Advanced Studies in Theoretical Physics
Vol. 12, 2018, no. 8, 381 - 387
HIKARI Ltd, www.m-hikari.com
https://doi.org/10.12988/astp.2018.8731

Modified Geroch Functional and Submanifold Stability in Higher Dimension

Flinn C. Rajabaycolle[1,2] Fiki T. Akbar[1], Jusak S. Kosasih[1] and Bobby E. Gunara[1]

[1] Theoretical Physics Laboratory
Theoretical High Energy Physics and Instrumentation Research Group
Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung, Jl. Ganesha 10 Bandung 40132, Indonesia

[2] Department of Physics, Faculty of Mathematics and Natural Sciences
Universitas Cenderawasih
Jl. Kampwolker Kampus Uncen Baru Waena-Jayapura 99351, Indonesia

Copyright © 2018 Flinn C. Rajabaycolle, Fiki T. Akbar, Jusak S. Kosasih and Bobby E. Gunara. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

In this paper, we study critical points of so called modified Geroch functional (MGF) in higher dimension, $n > 2$, which can be viewed as critical points of the area functional of one codimensional submanifold. Further analysis at the level of second order variation of MGF we obtain that the eigenvalue of the Jacobi operator has upper or lower bounds that depends on the 'cosmological' constant of the submanifold.

1 Introduction

There is still no satisfactory definition of mass in general relativity. It does not arise as naturally as in classical physics. In Newtonian theory, we can find the mass by simply integrating the mass density over the volume, while in general relativity there is no mass density of the gravitational field. The gravitational field itself can be constructed from the first derivatives of the metric. However, when the spacetime is asymptotically flat, mass can be defined well. Because of this problem, one has to find a definition of mass at quasi-local level. Several attempts have been made for definition of quasi-local mass and some of them are the Hawking mass and the Geroch mass.

In 1968, the Hawking mass was first found by Hawking [[2]] in his paper about gravitational radiation. It can be described as a measure of the bending of ingoing and outgoing rays of light that are orthogonal to a two-sphere. Let a two-surface Σ is embedded to a three-manifold M, $\Sigma \subset M$, then the Hawking mass $m_H(\Sigma)$ is defined to be

$$m_H(\Sigma) := \sqrt{\frac{S}{16\pi}}\left(1 + \frac{1}{16\pi}\int_\Sigma \rho_+\rho_- d\sigma\right), \tag{1}$$

where ρ_+ and ρ_- are the expression of outgoing and ingoing null geodesics, respectively, and $S = \text{Area}(\Sigma)$.

The Geroch mass was introduced by Geroch [[3]] in 1973. It was derived from the Hawking mass [[4]] and defined as

$$m_G(\Sigma) := \frac{1}{16\pi}\sqrt{\frac{S}{16\pi}}\int_\Sigma (2\tilde{R} - H^2) d\sigma, \tag{2}$$

with

$$H = \tilde{g}^{ij} K_{ij}, \tag{3}$$

and

$$K_{ij} = g_{\mu\nu}\Gamma^{\mu}{}_{ij}\hat{n}^{\nu}, \tag{4}$$

where indices i and j are used for spatial components of the submanifold Σ, indices μ and ν are used for spatial components of the ambient manifold M, $\tilde{R}$ is the scalar curvature of Σ, H is the mean curvature (extrinsic curvature) of Σ, K_{ij} is the second fundamental form of Σ, and $\hat{n}^\nu$ is the normalized normal vector or can be written as $\hat{n}^\nu = n^\nu/|n|$. Geroch [[3]] then proved that mass (2) under inverse mean curvature flow (IMCF) is always non-negative (positive mass theorem) using a functional

$$f_G(\Sigma) := \int_\Sigma (2\tilde{R} - H^2) d\sigma, \tag{5}$$

by computing its first variation with respect to a parameter t. As shown by Geroch [[3]], when $f_G(\Sigma) \geq 0$ for all t, then $m_G(H) \geq 0$ as well. This functional is called Geroch functional. Hence, we can write Eq. (2) to be

$$m_G(\Sigma) := \frac{1}{16\pi}\sqrt{\frac{S}{16\pi}} f_G(\Sigma). \tag{6}$$

Furthermore, in 1977 Jang and Wald [[5]] made a slight modification of Geroch's positive mass argument, and using the Geroch functional (5) to prove the Riemannian Penrose inequality (RPI), where RPI states that if m is the ADM mass and S is the area of the horizon of the black hole, then RPI is defined as [[6]]

$$m \geq \sqrt{\frac{S}{16\pi}}. \tag{7}$$

In 2012, Maximo and Nunes [[1]] then introduced another modification of the Hawking mass. Inspired by the work of Geroch [[3]], Jang and Wald [[5]], Maximo and Nunes [[1]], we will define a functional called modified Geroch functional (MGF). It has already been known that to find the critical point of mass, one can use the first variation of mass. In the present paper, instead of using mass, we will use MGF and compare our result to [[1]].

Let $\Sigma^n \subset M^{n+1}$ be an n-dimensional compact submanifold Σ^n embedded in an $(n+1)$-dimensional manifold M^{n+1} which could be compact or noncompact. We write MGF on Σ^n is defined as

$$f_{MG}(\Sigma^n) := \int_{\Sigma^n} (2\tilde{R} - H^2 - \lambda) d\sigma \ , \tag{8}$$

where λ is a real constant. In our previous work [[7]], we have studied the monotonicity of mass under IMCF using mass formula which is similar to (8). We find that the Hawking mass related to (8) is non-decreasing under IMCF. In this paper, we will examine the functional (8) on the minimal subamifold Σ^n. Our result shows that the eigenvalue of the Jacobi operator has upper or lower bounds that depends on λ and the 'cosmological' constant of Σ^n.

The structure of the paper is as follows. In Section 2, we calculate the first variation of MGF to study its critical point. In Section 3 , we discuss the relation between the functional (8) and submanifold stability. We add a complete form of the second variation of (8) in the appendix.

2 First Variation and Critical Point of MGF

In this section we calculate the first variation of MGF (8). First of all, let $\Sigma^n \subset M^{n+1}$ be a compact submanifold. Using the flow equation

$$\frac{\partial \mathcal{F}(x,t)}{\partial t} = \phi(x,t)\hat{n}(x,t), \text{ for each } x \in \Sigma^n \text{ and } \phi \in C^\infty(\Sigma^n), \tag{9}$$

with $\phi(x,t)$ is some real function, $t \geq 0$, $\mathcal{F}$ is a smooth function $\mathcal{F} : \Sigma^n \to M^{n+1}$, and $\hat{n}$ is an outward pointing unit normal vector of Σ^n_t in M^{n+1}, the first variation of area has the form

$$\frac{d}{dt}(d\sigma_t)|_{t=0} = -\phi H d\sigma \ , \tag{10}$$

where H is the mean curvature. Then, the first derivative of H is given by

$$\frac{dH}{dt}(0) = \Delta\phi + \mathrm{Ric}(\hat{n},\hat{n})\phi + |K|^2\phi \ , \tag{11}$$

with $\mathrm{Ric}(\hat{n},\hat{n})$ is related to the curvature $\tilde{R}$ of Σ^n and the curvature R of M^{n+1} via Gauss equation

$$\mathrm{Ric}(\hat{n},\hat{n}) = \frac{R}{2} - \frac{\tilde{R}}{2} + \frac{H^2}{2} - \frac{|K|^2}{2} \ , \tag{12}$$

where $|K|^2 := K^{ij}K_{ij}$. Then, the first variation of MGF (8) can be written down as

$$\frac{d}{dt}f_{MG}(\Sigma^n_t)|_{t=0} = -\int_{\Sigma^n} 2\phi(\Delta H + \alpha(0)H - 2\tilde{R}_{ij}K^{ij})d\sigma \ , \tag{13}$$

with

$$\alpha(0) = \frac{R}{2} + \frac{\tilde{R}}{2} + \frac{|K|^2}{2} - \frac{\lambda}{2} \ , \tag{14}$$

where we have used [[**?**]]

$$\frac{d}{dt}\left(\int_{\Sigma^n} 2\tilde{R}d\sigma_t\right)|_{t=0} = \int_{\Sigma^n}(-2\tilde{R}\phi H + 4\phi\tilde{R}_{ij}K^{ij})d\sigma \ . \tag{15}$$

Note that throughout this paper we take (13) to be

$$\frac{d}{dt}f_{MG}(\Sigma^n_t)|_{t=0} \geq 0 \ . \tag{16}$$

Now, suppose both n-submanifold Σ^n and $(n+1)$-manifold M^{n+1} are Einstein, so that their Ricci tensors can be written as

$$\begin{aligned} R_{\mu\nu} &= \Lambda g_{\mu\nu} \ , \\ \tilde{R_{ij}} &= \tilde{\Lambda}\tilde{g}_{ij} \ , \end{aligned} \tag{17}$$

for some constants Λ and $\tilde{\Lambda}$ which folows

$$\frac{d}{dt}f(\Sigma^n_t)|_{t=0} = -2\int_{\Sigma^n}\phi(\Delta H + QH)d\sigma \ , \tag{18}$$

with

$$Q = \frac{(n+1)\Lambda}{2} + \frac{(n-4)\tilde{\Lambda}}{2} + \frac{|K|^2}{2} - \frac{\lambda}{2}. \tag{19}$$

If the submanifold Σ^n is minimal, i.e., $H = 0$, then it can also be viewed as the critical point of (8).

3 MGF and Submanifold Stability

In this section we study the stability of n-dimensional submanifold Σ^n embedded in (n+1)-dimensional Einstein manifold M^{n+1}, $\Sigma^n \subset M^{n+1}$, from the second variation of area.

First of all, we discuss the notion of submanifold stability using the second variation of area

$$\frac{d^2}{dt^2} d\sigma_t|_{t=0} = -\frac{d\phi}{dt} H d\sigma - \phi L\phi \; d\sigma + \phi^2 H^2 d\sigma \;, \tag{20}$$

where L is the Jacobi operator defined as

$$L := \Delta + \mathrm{Ric}(\hat{n}, \hat{n}) + |K|^2 \;. \tag{21}$$

A minimal submanifold Σ^n is said to be stable if the second variation of area (20) is non-negative which means

$$-L\phi = \xi\phi \geq 0 \;, \tag{22}$$

where ξ is the eigenvalue of L.

Next, we write down the second varitaion of MGF (8) on the minimal surface Σ^n, that is,

$$\frac{d^2}{dt^2} f(\Sigma^n_t)|_{t=0} = -\int_{\Sigma^n} \left(2n\tilde{\Lambda}\phi L\phi - 4\tilde{\varpi}\phi - \lambda\phi L\phi \; + 2(L\phi)^2 \right) d\sigma \;, \tag{23}$$

for all $\phi \in C^\infty(\Sigma^n)$ where

$$\tilde{\varpi} := \frac{1}{2} K^{ij} \left(\nabla^l \nabla_i (\mathcal{K}_{jl}) + \nabla^l \nabla_j (\mathcal{K}_{il}) - \Delta(\mathcal{K}_{ij}) \right) - 2\tilde{\Lambda} \left(\phi |K|^2 - \Delta\phi + \phi \mathrm{Ric}(\hat{n}, \hat{n}) \right) \;, \tag{24}$$

with $\mathcal{K}_{ij} := -2\phi K_{ij}$. Taking $\phi = 1$ and assuming that K_{ij} to be parallel, i.e. $\nabla_k K_{ij} = 0$, we find that the functional f_{MG} has a local maximum if

$$\xi > \lambda - 2(n+4)\tilde{\Lambda} \;, \tag{25}$$

while it has a local minimum if

$$\xi < \lambda - 2(n+4)\tilde{\Lambda} \;. \tag{26}$$

Both cases occur for $\lambda > 2(n+4)\tilde{\Lambda}$ since $\xi \geq 0$.

Acknowledgements. This work is funded by P3MI ITB 2018 and PDUPT Kemenristekdikti 2018.

Appendix Second Variation of MGF

In this section we calculate the second variation of MGF as follows

$$\frac{d^2}{dt^2}f(\Sigma_t) = \frac{d^2}{dt^2}\bigg(\int_\Sigma 2\tilde{R}d\sigma_t\bigg) - \int_\Sigma \frac{d^2(H^2+\lambda)}{dt^2}d\sigma_t - \int_\Sigma (H^2+\lambda)\frac{d^2}{dt^2}d\sigma_t. \tag{27}$$

From equation (15) we can compute its second variation:

$$\frac{d^2}{dt^2}\bigg(\int_\Sigma 2\tilde{R}d\sigma_t\bigg) = \beta(t) = \int_\Sigma -2\frac{d\phi}{dt}(\tilde{R}H - 2\tilde{R_{ij}}K^{ij})d\sigma_t + \int_\Sigma 2\phi\bigg(\tilde{\kappa}H + \tilde{R}L\phi - 2\tilde{\varpi}\bigg)d\sigma_t + \int_\Sigma 2\phi(\tilde{R}H - 2\tilde{R_{ij}}K^{ij})\frac{d}{dt}d\sigma_t, \tag{28}$$

where

$$\frac{d\tilde{R_{ij}}}{dt} = -\frac{1}{2}\bigg(\Delta_L\mathcal{K}_{ij} + \nabla_i\nabla_j\mathcal{K} + \nabla_j(\delta\mathcal{K})_j + \nabla_j(\delta\mathcal{K})_i\bigg), \tag{29}$$

$$\frac{dK_{ij}}{dt} = \phi K_{ik}K^{k_j} - \nabla_i\nabla_j\phi - \phi R_{iooj}, \tag{30}$$

$$\tilde{\varpi} = \frac{d}{dt}(\tilde{R}_{ij}K^{ij}) = \mathcal{K}^{ik}g^{jl}\tilde{R}_{kl} + \mathcal{K}^{jl}g^{ik}\tilde{R}_{kl} + \frac{1}{2}K^{kl}\bigg(-\Delta_L(\mathcal{K}_{kl}) - \nabla_k\nabla_l\mathcal{K} + \nabla_k(\delta\mathcal{K})_l + \nabla_l(\delta\mathcal{K})_k\bigg) - 2\tilde{R}^{ij}\bigg(\phi K_{ik}K^k_j - \nabla_i\nabla_j\phi - \phi R_{iooj}\bigg), \tag{31}$$

$$\tilde{\kappa} = \frac{d\tilde{R}}{dt} = 2\Delta(\phi H) - 2\nabla^i\nabla^j(\phi K_{ij}) - 2\phi\tilde{R_{ij}}K^{ij}, \tag{32}$$

with

$$\mathcal{K} = -2\phi H, \tag{33}$$

$$\mathcal{K}_{ij} = -2\phi K_{ij}, \tag{34}$$

$$\mathcal{K}^{ij} = 2\phi K^{ij}, \tag{35}$$

$$(\delta\mathcal{K})_i = -\tilde{g}^{jp}\nabla_j(2\phi K_{pi}), \tag{36}$$

$$(\Delta_L\mathcal{K})_{ij} = \Delta\mathcal{K}_{ij} + 2g^{qp}R^r_{qij}\mathcal{K}_{ij} - \tilde{g}^{qp}R_{ip}\mathcal{K}_{qj} - \tilde{g}^{qp}R_{jp}\mathcal{K}_{iq}. \tag{37}$$

We then calculate the second variation of $(H^2+\lambda)$

$$\frac{d^2(H^2+\lambda)}{dt^2} = 2(L\phi)^2 + 2HL'\phi. \tag{38}$$

Substitute, (38), (28) and (20) to (27), we find the second variation of MGF: with $\beta = \frac{d^2}{dt^2}\left(\int_\Sigma 2\tilde{R}d\sigma_t\right)$ is given in (28).

References

[1] D. Maximo and I. Nunes, 2012 Hawking Mass and Local Rigidity of Minimal Two-Spheres in Three Manifolds, *Communications in Analysis and Geometry*, **21** (2013), 409-433.
https://doi.org/10.4310/cag.2013.v21.n2.a6

[2] S. Hawking, Gravitational Radiation in an Expanding Universe, *J. Math. Phys.*, **9** (1968), 598-604. https://doi.org/10.1063/1.1664615

[3] R. Geroch, Energy Extraction, *Ann. New York Acad. Sci.*, **224** (1973), 108-117. https://doi.org/10.1111/j.1749-6632.1973.tb41445.x

[4] L. B. Szabados, Quasi-Local Energy-Momentum and Angular Momentum in General Relativity, *Living Rev. Relativ.*, **12** (2009), 4.
https://doi.org/10.12942/lrr-2009-4

[5] P. S. Jang and R. M. Wald, The positive energy conjecture and the cosmic censor hypothesis, *J. Math. Phys.*, **18** (1977), 41-44.
https://doi.org/10.1063/1.523134

[6] R. Penrose, Naked Singularities, *Annals of the New York Academy of Sciences* **224** (1973), no. 1, 125-134.
https://doi.org/10.1111/j.1749-6632.1973.tb41447.x

[7] F. Radjabaycolle, F. Akbar and B. Gunara, Modified Hawking Mass and Geroch Monotonicity on 2-Surfaces, *J. Phys.: Conf. Ser.*, **739** (2016), 012046. https://doi.org/10.1088/1742-6596/739/1/012046

Received: August 7, 2018; Published: December 18, 2018