THEORY OF LONGITUDINAL INSTABILITY FOR BUNCHED ELECTRON AND PROTON BEAMS

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Summary

We discuss an original approach for the treatment of the longitudinal stability of high-intensity proton and electron bunches. The general analysis is divided in three steps. First, we search for a stationary bunch distribution which is matched to the external RF forces as well as to the current dependent induced fields. We question the existence of such distribution. Second, we check the stability of the stationary solution by applying a small perturbation and observing whether this is initially damped or not. At this point a stability condition is derived in terms of current, surrounding impedance and bunch size. In the last step one should question what happens to the beam in case the stability condition is not satisfied. The problem here is the determination of the final bunch configuration. We will not deal much with this step. We observe that the "overshoot formula" which is derived from numerical calculation[1] is usually applied to proton bunches[2], whereas commonly the assumption is made an electron bunch matches always its size to the stability condition[3].

The originality of our approach stays in the combination of the three steps. All previous theories either consider only the first step[4] or combine the second and third ones but disregard the first[2,3,5]. Sometimes, in the latter case, the modification of the potential well is introduced ad hoc[3].

Our theory applies to the case of a real frequency independent impedance.

The Stationary Distribution

The starting point is the Fokker-Plank equation

$$\frac{\partial\psi}{\partial t} + \dot\phi\frac{\partial\psi}{\partial\phi} + \dot w\frac{\partial\psi}{\partial w} = \frac{\psi}{\tau} + \frac{w}{\tau}\frac{\partial\psi}{\partial w} + D\frac{\partial^2\psi}{\partial w^2} \qquad (1)$$

where ψ is the particle distribution in ϕ and w, ϕ the phase angle in RF radians unit and w the angular momentum canonically conjugated to ϕ. In the case of a proton bunch, $\tau\to\infty$ and $D\to 0$, and Eq. (1) reduce to the Vlasov equation.

The equations of motion are

$$\dot\phi = \partial H/\partial w \quad \text{and} \quad \dot w = F_p - \partial H/\partial\phi \ .$$

With a proper scaling it is possible to give the dimension of a frequency to w and to write for the Hamiltonian

$$H = \frac{1}{2}w^2 + U_{ext}(\phi) + U_{sc}(\phi)$$

where U_{ext} is the external RF potential function and U_{sc} the beam induced potential function in the "stationary" configurations. Finally F_p is the field induced by a perturbation around the bunch.

A stationary solution does not depend explicitly on the time, so that

$$F_p = 0 \quad \text{and} \quad \partial\psi/\partial t = 0 \ .$$

For a proton bunch one has simply

$$\dot\phi\frac{\partial\psi}{\partial\phi} + \dot w\frac{\partial\psi}{\partial w} = 0.$$

The general solution of this equation is any function of the Hamiltonian H. Thus for the case of a proton bunch there is an infinite variety of stationary distributions, whereas in the case of an electron bunch there is only one

$$\psi = Ce^{-aH} \qquad (2)$$

where $a = 1/\tau D$ and C is calculated by taking

$$\iint\psi\, d\phi\, dw = 1.$$

Observe that the electron bunch width does not depend on the current but the normalization constant does. Also (2) can be regarded as a special case of a proton bunch.

Let $g(\phi) = \int\psi dw$ denote the longitudinal distribution. We can write

$$U_{sc} = -e\eta hI\int^{\phi} R(\phi-\phi')g(\phi')d\phi'$$

where $\eta = h\alpha\omega_o^2/E$, α is the momentum compaction factor, h the RF harmonic number, ω_o the angular revolution frequency and E the total energy. I is the bunch average current and R is a Kernel function with the dimension of an impedance. One easily obtains by integrating both sides of (2) the following integral equation for $g(\phi)$

$$g(\psi) = \sqrt{\frac{2\pi}{a}}\, Ce^{-aU_{ext} + e\eta ahI\int^{\phi} R(\phi-\phi')g(\phi')d\phi'} \ . \qquad (3)$$

The degree of difficilty for the solution of this equation depends on the choice for $R(\phi-\phi')$. Eq. (3) has always a physical solution(*) for R=Z, real constant

$$g(\phi) = \frac{Ce^{-aU_{ext}}}{\sqrt{\frac{a}{2\pi}} - ae\eta hIZC\int^{\phi} e^{-aU_{est}}\, d\phi} \ . \qquad (4)$$

In the parabolic approximation $U_{ext} = \Omega_o^2\phi^2/2$ with Ω_o the angular frequency of the phase oscillations for I = 0, we derive

$$C = \frac{a\Omega_o}{2\pi}\,\frac{\tanh\,(ae\eta hIZ/2)}{ae\eta hIZ/2} \ . \qquad (5)$$

The distribution (4) is an asymmetric bell function with a long, backward tail.

Proton Bunches and Electron Bunches

When a stationary solution exists the question arises whether it is stable or not. The answer is found by adding a perturbation. Below some current the perturbation is damped. The threshold value will depend on machine parameters and bunch size. Above the threshold there is an instability. Proton bunches likely have more than one stationary distribution. We have seen one above with the form (2). But it is obvious that one can repeat the same exercise for

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any form. If a particular distribution like (2) becomes unstable above some current, the proton bunch will change its initial distribution to another one more stable. On the other hand the electrons have only one stationary distribution (2), and when this becomes unstable there is no other one to replace it. Thus what happens afterwards? One should look then for a time dependent solution of (1) which is bounded, namely "quasi-stationary". We will not deal further with this problem except observing that the electron bunch will continuously change its size and shape under the two opposite effects of the instability and the synchrotron radiation. Then one should be able to recognize an average beam size which (and this is our assumption) should match the threshold condition.

The Effect of a Perturbation

Write the general solution of (1) as $\psi = \psi_s + \psi_p$ = stationary distribution function + perturbation function. It is always possible to operate a transformation of variables from ϕ, w to H, K where H is the Hamiltonian and

$$K = t - \frac{1}{\sqrt{2}}\int_0^{\phi} \frac{d\phi}{\sqrt{H-U_{ext}-U_{sc}}}$$

H and K are canonically conjugated and are invariants in absence of the perturbation. Quite generally we can write

$$K = t - \frac{\nu(\phi,H,I)}{\Omega_s(H,I)}$$

where ν is an angle variable along a trajectory and Ω_s is the angular frequency on the trajectory. The following relation holds quite generally

$$\dot{\phi}\; d\nu/d\phi = \Omega_s \;.$$

Take for the perturbation

$$\psi_p = A_p(H)\; \exp i(p\nu - \Omega t) \qquad (6)$$

where Ω is a collective complex angular frequency. This represents a wave traveling around the contour of the bunch with mode number p. In the case of constant and real impedance we have

$$F_p = eZI\eta h \int \psi_p \; dw \;. \qquad (7)$$

Insertion of (6) and (7) in (1) gives

$$F_p \frac{\partial\psi}{\partial w} - i(\Omega - p\Omega_s)\psi_p = \frac{\psi_p}{\tau} + \frac{w}{\tau}\frac{\partial\psi_p}{\partial w} + D\frac{\partial^2\psi_p}{\partial w^2} \qquad (8)$$

where we have neglected a second order term. To resolve (8) we carry first an average over one phase oscillation. Multiply both sides by $\exp(i\Omega t - ip\nu)$ and integrate over ν. The average is taken at constant H. In the approximation the total potential $U_{ext} + U_{sc}$ is nearly parabolic, we obtain

$$Q_p\psi_s' - i(\Omega - p\Omega_s)A_p = \frac{A_p}{\tau} + \frac{H}{\tau}A_p' + D(A_p' + HA_p'' - \frac{p^2 A_p}{4H}) \qquad (9$$

where ' = d/dH and

$$Q_p = \frac{eZI\eta h}{2\pi}\int\{\int A_p e^{ip\nu}\, dw\}\, e^{-ip\nu}\, wd\eta . \qquad (10)$$

Except for few trivial cases, this double integral is of difficult solution. The following approximation can be useful

$$Q_p = eZI\eta h \int A_p \; dH \;. \qquad (11)$$

Proton Bunches

Take $\tau \to \infty$ and $D \to 0$. From (9) we derive

$$A_p = -iQ_p \frac{\psi_s'}{\Omega - p\Omega_s} \;. \qquad (12)$$

For a proton bunch the dependence of the stationary distribution on the current is not essential so that ψ_s is completely arbitrary. Take for instance a water bag distribution

$$\psi_s' = -\frac{\Omega_o}{2\pi H_o}\,\delta(H-H_o)$$

then also A_p is a delta-function and Q_p has to be calculated only at $H=H_o$. Inserting (12) in (10) gives

$$\Omega = p\Omega_s + i\; eZI\eta h\; \Omega_o/2\pi H_o .$$

One has stability when the imaginary part of Ω is positive. The same result can be obtained also by making use of (11). Then in the more general case inserting (12) in (11), one obtains the following dispersion relation

$$1 = -ieZI\eta h\int \frac{\psi_s' dH}{\Omega - p\Omega_s} \;. \qquad (13)$$

This is our main result for the proton bunch theory. It disagrees with Sacherer's findings.[7] Hereward[5] assumes there is always only one wave traveling around the contour of the bunch. On top the perturbation is damped and antidamped on the bottom. Then he infers that over one phase oscillation the total effect is zero and an individual bunch cannot be unstable. So doing he believes to carry out an analogy with the coasting beam theory where the perturbation has two wave components, one "slow" and one "fast", one traveling on top and the other on the bottom of the beam, one wave damped and the other antidamped. We believe the analogy exists but was not properly applied. Also for a bunched beam there are two wave components moving in opposite directions around the contour of the bunch, one on the outside and the other on the inside. This can be easily seen by taking a hollow bunch. In the limit $\psi_s = \delta\,(H-H_o)/2\pi$ and $\Omega = \Omega_o + \Omega_s' H$, Eq. (13) gives

$$\Omega = p\Omega_o \pm \sqrt{ieZI\eta p\; \Omega_s'\Omega_o/2\pi}$$

amely two solutions which correspond to the two wave components, one slow and the other fast, one stable and the other unstable.

The dispersion relation (13) is similar to the me for a coasting beam and can be solved with the usual techniques.[8] If one takes the distribution (2) then the following stability condition is derived

$$|Z/p| < \frac{0.462\ \Delta\Omega_s}{eI\ \eta Ch} \tag{14}$$

where $\Delta\Omega_s = |\Omega_s'/a|$ is the angular synchrotron frequency spread in the bunch. This can be expressed in terms of the full width ΔE at half of the maximum of the energy distribution. Eq. (14) becomes

$$|Z_{eff}/p| < 0.861(\frac{\alpha E}{eI})(\frac{\Delta E}{E})^2 B \tag{15}$$

where B is the ratio of the bunch length to the circumference. The stability condition (15) is strikingly similar to the one obtained by Boussard[2] applying coasting beam theory to a bunched beam. In our case

$$Z_{eff} = Z/Bh^2 .$$

The concept of efficient impedance Z_{eff} was introduced first by Sacherer[2] and later by Messerschmid and Month[10]. Likely Eq. (15) can be applied to a more general definition of Z_{eff} in this sense.

Electron Bunches

One has to solve Eq. (9) combined to either (10) or (11). In the approximation Q_p is constant, for instance, by making use of (11), by letting

$$y = A_p/a\tau CQ_p \quad , \quad \Delta = \tau(\Omega - p\Omega_o)$$

$$\mu = \tau p\Omega_s'/a \quad \text{and} \quad x = aH$$

we change Eq. (9) to

$$\frac{dy^2}{dx^2} + (1 + \frac{1}{x})\frac{dy}{dx} + (\frac{1}{x} - \frac{p^2}{4x^2} + i\frac{\Delta}{x} - i\mu)y = -\frac{e^{-x}}{x}. \tag{16}$$

This equation can be solved but it is not trivial. It involves the Kummer's functions. A dispersion relation is obtained from (11)

$$1 = eZI\eta h\tau C\int_0^\infty y dx \tag{17}$$

where y is a particular solution of (16) which goes to zero at least as e^{-x} for $x \to \infty$.

The perturbation function y is a complex quantity with real part y_r and imaginary part y_i. Eq. (16) actually corresponds to a system of two real second-order differential equations. There are three parameters: The mode number p, the spread factor μ and the complex shift $\Delta = \Delta_r + i\Delta_i$. To obtain a stability condition one sets $\Delta_i \to 0^-$ and solves (16) for a given p and μ. From (17) we derive $\int y_i dx = 0$ which is satisfied by some value of Δ_r. When this value is known, calling $G_p(\mu) = \int F_r dx$ we obtain the following stability criterion

$$1 > eZI\eta h\tau CG_p(\mu) . \tag{18}$$

By comparing (18) to (14) we obtain in the limit $\tau \to \infty$

$$\tau G_p(\mu) \rightarrow 2.16\ \tau/|\mu| . \tag{19}$$

When $\mu = 0$, which is the case of zero frequency spread, one gets $\Delta_r = 0$. This gives $y_i = 0$ identically and y_r is obtained by solving

$$\frac{d^2y_r}{dx^2} + (1+\frac{1}{x})\frac{dy_r}{dx} + (\frac{1}{x} - \frac{p^2}{4x^2})y_r = -\frac{e^{-x}}{x}$$

which is a non-trivial equation.

Combining (18) to (5), the stability criterion becomes

$$\frac{1}{\pi}\tau\Omega_o\ G_p(\mu)\ \tanh(\frac{aeIZ\eta}{e}) < 1 . \tag{20}$$

Eq. (20) is our main result. Observe that in principle μ depends on the bunch current, but probably this can be ignored. The function $G_p(\mu)$ can be determined only by solving (16). Temporarily one can use (19). When $2.16\Omega_o >> \pi|p|\Delta\Omega_s$ which is usually the case, Eq. (20) can be transformed to

$$I < 0.485\ \frac{|p|h^3\alpha^3 E}{eZ\ \nu_s}\ (\frac{\sigma_E}{E})^4 \tag{21}$$

where ν_s is the number of phase oscillations per turn and σ_E the natural rms width of the bunch. For a machine like PETRA, for instance, taking $Z = 60\ \kappa\Omega$ and $\nu_s = 0.01$ at 7 GeV, one obtains a safe threshold current of 34 mA for $p = 2$. At the same time the observations on SPEAR II[11] can be explained by taking $Z \sim 10\ \kappa\Omega$.

Electron Bunch Widening

According to our assumption, Eq. (20) is to be used also to calculate the bunch width when the current is larger than the threshold value. Temporarily one can replace (20) with (21). Denoting with R the ratio of the new bunch width to the natural one, one has

$$R = 7.86\ (IZ\nu_s{}^2\rho^2/|p|h^3\alpha^3E^5)^{1/4}$$

where IZ is in volts, E in GeV and ρ, the bending radius, in meters. The dependence with the energy and the RF voltage seems to be in agreement with the experimental observations[11]. But the current dependence power is 1/4 instead of 1/3 as observed. The correct current dependence can be obtained likely going back to (20) after having properly calculated the function $G_p(\mu)$. Observe that, eventually, the condition (20) shows a saturation level for the bunch width.

(*)The solution of (3) must be physically acceptable; for instance, it should be positive for any value of ϕ. Usually the existence of a physical solution is taken for granted[4]. Though attempts to solve numerically (3) showed a change of sign of $g(\phi)$ above some current[6].

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