

THE ^{116}Te NUCLEUS AS A CANDIDATE FOR U(5) DYNAMICAL SYMMETRY

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The experimental energy ratio $R_{4_1^+/2_1^+} \cong 2$ suggests ^{116}Te as a prototypical vibrational nucleus. To test this hypothesis and also consider the possibility of deformation signatures in this nucleus, the energy spectra and energy surface are derived using the Interacting Boson Model including Configuration Mixing (IBMCM) and also an SU(1,1)-based transitional Hamiltonian in the both IBM 1 and 2 versions between U(5) and SO(6) dynamical limits. Both models reproduced the experimental energy levels by acceptable accuracy when the deformation effect is neglected. In addition, the results for control parameters of transitional Hamiltonians and the shape of energy surface propose an exactly U(5)-like structure for this nucleus.

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1. Introduction

The vibrational nuclei are the subject of many recent studies due to the large amount of data which has been accumulated for these nuclei [1–14]. These data allow an exact calculation of phonon and multi-phonon structures, anharmonicities in vibrational spectra, and different theoretical approaches to understand this class of nuclei. On the other hand, the concept of deformed shapes and the appearance of different shapes in a given nucleus which lead to the shape coexistence phenomena force us to consider the effect of other symmetries in nuclei which are known as the best candidates for a special symmetry limit [1–10].

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The most commonly used framework to consider dynamical symmetries and related topics such as quantum phase transition and shape coexistence is the Interacting Boson Model [11–25]. This model in the s - and d -bosons version describes the nuclear structure of even–even nuclei within the $U(6)$ symmetry, possessing $U(5)$, $SU(3)$ and $O(6)$ dynamical symmetry limits. These descriptions point out that there is a first order shape phase transition between $U(5)$ and $SU(3)$ limits and a second order shape phase transition between $U(5)$ and $O(6)$ limits. The analytical description of nuclear structure at the critical point of phase transitions has attracted extensive interest in the recent decades. One has to employ some complicated numerical methods to diagonalize the transitional Hamiltonian in these situations but Pan and Draayer in Refs. [11, 12] have proposed a new solution which is based on affine $SU(1,1)$ -algebraic technique and explores the properties of nuclei classified in the $U(5) \leftrightarrow SO(6)$ transitional region of IBM.

It was long believed that the tellurium isotopes were good examples of the quadrupole vibrational nuclei, namely $U(5)$ nuclei [13–34]. However, during the last few years, new experimental data and calculations have led to a modified picture of these nuclei. By using the collective models in describing the structure of tellurium isotopes [13], these nuclei can be considered to be soft with regard to the γ deformation with an almost maximum effective triaxiality of $\gamma \approx 30^\circ$. This means that the tellurium isotopes appear to evolve from the $U(5)$ to $O(6)$ -like structure in the IBM classification. On the other hand, if we use the results of the Casten *et al.* [3] which suggest the energy ratio as $R_{4_1^+/2_1^+} \cong 2.00$ for spherical nuclei corresponding with the $U(5)$ symmetry, the ^{116}Te fulfills exactly this criterion, *e.g.* $R_{4_1^+/2_1^+} \cong 2.003$.

In this study, we have focused on the ^{116}Te nucleus to consider the validity of $R_{4_1^+/2_1^+} \cong 2.00$ measure to assume a nucleus as a spherical one. We have used two formalisms which contain the mixing of spherical and axially-deformed symmetries, namely interacting boson model including configuration mixing (IBMCM) and $SU(1,1)$ -based transitional Hamiltonian [35, 36] to consider the energy spectra of this nucleus. Moreover, the catastrophe theory formalism [37–44] is used to determine the exact values of control parameter and the energy surface of ^{116}Te .

2. Theoretical framework

The Interacting Boson Model (IBM) describes the collective properties of several medium- and heavy-mass nuclei via algebraic methods. IBM has three dynamical symmetries of $U(5)$, $SU(3)$ and $O(6)$ which correspond to harmonic vibrator, axial rotor and γ -unstable rotor as the geometrical analogues, respectively [1–3]. Iachello in Refs. [1, 2] introduced a new set of dynamical symmetries, *i.e.* $E(5)$ and $X(5)$, for nuclei which are located at

the critical point of transitional regions in addition to three dynamical symmetry limits. These new symmetries are used to an analytical description of nuclear structure at the critical point of phase transition. The $E(5)$ symmetry describes a second order phase transition between $U(5)$ and $O(6)$ symmetries of IBM [13–49].

2.1. Investigation of symmetry mixing by using IBMCM formalism

In many nuclei, the coexistence of two quite different structures in the same energy region is manifested in the experimental data. In addition, the low-lying excited 0^+ states close in energy to the 0^+ ground state are known as the signature of symmetry mixing. Generally, there is a configuration mixing between these structures. A particularly striking example of this effect is observed in the Hg and Pt isotopes. In terms of the nuclear shell model [6–11], the emergence of low-lying excited 0^+ states can be traced back to multi-particle–multi-hole excitations. The residual interaction between the valence protons and neutrons becomes subsequently enhanced, leading to the lowering the excited 0^+ energies such as condition which are available for Hg [22] isotopes or for light nuclei such as Te isotopes [46]. The second method consists in describing the general features of the two different configurations in terms of two different IBM calculations, and then mixing the results of these two calculations using an appropriate IBM mixing Hamiltonian [22]. Due to this technique, they made first two different IBA calculations for the vibrational and the rotational part of the energy spectrum, respectively. Then they mixed the two configurations using a specifically chosen mixing-Hamiltonian to obtain the final spectrum.

3. Results and discussion

This section consists of three subsections, *(i)* the framework of IBMCM, *(ii)* transitional Hamiltonian based on affine $SU(1,1)$ algebra, and *(iii)* calculation of energy surfaces. In the following, each of the subsections will be examined.

3.1. The framework of IBMCM

Specifically, $0p-0h$, $2p-2h$, $4p-4h \dots$ shell-model configurations correspond to systems of N , $N + 2$, $N + 4, \dots$ interacting bosons which are simultaneously treated and possibly mixed in configuration-mixed version of IBM. In IBM [17–22], the regular (reg) $0h-0p$ states are described in terms of N bosons, while the intruder $2p-2h$ and $4p-4h$ states require $N + 2$ and $N + 4$ bosons. IBMCM allows the simultaneous treatment and mixing of several boson configurations which correspond to different particle–hole

(p - h) shell-model excitations [25–30]. On the basis of intruder spin symmetry [28, 29], no distinction is made between particle and hole bosons. Hence, the model space which includes the regular proton $2h$ configurations and a number of valence neutrons outside of the $N = 82$ closed shell as well as the proton $4h$ - $2p$ configurations and the same number of valence neutrons, correspond to an $[N] \oplus [N+2]$ boson space. Consequently, the Hamiltonian for two configuration mixings can be written as [22]

$$\hat{H} = \hat{P}_N^\dagger \hat{H}_{ecqf}^N \hat{P}_N + \hat{P}_{N+2}^\dagger \left(\hat{H}_{ecqf}^{N+2} + \Delta^{N+2} \right) \hat{P}_{N+2} + \hat{V}_{\text{mix}}^{N,N+2}, \quad (1)$$

where $\hat{P}_N$ and $\hat{P}_{N+2}$ are projection operators onto the $[N]$ and the $[N+2]$ boson spaces, respectively, $\hat{V}_{\text{mix}}^{N,N+2}$ describes the mixing between the $[N]$ and the $[N+2]$ boson subspaces, and

$$\hat{H}_{ecqf}^i = \varepsilon_i \hat{n}_d + \kappa'_i \hat{L} \cdot \hat{L} + \kappa_i \hat{Q}(\chi_i) \cdot \hat{Q}(\chi_i)$$

is the extended consistent- Q Hamiltonian (ECQF) with $i = N, N+2$, $\hat{n}_d$ the d boson number operator,

$$\hat{L}_\mu = \left[d^\dagger \times \tilde{d} \right]_\mu^{(1)}$$

is the angular momentum operator, and

$$\hat{Q}_\mu(\chi_i) = \left[s^\dagger \times \tilde{d} + d^\dagger \times s \right]_\mu^{(2)} + \chi_i \left[d^\dagger \times \tilde{d} \right]_\mu^{(2)}$$

is the quadrupole operator. We are not considering the most general IBM Hamiltonian in each Hilbert space, $[N]$ and $[N+2]$, but we are restricting to an ECQF formalism [22] in each subspace. This approach has been shown to be a rather good approximation in many calculations and, in particular, in two recent papers describing the Pt isotopes [22, 23]. The parameter Δ^{N+2} can be associated with the energy needed to excite two proton particles across the $Z = 82$ shell gap, giving rise to $2p$ - $2h$ excitations, corrected for the pairing interaction gain and including monopole effects [22]. The operator $\hat{V}_{\text{mix}}^{N,N+2}$ describes the mixing between the N and the $N+2$ configurations, and is defined as

$$\hat{V}_{\text{mix}}^{N,N+2} = \omega_0^{N,N+2} \left(s^\dagger \times s^\dagger + s \times s \right) + \omega_2^{N,N+2} \left(d^\dagger \times d^\dagger + \tilde{d} \times \tilde{d} \right)^{(0)},$$

where the e_i ($i = N, N+2$) are the effective boson charges and $\hat{Q}_\mu(\chi_i)$ is the quadrupole operator. García-Ramos and Heyde [22] have used the following wave-function:

$$\psi(k, JM) = \sum_i a_i^k(J; N) \psi((s, d)_i^N; JM) + \sum_j b_j^k(J; N+2) \psi((s, d)_j^{N+2}; JM) \quad (2)$$

to determine the parameters appearing in the IBMCM Hamiltonian as well as in the $\hat{T}(E2)$ operator. In the above-mentioned relation, where k , i , and j are rank numbers. The weight of the wave function contained within the $[N]$ -boson subspace can then be defined as the sum of the squared amplitudes $\omega^k(J, N) \equiv \sum_i |a_i^k(J; N)|^2$. Likewise, one obtains the content in the $[N+2]$ -boson subspace. García-Ramos and Heyde [22] considered the Hg isotopic chain in their analyses and extracted the constants of Eq. (1) in such a way that some of these quantities were kept fixed for all members of isotopic chain. They have also considered some states and limited number of transitions but we have used all the available experimental data for energy levels. We have tried to obtain the best possible agreement with the experimental data for the excitation energies. Using the expression of the IBMCM Hamiltonian, as given in Eq. (1), in the most general case, 10 parameters show up. Following the method introduced by García-Ramos and Heyde in Ref. [22], we imposed a constraint of obtaining parameters change smoothly in passing from isotope to isotope. Note also that we constrained $\varepsilon_{N+2} = 0$, $\kappa'_N = 0$ at the first step of calculations. We have explored the validity of this assumption and we have found very little improvement in the value of quality measure of fitting processes

$$\chi^2 = \frac{1}{N - N'} \sum_{i=1}^N \frac{|X_{i,\text{exp}} - X_{i,\text{th}}|^2}{\sigma_i^2},$$

where N is the number of experimental data, N' is the number of parameter of IBMCM model, $\chi_{i,\text{exp}}$ is the experimental energy of a given state, and σ_i is an error assigned to each $\chi_{i,\text{exp}}$ point. On the other hand, we have kept the value that is needed to create an extra particle-hole pair constant. This fixed value yields from strong similarity between experimental energy spectra and switching off the mixing term and shifting this fixed value. We performed a set of exploratory calculations with different set of these quantities and found that best agreement corresponds to $\omega_0^{N,N+2} = 18$ and $\omega_2^{N,N+2} = 7.45$ (in keV). The minimization is carried out using ε_N , ε_{N+2} , κ_N , κ'_N , χ_N , κ_{N+2} , κ'_{N+2} , χ_{N+2} as free parameters and other quantities as fixed ones. We minimize the χ^2 function by using the package Minuit [25] which allows us to minimize any multi-variable function. In this way, we obtain $\varepsilon_N = 460.1$, $\varepsilon_{N+2} = 74.3$, $\kappa_N = -0.06$, $\kappa'_N = 0.47$, $\chi_N = 0.02$, $\kappa_{N+2} = -0.01$, $\kappa'_{N+2} = 0.02$, $\chi_{N+2} = -0.85$, $\Delta^{N+2} = 3140$, $\omega_0^{N,N+2} = 18$

and $\omega_2^{N,N+2} = 7.45$ (all of them are expressed in keV unit). We used these quantities in calculation of energy levels presented in Fig. 1 and compared them with experimental counterparts. We plotted all levels which appear for the first time in energy spectra in the first column (it is not ground band). Other columns contain the second and third levels with the same spin-parity assignment.

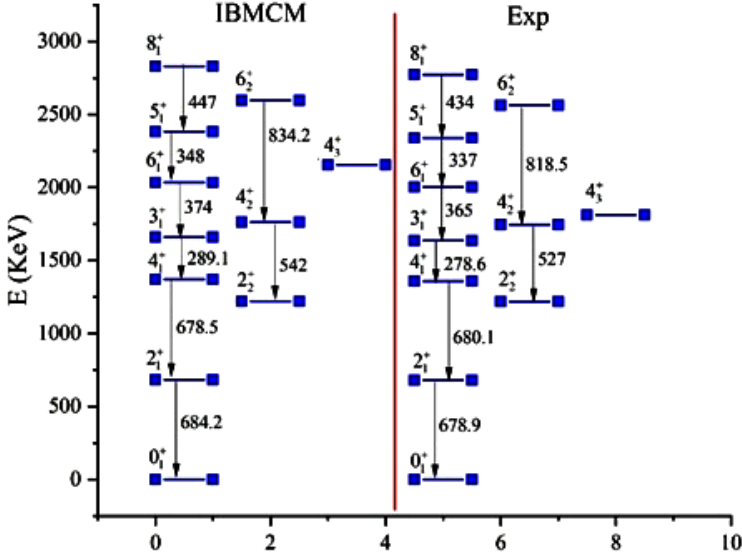


Fig. 1. Predictions of IBMCM for energy levels and their experimental counterparts in the ^{116}Te nucleus. Numbers between levels describe the distance between each other.

Energy spectra obtained using this technique are generally in good agreement with the experimental data and indicate the elegance of extraction procedure presented in this technique and they suggest the success of guess in parameterization. In the IBMCM formalism, ε_i and κ_i are considered as control parameters of the model which describe the effect of $\hat{n}_d$ and $\hat{Q}(\chi_i) \cdot \hat{Q}(\chi_i)$ terms in Hamiltonian. Different κ'_i values which describe the effect of quadrupole interactions in Hamiltonian yield small negative values near zero. These results derived via extraction process suggest the main role of $\hat{n}_d$ term, namely, this nucleus corresponds with the U(5) symmetry. This means that our idea to consider ^{116}Te as a candidate for U(5) symmetry can be approved. The negative values may depend on the type of asymmetry of the electronic charge distribution but with the lack of enough experimental data on dipole magnetic transition rates, it is impossible to discuss the origin of this behavior. One may expect intruder configurations appearing in nuclei located in the transitional regions, by construction at an excitation energy

that is much higher than the regular configurations. This is so because the large energy is needed to create the $2p$ - $2h$ excitation across the $Z = 50$ closed shell. In the case of the ^{116}Te nucleus, Δ^{N+2} is about 3140 keV, but according to the results of Ref. [22], the single-particle energy cost has to be corrected due to the strong pairing energy gain when forming two extra 4^+ coupled (particle and hole) pairs, and also the quadrupole energy gain when opening up the proton shell, as well as by the monopole correction caused by a change in the single-particle energy gap at $Z = 50$ as a function of the neutron number. In some cases, specifically around the mid-shell point at $N = 66$, the energy gain through these correlations can become so large that the intruder configurations are located below the energy of the regular configurations. In this case, one speaks about “islands of inversion” [18]. In $U(6)$ -based algorithms of IBM, we must diagonalize the considered Hamiltonian by some complicated numerical calculation. Pan and Draayer [11, 12] have proposed a new method based on the affine $SU(1,1)$ Lie algebra to exhibit the properties of nuclei which are located in the $U(5) \leftrightarrow SO(6)$ transitional region. The details of this model are available in Refs. [11, 12]. Here, we briefly outline the basic Ansatz and summarize the results.

3.2. Transitional Hamiltonian based on affine $SU(1,1)$ algebra

The Lie algebra corresponding to the $SU(1,1)$ group is generated by S^ν , $\nu = 0$ and $\pm$, which satisfies the following commutation relations:

$$[S^0, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = -2S^0. \quad (3)$$

On the other hand, the infinite dimensional $SU(1,1)$ algebra is generated using [11, 12]

$$S_n^\pm = c_s^{2n+1} S^\pm(s) + c_d^{2n+1} S^\pm(d), \quad S_n^0 = c_s^{2n} S^0(s) + c_d^{2n} S^0(d), \quad (4)$$

where c_s and c_d are real parameters and n can be $0, \pm 1, \pm 2, \dots$. The commutation relations of these operators are

$$[S_m^0, S_n^\pm] = \pm S_{m+n}^\pm, \quad [S_m^+, S_n^-] = -2S_{m+n+1}^0. \quad (5)$$

These operators $\{S_m^\mu, \mu = 0, +, -; \pm 1, \pm 2, \dots\}$ generate an affine Lie algebra $SU(1,1)$ without central extension. A transitional Hamiltonian can be set by using the generators of $SU(1,1)$ algebra to describe the transitional region between $U(5)$ and $SO(6)$ limits [11, 12]

$$\hat{H} = g S_0^+ S_0^- + \varepsilon S_1^0 + \gamma \hat{C}_2(SO(5)) + \delta \hat{C}_2(SO(3)), \quad (6)$$

where g, ε, γ and δ are real parameters, $\hat{C}_2(SO(3))$ and $\hat{C}_2(SO(5))$ denote the Casimir operators of these groups. If we consider $c_s = c_d$, Hamiltonian (6)

is similar to $SO(6)$ Hamiltonian and would be the same as $U(5)$ Hamiltonian when $c_s = 0$ and $c_d \neq 0$. Therefore, the $U(5) \leftrightarrow SO(6)$ transitional region can be described by $c_s \neq c_d \neq 0$ condition. In our calculation, we take $c_d (= 1)$ constant value and c_s vary between 0 and c_d .

One can use the Fourier–Laurent expansion of eigenstates and $SU(1,1)$ generators in terms of unknown c -number parameters x_i with $i = 1, 2, \dots, k$ to determine eigenstates of Hamiltonian (6). We can consider the eigenstates as [11, 12]

$$|k; \nu_s \nu n_{\Delta} LM\rangle = \sum_{n_i \in \mathbb{Z}} a_{n_1} a_{n_2} \dots a_{n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} S_{n_1}^+ S_{n_2}^+ \dots S_{n_k}^+ |lw\rangle. \quad (7)$$

The analytical behavior of wave functions suffices to consider x_i near zero. With using the commutation relations between the generators of $SU(1,1)$ algebra, wave functions are

$$|k; \nu_s \nu n_{\Delta} LM\rangle = N S_{x_1}^+ S_{x_2}^+ \dots S_{x_k}^+ |lw\rangle,$$

N is the normalization factor and

$$S_{x_i}^+ = \frac{c_s}{1 - c_s^2 x_i} S^+(s) + \frac{c_d}{1 - c_d^2 x_i} S^+(d). \quad (8)$$

The following set of equations makes it possible to determine c -numbers of x_i variables:

$$\frac{\varepsilon}{x_i} = \frac{g c_s^2 \left(\nu_s + \frac{1}{2} \right)}{1 - c_s^2 x_i} + \frac{g c_d^2 \left(\nu + \frac{5}{2} \right)}{1 - c_d^2 x_i} - \sum_{i \neq j} \frac{2}{x_i - x_j}, \quad \text{for } i = 1, 2, \dots, k. \quad (9)$$

Now, we can express the eigenvalues of Hamiltonian (9), *i.e.* $E^{(k)}$, as [11, 12]

$$\begin{aligned} E^{(k)} &= h^{(k)} + \gamma \nu (\nu + 3) + \delta L (L + 1) + \varepsilon \Lambda_1^0, \\ \Lambda_1^0 &= \frac{1}{2} \left[c_s^2 \left(\nu_s + \frac{1}{2} \right) + c_d^2 \left(\nu + \frac{5}{2} \right) \right], \end{aligned} \quad (10)$$

where

$$h^{(k)} = \sum_{i=1}^k \frac{\varepsilon}{x_i}. \quad (11)$$

The quantum number k is related to total boson number N by

$$N = 2k + \nu_s + \nu.$$

To get the energy levels of this model, a set of nonlinear Bethe-Ansatz equations (BAE) with k unknowns for k -pair excitations has been solved. First, we have changed the variables as

$$\varepsilon = \frac{\varepsilon}{g} (g = 1 \text{ keV [11, 12]}), \quad c = \frac{c_s}{c_d} \leq 1, \quad y_i = c_d^2 x_i.$$

Thus, the new form of Eq. (9) is

$$\frac{\varepsilon}{y_i} = \frac{c^2 (\nu_s + \frac{1}{2})}{1 - c^2 y_i} + \frac{(\nu + \frac{5}{2})}{1 - y_i} - \sum_{i \neq j} \frac{2}{y_i - y_j}, \quad \text{for } i = 1, 2, \dots, k. \quad (12)$$

We solved Eq. (12) with definite values of c and ε for $i = 1$ to determine the roots of Bethe-Ansatz equations (BAE) with specified values of ν_s and ν . These values are suggested from the experimental signature of nuclear shapes where for nuclei with $R_{4_1^+/2_1^+} \cong 2.00$, the first offer is $c = 0$ and $\varepsilon = 900$. Then, we used the “Find root” command in **Maple 17** software to get all y_i ’s. **Maple 17** is a mathematics-based software and service widely used for educational, engineering and research aims developed by the Maplesoft Company. We extracted the best set of Hamiltonian’s parameters, *i.e.* γ and δ , in comparison with the available experimental data [45–49] for excitation energies of selected states 0_1^+ , 2_1^+ , 4_1^+ , 0_2^+ , 2_2^+ , 4_2^+ and *etc.*, *e.g.* 12 levels up to 2_4^+ .

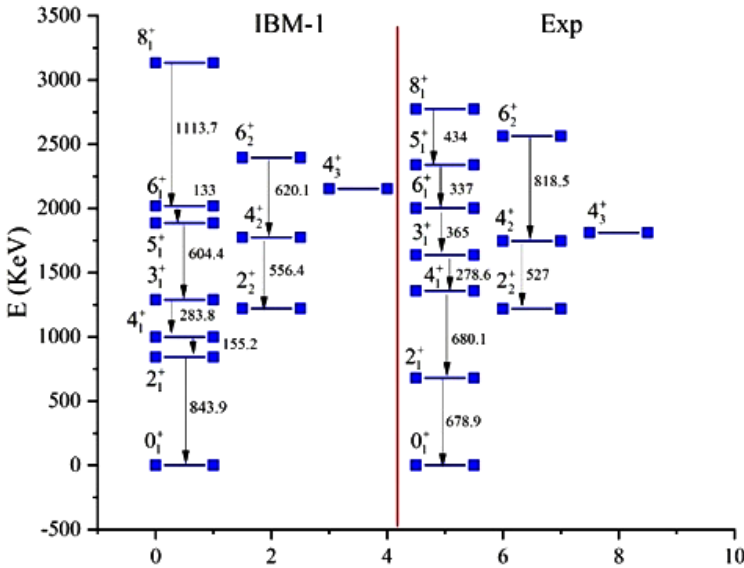


Fig. 2. Predictions of IBM 1 for energy levels of the ^{116}Te nucleus. Numbers between levels describe the distance between each other.

In summary, we have extracted γ and δ externally from empirical evidences and other quantities of Hamiltonian, namely c and ε , would be determined through the minimization of σ , $\sigma = (\frac{1}{N_{\text{tot}}} \sum_{i,\text{tot}} |E_{\text{exp}}(i) - E_{\text{cal}}(i)|^2)^{1/2}$. These processes were handled several times to produce best agreement between theoretical predictions and experimental counterparts for different energy levels.

A detailed comparison between the predictions of IBM 1 and experimental counterparts are presented in Fig. 2, while the parameters of energy eigenvalues, Eqs. (10)–(11), are listed in Table I.

TABLE I

Parameters of transitional Hamiltonian, Eq. (9). Experimental values are taken from Refs. [45, 49].

k	ν	L	E_{exp} [keV]	E_{th} [keV]	$ E_{\text{th}} - E_{\text{exp}} $	REP = $\left \frac{E_{\text{exp}} - E_{\text{th}}}{E_{\text{exp}}} \right \times 100$
4	0	0_1^+	0	0	0	0
3	2	2_1^+	678.92	843.9	165.0	19.55
3	1	2_2^+	1219	1220.1	1.1	0.08
2	3	3_1^+	1637.59	1282.9	354.7	27.64
3	2	4_1^+	1359	999.1	359.9	36.02
2	4	4_2^+	31746	0.5	1776.5	1.72
2	3	4_3^+	1811.77	2152.8	341.0	15.84
2	4	5_1^+	2339.87	1887.3	452.5	23.97
2	4	6_1^+	2002.24	2020.3	18.1	0.89
2	3	6_2^+	2564.5	2396.6	168.0	7.01
2	4	8_1^+	32773.1	3134	60.9	11.51
σ [Minimum variation]						263.42
ε [keV]						810.0442
c_s						0.0151
γ [keV]						-43.5709
δ [keV]						11.0824

These quantities describe the best agreement between the calculated energy levels and their experimental counterparts taken from Ref. [49], *i.e.* minimum values for σ . The result for control parameter, *e.g.* $c_s = 0.01$, suggests a pure vibrational symmetry in the ^{116}Te nucleus and confirms our idea to consider this nucleus as a candidate for U(5) symmetry. IBM 1 confirms our idea to consider ^{116}Te as a candidate for U(5) dynamical symmetry but as it is shown in Fig. 2, this formalism predicts some intruder states.

Different studies [21–29] reported such intruder states in Cd, Te isotopes chains located the closed shells. This means the normal vibrational construction would not explore the observed data for two phonon triplet states, $2p\text{--}4h$ excitation. Different authors suggested to use the IBM 2 predictions to calculate separately the normal and intruder states.

To this aim and for a complete description of intruder levels in the Te nucleus, we have used the same formalism to extend the IBM 2 calculation via $SU(1,1)$ Lie algebra. Details are available in Refs. [11, 12] and we explore the final results. In the IBM 2 case, the Hamiltonian can be considered as

$$\begin{aligned} \hat{H} = & gS_0^+S_0^- + \varepsilon S_1^0 + \gamma_1 \hat{C}_2(\text{SO}_\pi(5)) + \gamma_2 \hat{C}_2(\text{SO}_\nu(5)) + \delta_1 \hat{C}_2(\text{SO}_\pi(3)) \\ & + \delta_2 \hat{C}_2(\text{SO}_\nu(3)) + \delta \hat{C}_2(\text{SO}(3)) \end{aligned} \quad (13)$$

and similar to Eq. (5), the operators of such algebra are

$$\begin{aligned} S_n^\pm &= \sum_t c_{s;t}^{2n+1} S^\pm(s;t) + c_{d;t}^{2n+1} S^\pm(d;t), \\ S_n^0 &= \sum_t c_{s;t}^{2n} S^0(s;t) + c_{d;t}^{2n} S^0(d;t). \end{aligned} \quad (14)$$

The sum is carried over proton, π , and neutron, ν , indices. The eigenstates of Eq. (14) can be expressed as

$$|k; \beta; \nu_s^\pi, \nu_s^\nu, \nu^\pi, \nu^\nu; n_\Delta^\pi L_\pi, n_\Delta^\nu L_\nu; LM\rangle = N S_{x_1}^+ S_{x_2}^+ \dots S_{x_k}^+ |lw\rangle, \quad (15)$$

where $2k = N_\pi + N_\nu - \nu_s^\pi - \nu_s^\nu - \nu^\pi - \nu^\nu$ and

$$S_{x_i}^+ = \sum_t \frac{c_{s;t}}{1 - c_{s;t}^2 x_i} S^+(s;t) + \frac{c_{d;t}}{1 - c_{d;t}^2 x_i} S^+(d;t). \quad (16)$$

We have c -number x_i parameters which satisfy a set of equations as follows:

$$\begin{aligned} \frac{\varepsilon}{x_i} &= \sum_t g \left(\frac{c_{s;t}^2 (\nu_s^t + \frac{1}{2})}{1 - c_{s;t}^2 x_i} + \frac{c_{d;t}^2 (\nu^t + \frac{5}{2})}{1 - c_{d;t}^2 x_i} \right) - \sum_{i \neq j} \frac{2}{x_i - x_j}, \\ &\text{for } i = 1, 2, \dots, k. \end{aligned} \quad (17)$$

Energy spectra in the IBM 2 version of transitional Hamiltonian can be expressed as

$$\begin{aligned} E^{(k)} &= \sum_{i=1}^k \frac{\varepsilon}{x_i} + \gamma_1 \nu^\pi (\nu^\pi + 3) + \gamma_2 \nu^\nu (\nu^\nu + 3) + \delta_1 L_\pi (L_\pi + 1) \\ &\quad + \delta_2 L_\nu (L_\nu + 1) + \delta L(L + 1) + \varepsilon A_1^0, \\ A_1^0 &= \sum_t \frac{1}{2} \left[c_{s;t}^2 \left(\nu_s^t + \frac{1}{2} \right) + c_{d;t}^2 \left(\nu^t + \frac{5}{2} \right) \right]. \end{aligned} \quad (18)$$

We extracted the parameters of transitional Hamiltonian similar to the IBM 1 case. We have supposed $c_d = 1$ and then Eq. (17) have been solved for the $i = 1$ case with definite values of c and ε . Other parameters of Hamiltonian, namely δ_1 , δ_2 , δ , γ_1 and γ_2 , have been extracted from empirical data available for isotopic chain, and we would repeat these processes with different values of considered quantities to obtain the smallest σ values. Results for the parameters of Hamiltonian in the IBM 2 version are presented in Table II. In addition, a comparison between the IBM 2 predictions and the experimental counterpart for the ^{116}Te nucleus is given in Fig. 3.

TABLE II

Parameters of transitional Hamiltonian in the IBM 2 version. Experimental values are taken from Refs. [45, 49].

k	L	E_{exp} [keV]	E_{th} [keV]	$ E_{\text{th}} - E_{\text{exp}} $	REP = $\left \frac{E_{\text{exp}} - E_{\text{th}}}{E_{\text{exp}}} \right \times 100$
3	0_1^+	0	0	0	0
2	2_1^+	678.92	839	160.1	19.08
2	2_2^+	1219.7	918.4	300.6	32.73
2	3_1^+	1637.59	1419.9	217.6	15.33
2	4_1^+	1359	1518	159	10.47
2	4_2^+	1746	1589.6	156.4	9.84
2	4_3^+	1811.77	1905.9	94.1	4.94
1	5_1^+	2339.87	2154.2	185.7	8.62
2	6_1^+	2002.24	1802.8	199.4	11.06
2	6_2^+	2564.5	2310	243.5	10.49
2	8_1^+	2773.1	2467.6	305.5	12.37
σ [Minimum variation]					201.99
ε [keV]					398.6689
c_s					0.0418
γ_1 [keV]					30.4825
γ_2 [keV]					-188.7960
δ_1 [keV]					-11.8434
δ_2 [keV]					68.2475
δ [keV]					24.6055

IBM 2 suggests more exact results, *i.e.* minimum σ value, in comparison with experimental data and also the results of IBM 1. This means that one can consider the IBM 2 framework to describe energy spectra of the ^{116}Te nucleus. These results suggest the accuracy of this formalism and

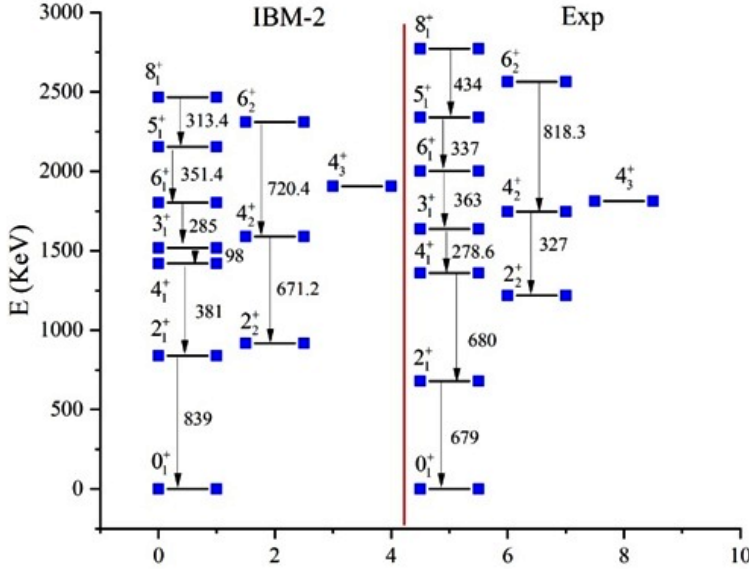


Fig. 3. Predictions of IBM 2 for energy levels of the ^{116}Te nucleus. Numbers between levels describe the distance between each other.

extraction process for such a nucleus. The results of the IBM 2 calculation confirm our idea of the effect of pairing to reduce the distance between theoretical predictions and experimental counterparts. In comparison with the predictions of IBM 1, we reach low uncertainty but our result for the c_s value is more than IBM 1 counterparts. This result for the control parameter of the model confirms the $U(5)$ -like structure but suggests a trivial effect due to the axial deformation on its structure.

To get the exact value of the control parameter, we have employed the catastrophe theory in combination with the coherent state formalism in the both IBM 1 and 2 which make it possible to get energy surfaces and determine the exact values of the control parameter.

4. Energy surfaces

The geometric configuration of IBM can be described in the framework of a coherent state. The coherent state [37–44] connects the algebraic and geometric descriptions of three dynamical symmetry limits of IBM and also allows the study of transitions among them. By using this formalism, one can evaluate the ground-state energy as a function of shape variables β and γ , *i.e.* the deformation parameters [37], similar to what have been done for $U(5) \leftrightarrow SO(6)$ and $U(5) \leftrightarrow SU(3)$ transitional regions [35, 36]. The clas-

sical limit corresponding to Hamiltonian (9) is obtained by considering its expectation value in the coherent state [37–41]

$$|N, \alpha_m\rangle = \left(s^\dagger + \sum_m \alpha_m d_m^\dagger \right)^N |0\rangle, \quad (19)$$

where $|0\rangle$ is the boson vacuum state, $s^\dagger$ and $d^\dagger$ are the boson operators of IBM, and parameter α_m can be related to the deformation collective parameters [37]

$$\alpha_0 = \beta \cos \gamma, \quad \alpha_{\pm 1} = 0, \quad \alpha_{\pm 2} = \frac{\beta}{\sqrt{2}} \cos \gamma. \quad (20)$$

In the IBM 2 framework, the most general form of coherent state is [42–44]

$$|N_\pi, N_\nu, \beta_\pi, \gamma_\pi, \beta_\nu, \gamma_\nu, \phi, \theta, \psi\rangle = \frac{1}{\sqrt{(N_\pi)!(N_\nu)!}} R(\theta, \phi, \psi) \left(\Gamma_\pi^\dagger \right)^{N_\pi} \left(\Gamma_\nu^\dagger \right)^{N_\nu} |0\rangle, \quad (21)$$

where

$$\Gamma_\rho^\dagger = \frac{\left[s_\rho^\dagger + \beta_\rho \cos \gamma_\rho d_{\rho,0}^\dagger + \frac{1}{\sqrt{2}} \beta_\rho \sin \gamma_\rho \left(d_{\rho,2}^\dagger + d_{\rho,-2}^\dagger \right) \right]}{\sqrt{1 + \beta_\rho^2}}, \quad (22)$$

and the Euler angles (θ, ϕ, ψ) define the orientation of deformation variables (β_π, γ_π) for proton bosons and (β_ν, γ_ν) for neutron bosons as has been shown in Ref. [42]. In the absence of hexadecupole interactions, the Euler angles can be taken equal to zero. The energy surface would be determined by means of

$$E = \frac{\langle N, \alpha_m | H | N, \alpha_m \rangle}{\langle N, \alpha_m | N, \alpha_m \rangle}. \quad (23)$$

Then, the energy surfaces from each part of transitional Hamiltonian can be written as

$$\langle g S_0^+ S_0^- \rangle = \frac{g}{4} \left(\frac{N_\rho (N_\rho - 1)}{(1 + \beta_\rho^2)^2} \right) (c_s^2 + 2c_s c_d \beta_\rho^2 + c_d^2 \beta_\rho^4), \quad (24)$$

$$\langle \varepsilon S_0^1 \rangle = \frac{\varepsilon c_s^2}{4} \left(\frac{2N_\rho}{1 + \beta_\rho^2} + 1 \right) + \frac{\varepsilon c_d^2}{4} \left(\frac{2N_\rho \beta_\rho^2}{1 + \beta_\rho^2} + 5 \right), \quad (25)$$

$$\langle \gamma \hat{C}_2(\text{SO}_\rho(5)) \rangle = 2 \frac{\gamma N_\rho \beta_\rho^2}{1 + \beta_\rho^2}, \quad (26)$$

$$\langle \delta \hat{C}_2(\text{SO}_\rho(3)) \rangle = \frac{3}{5} \frac{\delta_\rho N_\rho \beta_\rho^2}{1 + \beta_\rho^2}. \quad (27)$$

These yields the energy surfaces in the IBM 1 framework as

$$E(\beta, \gamma) = \frac{g}{4} \left(\frac{N(N-1)}{(1+\beta^2)^2} \right) (c_s^2 + 2c_s c_d \beta^2 + c_d^2 \beta^4) + \frac{\varepsilon c_s^2}{4} \left(\frac{2N}{1+\beta^2} + 1 \right) + \frac{\varepsilon c_d^2}{4} \left(\frac{2N\beta^2}{1+\beta^2} + 5 \right) + 2 \frac{\gamma N \beta^2}{1+\beta^2} + \frac{3}{5} \frac{\delta N \beta^2}{1+\beta^2}, \quad (28)$$

and similarly, we can get the energy surfaces in the IBM 2 framework as

$$E(\beta, \gamma) = \frac{g}{4} \left(\frac{N_\rho(N_\rho-1)}{(1+\beta_\rho^2)^2} \right) (c_s^2 + 2c_s c_d \beta_\rho^2 + c_d^2 \beta_\rho^4) + \frac{\varepsilon c_s^2}{4} \left(\frac{2N_\rho}{1+\beta_\rho^2} + 1 \right) + \frac{\varepsilon c_d^2}{4} \left(\frac{2N_\rho \beta_\rho^2}{1+\beta_\rho^2} + 5 \right) + 2 \frac{\gamma_1 N_\pi \beta_\pi^2}{1+\beta_\pi^2} + 2 \frac{\gamma_2 N_\nu \beta_\nu^2}{1+\beta_\nu^2} + \frac{3}{5} \frac{\delta_1 N_\pi \beta_\pi^2}{1+\beta_\pi^2} + \frac{3}{5} \frac{\delta_2 N_\nu \beta_\nu^2}{1+\beta_\nu^2} + \frac{3}{5} \frac{\delta N_\rho \beta_\rho^2}{1+\beta_\rho^2}. \quad (29)$$

To analyze the energy surfaces within the catastrophe theory formalism, we have determined the critical points of the energy surfaces. The following algebraic equation yields the variable β (in the IBM 1 formalism and the procedure is similar to IBM 2), and we will denote the final result as

$$\frac{\partial E}{\partial \beta} = \frac{\beta}{(1+\beta^2)^3} \left[gN(N-1)(c_s + c_d)(c_d - c_s)\beta^2 + 2 \left(\frac{N}{2} \varepsilon c_d^2 + 2\gamma N + \frac{3}{5} \delta N - \frac{N}{2} \varepsilon c_s^2 \right) (1+\beta^2) \right]. \quad (30)$$

The critical points of transitional region would be obtained by this equation. This expression shows that $\beta = 0$ is a critical point for any values of the parameters of the energy surfaces and is the fundamental root. The Taylor series expansion of the energy surfaces around this fundamental root is given by

$$E(\beta) = \frac{g}{4} N(N-1) c_s^2 + \frac{N}{2} \varepsilon c_s^2 + \frac{1}{4} \varepsilon (c_s^2 + 5c_d^2) + \frac{1}{2} \left[N(N-1) g c_s (c_d - c_s) + N \left(\varepsilon (c_d^2 - c_s^2) + \frac{6}{5} \delta + 4\gamma \right) \right] \beta^2 + \left[\frac{3}{4} N(N-1) g c_s^2 - N(N-1) g c_s c_d + \frac{1}{4} N(N-1) g c_d^2 + \frac{1}{2} N \alpha (c_s^2 - c_d^2) - \frac{3}{5} N \delta - 2N\gamma \right] \beta^4 + O(5) + \dots, \quad (31)$$

or can be rewritten in the form of

$$E(\beta) = A + A'\beta^2 + A''\beta^4 + \dots, \quad (32)$$

while the coefficients are given by

$$A = \frac{g}{4}N(N-1)c_s^2 + \frac{N}{2}\varepsilon c_s^2 + \frac{1}{4}\varepsilon(c_s^2 + 5c_d^2), \quad (33)$$

$$A' = \frac{1}{2} \left[N(N-1)gc_s(c_d - c_s) + N \left(\varepsilon(c_d^2 - c_s^2) + \frac{6}{5}\delta + 4\gamma \right) \right],$$

for IBM 1, (34)

$$A' = \frac{1}{2} \left[N(N-1)gc_s(c_d - c_s) + N \left(\varepsilon(c_d^2 - c_s^2) + \frac{6}{5}(\delta_1 + \delta_2) + 4(\gamma_1 + \gamma_2 + \gamma) \right) \right],$$

for IBM 2, (35)

$$A'' = \left(\frac{3}{4}N(N-1)gc_s^2 - N(N-1)gc_sc_d + \frac{1}{4}N(N-1)gc_d^2 + \frac{1}{2}N\varepsilon(c_s^2 - c_d^2) - \frac{3}{5}N\delta - 2N\gamma \right), \quad \text{for IBM 1,} \quad (36)$$

$$A'' = \frac{3}{4}N(N-1)gc_s^2 - N(N-1)gc_sc_d + \frac{1}{4}N(N-1)gc_d^2 + \frac{1}{2}N\varepsilon(c_s^2 - c_d^2) - \frac{3}{5}(N_\pi\delta_1 + N_\nu\delta_2) - 2(N\gamma + N_\pi\gamma_1 + N_\nu\gamma_2), \quad \text{for IBM 2.} \quad (37)$$

We must determine the bifurcation set, the locus of the points in the space of control parameters at which a transition occurs from one local minimum to another [8], to identify the exact value of control parameter for each nucleus. With using the $\det(H) = 0$ condition, H is the matrix of the second derivate of the energy surface at the critical point, which became $\partial^2 E / \partial \beta^2 = 0$ in the case of a function of one variable [39]. One gets the expression

$$c_s = \frac{(N-1)c_d + \sqrt{g^2(N-1)^2c_d^2 + 4[g(N-1) + \varepsilon][\varepsilon c_d^2 + 4\gamma + \frac{6}{5}\delta]}}{2(g(N-1) + \varepsilon)}. \quad (38)$$

In the IBM 1 formalism and similarly in the IBM 2 formalism, we get the following result:

$$c_s = \frac{g(N-1)c_d + \sqrt{g^2(N-1)^2c_d^2 + 4[g(N-1) + \varepsilon][\varepsilon c_d^2 + 4(\gamma_1 + \gamma_2) + \frac{6}{5}(\delta_1 + \delta_2 + \delta)]}}{2(g(N-1) + \varepsilon)}. \quad (39)$$

By using $N = 9$, *e.g.* the number of bosons in this nucleus, and other parameters as presented in Tables I and II for IBM 1 and 2, respectively, and

inserting it into Eqs. (38)–(39), we get $c_s = 0.02$ and 0.017 as the control parameters of model. These results, similar to what have been determined via extraction process, suggest ^{116}Te as a candidate for spherical symmetry. Moreover, we have presented the energy surfaces of the ^{116}Te nucleus. In the transition from $U(5)$ to the $SO(6)$ limit, the evolution of energy surface goes from a pure β^2 to a combination of β^2 and β^4 that has a deformed minimum [37]. Figure 4 shows the energy surface which is plotted as a function of β and the predictions of IBM 1 and 2. In addition, the small effect of quadrupole interactions in this nucleus, which are yielded by the IBMCM formalism, has not any significant effect on energy surfaces.

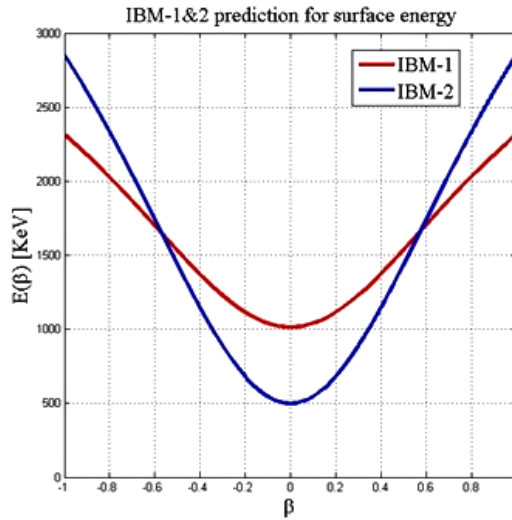


Fig. 4. The energy surfaces of the ^{116}Te nucleus as a function of order parameter β . We have used the predictions of IBM 1 and 2 to evaluate energy surfaces.

From these figures and tables, one can conclude that the calculated energy spectra in this approach are generally in good agreement with the experimental data. Our results indicate the elegance of the extraction procedures in IBMCM and IBM 2 formalisms and they suggest the success of estimation processes.

5. Conclusion

As a conclusion of this study, our models, IBMCM and transitional Hamiltonian in the $SU(1,1)$ framework in the both IBM 1 and 2 versions, describe the energy spectra of the ^{116}Te nucleus. What is more, the control parameter of both models has values which propose the dominant role of spherical symmetry in the ^{116}Te nucleus. This result suggests a spherical-like shape for this nucleus and offers it as a candidate for vibrational symmetry,

U(5) limit, of IBM. This result corresponds to the experimental energy ratio and the shape of energy surface of this nucleus. One may relate the U(5)-like structure of ^{116}Te to the occurrence of sub-closed shell in $N = 64$. The obtained results in this study confirm that this technique is worth extending for investigating the nuclear structure of other nuclei existing around the mass of $A \sim 120$.

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