curvature term, we get

$$\begin{aligned}
 (6.31) \quad \langle FV_{(\alpha,0)}^{g'}(\mathbf{x})V_{(0,\mathbf{m})}^{g'}(\mathbf{v}) \rangle_{\Sigma,g'} &= e^{\frac{1-6Q^2}{96\pi} \int_{\Sigma} (|\mathrm{d}\rho|_g^2 + 2K_g \rho) \mathrm{d}v_g - \sum_j \Delta \alpha_j \rho(x_j) - \sum_j \frac{R^2 m_j^2}{4} \rho(z_j)} \\
 &\quad \times \left(\frac{V_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2q}} e^{-\frac{\|\omega_{\mathbf{k}}\|_2^2}{4\pi} - \frac{\|\nu_{\mathbf{z},\mathbf{m}}^h\|_{g,0}^2}{4\pi} - \frac{\langle \omega_{\mathbf{k}}, \nu_{\mathbf{z},\mathbf{m}}^h \rangle_2}{2\pi}} \\
 &\quad \int_{\mathbb{T}_R} e^{-\frac{Q}{2}(\sum_j \alpha_j - \chi(\Sigma)Q)m_g(\rho)} H_1(c, \mathbf{k}) \mathrm{d}c,
 \end{aligned}$$

with

$$\begin{aligned}
 H_1(c, \mathbf{k}) &= \mathbb{E} \left[V_{(\alpha,0)}^g(\phi_g, \mathbf{x}) F(\phi_g - \frac{iQ}{2}(\rho - m_g(\rho))) \right. \\
 &\quad \left. \times e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\mathrm{reg}} K_g \phi_g \mathrm{d}v_g - \mu e^{-\frac{Q\beta}{2} m_g(\rho)} M_{\beta}^g(\phi_g, \Sigma)} \right].
 \end{aligned}$$

In the case of standard Liouville theory, the further terms involving $m_g(\rho)$ are absorbed thanks to invariance of the Lebesgue measure under translations. This argument fails to work here: indeed it would require the Lebesgue measure (on the circle) to be invariant under complex shifts $c \rightarrow c + ia$ for a real, which of course does not hold. We explain now how it works and, basically, translation invariance of the Lebesgue measure is replaced by a Fourier type argument. The function F is a linear combination of the form (6.14). So it is enough to consider F of the form $F(c, \phi) = e^{inc/R} P_k(\phi) G(e^{\frac{i}{R} I_{x_0}})$ (writing I_{x_0} as a shortcut for $I_{x_0}(\omega_{\mathbf{k}} + \nu_{\mathbf{z},\mathbf{m}}^h)$). Expand now the term

$$e^{-\mu e^{-\frac{\beta Q}{2} m_g(\rho)} M_{\beta}^g(\phi_g, \Sigma)} = \sum_{p=0}^{\infty} (-1)^p \frac{\mu^p}{p!} e^{ip\beta c} e^{-p\frac{\beta Q}{2} m_g(\rho)} M_{\beta}^g(X_g + I_{x_0}, \Sigma)^p$$

and plug this relation into (6.31). Performing the c -integral preserves only at most one term in the summation over p , i.e. the term corresponding to $\frac{1}{R}n + \sum_j \alpha_j - \chi(\Sigma)Q + p\beta = 0$, if it exists. As a side remark, notice that this argument also shows that for $F(c, \phi) = e^{inc/R} P_k(\phi) G(e^{\frac{i}{R} I_{x_0}})$ we have

$$(6.32) \quad \forall p \in \mathbb{N}_0, \quad \frac{1}{R}n + \sum_j \alpha_j - \chi(\Sigma)Q + p\beta \neq 0 \implies \langle FV_{(\alpha,0)}^{g'}(\mathbf{x})V_{(0,\mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma,g} = 0.$$

For this p , the contribution of all the terms involving $m_g(\rho)$ is a multiplicative factor given by

$$\exp\left(-\frac{Q}{2} m_g(\rho) \left(\sum_j \alpha_j - \chi(\Sigma)Q + \frac{1}{R}n + \beta p\right)\right).$$

But the condition above on p implies that this term equals 1. Therefore we end up with the final expression

$$\begin{aligned} \langle FV_{(\alpha,0)}^{g'}(\mathbf{x})V_{(0,\mathbf{m})}^{g'}(\mathbf{v}) \rangle_{\Sigma,g'} &= e^{\frac{1-6Q^2}{96\pi} \int_{\Sigma} (|d\rho|_g^2 + 2K_g \rho) dv_g - \sum_j \Delta_{\alpha_j} \rho(x_j) - \sum_j \frac{R^2 m_j^2}{4} \rho(z_j)} \\ &\times \left(\frac{V_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2q}} e^{-\frac{\|\omega_{\mathbf{k}}\|_g^2}{4\pi} - \frac{\|\nu_{z,\mathbf{m}}^h\|_{g,0}^2}{4\pi} - \frac{\langle \omega_{\mathbf{k}}, \nu_{z,\mathbf{m}}^h \rangle_{g,0}}{2\pi}} \\ &\times \int_{\mathbb{T}_R} H_2(c, \mathbf{k}) dc, \end{aligned}$$

with

$$H_2(c, \mathbf{k}) = \mathbb{E} \left[V_{(\alpha,0)}^g(\phi_g, \mathbf{x}) F(\phi_g - \frac{iQ}{2} \rho) e^{-\frac{iQ}{4\pi} \int_{\Sigma} K_g \phi_g dv_g - \mu M_{\beta}^g(\phi_g, \Sigma)} \right].$$

(3) Diffeomorphism invariance. We turn now to the diffeomorphism invariance. We consider (6.29) in the metric ψ^*g and we want to reformulate it in the metric g . For this, several observations are needed. First, as orientation preserving diffeomorphism preserves canonical basis, the natural choice of homology basis for (6.29) in the metric ψ^*g is $\psi^*\sigma$, with dual basis $\psi^*\omega_1, \dots, \psi^*\omega_{2q}$. Then $I_{x_0}^{\psi^*\sigma}(\psi^*\omega_{\mathbf{k}}) = I_{\psi(x_0)}^{\sigma}(\omega_{\mathbf{k}}) \circ \psi$. Similarly for the magnetic operators, the defect graph $\mathcal{D}_{v,\xi}$ is mapped by ψ to $\mathcal{D}_{\psi_*v, \psi \circ \xi}$. Thus we deduce that $I_{x_0}^{\xi}(\psi^*\nu_{z,\mathbf{m}}^h) = I_{\psi(x_0)}^{\psi \circ \xi}(\nu_{z,\mathbf{m}}^h) \circ \psi$. Then we note the standard relations

$$G_{\psi^*g}(x, y) = G_g(\psi(x), \psi(y)), \quad K_{\psi^*g}(x) = K_g(\psi(x)), \quad X_{\psi^*g} \stackrel{\text{law}}{=} X_g \circ \psi.$$

In particular $W_{\psi^*g}(x) = W_g(\psi(x))$ and, combining with the relations just above for the primitives $I_{x_0}^{\psi^*\sigma}(\psi^*\omega_{\mathbf{k}})$ and $I_{x_0}^{\xi}(\psi^*\nu_{z,\mathbf{m}}^h)$, we also obtain $M_{\beta}^{\psi^*g}(\phi_{\psi^*g} + u_{0,0}^{\psi^*g,x}, \Sigma) = M_{\beta}^{\psi^*g}(\phi_g + u_{0,0}^{g,\psi(x)}, \Sigma)$, where we have made explicit the dependence on g, \mathbf{x} of $u_{0,0}$ in the notations. Combining again with Lemma 4.4 for the curvature term, we get the result.

(4) Spins. The spin property results from Corollary 6.9 since the regularised electric operators are in $\mathcal{E}_R^{\mathbf{m}}(\Sigma)$. \square

6.2.3. *Electro-magnetic operators.* We complete this section with the operators that will be of utmost importance to describe the spectrum of this path integral: the electro-magnetic operators. Basically they are obtained by merging the positions \mathbf{x} and \mathbf{z} in Theorem 6.11. So, the setup is the same as previously with the further condition that the numbers of electric or magnetic charges are the same, i.e. $n_e = n_m$.

The path integral with electro-magnetic operators is defined by the limit

$$(6.33) \quad \langle FV_{(\alpha,\mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma,g} := \lim_{t \rightarrow 1} \langle FV_{(\alpha,0)}^g(\mathbf{x}(t))V_{(0,\mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma,g}$$

for $F \in \mathcal{E}_R^{\mathbf{m}}(\Sigma)$ (with $\mathbf{x} = \mathbf{z}$), where $\mathbf{x}(t) = (x_1(t), \dots, x_{n_m}(t))$ with $t \in [0, 1] \mapsto x_j(t)$ being any C^1 curve such that $x_j(1) = z_j$ and $\dot{x}_j(1) = v_j$. Indeed, the quantity in the right hand side only has a limit when $x_j \rightarrow z_j$ along a fixed direction, because of the winding around the points \mathbf{z} . This is why we need to fix a direction v_j when x_j approaches z_j .

Theorem 6.13. *Under the conditions (6.18), the limit (6.33) exists. Moreover the mapping $F \in \mathcal{E}_R^{\mathbf{m}}(\Sigma) \mapsto \langle FV_{(\alpha,\mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma,g}$ satisfies the following properties:*

- (1) **Existence:** *It is well-defined and extends to $F \in \mathcal{L}_{\alpha,\mathbf{m}}^{\infty,p}(H^s(\Sigma))$ for $s < 0$. For $F = 1$, it gives the correlation functions $\langle V_{(\alpha,\mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma,g}$.*

(2) **Conformal anomaly:** let g, g' be two conformal metrics on the closed Riemann surface Σ with $g' = e^\rho g$ for some $\rho \in C^\infty(\Sigma)$. Then we have

(6.34)

$$\frac{\langle FV_{(\alpha, \mathbf{m})}^{g'}(\mathbf{v}) \rangle_{\Sigma, g'}}{\langle F(\cdot - \frac{iQ}{2}\rho)V_{(\alpha, \mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma, g}} = \exp\left(\frac{c_{\text{IL}}}{96\pi} \int_{\Sigma} (|d\rho|_g^2 + 2K_g \rho) dv_g - \sum_{j=1}^{n_\epsilon} \Delta_{(\alpha_j, m_j)} \rho(z_j)\right),$$

where the conformal weights $\Delta_{(\alpha, m)}$ are given by (6.20) and the central charge is $c_{\text{IL}} := 1 - 6Q^2$.

(3) **Diffeomorphism invariance:** let $\psi : \Sigma' \rightarrow \Sigma$ be an orientation preserving diffeomorphism. Then

$$\langle F(\phi_{\psi^*g})V_{(\alpha, \mathbf{m})}^{\psi^*g}(\mathbf{v}) \rangle_{\Sigma', \psi^*g} = \langle F(\phi_g \circ \psi)V_{(\alpha, \mathbf{m})}^g(\psi_*\mathbf{v}) \rangle_{\Sigma, g}.$$

(4) **Spins:** with $r_\partial \mathbf{v} := (r_{\theta_1} v_1, \dots, r_{\theta_{n_m}} v_{n_m})$, then

$$\langle FV_{(\alpha, \mathbf{m})}^g(r_\partial \mathbf{v}) \rangle_{\Sigma, g} = e^{iR\langle \alpha, \mathbf{m}, \theta \rangle - iQR\langle \mathbf{m}, \theta \rangle} \langle FV_{(\alpha, \mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma, g},$$

where the vector $\alpha \cdot \mathbf{m}$ has entries $\alpha_j m_j$ for $j = 1, \dots, n_m$.

Proof. The proof consists in taking the limit in (6.29) as $(x_j(t), \dot{x}_j(t)) \rightarrow (z_j, v_j)$ when $t \rightarrow 1$. The properties of the path integral then result from taking the limit in the related properties of Theorem 6.11. The crucial argument in the proof is the following: since the 1-form $\nu_{z, \mathbf{m}}^h$ is of the form $m_j R d\theta$ in local radial coordinates $z - z_j = re^{i\theta}$ near z_j (see Proposition 3.10), then the function $e^{\frac{i}{R} I_{x_0}^\xi(\nu_{z, \mathbf{m}}^h)(x)}$ has a limit when $(x_j(t), \dot{x}_j(t)) \rightarrow (z_j, v_j)$ as $t \rightarrow 1$. An immediate consequence is the convergence of all terms of the form $e^{i\alpha_j I_{x_0}^\xi(\nu_{z, \mathbf{m}}^h)(x_j(t))}$ as $(x_j(t), \dot{x}_j(t)) \rightarrow (z_j, v_j)$. This makes the convergence obvious for all the prefactors in the expression (6.29). It then remains to focus on the integral. To get the argument simpler, we can choose the cohomology basis to consist of harmonic 1-forms ω_k ; in particular the term $e^{-\frac{1}{2\pi} \langle dX_g, \omega_k \rangle_2} = 1$ in (6.29). In (6.29), the terms involving F and the curvature depend on \mathbf{x} (recall that this dependence is hidden in $u_{0,0}$) and converge towards their value at \mathbf{z} (in the direction \mathbf{v}) in L^p for the measure $e^{-\frac{1}{4\pi} \|\omega_k\|_2^2} \delta_{\mathbf{k}} \otimes \mathbb{P} \otimes dc$. Using Hölder inequality, it remains to investigate the interaction term. Let us write $M(\mathbf{x})$ as a shortcut for the random variable $M_\beta^g(\phi_g + u_{0,0}, \Sigma)$ at \mathbf{x} , and $M(\mathbf{z})$ for this random variable evaluated at \mathbf{z} . Therefore we have to show that

$$\sup_{\mathbf{k} \in \mathbb{Z}^{2g}} \sup_{c \in \mathbb{R}/2\pi\mathbb{R}\mathbb{Z}} \mathbb{E}[|e^{-\mu M(\mathbf{x}(t))} - e^{-\mu M(\mathbf{z})}|^q] \rightarrow 0$$

as $t \rightarrow 1$, for $q > 1$. Using Hölder inequality (taking q slightly larger) and Proposition 5.3, this amounts to showing that, as $t \rightarrow 1$,

$$\sup_{\mathbf{k} \in \mathbb{Z}^{2g}} \sup_{c \in \mathbb{T}_R} \mathbb{E}[|e^{|\mu| |M(\mathbf{x}(t)) - M(\mathbf{z})|} - 1|^q] \rightarrow 0.$$

Using super-additivity of the mapping $x \mapsto x^q$ (for $q > 1$), this amounts to showing that

$$\sup_{\mathbf{k} \in \mathbb{Z}^{2g}} \sup_{c \in \mathbb{T}_R} \mathbb{E}[e^{A|M(\mathbf{x}(t)) - M(\mathbf{z})|} - 1] \rightarrow 0$$

for any $A > 0$. Using Proposition 5.3, we see that the above statement follows from Lemma 6.14 the proof of which is deferred to Appendix B:

Lemma 6.14. *We set $u_{x(t)}(y) := \sum_j \alpha_j G_g(y, x_j(t)) + I_{x_0}(\omega_{\mathbf{k}} + \nu_{z,\mathbf{m}}^h)(y)$. Then we have as $t \rightarrow 1$*

$$\int_{\Sigma} \int_{\Sigma} |e^{-\beta u_{x(t)}(y)} - e^{-\beta u_z(y)}| \cdot |e^{-\beta u_{x(t)}(y')} - e^{-\beta u_z(y')}| e^{\beta^2 G_g(y,y')} \, dv_g(y) \, dv_g(y') \rightarrow 0,$$

uniformly in \mathbf{k} .

Note that, in comparison to Theorem 6.11, the spin has a further term $e^{iR(\alpha \cdot \mathbf{m}, \theta)}$, which comes from the term $e^{i\alpha_j I_{x_0}(\nu_{z,\mathbf{m}}^h)}$ appearing due to the shift of the GFF by $I_{x_0}(\nu_{z,\mathbf{m}}^h)$ in the electric operators: if the vector v_j is rotated by an angle θ_j , and because the 1-form $\nu_{z,\mathbf{m}}^h$ is of the form $m_j R d\theta$ in local radial coordinates $z - z_j = re^{i\theta}$ near z_j , then

$$\begin{aligned} & \lim_{(x_j(t), \dot{x}_j(t)) \rightarrow (z_j, r_{\theta_j} v_j)} e^{i\alpha_j I_{x_0}(\nu_{z,\mathbf{m}}^h)(x_j(t))} \\ &= e^{i\alpha_j m_j R \theta_j} \lim_{(x_j(t), \dot{x}_j(t)) \rightarrow (z_j, v_j)} e^{i\alpha_j I_{x_0}(\nu_{z,\mathbf{m}}^h)(x_j(t))}. \end{aligned}$$

This ends the proof of Theorem 6.13. □

7. THREE POINT CORRELATION FUNCTIONS ON THE RIEMANN SPHERE

In the rational case, the condition (6.18) guarantees the existence of correlation functions. The condition $\alpha_j \in \frac{1}{R}\mathbb{Z}$ is however slightly misleading, for (6.32) implies that these correlation functions vanish if $\sum_j \alpha_j - \chi(\Sigma)Q \in \frac{1}{R}\mathbb{N}$. Therefore the region of interest where the correlations are non-trivial is

$$(7.1) \quad \sum_{j=1}^{n_m} \alpha_j - \chi(\Sigma)Q \in -\beta\mathbb{N},$$

which, together with the condition $\alpha_j > Q$ for all j , shows that the non-trivial correlation functions on the Riemann sphere have $n_m \geq 3$ (recall that $\chi(\Sigma) = 2$ for the Riemann sphere). We thus recover the standard fact for CFTs that 3 point correlation functions on the Riemann sphere are of special importance. Below, we will relate them to twisted Dotsenko-Fateev type integrals.

Symmetry reduction. We identify the Riemann sphere with the extended complex plane $\hat{\mathbb{C}}$ by stereographic projection. On the sphere, every metric is (up to diffeomorphism) conformal to the special metric $g_0 = |z|_+^{-4} |dz|^2$ with $|z|_+ = \max(|z|, 1)$. Applying the transformation rules of Theorem 6.13 (namely items (3) and (4)), one gets, with some straightforward computations, that the correlation functions are *conformally covariant*. More precisely, if $g = e^{\omega(z)} g_0 = g(z) |dz|^2$ is a conformal metric and if z_1, \dots, z_{n_m} are n_m distinct points in $\hat{\mathbb{C}}$, with associated unit tangent vectors v_1, \dots, v_{n_m} , then for a Möbius map $\psi(z) = \frac{az+b}{cz+d}$ (with $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$)

$$(7.2) \quad \langle V_{(\alpha, \mathbf{m})}^g(\psi_* \mathbf{v}) \rangle_{\hat{\mathbb{C}}, g} = \prod_{j=1}^{n_m} \left(\frac{|\psi'(z_j)|^2 g(\psi(z_j))}{g(z_j)} \right)^{-\Delta_{(\alpha_j, m_j)}} \langle V_{(\alpha, \mathbf{m})}^g(\mathbf{v}) \rangle_{\hat{\mathbb{C}}, g}.$$

Now we specify to the case $n_m = 3$. Without loss of generality, we may assume that the magnetic charges satisfy $m_1 \leq m_2 \leq m_3$. The Möbius covariance implies

in particular that the three point functions ($n_m = 3$) are determined up to a constant denoted $C_{\beta,\mu}(\alpha, \mathbf{m})$, called the *structure constant*:

$$(7.3) \quad \begin{aligned} \langle \prod_{j=1}^3 V_{(\alpha_j, m_j)}^g(z_j, v_j) \rangle_{\hat{C}, g} &= e^{\frac{c_{\mathbb{L}}}{96\pi} \int_{\Sigma} (|d\omega|_{g_0}^2 + 2K_{g_0} \omega) dv_{g_0}} e^{-iQR \sum_{j=1}^3 m_j \arg(v_j)} \\ &\times P_{\alpha, \mathbf{m}}(\mathbf{z}) \prod_{j=1}^3 g(z_j)^{-\Delta(\alpha_j, m_j)} C_{\beta, \mu}(\alpha, \mathbf{m}) \end{aligned}$$

with

$$\begin{aligned} P_{\alpha, \mathbf{m}}(\mathbf{z}) &:= |z_1 - z_3|^{2(\Delta(\alpha_2, m_2) - \Delta(\alpha_1, m_1) - \Delta(\alpha_3, m_3))} \\ &\times |z_2 - z_3|^{2(\Delta(\alpha_1, m_1) - \Delta(\alpha_2, m_2) - \Delta(\alpha_3, m_3))} |z_1 - z_2|^{2(\Delta(\alpha_3, m_3) - \Delta(\alpha_1, m_1) - \Delta(\alpha_2, m_2))}. \end{aligned}$$

In particular, by taking $(z_1, z_2, z_3) = (0, 1, \infty)$, we have

$$C_{\beta, \mu}(\alpha, \mathbf{m}) = \langle V_{(\alpha_1, m_1)}^{g_0}(0, e_1) V_{(\alpha_2, m_2)}^{g_0}(1, e_1) V_{(\alpha_3, m_3)}^{g_0}(\infty, e_1) \rangle_{\hat{C}, g_0}.$$

We will compute the 3 point function in the metric g_0 to deduce the structure constants. For this we assume $z_1, z_2, z_3 \in \mathbb{R}$ with $z_1 < z_2 < z_3$ and $v_1 = v_2 = v_3 = e_1$ with $e_1 = \partial_x$ if one uses the coordinate $z = x + iy$ on \mathbb{C} . In that case, we choose the branch cut to be $[z_1, z_3]$ with defect graph $z_1 \rightarrow z_2 \rightarrow z_3$. The 1-form $\nu_{z, \mathbf{m}}$ is given by

$$\begin{aligned} \nu_{z, \mathbf{m}}^h(z) &= \text{Im} \left(\frac{Rm_1(z_1 - z_2) dz}{(z - z_1)(z - z_2)} + \frac{Rm_3(z_3 - z_2) dz}{(z - z_2)(z - z_3)} \right) \\ &= \text{Im} \left(\left(\frac{Rm_1}{z - z_1} + \frac{Rm_2}{z - z_2} + \frac{Rm_3}{z - z_3} \right) dz \right). \end{aligned}$$

The primitive is then $I_0(\nu_{z, \mathbf{m}}^h)(z) = -Rm_1 \arg\left(\frac{z_2 - z}{z_1 - z}\right) + Rm_3 \arg\left(\frac{z_3 - z}{z_2 - z}\right)$ (which vanishes on the half-line $]z_3, +\infty[$). By an abuse of notations, we will denote by $I_0(\nu_{z, \mathbf{m}}^h)(z_j) := \lim_{t \rightarrow 1} I_0(\nu_{z, \mathbf{m}}^h)(x_j(t))$ where $x_j(t) = z_j - 1 + t$ is converging to z_j in the direction v_j as $t \rightarrow 1$. Let us compute $\|\nu_{z, \mathbf{m}}^h\|_{g_0, 0}^2$. We claim:

Lemma 7.1. *We have*

$$e^{-\frac{1}{4\pi} \|\nu_{z, \mathbf{m}}^h\|_{g_0, 0}^2} = \prod_{i < j} |z_i - z_j|^{R^2 m_i m_j} \times \prod_{j=1}^3 g_0(z_j)^{-\frac{R^2 m_j^2}{4}}.$$

Proof. We have

$$\|\nu_{z, \mathbf{m}}^h\|_{g_0, 0}^2 = \sum_{i, j=1}^3 \text{FP}_{\epsilon=0} C_{ij}(\epsilon)$$

with FP denoting finite part as $\epsilon \rightarrow 0$ and $C_{ij}(\epsilon)$ are integrals (for $z = x + iy$)

$$C_{ij}(\epsilon) = \int_{\mathbb{C} \setminus B_{g_0}(z, \epsilon)} \frac{R^2 m_i m_j}{(z - z_i)(\bar{z} - z_j)} dx dy,$$

where $B_{g_0}(z, \epsilon) = \cup_{j=1}^3 B_{g_0}(z_j, \epsilon) \cup \{|z| > 1/\epsilon\}$ with $B_{g_0}(z_j, \epsilon)$ the geodesic ball for g_0 centered at z_j and radius $\epsilon > 0$. For $\epsilon > 0$ small, if $|z_j| < 1$ we have $B_{g_0}(z_j, \epsilon) = \{z \mid |z - z_j| < \epsilon\}$ and if $|z_j| > 1$, $B_{g_0}(z_j, \epsilon) = \{z \mid |z - z_j| < \epsilon|zz_j|\}$. Notice that when

$i \neq j$, the finite part of the integral above reduces to considering the domain $|z| < 1/\epsilon$ since the integrand is $L_{\text{loc}}^1(\mathbb{C})$. Using radial coordinates $z = re^{i\theta}$, we get

$$C_{ij}(\epsilon) = \int_0^{1/\epsilon} \int_0^{2\pi} \frac{R^2 m_i m_j}{(r + e^{i\theta}(z_i - z_j))} dr d\theta = 2\pi R^2 m_i m_j (\log(1/\epsilon) - \log|z_i - z_j|),$$

thus $\text{FP}_{\epsilon \rightarrow 0} C_{ij}(\epsilon) = -2\pi R^2 m_i m_j \log|z_i - z_j|$. When $i = j$, we obtain if $|z_j| < 1$

$$\begin{aligned} \text{FP}_{\epsilon=0} C_{jj}(\epsilon) &= R^2 m_j^2 \text{FP}_{\epsilon=0} \left(\int_{\frac{1}{\epsilon} > |z|, |z-z_j| > 1} \frac{dx dy}{|z-z_j|^2} + \int_{\epsilon < |z-z_j| < 1} \frac{dx dy}{|z-z_j|^2} \right) \\ &= R^2 m_j^2 \text{FP}_{\epsilon=0} \int_{\epsilon|1+zz_j| < |z| < 1} \frac{1}{|z|^2} dx dy \\ &= R^2 m_j^2 \lim_{\epsilon \rightarrow 0} \left(-2\pi \log 2 + \int_{\epsilon|1+zz_j| < |z| < 2\epsilon} \frac{1}{|z|^2} dx dy \right) = 0, \end{aligned}$$

while if $|z_j| > 1$

$$\begin{aligned} \text{FP}_{\epsilon=0} C_{jj}(\epsilon) &= R^2 m_j^2 \text{FP}_{\epsilon=0} \int_{\epsilon|zz_j| < |z-z_j| < 1} \frac{1}{|z-z_j|^2} dx dy \\ &= R^2 m_j^2 \lim_{\epsilon \rightarrow 0} \int_{\epsilon|zz_j| < |z-z_j| < \epsilon} \frac{1}{|z-z_j|^2} dx dy \\ &= -R^2 m_j^2 \lim_{\epsilon \rightarrow 0} \left(2\pi \int_{\epsilon}^{\epsilon|z_j|^2} \frac{dr}{r} \right) = -4\pi \log|z_j|. \end{aligned}$$

Our claim follows. □

Now we compute the three point function in the metric g_0 using (6.17) (recall also (6.15)) and (6.33) with the following caveat: instead of using the GFF X_{g_0} in the metric g_0 , we can use a shift in the zero mode to use instead the GFF

$$X_0 := X_{g_0} - \frac{1}{2\pi} \int_0^{2\pi} X_{g_0}(e^{i\theta}) d\theta,$$

which has covariance:

$$(7.4) \quad G_0(z, z') = \log \frac{1}{|z-z'|} + \log|z|_+ + \log|z'|_+, \quad (\text{thus } W_0(z) = 0).$$

The Liouville field is thus $\phi_0 = c + X_0$. We deduce that (here we take the limit $t \rightarrow 1$ in the expression (6.29) for correlation functions as explained in the proof of Theorem 6.13)

$$\begin{aligned} \langle \prod_{j=1}^3 V_{(\alpha_j, m_j)}(z_j, v_j) \rangle_{\mathbb{C}, g_0} &= C_{g_0} \prod_{i < j} |z_i - z_j|^{R^2 m_i m_j} \prod_{j < j'} e^{-\alpha_j \alpha_{j'} G_0(z_j, z_{j'})} \\ (7.5) \quad &\times e^{-\frac{1}{4\pi} \|v_{z, m}^h\|_{g_0, 0}^2 + i \sum_{j=1}^3 \alpha_j I_0(v_{z, m}^h)(z_j)} \\ &\times \int_0^{2\pi R} e^{i(\sum_j \alpha_j - 2Q)c} H(c) dc \end{aligned}$$

with

$$H(c) = \mathbb{E} \left[e^{-\mu e^{i\beta c} \int_{\mathbb{C}} F(x, z) (\prod_j |z_j|_+^{-\beta \alpha_j}) M_{\beta}^{g_0}(\phi_0, dx)} \right],$$

$\mathbf{z} = (z_1, z_2, z_3)$, and if $\zeta(z)$ is Riemann's zeta function,

$$(7.6) \quad \begin{aligned} F(x, \mathbf{z}) &= \prod_{j=1}^3 \left(\frac{|x - z_j|}{|x|_+} \right)^{\beta \alpha_j} e^{i\beta I_0(v_{z,m}^h)(x)}, \\ C_{g_0} &:= \left(\frac{V_{g_0}(\phi_0, \hat{C})}{\det'(\Delta_{g_0})} \right)^{\frac{1}{2}} = 2^{\frac{2}{3}} \sqrt{\pi} e^{-\frac{1}{12} + 2\zeta'(-1)}. \end{aligned}$$

The last equality follows from [31, Proof of Corollary 11.5]. Set $s := \frac{2Q - \sum_j \alpha_j}{\beta} \in \mathbb{N}$. By performing the series expansion of the exponential in the expectation and computing the Fourier coefficients, we get

$$\begin{aligned} \langle \prod_{j=1}^3 V_{(\alpha_j, m_j)}(z_j, v_j) \rangle_{\hat{C}, g_0} &= 2\pi R C_{g_0} \frac{(-\mu)^s}{s!} \mathbb{E} \left[\left(\int_{\mathbb{C}} F(x, \mathbf{z}) M_{\beta}^{g_0}(dx) \right)^s \right] \prod_j |z_j|_+^{4\Delta(\alpha_j, m_j)} \\ &\quad \times \prod_{j < j'} |z_j - z_{j'}|^{\alpha_j \alpha_{j'} + R^2 m_j m_{j'}} e^{-i\pi \alpha_2 R m_1 + i\pi \alpha_3 R m_3}. \end{aligned}$$

Note that (if $\beta = \frac{p}{R}$)

$$e^{i\beta I_0(v_{z,m}^h)(x)} = \left(\frac{(z_2 - x)(\bar{z}_1 - \bar{x})}{(\bar{z}_2 - \bar{x})(z_1 - x)} \right)^{-\frac{pm_1}{2}} \left(\frac{(z_3 - x)(\bar{z}_2 - \bar{x})}{(z_2 - x)(\bar{z}_3 - \bar{x})} \right)^{\frac{pm_3}{2}}.$$

These computations make completely explicit the z_1, z_2, z_3 dependence.

Structure constant. Now we wish to fix the values of (z_1, z_2, z_3) to be $(0, 1, \infty)$ where the related correlation functions are defined by

$$\begin{aligned} &\langle V_{(\alpha_1, m_1)}(0, e_1) V_{(\alpha_2, m_2)}(1, e_1) V_{(\alpha_3, m_3)}(\infty, e_1) \rangle_{\hat{C}, g_0} \\ &= \lim_{|z_3| \rightarrow \infty} \langle V_{(\alpha_1, m_1)}(0, e_1) V_{(\alpha_2, m_2)}(1, e_1) V_{(\alpha_3, m_3)}(z_3, e_1) \rangle_{\hat{C}, g_0}, \end{aligned}$$

in which case the above relation becomes for $s \in \mathbb{N}$

$$\begin{aligned} &\langle V_{(\alpha_1, m_1)}(0, e_1) V_{(\alpha_2, m_2)}(1, e_1) V_{(\alpha_3, m_3)}(\infty, e_1) \rangle_{\hat{C}, g_0} \\ &= C_{g_0} 2\pi R \frac{(-\mu)^s}{s!} \mathbb{E} \left[\left(\int_{\mathbb{C}} \frac{|x|^{\beta \alpha_1} |1 - x|^{\beta \alpha_2}}{|x|_+^{\beta \bar{\alpha}}} \left(\frac{x}{\bar{x}} \right)^{\frac{pm_1}{2}} \left(\frac{1 - x}{1 - \bar{x}} \right)^{\frac{p(-m_1 - m_3)}{2}} M_{\beta}^{g_0}(\phi_0, dx) \right)^s \right] \\ &= C_{g_0} 2\pi R \frac{(-\mu)^s}{s!} \mathbb{E} \left[\left(\int_{\mathbb{C}} \frac{x^{\Delta_1} \bar{x}^{\bar{\Delta}_1} (1 - x)^{\Delta_2} (1 - \bar{x})^{\bar{\Delta}_2}}{|x|_+^{\beta \bar{\alpha}}} M_{\beta}^{g_0}(\phi_0, dx) \right)^s \right], \end{aligned}$$

where we have set $\bar{\alpha} = \sum_j \alpha_j$, $\Delta_1 = \frac{\beta \alpha_1 + pm_1}{2}$, $\bar{\Delta}_1 = \frac{\beta \alpha_1 - pm_1}{2}$, $\Delta_2 = \frac{\beta \alpha_2 + pm_2}{2}$, and $\bar{\Delta}_2 = \frac{\beta \alpha_2 - pm_2}{2}$. This expression can be expanded as a multiple integral

$$\begin{aligned} &\langle V_{(\alpha_1, m_1)}(0, e_1) V_{(\alpha_2, m_2)}(1, e_1) V_{(\alpha_3, m_3)}(\infty, e_1) \rangle_{\hat{C}, g_0} \\ &= C_{g_0} 2\pi R \frac{(-\mu)^s}{s!} \mathbb{E} \left[\int_{\mathbb{C}^s} \prod_{j=1}^s \frac{x_j^{\Delta_1} \bar{x}_j^{\bar{\Delta}_1} (1 - x_j)^{\Delta_2} (1 - \bar{x}_j)^{\bar{\Delta}_2}}{|x_j|_+^{\beta \bar{\alpha}}} \prod_{j=1}^s M_{\beta}^{g_0}(\phi_0, dx_j) \right] \\ &= C_{g_0} 2\pi R \frac{(-\mu)^s}{s!} \int_{\mathbb{C}^s} \prod_{j=1}^s x_j^{\Delta_1} \bar{x}_j^{\bar{\Delta}_1} (1 - x_j)^{\Delta_2} (1 - \bar{x}_j)^{\bar{\Delta}_2} \prod_{j < j'} |x_j - x_{j'}|^{\beta^2} dx_1 \dots dx_s \\ &=: C_{g_0} 2\pi R (-\mu)^s \mathcal{J}(\beta, \alpha_1, \alpha_2, m_1, m_2, s). \end{aligned}$$

In the last but one line, we have used the fact that

$$\begin{aligned} & \mathbb{E} \left[\prod_{j=1}^2 e^{i\beta X_{g_0}(x_j) + \frac{\beta^2}{2} \mathbb{E}[X_{g_0}(x_j)^2]} \right] g_0(x_1) \dots g_0(x_s) dx_1 \dots dx_s \\ &= \prod_{j < j'} e^{-\beta^2 G_0(x_j, x_{j'})} g_0(x_1) \dots g_0(x_s) dx_1 \dots dx_s \\ &= \prod_{j < j'} |x_j - x_{j'}|^{\beta^2} \prod_j |x_j|_+^{-\beta^2(s-1)-4} dx_1 \dots dx_s \\ &= \prod_{j < j'} |x_j - x_{j'}|^{\beta^2} \prod_j |x_j|_+^{\beta\alpha} dx_1 \dots dx_s. \end{aligned}$$

The pure electric case and the imaginary DOZZ. In the case when there are no magnetic charges, i.e. $m_1 = m_2 = 0$, this expression corresponds to the famous Dotsenko-Fateev integral [19], whose value is known (in case $m_1 = m_2 = 0$)

(7.7)

$$J(\beta, \alpha_1, \alpha_2, 0, 0, s) = \left(\frac{\pi}{\ell(\frac{\beta^2}{4})} \right)^s \frac{\prod_{j=1}^s \ell(j\frac{\beta^2}{4})}{\ell(\frac{\beta^2}{4}) \prod_{j=0}^{s-1} \ell(-\frac{\beta\alpha_1}{2} - j\frac{\beta^2}{4}) \ell(-\frac{\beta\alpha_2}{2} - j\frac{\beta^2}{4}) \ell(-\frac{\beta\alpha_3}{2} - j\frac{\beta^2}{4})}$$

with the convention that $\ell(x) = \Gamma(x)/\Gamma(1-x)$. This expression coincides with the imaginary DOZZ formula (see [70]), which is actually an analytic extension of this expression to all possible values of $\alpha_1, \alpha_2, \alpha_3$. For this, we introduce the special function $Y_{\frac{\beta}{2}}(z)$ defined for $0 < \text{Re}(z) < \tilde{Q} := \frac{\beta}{2} + \frac{2}{\beta}$ by the formula

$$(7.8) \quad \log Y_{\frac{\beta}{2}}(z) = \int_0^\infty \left((\frac{\tilde{Q}}{2} - z)^2 e^{-t} - \frac{(\sinh((\frac{\tilde{Q}}{2} - z)\frac{t}{2}))^2}{\sinh(\frac{t\beta}{4}) \sinh(\frac{t}{\beta})} \right) \frac{dt}{t}.$$

The function $Y_{\frac{\beta}{2}}$ can be analytically continued to \mathbb{C} because it satisfies remarkable functional relations for $z \in \mathbb{C}$

(7.9)

$$Y_{\frac{\beta}{2}}(z + \frac{\beta}{2}) = \frac{\Gamma(\frac{\beta}{2}z)}{\Gamma(1 - \frac{\beta}{2}z)} (\frac{\beta}{2})^{1-\beta z} Y_{\frac{\beta}{2}}(z), \quad Y_{\frac{\beta}{2}}(z + \frac{2}{\beta}) = \frac{\Gamma(\frac{2}{\beta}z)}{\Gamma(1 - \frac{2}{\beta}z)} (\frac{\beta}{2})^{\frac{4}{\beta}z-1} Y_{\frac{\beta}{2}}(z).$$

The function $Y_{\frac{\beta}{2}}$ has no poles in \mathbb{C} and the zeros of $Y_{\frac{\beta}{2}}$ are simple (if $\beta^2 \notin \mathbb{Q}$) and given by the discrete set $(-\frac{\beta}{2}\mathbb{N} - \frac{2}{\beta}\mathbb{N}) \cup (\tilde{Q} + \frac{\beta}{2}\mathbb{N} + \frac{2}{\beta}\mathbb{N})$. The imaginary DOZZ formula $C_\beta^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3)$ is the following expression

$$(7.10) \quad C_\beta^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3) = \left(\frac{\beta}{2} \right)^{2Q^2 - Q\bar{\alpha}} \left(\frac{\pi}{\ell(\frac{\beta^2}{4})} \right)^{\frac{2Q-\bar{\alpha}}{\beta}} \times \frac{Y_{\frac{\beta}{2}}(Q + \frac{\beta}{2} - \frac{\alpha_1 + \alpha_2 + \alpha_3}{2}) Y_{\frac{\beta}{2}}(\frac{\beta + \alpha_1 - \alpha_2 - \alpha_3}{2}) Y_{\frac{\beta}{2}}(\frac{\beta + \alpha_2 - \alpha_1 - \alpha_3}{2}) Y_{\frac{\beta}{2}}(\frac{\beta + \alpha_3 - \alpha_1 - \alpha_2}{2})}{Y_{\frac{\beta}{2}}(\frac{\beta}{2}) Y_{\frac{\beta}{2}}(\frac{2}{\beta} + \alpha_1) Y_{\frac{\beta}{2}}(\frac{2}{\beta} + \alpha_2) Y_{\frac{\beta}{2}}(\frac{2}{\beta} + \alpha_3)},$$

where $\bar{\alpha} = \alpha_1 + \alpha_2 + \alpha_3$. Then we claim:

Proposition 7.2. *Assume $m_i = 0$ and $\alpha_i > Q$ for $i = 1, 2, 3$, and (7.1). Then*

$$\begin{aligned} & \langle V_{(\alpha_1, m_1)}(0, e_1) V_{(\alpha_2, m_2)}(1, e_1) V_{(\alpha_3, m_3)}(\infty, e_1) \rangle_{\hat{c}, g_0} \\ &= C_{g_0} 2\pi R (-\mu)^{\frac{2Q - \sum_i \alpha_i}{\beta}} C_{\beta}^{\text{DOZZ}}(\alpha_1, \alpha_2, \alpha_3), \end{aligned}$$

with C_{g_0} given in (7.6).

Proof. To prove this formula, we see the big fraction in (7.10) as a product of 4 fractions (the n -th fraction is made up of the n -th term in the numerator divided by the n -th term in the denominator). Then we apply the $\beta/2$ -shift relation (7.9) s times for each fraction to get the result. \square

General case. The case including the presence of magnetic charges was studied in [55]. Let us introduce the complex Gamma function

$$\Gamma^{\mathbb{C}}(a|a') := \frac{1}{\pi} \int_{\mathbb{C}} z^{a-1} \bar{z}^{a'-1} e^{2i\text{Re}(z)} \frac{dz \wedge d\bar{z}}{2i},$$

where a, a' denotes a pair of complex numbers such that $a - a'$ is an integer. The integral converges if $0 < \frac{1}{2}\text{Re}(a + a') < 1$, and admits a meromorphic extension to each complex line $a - a' = k$. Furthermore $\Gamma^{\mathbb{C}}(a|a') = \frac{1}{\pi} i^{a'-a} \Gamma(a) \Gamma(a') \sin(\pi a')$. Then we have [55, Corollary 1.3]

$$\begin{aligned} \mathcal{J}(\beta, \alpha_1, \alpha_2, m_1, m_2, s) &= \prod_{j=1}^s \frac{\Gamma(\Delta_1 + 1 + (j-1)\frac{\beta^2}{4}) \Gamma(\bar{\Delta}_1 + 1 + (j-1)\frac{\beta^2}{4})}{\Gamma(\Delta_1 + \Delta_2 + 2 + (s+j-2)\frac{\beta^2}{4})} \\ &\times \frac{\sin(\pi(\bar{\Delta}_1 + (j-1)\frac{\beta^2}{4})) \sin(\pi(\bar{\Delta}_2 + (j-1)\frac{\beta^2}{4})) \sin(\frac{\pi j \beta^2}{4})}{\sin(\pi(\bar{\Delta}_1 + \bar{\Delta}_2 + (s+j-2)\frac{\beta^2}{4})) \sin(\frac{\pi \beta^2}{4})} \\ &\times \frac{\Gamma(\Delta_2 + 1 + (j-1)\frac{\beta^2}{4}) \Gamma(\bar{\Delta}_2 + 1 + (j-1)\frac{\beta^2}{4}) \Gamma(\frac{j \beta^2}{4})^2}{\Gamma(\bar{\Delta}_1 + \bar{\Delta}_2 + 2 + (s+j-2)\frac{\beta^2}{4}) \Gamma(\frac{\beta^2}{4})^2}. \end{aligned}$$

In case $m_1 = m_2 = 0$, using the formula $\Gamma(x)\Gamma(1-x) = \pi/\sin(\pi x)$ and the relation $\Delta_1 + \Delta_2 + 2 = \beta^2(1-s)/2 - \beta\alpha_3/2$, we recover the formula (7.7). Furthermore, the same relations show that the structure constant appears as a square root of a product of imaginary DOZZ formula. More precisely

Proposition 7.3. *Assume $\sum_{i=1}^3 m_i = 0$ and $\alpha_i > Q$ for $i = 1, 2, 3$, and (7.1). Then for C_{g_0} given in (7.6), we have*

$$\begin{aligned} & \langle V_{(\alpha_1, m_1)}(0, e_1) V_{(\alpha_2, m_2)}(1, e_1) V_{(\alpha_3, m_3)}(\infty, e_1) \rangle_{\hat{c}, g_0}^2 \\ &= C_{g_0}^2 4\pi^2 R^2 \mu^{\frac{2}{\beta}(2Q - \sum \alpha_i)} C_{\beta}^{\text{DOZZ}}(\alpha_1 + m_1 R, \alpha_2 + m_2 R, \alpha_3 + m_3 R) \\ &\times C_{\beta}^{\text{DOZZ}}(\alpha_1 - m_1 R, \alpha_2 - m_2 R, \alpha_3 - m_3 R). \end{aligned}$$

This specific form for the structure constants was already pointed out in [22, 54] in similar context and under some conditions.

8. SEGAL'S AXIOMS

In this section, we prove that our path integral satisfies Segal's axioms for CFT. These axioms are based on the notion of amplitudes: basically an amplitude corresponds to the path integral defined on surfaces with parametrised boundary, with a conditioning on the boundary values of the Liouville field. These amplitudes can then be paired together where the pairing of amplitudes of two surfaces with boundary corresponds to the amplitude of the glued surfaces along the corresponding boundaries. The pairing involves an integration over some functional space, and is associated to a Hilbert space (Section 8.1). Such amplitudes serve to decompose the partition function or correlation functions as a multiple pairing of elementary amplitudes (basically amplitudes of pairs of pants or variants with marked points). This is the philosophy that Segal developed to encode the conformal bootstrap in a geometrical setup.

In order to define the amplitudes properly, we shall need the gluing formalism for Riemann surfaces explained in Section 3.3, and a few facts on Dirichlet-to-Neumann maps recalled in Section 8.2. Next, we give the definition of amplitudes and then study the main properties of the defect graph associated to the electro-magnetic operators for amplitudes in Section 8.3. This is all we need to give the definition of amplitudes in Section 8.4. Finally we shall prove the Segal axioms in Section 8.5.

8.1. Hilbert space. A generic (real valued) Fourier series on the unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\} \simeq \mathbb{R}/2\pi\mathbb{Z}$ will be decomposed into its constant mode c and orthogonal part

$$(8.1) \quad \tilde{\varphi} = c + \varphi, \quad \varphi(\theta) = \sum_{n \neq 0} \varphi_n e^{in\theta}$$

with $(\varphi_n)_{n \neq 0}$ its other Fourier coefficients, which will be themselves parametrised by $\varphi_n = \frac{x_n + iy_n}{2\sqrt{|n|}}$ and $\varphi_{-n} = \frac{x_n - iy_n}{2\sqrt{|n|}}$ for $n > 0$, with $x_n, y_n \in \mathbb{R}$. The Sobolev space $H^s(\mathbb{T})$ is the space of such Fourier series equipped with the norm

$$(8.2) \quad \|\varphi\|_{H^s(\mathbb{T})}^2 := \sum_{n \in \mathbb{Z}} |\varphi_n|^2 (|n| + 1)^{2s} < \infty.$$

In the case when the coefficients x_n, y_n are random variables given by i.i.d. standard real Gaussians with 0 mean and variance 1, then the series (8.1) converges in $H^s(\mathbb{T})$ with $s < 0$. Such a random series arises naturally when considering the restriction of the whole plane GFF to the unit circle. Now we want to add a further equivariant part to the series (8.1), and for this we will proceed in a way similar to Section 3.10. Consider the space $H_{\mathbb{Z}}^s(\mathbb{R})$ to be the space of real-valued distributions u on \mathbb{R} such that their restriction on any finite size open interval belongs to $H^s(\mathbb{R})$ and such that

$$\forall n \in \mathbb{Z}, \quad u(\theta + 2\pi n) - u(\theta) \in 2\pi R\mathbb{Z}.$$

If one restricts to smooth functions in this space, this amounts precisely to the space of smooth functions u on \mathbb{R} such that $e^{\frac{i}{R}u}$ descends to a smooth function on \mathbb{T} . As in Section 3.10 we get an identification (with $\pi : \mathbb{R} \rightarrow \mathbb{R}/2\pi R\mathbb{Z}$ being the projection)

$$\mathbb{Z} \times H^s(\mathbb{T}) \mapsto H_{\mathbb{Z}}^s(\mathbb{R}), \quad (k, \tilde{\varphi}) \mapsto \pi^* \tilde{\varphi}(\theta) + kR\theta$$

and k corresponds to the degree of the map $e^{\frac{i}{R}u} : \mathbb{T} \rightarrow \mathbb{T}$. Below, we will write $\tilde{\varphi}$ instead of $\pi^* \tilde{\varphi}$ and implicitly identify $\tilde{\varphi}$ with its periodic lift to \mathbb{R} .

Consider now the probability space

$$(8.3) \quad \Omega_{\mathbb{T}} = (\mathbb{R}^2)^{\mathbb{N}^*}$$

equipped with the cylinder sigma-algebra $\Sigma_{\mathbb{T}} = \mathcal{B}^{\otimes \mathbb{N}^*}$ (\mathcal{B} stands for the Borel sigma-algebra on \mathbb{R}^2) and the Gaussian product measure

$$(8.4) \quad \mathbb{P}_{\mathbb{T}} := \bigotimes_{n \geq 1} \frac{1}{2\pi} e^{-\frac{1}{2}(x_n^2 + y_n^2)} dx_n dy_n.$$

The push-forward of $\mathbb{P}_{\mathbb{T}}$ by the random variable $(x_n, y_n)_{n \in \mathbb{N}} \rightarrow \varphi$ defined in (8.1) induces a measure, still denoted $\mathbb{P}_{\mathbb{T}}$ by a slight abuse of notation, that is supported on $H^s(\mathbb{T})$ for any $s < 0$ in the sense that $\mathbb{P}_{\mathbb{T}}(\varphi \in H^s(\mathbb{T})) = 1$. We next equip the space \mathbb{Z} with the discrete sigma-algebra and the counting measure $\mu_{\mathbb{Z}} = \sum_{k \in \mathbb{Z}} \delta_k$. Then we consider the Hilbert space

$$\mathcal{H} := L^2((\mathbb{R}/2\pi R\mathbb{Z}) \times \mathbb{Z} \times \Omega_{\mathbb{T}}, \mu_0),$$

where the underlying measure is given by

$$\mu_0 := dc \otimes \mu_{\mathbb{Z}} \otimes \mathbb{P}_{\mathbb{T}}$$

and Hermitian product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, with dc the Lebesgue measure on $\mathbb{R}/2\pi R\mathbb{Z}$. The random variable (with φ defined in (8.1))

$$(c, k, (x_n, y_n)_{n \in \mathbb{N}}) \in \mathbb{R} \times \mathbb{Z} \times \Omega_{\mathbb{T}} \mapsto \tilde{\varphi}^k(\theta) := c + kR\theta + \varphi(\theta) \in H_{\mathbb{Z}}^s(\mathbb{R})$$

induces, by push-forward of $dc \otimes \mu_{\mathbb{Z}} \otimes \mathbb{P}_{\mathbb{T}}$, a measure on $H_{\mathbb{Z}}^s(\mathbb{R})$. This measure is invariant by translation $\tau_R : u \mapsto u + 2\pi R$ on $H_{\mathbb{Z}}^s(\mathbb{R})$ and descends to the quotient $H_{\mathbb{Z}}^s(\mathbb{R})/\langle \tau_R \rangle$ by the group generated by τ_R , and it provides an isomorphism of measure space with $((\mathbb{R}/2\pi R\mathbb{Z}) \times \mathbb{Z} \times \Omega_{\mathbb{T}}, \mu_0)$. By abuse of notation, we also call μ_0 the measure on $H_{\mathbb{Z}}^s(\mathbb{R})/\langle \tau_R \rangle$, and this means that we can also rewrite our Hilbert space as

$$\mathcal{H} \simeq L^2(H_{\mathbb{Z}}^s(\mathbb{R})/\langle \tau_R \rangle, \mu_0).$$

Concretely, this allows to integrate continuous (or measurable) functionals $F : H_{\mathbb{Z}}^s(\mathbb{R}) \rightarrow \mathbb{R}^+$ that are τ_R periodic by the expression

$$\int_{H_{\mathbb{Z}}^s(\mathbb{R})/\langle \tau_R \rangle} F(u) d\mu_0(u) := \int_0^{2\pi R} \sum_{k \in \mathbb{Z}} \mathbb{E}[F(c + kR\theta + \varphi(\theta))] dc.$$

8.2. Dirichlet-to-Neumann map. Let Σ be a compact Riemann surface with real analytic boundary $\partial\Sigma = \sqcup_{j=1}^b \partial_j \Sigma$ consisting of b closed simple curves, which do not intersect each other (here b could possibly be equal to 0 in case $\partial\Sigma = \emptyset$) and parametrised by $\zeta_j : \mathbb{T} \rightarrow \partial_j \Sigma$ as in Section 3.2. We consider a metric g on Σ so that each boundary component has length 2π ; except when mentioned, g is not assumed to be admissible. We denote by $d\ell_g$ the Riemannian measure on $\partial\Sigma$ induced by g .

In what follows, boundary data for amplitudes will be encoded in a family of fields $\tilde{\varphi} = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_b) \in (H^s(\mathbb{T}))^b$, in which case the notation (8.1) referring to the j -th field will be augmented with an index j , namely, $c_j, \varphi_{j,n}, x_{j,n}$ or $y_{j,n}$. We still denote by $\langle \cdot, \cdot \rangle$ the distributional pairing between $(H^s(\mathbb{T}))^b$ and $(H^{-s}(\mathbb{T}))^b$ normalised so that $\langle 1, u \rangle = \frac{1}{2\pi} \int u d\theta$ if $u \in C^\infty(\mathbb{T})$.

For such a field $\tilde{\varphi} = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_b) \in (H^s(\mathbb{T}))^b$ with $s \in \mathbb{R}$, we will write $P\tilde{\varphi}$ for the harmonic extension of $\tilde{\varphi}$, that is $\Delta_g P\tilde{\varphi} = 0$ on $\Sigma \setminus \bigcup_j \partial_j \Sigma$ with boundary values

$P\tilde{\varphi}|_{\partial_j\Sigma} = \tilde{\varphi}_j \circ \zeta_j^{-1}$ for $j = 1, \dots, \mathfrak{b}$. The boundary value has to be understood in the following weak sense: for all $u \in C^\infty(\mathbb{T})$, if ζ_j is the (analytic extension to a small annulus around \mathbb{T} of the) parametrisation of $\partial_j\Sigma$

$$\lim_{r \rightarrow 1^-} \int_0^{2\pi} P\tilde{\varphi}(\zeta_j(re^{i\theta}))u(e^{i\theta}) \, d\theta = 2\pi\langle \tilde{\varphi}_j, u \rangle.$$

The definition of our amplitudes will involve the Dirichlet-to-Neumann operator (DN map for short). Recall that the DN map $\mathbf{D}_\Sigma : C^\infty(\mathbb{T})^{\mathfrak{b}} \rightarrow C^\infty(\mathbb{T})^{\mathfrak{b}}$ is defined as follows: for $\tilde{\varphi} \in C^\infty(\mathbb{T})^{\mathfrak{b}}$

$$\mathbf{D}_\Sigma \tilde{\varphi} = (-\partial_\nu P\tilde{\varphi}|_{\partial_j\Sigma} \circ \zeta_j)_{j=1, \dots, \mathfrak{b}},$$

where ν is the inward unit normal vector field to \mathcal{C}_j . Note that \mathbf{D}_Σ is a non-negative symmetric operator with kernel $\ker \mathbf{D}_\Sigma = \mathbb{R}\tilde{\mathbf{1}}$ where $\tilde{\mathbf{1}} = (1, \dots, 1)$. Indeed, by Green's formula, for $\tilde{\varphi} \in C^\infty(\mathbb{T})^{\mathfrak{b}}$

$$(8.5) \quad \int_\Sigma |dP\tilde{\varphi}|_g^2 \, dv_g = 2\pi\langle \tilde{\varphi}, \mathbf{D}_\Sigma \tilde{\varphi} \rangle.$$

Consider \mathfrak{b}' parametrised analytic closed simple non-overlapping curves $\zeta'_j : \mathbb{T} \rightarrow \mathcal{C}'_j$ in the interior of Σ and denote $\mathcal{C}' := \sqcup_{j=1}^{\mathfrak{b}'} \mathcal{C}'_j$. For a field $\tilde{\varphi} = (\tilde{\varphi}_1, \dots, \tilde{\varphi}_{\mathfrak{b}'}) \in (H^s(\mathbb{T}))^{\mathfrak{b}'}$ with $s \in \mathbb{R}$, we will write $P_{\mathcal{C}'}\tilde{\varphi}$ for the harmonic extension $\Delta_g P_{\mathcal{C}'}\tilde{\varphi} = 0$ on $\Sigma \setminus \mathcal{C}'$ with boundary value 0 on $\partial\Sigma$ and equal to $\tilde{\varphi}_j \circ \zeta'_j$ on \mathcal{C}'_j for $j = 1, \dots, \mathfrak{b}'$. The DN map $\mathbf{D}_{\Sigma, \mathcal{C}'} : C^\infty(\mathbb{T})^{\mathfrak{b}'} \rightarrow C^\infty(\mathbb{T})^{\mathfrak{b}'}$ associated to \mathcal{C}' is defined as the jump at \mathcal{C}' of the harmonic extension: for $\tilde{\varphi} \in C^\infty(\mathbb{T})^{\mathfrak{b}'}$

$$(8.6) \quad \mathbf{D}_{\Sigma, \mathcal{C}'} \tilde{\varphi} := -((\partial_{\nu_-} P_{\mathcal{C}'}\tilde{\varphi})|_{\mathcal{C}'_j} + (\partial_{\nu_+} P_{\mathcal{C}'}\tilde{\varphi})|_{\mathcal{C}'_j})_{j=1, \dots, \mathfrak{b}'}$$

Here ∂_{ν_\pm} denote the two inward normal derivatives along \mathcal{C}'_j from the right and from the left. Since $P_{\mathcal{C}'}(\tilde{\varphi})$ is not C^1 at \mathcal{C}' but it is piecewise smooth, the two normal derivatives are well-defined. The operator $\mathbf{D}_{\Sigma, \mathcal{C}'}$ is invertible and the Schwartz kernel of $\mathbf{D}_{\Sigma, \mathcal{C}'}^{-1}$ is (see e.g. the proof of [15, Theorem 2.1])

$$(8.7) \quad \mathbf{D}_{\Sigma, \mathcal{C}'}^{-1}(y, y') = \frac{1}{2\pi} G_{g, D}(y, y'), \quad y \neq y' \in \mathcal{C}'.$$

For respectively $k = \mathfrak{b}$ or $k = \mathfrak{b}'$, we let $\mathbf{D} : H^1(\mathbb{T})^k \rightarrow L^2(\mathbb{T})^k$ defined by

$$(8.8) \quad \forall \tilde{\varphi} \in C^\infty(\mathbb{T}; \mathbb{R})^k, \quad \langle \mathbf{D}\tilde{\varphi}, \tilde{\varphi} \rangle_2 := 2 \sum_{j=1}^k \sum_{n>0} n |\varphi_{j,n}|^2 = \frac{1}{2} \sum_{j=1}^k \sum_{n>0} ((x_{j,n})^2 + (y_{j,n})^2),$$

and we define the operators on $C^\infty(\mathbb{T})^{\mathfrak{b}}$ and $C^\infty(\mathbb{T})^{\mathfrak{b}'}$

$$(8.9) \quad \tilde{\mathbf{D}}_\Sigma := \mathbf{D}_\Sigma - \mathbf{D}, \quad \tilde{\mathbf{D}}_{\Sigma, \mathcal{C}'} := \mathbf{D}_{\Sigma, \mathcal{C}'} - 2\mathbf{D}.$$

We recall from [32, Lemma 4.1] that the operators $\tilde{\mathbf{D}}_\Sigma$ and $\tilde{\mathbf{D}}_{\Sigma, \mathcal{C}'}$ are smoothing operators in the sense that they are operators with smooth Schwartz kernel that are bounded for all $s, s' \in \mathbb{R}$ as maps

$$(H^s(\mathbb{T}))^{\mathfrak{b}} \rightarrow (H^{s'}(\mathbb{T}))^{\mathfrak{b}}, \text{ respectively } (H^s(\mathbb{T}))^{\mathfrak{b}'} \rightarrow (H^{s'}(\mathbb{T}))^{\mathfrak{b}'}$$

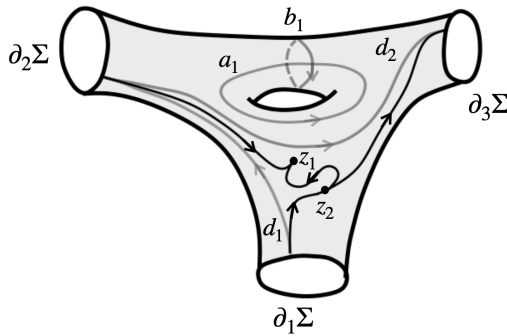


FIGURE 15. Homology on open surfaces. In gray, relative homology. In black, defect graph.

8.3. Curvature terms in the case with boundary. In what follows, we consider a Riemann surface Σ with an analytic parametrisation $\zeta = (\zeta_1, \dots, \zeta_b)$ of the boundary with $\zeta_j : \mathbb{T} \rightarrow \partial_j\Sigma$ where $\partial_1\Sigma, \dots, \partial_b\Sigma$ are the $b > 0$ boundary components of Σ . We consider the following data that we call *geometric data* of Σ :

Geometric data of Σ :

(i) Let g be an admissible metric on Σ , let x_0 be a base point chosen to be on the boundary $\partial\Sigma$ and distinct from $p_j := \zeta_j(1)$ for $j = 1, \dots, b$. Let $\mathbf{z} := (z_1, \dots, z_{n_m})$ be some marked points in its interior Σ° and let $\mathbf{v} = ((z_1, v_1), \dots, (z_{n_m}, v_{n_m})) \in (T\Sigma)^{n_m}$ be unit tangent vectors at these points, to which we attach magnetic charges $\mathbf{m} = (m_1, \dots, m_{n_m}) \in \mathbb{Z}^{n_m}$.

(ii) Let us fix a canonical geometric basis $\sigma := (\sigma_1, \dots, \sigma_{2g+b-1})$ of the relative homology $\mathcal{H}_1(\Sigma, \partial\Sigma)$, as defined below (3.9), consisting of $2g$ interior cycles $(\sigma_1, \dots, \sigma_{2g}) = (a_1, b_1, \dots, a_g, b_g)$ satisfying the intersection pairings

$$\iota(a_j, b_i) = \delta_{ij}, \quad \iota(a_i, a_j) = 0, \quad \iota(b_i, b_j) = 0,$$

and $b-1$ arcs $(\sigma_{2g+1}, \dots, \sigma_{2g+b-1}) = (d_1, \dots, d_{b-1})$ with endpoints on the boundary and no intersection with $\cup_j (a_j \cup b_j)$. We consider a basis $\omega_1^c, \dots, \omega_{2g+b-1}^c$ of $\mathcal{H}_R^1(\Sigma, \partial\Sigma)$ dual to σ , made of closed forms. Recall from Lemma 3.5 that these forms can be chosen to be compactly supported inside Σ° , and we shall use such compactly supported basis to prove the gluing property of amplitudes. We ask the arc d_j to have endpoints at $p_j = \zeta_j(1) \in \partial_j\Sigma$ and $p_{j+1} = \zeta_{j+1}(1) \in \partial_{j+1}\Sigma$ while making an (non-oriented) angle $\pi/2$ with $\partial_j\Sigma$ at the endpoints. The orientation of d_j can be taken either way, this will not play any role later. Then for each $\mathbf{k}^c = (k_1^c, \dots, k_{2g+b-1}^c) \in \mathbb{Z}^{2g+b-1}$, the form $\omega_{\mathbf{k}^c}^c := \sum_{j=1}^{2g+b-1} k_j^c \omega_j^c$ satisfies $\int_{\sigma_i} \omega_{\mathbf{k}^c}^c = 2\pi k_i^c R$. See Figure 15.

(iii) We encode the structure of the absolute cohomology and the magnetic operators in the closed 1-forms $\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}$ of Proposition 3.10 using the basis of $\mathcal{H}_1(\Sigma)$

$$(a_1, b_1, \dots, a_g, b_g, \partial_1\Sigma, \dots, \partial_{b-1}\Sigma),$$

where $\nu_{\mathbf{z},\mathbf{k}}$ are labeled by $\mathbf{k} = (k_1, \dots, k_b) \in \mathbb{Z}^b$ satisfying

$$(8.10) \quad \frac{1}{2\pi R} \int_{\partial_j \Sigma} \nu_{\mathbf{z},\mathbf{k}} = \zeta_j k_j, \quad \sum_{j=1}^{n_m} m_j + \sum_{j=1}^b \zeta_j k_j = 0,$$

where we recall that $\zeta_j = -1$ if the boundary is outgoing and $\zeta_j = 1$ if it is incoming. The choice of such a form is not unique: indeed the difference of two such forms is an exact 1-form df with $\partial_\nu f|_{\partial \Sigma} = 0$ (see Lemma 3.4). By possibly adding such an exact form, we can require that the form $\nu_{\mathbf{z},\mathbf{k}}$ takes values in $2\pi R\mathbb{Z}$ along the boundary-to-boundary arcs:

$$(8.11) \quad \forall j = 2g + 1, \dots, 2g + b - 1, \quad \int_{\sigma_j} \nu_{\mathbf{z},\mathbf{k}} \in 2\pi R\mathbb{Z}.$$

(iv) Next we construct a defect graph associated to the 1-forms $\nu_{\mathbf{z},\mathbf{k}}$ as follows. The construction, detailed below, is similar to the case of closed Riemann surfaces except that we will see the point $\zeta_j(1) \in \partial_j \Sigma$ as extra marked points with a magnetic charge $\zeta_j k_j$ assigned. So, for notational simplifications, we set $z_{n_m+j} = \zeta_j(1)$ and $m_{n_m+j} = \zeta_j k_j$ for $j = 1, \dots, b$. For $j > n_m$, we also choose $v_j \in T_{z_j} \Sigma$ to be unit normal vector to $\partial \Sigma$, it can be either inward $v_j = \nu$ or outward $v_j = -\nu$, and the orientation \pm will play a role.

We then associate the total magnetic charges (which are now \mathbf{k} dependent) defined by

$$(8.12) \quad \mathbf{m}(\mathbf{k}) = (m_1(\mathbf{k}), \dots, m_{n_m+b}(\mathbf{k})) := (m_1, \dots, m_{n_m}, \zeta_1 k_1, \dots, \zeta_b k_b).$$

Definition 8.1 (Defect graph). We consider a family of $n_m + b - 1$ arcs as follows:

- these arcs are indexed by $p \in \{1, \dots, n_m + b - 1\}$, are simple and do not intersect except eventually at their endpoints,
- each arc is a smooth oriented curve $\xi_p : [0, 1] \rightarrow \Sigma$ with endpoints $\xi_p(0) = z_j$ and $\xi_p(1) = z_{j'}$ for $j \neq j'$.
- these arcs satisfy $\dot{\xi}_p(0) = \lambda_{p,j} v_j$ and $\dot{\xi}_p(1) = \lambda_{p,j'} v_{j'}$ for some $\lambda_{p,j} > 0$ and $\lambda_{p,j'} > 0$ if $\xi_p(1) \notin \partial \Sigma$, while $\lambda_{p,j'} < 0$ if $\xi_p(1) \in \partial \Sigma$.
- consider the oriented graph with vertices \mathbf{z} and edges $(z_j, z_{j'})$, if there is an oriented arc with basepoint z_j and endpoint $z_{j'}$. This graph must be connected and without cycle, that is, there is no sequence of edges $(z_{j_1}, z_{j_2}), \dots, (z_{j_k}, z_{j_{k+1}})$ with $j_1 = j_{k+1}$.

In what follows, the union $\mathcal{D}_{\mathbf{v},\xi} := \bigcup_{p \in \{1, \dots, n_m + b - 1\}} \xi_p([0, 1])$ will be called the defect graph associated to \mathbf{v} and the collection of arcs $\xi := (\xi_1, \dots, \xi_{n_m + b - 1})$. See Figure 16.

Notice that the graph $\mathcal{D}_{\mathbf{v},\xi}$ is contractible to a point. The form $\nu_{\mathbf{z},\mathbf{k}}$ is exact on $\Sigma \setminus \mathcal{D}_{\mathbf{v},\xi}$ (the integral of this 1-form vanishes over any cycle in $\Sigma \setminus \mathcal{D}_{\mathbf{v},\xi}$) so that we can consider the primitive $I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{k}})$ on $\Sigma \setminus \mathcal{D}_{\mathbf{v},\xi}$. As in the closed case (recall (4.4)), we assign to each arc ξ_p a value $\kappa(\xi_p) \in 2\pi R\mathbb{Z}$, corresponding to the difference of the values of $I_{x_0}^\xi(\nu_{\mathbf{z},\mathbf{k}})$ on both sides of the arc. The value $\kappa(\xi_p)$ is defined by first gluing disks \mathcal{D}_j to $\partial_j \Sigma$ for each j in order to be in the setting of a closed surface and then by using the same definition as in the closed case, see (4.4).

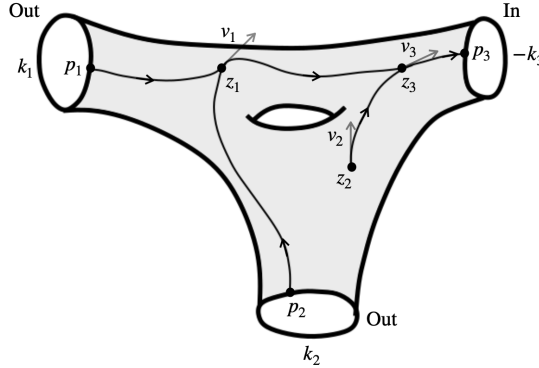


FIGURE 16. Example of defect graph (black) in case $k_2 \leq k_1 \leq m_1 \leq m_2 \leq m_3 \leq -k_3$

The regularised curvature terms are then defined by the same formula as (4.2) and (4.5):

$$(8.13) \quad \int_{\Sigma_\sigma}^{\text{reg}} K_g I_{x_0}^\sigma(\omega_{\mathbf{k}^c}^c) dv_g := \int_{\Sigma_\sigma} I_{x_0}^\sigma(\omega_{\mathbf{k}^c}^c) K_g dv_g + 2 \sum_{j=1}^g \left(\int_{a_j} \omega_{\mathbf{k}^c}^c \int_{b_j} k_g d\ell_g - \int_{b_j} \omega_{\mathbf{k}^c}^c \int_{a_j} k_g d\ell_g \right),$$

$$(8.14) \quad \int_{\Sigma}^{\text{reg}} I_{x_0}^\xi(\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}) K_g dv_g := \int_{\Sigma \setminus \mathcal{D}_{\nu, \xi}} I_{x_0}^\xi(\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}) K_g dv_g - 2 \sum_{p=1}^{n_m + b - 1} \kappa(\xi_p) \int_{\xi_p} k_g d\ell_g,$$

where k_g is the geodesic curvature as defined in (4.3). Remark that in the expression (8.13), there is no boundary term involving the arcs d_j or the boundary cycles c_j : the reason is that the curves c_j will be chosen to be geodesics (since our metrics are admissible) and that $\iota_{\partial \Sigma}^* \omega_{\mathbf{k}^c}^c = 0$ (thus $\int_{\partial_j \Sigma} \omega_{\mathbf{k}^c}^c = 0$) so that the natural boundary terms that one could add actually vanish:

$$\int_{\partial_j \Sigma} \omega_{\mathbf{k}^c}^c \int_{d_j} k_g d\ell_g - \int_{d_j} \omega_{\mathbf{k}^c}^c \int_{c_j} k_g d\ell_g = 0.$$

With the same proof as Lemma 4.3, we have

Lemma 8.2. *Let $\hat{g} = e^\rho g$ with $\partial_\nu \rho|_{\partial \Sigma} = 0$.*

(1) *if $\omega \in \mathcal{H}_R^1(\Sigma, \partial \Sigma)$ with compact support in Σ° , then*

$$\int_{\Sigma_\sigma}^{\text{reg}} I_{x_0}^\sigma(\omega) K_{\hat{g}} dv_{\hat{g}} = \int_{\Sigma_\sigma}^{\text{reg}} I_{x_0}^\sigma(\omega) K_g dv_g + \langle d\rho, \omega \rangle_2.$$

(2) *the regularised magnetic curvature term defined by (8.14) satisfies*

$$\int_{\Sigma}^{\text{reg}} I_{x_0}^\xi(\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}) K_{g'} dv_{g'} = \int_{\Sigma}^{\text{reg}} I_{x_0}^\xi(\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}) K_g dv_g + \langle d\rho, \nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}} \rangle_2.$$

Similarly to Lemma 4.5, we also have

Lemma 8.3. *Let $\sigma = (\sigma_j)_{j=1, \dots, 2g+b-1}$ and $\sigma' = (\sigma'_j)_{j=1, \dots, 2g+b-1}$ be canonical geometric bases as described above and $\omega \in \mathcal{H}_R^1(\Sigma, \partial\Sigma)$ with compact support in Σ° . Then the following identity holds true*

$$(8.15) \quad \int_{\Sigma_\sigma}^{\text{reg}} K_g I_{x_0}^\sigma(\omega) \, dv_g - \int_{\Sigma_{\sigma'}}^{\text{reg}} K_g I_{x_0}^{\sigma'}(\omega) \, dv_g \in 8\pi^2 RZ.$$

Proof. First, we show that we can change the arcs $(d_j)_j$ relating boundary circles of Σ in a way that each d_j has its endpoints on $\partial_1 \Sigma$ and $\partial_{j+1} \Sigma$ without changing the $(a_j, b_j)_j$, and so that (8.15) holds if σ' is the changed graph. It suffices to show that when only one arc d_j is modified to another arc d'_j such that d_j and d'_j both have one endpoint on, say, the boundary circle $\partial_\ell \Sigma$ then the formula (8.15) holds. Since $\int_{\partial_i \Sigma} \omega = 0$ for each $i = 1, \dots, b$ we claim that $I_{x_0}^{\sigma'}(\omega) = I_{x_0}^\sigma(\omega)$ on $\Sigma \setminus \cup_{j=1}^g (a_j \cup b_j)$. Indeed, any loop γ_{x_0} passing by x_0 and not intersecting $\cup_{j=1}^g (a_j \cup b_j)$ must be a linear combination of the boundary circles c_j in $\mathcal{H}_1(\Sigma)$: to see this, it suffices to glue a disk to each boundary circle, we obtain a closed surface $\hat{\Sigma}$ of genus g , and notice that the intersection number of γ_{x_0} with each element of a basis of $\mathcal{H}_1(\hat{\Sigma})$ is 0 (recall that the intersection number is a symplectic form, thus non-degenerate), thus $[\gamma_{x_0}] = 0$ in $\mathcal{H}_1(\hat{\Sigma})$. We deduce that $\int_{\Sigma_\sigma}^{\text{reg}} K_g I_{x_0}^\sigma(\omega) \, dv_g = \int_{\Sigma_{\sigma'}}^{\text{reg}} K_g I_{x_0}^{\sigma'}(\omega) \, dv_g$ and, applying this inductively, this shows that the arcs d_j for $j = 1, \dots, b-1$ can be modified in a way that each d_j has its endpoints on $\partial_1 \Sigma$ and $\partial_{j+1} \Sigma$ without changing the $(a_j, b_j)_{j=1 \dots g}$. Next, we can double the surface $\Sigma^{\#2} = \Sigma \# \Sigma$, view $(\sigma_j)_{j \leq 2g}$ as cycles of $\Sigma^{\#2}$ contained in the right copy of σ , and let τ denote the natural involution on $\Sigma^{\#2}$. The cycles

$$(8.16) \quad (a_1, b_1, \dots, a_g, b_g, -\tau(a_1), \tau(b_1), \dots, -\tau(a_g), \tau(b_g)) \in \mathcal{H}_1(\Sigma^{\#2})$$

form a part of a geometric symplectic basis of $\mathcal{H}_1(\Sigma^{\#2})$, which can be completed into a full geometric symplectic basis denoted $\sigma^\#$ by adding $b-1$ cycles $c_1 = \partial_2 \Sigma, \dots, c_{b-1} = \partial_b \Sigma$ coming from boundary cycles of Σ and $b-1$ non-intersecting cycles d_1, \dots, d_{b-1} with intersection pairing $\iota(c_i, d_j) = \delta_{ij}$ and d_j not intersecting any a_i, b_i . Let us extend ω by 0 from the left copy of Σ to $\Sigma^{\#2}$. Since the integral $\int_{c_i} \omega = 0$ for all i and since c_i are also geodesic curves, the geodesic curvature terms in the regularised integral on $\Sigma^{\#2}$ coming from the cycles are 0 except for those in (8.16). The regularised integral of ω on $\Sigma^{\#2}$ satisfies

$$\int_{\Sigma^{\#2}}^{\text{reg}} K_g I_{x_0}^{\sigma^\#}(\omega) \, dv_g = \int_{\Sigma_\sigma}^{\text{reg}} K_g I_{x_0}^\sigma(\omega) \, dv_g.$$

We can then apply Lemma 4.5 on $\Sigma^{\#2}$ to a change of geometric symplectic basis which preserves all the cycles contained in the left copy of $\Sigma^{\#2}$ and maps $(a_i, b_i)_{i \leq g}$ to other elements $(a'_i, b'_i)_{i \leq g}$ contained in the interior of the right copy of Σ . This proves the desired result. □

The invariance by diffeomorphism is proved in the same exact way as Lemma 4.4 and reads as follows:

Lemma 8.4. *For $\psi : \Sigma \rightarrow \Sigma$ an orientation preserving diffeomorphism and σ a canonical geometric basis of $\mathcal{H}_1(\Sigma, \partial\Sigma)$, let $\psi(\sigma)$ be the image canonical geometric basis of*

$\mathcal{H}_1(\Sigma, \partial\Sigma)$. Let $x_0 \in \Sigma$ and $\omega \in \mathcal{H}_R^1(\Sigma, \partial\Sigma)$ with compact support in Σ° . The following identity then holds true

$$\int_{\Sigma_\sigma}^{\text{reg}} K_g I_{x_0}^\sigma(\omega) \, dv_g = \int_{\Sigma_{\psi(\sigma)}}^{\text{reg}} K_{\psi_*g} I_{\psi(x_0)}^{\psi(\sigma)}(\psi_*\omega) \, dv_{\psi_*g}.$$

Now we state all the main properties of defect graphs needed in the sequel. We claim

Lemma 8.5. *The magnetic regularised curvature term only depends on the points $\mathbf{z} \in \Sigma^{n_m+b}$, the charges $\mathbf{m} \in \mathbb{Z}^{n_m+b}$ and the unit tangent vectors \mathbf{v} but not on the defect graph (i.e. the choice of arcs constructed from this data).*

Proof. By gluing disks \mathcal{D}_j to each boundary circle $\partial_j\Sigma$, with an admissible metric on each disk, we see that the problem just reduces to the case of a closed manifold, thus to Lemma 4.9. □

Lemma 8.6. *Let $\psi : \Sigma' \rightarrow \Sigma$ be an orientation preserving diffeomorphism. The regularised magnetic curvature term defined by (8.14) satisfies the relation*

$$\int_{\Sigma'}^{\text{reg}} I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m},\mathbf{k}})K_g \, dv_g = \int_{\Sigma}^{\text{reg}} I_{\psi(x_0)}^{\psi \circ \xi}(\psi_*\nu_{\mathbf{z},\mathbf{m},\mathbf{k}})K_{\psi_*g} \, dv_{\psi_*g}.$$

Proof. The proof is the same as that of Lemma 4.4. □

Now we treat the additivity of regularised curvature integrals. First we treat the case when we glue two Riemann surfaces:

Lemma 8.7. *Consider two Riemannian surfaces (Σ_i, g_i) for $i = 1, 2$ with admissible metrics, with $b_i > 0$ boundary components and genus g_i for $i = 1, 2$ and with analytic parametrisations ζ_i of their respective boundary. Assume $\partial_{b_1}\Sigma_1$ is outgoing and $\partial_{b_2}\Sigma_2$ is incoming and let $\Sigma = \Sigma_1 \# \Sigma_2$ be the glued surface obtained by identifying $\partial_{b_1}\Sigma_1$ with $\partial_{b_2}\Sigma_2$ and g be the metric induced by g_1, g_2 . Consider the geometric data (i), (ii), (iii), (iv) described above for Σ_1 and for Σ_2 , and denote them with either a subscript i or a superscript i when they are associated to Σ_i , for example $\sigma_i, \mathbf{k}_i^c, \xi_i, x_0^i, \nu_{z_i, m_i, k_i}^i, \omega_{k_i^c}^{i,c}$ etc.*

We choose these defect graphs by imposing that there is only one arc ξ_{1j_0} having $\zeta_{1b_1}(1)$ as endpoint on Σ_1 and only one arc $\xi_{2j'_0}$ having $\zeta_{2b_2}(1)$ as endpoint on Σ_2 . We assume that the marked point in $\{\mathbf{z}_1\}$ that belongs to $\partial_{b_1}\Sigma_1$ is glued to the marked point in $\{\mathbf{z}_2\}$ that belongs to $\partial_{b_2}\Sigma_2$ and that the associated tangent vectors to the defect graphs of Σ_1 and Σ_2 match (i.e. one is inward normal in Σ_1 and the other outward normal in Σ_2 , or conversely), so that the defect graph of Σ_1 glues well with the defect graph of Σ_2 . Assume $k_{1b_1} = k_{2b_2}$ and set $\mathbf{z} := (\mathbf{z}_1, \mathbf{z}_2)$, $\mathbf{m} := (\mathbf{m}_1, \mathbf{m}_2)$ and $\mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2)$. Then we obtain a defect graph $\mathcal{D}_{\mathbf{v}, \xi}$ on Σ by gathering all the arcs in ξ_1 but ξ_{1j_0} , all the arcs in ξ_2 but $\xi_{2j'_0}$, and the arc ξ obtained by gluing the arcs ξ_{1j_0} and $\xi_{2j'_0}$ with orientation. Furthermore, if $\mathbf{k} := (\mathbf{k}_1^-, \mathbf{k}_2^-)$ where $\mathbf{k}_i^- \in \mathbb{Z}^{b_i-1}$ stands for the vector \mathbf{k}_i with its b_i -th component removed and if one glues together the 1-forms ν_{z_i, m_i, k_i}^i , $i = 1, 2$, to get the 1-form $\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}$ on Σ , then

$$\int_{\Sigma}^{\text{reg}} I_{x_0}^{\xi}(\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}})K_g \, dv_g = \int_{\Sigma_1}^{\text{reg}} I_{x_0^1}^{\xi_1}(\nu_{z_1, m_1, k_1}^1)K_{g_1} \, dv_{g_1} + \int_{\Sigma_2}^{\text{reg}} I_{x_0^2}^{\xi_2}(\nu_{z_2, m_2, k_2}^2)K_{g_2} \, dv_{g_2},$$

where x_0^1 and x_0^2 are two base points respectively on $\partial_{b_1}\Sigma_1 \subset \Sigma_1$ and $\partial_{b_2}\Sigma_2 \subset \Sigma_2$, which we assume are identified under gluing to $x_0 = x_0^1 = x_0^2 \in \Sigma$. Similarly, let $\sigma = \sigma_1 \# \sigma_2$

be the gluing of the canonical geometric bases σ_1 and σ_2 from Lemma 3.7, assume that $\omega_{\mathbf{k}_i^c}^{i,c}$ are compactly supported in Σ_i° and let $\omega_{\mathbf{k}^c} := \omega_{\mathbf{k}_1^c}^{1,c} + \omega_{\mathbf{k}_2^c}^{2,c}$ for $\mathbf{k}^c = (\mathbf{k}_1^c, \mathbf{k}_2^c)$. Then the following holds true

$$(8.18) \quad \int_{\Sigma}^{\text{reg}} I_{x_0}^{\sigma}(\omega_{\mathbf{k}^c}^c) K_g \, dv_g = \int_{\Sigma_1}^{\text{reg}} I_{x_0^1}^{\sigma_1}(\omega_{\mathbf{k}_1^c}^{1,c}) K_{g_1} \, dv_{g_1} + \int_{\Sigma_2}^{\text{reg}} I_{x_0^2}^{\sigma_2}(\omega_{\mathbf{k}_2^c}^{2,c}) K_{g_2} \, dv_{g_2}.$$

Proof. The fact that $\mathcal{D}_{\mathbf{v},\xi}$ is a defect graph is clear. Since ξ is obtained by gluing of the arcs ξ_{1j_0} and $\xi_{2j'_0}$, then $I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m},\mathbf{k}})|_{\Sigma_i} = I_{x_0^i}^{\xi_i}(\nu_{z_i,\mathbf{m}_i,\mathbf{k}_i}^i)$ (indeed the branch cuts remain unchanged). What is less straightforward is the additivity of regularised curvature integrals. It results from a simple observation concerning the computation of the coefficients $\kappa(\xi_{ip})$ on Σ_i . Let us begin with Σ_1 (Σ_2 is similar). To compute the coefficient $\kappa(\xi_{1p})$ of the arc ξ_{1p} (for $p \neq j_0$) in the defect graph associated to Σ_1 , recall (see after Definition 8.1) that we consider a positively oriented closed contractible curve $\alpha_x : [0, 1]$ in the closed surface $\hat{\Sigma}_1$ (obtained by gluing disks at the boundary components of Σ_1), with $x = \alpha_x(0) = \xi_{1p}(t) = \alpha_x \cap \xi$ for some t and $\dot{\alpha}_x(0) = J \dot{\xi}_{1p}(t)$; note that α_x bounds a disk D_α with positively oriented boundary. If D_α does not contain $\zeta_{1b_1}(1)$, this means the same curve α_x can be used to compute $\kappa(\xi_{1p})$, both when the arc ξ_{1p} is seen as an element of the defect graph of Σ_1 and Σ , hence it takes the same value in both cases. If the point $\zeta_{1b_1}(1)$ belongs to D_α , then its contribution to $\kappa(\xi_{1p})$ is $-2\pi R k_{1b_1}$ (recall that $\partial_{b_1} \Sigma_1$ is outgoing). Next, if we compute the coefficient $\kappa(\xi_{1p})$ for the arc ξ_{1p} seen as an arc in the defect graph $\mathcal{D}_{\mathbf{v},\xi}$ on Σ , the curve α_x bounds a disk containing all the points of the defect graphs located in Σ_2 as well, producing a total contribution $2\pi R(\sum_{j=1}^{n_m^{(2)}} m_{2j} + \sum_{j=1}^{b_2-1} \zeta_{2j} k_{2j})$, which by (8.10) is equal to $-2\pi R k_{2b_2} = -2\pi R k_{1b_1}$. Hence $\kappa(\xi_{1p})$ has the same value when viewed in the graph $\mathcal{D}_{\mathbf{v},\xi}$ or in $\mathcal{D}_{\mathbf{v}_1,\xi_1}$. The situation is slightly different if we compute $\kappa(\xi_{1j_0})$ because of the orientation of $\xi|_{\Sigma_1}$. Of course, if this orientation is the same as ξ_{1j_0} then the argument is the same as above. If the orientation is reversed, then the sum defining $\kappa(\xi)$ in $\mathcal{D}_{\mathbf{v},\xi}$ involves the complement of the charges used to compute $\kappa(\xi_{1j_0})$ in Σ_1 , and since the total sum of all charges is 0, this means that $\kappa(\xi|_{\Sigma_1}) = -\kappa(\xi_{1j_0})$. But changing the orientation also changes the sign of the geodesic curvature so that

$$\kappa(\xi_{1j_0}) \int_{\xi_{1j_0}} k_g \, d\ell_g = \kappa(\xi|_{\Sigma_1}) \int_{\xi|_{\Sigma_1}} k_g \, d\ell_g.$$

The argument is the same on Σ_2 , and all in all, this proves the relation (8.17).

The identity (8.18) is clear, using that $\omega_{\mathbf{k}_i^c}^{i,c}$ vanishes near the boundaries $\partial\Sigma_i$. □

It remains to consider the case of self-gluing. We claim:

Lemma 8.8. *Consider a Riemann surface Σ with $\mathfrak{b} \geq 2$ boundary components and with analytic parametrisations ζ of their respective boundary. We consider marked points $\mathbf{z}_1 := (z_1, \dots, z_{n_m})$ on Σ and associated respective magnetic charges $\mathbf{m} := (m_1, \dots, m_{n_m})$ and unit tangent vectors $\mathbf{v} = ((z_1, v_1), \dots, (z_n, v_{n_m})) \in (T\Sigma)^{n_m}$ at the points z_j . We assume $\partial_{b-1}\Sigma$ is outgoing and $\partial_b\Sigma$ is incoming. Finally we consider $\mathbf{k} \in \mathbb{Z}^{\mathfrak{b}}$ and the 1-forms $\nu_{\mathbf{z},\mathbf{m},\mathbf{k}}$ on Σ given by Proposition 3.10 such that (recall that $\zeta_j = -1$ is the boundary $\partial_j\Sigma$ is*

outgoing and $\zeta_j = 1$ if it is incoming)

$$\begin{aligned} \forall j = 1, \dots, \mathfrak{b}, \quad & \int_{\partial_j \Sigma} \nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}} = 2\pi R \zeta_j k_j, \\ \forall j = 1, \dots, n_m, \quad & \int_{d_g(z_j, \cdot) = \epsilon} \nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}} = 2\pi R m_j, \\ & \sum_{j=1}^{n_m} m_j + \sum_{j=1}^{\mathfrak{b}} \zeta_j k_j = 0 \end{aligned}$$

and a defect graph $\mathcal{D}_{\mathbf{v}, \xi}$ on Σ . We choose this defect graph by imposing that

- there is only one arc ξ_{j_0} having $\zeta_{\mathfrak{b}}(1)$ as endpoint, and that this arc has $\zeta_{\mathfrak{b}-1}(1)$ as other endpoint, this arc is oriented according to the tangent vectors at the marked points $\zeta_{\mathfrak{b}}(1)$ and $\zeta_{\mathfrak{b}-1}(1)$ (one must be inward normal to $\partial \Sigma$, the other outward normal),
- if $\mathfrak{b} + n_m > 2$ there are exactly two arcs with $\zeta_{\mathfrak{b}-1}(1)$ as endpoint (thus including ξ_{j_0}). We call $\xi_{j'_0}$ this second arc.

Let $\Sigma^\#$ be the Riemann surface obtained by self-gluing Σ by identifying $\partial_{\mathfrak{b}-1} \Sigma$ and $\partial_{\mathfrak{b}} \Sigma$ using the corresponding parametrisations, and we assume $k_{\mathfrak{b}-1} = k_{\mathfrak{b}}$. Then we can obtain a defect graph $\mathcal{D}_{\mathbf{v}, \xi^\#}$ on $\Sigma^\#$ by taking all the arcs in ξ and removing the arcs having $\zeta_{\mathfrak{b}-1}(1)$ or $\zeta_{\mathfrak{b}}(1)$ as endpoints. The curves ξ_{j_0} and $\zeta_{\mathfrak{b}-1}$ intersect in 1 point. Let $\sigma^\#$ be a basis of the relative homology on $\Sigma^\#$ having both ξ_{j_0} and $\zeta_{\mathfrak{b}-1}$ has interior cycles.

We write $\mathbf{k} \in \mathbb{Z}^{\mathfrak{b}}$ as $\mathbf{k} = (\mathbf{k}_-, k_{\mathfrak{b}-1}, k_{\mathfrak{b}}) \in \mathbb{Z}^{\mathfrak{b}-2} \times \mathbb{Z} \times \mathbb{Z}$. Then $\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}$ can be split as $\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}} = \nu_{\mathbf{z}, \mathbf{m}, (\mathbf{k}_-, 0, 0)} + \nu_{\mathbf{z}, \mathbf{0}, (0, k_{\mathfrak{b}}, k_{\mathfrak{b}})}$, and each 1-form involved in this expression makes sense as 1-form on $\Sigma^\#$ by Lemma 3.8. Then

$$(8.19) \quad \begin{aligned} \int_{\Sigma}^{\text{reg}} I_{x_0}^{\xi}(\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}) K_g d\nu_g &= \int_{\Sigma^\#}^{\text{reg}} I_{x_0}^{\xi^\#}(\nu_{\mathbf{z}, \mathbf{m}, (\mathbf{k}_-, 0, 0)}) K_g d\nu_g \\ &+ \int_{\Sigma^\#}^{\text{reg}} I_{x_0}^{\sigma^\#}(\nu_{\mathbf{z}, \mathbf{0}, (0, k_{\mathfrak{b}}, k_{\mathfrak{b}})}) K_g d\nu_g. \end{aligned}$$

Proof. We first treat the case when $\mathfrak{b} + n_m > 2$. It is obvious that $\mathcal{D}_{\mathbf{v}, \xi^\#}$ is a defect graph associated to the charges \mathbf{m} for the points in $\{\mathbf{z}\}$ and \mathbf{k}_- for the boundary components of $\Sigma^\#$. Recall that ξ_{j_0} is the arc with basepoint $\zeta_{\mathfrak{b}}(1)$ and endpoint $\zeta_{\mathfrak{b}-1}(1)$. Let us call $\xi_{j'_0}$, $j'_0 \neq j_0$, the other arc with basepoint/endpoint $\zeta_{\mathfrak{b}-1}(1)$. Let us assume for convenience that all the arcs in the defect graph $\mathcal{D}_{\mathbf{v}, \xi^\#}$ keep the same label as in $\mathcal{D}_{\mathbf{v}, \xi}$. Therefore the arcs in the defect graph $\mathcal{D}_{\mathbf{v}, \xi^\#}$ are labelled with $p = 1, \dots, n_m + \mathfrak{b} - 1$ with $p \neq j_0, j'_0$.

Let us compute the coefficients $\kappa(\xi_p^\#)$. Then $\kappa(\xi_{j_0}) = -k_{\mathfrak{b}}$ (since $\zeta_{\mathfrak{b}}(1)$ is an end of the graph tree) and $\kappa(\xi_{j'_0}) = 0$ (because $k_{\mathfrak{b}-1} = k_{\mathfrak{b}}$). For $p \neq j_0, j'_0$, we have $\kappa(\xi_p^\#) = \kappa(\xi_p)$ because the structure for the defect graph $\mathcal{D}_{\mathbf{v}, \xi}$ we chose imposes that each time we meet $\zeta_{\mathfrak{b}-1}(1)$ when following counterclockwise the contour of the graph, we also meet

$\zeta_b(1)$, for a total contribution of both points given by $k_{b-1} - k_b = 0$. Therefore

$$\begin{aligned} \int_{\Sigma}^{\text{reg}} I_{X_0}^{\xi}(\nu_{z,m,k})K_g dv_g &= \int_{\Sigma \setminus \mathcal{D}_{v,\xi}} I_{X_0}^{\xi}(\nu_{z,m,k})K_g dv_g - 2 \sum_{p=1}^{n_m+b-1} \kappa(\xi_p) \int_{\xi_p} k_g d\ell_g \\ &= \int_{\Sigma \setminus \mathcal{D}_{v,\xi}} (I_{X_0}^{\xi}(\nu_{z,m,(k_-,0,0)}) + I_{X_0}^{\xi}(\nu_{z,0,(0,k_b,k_b)}))K_g dv_g \\ &\quad - 2 \sum_{p=1, p \neq j_0, j'_0}^{n_m+b-1} \kappa(\xi_p) \int_{\xi_p} k_g d\ell_g - 2 \sum_{p=j_0, j'_0} \kappa(\xi_p) \int_{\xi_p} k_g d\ell_g \\ &= \int_{\Sigma \setminus \mathcal{D}_{v,\xi^\#}} I_{X_0}^{\xi^\#}(\nu_{z,m,(k_-,0,0)})K_g dv_g - 2\kappa(\xi_{j_0}) \int_{\xi_{j_0}} k_g d\ell_g \\ &\quad + \int_{\Sigma_\sigma} I_{X_0}^{\sigma^\#}(\nu_{z,0,(0,k_b,k_b)})K_g dv_g \\ &\quad - 2 \sum_{p=1, p \neq j_0, j'_0}^{n_m+b-1} \kappa(\xi_p^\#) \int_{\xi_p} k_g d\ell_g \\ &= \int_{\Sigma}^{\text{reg}} (I_{X_0}^{\xi^\#}(\nu_{z,m,(k_-,0,0)}) + I_{X_0}^{\sigma^\#}(\nu_{z,0,(0,k_b,k_b)}))K_g dv_g, \end{aligned}$$

where we have used in the last line the fact that the 1-form $\nu_{z,m,(0,k_b,k_b)}$ on $\Sigma^\#$ possesses no trivial winding only around the cycle $\partial_{b-1}\Sigma$. Therefore the regularising term in expression (4.2) reduces to

$$-2\chi_{\nu_{z,0,(0,k_b,k_b)}}(\zeta_{b-1}) \int_{\xi_{j_0}} k_g d\ell_g = -2k_b \int_{\xi_{j_0}} k_g d\ell_g = 2\kappa(\xi_{j_0}) \int_{\xi_{j_0}} k_g d\ell_g.$$

Now we treat the case when $b + n_m = 2$, in which case the defect graph is made up of only one arc ξ_{j_0} and the defect graph on $\Sigma^\#$ is then empty. We can reproduce the above computation with $\nu_{z,m,(k_-,0,0)} = 0$ to get the result. \square

8.4. Definition of the amplitudes. We are now in position to define the amplitudes. In the case of closed Riemann surfaces, amplitudes basically correspond to the path integral with electro-magnetic operators (6.33). In the case of Riemann surfaces with b boundary components, the amplitudes will be a functional of the boundary fields

$$(\tilde{\varphi}_1^{k_1}, \dots, \tilde{\varphi}_b^{k_b}) := (k_1 R\theta + \tilde{\varphi}_1, \dots, k_b R\theta + \tilde{\varphi}_b) \in (H_{\mathbb{Z}}^s(\mathbb{T}))^b.$$

Below we shall identify a pair $(k, \tilde{\varphi}) \in \mathbb{Z} \times H^s(\mathbb{T})$ with the field $kR\theta + \pi^* \tilde{\varphi}(\theta) \in H_{\mathbb{Z}}^s(\mathbb{T})$ and we use the notation

$$\begin{aligned} (\mathbf{k}, \tilde{\varphi}) &:= (k_1, \dots, k_b, \tilde{\varphi}_1, \dots, \tilde{\varphi}_b) \in \mathbb{Z}^b \times (H^s(\mathbb{T}))^b, \\ \tilde{\varphi}^{\mathbf{k}} &:= (\tilde{\varphi}_1^{k_1}, \dots, \tilde{\varphi}_b^{k_b}) \in (H_{\mathbb{Z}}^s(\mathbb{T}))^b. \end{aligned}$$

We notice that we recover the fields $\tilde{\varphi}$ from $\tilde{\varphi}^{\mathbf{k}}$ by setting $\mathbf{k} = \mathbf{0} = (0, \dots, 0)$, i.e.

$$\tilde{\varphi} = \tilde{\varphi}^{\mathbf{0}} \in H^s(\mathbb{T})^b.$$

Recall from Section 8.2 that $P\tilde{\varphi}$ is the harmonic extension of $\tilde{\varphi} \in H^s(\mathbb{T})^b$. The definition of amplitudes will also involve the 1-form $\nu_{z,m,k}$ of Proposition 3.10. Recall that this 1-form is not in L^2 , which is why we introduced its regularised norm (see Lemma 3.11).

By extension and for notational simplicity, we define the regularised norm of $\nu_{z,m,k} + \omega$, where ω is a smooth 1-form on Σ , by

$$(8.20) \quad \|\nu_{z,m,k} + \omega\|_{g,0}^2 := \|\nu_{z,m,k}\|_{g,0}^2 + 2\langle \nu_{z,m,k}, \omega \rangle_2 + \|\omega\|_2^2.$$

Finally we need to introduce a space of reasonable test functions for our path integral in the case when Σ has a non-empty boundary. If $u \in H^s_\Gamma(\widetilde{\Sigma}_z)$ is an equivariant map, to each element $\gamma \in \Gamma$ descending to the homology class $[\partial_j \Sigma]$ there is an associated $k_j \in \mathbb{Z}$ such that $\gamma^*u - u = 2\pi Rk_j$, to each element γ descending to the homology class corresponding to a small oriented circle around z_j we have $\gamma^*u - u = 2\pi Rm_j$ for some $m_j \in \mathbb{Z}$, and to each element γ descending to the homology class σ_j of an interior cycle (i.e. belonging to $([a_i], [b_i])_i$) we have $\gamma^*u - u = 2\pi Rk_j^c$ for some $k_j^c \in \mathbb{Z}$. Completing \mathbf{k}^c by choosing $k_j^c \in \mathbb{Z}$ for $j > 2g$ we see that

$$u - I_{x_0}(\omega_{\mathbf{k}^c}^c) - I_{x_0}(\nu_{z,m,k})$$

descends to $\Sigma_z = \Sigma \setminus \{z\}$ as an $f \in H^s(\Sigma_z)$ distribution:

$$(8.21) \quad u = \pi^* f + I_{x_0}(\omega_{\mathbf{k}^c}^c) + I_{x_0}(\nu_{z,m,k}),$$

where $\pi : \widetilde{\Sigma}_z \rightarrow \Sigma_z$ is the projection from the universal cover of Σ_z to the base. However, the distribution f depends on our choice of $k_j^c \in \mathbb{Z}$ for $j > 2g$, i.e. the decomposition is not unique. More precisely, the morphism $(\mathbf{k}^c, \mathbf{k}, \mathbf{m}) \mapsto [\omega_{\mathbf{k}^c}^c + \nu_{z,m,k}] \in \mathcal{H}_R^1(\Sigma_z)$ has kernel $\{k_j^c = 0, \forall j \leq 2g\} \simeq \mathbb{Z}^{b-1}$, which comes from the kernel of the natural map $\mathcal{H}_R^1(\Sigma_z, \partial\Sigma) \rightarrow \mathcal{H}_R^1(\Sigma_z)$. For fixed \mathbf{m} , we consider the space $\mathcal{E}_R^{\mathbf{m}}(\Sigma)$ of functionals F , defined on $H^s_\Gamma(\widetilde{\Sigma}_z)$ for $s < 0$ fixed as follows: on the connected component of $H^s_\Gamma(\widetilde{\Sigma}_z)$ corresponding to the cohomology class $[\omega_{\mathbf{k}^c}^c + \nu_{z,m,k}] \in \mathcal{H}_R^1(\Sigma_z)$, we ask that when writing our distributions u under the form (8.21) for some \mathbf{k}^c, f , then F has the form

$$(8.22) \quad F(u) = P(f)G(e^{\frac{i}{R}(I_{x_0}(\omega_{\mathbf{k}^c}^c) + I_{x_0}(\nu_{z,m,k}))}),$$

where P is a polynomial depending on $(\mathbf{k}^c, \mathbf{k}, \mathbf{m})$ of the form $P(\langle f, g_1 \rangle, \dots, \langle f, g_{m_n} \rangle)$ where g_1, \dots, g_{m_n} belong to $H^{-s}(\Sigma)$ (hence $s < 0$), and G is continuous and bounded on $C^0(\Sigma, \mathbb{S}^1)$, also depending on $(\mathbf{k}^c, \mathbf{k}, \mathbf{m})$. Let us check that the class $\mathcal{E}_R^{\mathbf{m}}(\Sigma)$ is well-defined, i.e. it does not depend on the choice of \mathbf{k}^c and on the choice of representative in $[\omega_{\mathbf{k}^c}^c + \nu_{z,m,k}] \in \mathcal{H}_R^1(\Sigma_z)$. Indeed, another representative in the class, with the property of being of the form $\hat{\omega}_{\mathbf{k}^c}^c + \hat{\nu}_{z,m,k}$ with $\hat{\nu}_{z,m,k}$ as in Proposition 3.10 and $\hat{\omega}_{\mathbf{k}^c}^c$ as in Lemma 3.5, must be of the form $dh_{\mathbf{k}^c, \mathbf{k}, \mathbf{m}} + \omega_{\mathbf{k}^c}^c + \nu_{z,m,k}$ for some smooth function $h_{\mathbf{k}^c, \mathbf{k}, \mathbf{m}}$ on Σ locally constant near $\partial\Sigma$. This amounts to replacing f by $f + h_{\mathbf{k}^c, \mathbf{k}, \mathbf{m}}$ and, as in Section 6.1, it is direct to see that F written in the new decomposition of u has the right form for some new polynomials \hat{P}_n and function \hat{G} .

Let $u_z^0(x) := \sum_{j=1}^{n_m} i\alpha_j G_{g,D}(x, z_j)$. Next we define the seminorm on $\mathcal{E}_R^{\mathbf{m}}(\Sigma)$

$$\begin{aligned} \|F\|_{\mathcal{L}_{\alpha, \mathbf{m}}^{\infty, p}} := \sup_{\mathbf{k}} \left(\mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_{g,D}, \nu_{z,m,k} + \omega_{\mathbf{k}^c}^c \rangle - \frac{1}{4\pi} \|df_{\mathbf{k}}\|_2^2} \right. \right. \\ \left. \left. \times |F(X_{g,D} + P\tilde{\varphi} + u_z^0 + I_{x_0}(\omega_{\mathbf{k}^c}^c) + I_{x_0}(\nu_{z,m,k}))|^p \right] \right)^{\frac{1}{p}}, \end{aligned}$$

where $(1 - \Pi_1^c)\omega_{\mathbf{k}^c}^c = df_{\mathbf{k}^c}$ with Π_1^c is the projection on the space of harmonic 1-forms with relative boundary condition. With the same argument as in Lemma 6.2, we have:

Lemma 8.9. *The seminorm $\|F\|_{\mathcal{L}_{\alpha, \mathbf{m}}^{\infty, p}}$ does not depend on the choice of cohomology basis.*

Definition 8.10 (Amplitudes).

(A) Let $\partial\Sigma = \emptyset$. We suppose that the condition (6.18) holds. For F continuous bounded function on $\mathcal{E}_R^{\mathbf{m}}(\Sigma)$ for some $s < 0$, we define

$$(8.23) \quad \mathcal{A}_{\Sigma,g,v,\alpha,\mathbf{m}}(F) := \langle F(\phi_g) V_{(\alpha,\mathbf{m})}^g(\mathbf{v}) \rangle_{\Sigma,g}$$

using (6.33). If $F = 1$ then the amplitude is just the correlation function and will simply be denoted by $\mathcal{A}_{\Sigma,g,v,\alpha,\mathbf{m}}$.

(B) If $\partial\Sigma$ has $\mathfrak{b} > 0$ boundary components, consider a set of geometric data of Σ , as described above. Then the amplitude $\mathcal{A}_{\Sigma,g,v,\alpha,\mathbf{m},\zeta}$ is a function

$$(F, \tilde{\varphi}^{\mathbf{k}}) \in \mathcal{E}_R^{\mathbf{m}}(\Sigma) \times (H_{\mathbb{Z}}^s(\mathbb{R}))^{\mathfrak{b}} \mapsto \mathcal{A}_{\Sigma,g,v,\alpha,\mathbf{m},\zeta}(F, \tilde{\varphi}^{\mathbf{k}}),$$

that depends on a marked point x_0 in $\partial\Sigma$. It is defined as follows

$$(8.24) \quad \begin{aligned} &\mathcal{A}_{\Sigma,g,v,\alpha,\mathbf{m},\zeta}(F, \tilde{\varphi}^{\mathbf{k}}) \\ &:= \delta_0 \left(\sum_{j=1}^{n_m+\mathfrak{b}} m_j(\mathbf{k}) \lim_{t \rightarrow 1} \lim_{\epsilon \rightarrow 0} \sum_{\mathbf{k}^c \in \mathbb{Z}^{2q+\mathfrak{b}-1}} e^{-\frac{1}{4\pi} \|\nu_{z,\mathbf{m},\mathbf{k}} + \omega_{\mathbf{k}^c}^c\|_{g,0}^2} Z_{\Sigma,g} \mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}) \right. \\ &\left. \times \mathbb{E} \left[e^{-\frac{\langle dX_{g,D} + dP\tilde{\varphi}, \nu_{z,\mathbf{m},\mathbf{k}} + \omega_{\mathbf{k}^c}^c \rangle}{2\pi}} F(\phi_g) \prod_{j=1}^{n_m} V_{\alpha_j,g,\epsilon}(x_j(t)) e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g \phi_g \, dv_g - \mu M_{\beta}^g(\phi_g, \Sigma)} \right] \right), \end{aligned}$$

where δ_0 is the Dirac mass at 0, $t \in [0, 1] \mapsto x_j(t)$ is a C^1 -curve so that, as $t \rightarrow 1$, $(x_j(t), \dot{x}_j(t)) \rightarrow (z_j, \nu_j)$, the Liouville field is $\phi_g = X_{g,D} + P\tilde{\varphi} + I_{x_0}^{\xi}(\nu_{z,\mathbf{m},\mathbf{k}}) + I_{x_0}^{\sigma}(\omega_{\mathbf{k}^c}^c)$, the expectation \mathbb{E} is over the Dirichlet GFF $X_{g,D}$, $M_{\beta}^g(\phi_g, \Sigma)$ is defined as a limit in (5.4), $m_j(\mathbf{k})$ is defined in (8.12), and $Z_{\Sigma,g}$ is the normalisation constant

$$(8.25) \quad Z_{\Sigma,g} := \det(\Delta_{g,D})^{-\frac{1}{2}}.$$

The regularised curvature term is

$$(8.26) \quad \begin{aligned} \int_{\Sigma}^{\text{reg}} K_g \phi_g \, dv_g &:= \int_{\Sigma} K_g (X_{g,D} + P\tilde{\varphi}) \, dv_g + \int_{\Sigma}^{\text{reg}} K_g I_{x_0}^{\sigma}(\omega_{\mathbf{k}^c}^c) \, dv_g \\ &+ \int_{\Sigma}^{\text{reg}} K_g I_{x_0}^{\xi}(\nu_{z,\mathbf{m},\mathbf{k}}) \, dv_g \end{aligned}$$

and $\mathcal{A}_{\Sigma,g}^0(\tilde{\varphi})$ is the free field amplitude defined as (recall $\mathbf{D}_{\Sigma} - \mathbf{D}$ is smoothing)

$$(8.27) \quad \mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}) = e^{-\frac{1}{2} \langle \tilde{\varphi}, (\mathbf{D}_{\Sigma} - \mathbf{D}) \tilde{\varphi} \rangle}.$$

When $F = 1$, we will simply write $\mathcal{A}_{\Sigma,g,v,\alpha,\mathbf{m},\zeta}(\tilde{\varphi}^{\mathbf{k}})$.

Let us make a couple of remarks on the definition of the amplitudes for what follows.

Remark 8.11.

- (1) We first remark that if $\partial\Sigma \neq \emptyset$, the amplitude depends on the marked point x_0 . When this dependence needs to be precised, we shall add the upperscript x_0 to the amplitude, i.e. we will write

$$\mathcal{A}_{\Sigma,g,v,\alpha,\mathbf{m},\zeta}^{x_0}(F, \tilde{\varphi}^{\mathbf{k}}).$$

- (2) The amplitude does not depend on the choice of orientation of the arcs d_j , since such a change just amounts to making a resummation in the sum $\sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}}$ appearing in the definition of the amplitude.
- (3) The existence of the limit for the definition of amplitudes is not obvious. Yet it can be carried out along the lines of the argument in Theorems 6.11 and 6.13, with slight adaptations due to the presence of a boundary.

We now state the main properties of the amplitudes in the rational case, as well as their gluing properties in Section 8.5. The proofs are postponed to Section 8.8.

Proposition 8.12. *If $\partial\Sigma$ has $\mathfrak{b} > 0$ connected components, then the amplitudes satisfy:*

- *The limit (8.24) is well-defined for $F \in \mathcal{E}_R^m(\Sigma)$ in $\mu_0^{\otimes \mathfrak{b}}$ -measure and belongs to $\mathcal{H}^{\otimes \mathfrak{b}}$.*
- *The amplitudes do not depend on the choice of the relative homology basis σ , nor on the choice of the relative cohomology basis $(\omega_j^c)_j$ dual to σ .*
- *The amplitudes do not depend on the choice of the 1-form $\nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}}$ in the absolute cohomology, satisfying the conditions of Proposition 3.10 and (8.11).*
- **Conformal anomaly:** *let g, g' be two conformal admissible metrics on Σ with $g' = e^\rho g$ for some $\rho \in C^\infty(\Sigma)$, vanishing on $\partial\Sigma$. Then we have*

$$(8.28) \quad \mathcal{A}_{\Sigma, g', \nu, \alpha, \mathbf{m}, \zeta}(F, \tilde{\varphi}^{\mathbf{k}}) = e^{\frac{c_{\text{IL}}}{96\pi} \int_{\Sigma} (|d\rho|_g^2 + 2K_g \rho) dv_g - \sum_{j=1}^{n_m} \Delta(\alpha_j, m_j) \rho(z_j)} \times \mathcal{A}_{\Sigma, g, \nu, \alpha, \mathbf{m}, \zeta}(F(\cdot - \frac{iQ}{2}\rho), \mathbf{k}, \tilde{\varphi}),$$

where the conformal weights $\Delta_{\alpha, m}$ are given by (6.20) and the central charge is $c_{\text{IL}} := 1 - 6Q^2$.

- **Diffeomorphism invariance:** *let $\psi : \Sigma' \rightarrow \Sigma$ be an orientation preserving diffeomorphism. Then*

$$\mathcal{A}_{\Sigma', \psi^*g, \nu, \alpha, \mathbf{m}, \zeta}^{x_0}(F, \tilde{\varphi}^{\mathbf{k}}) = \mathcal{A}_{\Sigma, g, \psi_*\nu, \alpha, \mathbf{m}, \psi \circ \zeta}(F_\psi, \tilde{\varphi}^{\mathbf{k}}),$$

where $F_\psi(\phi) := F(\phi \circ \psi)$.

- **Spins:** *with $r_\theta \mathbf{v} := ((z_1, r_{\theta_1} v_1), \dots, (z_{n_m}, r_{\theta_{n_m}} v_{n_m}))$, then*

$$\mathcal{A}_{\Sigma, g, r_\theta \nu, \alpha, \mathbf{m}, \zeta}(F, \tilde{\varphi}^{\mathbf{k}}) = e^{iR(\alpha \cdot \mathbf{m}, \theta) - iQR(\mathbf{m}, \theta)} \mathcal{A}_{\Sigma, g, r_\theta \nu, \alpha, \mathbf{m}, \zeta}(F, \tilde{\varphi}^{\mathbf{k}}).$$

8.5. Gluing of surfaces and amplitudes: Statements of the results. Let us consider $(\Sigma_1, g_1, \mathbf{z}_1, \zeta_1)$ and $(\Sigma_2, g_2, \mathbf{z}_2, \zeta_2)$ two admissible surfaces with $\partial\Sigma_i \neq \emptyset$ for $i = 1, 2$. We enumerate the boundary components of $\partial\Sigma_i$ as $\partial_j \Sigma_i$ for $j = 1, \dots, \mathfrak{b}_i$. Assume that $\mathcal{C}_1 := \partial_{\mathfrak{b}_1} \Sigma_1$ is outgoing and $\mathcal{C}_2 := \partial_{\mathfrak{b}_2} \Sigma_2$ is incoming, and that $x_0^i \in \partial_{\mathfrak{b}_i} \Sigma_i$ for $i = 1, 2$. Then we can glue the two surfaces (Σ_i, g_i) , $i = 1, 2$ by identifying $\mathcal{C}_1 \sim \mathcal{C}_2$ using the parametrisations. This forms an admissible surface denoted $(\Sigma, g, \mathbf{z}, \zeta)$, with $\mathfrak{b} = \mathfrak{b}_1 + \mathfrak{b}_2 - 2$ boundary components. We assume that $x_0^1 = x_0^2$ on the glued surface.

At the marked points $\mathbf{z}_i = (z_{i1}, \dots, z_{in_m^i})$ on Σ_i we choose unit vectors $\mathbf{v}_i = (v_{i1}, \dots, v_{in_m^i})$ and we attach some weights $\alpha_i = (\alpha_{i1}, \dots, \alpha_{in_m^i})$ and magnetic charges $\mathbf{m}_i = (m_{i1}, \dots, m_{in_m^i})$. We use the notation

$$\mathbf{z} := (\mathbf{z}_1, \mathbf{z}_2), \quad \alpha := (\alpha_1, \alpha_2), \quad \mathbf{m} := (\mathbf{m}_1, \mathbf{m}_2), \quad \mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2)$$

and denote by ζ the collection of parametrisations of the boundaries $\partial_j \Sigma_1$ with $j = 1, \dots, \mathfrak{b}_1 - 1$ and $\partial_j \Sigma_2$ with $j = 1, \dots, \mathfrak{b}_2 - 1$. The surface Σ thus has an analytic boundary consisting of the curves $\partial_j \Sigma_1$ with $j = 1, \dots, \mathfrak{b}_1 - 1$ and $\partial_j \Sigma_2$ with $j = 1, \dots, \mathfrak{b}_2 - 1$. We

denote by $\mathcal{C} = \mathcal{C}_1 = \mathcal{C}_2$ the glued curve on Σ . Boundary conditions on Σ will thus be written as couples $(\tilde{\varphi}_1^{k_1}, \tilde{\varphi}_2^{k_2}) \in (H_{\mathbb{Z}}^s(\mathbb{R}))^{b_1-1} \times (H_{\mathbb{Z}}^s(\mathbb{R}))^{b_2-1}$. Similarly, for $i = 1, 2$, the surface Σ_i has analytic boundary made up of the curves $\partial_j \Sigma_i$ for $j = 1, \dots, b_i - 1$ and \mathcal{C}_i and boundary conditions on Σ_i will thus be written as couples $(\tilde{\varphi}_i^{k_i}, \tilde{\varphi}^k) \in (H_{\mathbb{Z}}^s(\mathbb{R}))^{b_i-1} \times H_{\mathbb{Z}}^s(\mathbb{R})$.

In the rational case, the corresponding amplitudes compose as follows:

Proposition 8.13. *Let F_1, F_2 respectively in $\mathcal{E}_R^{m_1}(\Sigma_1)$ and $\mathcal{E}_R^{m_2}(\Sigma_2)$ and let us denote by $F_1 \otimes F_2$ the functional on $\mathcal{E}_R^m(\Sigma, g)$ defined by*

$$F_1 \otimes F_2(\phi_g) := F_1(\phi_g|_{\Sigma_1})F_2(\phi_g|_{\Sigma_2}).$$

The following holds true

$$\begin{aligned} \mathcal{A}_{\Sigma, g, v, \alpha, m, \zeta}(F_1 \otimes F_2, \tilde{\varphi}_1^{k_1}, \tilde{\varphi}_2^{k_2}) = & C \int_{H_{\mathbb{Z}}^s(\mathbb{R})/(\tau_R)} \mathcal{A}_{\Sigma_1, g_1, v_1, \alpha_1, m_1, \zeta_1}(F_1, \tilde{\varphi}_1^{k_1}, \tilde{\varphi}^k) \\ & \times \mathcal{A}_{\Sigma_2, g_2, v_2, \alpha_2, m_2, \zeta_2}(F_2, \tilde{\varphi}_2^{k_2}, \tilde{\varphi}^k) d\mu_0(\tilde{\varphi}^k), \end{aligned}$$

where $C = \frac{1}{(\sqrt{2\pi})}$ if $\partial\Sigma \neq \emptyset$ and $C = \sqrt{2}$ if $\partial\Sigma = \emptyset$.

The proof of Proposition 8.13 will be done in the next sections. We remark that, in the case of a curve disconnecting the surface, the summation over k in the Hilbert space will always reduce to a Dirac mass at $k = 0$.

The situation will be different for self-gluing (or non-disconnecting curve), which we focus on now. Let (Σ, g, z, ζ) be an admissible surface with b boundary components such that the boundary contains an outgoing boundary component $\partial_{b-1}\Sigma \subset \partial\Sigma$ and an incoming boundary component $\partial_b\Sigma \subset \partial\Sigma$. We glue these two boundary components to produce the surface denoted $(\Sigma^\#, g, z, \zeta_\#)$. In this context, we will write the various fields living on the boundary components of Σ as $(\tilde{\varphi}_\#^k, \tilde{\varphi}_{b-1}^{k_{b-1}}, \tilde{\varphi}_b^{k_b}) \in (H_{\mathbb{Z}}^s(\mathbb{R}))^{b-2} \times H_{\mathbb{Z}}^s(\mathbb{R}) \times H_{\mathbb{Z}}^s(\mathbb{R})$ where $\tilde{\varphi}_{b-1}^{k_{b-1}}$ corresponds to $\partial_{b-1}\Sigma$, $\tilde{\varphi}_b^{k_b}$ corresponds to $\partial_b\Sigma$, and $\tilde{\varphi}_\#^k$ corresponds to the boundary components of $\Sigma^\#$. The location of the base point x_0 could be on any boundary component and we still denote by g the glued metric on $\Sigma^\#$.

Proposition 8.14. *For $F \in \mathcal{E}_R^m(\Sigma)$, the following holds true*

$$\mathcal{A}_{\Sigma^\#, g, v, \alpha, m, \zeta_\#}(F, \tilde{\varphi}_\#^k) = C \int_{H_{\mathbb{Z}}^s(\mathbb{R})/(\tau_R)} \mathcal{A}_{\Sigma, g, v, \alpha, m, \zeta}(F, \tilde{\varphi}_\#^k, \tilde{\varphi}^k, \tilde{\varphi}^k) d\mu_0(\tilde{\varphi}^k),$$

where $C = \frac{1}{\sqrt{2\pi}}$ if $\partial\Sigma \neq \emptyset$ and $C = \sqrt{2}$ if $\partial\Sigma = \emptyset$.

The proof of Proposition 8.14 will be done in the next sections.

The gluing of amplitudes is a property that is valid for the compactified boson, i.e. the \mathbb{T} -valued free field. So we will first state a general lemma for the gluing of the compactified boson as a starting point. This will allow us to establish that Liouville amplitudes are in L^2 and we will later deduce the gluing for Liouville amplitudes. This will not be a straightforward consequence because of subtleties related to the curvature term.

8.6. Gluing for the compactified boson. We consider the setup drawn for amplitudes with the difference that we will consider amplitudes where $\alpha = 0, \mu = 0$ and $Q = 0$. Because Q is set to 0, the results in this subsection do not depend on the rational/irrational case. Concretely, we consider

$$(8.29) \quad \begin{aligned} \mathcal{A}_{\Sigma, g, z, m, \zeta}^0(F, \tilde{\varphi}^{\mathbf{k}}) &:= \delta_0 \left(\sum_{j=1}^{n_m + b} m_j(\mathbf{k}) \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}} e^{-\frac{1}{4\pi} \|\nu_{z, m, \mathbf{k}} + \omega_{\mathbf{k}^c}^c\|_{g, 0}^2} \right. \\ &\quad \left. \times Z_{\Sigma, g} \mathcal{A}_{\Sigma, g}^0(\tilde{\varphi}) \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_{g, D} + dP\tilde{\varphi}, \nu_{z, m, \mathbf{k}} + \omega_{\mathbf{k}^c}^c \rangle} F(\phi_g) \right] \right), \end{aligned}$$

where expectation is taken with respect to the Dirichlet GFF and F can be any measurable positive functional, or any measurable functional such that

$$\mathcal{A}_{\Sigma, g, v, m, \zeta}^0(|F|, \tilde{\varphi}^{\mathbf{k}}) < \infty,$$

$\mu_0^{\otimes b}(\tilde{\varphi}^{\mathbf{k}})$ -almost surely (condition that we will shortcut as *integrable*). Such expression thus perfectly makes sense and we note that there is no dependence of this amplitude with respect to \mathbf{v} . We will further require F to be $2\pi R$ -periodic, meaning that $F(\phi_g + 2\pi nR) = F(\phi_g)$ for all $n \in \mathbb{Z}$.

We first claim that these amplitudes glue as prescribed by Segal. Under the conditions stated just before Proposition 8.13, we claim

Proposition 8.15. *Let F_1, F_2 be periodic measurable positive or periodic integrable respectively on Σ_1 and Σ_2 and let us denote by $F_1 \otimes F_2$ the functional on the glued surface Σ defined by*

$$F_1 \otimes F_2(\phi_g) := F_1(\phi_g|_{\Sigma_1})F_2(\phi_g|_{\Sigma_2}).$$

Then the following holds true

$$\begin{aligned} \mathcal{A}_{\Sigma, g, z, m, \zeta}^0(F_1 \otimes F_2, \tilde{\varphi}_1^{\mathbf{k}_1}, \tilde{\varphi}_2^{\mathbf{k}_2}) \\ = C \int_{H_{\mathbb{Z}}^2(\mathbb{R})/(\tau_R)} \mathcal{A}_{\Sigma_1, g_1, z_1, m_1, \zeta_1}^0(F_1, \tilde{\varphi}_1^{\mathbf{k}_1}, \tilde{\varphi}^{\mathbf{k}}) \mathcal{A}_{\Sigma_2, g_2, z_2, m_2, \zeta_2}^0(F_2, \tilde{\varphi}_2^{\mathbf{k}_2}, \tilde{\varphi}^{\mathbf{k}}) d\mu_0(\tilde{\varphi}^{\mathbf{k}}), \end{aligned}$$

where $C = \frac{1}{(\sqrt{2\pi})}$ if $\partial\Sigma \neq \emptyset$ and $C = \sqrt{2}$ if $\partial\Sigma = \emptyset$.

Proof. We split the proof in two cases depending on whether the glued surface Σ has a trivial boundary (case 1) or not (case 2). We write $n_m = n_m^1 + n_m^2$ and g for $g_1 + g_2$.

(1) Assume first $\partial\Sigma = \emptyset$. Then $\mathfrak{b}_1 = \mathfrak{b}_2 = 1$. Let us call σ_i a canonical geometric basis of the relative homology on Σ_i and note that this basis contains no boundary-to-boundary arcs, only interior cycles. Since the glued curve \mathcal{C} is homologically trivial in Σ_1, Σ_2 or Σ , the family $\sigma := \sigma_1 \cup \sigma_2$ forms a homology basis on Σ (see Figure 17). Let $\omega_1^{1,c}, \dots, \omega_{2g_1}^{1,c}$ be a cohomology basis of $\mathcal{H}^1(\Sigma_i, \partial\Sigma_i)$ dual to σ_i made of closed forms that are compactly supported inside Σ_i^o . Since they are compactly supported, all these forms can be obviously extended to Σ by prescribing their value to be 0 on $\Sigma \setminus \Sigma_i$. Then $\omega_1^{1,c}, \dots, \omega_{2g_1}^{1,c}, \omega_1^{2,c}, \dots, \omega_{2g_2}^{2,c}$ is a basis of $\mathcal{H}^1(\Sigma)$ made of closed forms, dual to σ . For $\mathbf{k}^c := (\mathbf{k}_1^c, \mathbf{k}_2^c) \in \mathbb{Z}^{2g_1} \times \mathbb{Z}^{2g_2}$, we set $\omega_{\mathbf{k}^c} := \omega_{\mathbf{k}_1^c}^{1,c} + \omega_{\mathbf{k}_2^c}^{2,c}$.

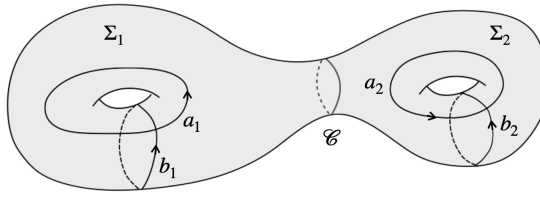


FIGURE 17. Case $\partial\Sigma = \emptyset$ with $\Sigma = \Sigma_1 \# \Sigma_2$ cut along \mathcal{C}

Next we consider the 1-forms $\nu_{z_1, \mathbf{m}_1, k_1}^1, \nu_{z_2, \mathbf{m}_2, k_2}^2$ respectively on Σ_1, Σ_2 given by Proposition 3.10. Notice that its proof (based on Lemma 3.6) shows that we can choose the forms $\nu_{z_i, \mathbf{m}_i, k_i}^i$ to be equal to $-\zeta_i k_i R d\theta$ near $\partial\Sigma_i$, in the chart $\omega_j : U_j \rightarrow \mathbb{A}_\delta$ associated to $\partial\Sigma_i$. Note that, in order for the amplitudes on Σ_1, Σ_2 to be non-zero, we must have $k_1 = \sum_{j=1}^{n_m^1} m_{1j}$ and $k_2 = -\sum_{j=1}^{n_m^2} m_{2j}$. Furthermore, for the amplitude on Σ to exist, we must have $\sum_{j=1}^{n_m^1} m_{1j} + \sum_{j=1}^{n_m^2} m_{2j} = 0$. All in all, $k_1 = k_2 = \sum_{j=1}^{n_m^1} m_{1j}$. In particular $\nu_{z_2, \mathbf{m}_2, k_2}^2$ is a smooth extension of $\nu_{z_1, \mathbf{m}_1, k_1}^1$ (viewed as a form on $\Sigma \setminus \Sigma_2$) to Σ , which means that we get a smooth closed 1-form $\nu_{z, \mathbf{m}}$ on Σ_z with winding m_{ij} around the point z_{ij} , for $i = 1, 2$ and $j = 1, \dots, n_m^i$. The defect graph \mathcal{D}_{v, ξ_1} on Σ_1 is chosen so that only one arc has the boundary point $\zeta_{11}(1)$ as endpoint, and similarly for Σ_2 . We get a defect graph $\mathcal{D}_{v, \xi}$ by gluing the two defect graphs (see Lemma 8.7), i.e. we keep all the arcs in Σ_1, Σ_2 that do not have endpoint on the boundary and we form one arc out of the two arcs with an endpoint on the boundary (one on Σ_1 and one on Σ_2).

On Σ , the path integral can be expressed, for all positive or integrable F , as

$$\begin{aligned} \mathcal{A}_{\Sigma, g, z, \mathbf{m}, \zeta}^0(F) &:= \left(\frac{v_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} \|\omega_{\mathbf{k}^c} + \nu_{z, \mathbf{m}}\|_{g, 0}^2} \\ &\times \int_{\mathbb{T}_R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}^c} + \nu_{z, \mathbf{m}} \rangle} F(\phi_g) \right] dc, \end{aligned} \tag{8.30}$$

where $\phi_g = c + X_g + I_{x_0}^\sigma(\omega_{\mathbf{k}^c}) + I_{x_0}^\xi(\nu_{z, \mathbf{m}})$. Let now X_1 and X_2 be two independent Dirichlet GFF respectively on Σ_1 and Σ_2 . We assume that they are both defined on Σ by setting $X_i = 0$ outside of Σ_i . Then we have the following decomposition in law (see Proposition 5.1)

$$X_g \stackrel{\text{law}}{=} X_1 + X_2 + P\mathbf{X} - c_g,$$

where \mathbf{X} is the restriction of the GFF X_g to the glued boundary component \mathcal{C} expressed in parametrised coordinates, i.e. $\mathbf{X} = X_g|_{\mathcal{C}} \circ \zeta_{\mathcal{C}}$ with $\zeta_{\mathcal{C}}$ the parametrisation of \mathcal{C} , $P\mathbf{X}$ is its harmonic extension to Σ and $c_g := \frac{1}{v_g(\Sigma)} \int_{\Sigma} (X_1 + X_2 + P\mathbf{X}) dv_g$. We can then plug this decomposition into the path integral (8.30) and then shift the c -integral by c_g (i.e. $c \mapsto c + c_g$) to get

$$\begin{aligned} \mathcal{A}_{\Sigma, g, z, \mathbf{m}, \zeta}^0(F_1 \otimes F_2) &:= \left(\frac{v_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} \|\omega_{\mathbf{k}^c} + \nu_{z, \mathbf{m}}\|_{g, 0}^2} \\ &\times \int_{\mathbb{T}_R} \mathbb{E} [\mathcal{B}_1(c, \mathbf{X}, \omega_{\mathbf{k}_1^c}^{1, c} + \nu_{z_1, \mathbf{m}_1, k_1}^1) \mathcal{B}_2(c, \mathbf{X}, \omega_{\mathbf{k}_2^c}^{2, c} + \nu_{z_2, \mathbf{m}_2, k_2}^2)] dc, \end{aligned} \tag{8.31}$$

where we have set

$$\mathcal{B}_i(c, \varphi, \omega) = \mathbb{E}\left[e^{-\frac{1}{2\pi}\langle dX_i + dP\varphi, \omega \rangle} F_i(\phi_i)\right]$$

with $\phi_i := c + X_i + P\varphi + I_{X_0}^i(\omega)$ and expectation is over the GFF X_i . Here we have used a shortcut notation $I_{X_0}^i(\omega)$: it means that for $\omega = \omega_{\mathbf{k}_i^c}^{i,c} + \nu_{\mathbf{z}_i, \mathbf{m}_i, k_i}^i$ we have $I_{X_0}^i(\omega) = I_{X_0}^{\sigma_i}(\omega_{\mathbf{k}_i^c}^{i,c}) + I_{X_0}^{\xi_i}(\nu_{\mathbf{z}_i, \mathbf{m}_i, k_i}^i)$.

Now we make a further shift in the c -variable in the expression above to subtract the mean $m_c(\mathbf{X}) := \frac{1}{2\pi} \int_0^{2\pi} \mathbf{X}(e^{i\theta}) d\theta$ to the field \mathbf{X} . As a consequence we can replace the law $\mathbb{P}_{\mathbf{X}}$ of \mathbf{X} in (8.31) (expectation is there w.r.t. $\mathbb{P}_{\mathbf{X}}$) by the law $\mathbb{P}_{\mathbf{X} - m_c(\mathbf{X})}$ of the recentered field $\mathbf{X} - m_c(\mathbf{X})$ so that we end up with (using the description of the law of $X - m_c(\mathbf{X})$ proved in [31, eq (5.14)] together with the computation of determinants [31, eq (5.15)])

$$\begin{aligned} \mathcal{A}_{\Sigma, \mathbf{g}, \mathbf{z}, \mathbf{m}, \zeta}^0(F) &= \sqrt{2} Z_{\Sigma_1, \mathbf{g}_1} Z_{\Sigma_2, \mathbf{g}_2} \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} \|\omega_{\mathbf{k}^c} + \nu_{\mathbf{z}, \mathbf{m}}\|_{\mathbf{g}, 0}^2} \\ &\times \int_{\mathbb{T}_{\mathbb{R}}} \int \mathcal{B}_1(c, \varphi, \omega_{\mathbf{k}_1^c}^{1,c}) \mathcal{B}_2(c, \varphi, \omega_{\mathbf{k}_2^c}^{2,c}) e^{-\frac{1}{2} \langle \varphi, \tilde{\mathbf{D}}_{\Sigma, c} \varphi \rangle} d\mathbb{P}_{\mathbb{T}}(\varphi) dc. \end{aligned}$$

Here we recall that $Z_{\Sigma_i, \mathbf{g}_i}$ and $\tilde{\mathbf{D}}_{\Sigma, c}$ were defined in (8.25) and (8.9). Next we observe that

$$\|\omega_{\mathbf{k}^c} + \nu_{\mathbf{z}, \mathbf{m}}\|_{\mathbf{g}, 0}^2 = \|\omega_{\mathbf{k}_1^c}^{1,c} + \nu_{\mathbf{z}_1, \mathbf{m}_1, k_1}^1\|_{\mathbf{g}_1, 0}^2 + \|\omega_{\mathbf{k}_2^c}^{2,c} + \nu_{\mathbf{z}_2, \mathbf{m}_2, k_2}^2\|_{\mathbf{g}_2, 0}^2$$

and that $\exp(-\frac{1}{2} \langle \varphi, \tilde{\mathbf{D}}_{\Sigma, c} \varphi \rangle) = \mathcal{A}_{\Sigma_1, \mathbf{g}_1}^0(\tilde{\varphi}) \mathcal{A}_{\Sigma_2, \mathbf{g}_2}^0(\tilde{\varphi})$ (whatever the value of c is, since $\mathbf{D}_{\Sigma, c} \mathbf{1} = \mathbf{D} \mathbf{1} = 0$). Also, note that, in the statement of Proposition 8.15, summation over k in the measure μ_0 reduces to $k = k_1 = k_2 = \sum_{j=1}^{n_{\mathbf{m}}} m_{1j}$. This completes the proof of the first case.

(2) Assume now $\partial\Sigma \neq \emptyset$. In that case, Σ_1 or Σ_2 has at least 2 boundary connected components and we will assume (even if it means re-labelling the surfaces) that this will be the case for Σ_1 . Let us take

$$\sigma_i = (a_{i1}, b_{i1}, \dots, a_{i\mathfrak{g}_i}, b_{i\mathfrak{g}_i}, d_{i1}, \dots, d_{i(b_i-1)})$$

a canonical geometric basis of $\mathcal{H}_1(\Sigma_i, \partial\Sigma_i)$ (see Figure 18). Let $\omega_1^{i,c}, \dots, \omega_{2\mathfrak{g}_i + b_i - 1}^{i,c}$ be a basis of $\mathcal{H}_R^1(\Sigma_i, \partial\Sigma_i)$ dual to σ_i made of closed forms that are compactly supported inside Σ_i° . Since they are compactly supported, all these forms can be obviously extended to Σ by prescribing their value to be 0 on $\Sigma \setminus \Sigma_i$.

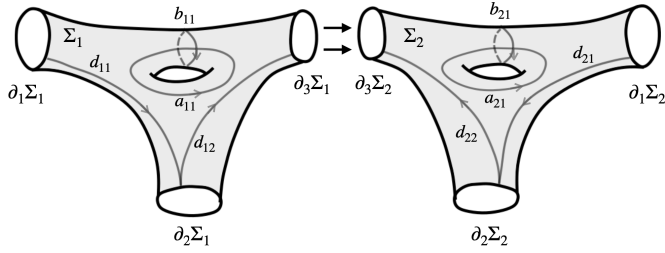
• We first consider the case where Σ_2 has $\mathfrak{b}_2 \geq 2$ boundary components. By Lemma 3.7, we get a basis of the relative homology $\mathcal{H}_1(\Sigma, \partial\Sigma)$

$$\sigma = \sigma_1 \# \sigma_2$$

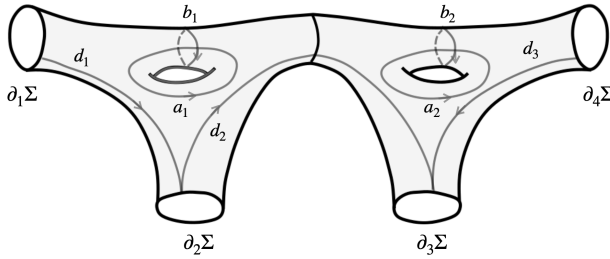
by gathering the curves for $i = 1, 2$: a_{ij}, b_{ij} for $j = 1, \dots, \mathfrak{g}_i$, d_{ij} for $j = 1, \dots, \mathfrak{b}_i - 2$, and finally the curve $d_{1(b_1-1)} - d_{2(b_2-1)}$ (see Figure 18). Then

$$\omega_1^{1,c}, \dots, \omega_{2\mathfrak{g}_1 + b_1 - 1}^{1,c}, \omega_1^{2,c}, \dots, \omega_{2\mathfrak{g}_2 + b_2 - 2}^{2,c}$$

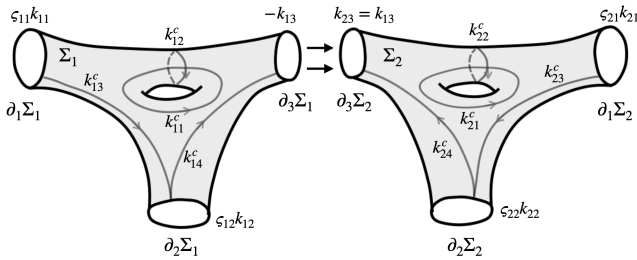
is a basis of $\mathcal{H}_R^1(\Sigma, \partial\Sigma)$ made of closed forms, dual to σ and compactly supported. For $\mathbf{k}^c := (\mathbf{k}_1^c, \mathbf{k}_2^c) \in \mathbb{Z}^{2\mathfrak{g}_1 + b_1 - 1} \times \mathbb{Z}^{2\mathfrak{g}_2 + b_2 - 2}$, we set $\omega_{\mathbf{k}^c}^c := \omega_{\mathbf{k}_1^c}^{1,c} + \omega_{(\mathbf{k}_2^c, 0)}^{2,c}$.



(A) Disconnected surface. In gray, the relative homology on both surfaces.



(B) Glued surface. In gray, the relative homology built from those on Σ_1 and Σ_2 .



(C) Assignment of values of k -parameters to each homology curve

FIGURE 18. Case $\partial\Sigma \neq \emptyset$

Now we focus on the absolute cohomology and magnetic 1-forms. Notice first that in case $\sum_{j=1}^{b_1-1} \varsigma_{1j} k_{1j} + \sum_{j=1}^{b_2-1} \varsigma_{2j} k_{2j} + \sum_{j=1}^{n_m^1} m_{1j} + \sum_{j=1}^{n_m^2} m_{2j} \neq 0$, both sides in the gluing statement vanish (because of δ -masses in the definition of amplitudes) so that equality obviously holds. So the case of interest is

$$(8.32) \quad \sum_{j=1}^{b_1-1} \varsigma_{1j} k_{1j} + \sum_{j=1}^{b_2-1} \varsigma_{2j} k_{2j} + \sum_{j=1}^{n_m^1} m_{1j} + \sum_{j=1}^{n_m^2} m_{2j} = 0,$$

in which case the only contributing term in the summation over k in the Hilbert space comes from the case when

$$\sum_{j=1}^{n_m^1} m_{1j} + \sum_{j=1}^{b_1} \varsigma_{1j} k_{1j} = 0 \quad \sum_{j=1}^{n_m^2} m_{2j} + \sum_{j=1}^{b_2} \varsigma_{2j} k_{2j} = 0$$

and (8.32), which implies in particular $k_{1b_1} = k_{2b_2}$. So we fix $k_{1b_1} = k_{2b_2}$ such that these conditions are fulfilled. Under these conditions, the forms $\nu_{z_i, \mathbf{m}_i, (\mathbf{k}_i, k_{ib_i})}^i$ (for $i = 1, 2$) satisfy

$$(8.33) \quad \begin{aligned} \int_{\partial_j \Sigma_i} \nu_{z_i, \mathbf{m}_i, (\mathbf{k}_i, k_{ib_i})}^i &= 2\pi R \zeta_{ij} k_{ij}, \quad \text{for } j = 1, \dots, b_i, \\ \int_{\partial_{b_1} \Sigma_1} \nu_{z_1, \mathbf{m}_1, (\mathbf{k}_1, k_{1b_1})}^1 &= - \int_{\partial_{b_2} \Sigma_2} \nu_{z_2, \mathbf{m}_2, (\mathbf{k}_2, k_{2b_2})}^2, \\ \int_{\partial \mathcal{D}_{ij}} \nu_{z_i, \mathbf{m}_i, (\mathbf{k}_i, k_{ib_i})}^i &= 2\pi R m_{ij}, \end{aligned}$$

where \mathcal{D}_{ij} is a small disk containing z_{ij} (and of course all integrals along interior cycles vanish). The second condition (and the fact that both forms involved are of the form $\zeta_{ib_i} k_{ib_i} R d\theta$ over a neighborhood of the boundary $\partial_{b_i} \Sigma_i$, see Proposition 3.10) allows us to glue together these forms to get a closed 1-form on Σ by the relation $\nu_{z, \mathbf{m}, \mathbf{k}} = \nu_{z_1, \mathbf{m}_1, (\mathbf{k}_1, k_{1b_1})}^1 \mathbf{1}_{\Sigma_1} + \nu_{z_2, \mathbf{m}_2, (\mathbf{k}_2, k_{2b_2})}^2 \mathbf{1}_{\Sigma_2}$, for $\mathbf{k} := (\mathbf{k}_1, \mathbf{k}_2) \in \mathbb{Z}^{b_1-1} \times \mathbb{Z}^{b_2-1}$, satisfying our basic conditions. Finally we consider two defect graphs on Σ_1 and Σ_2 as in Lemma 8.7, and glue them to get a defect graph on Σ . The amplitude on Σ then reads

$$\begin{aligned} &\mathcal{A}_{\Sigma, g, z, \mathbf{m}, \zeta}^0(F_1 \otimes F_2, \tilde{\varphi}_1^{\mathbf{k}_1}, \tilde{\varphi}_2^{\mathbf{k}_2}) \\ &= \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b_1+b_2-3}} e^{-\frac{1}{4\pi} \|\nu_{z, \mathbf{m}, \mathbf{k}} + \omega_{\mathbf{k}^c}^c\|_{g,0}^2} Z_{\Sigma, g} \mathcal{A}_{\Sigma, g}^0(\tilde{\varphi}_1, \tilde{\varphi}_2) \mathcal{B}_{\Sigma, g}(F_1 \otimes F_2, \tilde{\varphi}_1, \tilde{\varphi}_2, \mathbf{k}, \mathbf{k}^c) \end{aligned}$$

with

$$\mathcal{B}_{\Sigma, g}(F_1 \otimes F_2, \tilde{\varphi}_1, \tilde{\varphi}_2, \mathbf{k}, \mathbf{k}^c) := \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_{g,D} + dP\tilde{\varphi}, \nu_{z, \mathbf{m}, \mathbf{k}} + \omega_{\mathbf{k}^c}^c \rangle} F_1 \otimes F_2(\phi_g) \right],$$

where the Liouville field is $\phi_g = X_{g,D} + P(\tilde{\varphi}_1, \tilde{\varphi}_2) + I_{x_0}^\zeta(\nu_{z, \mathbf{m}, \mathbf{k}}) + I_{x_0}^\zeta(\omega_{\mathbf{k}^c}^c)$, the expectation \mathbb{E} is over the Dirichlet GFF $X_{g,D}$ on Σ , $P(\tilde{\varphi}_1, \tilde{\varphi}_2)$ stands for the harmonic extension to Σ of the boundary fields $\tilde{\varphi}_1, \tilde{\varphi}_2$, which stand respectively for the boundary conditions on the remaining (i.e. unglued) components of $\partial \Sigma_1$ and $\partial \Sigma_2$, namely

$$\Delta_g P(\tilde{\varphi}_1, \tilde{\varphi}_2) = 0 \quad \text{on } \Sigma, \quad P(\tilde{\varphi}_1, \tilde{\varphi}_2)|_{\partial_j \Sigma_i} = \tilde{\varphi}_{ij} \circ \zeta_{ij}^{-1} \text{ for } j < b_i.$$

Let now $X_1 := X_{g_1, D}$ and $X_2 := X_{g_2, D}$ be two independent Dirichlet GFF respectively on Σ_1 and Σ_2 . Then we have the following decomposition in law (see Proposition 5.1)

$$X_{g, D} \stackrel{\text{law}}{=} X_1 + X_2 + P\mathbf{Y},$$

where \mathbf{Y} is the restriction of $X_{g, D}$ to the glued boundary component \mathcal{C} expressed in parametrised coordinates, i.e. $\mathbf{Y} = X_{g, D}|_{\mathcal{C}} \circ \zeta_{1b_1}$, and $P\mathbf{Y}$ is its harmonic extension to Σ vanishing on $\partial \Sigma$, which is non-empty. We stress that, since $X_{g, D}$ is only a distribution, making sense of \mathbf{Y} is not completely obvious but, using the parametrisation, this can be done in the same way as making sense of the restriction of the GFF to a circle: since this is a standard argument, we do not elaborate more on this point. Finally we denote by h_e the restriction of the harmonic function $P(\tilde{\varphi}_1, \tilde{\varphi}_2)$ to \mathcal{C} in parametrised coordinates

$$h_e := P(\tilde{\varphi}_1, \tilde{\varphi}_2)|_{\mathcal{C}} \circ \zeta_{1b_1}.$$

Observe now the trivial fact that, on Σ_i ($i = 1, 2$), the function $P(\tilde{\varphi}_1, \tilde{\varphi}_2) + P\mathbf{Y}$ is harmonic with boundary values (expressed in parametrised coordinates on Σ_i) $\tilde{\varphi}_{ij}$ on $\partial_j \Sigma_i$ for $j < \mathfrak{b}_i$ and $(\mathbf{Y} + h_e)$ on \mathcal{C} . Thus we get, using Lemma 8.7,

$$\begin{aligned} & \mathcal{B}_{\Sigma, g}(F_1 \otimes F_2, \tilde{\varphi}_1, \tilde{\varphi}_2, \mathbf{k}, \mathbf{k}^c) \\ &= \int \mathcal{B}_{\Sigma_1, g_1}(F_1, \tilde{\varphi}_1, \tilde{\varphi} + h_e, \mathbf{k}_1, k_{1\mathfrak{b}_1}, \mathbf{k}_1^c) \mathcal{B}_{\Sigma_2, g_2}(F_2, \tilde{\varphi}_2, \tilde{\varphi} + h_e, \mathbf{k}_2, k_{2\mathfrak{b}_2}, \mathbf{k}_2^c) \mathbb{P}_{\mathbf{Y}}(d\tilde{\varphi}) \end{aligned}$$

with $\mathbb{P}_{\mathbf{Y}}$ the law of \mathbf{Y} and, for $i = 1, 2$,

$$(8.34) \quad \mathcal{B}_{\Sigma_i, g_i}(F_i, \tilde{\varphi}_i, \tilde{\varphi}, \mathbf{k}_i, k_{i\mathfrak{b}_i}, \mathbf{k}_i^c) := \mathbb{E}\left[F_i(\phi_i) e^{-\frac{1}{2\pi} \langle dX_i + dP(\tilde{\varphi}_i, \tilde{\varphi}), \nu_{z_i, \mathfrak{m}_i, (\mathbf{k}_i, k_{i\mathfrak{b}_i})}^i + \omega_{\mathbf{k}_i^c}^{i,c} \rangle}\right],$$

where \mathbb{E} is taken with respect to the Dirichlet GFF X_i on Σ_i ,

$$\phi_i = X_i + P(\tilde{\varphi}_i, \tilde{\varphi}) + I_{x_0}^{\sigma_i}(\omega_{\mathbf{k}_i^c}^{i,c}) + I_{x_0}^{\xi_i}(\nu_{z_i, \mathfrak{m}_i, (\mathbf{k}_i, k_{i\mathfrak{b}_i})}^i)$$

(here with an abuse of notations we identify \mathbf{k}_2^c with $(\mathbf{k}_2^c, 0)$ and $P(\tilde{\varphi}_i, \tilde{\varphi})$ stands for the harmonic extension on Σ_i of the boundary fields $\tilde{\varphi}_i, \tilde{\varphi}$ respectively on $\partial \Sigma_i \setminus \mathcal{C}$ and \mathcal{C} .)

Now we use Lemma [31, Lemmas 5.3 & 5.4] to get

$$\mathbb{P}_{\mathbf{Y}+h_e}(d\tilde{\varphi}) := \frac{1}{\sqrt{2\pi}} \frac{Z_{\Sigma_1, g_1} Z_{\Sigma_2, g_2}}{\sqrt{2\pi} Z_{\Sigma, g}} \frac{\mathcal{A}_{\Sigma_1, g_1}^0(\tilde{\varphi}_1, c + \varphi) \mathcal{A}_{\Sigma_2, g_2}^0(\tilde{\varphi}_2, c + \varphi)}{\mathcal{A}_{\Sigma, g}^0(\tilde{\varphi}_1, \tilde{\varphi}_2)} dc \otimes \mathbb{P}_{\mathbb{T}}(d\varphi).$$

Thus we obtain

$$\begin{aligned} & \mathcal{A}_{\Sigma, g, z, \mathfrak{m}, \zeta}^0(F_1 \otimes F_2, \tilde{\varphi}_1^{\mathbf{k}_1}, \tilde{\varphi}_2^{\mathbf{k}_2}) \\ &= \sum_{\mathbf{k}^c \in \mathbb{Z}^{2q+b-1}} e^{-\frac{1}{4\pi} \|\nu_{z, \mathfrak{m}, \mathbf{k}} + \omega_{\mathbf{k}^c}^c\|_{g,0}^2} Z_{\Sigma_1, g_1} Z_{\Sigma_2, g_2} \int_{\mathbb{R}} H_0(c, \mathbf{k}^c) \frac{dc}{\sqrt{2\pi}} \\ &= \sum_{\substack{\mathbf{k}^c \in \mathbb{Z}^{2q+b-1} \\ n \in \mathbb{Z}}} e^{-\frac{1}{4\pi} \|\nu_{z, \mathfrak{m}, \mathbf{k}} + \omega_{\mathbf{k}^c}^c\|_{g,0}^2} Z_{\Sigma_1, g_1} Z_{\Sigma_2, g_2} \int_0^{2\pi R} H_0(c + n2\pi R, \mathbf{k}^c) \frac{dc}{\sqrt{2\pi}}, \end{aligned}$$

where (the expectation \mathbb{E} is taken with respect to φ)

$$\begin{aligned} H_0(c, \mathbf{k}^c) &= \mathbb{E}\left[\mathcal{A}_{\Sigma_1, g_1}^0(\tilde{\varphi}_1, c + \varphi) \mathcal{A}_{\Sigma_2, g_2}^0(\tilde{\varphi}_2, c + \varphi) \right. \\ &\quad \times \mathcal{B}_{\Sigma_1, g_1}(F_1, \tilde{\varphi}_1, c + \varphi, \mathbf{k}_1, k_{1\mathfrak{b}_1}, \mathbf{k}_1^c) \\ &\quad \left. \times \mathcal{B}_{\Sigma_2, g_2}(F_2, \tilde{\varphi}_2, c + \varphi, \mathbf{k}_2, k_{2\mathfrak{b}_2}, \mathbf{k}_2^c)\right]. \end{aligned}$$

The last relation was obtained using the Chasles relation on the c -integral. Next, we introduce the harmonic functions P_n^i , for $i = 1, 2$ and $n \in \mathbb{Z}$, that are harmonic on Σ_i with boundary values $n2\pi R$ on \mathcal{C} and 0 on the other boundary components of Σ_i . Then we notice that, writing $\tilde{\varphi}$ for $c + \varphi$,

$$\begin{aligned} (8.35) \quad \mathcal{A}_{\Sigma_1, g_1}^0(\tilde{\varphi}_1, \tilde{\varphi} + n2\pi R) &= e^{-\frac{1}{2} \langle (\mathbf{D}_{\Sigma_1} - \mathbf{D})(\tilde{\varphi}_1, \tilde{\varphi} + n2\pi R), (\tilde{\varphi}_1, \tilde{\varphi} + n2\pi R) \rangle} \\ &= \mathcal{A}_{\Sigma_1, g_1}^0(\tilde{\varphi}_1, \tilde{\varphi}) e^{-\frac{1}{2\pi} \langle dP(\tilde{\varphi}_1, \tilde{\varphi}), dP_n^1 \rangle} e^{-\frac{1}{4\pi} \|dP_n^1\|_2^2} \end{aligned}$$

and similarly for $\mathcal{A}_{\Sigma_2, g_2}^0(\tilde{\varphi}_2, \tilde{\varphi} + n2\pi R)$. Let us write $\omega_{\mathbf{k}^c}^{1,c} = \omega_{\mathbf{k}^c}^{1,h} + du$ for some $u \in C^\infty(\Sigma_1)$ with $u|_{\partial \Sigma_1} = 0$ and $d^* \omega_{\mathbf{k}^c}^{1,h} = 0$ satisfies the relative condition $\iota_{\partial \Sigma_1}^* \omega_{\mathbf{k}^c}^{1,h} = 0$, then by Stokes formula on Σ_1

$$\langle \omega_{\mathbf{k}^c}^{1,c}, dP_n^1 \rangle_2 = \langle d^* \omega_{\mathbf{k}^c}^{1,h}, P_n^1 \rangle_2 + \langle u, d^* dP_n^1 \rangle_2 = 0.$$

Similarly we use (3.22) to write $\nu^1_{\mathbf{z}_1, \mathbf{m}_1, (\mathbf{k}^1, k_{b_1}^1)} = \nu^{1,h}_{\mathbf{z}_1, \mathbf{m}_1, (\mathbf{k}^1, k_{b_1}^1)} + du'$ for some $u' \in C^\infty(\Sigma_1)$ with $u'|_{\partial\Sigma} = 0$ and $d^* \nu^{1,h}_{\mathbf{z}_1, \mathbf{m}_1, (\mathbf{k}^1, k_{b_1}^1)} = 0$: then by Stokes and the fact that $\nu^1_{\mathbf{z}_1, \mathbf{m}_1, (\mathbf{k}^1, k_{b_1}^1)}$ satisfies the absolute boundary condition

$$\begin{aligned} \langle \nu^1_{\mathbf{z}_1, \mathbf{m}_1, (\mathbf{k}^1, k_{b_1}^1)}, dP_n^1 \rangle_2 &= \int_{\partial\Sigma_1} P_n^1 * \nu^{1,h}_{\mathbf{z}_1, \mathbf{m}_1, (\mathbf{k}^1, k_{b_1}^1)} \\ &= \int_{\partial\Sigma_1} P_n^1 \nu^1_{\mathbf{z}_1, \mathbf{m}_1, (\mathbf{k}^1, k_{b_1}^1)}(\nu) d\ell_{g_1} = 0. \end{aligned}$$

These two identities above imply on Σ_i with $g_i := g|_{\Sigma_i}$

$$(8.36) \quad \|\nu^i_{\mathbf{z}_i, \mathbf{m}_i, (\mathbf{k}^i, k_{b_i}^i)} + \omega_{\mathbf{k}^i, c}^{i,c}\|_{g_i, 0}^2 + \|dP_n^i\|_2^2 = \|\nu^i_{\mathbf{z}_i, \mathbf{m}_i, (\mathbf{k}^i, k_{b_i}^i)} + \omega_{\mathbf{k}^i, c}^{i,c} + dP_n^i\|_{g_i, 0}^2.$$

We deduce, by combining with (8.35), that

$$\begin{aligned} &\mathcal{A}_{\Sigma, g, \mathbf{z}, \mathbf{m}, \zeta}^0(F_1 \otimes F_2, \tilde{\varphi}_1^{\mathbf{k}_1}, \tilde{\varphi}_2^{\mathbf{k}_2}) \\ &= \sum_{\substack{\mathbf{k}^c \in \mathbb{Z}^{2q+b-1} \\ n \in \mathbb{Z}}} e^{-\frac{1}{4\pi}(\|\nu^1_{\mathbf{z}_1, \mathbf{m}_1, (\mathbf{k}_1, k_{1b_1})} + \omega_{\mathbf{k}_1^c}^{1,c} + dP_n^1\|_{g_1, 0}^2 + \|\nu^2_{\mathbf{z}_2, \mathbf{m}_2, (\mathbf{k}_2, k_{2b_2})} + \omega_{\mathbf{k}_2^c}^{2,c} + dP_n^2\|_{g_2, 0}^2)} \\ &\quad \times Z_{\Sigma_1, g_1} Z_{\Sigma_2, g_2} \int_0^{2\pi R} H_1(c, n, \mathbf{k}^c) \frac{dc}{\sqrt{2\pi}} \end{aligned}$$

with (expectation is over φ)

$$\begin{aligned} H_1(c, n, \mathbf{k}^c) &:= \mathbb{E}[\mathcal{A}_{\Sigma_1, g_1}^0(\tilde{\varphi}_1, \tilde{\varphi}) \mathcal{A}_{\Sigma_2, g_2}^0(\tilde{\varphi}_2, \tilde{\varphi}) \hat{\mathcal{B}}_{\Sigma_1, g_1}(F_1, \tilde{\varphi}_1, \tilde{\varphi}, \mathbf{k}_1, k_{1b_1}, \mathbf{k}_1^c, n) \\ &\quad \times \hat{\mathcal{B}}_{\Sigma_2, g_2}(F_2, \tilde{\varphi}_2, \tilde{\varphi}, \mathbf{k}_2, k_{2b_2}, (\mathbf{k}_2^c, 0), n)], \end{aligned}$$

and (expectation is over Dirichlet GFF X_i on Σ_i)

$$\begin{aligned} &\hat{\mathcal{B}}_{\Sigma_i, g_i}(F_i, \tilde{\varphi}_i, \tilde{\varphi}, \mathbf{k}_i, k_{ib_i}, \mathbf{k}_i^c, n) := \\ &\quad \mathbb{E}[F_i(\phi_i + P_n^i) e^{-\frac{1}{2\pi} \langle dX_i + dP(\tilde{\varphi}_i, \tilde{\varphi}), \nu^i_{\mathbf{z}_i, \mathbf{m}_i, (\mathbf{k}_i, k_{ib_i})} + \omega_{\mathbf{k}_i^c}^{i,c} + dP_n^i \rangle}]. \end{aligned}$$

Now the point is to see that the term dP_n^i encodes part of the relative cohomology on Σ_i . For this, let us first introduce the notation \mathbf{n}^i for the vector $(0, \dots, 0, n) \in \mathbb{Z}^{2q_i + b_i - 1}$. Since the 1-form dP_n^i is exact and since P_n^i takes the value $2\pi nR$ on \mathcal{C} and 0 on the other boundary components of Σ_i , we have

$$\begin{aligned} \int_{a_j^i} dP_n^i &= 0, & \int_{b_j^i} dP_n^i &= 0, \\ \int_{a_j^i} dP_n^i &= 0 \text{ for } j = 1, \dots, b_i - 1, & \int_{a_{b_i}^i} dP_n^i &= n2\pi R. \end{aligned}$$

The 1-form dP_n^i has therefore the same cycles/arcs as the 1-form $\omega_{\mathbf{n}^i}^{i,c}$. Thus, we have $dP_n^i = \omega_{\mathbf{n}^i}^{i,c} + df_{\mathbf{n}^i}^i$, for some smooth function $f_{\mathbf{n}^i}^i$ on Σ_i vanishing on the boundary $\partial\Sigma_i$, see Lemma 3.4. We can absorb the term $df_{\mathbf{n}^i}^i$ by means of the Girsanov transform. More precisely, on Σ_i ($i = 1, 2$), we apply the Girsanov transform to the term

$e^{-\frac{1}{2\pi}\langle dX_i, df_i^i \rangle - \frac{1}{4\pi}\|df_i^i\|_2^2}$, which has the effect of shifting the law of the GFF as $X_{g,D} \rightarrow X_{g,D} - f_{\mathbf{n}}^i$, to get that (note that $\langle dP(\tilde{\varphi}_i, \tilde{\varphi}), df_{\mathbf{n}}^i \rangle_2 = 0$ on Σ_i)

$$\begin{aligned} & e^{-\frac{1}{4\pi}\|\nu_{z_i, \mathbf{m}_i, (k_i, k_{i b_i})}^i + \omega_{\mathbf{k}_i^c}^{i,c} + dP_{\mathbf{n}}^i\|_{g_i, 0, \Sigma_i}^2} \hat{\mathcal{B}}_{\Sigma_i, g_i}(F_i, \tilde{\varphi}_i, \tilde{\varphi}, \mathbf{k}_i, k_{i b_i}, \mathbf{k}_i^c, n) \\ &= e^{-\frac{1}{4\pi}\|\nu_{z_i, \mathbf{m}_i, (k_i, k_{i b_i})}^i + \omega_{\mathbf{k}_i^c}^{i,c} + \omega_{\mathbf{n}^i}^{i,c}\|_{g_i, 0}^2} \hat{\mathcal{B}}'_{\Sigma_i, g_i}(F_i, \tilde{\varphi}_i, \tilde{\varphi}, \mathbf{k}_i, k_{i b_i}, \mathbf{k}_i^c, n), \end{aligned}$$

where

$$\begin{aligned} \hat{\mathcal{B}}'_{\Sigma_i, g_i}(F_i, \tilde{\varphi}_i, \tilde{\varphi}, \mathbf{k}^i, k_{b_i}^i, \mathbf{k}^{i,c}, n) := \\ \mathbb{E}\left[F_i(\phi_i + I_{X_0}^i(\omega_{\mathbf{n}^i}^{i,c}))e^{-\frac{1}{2\pi}\langle dX_i + dP(\tilde{\varphi}_i, \tilde{\varphi}), \nu_{z_i, \mathbf{m}_i, (k_i, k_{i b_i})}^i + \omega_{\mathbf{k}_i^c}^{i,c} + \omega_{\mathbf{n}^i}^{i,c} \rangle}\right]. \end{aligned}$$

In particular, note the relation $\omega_{\mathbf{k}^{i,c}}^{i,c} + \omega_{\mathbf{n}^i}^{i,c} = \omega_{\mathbf{k}^{i,c} + \mathbf{n}^i}^{i,c}$. Making next the change of variables $k_{2g_i + b_i - 1} + n \rightarrow k_{2g_i + b_i - 1}$ in the summation over \mathbf{k}^c , we end up with the gluing statement we claimed.

• Now we have to consider the case when Σ_2 has only $b_2 = 1$ boundary components, in which case we get a basis σ of the relative homology by gathering all the curves a_{ij}, b_{ij} (for $j = 1, \dots, g_i$), d_{1j} (for $j = 1, \dots, b_1 - 2$). Then $\omega_1^{1,c}, \dots, \omega_{2g_1 + b_1 - 2}^{1,c}, \omega_1^{2,c}, \dots, \omega_{2g_2}^{2,c}$ is a basis of $\mathcal{H}_R^1(\Sigma, \partial\Sigma)$ made up of closed forms, dual to σ and compactly supported. For $\mathbf{k}^c := (\mathbf{k}_1^c, \mathbf{k}_2^c) \in \mathbb{Z}^{2g_1 + b_1 - 2} \times \mathbb{Z}^{2g_2}$, we set $\omega_{\mathbf{k}^c}^c := \omega_{(\mathbf{k}_1^c, 0)}^{1,c} + \omega_{\mathbf{k}_2^c}^{2,c}$.

Regarding the absolute cohomology and magnetic 1-forms, the situation is somewhat simpler. The Dirac masses in the definition of amplitudes make it clear that the non-trivial case is (in other cases, both hands of the gluing statement are 0)

$$(8.37) \quad \sum_{j=1}^{b_1-1} \varsigma_{1j} k_{1j} + \sum_{j=1}^{n_m^1} m_{1j} + \sum_{j=1}^{n_m^2} m_{2j} = 0,$$

together with $\sum_{j=1}^{n_m^1} m_{1j} + \sum_{j=1}^{b_1} \varsigma_{1j} k_{1j} = 0$, $\sum_{j=1}^{n_m^2} m_{2j} + \varsigma_{21} k_{21} = 0$, which implies again $k_{1b_1} = k_{21}$. So we fix $k_{1b_1} = k_{21}$ such that these conditions are fulfilled. Under these conditions, the forms $\nu_{z_1, \mathbf{m}_1, (k^1, k_{1b_1})}^1$ and $\nu_{z_2, \mathbf{m}_2, k_{21}}^2$ satisfy still (8.33) (the only difference is that, now, $b_2 = 1$). The magnetic forms on Σ_1 and Σ_2 glue to form a magnetic 1-form on Σ by the relation $\nu_{z, \mathbf{m}, \mathbf{k}} = \nu_{z_1, \mathbf{m}_1, (k^1, k_{1b_1})}^1 \mathbf{1}_{\Sigma_1} + \nu_{z_2, \mathbf{m}_2, k_{21}}^2 \mathbf{1}_{\Sigma_2}$, for $\mathbf{k} := \mathbf{k}_1 \in \mathbb{Z}^{b_1-1}$. Finally we consider two defect graphs on Σ_1 and Σ_2 as in Lemma 8.7, and glue them to get a defect graph on Σ .

We can then follow the proof of the case $b_2 \geq 2$ with the difference that, on Σ_2 , the harmonic extension $P(\tilde{\varphi} + n2\pi R)$ is now trivial in the sense that it is equal to $P\tilde{\varphi} + n2\pi R$. Furthermore, the free field amplitude on Σ_2 is now

$$\mathcal{A}_{\Sigma_2, g_2}^0(\tilde{\varphi} + n2\pi R) = e^{-\frac{1}{2}\langle (\mathbf{D}_{\Sigma_1} - \mathbf{D})(\tilde{\varphi} + n2\pi R), \tilde{\varphi} + n2\pi R \rangle_2} = \mathcal{A}_{\Sigma_2, g_2}^0(\tilde{\varphi})$$

so that no change in the relative cohomology on Σ_2 is involved (note that adding $n2\pi R$ to ϕ_2 in $\hat{\mathcal{B}}_2$ does not change $\hat{\mathcal{B}}_2$ by periodicity). We still have a change of relative cohomology on Σ_1 (i.e. again with an extra dP_n) and it produces the summation over the 1-form that was absent on Σ , i.e. $\omega_{b_1-1}^{1,c}$. We obtain this way the gluing statement. \square

Now we focus on the case of self-gluing for our reduced form amplitudes. The setup is the same as that drawn just before Proposition 8.14. We claim

Proposition 8.16. *Let F be periodic positive or periodic integrable.*

$$\mathcal{A}_{\Sigma^\#,g,z,m,\zeta}^0(F, \tilde{\varphi}_\#^{\mathbf{k}}) = C \int \mathcal{A}_{\Sigma,g,z,m,\zeta}^0(F, \tilde{\varphi}_\#^{\mathbf{k}}, \tilde{\varphi}^k, \tilde{\varphi}^k) d\mu_0(\tilde{\varphi}^k),$$

where $C = \frac{1}{\sqrt{2\pi}}$ if $\partial\Sigma \neq \emptyset$ and $C = \sqrt{2}$ if $\partial\Sigma = \emptyset$.

Proof. We denote by \mathcal{C} the glued curve on Σ . Again, we split the proof in two parts depending whether $\Sigma^\#$ has a non-empty boundary or not.

(1) Assume first $\partial\Sigma^\# \neq \emptyset$. Let us call σ a basis of the relative homology on Σ . We choose this basis by taking $a_1, \dots, a_g, b_1, \dots, b_g, d_1, \dots, d_{b-1}$ where a_j, b_j are chosen inside Σ° such that the intersection numbers are given as in (3.4), and the d_j are non-intersecting simple curves (not closed) with the base point on $\partial_j\Sigma$ and the endpoint on $\partial_{j+1}\Sigma$, and each d_j is not intersecting any other curve of the basis (see Figure 19). Let $\omega_1^c, \dots, \omega_{2g+b-1}^c$ be a basis of $\mathcal{H}_R^1(\Sigma, \partial\Sigma)$ dual to σ made of closed forms that are compactly supported inside Σ° (hence can be viewed as compactly supported closed 1-forms on $\Sigma^\#$ too). We stress that the last boundary-to-boundary arc d_{b-1} joining $\partial_{b-1}\Sigma$ to $\partial_b\Sigma$ will form a cycle in the glued surface, and therefore will play a special role in what follows. For $\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}$, we set $\omega_{\mathbf{k}^c}^c := \sum_{j=1}^{2g+b-1} k_j^c \omega_j^c$ as usual.

Now we focus on the absolute cohomology and magnetic 1-forms. We discard first trivial cases: indeed, if $\mathbf{k} = (k_1, \dots, k_{b-2})$, notice that in case $\sum_{j=1}^{n_m} m_j + \sum_{j=1}^{b-2} \zeta_j k_j \neq 0$, both sides in the gluing statement vanish (because of Dirac masses) so that equality obviously holds. So the case of interest is

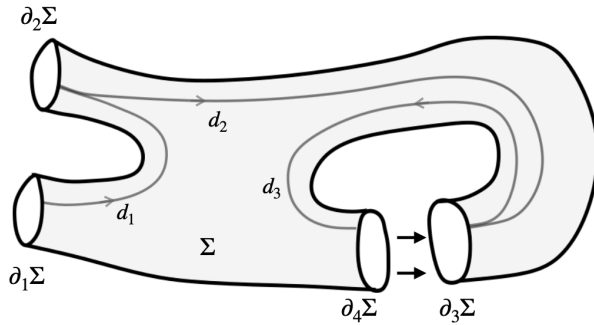
$$(8.38) \quad \sum_{j=1}^{n_m} m_j + \sum_{j=1}^{b-2} \zeta_j k_j = 0,$$

in which case the winding numbers around $\partial_{b-1}\Sigma$ and $\partial_b\Sigma$ must satisfy $k_{b-1} = k_b$. We will simply write k for $k_{b-1} = k_b$. Under these conditions, we consider the 1-form $\nu_{z,m,(k,k,k)}$ on Σ given by Lemma 3.10 satisfying

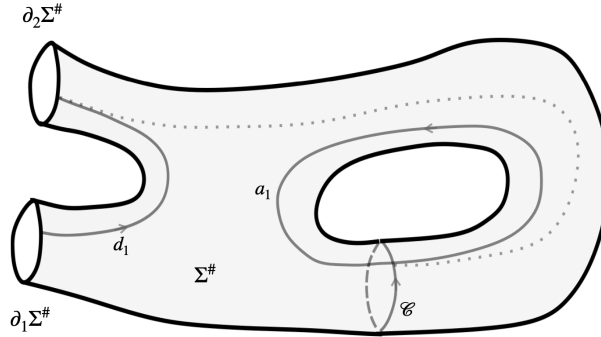
$$\forall j = 1, \dots, b-2, \int_{\partial_j\Sigma} \nu_{z,m,(k,k,k)} = 2\pi R \zeta_j k_j,$$

$$\int_{\partial_b\Sigma} \nu_{z,m,(k,k,k)} = - \int_{\partial_{b-1}\Sigma} \nu_{z,m,(k,k,k)} = 2\pi R k.$$

The last condition (and the fact that both forms involved are of the form $\pm k R d\theta$ over a neighborhood of the boundaries $\partial_{b-1}\Sigma$ and $\partial_b\Sigma$) allows us to self-glue (see Lemma 3.8) these forms by identifying $\partial_{b-1}\Sigma$ and $\partial_b\Sigma$ to get a closed 1-form on $\Sigma^\#$ denoted by $\nu_{z,m,(k,k)}$, satisfying our basic conditions for the definition of amplitude on the glued surface. Note that the curve \mathcal{C} will become a cycle on the glued surface, and therefore the form $\nu_{z,m,(k,k)}$, which has a winding k along this curve, will produce the missing part in the relative cohomology on the glued surface. For this we will split it as $\nu_{z,m,(k,k)} = \nu_{z,m,(k,0)} + \nu_{z,0,(0,k)}$, the last term producing 1-forms with winding k along \mathcal{C} and vanishing along any other cycle.



(A) Unglued surface Σ . In gray, the relative homology on Σ .



(B) Glued surface. The curve d_1 remains in $\mathcal{H}_1(\Sigma^\#, \partial\Sigma^\#)$ after gluing. The curve d_3 becomes a closed curve denoted a_1 in $\mathcal{H}_1(\Sigma^\#, \partial\Sigma^\#)$. The glued circle \mathcal{C} is a cycle on $\Sigma^\#$, with intersection number 1 with a_1 . The dotted arc $d_2 \in \mathcal{H}_1(\Sigma\partial\Sigma)$ has been removed to construct $\mathcal{H}_1(\Sigma^\#, \partial\Sigma^\#)$.

FIGURE 19. Case $\partial\Sigma \neq \emptyset$

As outlined above, we get now a basis $\sigma_\#$ of the relative homology on $\Sigma^\#$, which has $\mathfrak{b} - 2$ boundary components, by taking the cycles $a_1, \dots, a_g, b_1, \dots, b_g$ together with the cycles $d_{\mathfrak{b}-1}, \mathcal{C}$, and the boundary-to-boundary arcs $d_1, \dots, d_{\mathfrak{b}-3}$. Note that the arc $d_{\mathfrak{b}-2}$ has been discarded. A basis of the relative cohomology, dual to $\sigma_\#$, is now $\omega_1^c, \dots, \omega_{2g+\mathfrak{b}-3}^c, \omega_{2g+\mathfrak{b}-1}^c, \nu_{\mathbf{z}, \mathbf{0}, (0, k)}$ ($\omega_{2g+\mathfrak{b}-2}^c$ has been discarded since dual to $d_{\mathfrak{b}-2}$). Absolute cohomology and magnetic 1-forms are then encoded in the forms $\nu_{\mathbf{z}, \mathbf{m}, (\mathbf{k}, 0)}$ for $\mathbf{k} \in \mathbb{Z}^{\mathfrak{b}-2}$ with defect graph given by Lemma 8.8. Since we have removed the 1-form $\omega_{2g+\mathfrak{b}-2}^c$, we need to consider 1-forms $\omega_{\mathbf{k}^c}^c$ where the $(2g + \mathfrak{b} - 2)$ -th component of \mathbf{k}^c has been set to 0. We will thus need to consider the vector $(k_1^c, \dots, k_{2g+\mathfrak{b}-3}^c, 0, k_{2g+\mathfrak{b}-1}^c) \in \mathbb{Z}^{2g+\mathfrak{b}-1}$, which will be identified with $\mathbf{k}^c_- \in \mathbb{Z}^{2g+\mathfrak{b}-2}$. Also we consider the defect graph on $\Sigma^\#$ obtained from Lemma 8.8. The definition of the amplitude on $\partial\Sigma^\#$ then yields

$$\begin{aligned} \mathcal{A}_{\Sigma^\#, g, \mathbf{z}, \mathbf{m}, \zeta_\#}^0(F, \tilde{\varphi}_\#^{\mathbf{k}}) &:= \sum_{(\mathbf{k}^c_-, k) \in \mathbb{Z}^{2g+\mathfrak{b}-2} \times \mathbb{Z}} e^{-\frac{1}{4\pi} \|\nu_{\mathbf{z}, \mathbf{m}, (\mathbf{k}, 0)} + \omega_{\mathbf{k}^c_-}^c + \nu_{\mathbf{z}, \mathbf{0}, (0, k)}\|_{g, 0}^2} \\ &\times Z_{\Sigma^\#, g} \mathcal{A}_{\Sigma^\#, g}^0(\tilde{\varphi}_\#) \mathcal{B}_{\Sigma^\#, g}(F, \tilde{\varphi}_\#, \mathbf{k}, \mathbf{k}^c_-, k) \end{aligned}$$

with

$$\mathcal{B}_{\Sigma^\#,g}(F, \tilde{\varphi}_\#, \mathbf{k}, \mathbf{k}_-, k) := \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_{g,D} + dP\tilde{\varphi}_\#, \nu_{z,m,(k,0)} + \omega_{\mathbf{k}_-}^c + \nu_{z,0,(0,k)} \rangle} F(\phi_g) \right],$$

where the Liouville field is $\phi_g = X_{g,D} + P\tilde{\varphi}_\# + I_{x_0}^{\xi_\#}(\nu_{z,m,(k,0)}) + I_{x_0}^{\sigma_\#}(\omega_{\mathbf{k}_-}^c) + I_{x_0}^{\sigma_\#}(\nu_{z,0,(0,k)})$, the expectation \mathbb{E} is over the Dirichlet GFF $X_{g,D}$ on $\Sigma^\#$, $P\tilde{\varphi}_\#$ stands for the harmonic extension to $\Sigma^\#$ of the boundary fields $\tilde{\varphi}_\#$, which stand respectively for the boundary conditions on the remaining (i.e. unglued) components of $\partial\Sigma^\#$, namely

$$\Delta_g P\tilde{\varphi}_\# = 0 \quad \text{on } \Sigma^\#, \quad P\tilde{\varphi}_\#|_{\partial_j \Sigma^\#} = \tilde{\varphi}_{\#j} \circ \zeta_j^{-1} \text{ for } j \leq \mathfrak{b} - 2.$$

Let now X be an independent Dirichlet GFF on Σ . Then we have the following decomposition in law (see Proposition 5.1)

$$X_{g,D} \stackrel{\text{law}}{=} X + P\mathbf{Y},$$

where \mathbf{Y} is the restriction of $X_{g,D}$ to the glued boundary component \mathcal{C} expressed in parametrised coordinates, i.e. $\mathbf{Y} = X_{g,D}|_{\mathcal{C}} \circ \zeta$, and $P\mathbf{Y}$ is its harmonic extension to Σ vanishing on $\partial\Sigma^\#$, which is non-empty. Again we denote by $h_{\mathcal{C}}$ the restriction of the harmonic function $P\tilde{\varphi}_\#$ to \mathcal{C} in parametrised coordinates

$$h_{\mathcal{C}} := P\tilde{\varphi}_\#|_{\mathcal{C}} \circ \zeta$$

and, on Σ , the function $P\tilde{\varphi}_\# + P\mathbf{Y}$ is harmonic with boundary values (expressed in parametrised coordinates on Σ) $\tilde{\varphi}_{\#j}$ on $\partial_j \Sigma$ for $j < \mathfrak{b} - 2$ and $(\mathbf{Y} + h_{\mathcal{C}})$ on $\partial_{\mathfrak{b}-1} \Sigma$ and $\partial_{\mathfrak{b}} \Sigma$. Proceeding as in the proof of Proposition 8.15, using the law of the field \mathbf{Y} proved in [31, Lemma 5.3 and Lemma 5.4] and applying the Chasles relation, we arrive at the expression

$$\begin{aligned} \mathcal{A}_{\Sigma^\#,g,z,m,\zeta_\#}^0(F, \tilde{\varphi}_\#^{\mathbf{k}}) &= \sum_{(\mathbf{k}_-^c, k, n) \in \mathbb{Z}^{2\mathfrak{q}+\mathfrak{b}-2} \times \mathbb{Z} \times \mathbb{Z}} e^{-\frac{1}{4\pi} \|\nu_{z,m,(k,0)} + \omega_{\mathbf{k}_-}^c + \nu_{z,0,(0,k)}\|_{g,0}^2} Z_{\Sigma,g} \\ &\quad \times \int_0^{2\pi R} H_2(c + 2\pi nR, k, \mathbf{k}^c) \frac{dc}{\sqrt{2\pi}}, \end{aligned}$$

where (expectation \mathbb{E} is taken with respect to φ)

$$H_2(c, k, \mathbf{k}^c) := \mathbb{E} \left[\mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}_\#, c + \varphi, c + \varphi) \mathcal{B}_{\Sigma,g}(F, \tilde{\varphi}_\#, c + \varphi, c + \varphi, \mathbf{k}, \mathbf{k}_-, k) \right],$$

and (expectation \mathbb{E} is taken with respect to the Dirichlet GFF X on Σ)

$$\mathcal{B}_{\Sigma,g}(F, \tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}, \mathbf{k}, \mathbf{k}_-, k) := \mathbb{E} \left[F(\phi) e^{-\frac{1}{2\pi} \langle dX + dP(\tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}), \nu_{z,m,(k,k)} + \omega_{\mathbf{k}_-}^c \rangle} \right],$$

where $\phi = X + P(\tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}) + I_{x_0}^{\sigma}(\omega_{\mathbf{k}_-}^c) + I_{x_0}^{\xi}(\nu_{z,m,(k,k)})$ and $P(\tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi})$ stands for the harmonic extension to Σ of the boundary fields $\tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}$.

Next, we introduce the harmonic functions P_n , for $n \in \mathbb{Z}$, that are harmonic on Σ with boundary values $n2\pi R$ on both boundary components corresponding to \mathcal{C} in Σ , and 0 on the other boundary components of Σ . Then we notice that, writing $\tilde{\varphi}$ for $c + \varphi$,

$$\begin{aligned} &\mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}_\#, \tilde{\varphi} + n2\pi R, \tilde{\varphi} + n2\pi R) \\ &= e^{-\frac{1}{2} \langle \mathbf{D}_\Sigma - \mathbf{D} \rangle (\tilde{\varphi}_\#, \tilde{\varphi} + n2\pi R, \tilde{\varphi} + n2\pi R), (\tilde{\varphi}_\#, \tilde{\varphi} + n2\pi R, \tilde{\varphi} + n2\pi R)} \\ &= \mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}) e^{-\frac{1}{2\pi} \langle dP(\tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}), dP_n \rangle} e^{-\frac{1}{4\pi} \|dP_n\|_2^2}. \end{aligned}$$

Notice also, using the similar argument as for (8.36), that

$$\|\nu_{z,m,(k,k)} + \omega_{\mathbf{k}^-}^c\|_{g,0}^2 + \|dP_n\|_2^2 = \|\nu_{z,m,(k,k)} + \omega_{\mathbf{k}^-}^c + dP_n\|_{g,0}^2.$$

Therefore, one obtains

$$\begin{aligned} \mathcal{A}_{\Sigma^\#,g,z,m,\zeta^\#}^0(F, \tilde{\varphi}_\#^{\mathbf{k}}) &= \sum_k \sum_{(\mathbf{k}^-,n)} e^{-\frac{1}{4\pi}\|\nu_{z,m,(k,k)} + \omega_{\mathbf{k}^-}^c + dP_n\|_{g,0}^2} Z_{\Sigma,g} \\ &\times \int_0^{2\pi R} \mathbb{E}\left[\mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}) \hat{\mathcal{B}}_{\Sigma,g}(F, \tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}, \mathbf{k}, \mathbf{k}^-, k, n)\right] \frac{dc}{\sqrt{2\pi}}, \end{aligned}$$

where sums run respectively in \mathbb{Z} and $\mathbb{Z}^{2g+b-2} \times \mathbb{Z}$, and

$$\begin{aligned} \hat{\mathcal{B}}_{\Sigma,g}(F, \tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}, \mathbf{k}, \mathbf{k}^-, k, n) \\ := \mathbb{E}\left[F(\phi + P_n) e^{-\frac{1}{2\pi}\langle dX + dP(\tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}), \nu_{z,m,(k,k)} + \omega_{\mathbf{k}^-}^c + dP_n \rangle}\right]. \end{aligned}$$

Again, the point is now to relate dP_n to the relative cohomology: indeed, it encodes the 1-forms dual to the boundary-to-boundary arc d_{b-2} that we have previously removed. To see this, let us first introduce the notation \mathbf{n} for the vector $(0, \dots, n, 0) \in \mathbb{Z}^{2g+b-1}$. Since the 1-form dP_n is exact and since it takes the value 0 on $\partial_j \Sigma$ for $j \leq b-2$ and n on both $\partial_{b-1} \Sigma$ and $\partial_b \Sigma$, we have

$$\int_{a_j} dP_n = 0, \quad \int_{b_j} dP_n = 0, \quad \int_{d_j} dP_n = 0 \text{ for } j \neq b-2, \quad \int_{d_{b-2}} dP_n = n2\pi R.$$

The 1-form dP_n has therefore the same cycles/arcs as the 1-form $\omega_{\mathbf{n}}^c$. Thus, since $dP_n \in \mathcal{H}_R^1(\Sigma, \partial\Sigma)$ satisfies the relative boundary condition, we have $dP_n = \omega_{\mathbf{n}}^c + df_{\mathbf{n}}$, for some smooth function on Σ vanishing on the boundary $\partial\Sigma$. As in the proof of Proposition 8.15, we can replace dP_n by $\omega_{\mathbf{n}}^c + df_{\mathbf{n}}$ in the expression of $\hat{\mathcal{B}}_{\Sigma,g}$ and apply Girsanov to the term $e^{-\frac{1}{2\pi}\langle dX, df_{\mathbf{n}} \rangle - \frac{1}{4\pi}\|df_{\mathbf{n}}\|_2^2}$ to get that

$$\begin{aligned} \mathcal{A}_{\Sigma^\#,g,v,m,\zeta^\#}^0(F, \tilde{\varphi}_\#^{\mathbf{k}}) &= \sum_k \sum_{(\mathbf{k}^-,n)} e^{-\frac{1}{4\pi}\|\nu_{z,m,(k,k)} + \omega_{\mathbf{k}^-}^c + \omega_{\mathbf{n}}^c\|_2^2} Z_{\Sigma,g} \\ &\times \int_0^{2\pi R} \mathbb{E}\left[\mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}) \tilde{\mathcal{B}}_{\Sigma,g}(F, \tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}, \mathbf{k}, \mathbf{k}^-, k, n)\right] \frac{dc}{\sqrt{2\pi}}, \end{aligned}$$

where the sums run over \mathbb{Z} and $\mathbb{Z}^{2g+b-2} \times \mathbb{Z}$, and

$$\begin{aligned} \tilde{\mathcal{B}}_{\Sigma,g}(F, \tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}, \mathbf{k}, \mathbf{k}^-, k, n) \\ := \mathbb{E}\left[F(\phi + I_{x_0}^\sigma(\omega_{\mathbf{n}}^c)) e^{-\frac{1}{2\pi}\langle dX + dP(\tilde{\varphi}_\#, \tilde{\varphi}, \tilde{\varphi}), \nu_{z,m,(k,k)} + \omega_{\mathbf{k}^-}^c + \omega_{\mathbf{n}}^c \rangle}\right]. \end{aligned}$$

This means that, in the expression of $\tilde{\mathcal{B}}_{\Sigma,g}$, we have the relation $\omega_{\mathbf{k}^-}^c + \omega_{\mathbf{n}}^c = \omega_{\mathbf{k}^- + \mathbf{n}}^c$. Therefore, the relative cohomology term in $\tilde{\mathcal{B}}_{\Sigma,g}$ is $\omega_{\mathbf{k}^- + \mathbf{n}}^c$ and the summation $\sum_{(\mathbf{k}^-,n) \in \mathbb{Z}^{2g+b-2} \times \mathbb{Z}}$ thus corresponds to a sum over the whole relative cohomology basis on Σ . This proves the claim.

(2) Assume now $\partial\Sigma^\# = \emptyset$. The surface Σ has now two boundary components $\partial_1 \Sigma, \partial_2 \Sigma$, which we want to glue to get a surface $\Sigma^\#$. We take a basis σ of $\mathcal{H}_1(\Sigma, \partial\Sigma)$ made of $a_1, \dots, a_g, b_1, \dots, b_g, d_1$ where a_j, b_j are cycles chosen inside Σ° such that the intersection numbers are given as in (3.4), and d_1 is a non-intersecting simple curve

(not closed) with the base point on $\partial_1\Sigma$ and the endpoint on $\partial_2\Sigma$, and d_1 is not intersecting any other curve of the basis. Let $\omega_1^c, \dots, \omega_{2g+1}^c$ be a basis of $\mathcal{H}_R^1(\Sigma, \partial\Sigma)$ dual to σ made of closed forms that are compactly supported inside Σ° (hence can be viewed as closed 1-forms on $\Sigma^\#$ too). For $\mathbf{k}^c \in \mathbb{Z}^{2g+1}$, we set $\omega_{\mathbf{k}^c}^c := \sum_{j=1}^{2g+1} k_j^c \omega_j^c$ as usual.

We focus now on the absolute cohomology and magnetic forms. Again, we identify the only non-trivial case to be treated. It corresponds to

$$(8.39) \quad \sum_{j=1}^{n_m} m_j = 0, \quad k_1 = k_2.$$

As such, we will simply write k for $k_1 = k_2$. Next, we consider the closed 1-form $\nu_{\mathbf{z}, \mathbf{m}, (k, k)}$ with winding numbers given by

$$\int_{\partial_1\Sigma} \nu_{\mathbf{z}, \mathbf{m}, (k, k)} = -2\pi Rk, \quad \int_{\partial_2\Sigma} \nu_{\mathbf{z}, \mathbf{m}, (k, k)} = 2\pi Rk,$$

winding m_j around z_j , and 0 along any other interior cycle. The condition above (and the fact that both forms involved are of the form $\pm Rk d\theta$ over a neighborhood of the boundaries $\partial_1\Sigma$ and $\partial_2\Sigma$) allows us to self-glue this form by identifying $\partial_1\Sigma$ and $\partial_2\Sigma$ to get a closed 1-form on $\Sigma^\#$ denoted by $\nu_{\mathbf{z}, \mathbf{m}, k}$. Note that the curve \mathcal{C} will become a cycle on the glued surface, and therefore the form $\nu_{\mathbf{z}, \mathbf{m}, k}$, which has a winding k along this curve, will be part of the cohomology on the glued surface. For this, we split $\nu_{\mathbf{z}, \mathbf{m}, k}$ as $\nu_{\mathbf{z}, \mathbf{m}, 0} + \nu_{\mathbf{z}, 0, k}$.

Now we get a basis $\sigma_\#$ of homology on $\Sigma^\#$, which is a closed surface, by taking the cycles $a_1, \dots, a_g, b_1, \dots, b_g$ together with the cycles d_1, \mathcal{C} . A basis of cohomology, dual to $\sigma_\#$, is now $\omega_1^c, \dots, \omega_{2g+1}^c, \nu_{\mathbf{z}, 0, k}$. The magnetic 1-form on $\Sigma^\#$ is now $\nu_{\mathbf{z}, \mathbf{m}, 0}$.

Based on this, the rest of the proof is a combination of arguments already used so we just outline the proof. We condition the amplitude on $\Sigma^\#$ on the values of the GFF along \mathcal{C} and use the description of this law given by [31, eq (5.14)] (similar to proof of Proposition 8.13 case $\partial\Sigma = \emptyset$). We apply the Chasles relation on the c -integral, then we use

$$\mathcal{A}_{\Sigma, g}^0(\tilde{\varphi} + n2\pi R, \tilde{\varphi} + n2\pi R) = \mathcal{A}_{\Sigma, g}^0(\tilde{\varphi}, \tilde{\varphi})$$

because harmonic functions on Σ , worth $n2\pi R$ on both $\partial_1\Sigma$ and $\partial_2\Sigma$, must be constant and equal to $n2\pi R$. Therefore there is no contribution coming from the shift by $n2\pi R$ and we get this way the expression of the glued amplitude on Σ . Details are left to the reader. □

8.7. Amplitudes are L^2 . In this section, we shall prove that the Liouville amplitudes of surfaces with boundary are in the Hilbert space $\mathcal{H}^{\otimes b}$. For this, the strategy will be, roughly speaking, to prove that the correlation functions on the doubled surface correspond to the squared norm of the amplitude.

To prove this statement the first lemma we need focuses on the effect of reverting orientation on a given Riemann surface with or without boundary. So we consider the setup for the definition of amplitudes. We denote by Σ' the surface Σ but with orientation reversed. Lemma 8.17 directly follows from definitions:

Lemma 8.17. *We have the relation*

$$\mathcal{A}_{\Sigma, g, \mathbf{z}, \mathbf{m}, \zeta}^0(F, \tilde{\varphi}^{\mathbf{k}}) = \mathcal{A}_{\Sigma', g, \mathbf{z}, -\mathbf{m}, \zeta}^0(F, \tilde{\varphi}^{\mathbf{k}}).$$

Proof. In view of the properties of the form $\nu_{z,\mathbf{k},\mathbf{m}}^\Sigma$ on Σ and $\nu_{z,\mathbf{k},\mathbf{m}}^{\Sigma'}$ on Σ' given in Proposition 3.10, we observe that

$$\nu_{z,\mathbf{m},\mathbf{k}}^{\Sigma'} = \nu_{z,-\mathbf{m},\mathbf{k}}^\Sigma.$$

This directly implies the result by using the definition of (8.29) and the fact that all the other quantities involved in the free field amplitude are independent of the orientation of Σ . \square

8.7.1. *Regularised amplitudes.* We consider a surface Σ with non-empty boundary and all the data from the setup for amplitudes. For $\epsilon > 0$, we denote by $\mathcal{A}_{\Sigma,g,x,v,\alpha,m,\zeta}^\epsilon(F, \tilde{\varphi}^{\mathbf{k}})$ the regularised amplitudes:

$$\begin{aligned} \mathcal{A}_{\Sigma,g,x,v,\alpha,m,\zeta}^\epsilon(F, \tilde{\varphi}^{\mathbf{k}}) &:= \delta_0\left(\sum_{j=1}^{n_m+b} m_j\right) \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}} e^{-\frac{1}{4\pi} \|\nu_{z,\mathbf{m},\mathbf{k}} + \omega_{\mathbf{k}^c}^c\|_{g,0}^2} Z_{\Sigma,g} \mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}) \\ &\times \mathbb{E}\left[e^{-\frac{1}{2\pi} \langle dX_{g,D} + dP\tilde{\varphi} + \nu_{z,\mathbf{m},\mathbf{k}} + \omega_{\mathbf{k}^c}^c \rangle} F(\phi_g) \prod_{j=1}^{n_m} V_{\alpha_j,g,\epsilon}(x_j) e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g \phi_g \, dv_g - \mu M_{\beta}^g(\phi_g, \Sigma)}\right]. \end{aligned}$$

Recall that amplitudes are defined as the limit $\lim_{x \rightarrow z} \lim_{\epsilon \rightarrow 0}$ of this quantity, where the limit when $x_i \rightarrow z_i$ is understood as in (6.33) along a curve with a tangent vector given by v_i . We will essentially follow the argument in Theorems 6.11 and 6.13 to take these limits, with adaptations due to the presence of a boundary.

The first step is to adapt the imaginary Girsanov transform. Let us define

$$u_x^{\epsilon, \epsilon'}(y) := \sum_{j=1}^{n_m} i\alpha_j \mathbb{E}[X_{g,D,\epsilon}(x_j) X_{g,D,\epsilon'}(y)].$$

We write simply $u_x^\epsilon(y)$ for the function $u_x^{\epsilon, \epsilon'}(y)$ for $\epsilon' = 0$. We claim:

Lemma 8.18. *The following identity holds*

$$\begin{aligned} \mathcal{A}_{\Sigma,g,x,v,\alpha,m,\zeta}^\epsilon(F, \tilde{\varphi}^{\mathbf{k}}) &= \\ \delta_0\left(\sum_{j=1}^{n_m+b} m_j(\mathbf{k})\right) C_\epsilon &\sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}} e^{-\frac{1}{4\pi} \|\nu_{z,\mathbf{m},\mathbf{k}} + \omega_{\mathbf{k}^c}^c\|_{g,0}^2} Z_{\Sigma,g} \mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}) \prod_{j=1}^{n_m} e^{i\alpha_j P\tilde{\varphi}_\epsilon(z_j)} \\ \mathbb{E}\left[e^{-\frac{1}{2\pi} \langle dX_{g,D} + dP\tilde{\varphi} + du_x^\epsilon + \nu_{z,\mathbf{m},\mathbf{k}} + \omega_{\mathbf{k}^c}^c \rangle} F(\phi_g + u_x^\epsilon) e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g(\phi_g + u_x^\epsilon) \, dv_g - \mu M_{\beta}^g(\phi_g + u_x^\epsilon, \Sigma)}\right], \end{aligned}$$

where expectation is over the Dirichlet GFF and the constant C_ϵ is given by

$$C_\epsilon := e^{-\sum_{j < j'} \alpha_j \alpha_{j'} \mathbb{E}[X_{g,D,\epsilon}(x_j) X_{g,D,\epsilon}(x_{j'})]} \prod_{j=1}^{n_m} \epsilon^{-\frac{\alpha_j^2}{2}} e^{-\frac{\alpha_j^2}{2} \mathbb{E}[X_{g,D,\epsilon}(x_j)^2]}.$$

Proof. This lemma relies on the Girsanov transform applied to $\prod_{j=1}^{n_m} e^{i\alpha_j X_{g,D,\epsilon}(x_j)}$, hence performing a shift to the GFF by $X_g \rightarrow X_g + u_x^\epsilon$, with the variance term of this shift producing the constant C_ϵ (up to the terms $\epsilon^{-\frac{\alpha_j^2}{2}}$). Therefore it follows the argument explained in the beginning of the proof of Theorem 6.11. To reproduce the argument, we need the following tools:

- (1) $\tilde{\varphi}$ a.s., the mapping $f \mapsto \int_{\Sigma} f(x)M_{\beta}^g(X_{g,D} + P\tilde{\varphi} + I_{x_0}^{\sigma}(\omega_{\mathbf{k}^c}^c) + I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m},\mathbf{k}}), dx)$ is a distribution (in the sense of Schwartz) of order 2 and there exists some L^2 random variable D_{Σ} such that

$$\forall f \in C_c^{\infty}(\Sigma), \quad \left| \int_{\Sigma} f(x)M_{\beta}^g(X_{g,D} + P\tilde{\varphi} + I_{x_0}^{\sigma}(\omega_{\mathbf{k}^c}^c) + I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m},\mathbf{k}}), dx) \right| \leq D_{\Sigma} \|\Delta_g f\|_{\infty}.$$

- (2) we have the estimate

$$\mathbb{E} \left[\exp \left(\left| \int_{\Sigma} f(x)M_{\beta}^g(X_{g,D} + P\tilde{\varphi} + I_{x_0}^{\sigma}(\omega_{\mathbf{k}^c}^c) + I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m},\mathbf{k}}), dx) \right| \mid \tilde{\varphi} \right) \right] \leq C \|f\|_{\infty}$$

for some deterministic constant $C > 0$, where expectation is taken over the Dirichlet GFF.

To establish the first property, we can write $f(x) = \int_{\Sigma} \Delta f(y)G_{g,D}(x, y) dv_g(y)$ and then follow the argument in Lemma 6.12 to get that

$$D_{\Sigma} = \int_{\Sigma} \left| \int_{\Sigma} G_{g,D}(x, y)M_{\beta}^g(X_g + P\tilde{\varphi} + I_{x_0}^{\sigma}(\omega_{\mathbf{k}^c}^c) + I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m},\mathbf{k}}), dx) \right| dv_g(y),$$

which is $\tilde{\varphi}$ a.s. in L^2 in the expectation with respect to the Dirichlet GFF.

For the exponential moment estimate, this follows from Proposition 5.3 again. Note that the contribution from the harmonic extension is trivial using $|e^{i\beta P\tilde{\varphi}}| = 1$, and this is why we get a deterministic bound. \square

Now we claim:

Lemma 8.19. *The following convergence result holds $(dc \otimes \mathbb{P}_{\mathbb{T}})^{\otimes b}$ almost surely:*

$$\mathcal{A}_{\Sigma, g, \mathbf{x}, \mathbf{v}, \alpha, \mathbf{m}, \zeta}^{\epsilon}(F, \tilde{\varphi}^{\mathbf{k}}) \rightarrow \mathcal{A}_{\Sigma, g, \mathbf{v}, \alpha, \mathbf{m}, \zeta}(F, \tilde{\varphi}^{\mathbf{k}})$$

as $\epsilon \rightarrow 0$ and $\mathbf{x} \rightarrow \mathbf{z}$ in the direction $\mathbf{v} = ((z_1, \nu_1), \dots, (z_n, \nu_n)) \in (T\Sigma)^n$, where

(8.40)

$$\begin{aligned} & \mathcal{A}_{\Sigma, g, \mathbf{v}, \alpha, \mathbf{m}, \zeta}(F, \tilde{\varphi}^{\mathbf{k}}) \\ &= \delta_0 \left(\sum_{j=1}^{n_m + b} m_j(\mathbf{k}) Z_{\Sigma, g} \mathcal{A}_{\Sigma, g}^0(\tilde{\varphi}) e^{-\sum_{j < j'} \alpha_j \alpha_{j'} G_{g,D}(z_j, z_{j'})} \prod_{j=1}^{n_m} e^{i\alpha_j P\tilde{\varphi}(z_j) - \sum_j \frac{\alpha_j^2}{2} W_g(z_j)} \right. \\ & \times \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}} e^{-\frac{1}{4\pi} \|\nu_{\mathbf{z},\mathbf{m},\mathbf{k}} + \omega_{\mathbf{k}^c}^c\|_{g,0}^2} \prod_j e^{i\alpha_j (I_{x_0}^{\sigma}(\omega_{\mathbf{k}^c}^c) + I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m},\mathbf{k}}))(z_j)} \times \\ & \left. \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_{g,D} + dP\tilde{\varphi} + du_{\mathbf{z},\nu_{\mathbf{z},\mathbf{m},\mathbf{k}}} + \omega_{\mathbf{k}^c}^c \rangle} F(\phi_g + u_{\mathbf{z}}) e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g(\phi_g + u_{\mathbf{z}}) dv_g - \mu M_{\beta}^g(\phi_g + u_{\mathbf{z}}, \Sigma)} \right], \right. \end{aligned}$$

where $u_{\mathbf{z}}(x) = \sum_{j=1}^{n_m} i\alpha_j G_{g,D}(x, z_j)$ and the Liouville field is $\phi_g = X_{g,D} + P\tilde{\varphi} + I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m},\mathbf{k}}) + I_{x_0}^{\sigma}(\omega_{\mathbf{k}^c}^c)$. Expectation is over the Dirichlet GFF and the evaluation of $I_{x_0}^{\xi}(\nu_{\mathbf{z},\mathbf{m},\mathbf{k}})$ at the points z_j is done in the direction prescribed by the vectors \mathbf{v} .

Proof. Beware that the pairing $\langle du_{\mathbf{z}}, \nu_{\mathbf{z},\mathbf{m},\mathbf{k}} \rangle$ makes sense due to the form of the singularity at z_j : it is radial for the Green $du_{\mathbf{z}}$ whereas it is angular for $\nu_{\mathbf{z},\mathbf{m},\mathbf{k}}$. The limit in ϵ

is the same as in the proof of Theorems 6.11 and 6.13: denoting by $\Delta_{\epsilon, \mathbf{x}}$ the difference between regularised amplitudes, the argument of Theorems 6.11 and 6.13 leads to

$$|\Delta_{\epsilon, \mathbf{x}}| \leq \sum_{\mathbf{k} \in \mathbb{Z}^{2\mathfrak{b}}} \mathcal{A}_{\Sigma, g}^0(\tilde{\varphi}) e^{-\frac{1}{4\pi} \|\Pi_1^c \omega_{\mathbf{k}^c}^c\|_2^2 + C|\mathbf{k}|} e^{-\frac{1}{2\pi} \langle dP\tilde{\varphi}, \Pi_1^c \omega_{\mathbf{k}^c}^c \rangle} C_{\epsilon, \mathbf{x}, \mathbf{z}}(F, \tilde{\varphi})$$

for some constant $C_{\epsilon, \mathbf{x}, \mathbf{z}}(F, \tilde{\varphi})$ such that $\lim_{\mathbf{x} \rightarrow \mathbf{z}} \lim_{\epsilon \rightarrow 0} C_{\epsilon, \mathbf{x}, \mathbf{z}}(F, \tilde{\varphi}) = 0$, $\mu_0^{\otimes \mathfrak{b}}$ almost surely in $\tilde{\varphi}$. Here again, we complete the argument with the fact that $e^{-\frac{1}{2\pi} \langle dP\tilde{\varphi}, \Pi_1^c \omega_{\mathbf{k}^c}^c \rangle} \leq e^{C|\mathbf{k}|}$. □

We prove now that this expression makes sense as an element in $\mathcal{H}^{\otimes \mathfrak{b}}$.

Lemma 8.20. *Let (Σ, g) be an admissible surface with \mathfrak{b} boundary components and parametrisations ζ . Let $\mathbf{m} \in \mathbb{Z}^{n_{\mathfrak{m}}}$ and $\alpha \in (\frac{1}{R}\mathbb{Z})^{n_{\mathfrak{m}}}$ satisfying (6.18), $\mathbf{v} \in (T\Sigma)^{n_{\mathfrak{m}}}$. If $F \in \mathcal{E}_R^{\mathfrak{m}}$, the amplitudes defined in Definition 8.10 satisfy*

$$\mathcal{A}_{\Sigma, g, \mathbf{v}, \alpha, \mathbf{m}, \zeta}(F, \cdot) \in \mathcal{H}^{\otimes \mathfrak{b}}.$$

Proof. Consider another copy of Σ , call it Σ' , with reverted orientation. We can glue Σ to Σ' along the \mathfrak{b} boundary components (the i -th boundary component of Σ is glued to the corresponding i -th boundary component of Σ') to get the double surface $\Sigma^{\#2}$ without boundary, which comes equipped with an involution $\tau : \Sigma^{\#2} \rightarrow \Sigma^{\#2}$ mapping a point x in Σ to its copy in Σ' . The metric g also extends to $\Sigma^{\#2}$ to a symmetric metric under τ .

We want to prove that the amplitude $(\tilde{\varphi}, \mathbf{k}) \mapsto \mathcal{A}_{\Sigma, g, \mathbf{v}, \alpha, \mathbf{m}, \zeta}(F, \tilde{\varphi}^{\mathbf{k}})$ is in $\mathcal{H}^{\otimes \mathfrak{b}}$.

Let us consider the amplitude

$$\mathcal{A}_{\Sigma, g, \mathbf{z}, \mathbf{m}, \zeta}^0(F_{\mathbf{z}}, \tilde{\varphi}^{\mathbf{k}}),$$

where

$$F_{\mathbf{z}}(\phi) = e^{-\frac{1}{2\pi} \langle du_{\mathbf{z}}, \nu_{\mathbf{z}, \mathbf{m}, \mathbf{k}} + \omega_{\mathbf{k}^c}^c \rangle} |F(\phi + u_{\mathbf{z}})| |e^{-\mu M_{\beta}^g(\phi + u_{\mathbf{z}}, \Sigma)}|$$

and the function $u_{\mathbf{z}}$ is given by $u_{\mathbf{z}}(x) = \sum_{j=1}^{n_{\mathfrak{m}}} i\alpha_j G_{g, D}(x, z_j)$. Note that

$$\mathcal{A}_{\Sigma, g, \mathbf{z}, \mathbf{m}, \zeta}^0(F_{\mathbf{z}}, \tilde{\varphi}^{\mathbf{k}}) \geq 0.$$

Furthermore, given the formula for amplitudes (8.40), we have

$$|\mathcal{A}_{\Sigma, g, \mathbf{v}, \alpha, \mathbf{m}, \zeta}(F, \tilde{\varphi}^{\mathbf{k}})| \leq C \mathcal{A}_{\Sigma, g, \mathbf{v}, \mathbf{m}, \zeta}^0(F_{\mathbf{z}}, \tilde{\varphi}^{\mathbf{k}})$$

for some C , which takes into account all trivial factors (it may depend on \mathbf{z}, α).

We can then glue the amplitudes $\mathcal{A}_{\Sigma, g, \mathbf{z}, \mathbf{m}, \zeta}^0(F_{\mathbf{z}}, \tilde{\varphi}^{\mathbf{k}})$ and $\mathcal{A}_{\Sigma', g, \mathbf{z}, -\mathbf{m}, \zeta}^0(F_{\mathbf{z}}, \tilde{\varphi}^{\mathbf{k}})$ using Propositions 8.15 and 8.16 to get

$$C' \int \mathcal{A}_{\Sigma, g, \mathbf{z}, \mathbf{m}, \zeta}^0(F_{\mathbf{z}}, \tilde{\varphi}^{\mathbf{k}}) \mathcal{A}_{\Sigma', \tau^* g, \mathbf{z}, -\mathbf{m}, \zeta}^0(F_{\mathbf{z}}, \tilde{\varphi}^{\mathbf{k}}) d\mu_0^{\otimes \mathfrak{b}}(\tilde{\varphi}^{\mathbf{k}}) = \mathcal{A}_{\Sigma^{\#2}, g, \hat{\mathbf{z}}, \hat{\mathbf{m}}}^0(\hat{F}_{\mathbf{z}})$$

for some explicit constant C' appearing in Propositions 8.15 and 8.16, where $\hat{\mathbf{z}} = (\mathbf{z}, \tau(\mathbf{z}))$, and $\hat{\mathbf{m}} = (\mathbf{m}, -\mathbf{m})$. The functional $\hat{F}_{\mathbf{z}}(\phi)$ is given by

$$\hat{F}_{\mathbf{z}}(\phi) := e^{-\frac{1}{2\pi} \langle d\hat{u}_{\mathbf{z}}, \nu_{\mathbf{z}, \mathbf{m}} + \omega_{\mathbf{k}} \rangle} |F(\phi + \hat{u}_{\mathbf{z}}|_{\Sigma})| \cdot |F(\phi + \hat{u}_{\mathbf{z}}|_{\Sigma'})| \cdot |e^{-\mu M_{\beta}^g(\phi + \hat{u}_{\mathbf{z}}, \Sigma^{\#2})}|$$

and $\hat{u}_{\mathbf{z}} = u_{\mathbf{z}} \mathbf{1}_{\Sigma} + \tau^* u_{\mathbf{z}} \mathbf{1}_{\Sigma'}$. An argument similar to the proof of Theorem 6.13 shows that the amplitude $\mathcal{A}_{\Sigma^{\#2}, g, \hat{\mathbf{z}}, \hat{\mathbf{m}}}^0(\hat{F}_{\mathbf{z}})$ is finite (there, we have the Green function G_g on $\Sigma^{\#2}$

instead of the Dirichlet Green function in the definition of \hat{u}_z , but this is harmless to adapt). Therefore, using Lemma 8.17, we deduce

$$\begin{aligned} & \int |\mathcal{A}_{\Sigma, g, v, \alpha, m, \zeta}(F, \tilde{\varphi}^k)|^2 d\mu_0^{\otimes b}(\tilde{\varphi}^k) \\ & \leq C^2 \int |\mathcal{A}_{\Sigma, g, z, m, \zeta}^0(F_z, \tilde{\varphi}^k)|^2 d\mu_0^{\otimes b}(\tilde{\varphi}^k) \\ & = C^2 \int \mathcal{A}_{\Sigma, g, z, m, \zeta}^0(F_z, \tilde{\varphi}^k) \mathcal{A}_{\Sigma', \tau^*g, z, -m, \zeta}^0(F_z, \tilde{\varphi}^k) d\mu_0^{\otimes b}(\tilde{\varphi}(\mathbf{k})) \\ & = \frac{C^2}{C'} \mathcal{A}_{\Sigma^{\#2}, g, \hat{z}, \hat{m}}^0(\hat{F}_z) < +\infty. \end{aligned}$$

This shows that the amplitudes are in $\mathcal{H}^{\otimes b}$. Note that the regularised amplitudes are in $\mathcal{H}^{\otimes b}$ for the same reason. □

8.8. Proof of Propositions 8.13, 8.14 and 8.12. Now we focus on the proof of Propositions 8.13 and 8.14. We begin with the first one. Recall the setup is described just before the statement of Proposition 8.13. First we claim that the gluing property holds for ϵ -regularised amplitudes

(8.41)

$$\begin{aligned} & \mathcal{A}_{\Sigma^{\#}, g, x, v, \alpha, m, \zeta}^\epsilon(G_1 \otimes G_2, \tilde{\varphi}_1^{k_1}, \tilde{\varphi}_2^{k_2}) \\ & = C \int \mathcal{A}_{\Sigma_1, g_1, x_1, v_1, \alpha_1, m_1, \zeta_1}^\epsilon(G_1, \tilde{\varphi}_1^{k_1}, \tilde{\varphi}^k) \mathcal{A}_{\Sigma_2, g_2, x_2, v_2, \alpha_2, m_2, \zeta_2}^\epsilon(G_2, \tilde{\varphi}_2^{k_2}, \tilde{\varphi}^k) d\mu_0(\tilde{\varphi}^k), \end{aligned}$$

where $C = \frac{1}{(\sqrt{2\pi})}$ if $\partial\Sigma \neq \emptyset$ and $C = \sqrt{2}$ if $\partial\Sigma = \emptyset$. For this we apply Proposition 8.15

with $F_i(\phi) = G_i(\phi) \prod_{j=1}^{n_i^m} V_{\alpha_{ij}, g, \epsilon}(x_j) e^{-\frac{iQ}{4\pi} \int_{\Sigma_i}^{\text{reg}} K_g \phi_g dv_g - \mu M_\beta^g(\phi_g, \Sigma_i)}$ for $i = 1, 2$. Lemma 8.7 makes sure that

$$F_1 \otimes F_2(\phi) = G_1 \otimes G_2(\phi) \prod_{i=1,2} \left(\prod_{j=1}^{n_i^m} V_{\alpha_{ij}, g, \epsilon}(x_j) \right) e^{-\frac{iQ}{4\pi} \int_\Sigma^{\text{reg}} K_g \phi_g dv_g - \mu M_\beta^g(\phi_g, \Sigma)}.$$

In particular, we stress that it is not a priori straightforward, because of the regularisation, to see that the regularised curvature on Σ_1 and Σ_2 sum up to produce the regularised curvature on Σ ; this is the main outcome of Lemma 8.7. Then we claim:

Lemma 8.21. *We have the convergence*

$$\lim_{x \rightarrow z} \lim_{\epsilon \rightarrow 0} \mathcal{A}_{\Sigma, g, x, v, \alpha, m, \zeta}^\epsilon(F, \tilde{\varphi}^k) = \mathcal{A}_{\Sigma, g, v, \alpha, m, \zeta}(F, \tilde{\varphi}^k) \text{ in } \mathcal{H}^{\otimes b},$$

where x tends to z in the direction of v .

Proof. We already know that the convergence holds almost surely. Now we show that regularised amplitudes form a Cauchy sequence in $\mathcal{H}^{\otimes b}$. For this we consider the doubled surface $\Sigma^{\#2}$ as before equipped with its involution τ and we still write g for the symmetrized metric $g + \tau_*g$ induced by g on $\Sigma^{\#2}$. For $G \in \mathcal{E}_R^{\epsilon, m}(\Sigma)$, we denote by $\tau(G) \in \mathcal{E}_R^{\epsilon, m}(\tau(\Sigma))$ the functional $\tau(G)(\phi_g) := G(\phi_g \circ \tau)$. We consider the surfaces with boundary $\Sigma_1 = \Sigma$ and $\Sigma_2 = \tau(\Sigma)$. We consider the amplitudes $\mathcal{A}_{\Sigma, g, x, v, \alpha, m, \zeta}^\epsilon(G, \tilde{\varphi}^k)$ and

$\overline{\mathcal{A}_{\tau(\Sigma), \tau_*g, \tau(x), \tau_*\mathbf{v}, \alpha, -\mathbf{m}, \tau\circ\zeta}^{\epsilon'}(\tau(G), \tilde{\varphi}^{\mathbf{k}})}$, where \bar{a} means complex conjugate of a . We make now two observations. First, by Lemma 8.17, we have the relation

$$\overline{\mathcal{A}_{\tau(\Sigma), \tau_*g, \tau(x), \tau_*\mathbf{v}, \alpha, -\mathbf{m}, \tau\circ\zeta}^{\epsilon'}(\tau(G), \tilde{\varphi}^{\mathbf{k}})} = \overline{\mathcal{A}_{\Sigma, g, \mathbf{x}, \mathbf{v}, \alpha, \mathbf{m}, \zeta}^{\epsilon'}(G, \tilde{\varphi}^{\mathbf{k}})}.$$

Second, we have

$$\begin{aligned} & \overline{\mathcal{A}_{\tau(\Sigma), \tau_*g, \tau(x), \tau_*\mathbf{v}, \alpha, -\mathbf{m}, \tau\circ\zeta}^{\epsilon'}(\tau(G), \tilde{\varphi}^{\mathbf{k}})} \\ &= \delta_0 \left(\sum_{j=1}^{n_m} -m_j + \sum_{j=1}^b k_j \right) \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}} e^{-\frac{1}{4\pi} \|\nu_{\tau(z), -\mathbf{m}, \mathbf{k} + \omega_{\mathbf{k}^c}^c\|_{g,0}^2} Z_{\tau(\Sigma), \tau_*g} \mathcal{A}_{\tau(\Sigma), \tau_*g}^0(\tilde{\varphi})} \\ & \times \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_{\tau_*g, D} + dP\tilde{\varphi}, \nu_{\tau(z), -\mathbf{m}, \mathbf{k} + \omega_{\mathbf{k}^c}^c} \rangle} \tau(G)(\phi_{\tau_*g}) \right. \\ & \left. \prod_{j=1}^{n_m} V_{-\alpha_j, \tau_*g, \epsilon'}(\tau(x_j)) e^{\frac{iQ}{4\pi} \int_{\Sigma'}^{\text{reg}} K_{\tau_*g} \phi_{\tau_*g} dv_{\tau_*g} - \mu M_{-\beta}^{\tau_*g}(\phi_{\tau_*g}, \tau(\Sigma))} \right], \end{aligned}$$

which we glue together to form an amplitude on $\Sigma^{\#2}$ using Proposition 8.15:

$$\begin{aligned} & \mathcal{A}_{\Sigma^{\#2}, g, \hat{\mathbf{x}}, \hat{\mathbf{v}}, \hat{\alpha}, \hat{\mathbf{m}}}^{\epsilon, \epsilon'}(G \otimes \overline{\tau(G)}) \\ &= C \int \mathcal{A}_{\Sigma, g, \mathbf{x}, \mathbf{v}, \alpha, \mathbf{m}, \zeta}^{\epsilon}(\overline{G}, \tilde{\varphi}(\mathbf{k})) \overline{\mathcal{A}_{\tau(\Sigma), \tau_*g, \tau(x), \tau_*\mathbf{v}, \alpha, -\mathbf{m}, \tau\circ\zeta}^{\epsilon'}(\tau(G), \tilde{\varphi}^{\mathbf{k}})} d\mu_0^{\otimes b}(\tilde{\varphi}^{\mathbf{k}}), \end{aligned}$$

where $\hat{\mathbf{x}} := (\mathbf{x}, \tau(\mathbf{x}))$, $\hat{\mathbf{v}} := (\mathbf{v}, \tau_*\mathbf{v})$, $\hat{\alpha} = (\alpha, \alpha)$, $\hat{\mathbf{m}} = (\mathbf{m}, -\mathbf{m})$, and the amplitude type functional has the following expression

$$\begin{aligned} & \mathcal{A}_{\Sigma^{\#2}, g, \hat{\mathbf{x}}, \hat{\mathbf{v}}, \hat{\alpha}, \hat{\mathbf{m}}}^{\epsilon, \epsilon'}(G \otimes \tau(G)) \\ &= \left(\frac{V_g(\Sigma)}{\det'(\Delta_g)} \right)^{\frac{1}{2}} \sum_{\mathbf{k} \in \mathbb{Z}^{2g}} e^{-\frac{1}{4\pi} \|\omega_{\mathbf{k}}\|_g^2 - \frac{1}{4\pi} \|\nu_{z, \mathbf{m}}\|_{g,0}^2 - \frac{1}{2\pi} \langle \omega_{\mathbf{k}}, \nu_{z, \mathbf{m}} \rangle} \\ & \times \int_0^{2\pi R} \mathbb{E} \left[e^{-\frac{1}{2\pi} \langle dX_g, \omega_{\mathbf{k}} + \nu_{z, \mathbf{m}} \rangle} G \otimes \overline{\tau(G)}(\phi_g) V_{(\alpha, 0)}^{g, \epsilon}(\mathbf{x}) \overline{V_{(\alpha, 0)}^{g, \epsilon'}(\tau(\mathbf{x}))} \right. \\ & \left. \times e^{-\frac{iQ}{4\pi} \int_{\Sigma_1}^{\text{reg}} K_g \phi_g dv_g + \frac{iQ}{4\pi} \int_{\Sigma_2}^{\text{reg}} K_g \phi_g dv_g - \hat{M}_{\beta}^g(\phi_g, \Sigma^{\#2})} \right] dc, \end{aligned}$$

where $\phi_g := c + X_g + I_{x_0}^{\#}(\omega_{\mathbf{k}}) + I_{x_0}^{\#}(\nu_{z, \mathbf{m}})$, the potential is given by

$$\begin{aligned} & \hat{M}_{\beta}^{g, \epsilon}(h, dx) := \mu \epsilon^{-\frac{\beta^2}{2}} e^{i\beta h_{\epsilon}(x)} \mathbf{1}_{\Sigma_1}(x) dv_g(x) + \bar{\mu} \epsilon^{-\frac{\beta^2}{2}} e^{-i\beta h_{\epsilon}(x)} \mathbf{1}_{\Sigma_2}(x) dv_g(x), \\ & \hat{M}_{\beta}^g(\phi_g, \Sigma^{\#2}) = \lim_{\epsilon \rightarrow 0} \hat{M}_{\beta}^{g, \epsilon}(\phi_g, \Sigma^{\#2}), \end{aligned}$$

and $V_{(\alpha, 0)}^{g, \epsilon}(u, \mathbf{x})$ are electric type operators applied to the Liouville field. Note that the zero mode contributions from both curvature terms cancel out, hence the curvature term remains c -periodic. We can then follow the proof of Theorems 6.11 and 6.13 to take the limit as $\epsilon, \epsilon' \rightarrow 0$, then $\mathbf{x} \rightarrow \mathbf{z}$ in the direction of \mathbf{v} and obtain a limit that does not depend on choice of the families ϵ, ϵ' . We stress that the condition to follow these proofs remains $\alpha_j > Q$ since the electric insertions with point x in Σ_1 will create a singularity in Σ_1 , and the electric insertions at point $\tau(x)$ create a singularity only on Σ_2 where the potential has a reversed sign $-\beta$.

For the last part of the argument, we shortcut $\mathcal{A}_{\Sigma, g, \mathbf{x}, \mathbf{v}, \alpha, \mathbf{m}, \zeta}^\epsilon(F, \tilde{\varphi}^k)$ as $\mathcal{A}_x(\epsilon)$ and we denote by L the limit of $\langle \mathcal{A}_x(\epsilon), \mathcal{A}_x(\epsilon') \rangle_{\mathcal{H}^{\otimes b}} = \int \mathcal{A}_x(\epsilon) \overline{\mathcal{A}_x(\epsilon')} d\mu_0^{\otimes b}(\tilde{\varphi}^k)$. We have shown that

$$\begin{aligned} \|\mathcal{A}_x(\epsilon) - \mathcal{A}_x(\epsilon')\|_{\mathcal{H}^{\otimes b}}^2 &= \langle \mathcal{A}_x(\epsilon), \mathcal{A}_x(\epsilon) \rangle_{\mathcal{H}^{\otimes b}} - \langle \mathcal{A}_x(\epsilon), \mathcal{A}_x(\epsilon') \rangle_{\mathcal{H}^{\otimes b}} \\ &\quad - \langle \mathcal{A}_x(\epsilon'), \mathcal{A}_x(\epsilon) \rangle_{\mathcal{H}^{\otimes b}} + \langle \mathcal{A}_x(\epsilon'), \mathcal{A}_x(\epsilon') \rangle_{\mathcal{H}^{\otimes b}} \\ &\rightarrow L - L - L + L = 0 \end{aligned}$$

as $\epsilon, \epsilon' \rightarrow 0$, then $\mathbf{x} \rightarrow \mathbf{z}$ in the direction \mathbf{v} . Hence, the sequence $\mathcal{A}_x(\epsilon)$ is Cauchy in $\mathcal{H}^{\otimes b}$, and then converges in $\mathcal{H}^{\otimes b}$ towards the amplitude (since we know almost sure convergence already holds). \square

Now we complete the proof of Proposition 8.13. Back to (8.41), Lemma 8.21 shows that the regularised amplitudes on respectively Σ_1 and Σ_2 , in probability in $\tilde{\varphi}_1^{k_1}$ and $\tilde{\varphi}_2^{k_2}$, converge as a function of $\tilde{\varphi}^k$ in \mathcal{H} towards the limiting amplitudes. We can then pass to the limit $\epsilon \rightarrow 0$ and then $\lim_{\mathbf{x} \rightarrow \mathbf{z}}$ in (8.41) to get our claim.

The case of self-gluing, namely Proposition 8.14, is slightly more subtle. Indeed self-gluing can be seen as a partial trace and it is not clear that this partial trace makes sense in generality. In [31, Lemma 7.2], it is shown that the partial trace makes sense if we can show that amplitudes are composition of Hilbert-Schmidt operators. Observe that an annulus with Out/In boundary component can be seen as the integral kernel of some Hilbert-Schmidt operator $\mathcal{H} \rightarrow \mathcal{H}$, since we have proved that amplitudes are L^2 . Therefore, any (regularised) amplitude can be seen as a composition of Hilbert-Schmidt operators because, for any surface Σ with boundary, one can see Σ as the gluing of the surface obtained by removing from Σ small annuli around the boundary components and those annuli. The corresponding regularised amplitudes converge in L^2 towards the limiting amplitudes by Lemma 8.21. It is then straightforward to pass to the limit in $\epsilon \rightarrow 0$ and then $\lim_{\mathbf{x} \rightarrow \mathbf{z}}$ in the analog of (8.41) for the case of self-gluing to deduce Proposition 8.14. Note that in the case of self-gluing, the behaviour of the regularised curvature is treated in Lemma 8.8.

8.8.1. *Proof of Proposition 8.12.* First, we claim that the amplitude is $2\pi R\mathbb{Z}^b$ periodic in the $c \in \mathbb{R}^b$ variable, i.e. it can be viewed as a function of $(\mathbf{c}, \mathbf{k}, \varphi) \in (\mathbb{R}/2\pi R\mathbb{Z})^b \times \mathbb{Z}^b \times H^s(\mathbb{T})^b$. Indeed, assume $x_0 \in \partial_1 \Sigma$ for simplicity. First, consider the case where all the c_j are replaced by $c_j + 2\pi R$. All terms in the amplitude that are $2\pi R\mathbb{Z}$ -periodic functions of ϕ_g are clearly unchanged, the terms involving $dP\tilde{\varphi}$ and $\mathcal{A}_{\Sigma, g}^0(\tilde{\varphi})$ are also invariant by such change, and finally the curvature term becomes

$$e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g(\phi_g + 2\pi R) dv_g} = e^{-2\pi i Q \chi(\Sigma) R} e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g \phi_g dv_g} = e^{-\frac{iQ}{4\pi} \int_{\Sigma}^{\text{reg}} K_g \phi_g dv_g}.$$

This shows the invariance of the amplitude under such change. Next, consider the case where only one c_j is replaced by $c_j + 2\pi R$. First, if $j = 1$, by adding $-2\pi R$ to ϕ_g as above, we see that this reduces the problem to adding $-2\pi R$ to each of the c_j with $j > 1$. It thus suffices to prove invariance of the amplitude under each transformation $c_j \rightarrow c_j + 2\pi R$ for some $j > 1$. Let us define $W(h) := \mathcal{A}_{\Sigma, g, \mathbf{v}, \alpha, \mathbf{m}, \zeta}(F, \tilde{\varphi}^k + h)$ for h locally

constant on $\partial\Sigma$ and show $W(h_j) = W(0)$ if $h_j = 2\pi R\ell\mathbf{1}_{\partial_j\Sigma}$. We have

$$W(h_j) = \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}} \mathcal{A}_{\Sigma,g}^0(\tilde{\varphi} + h_j) e^{-\frac{1}{4\pi} \|\nu_{z,m,k} + \omega_{\mathbf{k}^c}^c\|_{g,0}^2} \times \mathbb{E}[e^{-\frac{1}{2\pi} \langle dX_{g,D} + dP\tilde{\varphi} + dPh_j, \nu_{z,m,k} + \omega_{\mathbf{k}^c}^c \rangle} H(\phi_g + Ph_j)],$$

where H is some functional and the expectation is in the Dirichlet GFF. Notice that

$$\mathcal{A}_{\Sigma,g}^0(\tilde{\varphi} + h_j) = e^{-\frac{1}{4\pi} \int_{\Sigma} |dPh_j|_g^2 dv_g - \frac{1}{2\pi} \langle dPh_j, dP\tilde{\varphi} \rangle} \mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}),$$

which gives that

$$W(h_j) = \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}} \mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}) e^{-\frac{1}{4\pi} \|\nu_{z,m,k} + \omega_{\mathbf{k}^c}^c + dPh_j\|_{g,0}^2} \times \mathbb{E}[e^{-\frac{1}{2\pi} \langle dX_{g,D} + dP\tilde{\varphi}, \nu_{z,m,k} + \omega_{\mathbf{k}^c}^c + dPh_j \rangle} H(\phi_g + Ph_j)].$$

The exact form dPh_j belongs to the same class as ω_{2g+j-1}^c in $\mathcal{H}_R^1(\Sigma, \partial\Sigma)$ (which is dual to $d_{j-1} \in \mathcal{H}_1(\Sigma, \partial\Sigma)$), thus $dPh_j = \omega_{2g+j-1}^c + df_j$ with f_j smooth function and $f_j|_{\partial\Sigma} = 0$. Using a resummation of \mathbf{k}^c , we have

$$\begin{aligned} W(h_j) &= \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}} \mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}) e^{-\frac{1}{4\pi} \|\nu_{z,m,k} + \omega_{\mathbf{k}^c}^c + df_j\|_{g,0}^2} \times \mathbb{E}[e^{-\frac{1}{2\pi} \langle dX_{g,D} + dP\tilde{\varphi}, \nu_{z,m,k} + \omega_{\mathbf{k}^c}^c + df_j \rangle} H(\phi_g + f_j)] \\ &= \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}} \mathcal{A}_{\Sigma,g}^0(\tilde{\varphi}) e^{-\frac{1}{4\pi} \|\nu_{z,m,k} + \omega_{\mathbf{k}^c}^c\|_{g,0}^2} \times \mathbb{E}[e^{-\frac{1}{2} \langle X_{g,D}, \Delta_g f_j \rangle} e^{-\frac{1}{2\pi} \langle d(X_{g,D} + f_j + P\tilde{\varphi}), \nu_{z,m,k} + \omega_{\mathbf{k}^c}^c \rangle} H(\phi_g + f_j)]. \end{aligned}$$

We can apply the Girsanov transform to absorb the shift by f_j on the Dirichlet GFF, since f_j vanishes at ∂M . This gives that $W(h_j) = W(0)$. More generally, the GFF expectation with respect to the Dirichlet GFF can absorb via the Girsanov transform any shift in the path integral by exact forms of the form df for some smooth function f vanishing along the boundary (we call these forms exact forms of Dirichlet type), which means that the path integral features invariance along the relative cohomology classes. For instance if we want to change the relative (co-)homology basis, this can be performed as in the proof of Proposition 6.4 because changing relative cohomology basis amounts to shifting the original basis by exact forms of Dirichlet type, up to the invariance of the curvature term with respect to homology basis. This point does not raise any difficulty.

What is more subtle is that the amplitudes are invariant under change of 1-forms in the absolute cohomology, i.e. the 1-form $\nu_{z,m,k}$. Let us consider another 1-form $\nu'_{z,m,k}$ satisfying our assumptions (i.e. Proposition 3.10 and (8.11)). Then the difference can be expressed (see Lemma 3.4) as $\nu_{z,m,k} - \nu'_{z,m,k} = df$ for some smooth function f on Σ such that $\partial_\nu f|_{\partial\Sigma} = 0$ with ν the unit vector normal to $\partial\Sigma$ (such exact forms are said to be of Neumann type). This type of shifts cannot be absorbed by the Dirichlet GFF. This is where our complete set of assumptions will be crucial. The forms $\nu_{z,m,k}$ are required to be of the form $Rkd\theta$ in local coordinates near the boundary components (and near the marked points), and this forces $df = 0$ near the boundary components.

So f is locally constant near the boundary components. From (8.11), it is then plain to check that one can decompose f as $f(x) = C + h(x) + \bar{f}(x)$, where h is smooth on Σ and constant near the boundary components/ marked points with values in $R\mathbb{Z}$, \bar{f} is smooth and vanishes on $\partial\Sigma$ and C is a constant fixed to have $f(x_0) = 0$. Next, we observe that there is $\mathbf{k}_0^c \in \mathbb{Z}^{2g+b-1}$ such that $dh = \omega_{\mathbf{k}_0^c}^c + df_{\mathbf{k}_0^c}$ for some exact form $df_{\mathbf{k}_0^c}$ of Dirichlet type by Lemma 3.4 (in fact only the boundary-to-boundary arcs matter, meaning $\mathbf{k}_0^c = (\mathbf{k}_0^{ic}, \mathbf{k}_0^{bc}) \in \mathbb{Z}^{2g} \times \mathbb{Z}^{b-1}$ with $\mathbf{k}_0^{ic} = 0$). Next we plug the relation $\nu_{z,\mathbf{m},\mathbf{k}} = \nu'_{z,\mathbf{m},\mathbf{k}} + df$ in the expression for amplitudes (8.24), we perform a change of variables $\mathbf{k}^c \rightarrow \mathbf{k}^c - \mathbf{k}_0^c$ in the summation $\sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}}$ to absorb the $\omega_{\mathbf{k}_0^c}^c$ -component of f . Finally the exponential

term in the expectation in (8.24) produces a term $e^{-\frac{1}{2\pi} \langle dX_{g,D}, d\bar{f} + df_{\mathbf{k}_0^c} \rangle}$, to which we apply the Girsanov transform. This absorbs the $\bar{f} + f_{\mathbf{k}_0^c}$ component of f and in conclusion we get the expression (8.24) where $\nu_{z,\mathbf{m},\mathbf{k}}$ has been replaced by $\nu'_{z,\mathbf{m},\mathbf{k}}$. Hence the invariance under changes of representatives of 1-form in the absolute cohomology.

Now we turn to the Weyl anomaly. We have to deal with a change of conformal metrics $g' = e^\rho g$ in the definition (8.24). Recall from (5.7) that $M_\beta^{g'}(X_{g',D}, dx) = e^{-\frac{\beta Q}{2} \rho(x)} M_\beta^g(X_{g,D}, dx)$ and from (5.6) that $V_{\alpha_i, g'}(x_i) = e^{-\frac{\alpha_i^2}{4} \rho(x_i)} V_{\alpha_i, g}(x_i)$ (the last identity making sense when inserted inside expectation values). Also, from the relation for curvatures $K_{g'} = e^{-\rho}(K_g + \Delta_g \rho)$ we deduce that the term involving the curvature in (8.24), given by (8.26), reads (recall that $X_{g',D} = X_{g,D}$)

$$\begin{aligned} \int_\Sigma^{\text{reg}} K_{g'} \phi_g dv_{g'} &= \int_\Sigma K_g(X_{g,D} + P\bar{\varphi}) dv_g + \int_\Sigma \Delta_g \rho(X_{g,D} + P\bar{\varphi}) dv_g \\ &\quad + \int_\Sigma^{\text{reg}} K_{g'} I_{x_0}^\sigma(\omega_{\mathbf{k}^c}^c) dv_{g'} + \int_\Sigma^{\text{reg}} K_{g'} I_{x_0}^\xi(\nu_{z,\mathbf{m},\mathbf{k}}) dv_{g'}. \end{aligned}$$

The last two terms can be expressed in the metric g thanks to Lemma 8.2. In the second term in the right hand side, the contribution of the harmonic extension is treated by the Green identity to produce (with ν the unit inward pointing normal at $\partial\Sigma$)

$$\int_\Sigma \Delta_g \rho P\bar{\varphi} dv_g = \int_{\partial\Sigma} \partial_\nu \rho P\bar{\varphi} d\ell_g = 0$$

since the fact that both g, g' are admissible entails that $\partial_\nu \rho$ must vanish. Therefore (8.24) expressed in the metric g' can be rewritten as

$$\begin{aligned} (8.42) \quad &\mathcal{A}_{\Sigma, g', \nu, \alpha, \mathbf{m}, \zeta}(F, \bar{\varphi}^k) \\ &:= \delta_0 \left(\sum_{j=1}^{n_m+b} m_j(\mathbf{k}) \right) \lim_{x \rightarrow z} \lim_{\epsilon \rightarrow 0} \sum_{\mathbf{k}^c \in \mathbb{Z}^{2g+b-1}} e^{-\frac{1}{4\pi} \|\nu_{z,\mathbf{m},\mathbf{k}} + \omega_{\mathbf{k}^c}^c\|_{g,0}^2} Z_{\Sigma, g'} \\ &\times \mathcal{A}_{\Sigma, g'}^0(\bar{\varphi}) e^{-\sum_{j=1}^{n_m} \frac{\alpha_j^2}{4} \rho(x_j) - \sum_{j=1}^{n_m} \frac{m_j^2 R^2}{4} \rho(z_j)} \\ &\times \mathbb{E} \left[e^{-i \frac{Q}{4\pi} \int_\Sigma \Delta_g \rho X_{g,D} dv_g} e^{-\frac{1}{2\pi} \langle dX_{g,D} + dP\bar{\varphi}, \nu_{z,\mathbf{m},\mathbf{k}} + \omega_{\mathbf{k}^c}^c \rangle} F(\phi_g) \right. \\ &\quad \left. \times \prod_{j=1}^{n_m} V_{\alpha_j, g, \epsilon}(x_j) e^{-\frac{iQ}{4\pi} \int_\Sigma^{\text{reg}} K_g \phi_g dv_g - \mu M_\beta^g(\phi_g + iQ\rho/2, \Sigma)} \right], \end{aligned}$$

where we have used Lemma 3.11 to switch the metric in the term involving the regularised norm. Notice that $\mathcal{A}_{\Sigma, g}^0(\tilde{\varphi}) = \mathcal{A}_{\Sigma, g'}^0(\tilde{\varphi})$ since the quadratic form $\langle \mathbf{D}_{\Sigma} \tilde{\varphi}, \tilde{\varphi} \rangle = \int_{\Sigma} |\nabla^g P \tilde{\varphi}|_g^2 dv_g$ depends only on the conformal class of g for $\tilde{\varphi}$ smooth (and using the density of $C^\infty(\partial\Sigma) \subset H^s(\partial\Sigma)$).

Now we apply the Girsanov transform to the exponential term $e^{-\frac{iQ}{4\pi} \int_{\Sigma} \Delta_g \rho X_{g,D} dv_g}$. Again, we have to be cautious because of the imaginary factor i ; we should perform an analytic continuation argument, which is possible here because $F \in \mathcal{E}_R(\Sigma)$, as in Proposition 6.4. We omit the details. The variance of this transform is given by (it is negative because of the i^2)

$$-\frac{Q^2}{16\pi^2} \int_{\Sigma} \Delta_g \rho(x) G_D(x, y) \Delta_g \rho(y) dv_g(x) dv_g(y) = -\frac{Q^2}{8\pi} \int_{\Sigma} |d\rho|_g^2 dv_g.$$

It has the effect of shifting the mean of the GFF $X_{g,D}$, i.e. $X_{g,D}$ becomes $X_{g,D} - i\frac{Q}{2}\rho$. Then we see that we get almost the result, up to the +1 in front of the Liouville functional $\frac{1-6Q^2}{96\pi} \int_{\Sigma} (|d\rho|_g^2 + 2K_g \rho) dv_g$. This +1 comes from the variations of the regularised determinant of Laplacian, here $Z_{\Sigma, g'}$. The Polyakov formula for the regularised determinant of Laplacian [59, section 1] implies that (for admissible g, g')

$$Z_{\Sigma, g'} = \det(\Delta_{g,D})^{-\frac{1}{2}} \exp\left(\frac{1}{96\pi} \int_{\Sigma} (|d\rho|_g^2 + 2K_g \rho) dv_g\right).$$

This completes the argument for the Weyl anomaly.

The spin property follows the same argument as in Corollary 6.9.

APPENDIX A. PROOF OF LEMMA 4.5

Using Lemma 4.3, it suffices to prove the result for one choice of conformal representative in the conformal class of $[g]$. By the result of Aubin [5], one can prescribe the scalar curvature $K_{\hat{g}}$ of a conformal metric $\hat{g} := e^\rho g$ (for some $\rho \in C^\infty(\Sigma)$) as long as $K_{\hat{g}} \leq 0$ with $\int_{\Sigma} K_{\hat{g}} dv_{\hat{g}} < 0$. We have thus reduced to studying the case where the curvature K_g can be assumed to be non-positive and $K_g = 0$ everywhere but on an arbitrarily small open set \mathcal{K} .

The link between a symplectic basis $([a_j], [b_j])_j$ and another one $([a'_j], [b'_j])_j$ is given by a matrix $A \in \text{Sp}(2g, \mathbb{Z})$. The group $\text{Sp}(2g, \mathbb{Z})$ is generated by four types of elements, called Burkhardt generators, see [23, Proof of Th. 6.4].

The first is the factor rotation $r_{j_1} \in \text{Sp}(2g, \mathbb{Z})$, which keeps the basis σ fixed except for the two elements $[a_{j_1}], [b_{j_1}]$ where $r_{j_1}([a_{j_1}]) = [b_{j_1}]$ and $r_{j_1}([b_{j_1}]) = -[a_{j_1}]$. In terms of our regularised integral, this amounts to checking that

$$\chi(\gamma_{a_{j_1}}) \int_{b_{j_1}} k_g d\ell_g - \chi(\gamma_{b_{j_1}}) \int_{a_{j_1}} k_g d\ell_g = \chi(\gamma_{b_{j_1}}) \int_{-a_{j_1}} k_g d\ell_g - \chi(\gamma_{-a_{j_1}}) \int_{b_{j_1}} k_g d\ell_g,$$

which is straightforward (here we write $-a_{j_1}$ for the curve a_{j_1} with reverse orientation).

The second Burkhardt generator is the factor swap s_{j_1, j_2} which is the identity on $[a_j], [b_j]$ for $j \notin \{k, \ell\}$ and satisfies $s_{j_1, j_2}([a_{j_1}]) = [a_{j_2}]$ and $s_{j_1, j_2}([b_{j_1}]) = [b_{j_2}]$, i.e. it swaps \mathcal{J}_{j_1} and \mathcal{J}_{j_2} in our geometric representation of Σ in Figure 20. For this elementary move, it is clear from the definition of the regularised integral that this is invariant.

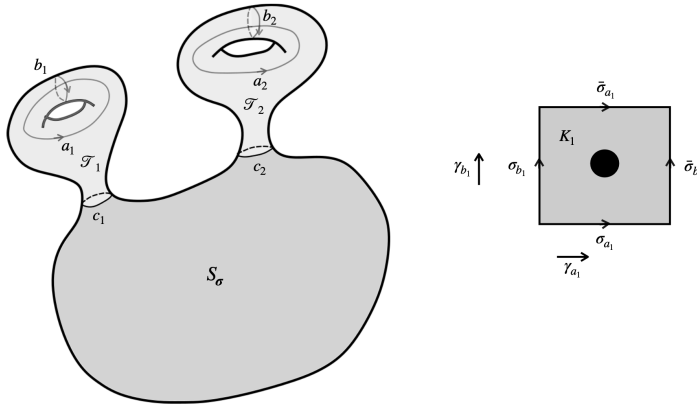


FIGURE 20. The surface decomposition and a geometric symplectic basis of $\mathcal{H}_1(\Sigma)$

The third Burkhardt generator is the transvection t_{j_1} , which preserves all $[a_j], [b_j]$ except for $t_k([a_{j_1}]) = [a_{j_1}] + [b_{j_1}]$. This is realized by a Dehn twist $T_{b_{j_1}}$ along b_{j_1} . Without loss of generality, we can assume that $j_1 = 1$ to simplify the notations. We consider the equivariant extension of $I_{x_0}^\sigma(\omega)$ from K_1 to $\tilde{\mathcal{T}}_1$ (which is the plane with the translated disks $D(e, \epsilon) + k$ removed, for $k \in \mathbb{Z}^2$) and still denote it by $I_{x_0}^\sigma(\omega)$. The new basis obtained by applying the transvection can be represented by simply changing the curve a_1 by a simple smooth curve a'_1 that we represent in K_1 by

$$\sigma_{a'_1}(t) = t + i\alpha(t) \quad \alpha(t) = 0 \text{ for } t \in [0, \epsilon], \quad \alpha(t) = 1 \text{ for } t \in [1 - \epsilon, 1]$$

for some $\epsilon > 0$ and $\alpha(t) \geq 0$ non-decreasing in the interval $t \in [\epsilon, 1 - \epsilon]$. We obtain a new fundamental domain K'_1 of \mathcal{T}_1 by considering the domain in $\tilde{\mathcal{T}}_1$ bounded by $\sigma_{a'_1}, \bar{\sigma}_{a'_1} := \gamma_{b_1}(\sigma_{a'_1})$ and the vertical lines $i\mathbb{R}$ and $1 + i\mathbb{R}$; see Figure 21.

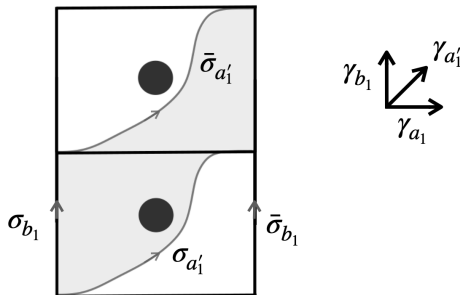


FIGURE 21. The domain K'_1 in gray and the new curves a'_1 lifted to $\tilde{\mathcal{T}}_1$

If σ' denotes the new canonical basis of $\mathcal{H}_1(\Sigma)$, we notice that $I_{x_0}^{\sigma'}(\omega)$ in K'_1 is equal to the equivariant extension of $I_{x_0}^\sigma(\omega)$. Let us denote by $D = K'_1 \setminus K_1$: we compute using

$$I_{x_0}^\sigma(\omega)(z+i) = I_{x_0}^\sigma(\omega)(z) + \chi(\gamma_{b_1}) \text{ for } z \in K_1$$

$$\begin{aligned} \int_{K'_1} K_g I_{x_0}^{\sigma'}(\omega) dv_g &= \int_{K'_1} K_g I_{x_0}^\sigma(\omega) dv_g = \int_{K_1 \cap K'_1} K_g I_{x_0}^\sigma(\omega) dv_g + \int_D K_g I_{x_0}^\sigma(\omega) dv_g \\ &= \int_{K_1} K_g I_{x_0}^\sigma(\omega) dv_g + \chi(\gamma_{b_1}) \int_D K_g dv_g \end{aligned}$$

and using Gauss-Bonnet,

$$\int_D K_g dv_g = 2\left(\int_{b_1} k_g d\ell_g + \int_{a_1} k_g d\ell_g - \int_{a'_1} k_g d\ell_g\right).$$

On the other hand, the difference of the boundary terms of $\int_{\Sigma_{\sigma'}}^{\text{reg}} K_g I_{x_0}^{\sigma'}(\omega) dv_g$ with that of $\int_{\Sigma_\sigma}^{\text{reg}} K_g I_{x_0}^\sigma(\omega) d\ell_g$ is given by

$$2(\chi(\gamma_{a_1}) + \chi(\gamma_{b_1})) \int_{b_1} k_g d\ell_g - 2\chi(\gamma_{b_1}) \int_{a'_1} k_g d\ell_g + 2\chi(\gamma_{b_1}) \int_{a_1} k_g d\ell_g - 2\chi(\gamma_{a_1}) \int_{b_1} k_g d\ell_g,$$

thus implying that

$$\int_{\Sigma_{\sigma'}}^{\text{reg}} K_g I_{x_0}^{\sigma'}(\omega) dv_g = \int_{\Sigma_\sigma}^{\text{reg}} K_g I_{x_0}^\sigma(\omega) dv_g.$$

It remains to deal with the case of the 4th Burkhardt generator, which is the factor mix f_{j_1, j_2} which preserves all $[a_j], [b_j]$ except for $j = j_1, j_2$ where

$$f_{j_1, j_2} : ([a_{j_1}], [b_{j_1}], [a_{j_2}], [b_{j_2}]) \mapsto ([a_{j_1}] - [b_{j_2}], [b_{j_1}], [a_{j_2}] - [b_{j_1}], [b_{j_2}]).$$

As before, without loss of generality we can assume that $j_1 = 1$ and $j_2 = 2$ to simplify the notations. We choose a curve a'_1 which represent the class $[a_1] - [b_2]$ and a curve a'_2 which represent the class $[a_2] - [b_1]$: they must have intersection pairing equal to 0 with all a_j, b_j except for the following j :

$$\begin{aligned} \iota(a'_1, a_1) &= 0, & \iota(a'_1, a_2) &= 1, & \iota(a'_1, b_1) &= 1, & \iota(a'_1, b_2) &= 0, \\ \iota(a'_2, a_1) &= 1, & \iota(a'_2, a_2) &= 0, & \iota(a'_2, b_1) &= 0, & \iota(a'_2, b_2) &= 1. \end{aligned}$$

We can assume that $S_\sigma = \hat{\mathbb{C}} \setminus \cup_{j=1}^g \mathcal{D}_j$ and $\mathcal{D}_1 = D(0, \epsilon), \mathcal{D}_2 = D(1, \epsilon)$. The curve a'_1 can then be chosen so that:

(1) its intersection with \mathcal{J}_1 lifts to $\sigma_{a'_1}(t) = i/2 + t$ for $t \in [0, 1/2 - \epsilon]$ and $\sigma_{a'_1}(t) = i/2 + (t - 1)$ for $t \in [3/2 + \epsilon, 2]$, and up to making a small deformation of the curve near $\sigma_{a'_1} \cap \partial\mathcal{D}'_1$, we can assume that this curve intersects $\partial\mathcal{D}'_1$ with an angle $\pi/2$;

(2) its intersection with S_σ decomposes into two pieces of curves $a'_1(t)$ for $t \in [1/2 - \epsilon, 1/2 + \epsilon]$ and $t \in [3/2 - \epsilon, 3/2 + \epsilon]$, with $a'_1(1/2 - \epsilon) = -i\epsilon/2, a'_1(1/2 + \epsilon) = 1 - i\epsilon/2$ and $a'_1(3/2 - \epsilon) = 1 + i\epsilon/2, a'_1(3/2 + \epsilon) = i\epsilon/2$, and the angle between a'_1 and $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$ is $\pi/2$;

(3) its intersection with \mathcal{J}_2 lifts to $\sigma_{a'_1}(t) = 1/2 + i(1 - t)$ for $t \in [1/2 + \epsilon, 1]$ and $\sigma_{a'_1}(t) = 1/2 + i(2 - t)$ for $t \in [1, 3/2 - \epsilon]$, and up to making a small deformation of the curve near $\sigma_{a'_1} \cap \partial\mathcal{D}'_2$ we can assume that this curve intersects $\partial\mathcal{D}'_2$ with an angle $\pi/2$. We define similarly a'_2 by reverting the role of a'_1 and a'_2 . See Figures 22 and 23.

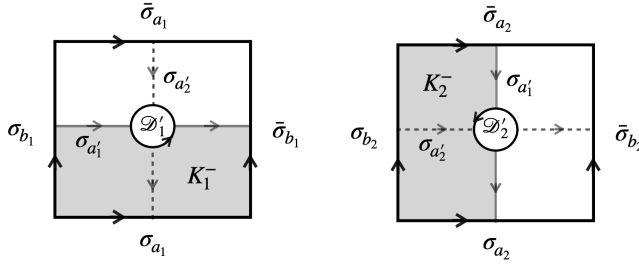


FIGURE 22. The domains K_1, K_2 and the new curves a'_1, a'_2 lifted to $\tilde{\mathcal{T}}_1, \tilde{\mathcal{T}}_2$

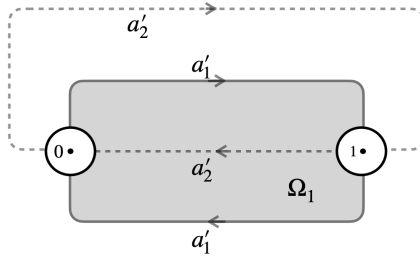


FIGURE 23. The domains S_σ near \mathcal{D}_1 and \mathcal{D}_2 , and the new curves a'_1, a'_2

For $j = 1, 2$, let us denote $\Omega_j \subset S_\sigma \cap \mathbb{C}$ the connected set bounded by the two connected components of $a'_j \cap S_\sigma$. We can freely choose the curves a'_1, a'_2 so that the set \mathcal{K} where the curvature is non-zero is contained in $S_\sigma^\circ \setminus (\Omega_1 \cup \Omega_2)$. We observe that $I_{x_0}^{\sigma'}(\omega)(x) = I_{x_0}^\sigma(\omega)(x)$ for $x \in S_\sigma \setminus (\Omega_1 \cup \Omega_2)$. We compute

$$\int_{\Sigma_\sigma} K_g I_{x_0}^{\sigma'}(\omega) \, dv_g = \int_{\mathcal{K}} K_g I_{x_0}^{\sigma'}(\omega) \, dv_g = \int_{\mathcal{K}} K_g I_{x_0}^\sigma(\omega) \, dv_g = \int_{\Sigma_\sigma} K_g I_{x_0}^\sigma(\omega) \, dv_g.$$

Let $K_1^- \subset K_1$ be the connected set bounded by $\sigma_{b_1}, \bar{\sigma}_{b_1}, \sigma_{a_1}, \sigma_{a_1}'$ and $K_2^- \subset K_2$ the connected set bounded by $\sigma_{b_2}, \sigma_{a_2}, \bar{\sigma}_{a_2}, \sigma_{a_2}'$. We compute using the Gauss-Bonnet formula in K_1^-

$$(A.1) \quad \int_{\sigma_{a_1}' \cap K_1} k_g \, dl_g = \pi + \int_{\sigma_{a_1}} k_g \, dl_g - \int_{\mathcal{C}'_1} k_g \, dl_g,$$

where \mathcal{C}'_1 is the semicircle $\partial \mathcal{D}'_1 \cap K_1^-$ oriented counter-clockwise. We next apply Gauss-Bonnet in the region $\Omega_1 \subset S_\sigma$ bounded by the two pieces of curves representing $a'_1 \cap S_\sigma$:

$$(A.2) \quad \int_{a'_1 \cap S_\sigma} k_g \, dl_g = \int_{\mathcal{C}_1} k_g \, dl_g + \int_{\mathcal{C}_2} k_g \, dl_g,$$

where, for $j = 1, 2$, \mathcal{C}_j is the semicircle $\Omega_1 \cap \partial \mathcal{D}_j$ oriented clockwise. Finally we apply Gauss-Bonnet in the domain K_2^- :

$$(A.3) \quad \int_{\sigma_{a_1}' \cap K_2} k_g \, dl_g = - \int_{\sigma_{b_2}} k_g \, dl_g - \int_{\mathcal{C}'_2} k_g \, dl_g + \pi,$$

where \mathcal{C}'_2 is the semicircle $\mathcal{C}'_2 = \partial\mathcal{D}'_2 \cap K_2^-$ oriented counter clockwise. We can use that $\int_{\mathcal{C}'_j} k_g d\ell_g = \int_{\mathcal{C}'_j} k_g d\ell_g$ for $j = 1, 2$, and summing (A.3), (A.2) and (A.1), we obtain

$$\int_{a'_1} k_g d\ell_g = 2\pi + \int_{a_1} k_g d\ell_g - \int_{b_2} k_g d\ell_g.$$

By symmetry, the same argument yields

$$\int_{a'_2} k_g d\ell_g = 2\pi + \int_{a_2} k_g d\ell_g - \int_{b_1} k_g d\ell_g.$$

We then obtain (with $\chi(a'_1) = \chi(a_1) - \chi(b_2)$ and $\chi(a'_2) = \chi(a_2) - \chi(b_1)$)

$$\begin{aligned} & - \sum_{j=1}^2 \chi(b_j) \int_{a'_j} k_g d\ell_g + \chi(a'_j) \int_{b_j} k_g d\ell_g + \sum_{j=1}^2 \chi(b_j) \int_{a_j} k_g d\ell_g - \chi(a_j) \int_{b_j} k_g d\ell_g \\ & = -(\chi(b_1) + \chi(b_2))2\pi. \end{aligned}$$

This shows that

$$\int_{\Sigma_{\sigma'}}^{\text{reg}} K_g I_{x_0}^{\sigma'}(\omega) dv_g = \int_{\Sigma_{\sigma}}^{\text{reg}} K_g I_{x_0}^{\sigma}(\omega) dv_g - 4\pi(\chi(b_1) + \chi(b_2)).$$

APPENDIX B. PROOF OF LEMMA 6.14

The term $I_{x_0}(\omega_{\mathbf{k}} + \nu_{\mathbf{z}, \mathbf{m}}^h)(y)$ being bounded on the complement of the defect graph, we can obviously get rid of this term. Then, observing that the Green function in isothermal local coordinates is of the form $-\ln|x - y| + f(x, y)$ for some smooth function f , it is plain to see that the control of our integral amounts to estimating the following quantities

$$\begin{aligned} & \int_{B(z_j, \delta)} \int_{B(z_{j'}, \delta)} |y - x_j(t)|^{\beta\alpha_j} (|x_{j'}(t) - y'|^{\beta\alpha_{j'}} - |z_{j'} - y'|^{\beta\alpha_{j'}}) \frac{dy dy'}{|y - y'|^{\beta^2}}, \\ & \int_{B(z_j, \delta)} \int_{B(z_{j'}, \delta)} |y - z_j|^{\beta\alpha_j} (|x_{j'}(t) - y'|^{\beta\alpha_{j'}} - |z_{j'} - y'|^{\beta\alpha_{j'}}) \frac{dy dy'}{|y - y'|^{\beta^2}} \end{aligned}$$

when $t \rightarrow 1$, for $\delta > 0$ small but fixed. We treat only the first integral because the second one is similar. If $j \neq j'$, then choosing $\delta > 0$ small enough so that the balls $B(z_j, \delta)$ and $B(z_{j'}, \delta)$ do not intersect, it is obvious to show that the integral converges to 0 because the on-diagonal term $|y - y'|^{-\beta^2}$ is bounded and $\beta\alpha_j > -2$ for all j . We can thus focus on the case $j = j'$ and, by invariance under translation, we can assume $z_j = 0$. We set

$$F_{\delta}(x) := \int_{B(0, \delta)} \int_{B(0, \delta)} |x - y|^{\beta\alpha_j} (|x - y'|^{\beta\alpha_j} - |y'|^{\beta\alpha_j}) \frac{dy dy'}{|y - y'|^{\beta^2}}.$$

We have to show that $F_{\delta}(x) \rightarrow 0$ as $x \rightarrow 0$. We have already seen that F_{δ} is finite using (6.27). Next, we want to show that F_{δ} is bounded. Indeed

$$\begin{aligned} |F_{\delta}(x)| & \leq \int_{B(0, \delta)} \int_{B(0, \delta)} |x - y|^{\beta\alpha_j} |x - y'|^{\beta\alpha_j} \frac{dy dy'}{|y - y'|^{\beta^2}} \\ & \quad + \int_{B(0, \delta)} \int_{B(0, \delta)} |x - y|^{\beta\alpha_j} |y'|^{\beta\alpha_j} \frac{dy dy'}{|y - y'|^{\beta^2}}. \end{aligned}$$

The first integral is bounded by an argument of translation invariance and we call $G_\delta(x)$ the second integral. For $|x| \geq \delta/4$ we claim that $G_\delta(x)$ is bounded by some δ dependent constant. Indeed we can split the y -integral in two parts depending on $|y| \leq \delta/8$ or $|y| \geq \delta/8$. On the first part, we can remove the first singularity $|x - y|^{\beta\alpha_j}$ since it is bounded and the remainder is then obvious to bound. On the second part, one of the two singularities involving y' is bounded (because $|y| \geq \delta/8$, y' cannot be close to both 0 and y) so that it is bounded in the same way.

Now we bound $G_\delta(x)$ for $|x| \leq \delta/2$. Using the same argument as above, we show that $G_\delta(x) - G_{\delta/2}(x)$ is bounded by some δ dependent constant and $G_{\delta/2}(x) = (1/2)^{2\beta\alpha_j - \beta^2 + 4} G_\delta(2x)$ by a change of variables. Therefore

$$G_\delta(x) \leq (1/2)^{2\beta\alpha_j - \beta^2 + 4} G_\delta(2x) + C_\delta.$$

Now we conclude that G_δ is bounded. Indeed, if $|x| \leq \delta/2$, we can find n such that $2^{-n-1}\delta < |x| \leq 2^{-n}\delta$. We can then iterate the previous relation to get $G_\delta(x) \leq (1/2)^{(2\beta\alpha_j - \beta^2 + 4)n} G_\delta(2^n x) + C_\delta \sum_{k=0}^{n-1} (1/2)^{(2\beta\alpha_j - \beta^2 + 4)k}$. Using that $G_\delta(x)$ is bounded for $|x| \geq \delta/4$, we deduce that G_δ is bounded.

Now we are back to the study of F_δ , which we know now to be bounded. Next, for $|x| \leq \delta/4$ we have

$$(B.1) \quad |F_\delta(x)| \leq \iint_{B(0, \delta/2)^2} |x - y|^{\beta\alpha_j} (|x - y'|^{\beta\alpha_j} - |y'|^{\beta\alpha_j}) \frac{dy dy'}{|y - y'|^{\beta^2}} + \iint_{B(0, \delta)^2 \setminus B(0, \delta/2)^2} |x - y|^{\beta\alpha_j} (|x - y'|^{\beta\alpha_j} - |y'|^{\beta\alpha_j}) \frac{dy dy'}{|y - y'|^{\beta^2}}.$$

The second integral in the rhs can be split in two parts depending on $|y| \geq \delta/2$, or $|y| \leq \delta/2$ and $|y'| \geq \delta/2$. On the first part, we can remove the first singularity $|x - y|^{\beta\alpha_j}$ since it is bounded and the remainder is then obvious to bound with the relation (B.2), which gives that the first part is less than $C(|x|^{\beta\alpha_j + 2} + |x|)$. On the second part, we use the mean value theorem to get that $||x - y'|^{\beta\alpha_j} - |y'|^{\beta\alpha_j}| \leq C|x|$, and the integral is less than $C|x|$ using invariance under translations. All in all

$$\iint_{B(0, \delta)^2 \setminus B(0, \delta/2)^2} |x - y|^{\beta\alpha_j} (|x - y'|^{\beta\alpha_j} - |y'|^{\beta\alpha_j}) \frac{dy dy'}{|y - y'|^{\beta^2}} \leq C(|x|^{\beta\alpha_j + 2} + |x|).$$

The first integral $\iint_{B(0, \delta/2)^2} \dots$ in (B.1) can be dealt with using a change of variables (dilation); it is equal to $2^{-(2\beta\alpha_j + 4 - \beta^2)} F_\delta(2x)$. So we have obtained

$$|F_\delta(x)| \leq 2^{-(2\beta\alpha_j + 4 - \beta^2)} F_\delta(2x) + C(|x|^{\beta\alpha_j + 2} + |x|).$$

We recall that $2\beta\alpha_j - \beta^2 + 4 > 0$. Iterating, we deduce that for $|x| \leq 2^{-n}\delta$,

$$F_\delta(x) \leq 2^{-(2\beta\alpha_j + 4 - \beta^2)(n-1)} F_\delta(2^{n-1}x) + C(|x|^{\beta\alpha_j + 2} + |x|) \sum_{k=0}^{n-2} 2^{-(2\beta\alpha_j + 4 - \beta^2)k}.$$

Next, for $|x| \leq \delta/4$, we can find n such that $\delta 2^{-n-1} < |x| \leq \delta 2^{-n}$. The above relation then yields (using that F_δ is bounded)

$$|F_\delta(x)| \leq C(|x|^{2\beta\alpha_j + 4 - \beta^2} + |x|^{\beta\alpha_j + 2} + |x|),$$

which completes the proof of the lemma, up to the following estimate.

We claim for all $|x| \leq \delta/4$:

$$(B.2) \quad \int_{B(0,\delta)} ||y' - x|^{\beta\alpha_j} - |y'|^{\beta\alpha_j}| dy' \leq C(|x|^{\beta\alpha_j+2} + |x|).$$

Let us call $H_\delta(x)$ this integral. It is plain to see that it is bounded. Next, we have

$$\begin{aligned} H_\delta(x) &\leq \int_{B(0,\delta/2)} ||y' - x|^{\beta\alpha_j} - |y'|^{\beta\alpha_j}| dy' \\ &\quad + \int_{B(0,\delta)\setminus B(0,\delta/2)} ||y' - x|^{\beta\alpha_j} - |y'|^{\beta\alpha_j}| dy' \\ &\leq 2^{-2-\beta\alpha_j} H_\delta(2x) + C|x|, \end{aligned}$$

where we have used that, for $|x| \leq \delta/4$,

$$\int_{B(0,\delta)\setminus B(0,\delta/2)} ||y' - x|^{\beta\alpha_j} - |y'|^{\beta\alpha_j}| dy' \leq C|x|,$$

following straightforwardly from the mean value theorem. Now, if $|x| \leq 2^{-n}\delta$, we can iterate the previous relation to get

$$H_\delta(x) \leq 2^{-(\beta\alpha_j+2)(n-1)} H_\delta(2^{n-1}x) + C|x| \sum_{k=0}^{n-2} 2^{-(\beta\alpha_j+2)k}.$$

Finally, for any $|x| \leq \delta/4$, we can find n such that $2^{-n-1}\delta < |x| \leq 2^{-n}\delta$. The previous relation then gives $|H_\delta(x)| \leq C(|x|^{\beta\alpha_j+2} + |x|)$ for some constant $C > 0$, eventually depending on δ .

ACKNOWLEDGMENTS

The authors wish to thank P. Bordereau, L. Eberhardt, J. Jacobsen, E. Peltola, S. Ribault, R. Santachiara, J. Teschner, B. Wu for fruitful discussions about this work. They are grateful to Y. Xie and Y. Xiao for noticing a few mistakes or missing arguments in the first version of the paper, and the two anonymous referees for their careful reading and suggestions which improved significantly the paper. Finally, special thanks are addressed to N. Seiberg: this work originates from discussions with him about the importance of constructing a path integral for minimal models.

REFERENCES

- [1] M. Ang, G. Cai, X. Sun, and B. Wu, *Integrability of conformal loop ensemble: imaginary DOZZ formula and beyond*, arXiv:2107.01788, 2024.
- [2] M. Ang, N. Holden, and X. Sun, *Integrability of SLE via conformal welding of random surfaces*, Comm. Pure Appl. Math. **77** (2024), no. 5, 2651–2707, DOI 10.1002/cpa.22180. MR4720219
- [3] M. Ang, G. Remy, and X. Sun, *The moduli of annuli in random conformal geometry*, Ann. Sci. Éc. Norm. Supér., **58** (2025), no. 4, 1037–1087.
- [4] M. Ang, G. Remy, X. Sun, and T. Zhu, *Derivation of all structure constants for boundary Liouville CFT*, arXiv:2305.18266, 2023.
- [5] T. Aubin, *Sur le problème de la courbure scalaire prescrite* (French, with English and French summaries), Bull. Sci. Math. **118** (1994), no. 5, 465–474. MR1305205
- [6] B. N. Bakalov and J. J. Villarreal, *Logarithmic vertex algebras*, Transform. Groups **29** (2024), no. 4, 1295–1357, DOI 10.1007/s00031-022-09759-z. MR4834512
- [7] S. Baxter, R. Kelland, and F. Wu, *Equivalence of the Potts model or Whitney polynomial with an ice-type model*, J. Phys. A Math. Gen. **9** (1976), no. 3.

- [8] A. Beilinson and V. Drinfeld, *Chiral algebras*, American Mathematical Society Colloquium Publications, vol. 51, American Mathematical Society, Providence, RI, 2004, DOI 10.1090/coll/051. MR2058353
- [9] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nuclear Phys. B **241** (1984), no. 2, 333–380, DOI 10.1016/0550-3213(84)90052-X. MR757857
- [10] N. Berestycki, B. Laslier, and G. Ray, *Dimers on Riemann surfaces II: Conformal invariance and scaling limit*, Probab. Math. Phys. **5** (2024), no. 4, 961–1037, DOI 10.2140/pmp.2024.5.961. MR4838986
- [11] N. Berestycki, B. Laslier, and G. Ray, *Dimers on Riemann surfaces I: Temperleyan forests*, Ann. Inst. Henri Poincaré D **12** (2025), no. 2, 363–444, DOI 10.4171/aihpd/193. MR4884943
- [12] C. Boutillier and B. de Tilière, *Loop statistics in the toroidal honeycomb dimer model*, Ann. Probab. **37** (2009), no. 5, 1747–1777, DOI 10.1214/09-AOP453. MR2561433
- [13] J. L. Cardy, *Conformal invariance and critical behavior*, Phys. A **140** (1986), no. 1-2, 219–224, DOI 10.1016/0378-4371(86)90225-6. Statphys 16 (Boston, Mass., 1986). MR873968
- [14] J. Cardy, *The $O(n)$ model on the annulus*, J. Stat. Phys. **125** (2006), no. 1, 1–21, DOI 10.1007/s10955-006-9186-8. MR2269559
- [15] G. Carron, *Déterminant relatif et la fonction Ξ* (French, with French summary), Amer. J. Math. **124** (2002), no. 2, 307–352. MR1890995
- [16] S. Y. Cheng, P. Li, and S. T. Yau, *On the upper estimate of the heat kernel of a complete Riemannian manifold*, Amer. J. Math. **103** (1981), no. 5, 1021–1063, DOI 10.2307/2374257. MR630777
- [17] F. David, A. Kupiainen, R. Rhodes, and V. Vargas, *Liouville quantum gravity on the Riemann sphere*, Comm. Math. Phys. **342** (2016), no. 3, 869–907, DOI 10.1007/s00220-016-2572-4. MR3465434
- [18] P. Di Francesco, H. Saleur, and J.-B. Zuber, *Relations between the Coulomb gas picture and conformal invariance of two-dimensional critical models*, J. Statist. Phys. **49** (1987), no. 1-2, 57–79, DOI 10.1007/BF01009954. MR923852
- [19] V. S. Dotsenko and V. A. Fateev, *Four-point correlation functions and the operator algebra in 2D conformal invariant theories with central charge $C \leq 1$* , Nuclear Phys. B **251** (1985), no. 5-6, 691–734, DOI 10.1016/S0550-3213(85)80004-3. MR789026
- [20] J. Dubédat, *Dimers and families of Cauchy-Riemann operators I*, J. Amer. Math. Soc. **28** (2015), no. 4, 1063–1167, DOI 10.1090/jams/824. MR3369909
- [21] B. Duplantier, J. Miller, and S. Sheffield, *Liouville quantum gravity as a mating of trees* (English, with English and French summaries), Astérisque **427** (2021), viii+257, DOI 10.24033/ast. MR4340069
- [22] B. Estienne and Y. Ikhlef, *Correlation functions in loop models*, arXiv:1505.00585.
- [23] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012. MR2850125
- [24] B. L. Feigin, A. M. Gainutdinov, A. M. Semikhatov, and I. Yu. Tipunin, *Logarithmic extensions of minimal models: characters and modular transformations*, Nuclear Phys. B **757** (2006), no. 3, 303–343, DOI 10.1016/j.nuclphysb.2006.09.019. MR2275182
- [25] S. Ferrara, A. F. Grillo, and R. Gatto, *Tensor representations of conformal algebra and conformally covariant operator product expansion*, Ann. Physics **76** (1973), 161–188, DOI 10.1016/0003-4916(73)90446-6. MR411394
- [26] J. Fjelstad, J. Fuchs, S. Hwang, A. M. Semikhatov, and I. Yu. Tipunin, *Logarithmic conformal field theories via logarithmic deformations*, Nuclear Phys. B **633** (2002), no. 3, 379–413, DOI 10.1016/S0550-3213(02)00220-1. MR1910269
- [27] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, Providence, RI, 2001, DOI 10.1090/surv/088. MR1849359
- [28] K. Gawędzki, *Lectures on conformal field theory*, Nucl. Phys. B **328** (1996), 733–752.
- [29] V. Gorbenko, S. Rychkov, and B. Zan, *Walking, weak first-order transitions, and complex CFTs*, J. High Energy Phys. **10** (2018), 108, front matter+48, DOI 10.1007/jhep10(2018)108. MR3879776
- [30] B. Gui, *Lectures on vertex operator algebras and conformal blocks*, arXiv:2305.03822.
- [31] C. Guillarmou, A. Kupiainen, R. Rhodes, and V. Vargas, *Segal’s axioms and bootstrap for Liouville theory*, arXiv:2112.14859, 2021.
- [32] C. Guillarmou, A. Kupiainen, R. Rhodes, and V. Vargas, *Conformal bootstrap in Liouville theory*, Acta Math. **233** (2024), no. 1, 33–194, DOI 10.4310/acta.2024.v233.n1.a2. MR4816634
- [33] C. Guillarmou and M. Mazzucchelli, *An introduction to geometric inverse problems*, Available on webpages of the authors, 2025. <https://www.imo.universite-paris-saclay.fr/~colin.guillarmou/publi.html>
- [34] C. Guillarmou, R. Rhodes, and V. Vargas, *Polyakov’s formulation of 2d bosonic string theory*, Publ. Math. Inst. Hautes Études Sci. **130** (2019), 111–185, DOI 10.1007/s10240-019-00109-6. MR4028515

- [35] V. Gurarie, *Logarithmic operators in conformal field theory*, Nuclear Phys. B **410** (1993), no. 3, 535–549, DOI 10.1016/0550-3213(93)90528-W. MR1256422
- [36] D. Harlow, J. Maltz, and E. Witten, *Analytic continuation of Liouville theory*, J. High Energy Phys. **12** (2011), 071, i, 104, DOI 10.1007/JHEP12(2011)071. MR2935613
- [37] Y. Ikhlef, J. L. Jacobsen, and H. Saleur, *Three-point functions in $c \leq 1$ Liouville theory and conformal loop ensembles*, Phys. Rev. Lett. **116** (2016), no. 13, 130601, 5, DOI 10.1103/PhysRevLett.116.130601. MR3600546
- [38] J. L. Jacobsen and J. Kondev, *Field theory of compact polymers on the square lattice*, Nuclear Phys. B **532** (1998), no. 3, 635–688, DOI 10.1016/S0550-3213(98)00571-9. MR1657026
- [39] V. Kac, *Vertex algebras for beginners*, 2nd ed., University Lecture Series, vol. 10, American Mathematical Society, Providence, RI, 1998, DOI 10.1090/ulect/010. MR1651389
- [40] D. Kapec and R. Mahajan, *Comments on the quantum field theory of the Coulomb gas formalism*, J. High Energy Phys. **4** (2021), Paper No. 136, 68, DOI 10.1007/jhep04(2021)136. MR4276830
- [41] R. Kenyon, *Conformal invariance of domino tiling*, Ann. Probab. **28** (2000), no. 2, 759–795, DOI 10.1214/aop/1019160260. MR1782431
- [42] R. Kenyon, *Dominos and the Gaussian free field*, Ann. Probab. **29** (2001), no. 3, 1128–1137, DOI 10.1214/aop/1015345599. MR1872739
- [43] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, *Fractal structure of 2D-quantum gravity*, Modern Phys. Lett. A **3** (1988), no. 8, 819–826, DOI 10.1142/S0217732388000982. MR947880
- [44] J. Kondev, *Liouville field theory of fluctuating loops*, Phys. Rev. Lett. **78** (1997), no. 23, 4320–4323.
- [45] J. Kondev and C. Henley, *Four-coloring model on the square lattice: a critical ground state*, Phys. Rev. B **52** (1995), no. 9, 6628–6639.
- [46] J. Kondev, J. de Gier, and B. Nienhuis, *Operator spectrum and exact exponents of the fully packed loop model*, J. Phys. A **29** (1996), no. 20, 6489–6504, DOI 10.1088/0305-4470/29/20/007. MR1421011
- [47] K. Kytölä, *SLE local martingales in logarithmic representations*, J. Stat. Mech. Theory Exp. **8** (2009), P08005, 39, DOI 10.1088/1742-5468/2009/08/p08005. MR2550068
- [48] K. Kytölä and D. Ridout, *On staggered indecomposable Virasoro modules*, J. Math. Phys. **50** (2009), no. 12, 123503, 51, DOI 10.1063/1.3191682. MR2582599
- [49] H. Lacoïn, R. Rhodes, and V. Vargas, *Complex Gaussian multiplicative chaos*, Comm. Math. Phys. **337** (2015), no. 2, 569–632, DOI 10.1007/s00220-015-2362-4. MR3339158
- [50] H. Lacoïn, R. Rhodes, and V. Vargas, *A probabilistic approach of ultraviolet renormalization in the boundary sine-Gordon model*, Probab. Theory Related Fields **185** (2023), no. 1-2, 1–40, DOI 10.1007/s00440-022-01174-5. MR4528966
- [51] M. Liu, E. Peltola, and H. Wu, *Uniform spanning tree in topological polygons, partition functions for SLE(8), and correlations in $c = -2$ logarithmic CFT*, Ann. Probab. **53** (2025), no. 1, 23–78, DOI 10.1214/24-aop1700. MR4852888
- [52] P. Mathieu and D. Ridout, *From percolation to logarithmic conformal field theory*, Phys. Lett. B **657** (2007), no. 1-3, 120–129, DOI 10.1016/j.physletb.2007.10.007. MR2406363
- [53] H. P. McKean Jr. and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry **1** (1967), no. 1, 43–69. MR217739
- [54] S. Migliaccio and S. Ribault, *The analytic bootstrap equations of non-diagonal two-dimensional CFT*, J. High Energy Phys. **5** (2018), 169, front matter+27, DOI 10.1007/jhep05(2018)169. MR3815018
- [55] Y. A. Neretin, *On the Dotsenko-Fateev complex twin of the Selberg integral and its extensions*, Ramanujan J. **64** (2024), no. 1, 37–55, DOI 10.1007/s11139-023-00804-3. MR4740257
- [56] B. Nienhuis, *Phase transitions and critical phenomena*, vol. 11, Academic Press, London, 1987.
- [57] R. Nivesvivat and S. Ribault, *Logarithmic CFT at generic central charge: from Liouville theory to the Q-state Potts model*, SciPost Phys. **10** (2021), no. 1, Paper No. 021, 37, DOI 10.21468/scipostphys.10.1.021. MR4230233
- [58] P. Nolin, W. Qian, X. Sun, and Z. Zhuang, *Backbone exponent for two-dimensional percolation*, arXiv:2309.05050.
- [59] B. Osgood, R. Phillips, and P. Sarnak, *Extremals of determinants of Laplacians*, J. Funct. Anal. **80** (1988), no. 1, 148–211, DOI 10.1016/0022-1236(88)90070-5. MR960228
- [60] P. A. Pearce, J. Rasmussen, and J.-B. Zuber, *Logarithmic minimal models*, J. Stat. Mech. Theory Exp. **11** (2006), P11017, 36, DOI 10.1088/1742-5468/2006/11/p11017. MR2280249
- [61] A. M. Polyakov, *Non-Hamiltonian approach to conformal quantum field theory* (Russian, with English summary), Ž. Eksper. Teoret. Fiz. **66** (1974), 23–42; English transl., Soviet Physics JETP **39** (1974), 10–18. MR395598

- [62] N. Read and H. Saleur, *Associative-algebraic approach to logarithmic conformal field theories*, Nuclear Phys. B **777** (2007), no. 3, 316–351, DOI 10.1016/j.nuclphysb.2007.03.033. MR2341025
- [63] S. Ribault and R. Santachiara, *Liouville theory with a central charge less than one*, J. High Energy Phys. **8** (2015), 109, front matter + 25, DOI 10.1007/JHEP08(2015)109. MR3400721
- [64] P. Ruelle, *Logarithmic conformal invariance in the Abelian sandpile model*, J. Phys. A **46** (2013), no. 49, 494014, 39, DOI 10.1088/1751-8113/46/49/494014. MR3146020
- [65] I. Rushkin, E. Bettelheim, I. A. Gruzberg, and P. Wiegmann, *Critical curves in conformally invariant statistical systems*, J. Phys. A **40** (2007), no. 9, 2165–2195, DOI 10.1088/1751-8113/40/9/020. MR2316324
- [66] R. Santachiara and J. Viti, *Local logarithmic correlators as limits of Coulomb gas integrals*, Nuclear Phys. B **882** (2014), 229–262, DOI 10.1016/j.nuclphysb.2014.02.022. MR3190469
- [67] V. Schomerus, *Rolling tachyons from Liouville theory*, J. High Energy Phys. **11** (2003), 043, 19, DOI 10.1088/1126-6708/2003/11/043. MR2039445
- [68] G. Segal, *The definition of conformal field theory*, Topology, geometry and quantum field theory, London Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004, pp. 421–577. MR2079383
- [69] E. Verlinde and H. Verlinde, *Chiral bosonization, determinants and the string partition function*, Nuclear Phys. B **288** (1987), no. 2, 357–396, DOI 10.1016/0550-3213(87)90219-7. MR896667
- [70] Al. B. Zamolodchikov, *A three-point function of minimal Liouville gravitation* (Russian, with Russian summary), Teoret. Mat. Fiz. **142** (2005), no. 2, 218–234, DOI 10.1007/s11232-005-0048-3; English transl., Theoret. and Math. Phys. **142** (2005), no. 2, 183–196. MR2141774

UNIVERSITÉ PARIS-SACLAY, CNRS, LABORATOIRE DE MATHÉMATIQUES D'ORSAY, 91405 ORSAY, RUE MICHEL MAGAT, 91405 ORSAY, FRANCE

Email address: colin.guillarmou@universite-paris-saclay.fr

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF HELSINKI, P.O. BOX 68, FIN-00014, HELSINKI, FINLAND

Email address: antti.kupiainen@helsinki.fi

AIX-MARSEILLE UNIVERSITÉ, CNRS, INSTITUT DE MATHÉMATIQUES DE MARSEILLE (I2M), 3 PLACE VICTOR HUGO CASE 19, 13331 MARSEILLE CEDEX 3

Email address: remi.rhodes@univ-amu.fr