3d-3d Correspondence for Seifert Manifolds

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To my family.

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ABSTRACT

In this dissertation, we investigate the 3d-3d correspondence for Seifert manifolds. This correspondence, originating from string theory and M-theory, relates the dynamics of three-dimensional quantum field theories with the geometry of three-manifolds.

We first start in Chapter 2 with the simplest cases and demonstrate the extremely rich interplay between geometry and physics even when the manifold is just a direct product $M_3 = \Sigma \times S^1$. In this particular case, by examining the problem from various vantage points, we generalize the celebrated relations between 1) the Verlinde algebra, 2) quantum cohomology of the Grassmannian, 3) Chern-Simons theory on $\Sigma \times S^1$ and 4) the index theory of the moduli space of flat connections to a completely new set of relations between 1) the "equivariant Verlinde algebra" for a complex group, 2) the equivariant quantum K-theory of the vortex moduli space, 3) complex Chern-Simons theory on $\Sigma \times S^1$ and 4) the equivariant index theory of the moduli space of Higgs bundles.

In Chapter 3 we move one step up in complexity by looking at the next simplest example of $M_3 = L(p, 1)$. We test the 3d-3d correspondence for theories that are labeled by lens spaces, reaching a full agreement between the index of the 3d $\mathcal{N} = 2$ "lens space theory" $T[L(p, 1)]$ and the partition function of complex Chern-Simons theory on $L(p, 1)$.

The two different types of manifolds studied in the previous two chapters also have interesting interactions. We show in Chapter 4 the equivalence between two seemingly distinct 2d TQFTs: one comes from the "Coulomb branch index" of the class $\mathcal{S}$ theory on $L(k, 1) \times S^1$, the other is the "equivariant Verlinde formula" on $\Sigma \times S^1$. We check this relation explicitly for $SU(2)$ and demonstrate that the $SU(N)$ equivariant Verlinde algebra can be derived using field theory via (generalized) Argyres-Seiberg dualities.

In the last chapter, we directly jump to the most general situation, giving a proposal for the 3d-3d correspondence of an arbitrary Seifert manifold. We remark on the huge class of novel dualities relating different descriptions of $T[M_3]$ with the same M_3 and suggest ways that our proposal could be tested.

PUBLISHED CONTENT AND CONTRIBUTIONS

This thesis is based on preprints [1], [2] and [3], which are adapted for Chapters 2, 3 and 4 respectively, as well as ongoing work such as [4] and [5]. Each of them is the outcome of active and fruitful interactions between me and my collaborators.

[1] Sergei Gukov and Du Pei. "Equivariant Verlinde formula from fivebranes and vortices" (2015). arXiv: 1501.01310 [hep-th]. URL: http://arxiv.org/abs/1501.01310.

[2] Du Pei and Ke Ye. "A 3d-3d appetizer" (2015). arXiv: 1503.04809 [hep-th]. URL: http://arxiv.org/abs/1503.04809.

[3] Sergei Gukov, Du Pei, Wenbin Yan, and Ke Ye. "Equivariant Verlinde algebra from superconformal index and Argyres-Seiberg duality" (2016). arXiv: 1605.06528 [hep-th]. URL: http://arxiv.org/abs/1605.06528/.

[4] Daniel Jafferis, Sergei Gukov, Monica Kang, and Du Pei. "Chern-Simons theory at fractional level" (work in progress).

[5] Jørgen Ellegaard Andersen and Du Pei. "Verlinde formula for Higgs bundle moduli spaces" (work in progress).

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Introduction

The 3d-3d correspondence is an elegant relation between 3-manifolds and three-dimensional quantum field theories [6–9]. The general spirit is that one can associate a 3-manifold M_3 with a 3d $\mathcal{N} = 2$ superconformal field theory $T[M_3; G]$, obtained by compactifying the 6d (2,0) theory on M_3

$$\begin{array}{c}\text{6d (2,0) theory on } M_3 \\ \rightsquigarrow \\ \text{3d } \mathcal{N} = 2 \text{ theory } T[M_3].\end{array} \tag{1.1}$$

In this procedure, the 6d theory is topologically twisted along M_3 to preserve $\mathcal{N} = 2$ supersymmetry in the remaining three flat directions. It is generally believed that $T[M_3]$ is independent of deformation of the metric on M_3, and is determined[1] by topology of M_3 along with the choice of a group G whose Lie algebra is of ADE type. Although the dictionary between the dynamics of $T[M_3; G]$ and the topological and geometrical properties of M_3 is incredibly rich [6, 8–12] and only partially explored, there are two very fundamental relations between M_3 and $T[M_3]$. Firstly, the moduli space of supersymmetric vacua of $T[M_3; G]$ on $\mathbb{R}^2 \times S^1$ is expected to be homeomorphic to the moduli space of flat $G_\mathbb{C}$-connections on M_3:

$$\mathcal{M}_{\text{SUSY}}(T[M_3; G]) \simeq \mathcal{M}_{\text{flat}}(M_3; G_\mathbb{C}). \tag{1.2}$$

Second, the partition function of $T[M_3]$ on lens space $L(k, 1)$ should be equal to the partition function of complex Chern-Simons theory on M_3 at level k [12, 13]:

$$Z_{T[M_3;G]}[L(k, 1)_b] = Z^{(k,\sigma)}_{\text{CS}}[M_3; G_\mathbb{C}]. \tag{1.3}$$

For $k = 0$, $L(k, 1) = S^1 \times S^2$, and the equation (3.2) maps the superconformal index of $T[M_3]$ to the partition function of complex Chern-Simons theory at level $k = 0$: [9]

$$\text{Index}_{T[M_3;G]}(q) = \text{Tr}\,(-1)^F q^{\frac{E+j_3}{2}} = Z^{(0,\sigma)}_{CS}[M_3; G_\mathbb{C}]. \tag{1.4}$$

Despite its beauty and richness, the 3d-3d correspondence has been haunted by many problems since its birth. For example, the theories $T_{\text{DGG}}[M_3]$ originally proposed in [8] miss many branches of flat connections and therefore fail even the most basic test (1.2). This problem was revisited and partially corrected in [14].

[1] To be more precise, it also depends on a choice of framing, which will play a relatively minor role in this thesis.

As for (1.3) and (1.4), prior to our work [2], there was simply no known proposal for $T[M_3]$ associated to *any* M_3 that passed these stronger tests. Even the very first non-trivial example of partition functions in Chern-Simons theory found in Witten's seminal paper [15],

$$Z_{\mathrm{CS}}[S^3; SU(2), k] = \sqrt{\frac{2}{k+2}} \sin\left(\frac{\pi}{k+2}\right), \tag{1.5}$$

had yet to find its home in the world of 3d $\mathcal{N} = 2$ theories.

This thesis summarizes the attempts made by the author and collaborators to fill this vacancy, bringing the 3d-3d correspondence back on a firm foundation. Our strategy is to start all over with the simplest class of three-manifolds—the Seifert manifolds. In fact, we have been mostly focusing on the *simplest Seifert manifolds* like $\Sigma \times S^1$ or lens spaces $L(k, 1)$ and analyzed in these cases the correspondence and its various applications.

The outline of the thesis is the following. In Chapter 2, we start with the problem of understanding complex Chern-Simons theory on $\Sigma \times S^1$ via 3d-3d correspondence. We show that complex Chern-Simons theory on $\Sigma \times S^1$ is equivalent to a topologically twisted supersymmetric theory and its partition function can be naturally regularized by turning on a mass parameter. We arrive at an extremely interesting 2d TQFT whose partition function gives "the equivariant Verlinde formula," generalizing the renowned Verlinde formula [16]. We offer many different ways of deriving this new formula by looking at the system from different angles, and, meanwhile, building bridges between this new 2d TQFT and many exciting subjects in mathematical physics such as quantum cohomology of vortex moduli spaces and quantization of the moduli space of Higgs bundles. This chapter is an adaptation of [1].

In Chapter 2, we have also obtained an explicit description of the "lens space theory" $T[L(p, 1)]$ as a by-product of the analysis of the equivariant Verlinde formula. And in Chapter 3, this $T[L(p, 1)]$ theory assumes a much bigger role, becoming the protagonist of the whole chapter, in which we test the 3d-3d correspondence for $L(p, 1)$. In this chapter, we first demonstrate a full agreement between the index of $T[L(p, 1)]$ and the partition function of complex Chern-Simons theory on $L(p, 1)$. In particular, for $p = 1$, we show how Witten's S^3 partition function of $SU(2)$ Chern-Simons theory arises from the index of a free theory. Then, we study $T[L(p, 1)]$ on the squashed three-sphere S^3_b. This enables us to see clearly, at the level of partition functions, to what extent $G_{\mathbb{C}}$ complex Chern-Simons theory can be thought of as two copies of Chern-Simons theory with compact gauge group G. This chapter is adapted from [2].

In Chapter 4, we relate the equivariant Verlinde formula with the 4d $\mathcal{N} = 2$ lens space index of class $\mathcal{S}$ theories [17–21]. Using the M-theory geometry, we first argue that two seemingly distinct 2d TQFTs—one being the "equivariant Verlinde formula" with the other coming from the "Coulomb branch index" of the class $\mathcal{S}$ theories—have to be equivalent. We also show that the gauge groups appearing on two sides are naturally G and its Langlands dual ${}^{L}G$. When G is not simply-connected, we provide a recipe of computing the index of $T[\Sigma, G]$ as summation over indices of $T[\Sigma, \widetilde{G}]$ with non-trivial background 't Hooft fluxes. Here $\widetilde{G}$ is the simply-connected group with the same Lie algebra. Then we check explicitly this relation between the Coulomb index and the equivariant Verlinde formula for $G = SU(2)$ or $SO(3)$. In the end, as an application of this newly found relation, we consider the more general case where G is $SU(N)$ or $PSU(N)$ and show that the equivariant Verlinde algebra can be derived using field theory via (generalized) Argyres-Seiberg duality. This chapter is adapted from [3].

In Chapter 5, the last in this dissertation, we consider the most general case of M_3 being an arbitrary Seifert manifold. Using the field theory counterpart of Dehn surgery, we show how to translate the Seifert invariants of M_3 into an explicit description of $T[M_3]$. This part is based on ongoing works including [4].

2.1 From the Verlinde formula to its equivariant version

In recent years, there has been a lot of work on realizing conformal theories in two dimensions and Chern-Simons theories with complex gauge groups on the world-volume of branes in string theory. Most of these constructions, though, focus on "non-compact" (irrational) theories. In particular, such a central element in two-dimensional CFT as the Verlinde formula [16] has not yet found its home in supersymmetric brane configurations.

The Verlinde formula is a simple and elegant expression for the number of conformal blocks in a 2d rational CFT on a Riemann surface Σ. The number depends only on the topology of Σ, an integer number k called the "level," and a choice of a compact Lie group G that in most of our discussion we assume to be simple. For instance, when Σ is a closed Riemann surface of genus g and $G = SU(2)$ the Verlinde formula reads:

$$\dim \mathcal{H}(\Sigma_g; SU(2)_k) \;=\; \left(\frac{k+2}{2}\right)^{g-1} \sum_{j=1}^{k+1} \left(\sin \frac{\pi j}{k+2}\right)^{2-2g} \tag{2.1}$$

This expression and its generalization to arbitrary G have a number of remarkable properties. First, for a fixed g, the expression on the right-hand side is actually a polynomial in k. Moreover, even though the coefficients of this polynomial are, in general, rational numbers, at every $k \in \mathbb{Z}$ it evaluates to a positive integer number (= number of conformal blocks).

The space $\mathcal{H}$ that appears in the Verlinde formula (2.1) can be also viewed as the Hilbert space associated to quantization of a symplectic manifold $(\mathcal{M}_{\text{flat}}(\Sigma; G), k\omega)$ that we briefly review in section 2.4. Despite many realizations of quantization problems in superstring theory and SUSY field theories [22–26], a simple quantization problem that leads to (2.1) has not been realized. In this chapter, we not only realize the Verlinde formula (2.1) as a partition function of a certain brane system, but we also propose its vast generalization based on the embedding in superstring theory.

In particular, we wish to re-create a "complexification" of the beautiful story that involves a number of exactly solvable theories, centered around the Verlinde formula:

$$\begin{aligned}
\dim \mathcal{H}(\Sigma; G, k) &= Z_{\text{CS}}(S^1 \times \Sigma) && (2.2\text{a})\\
&= \dim H^0(\mathcal{M}, \mathcal{L}) && (2.2\text{b})\\
&= \int_{\mathcal{M}} e^{c_1(\mathcal{L})} \wedge \text{Td}(\mathcal{M}) && (2.2\text{c})\\
&= Z_{G/G}(\Sigma) && (2.2\text{d})\\
&= Z_{\text{A-model}}(\text{Gr}(N, k)) && (2.2\text{e})\\
&= \dim \text{Hom}(\mathcal{B}', \mathcal{B}_{cc}) && (2.2\text{f})\\
&= \dim \text{Hom}(\widetilde{\mathcal{B}}', \widetilde{\mathcal{B}}_{cc}). && (2.2\text{g})
\end{aligned}$$

The first line here simply follows from the fact that the problem of quantizing $(\mathcal{M}_{\text{flat}}(\Sigma; G), k\omega)$ is what one encounters in Chern-Simons gauge theory. The latter theory is topological [15, 27] and, therefore, has trivial Hamiltonian $H = 0$, so that dimension of its Hilbert space can be computed via path integral on $S^1 \times \Sigma$. The second line is the result of geometric quantization of the moduli space $\mathcal{M} = \mathcal{M}_{\text{flat}}(\Sigma; G)$ of classical solutions with the prequantum line bundle $\mathcal{L}$, and (2.2c) follows from a further application of the Grothendieck-Riemann-Roch theorem.

Then, (2.2d) relates it to the partition function of the G/G gauged WZW model [28], and (2.2e) is based on the relation [29] to the partition function (more precisely, a certain correlation function) of the topological A-model on Σ with the Grassmannian target space $\text{Gr}(N, k)$. Finally, (2.2f) and (2.2g) follow from representing the Hilbert space $\mathcal{H}^{\text{CS}}$ as the space of open strings in the A-model [22] of complexification of $\mathcal{M}$, namely $\mathcal{M}_{\text{flat}}(\Sigma; G_{\mathbb{C}})$, and in the B-model [25] of its mirror $\mathcal{M}_{\text{flat}}(\Sigma; {}^L G_{\mathbb{C}})$, where ${}^L G$ denotes the GNO or Langlands dual group.

Unlike the classical phase space $\mathcal{M} = \mathcal{M}_{\text{flat}}(\Sigma; G)$, its complexification $\mathcal{M}_{\text{flat}}(\Sigma; G_{\mathbb{C}})$ is non-compact and, therefore, the corresponding Hilbert space $\mathcal{H}(\Sigma; G_{\mathbb{C}}, k)$ is infinite-dimensional. This fact is well known in the study of Chern-Simons theory with complex gauge group and all related problems where $\mathcal{M}_{\text{flat}}(\Sigma; G_{\mathbb{C}})$ shows up. Thus, it is unclear what the analogue of (2.1) and (2.2) might be if we naively replace a compact group G by its complexification $G_{\mathbb{C}}$. However, by identifying $\mathcal{M}_{\text{flat}}(\Sigma; G_{\mathbb{C}})$ with the Hitchin moduli space, we argue that the infinite-dimensional Hilbert space $\mathcal{H}(\Sigma; G_{\mathbb{C}}, k)$ comes equipped with a natural $\mathbb{Z}$-grading:

$$\mathcal{H}(\Sigma; G_{\mathbb{C}}, k) = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}_n \tag{2.3}$$

such that each graded piece, $\mathcal{H}_n$, is finite-dimensional. This allows us to introduce the graded dimension of $\mathcal{H}(\Sigma; G_{\mathbb{C}}, k)$, which we call the "equivariant Verlinde formula":

$$\dim_\beta \mathcal{H}(\Sigma; G_{\mathbb{C}}, k) \; := \; \sum_n t^n \dim \mathcal{H}_n, \tag{2.4}$$

where $t = e^{-\beta}$. We then generalize each line in (2.2) and, in particular, formulate several new TQFTs in three and two dimensions that compute the graded dimension (2.4). For example, for $G = SU(2)$, $g = 2$ and large enough k, the equivariant Verlinde formula gives

$$\begin{aligned} \dim_\beta \mathcal{H}(\Sigma; G_{\mathbb{C}}, k) = & \frac{1}{6}k^3 + k^2 + \frac{11}{6}k + 1 \\ & + \left(\frac{1}{2}k^3 + 3k^2 - \frac{1}{2}k - 3\right) t \\ & + \left(k^3 + 8k^2 - 3k + 6\right) t^2 \\ & + \left(\frac{5}{3}k^3 + 16k^2 - \frac{71}{3}k + 6\right) t^3 \\ & + \left(\frac{5}{2}k^3 + 29k^2 - \frac{109}{2}k + 63\right) t^4 \\ & + \ldots, \end{aligned} \tag{2.5}$$

where a careful reader can recognize (2.1) as the degree-0 piece, *i.e.*,

$$\mathcal{H}_0 \; = \; \mathcal{H}(\Sigma; G, k)\,. \tag{2.6}$$

Also, one can verify that the coefficient of t^n is always a positive integer, agreeing with its interpretation as dimension of $\mathcal{H}_n$. (In writing the t-expansion (2.5) we assumed that k is sufficiently large; the exact formula (**??**) is given in section 2.7 and always yields positive integer coefficients for all k.)

As we explain in the rest of this chapter, the equivariant Verlinde formula provides a connection between SUSY theories — *e.g.* realized on world-volume of various brane systems — and quantization, namely quantization of compact spaces, such as $\mathcal{M}_{\text{flat}}(\Sigma; G)$ and Bun_G, as well as their non-compact counterparts, such as $\mathcal{M}_{\text{flat}}(\Sigma; G_{\mathbb{C}})$ and the Hitchin moduli space. In particular, there are two 3d $\mathcal{N} = 2$ theories that will play an important role throughout the current chapter: the so-called "lens space theory" $T[L(k,1);\beta]$ and the mass deformation of a 3d $\mathcal{N} = 4$ sigma-model:

3d $\mathcal{N} = 2$ theory $T[L(k,1);\beta]$	3d $\mathcal{N} = 2$ theory $T[\Sigma \times S^1;\beta]$
super-Chern-Simons at level k with adjoint field Φ of mass β	sigma-model with target $\mathcal{M}_H$ and a real mass β for $U(1)_\beta$ flavor symmetry

(2.7)

To compute the equivariant Verlinde formula, the first theory needs to be put on $\Sigma \times S^1$ and topologically twisted, while the latter theory leads to an expression for (2.4) in terms of the equivariant integral over the Hitchin moduli space. The former is also equivalent to the IR limit of 3d $\mathcal{N} = 2$ SQCD with an adjoint multiplet that can be found on the world-sheet of half-BPS vortex strings. Thus, familiar vortex strings know about t-deformation of the Verlinde algebra!

Now we present a more detailed outline of this chapter of the dissertation and summary of the results.

Outline of this chapter

In section 2.2 we state the problem and introduce a one-parameter deformation of complex Chern-Simons theory on Seifert manifolds.

The two theories (2.7) are special cases of $T[M_3; \beta]$, where M_3 is an arbitrary Seifert manifold. As we explain in section 2.2, when M_3 is a Seifert manifold, the corresponding 3d $\mathcal{N} = 2$ theory $T[M_3]$ has a special flavor symmetry that we call $U(1)_\beta$. Turning on the real mass β for this flavor symmetry gives a family of 3d $\mathcal{N} = 2$ theories $T[M_3; \beta]$ which, via 3d-3d correspondence, provide a definition and natural regularization of complex Chern-Simons theory on M_3. Then, in section 2.2, we give the second, equivalent definition of complex Chern-Simons on M_3 as a standard topological twist of the 3d $\mathcal{N} = 2$ theory $T[L(k, 1); \beta]$ on M_3. (Evidence for this equivalence is presented in section 2.5.)

Section 2.3 relates exactly soluble theories described in this chapter to familiar brane constructions in type IIA and type IIB string theory. On one hand, it will give us a concrete description of the lens space theory $T[L(k, 1); \beta]$ as summarized in (2.7) and, on the other hand, will link our story to the classical problem about vortices on a plane, $\mathbb{R}^2 \cong \mathbb{C}$. Non-compactness of the plane leads to non-compactness of the vortex moduli space, which often is an obstacle in defining its topological and geometric invariants. This problem is easily cured in the equivariant setting, equivariant with respect to the rotation symmetry of the plane. In particular, it leads us to identify the equivariant quantum K-theory of the vortex moduli space with the "equivariant Verlinde algebra" for complex Chern-Simons theory (whose explicit form is described in section 2.7) and provides an analogue of (2.2e).

Section 2.4 gives a precise definition of the graded dimension (2.20) via 3d-3d correspondence and shows that it can be written as an equivariant integral over the Hitchin moduli space. This provides an analogue of (2.2c). The same graded dimension will be computed in other sections from a variety of different viewpoints.

In section 2.5 we demonstrate that β-deformed complex Chern-Simons theory is equivalent to a certain twist of 3d $\mathcal{N} = 2$ theory $T[L(k, 1); \beta]$,

$$\boxed{\begin{array}{c}\text{twist of } T[L(k,1);\beta] \text{ on} \\ \text{a Seifert manifold } M_3\end{array}} = \boxed{\begin{array}{c}\beta\text{-deformed complex} \\ \text{Chern-Simons on } M_3\end{array}}, \tag{2.8}$$

and compute its partition function (2.23) on $\Sigma \times S^1$ using the standard localization techniques. This gives a "three-dimensional" calculation of the equivariant Verlinde formula and, as such, can be regarded as a "complexification" of (2.2a).

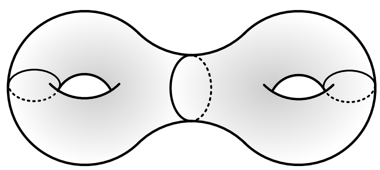


Figure 2.1: A genus-2 Riemann surface can be decomposed into two pairs of pants.

The goal of section 2.6 is to establish the analogue of (2.2d) for the graded dimension (2.4). We call the resulting 2d TQFT the "equivariant G/G gauged WZW model." In section 2.7, we formulate this theory as a set of gluing rules, by associating the "equivariant Higgs vertex" to each pair of pants, as in figure 2.1. In this section, we also discuss t-deformation and categorification of the Verlinde algebra.

In fact, 3d and 2d topological theories that compute (2.4) are part of a larger family of TQFTs labeled by $R \in \mathbb{Z}$. In three dimensions, R can be identified with the R-charge of the adjoint multiplet Φ in the twisted theory $T[L(k, 1); \beta]$. This leads to a generalization of (2.8). Via reduction to two dimensions, we obtain a large family of new TQFTs that generalize the gauged WZW model. Certain special values of R correspond to models that have been previously studied from different viewpoints.

From this perspective, sections 2.4 and 2.7 are all about the special case $R = 2$. Section 2.5 talks about general R, but most of the concrete formulae are written for $R = 2$. This is well compensated in section 2.6, whose main goal is to describe the family of 2d TQFTs on Σ for general R. Then, in section 2.6 we again focus on $R = 2$ that gives the equivariant G/G model (EGWZW) and whose partition function computes the equivariant Verlinde formula. Similarly, in section 2.6 we focus on $R = 0$ which gives the G/G gauged WZW-matter model (GWZWM).

In total, in this chapter we present *five* independent and concrete ways to compute the equivariant Verlinde formula:

- a three-dimensional computation in a topologically twisted 3d $\mathcal{N} = 2$ theory on $M_3 = \Sigma \times S^1$ (section 2.5);
- a computation based on 3d-3d correspondence that leads to an equivariant integral over the Hitchin moduli space $\mathcal{M}_H$ (sections 2.4 and 2.7);
- a two-dimensional computation in the equivariant G/G model on Σ (section 2.6);
- another two-dimensional calculation in the abelian 2d theory on the Coulomb branch (section 2.6);
- yet another two-dimensional calculation based on pair-of-pants decomposition of Σ and the "equivariant Higgs vertex" (section 2.7).

And we will even add to this list in later chapters.

It would be nice to add to this list a computation based on the 4d $\mathcal{N} = 2$ lens space index [20, 21]. Also, in section 2.3 we outline a generalization of (2.2e) that allows us to compute the equivariant Verlinde formula in the twisted theory on the vortex world-sheet. It would be nice to carry out the details of this approach and make contact with the equivariant vortex counting in [6].

2.2 Fivebranes on Riemann surfaces and 3-manifolds

Our starting point is the following configuration of M-theory fivebranes:

$$\begin{array}{llcccc} \text{space-time:} & L(k,1)_b & \times & T^*M_3 & \times & \mathbb{R}^2 \\ & & & \cup & & \\ N \text{ fivebranes:} & L(k,1)_b & \times & M_3\,. & & \end{array} \tag{2.9}$$

This is also used *e.g.* in 3d-3d correspondence. Here, M_3 is an arbitrary 3-manifold, embedded in a local Calabi-Yau 3-fold T^*M_3 as the zero section. As a result [30], the three-dimensional part of the fivebrane world-volume theory is topologically twisted. Namely, the topological twist along M_3 is the so-called Blau-Thompson twist [31, 32]. It preserves four real supercharges on the fivebrane world-volume, so that the effective theory in the remaining three dimensions of the fivebrane world-volume (which are not part of M_3) is 3d $\mathcal{N} = 2$ theory. This theory is usually denoted $T_N[M_3]$ since it depends on the number of fivebranes in (2.9) and on the choice of the 3-manifold M_3. (Sometimes, one simply writes $T[M_3]$ when the number of fivebranes is clear from the context, or denotes this theory $T[M_3; G]$.)

The effective 3d $\mathcal{N} = 2$ theory $T_N[M_3]$ can be further put on a curved background [33, 34], in particular on a squashed lens space $L(k, 1)_b$:

$$L(k,1)_b \;:=\; \{(z,w) \in \mathbb{C}^2 ,\; b^2|z|^2 + b^{-2}|w|^2 = 1\}/\mathbb{Z}_k \,, \tag{2.10}$$

where the action of $\mathbb{Z}_k$ is generated by $(z, w) \mapsto (e^{2\pi i/k}z, e^{-2\pi i/k}w)$. Then, reversing the order of compactification, it has been shown [13, 35] that the effective 3d theory on M_3 is the complex Chern-Simons theory, confirming the conjecture of [6, 8] (see also [7, 9, 12, 36, 37]).

Therefore, one can reduce the six-dimensional (2, 0) theory in two different ways, summarized by the following diagram:

$$\begin{array}{ccc} & \text{6d } (2,0) \text{ theory} \quad \text{on} \quad L(k,1)_b \times M_3 & \\ \swarrow & & \searrow \\ \text{3d } \mathcal{N}=2 \text{ theory} & & \text{complex Chern-Simons} \\ T[M_3] \text{ on } L(k,1)_b & & \text{theory on } M_3 \end{array} \tag{2.11}$$

The statement of 3d-3d correspondence is that physics of complex Chern-Simons theory on M_3 is encoded in the protected (supersymmetric) sector of the 3d $\mathcal{N} = 2$ theory $T[M_3]$. For instance, SUSY vacua of the theory $T[M_3]$ are in one-to-one correspondence with the complex flat connections on M_3. Various supersymmetric partition functions of $T[M_3; G]$ compute quantum $G_{\mathbb{C}}$ invariants of M_3, *e.g.* the vortex partition function (on $\mathbb{R}^2_\hbar \times S^1$) gives the perturbative partition function of complex Chern-Simons theory labeled by a flat connection α:

$$Z^{\text{vortex}}_{T_N[M_3]}(\hbar, \alpha) \;=\; Z^{\alpha}_{CS}(M_3; \hbar) \,. \tag{2.12}$$

Similarly, and closer to the setup in (2.9) that we shall use in this thesis, the partition function of 3d $\mathcal{N} = 2$ theory $T[M_3]$ on the squashed lens space is equal to the full partition function of complex Chern-Simons theory on M_3 at level $(k, \sigma = k\frac{1-b^2}{1+b^2})$:

$$Z_{T_N[M_3]}\,[L(k,1)_b] \;=\; Z^{(k,\sigma)}_{\text{CS}}\,[M_3; GL(N,\mathbb{C})] \;. \tag{2.13}$$

This correspondence, relating partition functions of a supersymmetric theory with those of a TQFT, is obviously a very interesting one. However, there is much to be understood on both sides. On the right-hand side, one basic problem is to produce a simple and effective technique to compute the partition function of complex Chern-Simons theory on arbitrary 3-manifolds (see [38–40] for some steps in this direction). On the "supersymmetric" left-hand side of the 3d-3d correspondence, the main problem is to develop tools for building the theory $T_N[M_3]$ associated with a given M_3. Previous attempts to tackle this problem either address only a certain sector of the theory $T_N[M_3]$ that does not capture all SUSY vacua / flat connections [8, 9] or build the full theory $T_N[M_3]$ only for particular 3-manifolds [14] and, therefore, are not systematic.

In particular, one motivation for our work is to understand $T_N[M_3]$ for Seifert 3-manifolds which, aside from the abelian case discussed in [41, sec. 2.2], escaped attention in 3d-3d correspondence. A Seifert manifold is the total space of a circle V-bundle over a two-dimensional, closed and orientable orbifold Σ,

$$S^1 \hookrightarrow M \xrightarrow{\pi} \Sigma \,. \tag{2.14}$$

Although almost all computations in this chapter can be easily generalized to arbitrary Seifert manifolds, for simplicity and concreteness we often carry out explicit computations in the basic example of a product $M_3 = \Sigma \times S^1$ explaining how generalizations can be achieved.

With $M_3 = \Sigma \times S^1$, the eleven-dimensional geometry (2.9) becomes:

$$\begin{array}{lccccccc} \text{symmetries:} & & & U(1)_N & & & & SU(2)_R \\ & & & \circlearrowleft & & & & \circlearrowleft \\ \text{space-time:} & L(k,1)_b & \times & T^*\Sigma & \times & S^1 & \times & \mathbb{R}^3 \\ & & & \cup & & & & \\ N \text{ fivebranes:} & L(k,1)_b & \times & \Sigma & \times & S^1 & & \end{array} \tag{2.15}$$

Now, one needs to do the topological twist only along a Riemann surface Σ which is embedded in the local Calabi-Yau 2-fold $T^*\Sigma$ in a supersymmetric way as the zero section. In particular, it preserves half of supersymmetry on the fivebrane world-volume, which now also includes the S^1 factor. Interpreting this S^1 as the M-theory circle, the above system of fivebranes reduces to N D4-branes, which carry maximally supersymmetric 5d super-Yang-Mills on their world-volume. A further reduction of 5d super-Yang-Mills on a Riemann surface with a partial topological twist along $\Sigma \subset T^*\Sigma$ requires gauge field and its superpartners to obey certain equations on Σ. This partial twist was studied exactly 20 years ago [42, 43] and the corresponding BPS equations turn out to be the Hitchin equations [44], so that the effective 3d $\mathcal{N} = 4$ theory is a sigma-model with Hitchin moduli space $\mathcal{M}_H(\Sigma; G)$ as the target. In recent years, this setup was also used in connection with the geometric Langlands program, AGT correspondence, *etc.*

To summarize, when $M_3 = \Sigma \times S^1$, the effective 3d theory $T_N[\Sigma \times S^1]$ has $\mathcal{N} = 4$ supersymmetry and the R-symmetry group is enhanced to $SU(2)_R \times SU(2)_N$. A subgroup of this R-symmetry group can be easily identified with isometries of the M-theory geometry: $SU(2)_R$ is the double cover of the rotation group $SO(3)$ acting on the last factor $\mathbb{R}^3$ in (2.15), while $U(1)_N$ (= a subgroup of $SU(2)_N$) acts on the cotangent fiber of $T^*\Sigma$.

One can introduce new parameters by weakly gauging these symmetries. We will be interested in a "canonical mass deformation" of $T[\Sigma \times S^1]$ which gives a $\mathcal{N} = 2$ theory that in (2.7) we denoted $T[\Sigma \times S^1; \beta]$. This deformation can be done to any 3d $\mathcal{N} = 4$ theory by regarding it as a 3d $\mathcal{N} = 2$ theory, whose R-symmetry group $U(1)_{R'}$ is generated by $j^3_N + j^3_R$, and weakly gauging $U(1)_\beta$ generated by $j^3_N - j^3_R$. Here we use $j^i_{N,R}, i = 1, 2, 3$ to denote the generators of $SU(2)_N \times SU(2)_R$.

Note, from the viewpoint of $\mathcal{N} = 2$ supersymmetry, $U(1)_\beta$ is a flavor symmetry that acts on the sigma-model target $\mathcal{M}_H(\Sigma; G)$ as

$$U(1)_\beta : \quad (A, \Phi) \mapsto (A, e^{i\theta}\Phi), \tag{2.16}$$

where each point in $\mathcal{M}_H(\Sigma; G)$ is represented by a Higgs bundle (A, Φ), see section 2.4 for a brief review. Weakly gauging this $U(1)_\beta$ symmetry deforms $\mathcal{N} = 4$ sigma-model with target $\mathcal{M}_H(\Sigma; G)$ to a $\mathcal{N} = 2$ theory $T[\Sigma \times S^1; \beta]$ with the same field content, but where half of the fields have (real) mass β. This deformation of $T[\Sigma \times S^1]$ can be realized in the brane geometry (2.15) by introducing Ω-background on both the two-dimensional cotangent fiber of $T^*\Sigma$ and on $\mathbb{R}^2 \subset \mathbb{R}^3$ with the equivariant parameters β and $-\beta$, respectively. We continue the discussion of the 3d $\mathcal{N} = 2$ theory $T[\Sigma \times S^1; \beta]$ in section 2.4.

Now, let us consider what this deformation means on the other side of the 3d-3d correspondence, *i.e.* for the complex Chern-Simons theory on M_3. When $M_3 = \Sigma \times S^1$ and $\beta = 0$ we have ordinary (undeformed) complex Chern-Simons theory, whose partition function on $\Sigma \times S^1$ computes the dimension of the Hilbert space associated to Σ:

$$Z_{\text{CS}}[\Sigma \times S^1; G_{\mathbb{C}}] \;=\; \dim \mathcal{H}_{\text{CS}}(\Sigma; G_{\mathbb{C}})\,. \tag{2.17}$$

The space $\mathcal{H}_{\text{CS}}(\Sigma; G_{\mathbb{C}})$ is infinite-dimensional due to non-compactness of the gauge group and one needs to regularize it in order to make sense of the above expression. We will do so by considering the graded dimension with respect to a $\mathbb{Z}$-grading on $\mathcal{H}_{\text{CS}}(\Sigma; G_{\mathbb{C}})$ induced by the circle action $U(1)_\beta$. We call the resulting TQFT the

5d	$SO(5)_L \times SO(5)_R$	field	$SO(2)_L$	$\times$ $U(1)_L$	$\times$ $U(1)_N$	$\times$ $U(1)_R$
A^{5d}	$(5,1)$	A	0	± 2	0	0
		A_0	0	0	0	0
		B	± 2	0	0	0
ϕ^{5d}	$(1,5)$	ϕ	0	0	± 2	0
		ϕ_0	0	0	0	0
		Y	0	0	0	± 2
λ^{5d}	$(4,4)$	λ	± 1	± 1	± 1	± 1

Table 2.1: The spectrum of 5d $\mathcal{N} = 2$ super-Yang-Mills theory on $S^2 \times \Sigma \times S^1$.

"β-deformed complex Chern-Simons theory". Note that the β-deformed complex Chern-Simons theory is well-defined not only on $\Sigma \times S^1$ but also on arbitrary Seifert manifolds since this is the class of 3-manifolds for which one finds the extra symmetry $U(1)_\beta$.

In order to understand how $U(1)_\beta$ acts on the fields of complex Chern-Simons theory, we can follow *e.g.* [13] and reduce the six-dimensional (2, 0) theory on the Hopf fiber of $L(k,1)$ to obtain 5d $\mathcal{N} = 2$ super-Yang-Mills theory on $S^2 \times (\Sigma \times S^1)$. The Lorentz and R-symmetry group $SO(5)_L \times SO(5)_R$ of the five-dimensional theory is broken down to

$$SO(5)_L \times SO(5)_R \;\to\; SO(2)_L \times SO(3)_L \times U(1)_N \times SU(2)_R\,. \tag{2.18}$$

Here $SO(2)_L$ is the Lorentz symmetry factor associated with S^2, while the second $SO(3)_L$ is the Lorentz factor associated with $\Sigma \times S^1$. If we choose the metric on Σ to be independent of S^1, the holonomy group is reduced from $SO(3)$ to $U(1)$. So in order to do the topological twist along Σ, we only need to use a $U(1)_L$ subgroup of $SO(3)_L$ and identify the new Lorentz group $U(1)'$ with the diagonal subgroup of $U(1)_L \times U(1)_N$. Also, the Ω-background picks out a $U(1)_R$ subgroup of $SU(2)_R$. In Table 2.1, we summarize how the fields in 5d super-Yang-Mills decompose and transform under $SO(2)_L \times U(1)_L \times U(1)_N \times U(1)_R$.

After the topological twist, the scalar ϕ becomes a one-form on Σ. In fact, $\mathcal{A} = A + i\phi$ and $\mathcal{A}_0 = A_0 + i\phi_0$ can be identified with the components of the connection of complex Chern-Simons theory along the Σ and S^1 directions, respectively. The $U(1)_\beta$ symmetry (2.16) does not act on A, A_0 or ϕ_0 but acts on ϕ by rotating its two components (ϕ_1, ϕ_2):

$$\theta \in U(1)_\beta \;:\; \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta \cdot \phi_1 - \sin\theta \cdot \phi_2 \\ \sin\theta \cdot \phi_1 + \cos\theta \cdot \phi_2 \end{pmatrix}. \tag{2.19}$$

As it is precisely ϕ, the imaginary part of the complex gauge connection, that gives rise to divergence in (2.17), one might hope that the $\mathbb{Z}$-grading of the Hilbert space $\mathcal{H}_{\text{CS}}(\Sigma; G_{\mathbb{C}})$ induced by $U(1)_\beta$ symmetry could provide the desired regularization. Indeed, as we show below, for each value of the $\mathbb{Z}$-grading, the corresponding component of the Hilbert space $\mathcal{H}_{\text{CS}}(\Sigma; G_{\mathbb{C}})$ is finite-dimensional, so that the partition function of the β-deformed complex Chern-Simons theory is a polynomial in $t = e^{-\beta}$ that gives the graded dimension (2.4) of the Hilbert space:

$$\dim_\beta \mathcal{H}_{\text{CS}}(\Sigma; G_{\mathbb{C}}) \;=\; Z_{\text{CS}}[\Sigma \times S^1; G_{\mathbb{C}}, \beta]\,. \tag{2.20}$$

The coefficient of t^n counts the dimension of the subspace that has eigenvalue n with respect to the symmetry $U(1)_\beta$.

In Chern-Simons theory with compact gauge group G, the Verlinde formula [16] is an explicit expression for $Z_{\text{CS}}[\Sigma \times S^1; G]$ and one of our primary goals in this chapter of the dissertation is to obtain its analog — which we call the "equivariant Verlinde formula" — for Chern-Simons theory with complex gauge group $G_{\mathbb{C}}$. In contrast to the Verlinde formula that depends on the choice of the gauge group G, level k, and topology of Σ, the equivariant Verlinde formula in addition depends also on β. Already at this stage one can anticipate some of its properties and behavior in different limits of β:

- When $\beta \to +\infty$, we expect the equivariant Verlinde formula to reduce to the usual Verlinde formula, because in this limit the only contribution to (2.20) comes from the singlet sector with respect to $U(1)_\beta$ and the contributions involving field ϕ, which is charged under this symmetry, are typically suppressed. Hence, in this limit the β-deformed complex Chern-Simons theory with gauge group $G_{\mathbb{C}}$ becomes Chern-Simons theory with compact gauge group G. So the equivariant Verlinde formula is a one-parameter deformation of the usual Verlinde formula.

- When $\beta \to 0$, we expect the equivariant Verlinde formula to be divergent because in this limit β will not provide any regularization for Chern-Simons theory with a complex gauge group $G_{\mathbb{C}}$.

Combining these two points together, one can view the equivariant Verlinde formula as an interpolation between the Verlinde formula with group G and with group $G_{\mathbb{C}}$.

Two different approaches to complex Chern-Simons theory

In general, there are two standard ways to preserve supersymmetry on a curved space M:

- **Deformation**. One way to preserve supersymmetry is to modify the supersymmetry algebra. An effective way of doing this is to couple the theory to supergravity and find consistent background values for these auxiliary fields [33]. This approach usually requires M to have non-trivial isometries.
- **Topological Twisting**. Another way is to perform a topological twist [45]. In a theory realized on world-volume of branes, this operation corresponds to embedding M as a calibrated submanifold in a special holonomy space [30]. This approach does not require M to have a symmetry.

Recall our eleven-dimensional geometry (2.15):

$$\begin{array}{lccccccc} N \text{ fivebranes:} & L(k,1)_b & \times & \Sigma & \times & S^1 & & \\ & & & \cap & & & & \\ \text{space-time:} & L(k,1)_b & \times & T^*\Sigma & \times & S^1 & \times & \mathbb{R}^3 \\ & & & \circlearrowright & & & & \circlearrowright \\ \text{symmetries:} & & & U(1)_N & & & & SU(2)_R \end{array} \tag{2.21}$$

We too have two possible ways to formulate the β-deformed complex Chern-Simons theory living on $\Sigma \times S^1$ as a topological theory with BRST symmetry. The first is to do "deformation", which means to reduce 6d $(2,0)$ theory on $L(k,1)$ as in [13], but now in the presence of the Ω-background. The second (and much easier) approach is to do a topological twist along $L(k,1)$, just like we did it along M_3.

In the eleven-dimensional geometry, this can be conveniently achieved by combining the $\mathbb{R}^3$ factor with $L(k,1)$ to obtain $T^*L(k,1)$. As the cotangent bundle of a lens space is trivial, there is no topological obstruction to doing so, although we do need to modify the metric of $\mathbb{R}^3$ so that the total space is Ricci-flat. In other words, now $L(k,1)$ is embedded as a special Lagrangian submanifold inside a local Calabi-Yau 3-fold:

$$\begin{array}{lccccc} N \text{ fivebranes:} & L(k,1)_b & \times & \Sigma & \times & S^1 \\ & \cap & & \cap & & \\ \text{space-time:} & T^*L(k,1)_b & \times & T^*\Sigma & \times & S^1 \\ & \circlearrowright & & \circlearrowright & & \\ \text{symmetries:} & SU(2)_R & & U(1)_N & & \end{array} \tag{2.22}$$

In order to introduce the equivariant parameter β, we need to single out an $\mathbb{R}^2_{-\beta}$ subspace of $\mathbb{R}^3$ to turn on the Ω-background. So, now we also need to specify how this $\mathbb{R}^2$ is fibered over $L(k,1)$. Lens spaces are particular examples of Seifert manifolds, and $L(k,1)$ is the total space of a degree k S^1-bundle over $\mathbb{CP}^1$. If we take $\mathbb{R}^2_{-\beta}$ to be the cotangent fiber of $\mathbb{CP}^1$, then the two sides of the 3d-3d correspondence are treated on equal footing[1] and this is exactly what we will do.

To summarize, the β-deformed complex Chern-Simons theory on $\Sigma \times S^1$ can be described as the topological twist of the 3d $\mathcal{N} = 2$ "lens space theory" $T[L(k,1);\beta]$, and our next task is to identify this theory and analyze its dynamics. Among other things, this gives another possible way to define the graded dimension (2.4) or the partition function of the β-deformed complex Chern-Simons theory (2.20):

$$\boxed{\dim_\beta \mathcal{H}_{\mathrm{CS}}(\Sigma; G_{\mathbb{C}}) \;=\; Z_{\mathrm{CS}}[\Sigma \times S^1; G_{\mathbb{C}}, \beta] \;=\; Z^{\mathrm{twisted}}_{T[L(k,1);\beta]}\left[\Sigma \times S^1\right]\;.} \tag{2.23}$$

In section 2.5 we present further evidence for the proposed relation (2.8) by calculating partition function and comparing with the prediction of the 3d-3d correspondence, *i.e.* with the equivariant integral over the Hitchin moduli space.

2.3 Branes and vortices

The theories studied in this chapter describe low-energy physics of certain brane configurations in type IIA and type IIB string theory. In particular, the type IIB brane configuration will help us identify the lens space theory $T[L(k,1);\beta]$ and its type IIA dual will make contact with the dynamics of vortices in 4d $\mathcal{N} = 2$ SQCD with a $U(k)$ gauge group.

"Lens space theory" $T[L(k,1)]$ from brane constructions

The reduction of the 6d (2,0) theory on $L(k,1)$ can be most easily performed by regarding this lens space as the total space of a $\mathbb{T}^2$ torus fibration over an interval. At each endpoint of the interval, the torus degenerates to a circle. The first homology group of the torus is

$$H_1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}, \tag{2.24}$$

generated by $[a]$ and $[b]$. Regarding the lens space $L(k,1)$ as a Hopf fibration over $\mathbb{CP}^1$ *à la* (2.14), we can also identify $[a]$ with the homology class of the Hopf fiber

[1] In this chapter we focus on the special case $M_3 = \Sigma \times S^1$, but it can be replaced with a more general Seifert manifold, which we will only discuss in the last chapter.

and $[b]$ with the latitude circle of the base $\mathbb{CP}^1$ which shrinks on both ends of the interval, see figure 2.2.

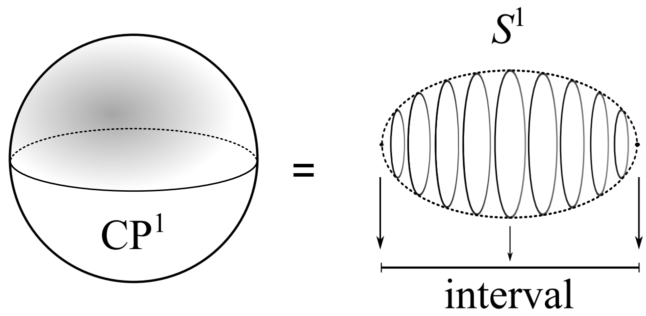


Figure 2.2: $\mathbb{CP}^1$ can be viewed as the total space of a circle fibration over an interval, with degenerate fibers at the endpoints of the interval.

Then, in representing $L(k, 1)$ as a $\mathbb{T}^2$-fibration over the interval, the vanishing cycle at one endpoint of the interval is homologous to $[b]$, whereas non-trivial topology of the Hopf fibration requires the vanishing cycle at the other endpoint of the interval to be $[b] + k[a]$. This torus fibration is illustrated in figure 2.3. Note, near the left endpoint of the interval, the base $\mathbb{CP}^1$ looks like a cigar and the total space of its cotangent bundle can be identified with a Taub-NUT space, such that $[b]$ is the S^1 fiber that vanishes at the Taub-NUT center.

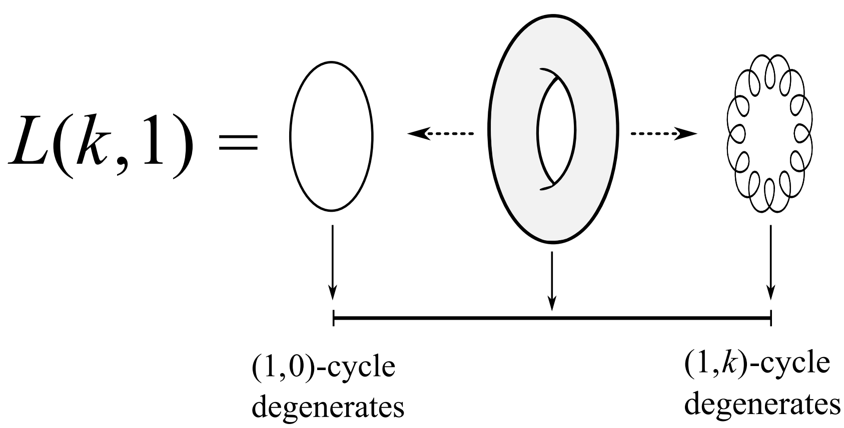


Figure 2.3: The lens space $L(k, 1)$ can be viewed as the total space of a 2-torus fibered over an interval. Near each endpoint of the interval, a particular cycle of the torus degenerates.

Now we are ready to reduce our 11-dimensional setup (2.22) on the torus $\mathbb{T}^2$.

Our choice of space-time coordinates is summarized in the following table:

space-time	0	1	2	3	4	5	6	7	8	9	10
M5	—	—	—	·	·	·	—	·	·	—	—
geometry	Σ		S^1	$\mathbb{R}^2_\beta$		$\mathbb{R}_{\text{Hopf}}$	Interval	$\mathbb{R}^2_{-\beta}$		$\mathbb{T}^2$	

We use (x^0, x^1, x^2) to parametrize $\Sigma \times S^1$, which for now we assume to be flat, until we are ready to implement the topological twist along Σ. We use (x^3, x^4) to parametrize the cotangent fiber $\mathbb{R}^2_\beta$ of Σ. And we let the Hopf fiber S^1 (a-cycle of the torus) to be parametrized by a periodic coordinate x^9 and its cotangent space $\mathbb{R}_{\text{Hopf}}$ to be parametrized by x^5. We use x^6 to be the coordinate on the interval base of the torus bundle, and (x^7, x^8) to be coordinates on the cotangent space $\mathbb{R}^2_{-\beta}$ of $\mathbb{CP}^1$, where $\mathbb{CP}^1$ is the base of the Hopf fibration. Lastly, we choose the b-cycle to be parametrized by x^{10}.

Type IIB brane configuration

We are going to use a famous duality between M-theory on a 2-torus and type IIB string theory on a circle, so that the $SL(2,\mathbb{Z})$ duality group of type IIB theory has a nice geometric interpretation as the mapping class group of the $\mathbb{T}^2$. What happens to M5-branes supported on $L(k,1)_b \times \Sigma \times S^1$ upon this reduction?

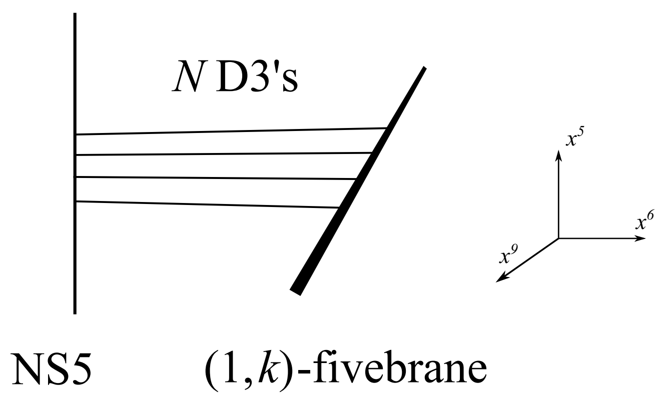


Figure 2.4: The NS5-D3-(1,k) brane system in type IIB string theory.

The fivebranes wrapping a torus give rise to a stack of N D3-branes and the boundary condition it satisfies demands that we have a NS5-brane on one side of the interval and a $(1,k)$-fivebrane on the other side of the interval:

space-time	0	1	2	3	4	5	6	7	8	9
N D3's	—	—	—	·	·	·	$\vdash\!\dashv$	·	·	·
NS5	—	—	—	—	—	—	·	·	·	·
$(1,k)$-brane	—	—	—	—	—	╲	·	·	·	╲

(2.25)

This brane configuration is illustrated in figure 2.4 and can be equivalently derived as follows.

As we pointed out earlier, near the left endpoint of the interval, the base $\mathbb{CP}^1$ looks like a cigar and the total space of its cotangent bundle can be identified with a Taub-NUT space, such that $[b]$ is the S^1 fiber that vanishes at the Taub-NUT center. Reducing M-theory on the circle fiber of the Taub-NUT space gives rise to a D6-brane, while N M5-branes become N D4-branes. In the coordinate system described above, the D6-brane is located at the Taub-NUT center:

$$x^6 = x^7 = x^8 = 0\,. \tag{2.26}$$

In other words, its world-volume spans the space-time directions 0123459. And it is easy to see that the D4-branes are extended along 01269. This is summarized in the table below:

space-time	0	1	2	3	4	5	6	7	8	9
D4	—	—	—	·	·	·	⊢	·	·	—
D6	—	—	—	—	—	—	·	·	·	—

Here we are looking at the geometry near the left endpoint of the interval x^6 so the D4-branes appear to be semi-infinite in the x^6 direction. Then, we perform a T-duality along the Hopf fiber direction parametrized by x^9. The D6-brane turns into a D5-brane with world-volume 012345, while the D4-brane becomes a D3-brane spanning 01236. For convenience we perform S-duality, which replaces D5 with an NS5-brane while leaving D3's invariant. We can perform a similar analysis near the right end-point of the interval and obtain N D3's ending on another NS5. But this picture at the right endpoint of the interval is in a different $SL(2,\mathbb{Z})$ duality frame of type IIB theory; in the original frame we will have a $(1,k)$-fivebrane instead of an NS5-brane. Also, the $(1,k)$-brane is rotated in the (x^5, x^9) plane

$$(1,k)\text{-brane}: \quad x^6 = x^7 = x^8 = 0, \quad kx^5 = x^9, \tag{2.27}$$

since it can be decomposed into an NS5-brane in 012345 and k D5-branes in 012349, as illustrated in figure 2.5.

To summarize, our M-theory setup (2.22) is dual to the type IIB brane configuration (2.25) illustrated in figures 2.4 and 2.5. In particular, 3d $\mathcal{N}=2$ lens space theory $T[L(k,1)]$ can be identified with the theory on D3-branes in figures 2.4 and 2.5. Besides an 3d $\mathcal{N}=2$ vector multiplet, it also contains an $\mathcal{N}=2$ chiral multiplet

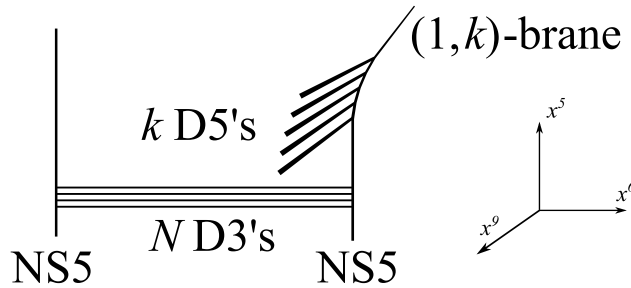


Figure 2.5: The (1,k)-brane in figure 2.4 is a bound state of an NS5-brane and k D5-branes.

Φ in the adjoint representation of the gauge group $G = U(N)$ that corresponds to the motion of D3-branes in directions x^3 and x^4. Weakly gauging the $U(1)_\beta$ symmetry (2.19) that rotates x^3 and x^4 gives a real mass β to Φ:

$$\delta S_{\text{mass}} = \int d^3x d^4\theta \, \Phi \, e^{\beta\theta^2} \, \Phi^\dagger \, . \tag{2.28}$$

Thus, we end up with the theory described in (2.7). (Here, β plays the role of mass parameter and, hence, is dimensionful. Starting from section 2.4, a dimensionless "equivariant parameter" β will also appear. As they are related simply by a $2\pi R_{S^1}$ factor, with R_{S^1} being the radius of the Seifert S^1 fiber, to avoid clutter we use the same symbol β for both quantities.)

Type IIA brane configuration

Our main application of the lens space theory $T[L(k,1);\beta]$ in this chapter is that its twisted partition function on $M_3 = \Sigma \times S^1$ gives the equivariant Verlinde formula. In particular, in sections 2.6 and 2.7 we will study the circle reduction of this theory to 2d TQFT on Σ. The latter is what we are going to call the equivariant G/G gauged WZW model and has a nice interpretation in our brane construction. This dimensional reduction corresponds to a T-duality along the S^1 direction parametrized by x^2. Upon this T-duality, N D3-branes in figure 2.5 transform into N D2-branes in directions 016, while k D5-branes turn into k D4-branes in directions 01349. The resulting type IIA brane configuration is shown in figure 2.6 and describes vortices in $U(k)$ four-dimensional SUSY gauge theory:

$$\frac{\text{2d } \mathcal{N} = (2,2) \text{ ``vortex theory'' on D2-branes}}{\overline{\begin{array}{c} U(N) \text{ SQCD with } k \text{ fundamental chiral multiplets} \\ \text{and one adjoint chiral multiplet } \Phi \text{ of mass } \beta \end{array}}} \tag{2.29}$$

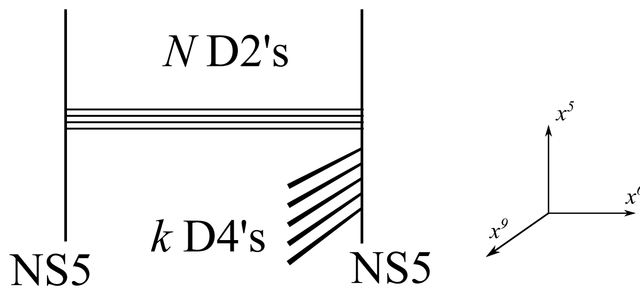


Figure 2.6: The NS5-D2-NS5-D4 brane system in Type IIA string theory obtained by dimensionally reducing the system in figure 2.4.

The type IIB and type IIA brane configurations in figures 2.5 and 2.6 will be extremely useful to us for analyzing 3d $\mathcal{N} = 2$ theory $T[L(k, 1); \beta]$ and its reduction to 2d, respectively. In particular, we can use either the UV or IR limit of these theories to study topological twist on a Riemann surface Σ. In the analogous problem that involves 4d $\mathcal{N} = 2$ gauge theory, the twist of the UV theory leads to Donaldson invariants, whereas topological twist of the IR limit leads to Seiberg-Witten equations. Similarly, we can obtain different expressions for the equivariant Verlinde formula (and equivariant Verlinde algebra) by implementing topological twist at different energy scales.

If we perform topological twist in the UV theory, we obtain a 3d TQFT discussed in section 2.5. On the other hand, if we follow 3d $\mathcal{N} = 2$ theory $T[L(k, 1); \beta]$ to the IR, then we do not even need to perform the topological twist: for generic values of $\beta \neq 0$ the theory has a mass gap and in the IR automatically flows to a TQFT that we call the equivariant G/G gauged WZW model. As we show next, there is yet another phase of the lens space theory $T[L(k, 1); \beta]$ that relates it to a classical problem about vortices.

Vortices and equivariant G/G gauged WZW model

Although exactly soluble field theories discussed in this chapter have a natural home in mathematical physics, they can be also realized in nature.

In particular, we claim that the low-energy effective theory of N vortices in 4d $\mathcal{N} = 2$ SQCD with $U(k)$ gauge group and Ω-background in the plane orthogonal to the vortex world-sheet is the equivariant G/G model. Furthermore, we claim that the equivariant Verlinde algebra (*i.e.* the algebra of loop operators in the β-deformed complex Chern-Simons theory), whose explicit form will be discussed in section 2.7, is given by the equivariant quantum K-theory of $\mathcal{V}_{N,k}$, the moduli space[2] of N $U(k)$ vortices on the plane $\mathbb{R}^2 \cong \mathbb{C}$. Here, the word "equivariant" means equivariant with respect to the rotation symmetry of the plane; this is precisely our symmetry $U(1)_\beta$. In the physics literature, this equivariant K-theory of vortex moduli spaces was first discussed in [6].

This provides an equivariant generalization of a beautiful story discovered by Witten [29] that relates the $\Sigma \times S^1$ partition function of $U(N)$ Chern-Simons theory at level k (*i.e.* the Verlinde formula) and the algebra of Wilson loops (*i.e.* the Verlinde algebra) to the quantum cohomology of the Grassmannian $Gr(N, N+k)$. Our equivariant generalization of this relation can be derived by starting with a "big theory":

$$\begin{aligned} &\text{3d } \mathcal{N}=2 \ U(N) \text{ super-Chern-Simons theory at level } \frac{k}{2} \\ + \quad & k \text{ chiral multiplets } Q_{A=1,\ldots,k} \text{ in the fundamental representation} \\ + \quad & 1 \text{ massive chiral multiplet } \Phi \text{ in the adjoint representation with mass } \beta. \end{aligned} \tag{2.30}$$

Because the gauge group is $U(N)$, we can turn on an FI parameter ζ and analyze the vacuum structure as a function of ζ. We will show that, as ζ varies, this theory interpolates between the lens space theory in (2.7) and 3d $\mathcal{N}=2$ sigma-model with the vortex moduli space $\mathcal{V}_{N,k}$ as the target and a potential that makes $\mathcal{V}_{N,k}$ effectively compact.

In order to analyze the vacuum structure of this theory, we need to study the scalar potential as a function of scalar fields, which are the following. Let σ_i, $i = 1, 2, \ldots, N$ be the eigenvalues of a scalar field σ in the $\mathcal{N}=2$ vector multiplet. The scalar components of Q_A will be denoted $q^i{}_A$ and assembled into a $N \times k$ matrix q. And the adjoint superfield Φ contains a $N \times N$ matrix of scalar fields $\varphi_i{}^j$.

A similar 3d $\mathcal{N}=2$ theory without the adjoint multiplet Φ was discussed in [46]. In the regime $\zeta < 0$ it has a unique supersymmetric vacuum where σ acquires an expectation value

$$\sigma = -\frac{\zeta}{k} \cdot \mathrm{Id}\,. \tag{2.31}$$

This gives a positive mass to all fundamental chiral multiplets Q_A. Integrating these chiral multiplets out leaves us with $\mathcal{N}=2$ $U(N)$ super-Chern-Simons theory at level k. On the other hand, if $\zeta > 0$ then one has $\sigma = 0$ and the D-term equation is

[2]Notice, that in the usual notation for the vortex moduli space k stands for the number of vortices, while N is the rank of the gauge group, whose role is reversed in our notations here.

now

$$\zeta \cdot \mathrm{Id} = qq^\dagger \,. \tag{2.32}$$

For $k \geq N$, the moduli space of solutions to this equation is the Grassmannian $\mathrm{Gr}(N,k)$ and, therefore, the low energy physics is described by the $\mathcal{N}=2$ Grassmannian sigma model. If one puts low-energy theories for both $\zeta<0$ and $\zeta>0$ on $\Sigma\times S^1$ and performs the topological twist, one arrives at the conclusion that $U(N)_{k-N}$ Chern-Simons theory[3] on $\Sigma\times S^1$ is equivalent to the topological A-model of $\mathrm{Gr}(N,k)$. Put in other words, the Verlinde algebra can be identified with the quantum cohomology ring of the Grassmannian. So this argument reproduces the main result of [29].

Now let us add the massive adjoint chiral multiplet Φ. For $\zeta \ll 0$, the supersymmetric vacuum characterized by

$$\sigma = -\frac{\zeta}{k}\cdot \mathrm{Id} \tag{2.33}$$

still exists and gives a mass to all the Q_A's. However, the mass of Φ still comes entirely from (2.28). Indeed, the only other potential contribution to the mass of its scalar component φ is the term

$$|[\sigma,\varphi]|^2 , \tag{2.34}$$

but the identity matrix commutes with any value of φ. A similar argument shows that fermions in the Φ multiplet also remain massless as σ gets a vev. Therefore, for $\zeta \ll 0$, after integrating out all the fundamental multiplets Q_A, the low-energy effective theory is described by $\mathcal{N}=2$ $U(N)_k$ super-Chern-Simons theory with an adjoint chiral superfield Φ of (real) mass β, which is precisely our 3d $\mathcal{N}=2$ theory $T_N[L(k,1);\beta]$. Hence, we showed that $T_N[L(k,1);\beta]$ can be identified with the $\zeta \ll 0$ phase of the "big theory" (2.30).

On the other hand, in the regime $\zeta>0$ the D-flatness condition of the theory (2.30) looks like

$$\zeta\cdot \mathrm{Id} = qq^\dagger + [\varphi,\varphi^\dagger]. \tag{2.35}$$

Therefore, the low-energy physics is described by an $\mathcal{N}=2$ sigma-model with the target space

$$\mathcal{V}_{N,k} \;\cong\; \left\{(q,\varphi)\middle|\zeta\cdot \mathrm{Id} = qq^\dagger + [\varphi,\varphi^\dagger]\right\}/U(N). \tag{2.36}$$

This space is conjectured by Hanany and Tong [47] to be homeomorphic to the moduli space $\mathcal{V}_{N,k}$ of N $U(k)$ vortices on $\mathbb{R}^2$. Hence, for $\zeta>0$ the low-energy

[3] $\mathcal{N}=2$ $U(N)_k$ super-Chern-Simons theory is equivalent to $U(N)_{k-N}$ bosonic Chern-Simons theory because integrating out gauginos in the adjoint representation shifts the level by $-N$.

physics of (2.30) is described by the $\mathcal{N} = 2$ sigma-model with the target space $\mathcal{V}_{N,k}$ and a potential

$$V = \frac{1}{2}\beta^2 |\varphi|^2 \tag{2.37}$$

that comes from the mass of Φ, *cf.* (2.28). Putting the low-energy theories for both $\zeta < 0$ and $\zeta > 0$ on $\Sigma \times S^1$ and performing the topological twist leads to the following conclusion:

The β-deformed complex Chern-Simons theory on $S^1 \times \Sigma$ is equivalent to a topological sigma-model to the vortex moduli space $\mathcal{V}_{N,k}$ equipped with the potential (2.37).

Note, one can perform different topological twists on Σ parametrized by different assignments of the R-charge to the adjoint multiplet Φ. This leads to a large family of quasi-topological theories in three dimensions, only one of which (for $R = 2$) happens to be related to complex Chern-Simons theory. It is interesting, though, to study the entire family of such theories, related to different variants of the equivariant quantum K-theory as shown here. Reduction of this family to 2d TQFTs labeled by $R \in \mathbb{Z}$ will be discussed in detail in section 2.6.

It would be interesting to derive the equivariant G/G model on the vortex world-sheet via the anomaly inflow [48] from 4d $\mathcal{N} = 2$ SQCD with $U(k)$ gauge group (and Ω-background in the plane orthogonal to the vortex world-sheet). A similar question for half-BPS surface operators in 4d gauge theory with $\mathcal{N} = 4$ supersymmetry was studied in [49].

2.4 Equivariant integration over Hitchin moduli space

In this section we consider the "supersymmetric" (*i.e.* left) side of the 3d-3d correspondence (2.11) when $M_3 = \Sigma \times S^1$ or, more generally, a Seifert manifold. This, in particular, will give the precise meaning to the graded dimension in (2.20) and show that it can be written as the equivariant integral over the Hitchin moduli space.

As explained in section 2.2 and summarized in (2.7), the 3d $\mathcal{N} = 2$ theory $T[\Sigma \times S^1; \beta]$ is a sigma-model with the Hitchin moduli space $\mathcal{M}_H$ as the target and has a real mass for the $U(1)_\beta$ flavor symmetry, whose action is described in (2.16) and (2.19).

Quantization of Hitchin moduli space

The dimension of the Hilbert space of Chern-Simons theory with compact gauge group G can be naturally expressed as an integral over the moduli space of flat connections $\mathcal{M}_{\text{flat}}$. Let A be a connection on the principal G-bundle over the Riemann surface Σ and F_A its curvature. Then the moduli space of flat connections is

$$\mathcal{M}_{\text{flat}}(\Sigma; G) = \{A | F_A = 0\} / \mathcal{G}, \tag{2.38}$$

where $\mathcal{G}$ is the group of gauge transformations. This space is equipped with a natural symplectic form [50]:

$$\omega = \frac{1}{4\pi^2} \int_\Sigma \text{Tr } \delta A \wedge \delta A, \tag{2.39}$$

where δ is the de Rham differential on $\mathcal{M}_{\text{flat}}$. With this particular normalization ω is the generator of the integral cohomology group $H^2(\mathcal{M}_{\text{flat}}, \mathbb{Z})$.

The classical phase space of Chern-Simons theory at level k on Σ is precisely the symplectic space

$$(\mathcal{M}_{\text{flat}}(\Sigma; G), k\omega), \tag{2.40}$$

and the Hilbert space $\mathcal{H}_{\text{CS}}(\Sigma; G, k)$ can be obtained by quantizing it [15]. In fact, $\mathcal{M}_{\text{flat}}(\Sigma; G)$ is a compact Käher space as the complex structure of Σ defines a complex structure on $\mathcal{M}_{\text{flat}}(\Sigma; G)$ that is compatible with ω. As a consequence, one can apply the technique of geometric quantization [51] to identify $\mathcal{H}_{\text{CS}}(\Sigma; G)$ with the space of holomorphic sections of a "prequantum line bundle" $\mathcal{L}^{\otimes k}$:

$$\mathcal{H}_{\text{CS}}(\Sigma; G, k) = H^0 \left(\mathcal{M}_{\text{flat}}(\Sigma; G), \mathcal{L}^{\otimes k} \right), \tag{2.41}$$

where $\mathcal{L}$ is the universal determinant line bundle with curvature ω. The index theorem, combined with the Kodaira vanishing theorem for the higher cohomology groups, relates the dimension of the Hilbert space to the index of a spin^c Dirac operator and then to an integral over $\mathcal{M}_{\text{flat}}(\Sigma; G)$:

$$\dim \mathcal{H}_{\text{CS}}(\Sigma; G, k) = \chi(\mathcal{M}_{\text{flat}}, \mathcal{L}^{\otimes k}) = \text{Index}(\not{\partial}_{\mathcal{L}^{\otimes k}}) = \int_{\mathcal{M}_{\text{flat}}} \text{Td}(\mathcal{M}_{\text{flat}}) \wedge e^{k\omega}, \tag{2.42}$$

where Td $(\mathcal{M}_{\text{flat}}(\Sigma; G))$ is the Todd class of $\mathcal{M}_{\text{flat}}(\Sigma; G)$.

Now let us consider Chern-Simons theory with complex gauge group $G_{\mathbb{C}}$. The classical phase space is a symplectic manifold

$$(\mathcal{M}_{\text{flat}}(\Sigma; G_{\mathbb{C}}) \cong \mathcal{M}_H(\Sigma; G),\ k\omega_I + \sigma\omega_K) \,. \tag{2.43}$$

Here $\mathcal{M}_H(\Sigma; G)$, later abbreviated as $\mathcal{M}_H$, is the moduli space of G-Higgs bundles over Σ [44]:

$$\mathcal{M}_H(\Sigma; G_{\mathbb{C}}) = \left\{ (A, \phi) \left| \begin{array}{rl} F_A - \phi\wedge\phi & = 0 \\ d_A\phi = d_A^\dagger\phi & = 0 \end{array} \right. \right\} / \mathcal{G}. \tag{2.44}$$

The adjoint-valued one-form $\phi \in \Omega^1(\Sigma, \mathfrak{g})$ is precisely our field ϕ that appeared earlier in Table 2.1. The Hitchin moduli space is hyper-Kähler: it comes equipped with three complex structures (I, J, K) and three real symplectic forms:

$$\omega_I = \frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}\, (\delta A\wedge\delta A - \delta\phi\wedge\delta\phi)\,, \tag{2.45}$$

$$\omega_J = \frac{1}{2\pi^2}\int_\Sigma \mathrm{Tr}\, (\delta A\wedge\star\delta\phi)\,, \tag{2.46}$$

$$\omega_K = \frac{1}{2\pi^2}\int_\Sigma \mathrm{Tr}\, (\delta A\wedge\delta\phi)\ . \tag{2.47}$$

This space can be viewed as a natural complexification of $\mathcal{M}_{\text{flat}}(\Sigma; G)$ and it is birationally equivalent to $T^*\mathcal{M}_{\text{flat}}$. The canonical determinant bundle $\mathcal{L}$ also extends naturally to a line bundle over $\mathcal{M}_H$ that we continue to call $\mathcal{L}$. The curvature of $\mathcal{L}$ is now ω_I. (This extension of $\mathcal{L}$ from $\mathcal{M}_{\text{flat}}(\Sigma; G)$ to $\mathcal{M}_H$ is one of the key elements in the "brane quantization" of the moduli space of flat connections [22].)

Just as in the quantization of (2.40), the quantization of (2.43) leads to a Hilbert space whose dimension can be formally expressed as an integral over $\mathcal{M}_H$ similar to (2.42):

$$\dim \mathcal{H}_{\text{CS}}(\Sigma; G_{\mathbb{C}}, k) = \int_{\mathcal{M}_H} \mathrm{Td}(\mathcal{M}_H)\wedge e^{k\omega_I + \sigma\omega_K}\ . \tag{2.48}$$

However, as the Hitchin moduli space is non-compact, the integral above is divergent, indicating that the Hilbert space associated with complex Chern-Simons theory is infinite-dimensional.

An interesting feature of the Hitchin moduli space is that it admits a circle action with compact fixed point loci which, anticipating a connection with an earlier discussion, we shall call $U(1)_\beta$. This action was used by Hitchin [44] to study topology of the moduli space of Higgs bundles and in the literature is sometimes referred to as "the Hitchin action". The corresponding vector field V on $\mathcal{M}_H$ is generated by the Hamiltonian:

$$\mu = \frac{1}{2\pi}\int_\Sigma \mathrm{Tr}\,(\phi\wedge\star\phi), \tag{2.49}$$

with the symplectic form ω_I:

$$\delta\mu = 2\pi\iota_V\omega_I. \tag{2.50}$$

Indeed, one can see that this action, rotating the Higgs fields ϕ, is exactly (2.19), which rotates the cotangent space of Σ where the field ϕ lives. Using this $U(1)_\beta$ action, we can regularize the divergent integral in (2.48) by converting it to an *equivariant integral*. First we define the equivariant differential associated with the Hamiltonian $U(1)_\beta$ action on $\mathcal{M}_H$:

$$D = \delta + 2\pi\beta\iota_V\,. \tag{2.51}$$

Here β is the generator of

$$H^*_{S^1}(pt) = H^*(\mathbb{CP}^\infty) = \mathbb{C}[\beta]\,, \tag{2.52}$$

assigned degree 2 in the equivariant cohomology to make D homogeneous. We have chosen β for this equivariant parameter, so that it can be identified with the mass parameter in the previous discussion. Then, the equivariantly closed extension of ω_I is

$$\widetilde{\omega}_I = \omega_I - \beta\mu, \tag{2.53}$$

which satisfies

$$D\widetilde{\omega}_I = 0\,. \tag{2.54}$$

Because ω_K is not invariant under $U(1)_\beta$, we set σ to zero in (2.48). In the original problem of quantizing (2.43) it means that we set the "imaginary part" of the complex Chern-Simons theory to zero. Since all the relevant characteristic classes have equivariant extensions, it is natural to replace the divergent integral (2.48) with an equivariant integral that computes the equivariant index:

$$\mathrm{Index}(\bar{\partial}_{\mathcal{L}^{\otimes k}}) = \int_{\mathcal{M}_H} \mathrm{ch}\left(\mathcal{L}^{\otimes k}\right) \wedge \mathrm{Td}(\mathcal{M}_H) \tag{2.55}$$

$$\updownarrow \tag{2.56}$$

$$\mathrm{Index}_{S^1}(\bar{\partial}_{\mathcal{L}^{\otimes k}};\beta) = \int_{\mathcal{M}_H} \mathrm{ch}\left(\mathcal{L}^{\otimes k},\beta\right) \wedge \mathrm{Td}(\mathcal{M}_H,\beta). \tag{2.57}$$

In particular, the equivariant Chern character

$$\mathrm{ch}(\mathcal{L}^{\otimes k},\beta) = \exp\left(k\widetilde{\omega}_I\right) = \exp\left(k\omega_I - k\beta\mu\right) \tag{2.58}$$

exponentially suppresses the contribution of parts far away from $\mathcal{M}_{\mathrm{flat}}(\Sigma;G) \subset \mathcal{M}_H$, where $\mu \gg 0$. Therefore, one may hope that a positive value of β provides the desired regularization of the naive expression (2.48).

Using the Atiyah-Bott localization formula [52] one can rewrite the right-hand side of (2.55) as an integral over the critical manifolds, F_d, of μ:

$$\mathrm{Index}_{S^1}(\bar{\partial}_{\mathcal{L}^{\otimes k}}, \beta) = \sum_{F_d} e^{-\beta k \cdot \mu(F_d)} \int_{F_d} \frac{\mathrm{Td}(F_d) \wedge e^{k\omega_I}}{\prod_i \left(1 - e^{-x_i - \beta n_i}\right)}, \tag{2.59}$$

which is manifestly convergent as all critical manifolds are compact. In the denominator we used the splitting principle to decompose the normal bundle of F_d into line bundles L_i whose equivariant Chern classes are $1 + x_i + \beta n_i$.

The equivariant index (2.59) is going to be our definition for the graded dimension of the Hilbert space of complex Chern-Simons theory (2.20):

$$\begin{aligned} \dim_\beta \mathcal{H}(\Sigma; G_{\mathbb{C}}, k) &= Z_{\mathrm{CS}}[\Sigma \times S^1; G_{\mathbb{C}}, k, \beta] \\ &= \mathrm{index}_{S^1}(\bar{\partial}_{\mathcal{L}^{\otimes k}}; \beta) = \int_{\mathcal{M}_H} \mathrm{Td}(\mathcal{M}_H, \beta) \wedge \exp(k\widetilde{\omega}_I)\,. \end{aligned} \tag{2.60}$$

Note, every quantity in this formula, except for the first one (*viz.* the partition function of complex Chern-Simons theory with β deformation) has precise mathematical definition and at this stage can in principle be computed directly. In section 2.7 we perform the equivariant integration explicitly in the case of $G = SU(2)$ for some punctured Riemann surfaces and obtain the $SU(2)$ "equivariant Verlinde algebra" generalizing the usual Verlinde algebra.

However, this direct approach becomes progressively more complicated as the rank of the gauge group gets larger and larger. Our goal is to evaluate (2.60) indirectly, using the 3d-3d correspondence (2.11) to compactify the fivebrane theory on $L(k, 1)$ first and then use string dualities of section 2.3 to derive the exact solution of the β-deformed complex Chern-Simons theory on $M_3 = \Sigma \times S^1$ (and, more generally, on Seifert manifolds). We hope that many alternative ways for computing the integral (2.60) presented in this thesis can shed light on the singularity structure of the moduli space of Higgs bundles (when the rank and the degree are not coprime).

Before we proceed, let us point out that in [53] a similar integral over $\mathcal{M}_H$ which computes the "equivariant volume",

$$\mathrm{Vol}_\beta(\mathcal{M}_H) = \int_{\mathcal{M}_H} \exp(\widetilde{\omega}_I), \tag{2.61}$$

was studied using the "topological Yang-Mills-Higgs model." This model was later analyzed in detail in [54, 55]. As the equivariant index is the K-theoretic lift of the equivariant volume, we expect the β-deformed complex Chern-Simons theory to share a lot of similarities with the Yang-Mills-Higgs model. In particular, it should have a BRST symmetry. One way to obtain a theory with BRST symmetry is to start with a supersymmetric theory and perform a topological twist. As we will see in the next section, this is indeed the case: the β-deformed complex Chern-Simons theory on $\Sigma \times S^1$ is equivalent to a topologically twisted 3d $\mathcal{N} = 2$ supersymmetric gauge theory.

2.5 β-deformed complex Chern-Simons

Complex Chern-Simons theory from topological twist

Since generic 3d $\mathcal{N} = 2$ theories have R-symmetry group $U(1)$ they cannot be twisted on general 3-manifolds with holonomy group $SO(3)$. However, if $M_3 = \Sigma \times S^1$ is equipped with a metric such that the $U(1)_S$ Seifert action rotating the S^1 factor is an isometry, then the holonomy group is reduced to $U(1)$ and one can perform a "semi-topological" twist for a 3d $\mathcal{N} = 2$ theory on M_3. After the twist, the resulting theory does not depend on the choice of metric, as long as $U(1)_S$ is still an isometry of that metric. Equivalently, upon the dimensional reduction on a circle fiber it gives truly topological theory in two dimensions. When M_3 is not $\Sigma \times S^1$ but still Seifert, equipped with a $U(1)_S$ invariant metric, one cannot do the same kind of topological twist to a 3d $\mathcal{N} = 2$ theory but can still put it on M_3 by deforming the supersymetry algebra. This is the approach taken by Källén in [56] for $\mathcal{N} = 2$ super-Chern-Simons theory and by Ohta and Yoshida in [57] for $\mathcal{N} = 2$ Chern-Simons-matter theories.

Here, we apply this to a particular 3d $\mathcal{N} = 2$ theory, namely $T[L(k,1);\beta]$ that one finds after the reduction of the 6d $(2,0)$ fivebrane theory on a lens space. As any other 3d $\mathcal{N} = 2$ theory, $T[L(k,1);\beta]$ can be twisted on $\Sigma \times S^1$ or defined on more general Seifert manifolds using deformed SUSY. Then, according to section 2.2, this theory on M_3 will be precisely the sought-after "β-deformed $G_{\mathbb{C}}$ complex Chern-Simons theory" at level k. At this stage, from the definition in section 2.2, we know the following three facts about this β-deformed $G_{\mathbb{C}}$ complex Chern-Simons theory at level k:

1. For $\beta \to +\infty$ it reduces to Chern-Simons theory with compact gauge group G at level k.
2. For $\beta \to 0$ it becomes Chern-Simons theory with non-compact gauge group $G_{\mathbb{C}}$.

3. For general β, we would expect the theory to produce the equivariant integral (2.59) over the Hitchin moduli space $\mathcal{M}_H$ if we put it on $\Sigma \times S^1$.

Now we demonstrate that 3d $\mathcal{N} = 2$ theory $T[L(k,1);\beta]$ twisted on $\Sigma \times S^1$ indeed satisfies all these criteria, thereby verifying (2.8). Then, in subsection 2.5, we compute its partition function (2.23) using localization.

The limit $\beta \to +\infty$ and compact group G

In the $\beta \to +\infty$ limit, the adjoint chiral multiplet Φ in $T[L(k,1);\beta]$ can be integrated out and it will produce a shift of the Chern-Simons level $k \to k' = k + h_{\mathfrak{g}}$, where $h_{\mathfrak{g}}$ is the dual Coxeter number of the Lie algebra $\mathfrak{g}$. Then we are left with $\mathcal{N} = 2$ super-Chern-Simons theory with gauge group G at level k'. This theory can be further reduced to pure bosonic Chern-Simons theory after integrating out gauginos $\lambda, \lambda^\dagger$ and bosonic fields σ, D. The functional determinant associated with gauginos is not well defined and one needs to regularize it. A standard way to do this is to add a Yang-Mills term to the theory and send the Yang-Mills coupling to infinity. Using this regularization, which is natural from the brane picture, the functional integral over gaugino fields will produce a further shift $k' \to k$, see *e.g.* [58].

Notice that expectation values of physical observables in Chern-Simons theory at level k usually depend on $k' = k + h_{\mathfrak{g}}$, which comes from gluon loops. Combined with this, there are in total three level-shifting effects, which are summarized below.

1. Integrating out $\mathcal{N} = 2$ adjoint chiral multiplet with large positive mass shifts the level by $+h_{\mathfrak{g}}$.

2. Integrating out gauginos in super-Chern-Simons theory shifts the level by $-h_{\mathfrak{g}}$.

3. Integrating over gauge fields to compute partition function or expectation values of physical observables effectively renormalizes the level by $+h_{\mathfrak{g}}$.

The effects of 1 and 2 cancel each other so that $T[L(k,1);\beta \to \infty]$ is equivalent to pure bosonic Chern-Simons theory at level k.

The limit $\beta \to 0$ and complex group $G_{\mathbb{C}}$

In this limit $T[L(k,1);\beta]$ is a superconformal theory and topological twist is crucial in order to produce a TQFT. (In general, a gapped theory is expected to flow to a TQFT in the infrared even without a topological twist.) The topological twist of $\mathcal{N} = 2$ super-Chern-Simons theory with general matter content on a Seifert manifold is discussed in [57]. In particular, on $\Sigma \times S^1$ a chiral multiplet will yield two BRST-multiplets (φ, ψ) and (χ, η). Here φ and η are bosons while ψ and χ are fermions. Regarded as fields on Σ, they are respectively sections of

$$\begin{aligned} (\varphi, \psi) &\in \Gamma\left[\Omega^0(L\mathfrak{g} \otimes \mathbb{C})\right], \\ (\chi, \eta) &\in \Gamma\left[\Omega^1(L\mathfrak{g})\right], \end{aligned} \tag{2.62}$$

where $L\mathfrak{g}$ is the Lie algebra of the loop group LG. Using the complex structure of the Riemann surface, one can decompose (χ, η) into $(1, 0)$-forms (χ_z, η_z) and $(0, 1)$-forms $(\chi_{\bar{z}}, \eta_{\bar{z}})$. Similarly, the components of a vector multiplet (A, λ, σ, D) now become $(A_z, A_{\bar{z}}, A_0, \lambda_z, \lambda_{\bar{z}}, \lambda_0, \sigma, D)$. (See appendix of [57] for definitions of these fields and their transformation rules.) In what follows, we will focus on the matter part which comes from the chiral multiplet Φ. The corresponding BRST transformations are[4]

$$\begin{aligned} Q\varphi &= \psi, & Q\psi &= -i\mathcal{D}_0\varphi - i\sigma\varphi, \\ Q\chi_z &= \eta_z, & Q\eta_z &= -i(\mathcal{D}_0 + \sigma)\chi_z + \beta\chi_z, \\ Q\chi_{\bar{z}} &= \eta_{\bar{z}}, & Q\eta_{\bar{z}} &= i(\mathcal{D}_0 + \sigma)\chi_{\bar{z}} + \beta\chi_{\bar{z}}. \end{aligned} \tag{2.63}$$

However, this is not the only possible twist of the 3d $\mathcal{N} = 2$ theory $T[L(k, 1); \beta]$. The twist described above corresponds to assigning R-charge[5] $R = 0$ for Φ. Since the new Lorentz group of the Riemann surface $U(1)'_L$ is taken to be the diagonal subgroup of $U(1)_L \times U(1)'_R$, this assignment makes the scalars φ remain scalar after the twist. As $T[L(k, 1); \beta]$ has no F-term interactions[6] and the $U(1)'_R$ R-charge assignment for Φ is unconstrained, nothing prevents us from considering more general integer values of R. In particular, what turns out to be related to complex Chern-Simons theory is the case of $R = 2$. When $R = 2$, the fields are sections of:

$$\begin{aligned} (\varphi, \psi) &\in \Gamma\left[\Omega^1(L\mathfrak{g})\right], \\ (\chi, \eta) &\in \Gamma\left[\Omega^0(L\mathfrak{g} \otimes \mathbb{C})\right], \end{aligned} \tag{2.64}$$

[4]Notation in [57] differs from ours by $z \leftrightarrow \bar{z}$. The notation used here is chosen to agree with that in gauged WZW-matter model, which will be discussed below.

[5]Our convention is such that the superspace coordinates θ has R-charge 1.

[6]Recall, that the real mass is given by a D-term.

and we will write them in components as $(\varphi_z, \varphi_{\overline{z}}, \psi_z, \psi_{\overline{z}}, \chi, \eta)$. The BRST transformations are:

$$\begin{aligned} Q\varphi_z &= \psi_z, \quad Q\psi_z = -(\mathcal{D}_0 + \sigma)\varphi_z + \beta\varphi_z\,, \\ Q\varphi_{\overline{z}} &= \psi_{\overline{z}}, \quad Q\psi_{\overline{z}} = (\mathcal{D}_0 + \sigma)\varphi_{\overline{z}} + \beta\varphi_{\overline{z}}\,, \\ Q\chi &= \eta, \qquad Q\eta = -i(\mathcal{D}_0 + \sigma)\chi\,. \end{aligned} \tag{2.65}$$

Now we describe the relation between this twisted SUSY theory and complex Chern-Simons theory, whose action at level $(k,\sigma) = (k,0)$ is

$$S^{(k,0)}_{\mathrm{CS}}(\underline{A}, \underline{\phi}) = \frac{k}{4\pi}\int \left(\underline{A}\wedge d\underline{A} + \frac{2}{3}\underline{A}\wedge\underline{A}\wedge\underline{A} - \underline{\phi}\, d_{\underline{A}}\underline{\phi}\right), \tag{2.66}$$

where $\underline{A} = A + A_0 dx^0$ and $\underline{\phi} = \phi + \phi_0 dx^0$ are gauge fields in 3d. We see that the part involving Higgs field ϕ, which will eventually be identified with the adjoint scalar Lagrangian in $T[L(k,1); \beta = 0]$, is well separated from the gauge field Lagrangian.

At this stage, there are two obvious disconnects with the twist of $\mathcal{N} = 2$ theory $T[L(k,1)]$. First of all, the $U(1)_\beta$ flavor symmetry is missing in complex Chern-Simons theory. Secondly, complex Chern-Simons theory is invariant under a larger gauge group $G_{\mathbb{C}}$. The two difficulties actually cancel each other as we will see next.

We first rewrite the action (2.66) in the geometry $\Sigma \times S^1$:

$$\begin{aligned} &S^{(k,0)}_{\mathrm{CS}}(A, \phi, A_0, \phi_0) = \\ &\frac{k}{4\pi}\int_{\Sigma\times S^1} \mathrm{Tr}\,\left(A\wedge D_0 A + 2A_0\wedge A\wedge A + 2A_0\wedge dA - 2\phi_0\wedge d_A\phi - \phi\wedge D_0\phi\right). \end{aligned} \tag{2.67}$$

Here D_0 is the covariant derivative along the S^1 fiber of the Seifert manifold or $\Sigma \times S^1$ in our basic example. The integral over ϕ_0 can be explicitly carried out and gives a delta function that implements the constraint

$$d_A\phi = 0\,. \tag{2.68}$$

After integrating out ϕ_0, the Lagrangian is invariant under $U(1)_\beta$, but the condition above is not. A natural way to cure this problem is to impose the gauge choice

$$d^\dagger_A\phi = 0\,. \tag{2.69}$$

Note, the above two equations are also two of the three Hitchin equations. After these steps, the only term in the Lagrangian that depends on ϕ is proportional to

$$\phi\wedge D_0\phi\,. \tag{2.70}$$

In the twisted 3d $\mathcal{N} = 2$ theory $T[L(k,1); \beta = 0]$, the whole matter part of the action is Q-exact, and nothing prevents us from changing it into another Q-exact term, such as

$$\frac{1}{2} Q \left(\varphi_z \wedge \psi_{\overline{z}} - \psi_z \wedge \varphi_{\overline{z}} \right) = \psi_z \wedge \psi_{\overline{z}} + \varphi_z \wedge D_0 \varphi_{\overline{z}}. \tag{2.71}$$

After integrating out ψ, gauginos λ, scalars σ and D, we obtain precisely the complex Chern-Simons action. (Notice, that the shifts of level caused by ψ and λ cancel each other.)

Equivariant Verlinde formula

The Verlinde formula is usually written as a sum over highest weight integrable representations of the loop group LG at level k (see *e.g.* (2.1) for $G = SU(2)$, in which case it is simply a sum over $j = 1, 2, \ldots, k+1$):

$$j \ \in \ \Lambda_{G,k} = \left(\frac{\Lambda_{\mathrm{wt}}}{W \times (k+h)\Lambda_{\mathrm{rt}}} \right)' . \tag{2.72}$$

Here $h_{\mathfrak{g}}$ is the dual Coxeter number of the Lie algebra $\mathfrak{g}$ and the prime means that the fixed points are removed. It is natural to expect that the equivariant Verlinde formula, defined as the partition function of the β-deformed complex Chern-Simons theory (2.20), takes a similar form.

Now, once we established the equivalence of the β-deformed complex Chern-Simons with the twist of 3d $\mathcal{N} = 2$ theory $T[L(k,1); \beta]$ described in the previous subsection, we can use the standard localization techniques to compute its partition function. Thus, one can follow *e.g.* the techniques of [57] to calculate the partition function of the β-deformed complex Chern-Simons theory not only on $\Sigma \times S^1$ but on any Seifert manifolds M_3, and with arbitrary R-charge assignment for adjoint chiral multiplet Φ. Here, for simplicity, we focus on the particular case of $R = 2$ and $M_3 = \Sigma \times S^1$. Generalization of the equivariant Verlinde formula to arbitrary value of $R \in \mathbb{Z}$ will be discussed in the next section from a 2d perspective.

Using the localization procedure described in [57], one can express the whole partition function as a path integral over two-dimensional abelian fields

$$\begin{aligned} Z^{\beta-\mathrm{CS}}[\Sigma; U(N), k, t] = \frac{1}{|W|} \int \mathcal{D}\sigma_a \mathcal{D}\lambda_a \mathcal{D}A_a \left[\prod_{\alpha} \left(1 - e^{2\pi i (\sigma_a - \sigma_b)} \right)^{1-h} \right] \Xi^{3\mathrm{d}} \\ \times \exp \left\{ i \int_{\Sigma} \left[\left((k+N)\sigma_a - \sum_{b=1}^{N} \sigma_b + \frac{N-1}{2} \right) F^a + \frac{k}{4\pi} \lambda_a \wedge \lambda_a \right] \right\}, \end{aligned} \tag{2.73}$$

where $(\sigma_a, \lambda_a, A_a), a = 1, 2, \ldots, N$ are fields living on Σ and valued in the Cartan of $\mathfrak{u}(N)$. The important factor $\Xi^{3\mathrm{d}}$ is the matter contribution to the path integral

$$\Xi^{3\mathrm{d}} = \frac{\det_\chi\left[-i\mathcal{L}_0 - \frac{\mathrm{ad}(2\pi\sigma)+i\beta}{\ell}\right]}{\det_\varphi\left[-i\mathcal{L}_0 - \frac{\mathrm{ad}(2\pi\sigma)+i\beta}{\ell}\right]}, \tag{2.74}$$

where $\mathcal{L}_0$ is the Lie derivative along the Seifert fiber and $\ell = 2\pi R_{S^1}$ is the circumference of the Seifert S^1 fiber. If we set $\Xi^{3\mathrm{d}}$ to a constant by sending β to infinity, the rest of the path integral is exactly the partition function of Chern-Simons theory on $\Sigma\times S^1$ and it gives the ordinary Verlinde formula. Hence, the functional determinant (2.74) contains interesting information about how the equivariant Verlinde formula depends on the deformation parameter β and we now evaluate it.

First we decompose χ and φ into Fourier modes

$$\begin{aligned} \chi(z,\bar{z},\theta) &= \sum_{m\in\mathbb{Z}} \chi_m(z,\bar{z})e^{-im\theta}, \\ \varphi(z,\bar{z},\theta) &= \sum_{m\in\mathbb{Z}} \varphi_m(z,\bar{z})e^{-im\theta}. \end{aligned} \tag{2.75}$$

These modes are sections of

$$\begin{aligned} \chi_m &\in \Gamma[\Omega^0(\Sigma, \mathfrak{g}\otimes\mathbb{C})], \\ \varphi_m &\in \Gamma[\Omega^1(\Sigma, \mathfrak{g})]. \end{aligned} \tag{2.76}$$

Then (2.74) can be decomposed into

$$\prod_{m\in\mathbb{Z}} \frac{\det_\chi[-i\mathcal{L}_0 - \mathrm{ad}\left(\frac{2\pi\sigma}{\ell}\right) - \frac{i\beta}{\ell}]}{\det_\varphi[-i\mathcal{L}_0 - \mathrm{ad}\left(\frac{2\pi\sigma}{\ell}\right) - \frac{i\beta}{\ell}]} = \prod_\alpha\prod_{m\in\mathbb{Z}}\left[-\frac{2\pi m}{\ell} - \alpha\left(\frac{2\pi\sigma}{\ell}\right) - \frac{i\beta}{\ell}\right]^{\mathrm{Index}\,\bar{\partial}_A|_{(\alpha)}}. \tag{2.77}$$

Here α runs over all roots of $\mathfrak{g}$. From this expression, it is easy to see that ℓ only enters as a normalization factor, in agreement with the TQFT nature of the β-deformed complex Chern-Simons theory.

After ignoring a normalization factor that does not depend on the deformation parameter β, the functional determinant is

$$\Xi^{3\mathrm{d}} = \prod_\alpha \left\{(\alpha(2\pi\sigma)+i\beta)\prod_{m=1}^{+\infty}\left[(2\pi m)^2 - (\alpha(2\pi\sigma)+i\beta)^2\right]\right\}^{1-h-\alpha(n)}. \tag{2.78}$$

Here we also used the index theorem

$$\mathrm{Index}\,\bar{\partial}_A|_{(\alpha)} = 1 - h - \alpha(n), \tag{2.79}$$

with the last term being the degree of the line bundle labeled by α:

$$\alpha(n) = \frac{1}{2\pi}\int_\Sigma \alpha_a F^a. \tag{2.80}$$

The infinite product over m gives a sine function:

$$\begin{aligned}\Xi^{\mathrm{3d}} &= \prod_\alpha \left[(\alpha(2\pi\sigma)+i\beta)\prod_{m=1}^{+\infty}(2\pi m)^2\cdot\left(1-\frac{(\alpha(2\pi\sigma)+i\beta)^2}{(2\pi m)^2}\right)\right]^{1-h-\alpha(n)} \\ &\propto \prod_\alpha\left[2\sin\left(\alpha(\pi\sigma)+\frac{i\beta}{2}\right)\right]^{1-h-\alpha(n)} = \prod_\alpha \left|1-e^{2\pi i\alpha(\sigma)-\beta}\right|^{1-h-\alpha(n)}. \end{aligned} \tag{2.81}$$

Introducing $t = e^{-\beta}$, we decompose the contribution of abelian fields (product over zero roots in $\prod_\alpha$) from that of non-abelian fields (product over non-zero roots):

$$\Xi^{\mathrm{3d}} = \Xi^{\mathrm{3d}}_{\mathrm{ab}}\cdot\Xi^{\mathrm{3d}}_{\mathrm{nab}}, \tag{2.82}$$

where the abelian functional determinant, modulo a normalization factor[7], is given by

$$\Xi^{\mathrm{3d}}_{\mathrm{ab}} = \frac{1}{(1-t)^{N(h-1)}}, \tag{2.83}$$

while the non-abelian contribution is

$$\Xi^{\mathrm{3d}}_{\mathrm{nab}} = \left[\prod_{\alpha\neq 0} M_\alpha(\sigma,t)\right]^{1-h-\alpha(n)}, \tag{2.84}$$

with

$$M_\alpha(\sigma,t) = 1 - te^{2\pi i\alpha(\sigma)}. \tag{2.85}$$

The non-abelian contribution $\Xi^{\mathrm{3d}}_{\mathrm{nab}}$ can be further decomposed into

$$\Xi^{\mathrm{3d}}_{\mathrm{nab}} = \prod_{\alpha\neq 0}[M_\alpha(\sigma,t)]^{1-h}\cdot e^{-\frac{1}{2\pi}\int_\Sigma \alpha(F)\log M_\alpha}. \tag{2.86}$$

The part that depends on F can be combined with another term in (2.73):

$$i\int_\Sigma\left((k+N)\sigma_a - \sum_{b=1}^N \sigma_b + \frac{N-1}{2}\right)F^a \tag{2.87}$$

to give

$$i\int_\Sigma \zeta_a F^a, \tag{2.88}$$

[7]We did not keep track of the overall normalization constant, but it can be easily restored by demanding that $\beta\to+\infty$ gives back the usual Verlinde formula.

where

$$\zeta_a(\sigma) = k\sigma_a - \frac{i}{2\pi}\sum_{b\neq a}\log\left(\frac{e^{2\pi i\sigma_a} - te^{2\pi i\sigma_b}}{te^{2\pi i\sigma_a} - e^{2\pi i\sigma_b}}\right). \tag{2.89}$$

Performing a functional integral over A_a and over non-zero modes of λ_a in (2.73) gives a collection of delta-functions requiring ζ_a to be an integer:

$$\sum_{l_a\in\mathbb{Z}}\delta(\zeta_a - l_a). \tag{2.90}$$

Then we integrate over σ_a's. The delta-functions produce a factor

$$\sum_{\{\sigma\}\in\{\text{Bethe}\}}\det\left|\frac{\partial\zeta_a}{\partial\sigma_b}\right|^{-1}. \tag{2.91}$$

Here $\{\text{Bethe}\}$ stands for the set of solutions to the following Bethe ansatz equations:

$$e^{2\pi ik\sigma_a}\prod_{b\neq a}\left(\frac{e^{2\pi i\sigma_a} - te^{2\pi i\sigma_b}}{te^{2\pi i\sigma_a} - e^{2\pi i\sigma_b}}\right) = 1, \quad \text{for all of } a = 1, 2, \ldots, N. \tag{2.92}$$

The set of solutions to the Bethe ansatz equations is acted upon by the Weyl group, and after modding out by this symmetry, the solutions are labeled by Young tableaux with at most N rows and k columns. Notice that the Bethe ansatz equations are the same for all choices of the R-charge assignment to the adjoint chiral multiplet Φ.

Further integrating over the zero modes of λ_a gives a factor

$$\left|\frac{\partial\zeta_a}{\partial\sigma_b}\right|^{h}. \tag{2.93}$$

Therefore, the partition function is

$$Z^{\beta-\text{CS}}(\Sigma; U(N), k, t) =$$
$$\sum_{\{\sigma\}\in\{\text{Bethe}\}}\left[\frac{1}{(1-t)^N}\det\left|\frac{\partial\zeta_a}{\partial\sigma_b}\right|\prod_{a\neq b}\frac{1}{\left(e^{2\pi i\sigma_a} - te^{2\pi i\sigma_b}\right)\left(e^{2\pi i\sigma_a} - e^{2\pi i\sigma_b}\right)}\right]^{h-1}. \tag{2.94}$$

This "equivariant Verlinde formula" enjoys many interesting properties, some of which extend the remarkable properties of the ordinary Verlinde formula, *cf.* (2.1). In the next section, we present yet another derivation of this formula, from the two-dimensional point of view. Furthermore, we extend it to an entire family parametrized by the choice of the R-charge assignment of Φ and then make various comments about this general result.

2.6 A new family of 2d TQFTs

In the previous section, we have seen that twisted 3d $\mathcal{N} = 2$ theory $T[L(k,1);\beta]$ on $\Sigma \times S^1$ can be viewed as a one parameter deformation of complex Chern-Simons theory and it provides a natural way to regularize the latter theory. In fact, there is an entire family of twisted theories labeled by $R \in \mathbb{Z}$, the R-charge of the adjoint multiplet Φ in 3d $\mathcal{N} = 2$ theory $T[L(k,1);\beta]$.

In this section, we wish to study dimensional reduction of this family to two dimensions. In particular, we find a new family of 2d TQFTs labeled by $R \in \mathbb{Z}$ that generalize the G/G gauged WZW model and compute their partition functions on an arbitrary Riemann surface Σ. In certain special cases, we can compare our results to the previous literature.

Equivariant G/G gauged WZW model

We know from section 2.3 that the low-energy dynamics of $T[L(k,1);\beta]$ is given by a topological sigma-model to the vortex moduli space with a potential. In the limit $\beta \to +\infty$, the effective target space of the sigma-model becomes the Grassmannian and the topological sigma-model is equivalent to the G/G gauged WZW model. Our next goal is to give an equivariant generalization of the gauged WZW model, which we call the "equivariant G/G gauged WZW model."

The Lagrangian formulation of this theory can be directly obtained by dimensional reduction of the β-deformed complex Chern-Simons theory on S^1, but we won't follow this approach. Instead, we write down the Lagrangian formulation of the equivariant G/G gauged WZW model and then show that it leads to the same partition function on Σ as the β-deformed complex Chern-Simons theory on $\Sigma \times S^1$.

The fields in the ordinary, non-equivariant G/G model are (A, λ, g), where A is the gauge field, $g \in \mathcal{G} \cong \mathrm{Map}(\Sigma, G)$ is a group-valued field, and λ is an auxiliary Grassmann 1-form in the adjoint representation that is required to make the BRST symmetry manifest. The BRST charge Q_g depends on g and takes the following form [59]:

$$\begin{aligned} Q_g A &= \lambda, \\ Q_g \lambda^{(1,0)} &= (A^g)^{(1,0)} - A^{(1,0)}, \\ Q_g \lambda^{(0,1)} &= -\left(A^{g^{-1}}\right)^{(0,1)} + A^{(0,1)}, \end{aligned} \tag{2.95}$$

where

$$A^g = g^{-1} A g + g^{-1} dg. \tag{2.96}$$

At level k, the action of the G/G model is

$$kS_{G/G}(A,\lambda,g) = kS_G(A,g) - ik\Gamma(A,g) + \frac{i}{4\pi}\int_\Sigma \mathrm{Tr}\,(\lambda\wedge\lambda), \tag{2.97}$$

with the first term on the right-hand side being the kinetic term

$$S_G(g,A) = -\frac{1}{8\pi}\int_\Sigma \mathrm{Tr}\,(g^{-1}d_A g\wedge\star g^{-1}d_A g), \tag{2.98}$$

and the second term being the topological term

$$\Gamma(g,A) = \frac{1}{12\pi}\int_B \mathrm{Tr}\left[\left(g^{-1}dg\right)^3\right] - \frac{1}{4\pi}\int_\Sigma \mathrm{Tr}\left(Adgg^{-1} + AA^g\right). \tag{2.99}$$

Here, B is a handlebody with $\partial B = \Sigma$.

Now we add the chiral multiplet

$$\Phi = \varphi + \theta_\pm\psi^\pm + \theta^2 F, \tag{2.100}$$

and perform the topological twist. In order to do this, just like in three dimensions, we need to assign R-charge R to the superfield Φ under $U(1)_V$. The brane construction discussed in section 2.3 naturally leads to $R = 2$, but one can consider more general situations, where R is an arbitrary integer.

Identifying the diagonal subgroup of $U(1)_L \times U(1)_V$ with the twisted Lorentz group makes φ a section of $\Omega^0(\Sigma, K^{R/2})$, $\psi^\pm$ a section of $\Omega^0\left(\Sigma, K^{(R-1\pm1)/2}\right)$, and F a section of $H^0(\Sigma, K^{R/2-1})$, where K is the canonical bundle of the Riemann surface Σ. So, after the twist we end up with two BRST-multiplets that come from Φ:

$$\begin{aligned} (\varphi,\psi=\psi^+) \quad &\in \quad \Gamma\left[\Omega^0(\Sigma,K^{R/2})\right], \\ (\chi=\psi^-,\eta=F) \quad &\in \quad \Gamma\left[\Omega^0(\Sigma,K^{R/2-1})\right], \end{aligned} \tag{2.101}$$

along with their complex conjugate $(\varphi^\dagger,\psi^\dagger)$ and $(\chi^\dagger,\eta^\dagger)$ from $\Phi^\dagger$.

For $R = 2$, the fields (χ,η) are scalars while (φ,ψ) are $(1,0)$-forms, which indeed corresponds to the geometry of M5-branes wrapped on $\Sigma \subset T^*\Sigma$. Similarly, for $R = 0$, the fields (φ,ψ) are scalars, while (χ,η) are $(0,1)$-forms. This choice of the R-charge corresponds to the geometry of $\Sigma\times\mathbb{C}$. We come back to the detailed discussion of these two choices after describing the family of 2d TQFTs labeled by arbitrary (even) integer values of R.

At this stage, one can proceed in many different ways to study this family of TQFT's parametrized by R. For example, one can take a "top-down approach" by starting with the UV Lagrangian of the $\mathcal{N} = (2,2)$ SQCD with a massive adjoint chiral superfield Φ and study the resulting model after topological twist using localization.[8] However, since our goal is to generalize the gauged WZW model, we would like to have an explicit Lagrangian formulation that resembles the gauged WZW model. In fact, this is already partially achieved in the literature. As it turns out, for $R = 0$, the theory becomes the G/G gauged WZW-matter model that was introduced in [60]. Here, we generalize the approach of [60] to formulate an entire family of such theories with a general value of R. We shall refer to this new TQFT as the "equivariant G/G model."

The fields of the equivariant G/G model with general R are $(A, \lambda, \varphi, \psi, \eta, \chi, g)$, where A, φ, η, g are bosons and the rest are fermions. The BRST charge $Q_{(g,t)}$ acts on the fields in the following way:

$$\begin{aligned}
&Q_{(g,t)}A = \lambda, \quad Q_{(g,t)}\lambda^{(1,0)} = (A^g)^{(1,0)} - A^{(1,0)}, \quad Q_{(g,t)}\lambda^{(0,1)} = -\left(A^{g^{-1}}\right)^{(0,1)} + A^{(0,1)},\\
&Q_{(g,t)}\varphi = \psi, \qquad Q_{(g,t)}\psi = t\,(\varphi^g) - \varphi, \qquad Q_{(g,t)}\psi^\dagger = -t\left(\varphi^\dagger\right)^{g^{-1}} + \varphi^\dagger,\\
&Q_{(g,t)}\chi = \eta, \qquad Q_{(g,t)}\eta = t\chi^g - \chi, \qquad Q_{(g,t)}\eta^\dagger = -t\left(\chi^\dagger\right)^{g^{-1}} + \chi^\dagger,\\
&Q_{(g,t)}g = 0,
\end{aligned} \tag{2.102}$$

where

$$\begin{aligned}
A^g &= g^{-1}Ag + g^{-1}dg,\\
\varphi^g &= g^{-1}\varphi g,\\
\chi^g &= g^{-1}\chi g\,.
\end{aligned} \tag{2.103}$$

The action of $Q_{(g,t)}$ in (2.102) is almost exactly the same as in [60], except that spins of fields (2.101) depend on R. Also, notice that our conventions here slightly differ from [60] by $\eta \leftrightarrow \eta^\dagger$ and $\chi \leftrightarrow \chi^\dagger$.

The square of the BRST charge $Q^2_{(g,t)} = \mathcal{L}_{(g,t)}$ defines a bosonic transformation on the space of fields and the action of the theory needs to be invariant under it. In the gauged WZW-matter model, the action consists of the original action of the gauged WZW model and a $Q_{(g,t)}$-exact term,

$$S_{\mathrm{GWZWM}} = S_{\mathrm{GWZW}} + Q_{(g,t)}(S'), \tag{2.104}$$

and the theory does not depend on S' as long as the latter satisfies

$$\mathcal{L}_{(g,t)}S' = 0. \tag{2.105}$$

[8] An example of this theory, for $R = 2$, is the world-volume theory on D2-branes in figure 2.6.

The freedom of choosing different forms of S' can be used to localize the partition function. In the equivariant G/G model with general R, the action also takes the form (2.104):

$$S_{R\text{-EGWZW}} = S_{\text{GWZW}} + Q_{(g,t)}(S'), \tag{2.106}$$

with S' obeying

$$\mathcal{L}_{(g,t)} S' = 0. \tag{2.107}$$

There are different ways to explain why the BRST transformation and the Lagrangian take this particular form. For example, one can start with the Lagrangian and BRST transformation of the β-deformed complex Chern-Simons theory and compactify on a circle to directly derive the equivariant G/G model. Or, one can start with the UV theory (2.29) in 2d and analyze the IR limit following [29]. Here we will follow a simplified version of the latter approach to illustrate that (2.102) and (2.106) — which may seem a little strange at a first glance — are, in fact, what one should expect.

The Lagrangian of the UV theory (2.29) consists of two parts. The first part is $\mathcal{N} = (2,2)$ $U(N)$ SQCD with k fundamental chiral multiplets, which in the IR flows to the gauged WZW model. In the IR, the field g is identified with the scalar component σ of the vector multiplet:

$$g \sim \sigma. \tag{2.108}$$

In analyzing the low-energy fate of the second term, we can assume $g = 1$. Then, only the mass term remains, and we have

$$\begin{gathered} S_{R\text{-EGWZW}}(A, \lambda, \varphi, \psi, \eta, \chi, g = 1) = \\ k S_{\text{GWZW}}(A, \lambda, g = 1) + \int d^2 z \left(m^2 \varphi \varphi^\dagger + m \psi \psi^\dagger \right). \end{gathered} \tag{2.109}$$

Indeed, the above action is invariant under $Q_{(1,t)}$ and the second term can be written as

$$\int d^2 z \left(m^2 \varphi \varphi^\dagger + m \psi \psi^\dagger \right) = Q_{(1,t)} S' = \int d^2 z \left[\frac{m}{2} Q_{(1,t)} \left(\varphi \psi^\dagger - \psi \varphi^\dagger \right) \right] \tag{2.110}$$

if we set the IR mass to be

$$m = 1 - t. \tag{2.111}$$

It is easy to verify that

$$\mathcal{L}_{(1,t)} S' = 0. \tag{2.112}$$

This simplified situation with $g = 1$ tells us that the form of the BRST-transformation (2.102), which has no derivative terms, and the form of the action (2.106), where the extra fields only enter via BRST exact terms, are indeed expected.

Now we proceed to find the partition function of the equivariant G/G model with $G = U(N)$ and general R. As one would expect, this theory shares a lot of similarities with the gauged WZW-matter model that corresponds to $R = 0$ and the localization computation is very similar, except that the spin assignments of various fields can be different. So, instead of repeating everything in section 3 of [60], we only sketch the computation and point out how these two theories are different. First we modify S' to be symmetric in the two BRST-multiplets (φ, ψ) and (χ, η) (*cf.* equation (3.15) and (3.16) in [60][9]):

$$\begin{aligned} S_{\text{matter}}(g, A, \varphi, \psi, \eta, \chi) &= Q_{(g,t)} S' \\ &= Q_{(g,t)} \left[\frac{1}{4\pi} \int_\Sigma \operatorname{Tr} \left(\varphi\psi^\dagger - \psi\varphi^\dagger + \chi\eta^\dagger - \eta\chi^\dagger \right) \right] \qquad (2.113) \\ &= \frac{1}{2\pi} \int_\Sigma \left\{ (\varphi - t\varphi^g, \varphi) + (\psi, \psi) + (\chi - t\chi^g, \chi) + (\eta, \eta) \right\} . \end{aligned}$$

Here $(\cdot, \cdot)$ stands for the inner product and its definition for each field is clear from the context.

Now, following [61], we perform the abelianization and integrate out the off-diagonal components of g, A and λ. After abelianization, g belongs to the Cartan torus, generated by H^a, $a = 1, 2, \ldots, N$:

$$g = \exp\left(2\pi i \sum_{a=1}^{N} \sigma_a H^a \right), \qquad (2.114)$$

and the fields (A, λ, g) are replaced by the abelian fields $(A_a, \lambda_a, \sigma_a)$. Notice that the principal $U(1)^N$-bundle may be non-trivial; it is characterized by the flux $(n_1, \ldots, n_N)$,

$$n_a = \frac{1}{2\pi} \int_\Sigma F_a, \qquad (2.115)$$

and we need to sum over all flux sectors. The theory after abelianization is a BF-model with B valued in the Cartan torus, coupled to the rest of the fields $(\varphi, \psi, \chi, \eta)$. As all these matter fields have Gaussian action, they can be integrated out explicitly. We first decompose them into the Cartan-Weyl basis that diagonalizes the adjoint

[9] We believe there should be no factor of k multiplying S_{matter} as appears in [60].

action of $g = e^{2\pi i \sigma}$:

$$\varphi = \sum_{a=1}^{N} \varphi_a H^a + \sum_{\alpha} \varphi_\alpha E^\alpha, \tag{2.116}$$

$$\chi = \sum_{a=1}^{N} \chi_a H^a + \sum_{\alpha} \chi_\alpha E^\alpha, \tag{2.117}$$

where the α's are the roots of $\mathfrak{su}(N)$ and

$$\mathrm{Ad}_{e^{2\pi i \sigma}}(E^\alpha) = e^{2\pi i \alpha(\sigma)} E^\alpha. \tag{2.118}$$

Upon this decomposition, the trivial adjoint $\mathfrak{u}(N)$ bundle now splits into a direct sum of line bundles $\mathbb{C}^N \oplus \bigoplus_\alpha V_\alpha$ and the fields φ_α and χ_α take values in

$$\varphi_\alpha \in \Gamma\left[\Omega^0(\Sigma, K^{R/2} \otimes V_\alpha)\right], \tag{2.119}$$

$$\chi_\alpha \in \Gamma\left[\Omega^0(\Sigma, K^{R/2-1} \otimes V_\alpha)\right]. \tag{2.120}$$

Integrating out matter fields valued in the Cartan gives a functional determinant

$$\Xi^{\text{2d}}_{\text{ab}} = \prod_{a}^{N} \frac{\mathrm{Det}_\chi(1-t)}{\mathrm{Det}_\varphi(1-t)}, \tag{2.121}$$

while integrating out the matter fields valued in the V_α's will leave us with another functional determinant:

$$\Xi^{\text{2d}}_{\text{nab}} = \prod_{\alpha>0} \frac{\mathrm{Det}_\chi\left[M_\alpha(\sigma,t)\cdot M_{-\alpha}(\sigma,t)\right]}{\mathrm{Det}_\varphi\left[M_\alpha(\sigma,t)\cdot M_{-\alpha}(\sigma,t)\right]}, \tag{2.122}$$

where, as in section 2.5,

$$M_\alpha(\sigma,t) = 1 - t e^{2\pi i \alpha(\sigma)}. \tag{2.123}$$

Since χ is fermionic, the functional determinant associated to it appears in the numerator, while the bosonic determinant for the fields φ appears in the denominator. Up to this point, everything is independent of the R-charge assignment of the chiral multiplet Φ and, in fact, all dependence on the choice of R is encoded in this functional determinant.

As χ_α and φ_α both contain two degrees of freedom, the numerator and the denominator almost cancel. They don't cancel completely because the number of zero modes is different for these two fields. This difference can be computed using the Hirzebruch-Riemann-Roch theorem:

$$\begin{aligned}&\dim \Omega^0(\Sigma, K^{R/2-1} \otimes V_\alpha) - \dim \Omega^0(\Sigma, K^{R/2} \otimes V_\alpha) \\ &\qquad = 1 - h + (1 - R/2)(2h - 2) - \alpha(n) = -\chi(\Sigma) \cdot \frac{1-R}{2} - \alpha(n).\end{aligned}$$

Here h is the genus, $\chi(\Sigma) = 2 - 2h$, and the last term $\alpha(n)$ is the degree of the line bundle V_α, which can be written as an integral

$$\alpha(n) = \frac{1}{2\pi} \int_\Sigma \alpha(F) = \frac{1}{2\pi} \int_\Sigma \alpha_a F^a. \tag{2.124}$$

As a result, the first functional determinant is simply

$$\Xi_{\text{ab}}(R) = \prod_{i=1}^{N} (1-t)^{-\chi(\Sigma)\frac{1-R}{2}} = (1-t)^{N(h-1)(1-R)}, \tag{2.125}$$

and the second functional determinant becomes

$$\begin{aligned}\Xi_{\text{nab}}(R) = \prod_{\alpha>0} \frac{\mathrm{Det}_{(1,0)}\left[M_\alpha(\sigma,t) \cdot M_{-\alpha}(\sigma,t)\right]}{\mathrm{Det}_0\left[M_\alpha(\sigma,t) \cdot M_{-\alpha}(\sigma,t)\right]} &= \prod_\alpha M_\alpha(\sigma,t)^{(h-1)(1-R)-\alpha(n)} \\ &= \prod_\alpha \left[M_\alpha(\sigma,t)\right]^{(h-1)(1-R)} \cdot e^{-\frac{1}{2\pi}\int_\Sigma \alpha(F)\log M_\alpha}.\end{aligned}$$

The partition function for general R is now (*cf.* (3.29) in [60])

$$\begin{aligned}Z^R = [\Sigma; U(N), k, t] = \frac{1}{|W|} \int \mathcal{D}\sigma_a \mathcal{D}\lambda_a \mathcal{D}A_a \left[\prod_\alpha \left(1 - e^{2\pi i(\sigma_a - \sigma_b)}\right)^{1-h}\right] \Xi_{\text{ab}} \Xi_{\text{nab}} \\ \times \exp\left\{ i \int_\Sigma \left[\left((k+N)\sigma_a - \sum_{b=1}^{N} \sigma_b + \frac{N-1}{2} \right) F^a + \frac{k}{4\pi} \lambda_a \wedge \lambda_a \right] \right\}.\end{aligned} \tag{2.126}$$

The F^a-dependent part of Ξ_{nab} combines with other terms in the exponent that are proportional to F^a to give

$$\zeta_a(\sigma) = k\sigma_a - \frac{i}{2\pi} \sum_{b \neq a} \log\left(\frac{e^{2\pi i \sigma_a} - t e^{2\pi i \sigma_b}}{t e^{2\pi i \sigma_a} - e^{2\pi i \sigma_b}} \right). \tag{2.127}$$

Integrating over A_a and over non-zero modes of λ_a gives a collection of delta-functions requiring ζ_a to be integral:

$$\sum_{l_a \in \mathbb{Z}} \delta(\zeta_a - l_a). \tag{2.128}$$

Then we integrate over the σ_a's. The delta-functions will produce a factor of

$$\sum_{\{\sigma\}\in\{\text{Bethe}\}} \det\left|\frac{\partial\zeta_a}{\partial\sigma_b}\right|^{-1}. \tag{2.129}$$

Here {Bethe} stands for the set of solutions to the following Bethe ansatz equations:

$$e^{2\pi i k\sigma_a}\prod_{b\neq a}\left(\frac{e^{2\pi i\sigma_a}-te^{2\pi i\sigma_b}}{te^{2\pi i\sigma_a}-e^{2\pi i\sigma_b}}\right)=1, \quad \text{for all of } a=1,2,\dots,N. \tag{2.130}$$

The set of solutions to the Bethe ansatz equations is acted upon by the Weyl group, and after the quotient by this symmetry, the solutions are labeled by Young tableaux with at most N rows and k columns. Notice that the Bethe ansatz equations are the same for all choices of R-charge assignment.

Further integrating over the zero modes of λ_a gives a factor

$$\left|\frac{\partial\zeta_a}{\partial\sigma_b}\right|^{h}. \tag{2.131}$$

Therefore, the partition function is

$$\boxed{Z^R(\Sigma;U(N),k,t)=\sum_{\substack{\{\sigma\}\in\\ \{\text{Bethe}\}}}\left[(1-t)^{N(1-R)}\det\left|\frac{\partial\zeta_a}{\partial\sigma_b}\right|\prod_{a\neq b}\frac{\left(e^{2\pi i\sigma_a}-te^{2\pi i\sigma_b}\right)^{1-R}}{e^{2\pi i\sigma_a}-e^{2\pi i\sigma_b}}\right]^{h-1}} \tag{2.132}$$

This is the partition function of the equivariant G/G model with a general R-charge assignment. Now we proceed to discuss two important cases $R=2$ and $R=0$.

$R = 2$ and the equivariant Verlinde formula

As we emphasized earlier, the brane constructions in section 2.3 naturally lead to $R = 2$, which is the case that we are mostly interested in. The corresponding 2d TQFT is the equivariant G/G model whose partition function gives the equivariant Verlinde formula:

$$Z_{\text{EGWZW}}(\Sigma;U(N),k,t)=\sum_{\{\sigma\}\in\{\text{Bethe}\}}\left[\frac{1}{(1-t)^N}\det\left|\frac{\partial\zeta_a}{\partial\sigma_b}\right|\prod_{a\neq b}\frac{1}{\left(e^{2\pi i\sigma_a}-te^{2\pi i\sigma_b}\right)\left(e^{2\pi i\sigma_a}-e^{2\pi i\sigma_b}\right)}\right]^{h-1}. \tag{2.133}$$

It has several nice properties:

- For $t = 0$ ($\beta \to +\infty$), the "equivariant Verlinde formula" turns into the ordinary Verlinde formula, as one can directly verify.

- In the limit $t \to 1$ ($\beta \to 0$), the equivariant Verlinde formula diverges as

$$Z \sim (1-t)^{-(h-1)\cdot \dim(G)}. \tag{2.134}$$

  This is indeed what one would expect from the geometry of the Hitchin moduli space, that (up to higher codimension strata) looks like $T^*\mathcal{M}_{\text{flat}}$. Notice, the order of the pole in the above formula, $(h-1)\cdot \dim(G)$, is precisely the complex dimension of the cotangent fiber, whose non-compactness causes the divergence of the equivariant integral (2.55) in the limit $t \to 1$.

- The equivariant Verlinde formula should be a power series with integer coefficients, because it is defined as the graded dimension of the Hilbert space of complex Chern-Simons theory, *cf.* (2.4) and (2.20). This is indeed the case, as we will explicitly verify for $G = SU(2)$ in section 2.7, where a cutting and gluing approach is developed to calculate the same partition function from basic building blocks that only involve rational functions of t that can be written as power series with integer coefficients.

- In the limit $k \to +\infty$, with $k \cdot \beta$ fixed, the equivariant Verlinde formula turns into the formula for the equivariant volume of $\mathcal{M}_H$, or equivalently, the partition function of the topological Yang-Mills-Higgs model in [53].

To the best of our knowledge, the equivariant Verlinde formula associated with the choice $R = 2$ is novel. In [54] and [55], a model named "generalized G/G gauged WZW model" was proposed. Although it shares some similarities with the equivariant G/G model, the BRST-transformation rules, the Bethe ansatz equations and the partition function are all different. It would be interesting to see what the geometric interpretation of the generalized G/G model is, as well as to study its embedding into critical string theory as we have done in section 2.3.

For the other special value of $R = 0$ we get the G/G gauged WZW-matter model of Okuda and Yoshida, which did appeared in the mathematical literature, albeit in a completely different form (as we explain next).

$R = 0$ and gauged WZW-matter model

For $R = 2$, the field χ is a scalar and φ is a 1-form. When $R = 0$, their spin assignments are reversed, *cf.* (2.101). Therefore, the h-dependent parts of the functional determinants are simply inverted when one goes from one case to the other:

$$\Xi_{\text{ab}}(R=2) = \frac{1}{\Xi_{\text{ab}}(R=0)} = \frac{1}{(1-t)^{N(h-1)}}, \tag{2.135}$$

$$\Xi'_{\text{nab}}(R=2) = \frac{1}{\Xi'_{\text{nab}}(R=0)} = \left[\prod_{\alpha} M_{\alpha}(\sigma,t)\right]^{1-h}. \tag{2.136}$$

Here

$$\Xi'_{\text{nab}}(R) = \left[\prod_{\alpha} M_{\alpha}(\sigma,t)\right]^{(h-1)(R-1)} \tag{2.137}$$

is the part of Ξ_{nab} that does not depend on F_a. So, the partition function of the G/G gauged WZW-matter model is

$$Z_{\text{GWZWM}}(\Sigma; U(N), k, t) = \sum_{\{\sigma\}\in\{\text{Bethe}\}} \left[(1-t)^N \det\left|\frac{\partial \zeta_a}{\partial \sigma_b}\right| \prod_{a\neq b} \frac{e^{2\pi i\sigma_a} - te^{2\pi i\sigma_b}}{e^{2\pi i\sigma_a} - e^{2\pi i\sigma_b}}\right]^{h-1}. \tag{2.138}$$

It was verified numerically in [60] that, for small values of k, N and h, the G/G gauged WZW-matter model gives a 2d TQFT whose corresponding Frobenius algebra is the "deformed Verlinde algebra" constructed by Korff in [62]. Korff's construction is motivated by the q-boson model and uses the cylindric generalization of skew Macdonald functions.

In fact, the partition function of the gauged WZW-matter model appeared in the mathematical literature even earlier! It can be identified with an index formula for the moduli stack of algebraic $G_{\mathbb{C}}$-bundles over Σ first conjectured by Teleman [63] and later proved by Teleman and Woodward [64]. As we mentioned earlier, considering the index associated to the prequantum line bundle $\mathcal{L}$ over $\text{Bun}_{G_{\mathbb{C}}}(\Sigma)$ — which is basically $\mathcal{M}_{\text{flat}}(\Sigma; G)$ away from stacky points — gives the Verlinde formula. Telemann and Woodward then considered higher rank bundles over $\text{Bun}_{G_{\mathbb{C}}}(\Sigma)$. In particular, they considered the following bundle:

$$\lambda_t(T\mathcal{M}) \otimes \mathcal{L}^{\otimes k} \in K^0(\mathcal{M}, \mathbb{Q})[t], \tag{2.139}$$

where λ_t stands for the total lambda class, defined as follows. For a vector bundle V over space X, let $\lambda^l(V)$ be the K^0-class of $\Lambda^l V$, then

$$\lambda_t(V) = 1 + t\lambda^1(V) + t^2\lambda^2(V) + \ldots \in K^0(X, \mathbb{Q})[t]. \tag{2.140}$$

One can explicitly check that, at least for $G = U(N)$, the index of this bundle can be identified with the partition function of the gauged WZW-matter model, modulo a sign convention for the equivariant parameter:

$$t_{\text{TW}} = -t_{\text{here}}\,. \tag{2.141}$$

Relation to Bethe/Gauge correspondence

In [65–67], Nekrasov and Shatashvili proposed a relation between integrable models and supersymmetric gauge theories with four supercharges. In this chapter of the dissertation, we are concerned with two types of 3d $\mathcal{N} = 2$ theories: $T[L(k,1);\beta]$ and $T[\Sigma \times S^1;\beta]$. Of course, these two theories are special cases of $T[M_3;\beta]$, where M_3 is an arbitrary Seifert manifold.

The theory $T[\Sigma \times S^1;\beta]$, which was the subject of section 2.4, does not have a Chern-Simons term; it is a canonical mass deformation of 3d $\mathcal{N} = 4$ theory. The relation between theories of this type and integrable models was also explored in [68, 69]. Here, we shall focus on the lens space theory $T[L(k,1);\beta]$.

Although Okuda and Yoshida [60] found a relation between the gauged WZW-matter model and the q-boson model, the connection to SUSY gauge theory was missing. The results of our work fill this gap. In particular, according to our discussion in section 2.5, the gauged WZW-matter model is precisely 3d $\mathcal{N} = 2$ theory $T[L(k,1);\beta]$ twisted on $\Sigma \times S^1$. This kind of scenario was discussed by Nekrasov and Shatashvili in [70], and we now embed $T[L(k,1);\beta]$ into the framework of Bethe/gauge correspondence following their work.

From the matter content (2.7) of $T[L(k,1);\beta]$, one can easily write down the effective twisted superpotential:

$$\widetilde{\mathcal{W}}_{\text{eff}}(\sigma) = (k+N)\pi i \sum_{a=1}^{N} \sigma_a^2 - \pi i \left(\sum_{a=1}^{N} \sigma_a\right)^2 + \frac{1}{2\pi i}\sum_{a\neq b} \mathrm{Li}_2\left[t e^{2\pi i(\sigma_a - \sigma_b)}\right] + (N-1)\pi i \sum_{a=1}^{N} \sigma_a. \tag{2.142}$$

The first two terms come from the Chern-Simons term of the 3d $\mathcal{N} = 2$ vector multiplet. One can directly see that, after integrating out W-bosons, the levels for the $SU(N)$ and the $U(1)$ parts of the $U(N)$ gauge group are now $k+N$ and k, respectively. The third term in (2.142) comes from the adjoint chiral multiplet in 3d. And, unsurprisingly,

$$t = e^{-\ell\beta}, \tag{2.143}$$

where ℓ is the circumference of the S^1 Seifert fiber. The last term in (2.142) also originates from 3d gauge fields:

$$2\pi i\langle\rho,\sigma\rangle = \pi i\sum_{a>b}(\sigma_a-\sigma_b) \sim \pi i(N-1)\sum_{a=1}^{N}\sigma_a. \tag{2.144}$$

Here ρ is the Weyl vector, and, in the last step, we have used the fact that the shift

$$\widetilde{\mathcal{W}}_{\text{eff}} \longrightarrow \widetilde{\mathcal{W}}_{\text{eff}} + 2\pi i\sum_{a}^{N} n_a\sigma_a \tag{2.145}$$

generates a symmetry of the 2d abelian system.

The Bethe ansatz equations are given by

$$\exp\left[\frac{\partial\widetilde{W}_{\text{eff}}}{\partial\sigma_a}\right] = e^{2\pi i\zeta_a} = 1, \qquad \text{for all } a = 1, 2, \ldots, N. \tag{2.146}$$

The topological action is

$$\int_\Sigma \left[\frac{\partial\widetilde{W}_{\text{eff}}}{\partial\sigma_a}F_a + \frac{1}{4\pi i}\frac{\partial^2\widetilde{W}_{\text{eff}}}{\partial\sigma_a\partial\sigma_b}\lambda^a\wedge\lambda^b + \mathcal{U}(\sigma)\mathcal{R}\right], \tag{2.147}$$

where the last term involves the Euler density $\mathcal{R}$ and the dilaton coupling

$$\mathcal{U}(\sigma) = \mathcal{U}_{\text{gauge}}(\sigma) + \mathcal{U}_{\text{matter}}(\sigma), \tag{2.148}$$

such that

$$\mathcal{U}_{\text{gauge}}(\sigma) = \sum_{\alpha}\log\left(1-e^{\alpha}(\sigma)\right), \tag{2.149}$$

and

$$\mathcal{U}_{\text{matter}}(\sigma) = \left(\frac{R-1}{2}\right)\text{Tr}_{\text{adj}}\left[\log\left(1-te^{-\sigma}\right)\right]. \tag{2.150}$$

Here and throughout the paper, R is the $U(1)_V$ R-charge assigned to Φ and, in fact, this is the only place where R enters our formula. For the two choices of R-charge assignment discussed in sections 2.6 we have:

$$\mathcal{U}(\sigma)^{R=0}_{\text{matter}} \;=\; -\,\mathcal{U}(\sigma)^{R=2}_{\text{matter}}\,. \tag{2.151}$$

Then, the partition function of the topologically twisted theory is written as a sum over solutions to the Bethe ansatz equations:

$$Z^R\left(T[L(k,1);\beta];\Sigma\times S^1\right) = \sum_{\{\sigma\}\in\{\text{Bethe}\}}\left(e^{-\mathcal{U}^R(\sigma)}\det\left|\frac{1}{2\pi i}\frac{\partial^2\widetilde{W}_{\text{eff}}}{\partial\sigma_a\partial\sigma_b}\right|\right)^{g-1}. \tag{2.152}$$

One can check that this expression indeed agrees with the partition functions obtained previously. In particular, it gives the equivariant Verlinde formula for $R = 2$ and the partition function of the gauged WZW-matter model for $R = 0$.

Each summand in the partition function of the twisted SUSY gauge theory should be mapped to the squared norm of a Bethe state on the integrable model side. (Bethe states have a natural normalization and their norms are physical quantities.) As was checked in [60], the summands in the partition function of the gauged WZW-matter model indeed correspond to the squared norms of Bethe states of the q-boson model. Naturally, this raises a series of questions: What about the partition functions of topological theories with $R \neq 0$? What is their meaning on the integrable model side? Is $R = 0$ "special"?

It would be also interesting to study (quantum) spectral curves for 3d $\mathcal{N} = 2$ theories $T[L(k,1);\beta]$ and $T[M_3;\beta]$ following [68, sec. 5]. The spectral curves for these theories are expected to be spectral curves of integrable systems related to the ones discussed here by spectral duality. In particular, it should provide a candidate for the spectral duality of the q-boson model, and it would be interesting to make contact with [71].

2.7 t-deformation and categorification of the Verlinde algebra

In previous sections, we focused on the partition function of the β-deformed complex Chern-Simons theory on $\Sigma \times S^1$ — the equivariant Verlinde formula — and have shown that it can be derived in at least three different ways (the first is intrinsically three-dimensional and the other two are two-dimensional):

1. Section 2.5: Starting with the 3d $\mathcal{N} = 2$ theory $T[L(k,1);\beta]$ one can perform a topological twist on $\Sigma \times S^1$ and compute the partition function using localization *à la* [57].

2. Section 2.6: One can first reduce twisted $T[L(k,1);\beta]$ to 2d to obtain the equivariant G/G gauged WZW model on Σ and apply localization techniques and compute its partition function as in [60].

3. Also in section 2.6: One can first compactify $T[L(k,1);\beta]$ on a circle and obtain the low-energy effective $\mathcal{N} = (2,2)$ abelian gauge theory governed by the twisted effective superpotential as a function on the Coulomb branch. Then one can twist this 2d theory and compute its partition function following [70].

Naturally, the next step is to go beyond the partition function and incorporate operators. Indeed, one would expect loop operators to play very interesting role in complex Chern-Simons theory, just as they do in ordinary Chern-Simons theory. Recall that in Chern-Simons theory with compact gauge group G, Wilson loops are labeled by integrable representations of the loop group LG and their fusion rules give the Verlinde algebra, which basically describes how the tensor product of two representations decomposes. Then one can ask what the analog of this story in the β-deformed complex Chern-Simons theory is.

It turns out that a finite β simply deforms the Verlinde algebra to what we call the "equivariant Verlinde algebra." For example, the usual fusion rule for $G = SU(2)$ at level $k = 9$ for two fundamental representations

$$\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \tag{2.153}$$

is deformed into

$$\mathbf{2} \otimes \mathbf{2} = \frac{1}{1-t^2}\mathbf{1} \oplus \frac{1}{1-t}\mathbf{3} \oplus \frac{t}{1-t}\mathbf{5} \oplus \frac{t^2}{1-t}\mathbf{7} \oplus \frac{t^3}{1-t^2}\mathbf{9}\,. \tag{2.154}$$

Clearly, in the limit $t \to 0$ ($\beta \to \infty$) one recovers (2.153). Expressions like (2.154) are ubiquitous in computations of refined BPS invariants and categorification of quantum group invariants [72, 73]. In fact, just like in those examples, each coefficient on the right-hand side of (2.154) is a graded dimension (2.4) of an infinite-dimensional vector space V_j that appears as a "coefficient" in the OPE of line operators in 3d:

$$\mathbf{2} \otimes \mathbf{2} = \bigoplus_j V_j \otimes (\mathbf{2j} + \mathbf{1}). \tag{2.155}$$

In other words, as explained *e.g.* in [74, 75], replacing $\Sigma \times S^1$ by $\Sigma \times \mathbb{R}$ leads to a categorification in the sense that numerical coefficients are replaced by vector spaces (whose dimensions are the numerical coefficients). In the present case, we obtain a categorification of the equivariant Verlinde algebra since the "coefficients" in the OPE of line operators on $\Sigma \times \mathbb{R}$ are indeed vector space, namely V_j in our case. In the present example, (2.155) is a categorification of (2.154) with

$$\begin{aligned} V_0 &= \mathbb{C}[x_0]\{0\}, \\ V_1 &= \mathbb{C}[x_1]\{0\}, \\ V_2 &= \mathbb{C}[x_2]\{1\}, \\ V_3 &= \mathbb{C}[x_3]\{2\}, \\ V_4 &= \mathbb{C}[x_4]\{3\}, \end{aligned} \tag{2.156}$$

where $\dim_\beta(x_0) = \dim_\beta(x_4) = 2$, $\dim_\beta(x_1) = \dim_\beta(x_2) = \dim_\beta(x_3) = 1$, and $\{n\}$ denotes the degree shift by n units.

Of course, one can obtain a deformed algebra such as (2.154) by computing the partition function on $\Sigma \times S^1$ with insertion of multiple loop operators that lie along the S^1 fiber direction, using similar localization techniques as in previous sections. This problem will be studied more systematically elsewhere. In this section, we will analyze a simplified version of this problem with $G = SU(2)$ using a completely different method. Namely, we evaluate the equivariant integrals over Hitchin moduli space directly for some simple Riemann surfaces and build the TQFT using cutting and gluing.

"Equivariant Higgs vertex"

In order to perform cutting and gluing, it is important to generalize everything to punctured Riemann surfaces. We use $\Sigma_{h,n}$ to denote a Riemann surface with genus h and n ramification points $p_1, p_2, \ldots, p_n \in \Sigma$. Here we only consider "tame" ramification discussed in detail in [74, 75]. Near each puncture p_r, the ramification data is specified by a triple denoted as[10] $(\alpha_I, \alpha_J, \alpha_K) \in \mathbf{T}^3$, where $\mathbf{T} = U(1)$ is the Cartan torus of $G = SU(2)$. However, our approach only applies directly to cases where $\alpha_J = \alpha_K = 0$, as $U(1)_\beta$, which we use to regularize the non-compactness of the moduli space, acts on $\alpha_J + i\alpha_K$ by multiplying it with a phase. In order to make it invariant under $U(1)_\beta$, we need to impose the condition $\alpha_J = \alpha_K = 0$. In the following, we simply use α_r to denote α_I associated with the ramification point p_r.

Then the moduli space of ramified Higgs bundles $\mathcal{M}_H(\Sigma_{h,n}; \alpha_1, \alpha_2, \ldots, \alpha_n)$ can be identified with the moduli space of flat $SL(2,\mathbb{C})$ connections over $\Sigma_{h,n}$ with boundary condition that near a puncture p_r, only the real part A of the connection $\mathcal{A} = A + i\phi$ develops a singularity

$$A \sim \alpha_r d\theta. \tag{2.157}$$

Equivalently, we demand the holonomy around each puncture p_r to be in the same conjugacy class as

$$e^{2\pi i \alpha_r \sigma^3} = \exp\left[2\pi i \begin{pmatrix} \alpha_r & 0 \\ 0 & -\alpha_r \end{pmatrix}\right]. \tag{2.158}$$

[10] This triple is denoted as (α, β, γ) in [74, 75]. Here we use a different notation to avoid confusion with the equivariant parameter β.

The action of the affine Weyl group on α's leaves the conjugacy class of the monodromy invariant. So without loss of generality, we assume all α_r's to live in the Weyl alcove $[0, \frac{1}{2}]$.

As in the unramified case, we can consider the problem of quantizing the moduli space of ramified Higgs bundles, $\mathcal{M}_H(\Sigma_{h,n}; \alpha_1, \alpha_2, \ldots, \alpha_n)$, with symplectic form $k\omega_I$ and our goal is to identify a 2d TQFT whose partition function is the dimension of the Hilbert space $\mathcal{H}(\Sigma_{h,n}; \alpha_1, \alpha_2, \ldots, \alpha_n)$. This TQFT — which we call $SU(2)$ "equivariant Verlinde TQFT" — is equivalent to the equivariant G/G model of section 2.6 specialized to the choice of $G = SU(2)$, but formulated in a different way, via cutting and gluing.

Any 2d TQFT can be formulated in a set of Atiyah-Segal axioms, which assign a Hilbert space V to a circle S^1 and an element in $\mathrm{Hom}(V^{\otimes n}, \mathbb{C})$ to a punctured Riemann surface $\Sigma_{h,n}$. In particular, if $n = 0$, the TQFT assigns to a genus-h Riemann surface an element in $\mathrm{Hom}(\mathbb{C}, \mathbb{C})$. This element is determined by the image of $1 \in \mathbb{C}$, which is precisely the partition function in physicists' language.

Two-dimensional TQFTs are particularly simple, as any Riemann surface, punctured or not, can be cut along circles to be decomposed into three basic ingredients: the cap, the cylinder and pair of pants, *cf.* figure 2.1. One only needs to determine how the TQFT functor acts on the three basic building blocks. If we find a basis e_μ (or in physicists' notation $\{\langle\mu|\}$) of V, then the TQFT assigns "metric" $\eta^{\mu\nu}$ to a cylinder, "fusion coefficients" $f^{\mu\nu\rho}$ to a pair of pants, and a distinguished state $e_\emptyset \in V$ to a cap. This is summarized in table 2.2.

Topological invariance requires the "equivariant Higgs vertex" $f^{\mu\nu\rho}$ to be symmetric in the three indices. Also, as a four-holes sphere can be decomposed into two pairs of pants in different ways, the fusion coefficients have to satisfy the commutativity relation:

$$f^{\mu_1\nu_1\rho_1}\eta_{\rho_1\rho_2}f^{\mu_2\nu_2\rho_2} = f^{\mu_1\nu_2\rho_1}\eta_{\rho_1\rho_2}f^{\mu_2\nu_1\rho_2}. \tag{2.159}$$

Here $\eta_{\rho_1\rho_2} = \left(\eta^{-1}\right)^{\rho_1\rho_2}$ is the inverse metric naturally defined on $V^{*\otimes 2}$. Using these properties, it is easy to prove that a 2d TQFT is equivalent to a commutative Frobenius algebra. For the equivariant Verlinde TQFT, the corresponding algebra is the "equivariant Verlinde algebra", the one parameter generalization of the Verlinde algebra that we alluded to.

Before figuring out what $V, \eta^{\mu\nu}, f^{\mu\nu\rho}$ and $e_\emptyset$ are, we first see what the prediction from the equivariant gauged WZW model looks like. First of all, the dimension of

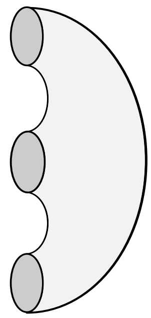
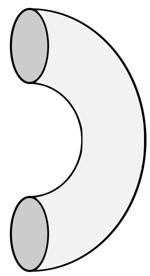
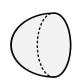
	$\rightsquigarrow$	$f^{\mu\nu\rho}$
	$\rightsquigarrow$	$\eta^{\mu\nu}$
	$\rightsquigarrow$	$e_{\emptyset}$

Table 2.2: Building blocks of a 2d TQFT.

V should be the number of solutions to the Bethe ansatz equations

$$\dim V = Z_{\text{EGWZW}}\left[\mathbb{T}^2; SU(2)\right] = \sum_{\{\text{Bethe}\}} 1. \tag{2.160}$$

The Bethe ansatz equations for $SU(2)$ can be obtained[11] by combining the two equations for $U(2)$,

$$e^{2\pi i k\sigma_1}\left(\frac{e^{2\pi i\sigma_1} - te^{2\pi i\sigma_2}}{te^{2\pi i\sigma_1} - e^{2\pi i\sigma_2}}\right) = 1, \tag{2.161}$$

$$e^{2\pi i k\sigma_2}\left(\frac{e^{2\pi i\sigma_2} - te^{2\pi i\sigma_1}}{te^{2\pi i\sigma_2} - e^{2\pi i\sigma_1}}\right) = 1, \tag{2.162}$$

into a single equation satisfied by

$$\sigma = \frac{1}{2}(\sigma_1 - \sigma_2) \in \left[0, \frac{1}{2}\right]. \tag{2.163}$$

So the Bethe ansatz equation for $SU(2)$ is simply

$$e^{4\pi i k\sigma}\left(\frac{e^{2\pi i\sigma} - te^{-2\pi i\sigma}}{te^{2\pi i\sigma} - e^{-2\pi i\sigma}}\right)^2 = 1. \tag{2.164}$$

[11]There are two ways of eliminating the $U(1)$ factor. Apart from the one described here, one can also set $\sigma_1 = -\sigma_2$. This corresponds to $U(2)/U(1) = SU(2)/\mathbb{Z}_2 = SO(3)$.

In the limit $\beta \to +\infty$ ($t \to 0$), the equivariant Verlinde TQFT becomes the ordinary Verlinde TQFT (*i.e.* G/G WZW model) and the Bethe ansatz equation becomes:

$$e^{4\pi i(k+2)\sigma} = 1. \tag{2.165}$$

There are $k+1$ solutions to this equation, namely:

$$\sigma_l = \frac{l+1}{2(k+2)}, \quad l = 0, 1, \ldots, k\,. \tag{2.166}$$

One can verify that this number of solutions is independent of β and will always be $k+1$. So, regardless of the value of β, the Hilbert space V of a 2d TQFT is always $k+1$-dimensional.

There is one subtle point that is worth mentioning. In the literature there is some confusion about the "end point contribution" to the Verlinde formula. Namely, $l = -1$ and $l = k+1$ also give valid solutions to the equation (2.165) and they indeed appear in localization computation (see *e.g.* [61]). However, their contribution is divergent if genus $h > 1$, and it was argued that they should be simply ignored. Our approach gives a different point of view on this issue. For any positive value of β, solutions σ_{-1} and σ_{k+1} are never inside the interval $[0, \frac{1}{2}]$ and, therefore, they never contribute to the equivariant Verlinde formula. When $\beta \to +\infty$, we have $\sigma_{-1} \to 0$ from the left and $\sigma_{k+1} \to \frac{1}{2}$ from the right. If we think of the ordinary Verlinde formula as the $\beta \to \infty$ limit of the equivariant Verlinde formula, then we should never include the contributions associated to σ_{-1} and σ_{k+1}. Similar phenomena happen when $\beta \to 0$. In that limit, σ_0 and σ_k move toward the endpoints of $[0, \frac{1}{2}]$. But as they will always be inside the interval, one should always include their full contributions.

The fact that V is finite dimensional is also expected from the geometry of the Hitchin moduli space. If there is no puncture, then $\mathcal{M}_H(\Sigma_{h,n}; SU(2))$ with symplectic form $k\omega_I$ is always quantizable. However, if we add punctures, $k\omega_I$ may not have integral periods over all 2-cycles of $\mathcal{M}_H(\Sigma_{h,n}; \alpha_1, \alpha_2, \ldots, \alpha_n)$, and this will be an obstruction to quantization. So, the α's need to satisfy certain integrality conditions that we now analyze.

In general, the moduli space of a ramified Higgs bundle can be conveniently viewed as a fibration of coadjoint orbits over the moduli space of unramified Higgs bundles. More concretely, in the case of $G_{\mathbb{C}} = SL(2,\mathbb{C})$, we have

$$\begin{array}{ccc} T^*\mathbb{CP}^1_{\alpha_1} \times \ldots \times T^*\mathbb{CP}^1_{\alpha_n} & \to & M_H(\Sigma_{h,n};\alpha_1,\alpha_2,\ldots,\alpha_n) \\ & & \downarrow \\ & & M_H(\Sigma_{h,n}), \end{array} \tag{2.167}$$

where $T^*\mathbb{CP}^1_{\alpha_r} = O_{\alpha_r}$ is the orbit of α_r in $\mathfrak{sl}(2,\mathbb{C})$ under adjoint action. Then integrality of the periods of $k\omega_I$ is translated to the following condition:

$$\int_{\mathbb{CP}^1_{\alpha_r}} k\omega_I = 2k\alpha_r \in \mathbb{Z}. \tag{2.168}$$

If we introduce

$$\lambda_r = 2k\alpha_r \in [0,k], \tag{2.169}$$

then there are $k+1$ possible values of λ_r for each puncture p_r, corresponding to $k+1$ states $\langle\lambda|$'s in V. And this indeed agrees with the prediction of the equivariant gauged WZW model. These states correspond to point-like defects on the Riemann surface, and from the three-dimensional point of view, these defects are Wilson loops along the S^1 fiber direction of $\Sigma \times S^1$.

Another prediction from physics is that the partition function of the equivariant Verlinde TQFT — or, equivalently, the value of the equivariant integral (2.55) over $\mathcal{M}_H$ — can be naturally written as a sum over solutions to the Bethe ansatz equation. A similar phenomenon was already pointed out back in [53], but it was never verified or properly understood. Next, we will construct the TQFT and see how the Bethe ansatz equation for $SU(2)$ naturally arises when one attempts to diagonalize the fusion rules.

Equivariant Verlinde algebra from Hitchin moduli space

In order to derive the "equivariant Higgs vertex" $f^{\lambda_1\lambda_2\lambda_3}$, we do the equivariant integration over the Hitchin moduli space, $\mathcal{M}_H(\Sigma_{0,3};\alpha_1,\alpha_2,\alpha_3)$, associated with the three-punctured sphere. The virtual dimension of this space is $2\times(3h-3+n) = 0$, so we expect it to be a collection of points which makes the equivariant integration very easy. We first consider the limit $\beta \to +\infty$. In this limit, the equivariant integral becomes an ordinary integral over the moduli space of $SU(2)$ connections and simply counts the number of points in $\mathcal{M} = \mathcal{M}_{\text{flat}}(\Sigma_{0,3}; SU(2),\alpha_1,\alpha_2,\alpha_3)$. In fact, this moduli space is either a point or empty. So the fusion coefficient $f^{\lambda_1\lambda_2\lambda_3}_{\beta\to+\infty}$ is either 1 or zero. One special thing about this zero-dimensional moduli space is that the quantizability condition is slightly more subtle, as the coadjoint orbits $\mathbb{CP}^1_{\alpha_i}$'s are no longer real 2-cycles. More precisely, in addition to requiring $(\alpha_1, \alpha_2, \alpha_3)$ to satisfy integrality condition

$$(\lambda_1, \lambda_2, \lambda_3) = 2k(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3, \tag{2.170}$$

one also needs to require $\lambda_1 + \lambda_2 + \lambda_3$ to be even. Then, the condition for $f^{\lambda_1\lambda_2\lambda_3}_{\beta\to+\infty}$ to be 1 is that $(\lambda_1, \lambda_2, \lambda_3)$ satisfies both the quantization condition and the "triangle inequality". We now explain the second condition more precisely, which is important for the equivariant generalization later.

When the quantization condition is satisfied, the triple $(\lambda_1, \lambda_2, \lambda_3)$ corresponds to an integer point in the cube $\{(x, y, z)|0 \leq x, y, z \leq k\}$. There is a tetrahedron inside this cube with four faces given by the following four equations:

$$\begin{aligned} d_0 &= \lambda_1 + \lambda_2 + \lambda_3 - 2k = 0, \\ d_1 &= \lambda_1 - \lambda_2 - \lambda_3 = 0, \\ d_2 &= \lambda_2 - \lambda_3 - \lambda_1 = 0, \\ d_3 &= \lambda_3 - \lambda_1 - \lambda_2 = 0. \end{aligned} \tag{2.171}$$

Define the distance of a point $(\lambda_1, \lambda_2, \lambda_3)$ to the tetrahedron faces as, see figure 2.7,

$$\Delta\lambda = \max(d_0, d_1, d_2, d_3). \tag{2.172}$$

We also define another quantity

$$\Delta\alpha = \frac{\Delta\lambda}{2k}. \tag{2.173}$$

If $\Delta\lambda \leq 0$, then the point is either inside the tetrahedron or on the boundary of it and $\mathcal{M}_{\text{flat}}(\Sigma_{0,3}; SU(2))$ is a point. If $\Delta\lambda > 0$, then the point is outside the tetrahedron and $\mathcal{M}_{\text{flat}}(\Sigma_{0,3}; SU(2))$ is empty. We call this condition "triangle inequality" for the following reason: when $d_1 > 0$ or $d_2 > 0$ or $d_3 > 0$, the three λ's won't be able to form a triangle. The situation $d_0 > 0$ corresponds to the case when the triangle is too large to live in $SU(2)$, which is a compact group.

Combining the quantization condition with the $\Delta\lambda \leq 0$ condition, we obtain the fusion coefficient in the $\beta \to +\infty$ limit:

$$f^{\lambda_1\lambda_2\lambda_3} = \begin{cases} 1 & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is even and } \Delta\lambda \leq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.174}$$

We now consider the case of finite β. The geometry of the relevant Hitchin moduli space $\mathcal{M}_H$ is described in detail in [76]. What differs from the $\beta \to +\infty$

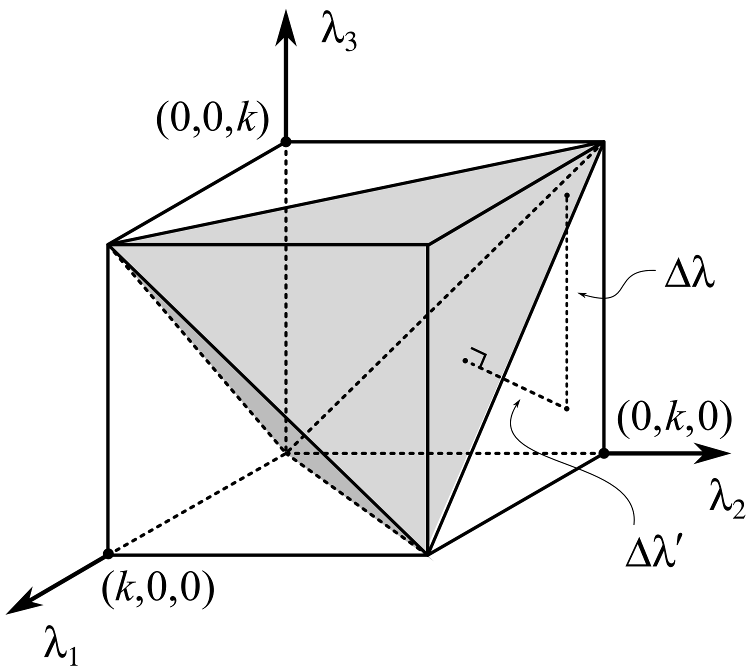


Figure 2.7: The “fusion tetrahedron” with definition of $\Delta\lambda$ and $\Delta\lambda' = \Delta\lambda/\sqrt{3}$.

case is that $\mathcal{M}_H(\Sigma_{0,3}; \alpha_1, \alpha_2, \alpha_3)$ is never empty and is always a point. This is consistent with a general property of moduli space of parabolic Higgs bundles: the topology only depends on the quasi-parabolic structure. Then, the equivariant integral

$$\int_{\mathcal{M}_H} \mathrm{Td}(\mathcal{M}_H, \beta) \wedge e^{k\widetilde{\omega}_I} \tag{2.175}$$

simply becomes

$$e^{-\beta k \mu_0} = e^{-\beta \Delta\lambda/2} = t^{\Delta\lambda/2}, \tag{2.176}$$

where

$$\mu_0 = \Delta\alpha \tag{2.177}$$

is the value of the moment map for $U(1)_\beta$ at that point [76]. So we have the fusion coefficients

$$f^{\lambda_1\lambda_2\lambda_3} = \begin{cases} 1 & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is even and } \Delta\lambda \leq 0, \\ e^{-\beta\Delta\lambda/2} & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is even and } \Delta\lambda > 0, \\ 0 & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is odd.} \end{cases} \tag{2.178}$$

The next thing one needs is the metric $\eta^{\mu\nu}$ associated to a cylinder. As the Hitchin moduli space $\mathcal{M}_H\,(\Sigma_{0,2}; SU(2))$ has negative virtual dimension, one needs to be careful when trying to make sense of the equivariant integral. Alternatively, one can deduce $\eta^{\mu\nu}$ from the equivariant Verlinde number associated with other

Riemann surfaces. For example, one can consider the four-holed sphere and do the integration over $\mathcal{M}_H\left(\Sigma_{0,4}; SU(2)\right)$. This moduli space is an elliptic surface with the elliptic fibration over $\mathbb{C}$, which is precisely the Hitchin fibration. The only singular fiber is the "nilpotent cone," the fiber over zero of the Hitchin base $\mathbb{C}$, and has Kodaira type I_0^* (or, affine D_4 in physicists' notation), see *e.g.* [75] for details. These nice properties make the equivariant integration easy to do. But instead of presenting the results of this computation, we directly give the form of the metric that is obtained by combining this result with the fusion coefficients:

$$\eta^{\lambda_1\lambda_2} = \mathrm{diag}\{1-t^2, \underbrace{1-t,\ 1-t,\ \ldots,\ 1-t,}_{k-1 \text{ entries that are all } (1-t)} 1-t^2\}. \tag{2.179}$$

Notice that because $\mathcal{M}_H\left(\Sigma_{0,2}; SU(2)\right)$ has virtual complex dimension -2, η has a first order zero when $t \to 1$, instead of a pole. Also, the metric is diagonal and only becomes the identity matrix when $t = 0$.

Once we know f and η, it is easy to find the state $\langle\emptyset|$ from the consistency of the gluing rules (attaching a cap to a pair of pants should give a cylinder):

$$f^{\mu\nu\emptyset} = \eta^{\mu\nu}. \tag{2.180}$$

And one finds

$$\langle\emptyset| = \langle 0| - t\langle 2|, \tag{2.181}$$

when $k \geq 2$. For $k = 1$ and $k = 0$, $\langle\emptyset| = \langle 0|$ and one can further verify that the Verlinde TQFT is not deformed by turning on β in these two cases.

Before proceeding further it is convenient to introduce a normalized basis $\{\langle\underline{\lambda}| = (\eta_{\lambda\lambda})^{1/2}\langle\lambda|\}$ in which the TQFT "metric" η is the identity. In this basis, the commutativity relation (2.159) becomes simply

$$f^{\lambda_1\underline{\mu\nu}} f^{\lambda_2\underline{\nu\xi}} \ = \ f^{\lambda_2\underline{\mu\nu}} f^{\lambda_1\underline{\nu\xi}}. \tag{2.182}$$

The above relation can be interpreted as the mutual commutativity of $k+1$ matrices $[f^0]$, $[f^1]$, ..., $[f^{k+1}]$, where

$$[f^{\mu}]^{\underline{\nu\rho}} = f^{\mu\underline{\nu\rho}}, \tag{2.183}$$

is a $(k+1)\times(k+1)$ matrix.

Now, that we have all the building blocks of the equivariant Verlinde TQFT, we can calculate any correlation function on any Riemann surface. However, the basis $\{\langle 0|, \langle 1|, \ldots \langle k|\}$, or its normalized version, is not the most convenient for this purpose. One would like to work in a different basis $\{\langle\widehat{0}|, \langle\widehat{1}|, \ldots, \langle\widehat{k}|\}$ where the fusion rules are diagonalized. Namely, in the normalized basis, the matrices $[f^0]$, $[f^1], \ldots, [f^k]$ are mutually commutative and simultaneously diagonalizable, with $\{\langle\widehat{0}|, \langle\widehat{1}|, \ldots, \langle\widehat{k}|\}$ being the set of eigenvectors. As the fusion coefficients $f^{\mu\nu\rho}$ are completely symmetric in the three indices, in the diagonal basis we have

$$f^{\widehat{\mu}\widehat{\nu}\widehat{\rho}} \sim \delta_{\widehat{\mu}\widehat{\nu}\widehat{\rho}}, \tag{2.184}$$

where δ_{abc} is the "3d Kronecker delta function" (equal to 1 when $a = b = c$ and zero otherwise).

Before attempting to find this new basis, we first briefly comment on its normalization. There are two possible choices: we can choose either

$$f^{\widehat{\mu}\widehat{\nu}\widehat{\rho}} = \delta_{\widehat{\mu}\widehat{\nu}\widehat{\rho}}, \tag{2.185}$$

or

$$\eta^{\widehat{\mu}\widehat{\nu}} = \delta^{\widehat{\mu}}_{\widehat{\nu}}. \tag{2.186}$$

If the first normalization is chosen, this basis is what mathematicians would call the "idempotent basis" of the equivariant Verlinde algebra and it coincides with the basis formed by "Bethe states." We will work with the second choice of normalization, where one does not need to distinguish between upper and lower indices.

Bethe Ansatz equation from the fusion rules

The standard way to find the eigenvectors of a set of commuting matrices is to first pick a linear combination of the matrices and to solve for the eigenvalues. At this point, one may (correctly) anticipate that the characteristic polynomial equation of a particular linear combination of $[f]$'s gives the Bethe ansatz equation. Indeed, this is true and that matrix is

$$[f_B] = [f_1] - t[f_3], \tag{2.187}$$

when $k \geq 3$. For $k = 2$, $[f_B] = [f_1]$ does not depend on β at all. As it turns out, β only appears in the normalization factor when $k = 2$, making this case uninteresting.

So from now on, we assume that $k \geq 3$. Written in the matrix form, f_B is

$$[f_B] = \begin{pmatrix} 0 & \sqrt{1+t} & 0 & 0 & 0 & \cdots & 0 \\ \sqrt{1+t} & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & & & \vdots \\ 0 & 0 & 1 & \ddots & \ddots & & 0 \\ 0 & 0 & & \ddots & 0 & 1 & 0 \\ \vdots & \vdots & & & 1 & 0 & \sqrt{1+t} \\ 0 & 0 & \cdots & 0 & 0 & \sqrt{1+t} & 0 \end{pmatrix}. \tag{2.188}$$

The characteristic polynomial equation for $[f_B]$ is

$$\det\left(x[I] - [f_B]\right) = 0, \tag{2.189}$$

where $[I]$ is the identity matrix of size $(k+1) \times (k+1)$. By expanding this determinant along the first and last columns, it is easy to find that

$$\det\left(x[I] - [f_B]\right) = x^2 A_{k-1} - 2x(1+t)A_{k-2} + (1+t)^2 A_{k-3}, \tag{2.190}$$

where A_n is a polynomial in x defined as the determinant of a $n \times n$ matrix

$$A_n = \det\begin{pmatrix} x & -1 & 0 & 0 & 0 & \cdots & 0 \\ -1 & x & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & x & -1 & & & \vdots \\ 0 & 0 & -1 & \ddots & \ddots & & 0 \\ 0 & 0 & & \ddots & x & -1 & 0 \\ \vdots & \vdots & & & -1 & x & -1 \\ 0 & 0 & \cdots & 0 & 0 & -1 & x \end{pmatrix}. \tag{2.191}$$

Using the initial condition $A_0 = 1$ and $A_1 = x$, along with the recursion relation

$$A_{n+1} = xA_n - A_{n-1} \tag{2.192}$$

that can be derived by expanding the determinant along the first column, one finds

$$A_n = \frac{\sin\left[2\pi(n+1)\sigma\right]}{\sin 2\pi\sigma}. \tag{2.193}$$

Here we made the following change of variables

$$x = 2\cos 2\pi\sigma. \tag{2.194}$$

Then, one finds the characteristic polynomial equation (2.189) to be

$$e^{4\pi ik\sigma}\left(\frac{e^{2\pi i\sigma}-te^{-2\pi i\sigma}}{te^{2\pi i\sigma}-e^{-2\pi i\sigma}}\right)^2 = 1\,. \tag{2.195}$$

This is exactly the Bethe ansatz equation (2.164) for the equivariant $SU(2)/SU(2)$ gauged WZW model!

For $0 < t < 1$, the equation (2.195) always has $k+1$ real solutions σ_l, $l = 0, 1, \ldots, k$ inside the interval $\left(0, \frac{1}{2}\right)$. So we can assume $\sigma_0 < \sigma_1 < \ldots\sigma_k$. As we mentioned previously, in the limit $t \to 0$, the Bethe ansatz equation (2.195) becomes

$$e^{4\pi i(k+2)\sigma} \;=\; 1\,. \tag{2.196}$$

And in the other limit $t \to 1$, the Bethe ansatz equation (2.195) becomes

$$e^{4\pi ik\sigma} \;=\; 1\,. \tag{2.197}$$

This agrees with the fact that the quantum shift of the level k in Chern-Simons theory with complex gauge group is zero [22].

There is another interesting property satisfied by the Bethe ansatz equation (2.195). If $\sigma \in (0, \frac{1}{2})$ is a solution to (2.195), then $\frac{1}{2} - \sigma$ is also a solution. So the $k+1$ roots $\{\sigma_l\}$ are naturally paired. As a consequence, if k is even, $\sigma_{k/2} = \frac{1}{4}$ is always a solution.

Now we have the eigenvalues $x_l = 2\cos 2\pi\sigma_l$ that are solutions to (2.195), and the next step is to find the eigenvectors $\langle\widehat{l}|$. In the normalized basis, they are $(k+1)\times 1$ matrices $[v_l]$ that can be obtained by solving the linear equation

$$[v_l] = x_l[v_l]. \tag{2.198}$$

It is easy to find

$$[v_l] = C_l\begin{pmatrix} \sqrt{1+t}\sin 2\pi\sigma_l \\ \sin 4\pi\sigma_l \\ \sin 6\pi\sigma_l - t\sin 2\pi\sigma_l \\ \sin 8\pi\sigma_l - t\sin 4\pi\sigma_l \\ \vdots \\ \sin 2k\pi\sigma_l - t\sin 2(k-2)\pi\sigma_l \\ \frac{\sin 2(k+1)\pi\sigma_l - t\sin 2(k-1)\pi\sigma_l}{\sqrt{1+t}} \end{pmatrix}. \tag{2.199}$$

Here C_l is a normalization factor

$$C_l^{-2} = \left|1 - te^{4\pi i\sigma_l}\right|^2 \cdot \frac{k+2}{2} + 2t\cos 4\pi\sigma_l - 2t^2 \,. \tag{2.200}$$

In the new basis, the fusion rules are:

$$f^{\widehat{\mu}\widehat{\nu}\widehat{\rho}} = N_{\widehat{\mu}}\delta_{\widehat{\mu}\widehat{\nu}\widehat{\rho}} \,. \tag{2.201}$$

Explicitly, the "eigenvalues of the fusion rules" N_l's are

$$N_l = \frac{1}{\sqrt{1-t} \cdot \sin 2\pi\sigma_l \, |1 - te^{4i\pi\sigma_l}|^2} \,. \tag{2.202}$$

In particular, from gluing $2h-2$ copies of pairs of pants (as in figure 2.1), it immediately follows that on a closed Riemann surface Σ_h the partition function is

$$\begin{aligned} Z(\Sigma_h; k, t) &= \sum_{l=0}^{k} N_l^{2h-2} \\ &= \frac{1}{(1-t)^{h-1}} \sum_{l=0}^{k} \left(\frac{k+2}{2} + \frac{2t\cos 4\pi\sigma_l - 2t^2}{|1 - te^{4\pi i\sigma_l}|^2}\right)^{h-1} \left(\frac{1}{\sin 2\pi\sigma_l \, |1 - te^{4\pi i\sigma_l}|}\right)^{2h-2} . \end{aligned}$$

We call this the "$SL(2,\mathbb{C})$ equivariant Verlinde formula". It is easy to check that for $t = 0$ it indeed reduces to the $SU(2)$ Verlinde formula. In the special case of $h = 0$, we have

$$Z(S^2; k, t) = \sum_{\widehat{l}=0}^{k} N_{\widehat{l}}^{-2} = \sum_{\widehat{l}=0}^{k} |\langle \underline{\widehat{l}} | \phi \rangle|^2 = \langle \phi | \phi \rangle = 1 - t^3. \tag{2.203}$$

For generic values of t, this formula gives a non-trivial identity satisfied by roots of the Bethe Ansatz equation.

To a n-punctured Riemann surface, the 2d TQFT functor assigns a vector in $(V^*)^{\otimes n}$:

$$Z(\Sigma_{h,n}; k, t) = \sum_{l=0}^{k} N_l^{2h-2+n} \langle \widehat{l} |^{\otimes n}. \tag{2.204}$$

$M_3 = L(p,1)$ and a "3d-3d appetizer"

3.1 Testing the 3d-3d correspondence

As was mentioned in the introduction, given a proposal for a theory $T[M_3]$ label by a particular three-manifold M_3, there is a set of "standard tests" to run. The the classical and perhaps the easiest test is to see whether the relation between moduli spaces is satisfied:

$$\mathcal{M}_{\mathrm{SUSY}}(T[M_3;G]) \simeq \mathcal{M}_{\mathrm{flat}}(M_3;G_{\mathbb{C}}). \tag{3.1}$$

If a theory passes (3.1), one can perform a quantum test to check whether partition functions match:

$$Z_{T[M_3;G]}[L(k,1)_b] = Z_{\mathrm{CS}}^{(k,\sigma)}[M_3;G_{\mathbb{C}}]. \tag{3.2}$$

The level of complex Chern-Simons theory has a real part k and an "imaginary part"[1] σ, and σ is related to the squashing parameter b of lens space $L(k,1)_b = S^3_b/\mathbb{Z}_k$ by [12, 13]:

$$\sigma = k \cdot \frac{1-b^2}{1+b^2}. \tag{3.3}$$

For $k=0$, $L(k,1) = S^1 \times S^2$, and (3.2) reduces to the relation between the index of $T[M_3]$ and the partition function of complex Chern-Simons theory at level $(0,\sigma)$ [9]

$$\mathrm{Index}_{T[M_3;G]}(q) = \mathrm{Tr}\,(-1)^F q^{\frac{E+j_3}{2}} = Z_{CS}^{(0,\sigma)}[M_3;G_{\mathbb{C}}]. \tag{3.4}$$

In Chapter 2, a candidate for the 3d theory $T[L(p,1)]$ was proposed and studied[2]:

$$T[L(p,1);G] \quad = \quad \boxed{\begin{array}{c}\text{3d } \mathcal{N}=2 \ G \text{ super-Chern-Simons theory at level } p \\ + \text{ adjoint chiral multiplet } \Phi\end{array}}\,. \tag{3.5}$$

This theory was used to produce Verlinde formula, the partition function of Chern-Simons theory on $S^1 \times \Sigma$, along with its "complexification" — the "equivariant Verlinde formula." Therefore, one may wonder whether this theory could also give the correct partition function of Chern-Simons theory on S^3,

$$Z_{\mathrm{CS}}[S^3;SU(2),k] = \sqrt{\frac{2}{k+2}}\sin\left(\frac{\pi}{k+2}\right) \tag{3.6}$$

[1] We use the quotation marks here because σ can be either purely imaginary or purely real as pointed out in [77].

[2] More precisely, this is the UV CFT that can flow to numerous different IR theories labelled by UV R-charges of Φ. The IR theory relevant for the 3d-3d relation is given by $R(\Phi)=2$.

and its complex analog,

$$Z_{\text{CS}}[S^3; SL(2,\mathbb{C}), \tau, \overline{\tau}] = \sqrt{\frac{4}{\tau\overline{\tau}}} \sin\left(\frac{2\pi}{\tau}\right) \sin\left(\frac{2\pi}{\overline{\tau}}\right). \tag{3.7}$$

Here we have used holomorphic and anti-holomorphic coupling constants

$$\tau = k + \sigma, \quad \overline{\tau} = k - \sigma. \tag{3.8}$$

Indeed, according to the general statement of the 3d-3d correspondence, $T[L(p,1)]$ needs to satisfy

$$Z_{T[L(p,1);G]}[L(k,1)_b] = Z_{\text{CS}}^{(k,\sigma)}[L(p,1); G_{\mathbb{C}}] \tag{3.9}$$

and

$$\text{Index}_{T[L(p,1);G]}(q) = \text{Tr}\,(-1)^F q^{\frac{E+j_3}{2}} = Z_{CS}^{(0,\sigma)}[L(p,1); G_{\mathbb{C}}]. \tag{3.10}$$

And if we take $p = 1$, the above relation states that the index of $T[S^3]$ should give the S^3 partition function of complex Chern-Simons theory. Even better, as there is a conjectured duality [78, 79] relating this theory to free chiral multiplets, one should be able to obtain (3.6) and (3.7) by simply computing the index of a free theory! This relation, summarized in diagrammatic form below,

$$\boxed{\begin{array}{c}\text{Chern-Simons}\\ \text{theory on } S^3\end{array}} \overset{\text{3d-3d}}{\longleftrightarrow} \boxed{\begin{array}{c}\text{Index of}\\ T[S^3]\end{array}} \overset{\text{duality}}{\longleftrightarrow} \boxed{\begin{array}{c}\text{free chiral}\\ \text{multiplets}\end{array}} \tag{3.11}$$

will be the subject of section 3.2. We start section 3.2 by proving the duality (at the level of superconformal index) in (3.11) for $G = U(N)$ and then "rediscover" the S^3 partition function of $U(N)$ Chern-Simons theory from the index of N free chiral multiplets. Then in section 3.3 we go beyond $p = 1$ and study theories $T[L(p,1)]$ with higher p. We check that the index of $T[L(p,1)]$ gives precisely the partition function of complex Chern-Simons theory on $L(p,1)$ at level $k = 0$. In addition, we discover that index of $T[L(p,1)]$ has some interesting properties. For example, when p is large,

$$\text{Index}_{T[L(p,1);U(N)]} = (2N-1)!! \tag{3.12}$$

is a constant that only depends on the choice of the gauge group. In the rest of section 3.3, we study $T[L(p,1)]$ on S_b^3 and use the 3d-3d correspondence to give predictions for the partition function of complex Chern-Simons theory on $L(p,1)$ at level $k = 1$.

3.2 Chern-Simons theory on S^3 and free chiral multiplets

According to the proposal (3.5), the theory $T[S^3]$ is $\mathcal{N}=2$ super-Chern-Simons theory at level $p=1$ with an adjoint chiral multiplet. If one takes the gauge group to be $SU(2)$, this theory was conjectured by Jafferis and Yin to be dual to a free $\mathcal{N}=2$ chiral multiplet [78]. The Jafferis-Yin duality has been generalized to higher rank groups by Kapustin, Kim and Park [79]. For $G=U(N)$, the statement of the duality is:

$$T[S^3]=\boxed{\begin{array}{c}U(N)_1\text{ super-Chern-Simons theory}\\ +\text{ adjoint chiral multiplet}\end{array}}\quad\overset{\text{duality}}{\longleftrightarrow}\quad\boxed{\begin{array}{c}N\text{ free chiral}\\ \text{multiplets}\end{array}}. \tag{3.13}$$

In Chapter 2, a similar duality was discovered[3]:

$$T[L(p,1)]=\boxed{\begin{array}{c}U(N)_p\text{ super-Chern-Simons theory}\\ +\text{ adjoint chiral multiplet}\end{array}}\quad\overset{\text{duality}}{\longleftrightarrow}\quad\boxed{\begin{array}{c}\text{sigma model to}\\ \text{vortex moduli space }\mathcal{V}_{N,p}\end{array}}. \tag{3.14}$$

Here,

$$\mathcal{V}_{N,p}\;\cong\;\left\{(q,\varphi)|\zeta\cdot\mathrm{Id}=qq^\dagger+[\varphi,\varphi^\dagger]\right\}/U(N), \tag{3.15}$$

with q being an $N\times p$ matrix, φ an $N\times N$ matrix and $\zeta\in\mathbb{R}^+$ the "size parameter," was conjectured to be the moduli space of N vortices in a $U(p)$ gauge theory [47]. For $p=1$, it is a well known fact that (see, *e.g.* [80])

$$\mathcal{V}_{N,1}\simeq\mathrm{Sym}^N(\mathbb{C})\simeq\mathbb{C}^N. \tag{3.16}$$

This is already very close to proving that $T[L(1,1);U(N)]=T[S^3;U(N)]$ is dual to N free chirals, with only one missing step. In order to completely specify the sigma model, one also needs to determine the metric on this space. A sigma model to $\mathbb{C}^N$ with the flat metric is indeed a free theory, but it is not obvious that the metric on $\mathcal{V}_{N,1}$ is flat[4]. However, as the superconformal index of a sigma model only depends on topological properties of the target space, one obtains that

$$\text{index of }T[S^3;U(N)]\;=\;\text{index of }N\text{ free chirals}, \tag{3.17}$$

proving the duality in (3.11) at the level of index. Combining (3.17) with the 3d-3d correspondence, one concludes that the index of free chirals equals the S^3 partition

[3] There, the adjoint chiral is usually assumed to be massive, which introduces an interesting "equivariant parameter" β. Here we are more concerned with the limit where that parameter is zero.

[4] $\mathcal{V}_{N,p}$ can be obtained using Kähler reduction from $\mathbb{C}^{N(N+p)}$ as in (3.15), and a Kähler metric is also inherited in this process. However, this metric on $\mathcal{V}_{N,p}$ is not protected from quantum corrections. The quantum metric is yet unknown to the best of our knowledge, but for the JY-KKP duality to be true, it should flow to a flat metric in the IR for $p=1$ — a somewhat surprising prediction.

functions of Chern-Simons theory. This is what we will explicitly demonstrate in this section.

Chern-Simons theory on the three-sphere

The partition function of $U(N)$ Chern-Simons theory on S^3 is

$$Z_{\mathrm{CS}}\left(S^3; U(N), k\right) = \frac{1}{(k+N)^{N/2}} \prod_{j=1}^{N-1} \left[\sin \frac{\pi j}{k+N}\right]^{N-j}. \tag{3.18}$$

For $N = 2$, this gives back (3.6) for $SU(2)$ (modulo a factor coming from the additional $U(1)$). It is convenient to introduce

$$q = e^{\frac{2\pi i}{k+N}}, \tag{3.19}$$

the variable commonly used for the Jones polynomial, and express (3.18) as (mostly) a polynomial in $q^{1/2}$ and $q^{-1/2}$:

$$Z_{\mathrm{CS}}\left(S^3; U(N), k\right) = C \cdot (\ln q)^{N/2} \prod_{j}^{N-1} \left[q^{j/2} - q^{-j/2}\right]^{N-j}. \tag{3.20}$$

Here C is a normalization factor that does not depend on q and such factors will be dropped in many later expressions without comment.

One can easily obtain the partition function for $GL(N, \mathbb{C})$ Chern-Simons theory by noticing that it factorizes into two copies of (3.18) at level $k_1 = \tau/2$ and $k_2 = \overline{\tau}/2$

$$Z_{\mathrm{CS}}\left(S^3; GL(N, \mathbb{C})\right) = (\ln q \ln \overline{q})^{N/2} \prod_{j=1}^{N-1} \left[q^{j/2} - q^{-j/2}\right]^{N-j} \left[\overline{q}^{-j/2} - \overline{q}^{j/2}\right]^{N-j}. \tag{3.21}$$

Here, in slightly abusive use of notation (*cf.* (3.19)),

$$q = e^{\frac{4\pi i}{\tau}}, \quad \overline{q} = e^{\frac{4\pi i}{\overline{\tau}}}. \tag{3.22}$$

Notice that the quantum shift of the level $k \to k + N$ in $U(N)$ Chern-Simons theory is absent in the complex theory [22, 77, 81]. Although (3.21) is almost a polynomial, it contains "$\ln q$" factors. So, at this stage, it is still somewhat mysterious how (3.21) can be obtained as the index of any supersymmetric field theory.

In (3.21) the level is arbitrary and the $k = 0$ case is naturally related to superconformal index of $T[S^3]$ (3.10). For $k = 0$,

$$q = e^{\frac{4\pi i}{\sigma}}, \quad \overline{q} = e^{-\frac{4\pi i}{\sigma}} = q^{-1}, \tag{3.23}$$

and

$$Z_{\text{CS}}^{(0,\sigma)}\left(S^3; GL(N,\mathbb{C})\right) = (\ln q)^N \prod_{j=1}^{N-1} \left[(1-q^j)(1-q^{-j})\right]^{N-j}. \tag{3.24}$$

This is the very expression that we want to reproduce from the index of free chiral multiplets.

Index of a free theory

The superconformal index of a 3d $\mathcal{N} = 2$ free chiral multiplet only receives contributions from the scalar component X, the fermionic component $\overline{\psi}$ and their ∂_+ derivatives. If we assume the R-charge of X to be r, then the R-charge of $\overline{\psi}$ is $1-r$ and the superconformal index of this free chiral is given by

$$\mathcal{I}_r(q) = \prod_{j=0}^{\infty} \frac{1-q^{1-r/2+j}}{1-q^{r/2+j}}. \tag{3.25}$$

In the j-th factor of the expression above, the numerator comes from fermionic field $\partial^j\overline{\psi}$ while the denominator comes from bosonic field $\partial^j X$. Here q is a fugacity variable that counts the charge under $\frac{E+j_3}{2} = R + j_3/2$ and it is the expectation of the 3d-3d correspondence [9] that this q is mapped to the "q" in (3.24), which justifies our usage of the same notation for two seemingly different variables. Now the only remaining problem is to decide what are the R-charges for the N free chiral multiplets.

The UV description of theory $T[L(p,1)]$ has an adjoint chiral multiplet Φ and in general one has the freedom of choosing the R-charge of Φ. Different choices give different IR fix points which form an interesting family of theories. As was argued in Chapter 2 using brane construction, the natural choice — namely the choice that one should use for the 3d-3d correspondence — is $R(\Phi) = 2$. For example, in order to obtain the Verlinde formula, it is necessary to choose $R(\Phi) = 2$ while other choices give closely related yet different formulae. As the N free chirals in the dual of $T[S^3; U(N)]$ are directly related to $\mathrm{Tr}\,\Phi$, $\mathrm{Tr}\,\Phi^2$, $\ldots$, $\mathrm{Tr}\,\Phi^N$, the choice of their R-charges should be

$$r_m = R(X_m) = 2m, \text{ for } m = 1, 2, \ldots, N. \tag{3.26}$$

The index for this assignment of R-charges — out of the unitarity bound — contains negative powers of q. However, this is not a problem at all because the UV R-charges are mixed with the $U(N)$ flavor symmetries, and q counts a combination of R- and flavor charges.

One interesting property of the index of a free chiral multiplet (3.25) is that it will vanish due to the numerator of the $(m-1)$-th factor:

$$1 - q^{m-r_m/2} = 0. \tag{3.27}$$

However, there is a very natural way of regularizing it and obtaining a finite result. Namely, we multiply the q-independent normalization coefficient $(r_m/2 - m)^{-1}$ to the whole expression and turn the vanishing term above into

$$\lim_{r_m \to 2m} \frac{1 - q^{m-r_m/2}}{r_m/2 - m} = \ln q. \tag{3.28}$$

And this is exactly how the "$\ln q$" factors on the Chern-Simons theory side arise. With this regularization

$$\mathcal{I}_{2m}(q) = \ln q \prod_{j=1}^{m-1} \left[\left(1 - q^{-j}\right) \left(1 - q^{j}\right) \right], \tag{3.29}$$

and the $2m-1$ factors come from the fermionic fields $\overline{\psi}_m, \partial\overline{\psi}_m, \ldots, \partial^{2m-2}\overline{\psi}_m$. The contribution of $\partial^{2m-1+l}\overline{\psi}_m$ will cancel with the bosonic field $\partial^l X$ as they have the same quantum number. The special log term comes from the field $\partial^{m-1}\overline{\psi}_m$, which has exactly $R + 2j_3 = 0$.

Then it is obvious that

$$\text{Index}_{T[S^3;U(N)]} = \prod_{m=1}^{N} \mathcal{I}_{2m}(q) = (\ln q)^N \prod_{j=1}^{N-1} \left[(1-q^j)(1-q^{-j}) \right]^{N-j} \tag{3.30}$$

is exactly the partition function of complex Chern-Simons theory on S^3 (3.24). For example, if $N = 1$,

$$\text{Index}_{T[S^3;U(1)]} = \mathcal{I}_2(q) = \ln q. \tag{3.31}$$

For $N = 2$,

$$\text{Index}_{T[S^3;U(2)]} = \mathcal{I}_2(q) \cdot \mathcal{I}_4(q) = (\ln q)^2 \, (1 - q^{-1})(1 - q). \tag{3.32}$$

To get the renowned S^3 partition function of the $SU(2)$ Chern-Simons theory, we just need to divide the $N = 2$ index by the $N = 1$ index and take the square root:

$$\sqrt{\frac{\text{Index}_{T[S^3;U(2)]}}{\text{Index}_{T[S^3;U(1)]}}} = \sqrt{\mathcal{I}_4(q)} = -i \cdot (\ln q)^{1/2} \left(q^{1/2} - q^{-1/2}\right). \tag{3.33}$$

For compact gauge group $SU(2)$, we substitute in

$$q = e^{\frac{2\pi i}{k+2}} \tag{3.34}$$

and up to an unimportant normalization factor, (3.33) is exactly

$$Z_{\text{CS}}(S^3; SU(2), k) = \sqrt{\frac{2}{k+2}} \sin \frac{\pi}{k+2}. \tag{3.35}$$

As almost anything in a free theory can be easily computed, one can go beyond index and check the following relation

$$Z_{N \text{ free chirals}}(L(k,1)_b) = Z_{CS}^{(k,\sigma)}(S^3; U(N)). \tag{3.36}$$

The left-hand side can be expressed as a product of double sine functions [82] and with the right choice of R-charges it becomes exactly the right-hand side, given by (3.18). As this computation is almost identical for what we did with index, we omit it here to avoid repetition.

Before ending this section, we comment on deforming the relation (3.11). In the formulation of $T[L(p,1)]$ in (3.5), there is a manifest $U(1)$ flavor symmetry that can be weakly gauged to give an "equivariant parameter" β. And the partition function of $T[L(p,1);\beta]$ should be related to the β-deformed complex Chern-Simons theory studied in Chapter 2:

$$Z_{T[L(p,1);\beta]}(L(k,1)) = Z_{\beta\text{-CS}}(L(p,1);k). \tag{3.37}$$

When $p = 1$, if the JY-KKP duality is true, this $U(1)$ flavor symmetry is expected to be enhanced to a $U(N)$ flavor symmetry of $T[S^3; U(N)]$ that is only visible in the dual description with N free chiral multiplets. Then one can deform $T[S^3]$ by adding N equivariant parameters $\beta_1, \beta_2, \ldots, \beta_N$. It is interesting to ask whether the Chern-Simons theory on S^3 naturally admits such an N-parameter deformation and whether one can have a more general matching.

$$\text{Index}_{T[S^3]}(q; \beta_1, \beta_2, \ldots, \beta_N) = Z_{\text{CS}}(S^3; q, \beta_1, \beta_2, \ldots, \beta_N). \tag{3.38}$$

As Chern-Simons theory on S^3 is dual to closed string on the resolved conifold [83, 84], it would also be interesting to understand whether similar deformation of the closed string amplitudes F_g exists.

3.3 3d-3d correspondence for lens spaces

In the previous section, we focused on $T[S^3]$ and found that it fits perfectly inside the 3d-3d correspondence. This theory is the special $p = 1$ limit of a general class (3.5) of theories $T[L(p, 1)]$ proposed in Chapter 2. In this chapter, we will test this proposal and see whether it stands well with various predictions of the 3d-3d correspondence. There are several tests to run on the proposed lens space theories (3.5). The most basic one is the correspondence between moduli spaces (3.1) that one can formulate classically without doing a path integral:

$$\mathcal{M}_{\text{SUSY}}\left(T[L(p, 1); U(N)]\right) \simeq \mathcal{M}_{\text{flat}}\left(L(p, 1); GL(N, \mathbb{C})\right). \tag{3.39}$$

And our first task in this section is to verify that this is indeed an equality.

$\mathcal{M}_{\text{SUSY}}$ vs. $\mathcal{M}_{\text{flat}}$

The moduli space of flat H-connections on a three manifold M_3 can be identified with the character variety:

$$\mathcal{M}_{\text{flat}}\left(M_3; H\right) \simeq \text{Hom}(\pi_1(M_3), H)/H. \tag{3.40}$$

As $\pi_1(L(p, 1)) = \mathbb{Z}_p$, this character variety is particularly simple. For example, if we take $H = U(N)$ or $H = GL(N, \mathbb{C})$ — the choice between $U(N)$ or $GL(N, \mathbb{C})$ does not even matter — this space is a collection of points labelled by Young tableaux with size smaller than $N \times p$. This is in perfect harmony with the other side of the 3d-3d relation where the supersymmetric vacua of $T[L(p, 1); U(N)]$ on $S^1 \times \mathbb{R}^2$ are also labeled by Young tableaux with the same constraint. We will now make this matching more explicit.

If we take the holonomy along the S^1 Hopf fiber of $L(p, 1)$ to be A, then

$$\mathcal{M}_{\text{flat}}\left(L(p, 1); GL(N, \mathbb{C})\right) \simeq \{A \in GL(N, \mathbb{C}) | A^p = \text{Id}\}/GL(N, \mathbb{C}). \tag{3.41}$$

First we can use the $GL(N, \mathbb{C})$ action to cast A into Jordan normal form. But in order to satisfy $A^k = \text{Id}$, A has to be diagonal, and each of its diagonal entries a_l has to be one of the p-th roots of unity:

$$a_l^p = 1, \text{ for all } l = 1, 2, \ldots, N. \tag{3.42}$$

One can readily identify this set of equations with the $t \to 1$ limit of the Bethe ansatz equations that determine the supersymmetric vacua of $T[L(p, 1); U(N)]$ on $S^1 \times \mathbb{R}^2$ derived in Chapter 2:

$$e^{2\pi i p \sigma_l} \prod_{m \neq l} \left(\frac{e^{2\pi i \sigma_l} - t e^{2\pi i \sigma_m}}{t e^{2\pi i \sigma_l} - e^{2\pi i \sigma_m}} \right) = 1, \quad \text{for all of } l = 1, 2, \ldots, N. \tag{3.43}$$

For $t = 1$, this equation is simply

$$e^{2\pi i p \sigma_l} = 1, \text{ for } l = 1, 2, \ldots, N. \tag{3.44}$$

And this is exactly (3.42) if one makes the following identification

$$a_l = e^{2\pi i \sigma_l}. \tag{3.45}$$

Of course this relation between a_l and σ_l is more than just a convenient choice. It can be derived using the brane construction of $T[L(p,1)]$. In fact, it just comes from the familiar relation in string theory between holonomy along a circle and positions of D-branes after T-duality. Indeed, in the above expression, the a_l's on the left-hand side label the $U(N)$-holonomy along the Hopf fiber, while the σ_l's on the right-hand side are coordinates on the Coulomb branch of $T[L(p,1)]$ after reduction to 2d, which exactly correspond to positions of N D2-branes.

$G_{\mathbb{C}}$ Chern-Simons theory from G Chern-Simons theory

The fact that $\mathcal{M}_{\text{flat}}$ is a collection of points is important for us to compute the partition function of complex Chern-Simons theory. Although there have been many works on complex Chern-Simons theory and its partition functions, starting from [77, 85] to perturbative invariant in [81, 86], state integral models in [12, 38, 87] and mathematically rigorous treatment in [88–90], what usually appear are certain subsectors of complex Chern-Simons theory, obtained from some consistent truncation of the full theory. In general, the *full* partition function of complex Chern-Simons theory is difficult to obtain, and requires proper normalization to make sense of. Some progress has been made toward understanding the full theory on Seifert manifolds in Chapter 2 using topologically twisted supersymmetric theories. However, if $\mathcal{M}_{\text{flat}}(M_3; G_{\mathbb{C}})$ is discrete and happens to be the same as $\mathcal{M}_{\text{flat}}(M_3; G)$, then one can attempt to construct the full partition function of the $G_{\mathbb{C}}$ Chern-Simons theory on M_3 from the G Chern-Simons theory. The procedure is the following. One first writes the partition function of the G Chern-Simons theory as a sum over flat connections:

$$Z^{\text{full}} = \sum_{\alpha \in \mathcal{M}} Z_\alpha. \tag{3.46}$$

And because the action of the $G_{\mathbb{C}}$ Chern-Simons theory

$$\begin{aligned} S =& \frac{\tau}{8\pi} \int \mathrm{Tr}\left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right) \\ &+ \frac{\overline{\tau}}{8\pi} \int \mathrm{Tr}\left(\overline{\mathcal{A}} \wedge d\overline{\mathcal{A}} + \frac{2}{3}\overline{\mathcal{A}} \wedge \overline{\mathcal{A}} \wedge \overline{\mathcal{A}}\right) \end{aligned} \tag{3.47}$$

is simply two copies of the G Chern-Simons theory action at level $k_1 = \tau/2$ and $k_2 = \overline{\tau}/2$, one would have

$$Z_\alpha(G_{\mathbb{C}}; \tau, \overline{\tau}) = Z_\alpha\left(G; \frac{\tau}{2}\right) Z_\alpha\left(G; \frac{\overline{\tau}}{2}\right), \tag{3.48}$$

if $\mathcal{A}$ and $\overline{\mathcal{A}}$ were independent fields. So, one would naively expect

$$Z^{\text{full}}(G_{\mathbb{C}}; \tau, \overline{\tau}) = \sum_{\alpha \in \mathcal{M}} Z_\alpha\left(G; \frac{\tau}{2}\right) Z_\alpha\left(G; \frac{\overline{\tau}}{2}\right). \tag{3.49}$$

But as $\mathcal{A}$ and $\overline{\mathcal{A}}$ are not truly independent, (3.49) is in general incorrect and one needs to modify it in a number of ways. For example, as mentioned before, the quantum shift of the level τ and $\overline{\tau}$ in $G_{\mathbb{C}}$ Chern-Simons theory is zero, so for $Z_\alpha(G)$ on the right-hand side, one needs to at least remove the quantum shift $k \to k + \check{h}$ in G Chern-Simons theory, where $\check{h}$ is the dual Coxeter number of $\mathfrak{g}$. There may be other effects that lead to relative coefficients between contributions from different flat connections α and the best one could hope for is

$$Z^{\text{full}}(G_{\mathbb{C}}; \tau, \overline{\tau}) = \sum_{\alpha \in \mathcal{M}} e^{iC_\alpha} Z'_\alpha\left(G; \frac{\tau}{2}\right) Z'_\alpha\left(G; \frac{\overline{\tau}}{2}\right), \tag{3.50}$$

where

$$Z'_\alpha\left(G; \frac{\tau}{2}\right) = Z_\alpha\left(G; \frac{\tau}{2} - \check{h}\right). \tag{3.51}$$

One way to see that (3.49) is very tenuous, even after taking care of the level shift, is by noticing that the left-hand side and the right-hand side behave differently under a change of framing. If the framing of the three-manifold is changed by s units, the left-hand side will pick up a phase factor

$$\exp\left[\varphi^{\text{fr.}}_{\mathbb{C}} \cdot s\right] = \exp\left[\frac{\pi i (c_L - c_R)}{12} \cdot s\right]. \tag{3.52}$$

Here c_L and c_R are the left- and right-moving central charges of the hypothetical conformal field theory that lives on the boundary of the complex Chern-Simons theory [77]:

$$(c_L, c_R) = \dim G \cdot \left(1 - \frac{2\check{h}}{\tau}, 1 + \frac{2\check{h}}{\overline{\tau}}\right). \tag{3.53}$$

The right-hand side of (3.49) consists of two copies of the Chern-Simons theory with compact gauge group G, so the phase from change of framing is

$$\exp\left[\varphi^{\text{fr.}} \cdot s\right] = \exp\left[\frac{\pi i}{12}\left(\frac{\tau/2 - \check{h}}{\tau/2} + \frac{\overline{\tau}/2 - \check{h}}{\overline{\tau}/2}\right) \dim G \cdot s\right]. \tag{3.54}$$

The two phases are in general different

$$\varphi^{\text{fr.}}_{\mathbb{C}} - \varphi^{\text{fr.}} = \frac{2\pi i \dim G}{12}. \tag{3.55}$$

So (3.49) has no chance of being correct at all and the minimal way of improving it is to add the phases, C_α, as in (3.50), which also transform under change of framing.

It may appear that the expression (3.50) is not useful unless one can find the values of the C_α's. However, as it turns out, for $k = 0$ (or equivalently $\tau = -\overline{\tau}$), all of the C_α's are constant, and (3.50) without the C_α's gives the correct partition function[5]. This may be closely related to the fact that for $k = 0$,

$$c_L - c_R = -2\check{h} \dim G \left(\frac{1}{\tau} + \frac{1}{\overline{\tau}}\right) = 0. \tag{3.56}$$

Superconformal index

We have shown that the proposal (3.5) for $T[L(p,1)]$ gives the right supersymmetric vacua and we shall now move to the quantum level and check the relation between the partition functions:

$$\text{Index}_{T[L(p,1);U(N)]}(q) = Z_{CS}\left(L(p,1); GL(N,\mathbb{C}), q\right). \tag{3.57}$$

We have already verified this for $p = 1$ in the previous section. Now we consider the more general case with $p \geq 1$.

The superconformal index of a 3d $\mathcal{N} = 2$ SCFT is given by [91]

$$\mathcal{I}(q, t_i) = \text{Tr}\left[(-1)^F e^{-\gamma(E-R-j_3)} q^{\frac{E+j_3}{2}} t^{f_i}\right]. \tag{3.58}$$

Here, the trace is taken over the Hilbert space of the theory on $\mathbb{R} \times S^2$. Because of supersymmetry, only BPS states with

$$E - R - j_3 = 0 \tag{3.59}$$

will contribute. As a consequence, the index is independent of γ and only depends on q and the flavor fugacities, t_i. For $T[L(p,1)]$, there is always a $U(1)$ flavor symmetry and we can introduce at least one parameter t. When this parameter is turned on, on the other side of the 3d-3d correspondence, complex Chern-Simons theory will become the "deformed complex Chern-Simons theory". This deformed version of Chern-Simons theory was studied on geometry $\Sigma \times S^1$ in Chapter 2 and

[5] "Correct" in the sense that it matches the index of $T[L(p,1)]$.

will be studied on more general Seifert manifolds in the last chapter and future publications. However, because in this chapter our goal is to *test* the 3d-3d relation (as opposed to using it to study the deformed Chern-Simons theory), we will usually turn off this parameter by setting $t = 1$, and compare the index $\mathcal{I}(q)$ with the partition function of the *undeformed* Chern-Simons theory, which is only a function of q, as in (3.24).

Viewing the index as the partition function on $S^1 \times_q S^2$ and using localization, (3.58) can be expressed as an integral over the Cartan $\mathbb{T}$ of the gauge group G [92]:

$$\mathcal{I} = \frac{1}{|\mathcal{W}|} \sum_m \int \prod_j \frac{dz_j}{2\pi i z_j} e^{-S_{\mathrm{CS}}(m)} q^{\epsilon_0/2} e^{ib_0(h)} t^{f_0} \exp\left[\sum_{n=1}^{+\infty} \frac{1}{n} \operatorname{Ind}(z_j^n, m_j; t^n, q^n)\right]. \tag{3.60}$$

Here $h, m \in \mathfrak{t}$ are valued in the Cartan subalgebra. Physically, e^{ih} is the holonomy along S^1 and is parametrized by z_i, which are coordinates on $\mathbb{T}$.

$$m = \frac{i}{2\pi} \int_{S^2} F \tag{3.61}$$

is the monopole number on S^2 and takes value in the weight lattice of the Langlands dual group ${}^L G$. $|\mathcal{W}|$ is the order of the Weyl group and the other quantities are:

$$\begin{aligned} b_0(h) &= -\frac{1}{2} \sum_{\rho \in \mathfrak{R}_\Phi} |\rho(m)|\, \rho(h), \\ f_0 &= -\frac{1}{2} \sum_{\rho \in \mathfrak{R}_\Phi} |\rho(m)|\, f, \\ \epsilon_0 &= \frac{1}{2} \sum_{\rho \in \mathfrak{R}_\Phi} (1-r)\, |\rho(m)| - \frac{1}{2} \sum_{\alpha \in \mathrm{ad}(G)} |\alpha(m)|, \\ S_{CS} &= ip \operatorname{tr}(mh), \end{aligned} \tag{3.62}$$

and

$$\begin{aligned} \operatorname{Ind}(e^{ih_j} = z_j, m_j; t; q) = &- \sum_{\alpha \in \mathrm{ad}(G)} e^{i\alpha(h)} q^{|\alpha(m)|} \\ &+ \sum_{\rho \in \mathfrak{R}_\Phi} \left[e^{i\rho(h)} t \frac{q^{|\rho(m)|/2 + r/2}}{1-q} - e^{-i\rho(h)} t^{-1} \frac{q^{|\rho(m)|/2 + 1 - r/2}}{1-q} \right] \end{aligned} \tag{3.63}$$

is the “single particle" index. $\mathfrak{R}_\Phi$ is the gauge group representation for all matter fields. Using this general expression, the index of $T[L(p,1); U(N)]$ can be expressed in the following form:

$$\mathcal{I}(q,t) = \sum_{m_1\geqslant\cdots\geqslant m_N\in\mathbb{Z}} \frac{1}{|\mathcal{W}_m|} \int \prod_j \frac{dz_j}{2\pi i z_j} \prod_i^N (z_i)^{2pm_i} \prod_{i\neq j}^N t^{-|m_i-m_j|/2} q^{-R|m_i-m_j|/4}$$
$$\left(1-q^{|m_i-m_j|/2}\frac{z_i}{z_j}\right) \prod_{i\neq j}^N \frac{\left(\frac{z_j}{z_i}t^{-1}q^{|m_i-m_j|/2+1-R/2};q\right)_\infty}{\left(\frac{z_i}{z_j}tq^{|m_i-m_j|/2+R/2};q\right)_\infty} \times \left[\frac{(t^{-1}q^{1-R/2};q)_\infty}{(tq^{R/2};q)_\infty}\right]^N . \tag{3.64}$$

Here we used the q-Pochhammer symbol $(z;q)_n = \prod_{j=0}^{n-1}(1-zq^j)$. $\mathcal{W}_m \subset \mathcal{W}$ is the stabilizer subgroup of the Weyl group that fixes $m \in \mathfrak{t}$ and R stands for the R-charge of the adjoint chiral multiplet and will be set to $R = 2$ — the choice that gives the correct IR theory.

In the previous section, we have found the index for $T[S^3]$ to be exactly equal to the S^3 partition function of Chern-Simons theory. There, we used an entirely different method by working with the dual description of $T[L(p,1);U(N)]$, which is a sigma model to the vortex moduli space $\mathcal{V}_{N,p}$. For $p = 1$, this moduli space is topologically $\mathbb{C}^N$ and the index of the sigma model is just that of a free theory. For $p \geq 2$, such a simplification will not occur and the index of the sigma model is much harder to compute[6]. In contrast, the integral expression (3.64) is *easier* to compute with larger p than with $p = 1$, because fewer topological sectors labelled by the monopole number m contribute. As we will see later, when p is sufficiently large, only the sector $m = (0,0,\ldots,0)$ gives non-vanishing contribution. So the two approaches of computing the index have their individual strengths and are complementary to each other.

Now, one can readily compute the index for any $T[L(p,1);G]$ and then compare $\mathcal{I}(q,t=1)$ with the partition function of the complex Chern-Simons theory on $L(p,1)$. We will first do a simple example with $G = SU(2)$, to illustrate some general features of the index computation.

[6] In general, it can be written as an integral of a characteristic class over $\mathcal{V}_{N,p}$ that one can evaluate using the Atiyah-Bott localization formula. Similar computations were done in two dimensions in, *e.g.* [6] and [93].

Index of $T[L(p,1);SU(2)]$

We will start with $p = 1$ and see how the answer from section 3.2 arises from the integral expression (3.64). In this case, (3.64) becomes

$$
\begin{aligned}
\mathcal{I} &= \sum_{m\in\mathbb{Z}} \int \frac{dz}{4\pi i z} e^{ihm} q^{-2|m|} \left(1 - q^{|m|} e^{ih}\right)^2 \left(1 - q^{|m|} e^{-ih}\right)^2 \prod_{k=0}^{+\infty} \frac{1 - q^{k+1-R/2}}{1 - q^{k+R/2}} \\
&= \sum_{m\in\mathbb{Z}} \int \frac{dz}{4\pi i z} z^m q^{-2|m|} \left(1 + q^{2|m|} - z q^{|m|} - z^{-1} q^{|m|}\right)^2 \left[(R-2)\ln q\right] \\
&= \sum_{m\in\mathbb{Z}} \int \frac{dz}{4\pi i z} z^m \left(q^{2|m|} + q^{-2|m|} + 4 - 2\left(z + \frac{1}{z}\right)\left(q^{|m|} + \frac{1}{q^{|m|}}\right) + \left(z^2 + \frac{1}{z^2}\right)\right) \\
&\qquad \times \left[(R/2 - 1)\ln q\right].
\end{aligned} \tag{3.65}
$$

As in section 3.2, the index will be zero if we naively take $R = 2$ because of the $1 - q^{1-r/2}$ factor in the infinite product. When $R \to 2$, the zero factor becomes

$$
1 - q^{1-R/2} = 1 - \exp\left[(1 - R/2)\ln q\right] \approx (R/2 - 1)\ln q. \tag{3.66}
$$

As in section 3.2, we can introduce a normalization factor $(R/2-1)^{-1}$ in the index to cancel the zero, making the index expression finite.

The integral in (3.65) is very easy to do and the index receives contributions from three different monopole number sectors

$$
\mathcal{I} = \frac{1}{2}\ln q\ (\mathcal{I}_{m=0} + \mathcal{I}_{m=\pm1} + \mathcal{I}_{m=\pm2}), \tag{3.67}
$$

with

$$
\mathcal{I}_{m=0} = \int \frac{dz}{2\pi i z}\left(q^0 + q^{-0} + 4\right) = 6, \tag{3.68}
$$

$$
\mathcal{I}_{m=\pm1} = -2\sum_{m=\pm1} \int \frac{dz}{2\pi i z} z^m \left(q^{|m|} + q^{-|m|}\right)\left(z + \frac{1}{z}\right) = -4(q + q^{-1}), \tag{3.69}
$$

and

$$
\mathcal{I}_{m=\pm2} = \sum_{m=\pm2} \int \frac{dz}{2\pi i z} z^m \left(z^2 + \frac{1}{z^2}\right) = 2. \tag{3.70}
$$

So the index is

$$
\begin{aligned}
\mathcal{I} &= \frac{1}{2}\ln q\left(6 - 4(q + q^{-1}) + 2\right) \\
&= -2\ln q\left(q^{1/2} - q^{-1/2}\right)^2.
\end{aligned} \tag{3.71}
$$

Modulo a normalization constant, this is in perfect agreement with results in section 3.2. Indeed, the square root of (3.71) is identical to (3.33) and reproduces the

S^3 partition function of the $SU(2)$ Chern-Simons theory,

$$Z_{CS}(S^3; SU(2), k) = \sqrt{\frac{2}{k+2}} \sin \frac{\pi}{k+2}, \tag{3.72}$$

once we set

$$q = e^{\frac{2\pi i}{k+2}}. \tag{3.73}$$

It is very easy to generalize the result (3.71) to arbitrary p. For general p, the index is given by

$$\begin{aligned} \mathcal{I} &= \frac{1}{2} \ln q \sum_{m \in \mathbb{Z}} \int \frac{dz}{2\pi i z} z^{pm} \\ &\times \left(q^{2|m|} + q^{-2|m|} + 4 - 2 \left(q^{|m|} + q^{-|m|} \right) \left(z + \frac{1}{z} \right) + \left(z^2 + \frac{1}{z^2} \right) \right). \end{aligned} \tag{3.74}$$

The only effect of p is to select monopole numbers that contribute. For example, if $p = 2$, only $m = 0$ and $m = \pm 1$ contribute to the index and we have

$$\mathcal{I}^{p=2} = \frac{1}{2} \ln q \left(\mathcal{I}_{m=0} + \mathcal{I}^{p=2}_{m=\pm 1} \right) = \frac{1}{2} \ln q \, (6 + 2) = 4 \ln q. \tag{3.75}$$

If $p > 2$, only the trivial sector is selected, and

$$\mathcal{I}(p > 2) = \frac{1}{2} \ln q \; \mathcal{I}_{m=0} = 3 \ln q. \tag{3.76}$$

This is a general feature of indices of the "lens space theory" and we will soon encounter this phenomenon with higher rank gauge groups.

The test for 3d-3d correspondence

We list the index of $T[L(p,1); U(N)]$, obtained using *Mathematica*, in table 3.1. Due to limitation of space and computational power, it contains results up to $N = 5$ and $p = 6$. The omnipresent $(\ln q)^N$ factors are dropped to avoid clutter, and after this every entry in table 3.1 is a Laurent polynomial in q with integer coefficients. Also, when the gauge group is $U(N)$, monopole number sectors are labeled by an N-tuple of integers $m = (m_1, m_2, \ldots, m_N)$ and a given sector can only contribute to the index if $\sum m_i = 0$.

From the table, one may be able to recognize the large p behavior for $U(3)$ and $U(4)$ similar to (3.75) and (3.76). Indeed, it is a general feature of the index $\mathcal{I}_{T[L(p,1);U(N)]}$ that fewer monopole number sectors contribute when p increases. In order for a monopole number $m = (m_1, \ldots, m_N)$ to contribute,

$$|pm_i| \leq 2N - 2 \tag{3.77}$$

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$
$U(2)$	$2(1-q)(1-q^{-1})$	4	3	3	3	3
$U(3)$	$6(1-q)^2(1-q^2)$ $(1-q^{-1})^2(1-q^{-2})$	$28-6q^{-2}-8q^{-1}$ $-8q-6q^2$	$23+2q^{-1}+2q$	16	15	15
$U(4)$	$24(1-q)^3(1-q^2)^2$ $(1-q^3)(1-q^{-1})^3$ $(1-q^{-2})^2(1-q^{-3})$	$504+$ $84q^{-4}-96q^{-3}$ $-80q^{-2}-160q^{-1}$ $-160q-80q^2$ $-96q^3+84q^4$	$204-30q^{-3}$ $-48q^{-2}-24q^{-1}$ $-24q-48q^2$ $-30q^3$	$188+10q^{-2}$ $+24q^{-1}+24q$ $+10q^2$	$121+$ $2q^{-1}+2q$	108
$U(5)$	$120(1-q)^4(1-q^2)^3$ $(1-q^3)^2(1-q^4)$ $(1-q^{-1})^4(1-q^{-2})^3$ $(1-q^{-3})^2(1-q^{-4})$	$12336+$ $120q^{-10}+192q^{-9}$ $-1080q^{-8}+48q^{-7}$ $+120q^{-6}+3792q^{-5}$ $-2016q^{-4}-1296q^{-3}$ $-3312q^{-2}-2736q^{-1}$ $-2736q-3312q^2$ $-1296q^3-2016q^4$ $+3792q^5+120q^6$ $+48q^7-1080q^8$ $+192q^9+120q^{10}$	$3988+$ $180q^{-6}+388q^{-5}$ $-294q^{-4}-932q^{-3}$ $-584q^{-2}-752q^{-1}$ $-752q-584q^2$ $-932q^3-294q^4$ $+388q^5+180q^6$	$2144-$ $240q^{-4}-320q^{-3}$ $-320q^{-2}-192q^{-1}$ $-192q-320q^2$ $-320q^3-240q^4$	$1897+$ $70q^{-3}+192q^{-2}$ $352q^{-1}+352q$ $+192q^2+70q^3$	$1188+$ $14q^{-2}+40q^{-1}$ $40q+14q^2$

Table 3.1: The superconformal index of the "lens space theory" $T[L(p,1),U(N)]$, which agrees with the partition function of $GL(N,\mathbb{C})$ Chern-Simons theory at level $k=0$ on lens space $L(p,1)$.

needs to be satisfied for all m_i. For large $p > 2N-2$, $\mathcal{I}$ only receives a contribution from the $m=0$ sector and becomes a constant:

$$\mathcal{I}(U(N), p > 2N-2) = \mathcal{I}_{m=(0,0,0,\ldots,0)} = (2N-1)!!\,. \tag{3.78}$$

For $p = 2N-2$, the index receives contributions from two sectors[7]:

$$\mathcal{I}(U(N), p = 2N-2) = \mathcal{I}_{m=(0,0,0,\ldots,0)} + \mathcal{I}_{m=(1,0,\ldots,0,-1)} = [(2N-1)!! + (2N-5)!!]\,. \tag{3.79}$$

While the $\ln q$ factors (that we have omitted) are artifacts of our scheme of removing zeros in $\mathcal{I}$, the constant coefficient $(2N-1)!!$ in (3.78) is counting BPS states. Then one can ask a series of questions: 1) What are the states or local operators that are being counted? 2) Why is the number of such operators independent of p when p is large?

Partition functions Z_{CS} of the complex Chern-Simons theory on lens spaces can also be computed systematically. Please see appendix A.1 for details of the method we use. For $k=0$, $G_{\mathbb{C}} = GL(N,\mathbb{C})$, the partition functions on $L(p,1)$ only depend

[7]Here, double factorial of a negative number is taken to be 1.

on $q = e^{4\pi i/\tau}$ as $\overline{q} = e^{4\pi i/\overline{\tau}} = q^{-1}$. After dropping a $(\ln q)^N$ factor as in the index case, it is again a polynomial. We have computed this partition function up to $N = 5$ and $p = 6$ and found a perfect agreement with the index in table 3.1.

From the point of view of the complex Chern-Simons theory, this large p behavior (3.78) seems to be even more surprising — it predicts that the partition functions of the complex Chern-Simons theory on $L(p, 1)$ at level $k = 0$ are constant when p is greater than twice the rank of the gauge group. One can then ask 1) why is this happening? And 2) what is the geometric meaning of this $(2N - 1)!!$ constant?

$T[L(p, 1)]$ **on** S^3_b

In previous sections, we have seen that the superconformal index of $T[L(p, 1)]$ agrees completely with the partition function of the complex Chern-Simons theory at level $k = 0$ given by (3.50) with trivial relative phases $C_\alpha = 0$:

$$Z(G_{\mathbb{C}}; \tau, \overline{\tau}) = \sum_{\alpha \in \mathcal{M}} Z'_\alpha\left(G; \frac{\tau}{2}\right) Z'_\alpha\left(G; \frac{\overline{\tau}}{2}\right), \tag{3.80}$$

for $G = U(N)$. But for more general k, one can no longer expect this to be true. We will now consider the S^3_b partition function of $T[L(p, 1)]$, which will give the partition function of the complex Chern-Simons theory at level [13]

$$(k, \sigma) = \left(1, \frac{1 - b^2}{1 + b^2}\right). \tag{3.81}$$

And we will examine for which choices of N and p that setting all phases $C_\alpha = 0$ becomes a mistake, by comparing the S^3_b partition function of $T[L(p, 1)]$ to the "naive" partition function (3.80) of the complex Chern-Simons theory at level $k = 1$ on $L(p, 1)$.

There are two kinds of squashed three-spheres breaking the $SO(4)$ isometry of the round S^3: the first one preserves $SU(2) \times U(1)$ isometry while the second one preserves $U(1) \times U(1)$ [94]. However, despite the geometry being different, the partition functions of 3d $\mathcal{N} = 2$ theories that one gets are the same [94–97]. In fact, as was shown in [98], three-sphere partition functions of $\mathcal{N} = 2$ theories only admit a one-parameter deformation. We will choose the "ellipsoid" geometry with the metric

$$ds_3^2 = f(\theta)^2 d\theta^2 + \cos^2\theta d\phi_1^2 + \frac{1}{b^4}\sin^2\theta d\phi_2^2, \tag{3.82}$$

where $f(\theta)$ is arbitrary and does not affect the partition function of the supersymmetric theory.

Using localization, partition function of a $\mathcal{N} = 2$ gauge theory on such an ellipsoid can be written as an integral over the Cartan of the gauge group [94, 96]. Consider an $\mathcal{N} = 2$ Chern-Simons-matter theory with gauge group being $U(N)$. A classical Chern-Simons term with level k contributes

$$Z_{\mathrm{CS}} = \exp\left(\frac{i}{b^2}\frac{k}{4\pi}\sum_{i=1}^{N}\lambda_i^2\right) \tag{3.83}$$

to the integrand. The one-loop determinant of $U(N)$ vector multiplet, combined with the Vandermonde determinant, gives

$$Z_{\mathrm{gauge}} = \prod_{i<j}^{N}\left(2\sinh\frac{\lambda_i-\lambda_j}{2}\right)\left(2\sinh\frac{\lambda_i-\lambda_j}{2b^2}\right). \tag{3.84}$$

A chiral multiplet in the representation $\mathfrak{R}$ gives a product of double sine functions:

$$Z_{\mathrm{matter}} = \prod_{\rho\in\mathfrak{R}} s_b\left(\frac{iQ}{2}(1-R)-\frac{\rho(\lambda)}{2\pi b}\right), \tag{3.85}$$

where $Q = b + 1/b$, R is the R-charge of the multiplet and the double sine function is defined as

$$s_b(x) = \prod_{p,q=0}^{+\infty}\frac{pb+qb^{-1}+\frac{Q}{2}-ix}{pb^{-1}+qb+\frac{Q}{2}+ix}. \tag{3.86}$$

Then we can express the S_b^3 partition function of $T[L(p,1)]$ using the UV description in (3.5) as

$$\begin{aligned} Z(T[L(p,1),U(N)],b) = \frac{1}{N!}\int\prod_{i}^{N}\frac{d\lambda_i}{2\pi}\exp\left(-\frac{i}{b^2}\frac{p}{4\pi}\sum_{i=1}^{N}\lambda_i^2\right)&\\ \times\prod_{i<j}^{N}\frac{4}{\pi^2}\left(\sinh\frac{\lambda_i-\lambda_j}{2}\right)^2\left(\sinh\frac{\lambda_i-\lambda_j}{2b^2}\right)^2&, \end{aligned} \tag{3.87}$$

which is a Gaussian integral. We list our results in table 3.2 and 3.3. A universal factor

$$\left(\frac{b}{ip}\right)^{N/2}\pi^{-N(N-1)} \tag{3.88}$$

is dropped in making these two tables.

If one compares results in table 3.2 and 3.3 with partition functions of complex Chern-Simons theory naively computed using (3.49), one will find a perfect agreement for $p = 1$ once the phase factor

$$\exp\left[\frac{\pi i(c_L-c_R)}{12}\cdot(3-p)\right] \tag{3.89}$$

p	$U(2)$	$U(3)$	$U(4)$
1	$2e^{-2i\pi b^2-\frac{2i\pi}{b^2}}$ $\left(1-e^{\frac{2i\pi}{b^2}}\right)\left(1-e^{2i\pi b^2}\right)$	$6e^{-8i\pi b^2-\frac{8i\pi}{b^2}}\left(1-e^{\frac{2i\pi}{b^2}}\right)^3\left(1+e^{\frac{2i\pi}{b^2}}\right)$ $\left(1-e^{2i\pi b^2}\right)^3\left(1+e^{2i\pi b^2}\right)$	$24e^{-20i\pi b^2-\frac{20i\pi}{b^2}}\left(1-e^{\frac{2i\pi}{b^2}}\right)^6\left(1+e^{\frac{2i\pi}{b^2}}\right)^2$ $\left(1+e^{\frac{2i\pi}{b^2}}+e^{\frac{4i\pi}{b^2}}\right)\left(1-e^{2i\pi b^2}\right)^6$ $\left(1+e^{2i\pi b^2}\right)^2\left(1+e^{2i\pi b^2}+e^{4i\pi b^2}\right)$
2	$2-2e^{-\frac{i\pi}{b^2}}-2e^{-i\pi b^2}$ $+2e^{-i\pi b^2-\frac{i\pi}{b^2}}$	$2e^{-4i\pi(b^2+b^{-2})}$ $\left(1-e^{\frac{2i\pi}{b^2}}\right)\left(1-e^{2i\pi b^2}\right)$ $\left(-6e^{\frac{i\pi}{b^2}}+3e^{\frac{2i\pi}{b^2}}-6e^{i\pi b^2}+3e^{2i\pi b^2}\right.$ $-4e^{i\pi(b^2+b^{-2})}+3e^{2i\pi(b^2+b^{-2})}$ $\left.-6e^{i\pi\left(b^2+2b^{-2}\right)}-6e^{i\pi\left(2b^2+b^{-2}\right)}+3\right)$	$8e^{-10i\pi\left(b^2+b^{-2}\right)}\left(1-e^{\frac{2i\pi}{b^2}}\right)^2\left(1-e^{2ib^2\pi}\right)^2$ $\left(3-9e^{\frac{i\pi}{b^2}}+9e^{\frac{2i\pi}{b^2}}-6e^{\frac{3i\pi}{b^2}}+9e^{\frac{4i\pi}{b^2}}-9e^{\frac{5i\pi}{b^2}}\right.$ $+3e^{\frac{6i\pi}{b^2}}-9e^{ib^2\pi}+9e^{2ib^2\pi}-6e^{3ib^2\pi}$ $+9e^{4ib^2\pi}-9e^{5ib^2\pi}+3e^{6ib^2\pi}-9e^{i\pi\left(b^2+b^{-2}\right)}$ $+27e^{2i\pi\left(b^2+b^{-2}\right)}-4e^{3i\pi\left(b^2+b^{-2}\right)}+27e^{4i\pi\left(b^2+b^{-2}\right)}$ $-9e^{5i\pi\left(b^2+b^{-2}\right)}+3e^{6i\pi\left(b^2+b^{-2}\right)}-27e^{i\pi\left(b^2+2b^{-2}\right)}$ $+27e^{2i\pi\left(b^2+2b^{-2}\right)}-6e^{3i\pi\left(b^2+2b^{-2}\right)}-6e^{i\pi\left(b^2+3b^{-2}\right)}$ $+9e^{2i\pi\left(b^2+3b^{-2}\right)}-27e^{i\pi\left(b^2+4b^{-2}\right)}-9e^{i\pi\left(b^2+5b^{-2}\right)}$ $-9e^{i\pi\left(b^2+6b^{-2}\right)}-18e^{i\pi\left(2b^2+3b^{-2}\right)}+9e^{2i\pi\left(2b^2+3b^{-2}\right)}$ $-27e^{i\pi\left(2b^2+5b^{-2}\right)}-18e^{i\pi\left(3b^2+2b^{-2}\right)}+9e^{2i\pi\left(3b^2+2b^{-2}\right)}$ $-18e^{i\pi\left(3b^2+4b^{-2}\right)}-6e^{i\pi\left(3b^2+5b^{-2}\right)}-18e^{i\pi\left(4b^2+3b^{-2}\right)}$ $-27e^{i\pi\left(4b^2+5b^{-2}\right)}-27e^{i\pi\left(5b^2+2b^{-2}\right)}-6e^{i\pi\left(5b^2+3b^{-2}\right)}$ $-27e^{i\pi\left(5b^2+4b^{-2}\right)}-9e^{i\pi\left(5b^2+6b^{-2}\right)}-9e^{i\pi\left(6b^2+5b^{-2}\right)}$ $-27e^{i\pi\left(2b^2+b^{-2}\right)}+27e^{2i\pi\left(2b^2+b^{-2}\right)}$ $-6e^{3i\pi\left(2b^2+b^{-2}\right)}-6e^{i\pi\left(3b^2+b^{-2}\right)}+9e^{2i\pi\left(3b^2+b^{-2}\right)}$ $\left.-27e^{i\pi\left(4b^2+b^{-2}\right)}-9e^{i\pi\left(5b^2+b^{-2}\right)}-9e^{i\pi\left(6b^2+b^{-2}\right)}\right)$
3	$2-2e^{-\frac{2i\pi}{3b^2}}-2e^{-\frac{2}{3}i\pi b^2}$ $-e^{-\frac{2i\pi}{3}(b^2+b^{-2})}$	$-3e^{-\frac{8i\pi}{3}\left(b^2+b^{-2}\right)}\times$ $\left(4e^{\frac{2i\pi}{3b^2}}+2e^{\frac{2i\pi}{b^2}}+2e^{\frac{8i\pi}{3b^2}}\right.$ $+4e^{\frac{2}{3}i\pi b^2}+2e^{2i\pi b^2}+2e^{\frac{8}{3}i\pi b^2}$ $-8e^{\frac{2i\pi}{3}\left(b^2+b^{-2}\right)}+4e^{2i\pi\left(b^2+b^{-2}\right)}$ $-2e^{\frac{8i\pi}{3}\left(b^2+b^{-2}\right)}+8e^{\frac{2i\pi}{3}\left(b^2+3b^{-2}\right)}$ $-4e^{\frac{2i\pi}{3}\left(b^2+4b^{-2}\right)}$ $+4e^{\frac{2i\pi}{3}\left(3b^2+4b^{-2}\right)}+4e^{\frac{2i\pi}{3}\left(4b^2+3b^{-2}\right)}$ $\left.+8e^{\frac{2\pi i}{3}\left(3b^2+b^{-2}\right)}-4e^{\frac{2\pi i}{3}\left(4b^2+\pi b^{-2}\right)}+1\right)$	$-6e^{-\frac{20i\pi}{3}\left(b^2+b^{-2}\right)}\left(1-e^{\frac{2i\pi}{b^2}}\right)\left(1-e^{2ib^2\pi}\right)$ $\left(1+6e^{\frac{2i\pi}{3b^2}}+5e^{\frac{2i\pi}{b^2}}+8e^{\frac{8i\pi}{3b^2}}+3e^{\frac{4i\pi}{b^2}}+4e^{\frac{14i\pi}{3b^2}}\right.$ $+6e^{\frac{2}{3}ib^2\pi}+5e^{2ib^2\pi}+8e^{\frac{8}{3}ib^2\pi}+3e^{4ib^2\pi}$ $+4e^{\frac{14}{3}ib^2\pi}-18e^{\frac{2i\pi}{3}\left(b^2+b^{-2}\right)}-2e^{\frac{4i\pi}{3}\left(b^2+b^{-2}\right)}$ $+25e^{2ipi\left(b^2+b^{-2}\right)}-28e^{\frac{8i\pi}{3}\left(b^2+b^{-2}\right)}-2e^{\frac{10i\pi}{3}\left(b^2+b^{-2}\right)}$ $+9e^{4i\pi\left(b^2+b^{-2}\right)}-4e^{\frac{14i\pi}{3}\left(b^2+b^{-2}\right)}-4e^{\frac{4i\pi}{3}\left(b^2+2b^{-2}\right)}$ $+15e^{2i\pi\left(b^2+2b^{-2}\right)}+30e^{\frac{2i\pi}{3}\left(b^2+3b^{-2}\right)}-24e^{\frac{2i\pi}{3}\left(b^2+4b^{-2}\right)}$ $+18e^{\frac{2i\pi}{3}\left(b^2+6b^{-2}\right)}-12e^{\frac{2i\pi}{3}\left(b^2+7b^{-2}\right)}$ $+24e^{\frac{4i\pi}{3}\left(2b^2+3b^{-2}\right)}+2e^{\frac{2i\pi}{3}\left(2b^2+5b^{-2}\right)}+4e^{\frac{2i\pi}{3}\left(2b^2+7b^{-2}\right)}$ $+24e^{\frac{4i\pi}{3}\left(3b^2+2b^{-2}\right)}+40e^{\frac{2i\pi}{3}\left(3b^2+4b^{-2}\right)}+20e^{\frac{2i\pi}{3}\left(3b^2+7b^{-2}\right)}$ $+40e^{\frac{2i\pi}{3}\left(4b^2+3b^{-2}\right)}+4e^{\frac{2i\pi}{3}\left(4b^2+5b^{-2}\right)}-20e^{\frac{2i\pi}{3}\left(4b^2+7b^{-2}\right)}$ $+2e^{\frac{2i\pi}{3}\left(5b^2+2b^{-2}\right)}+4e^{\frac{2i\pi}{3}\left(5b^2+4b^{-2}\right)}-4e^{\frac{2i\pi}{3}\left(5b^2+7b^{-2}\right)}$ $+12e^{\frac{2i\pi}{3}\left(6b^2+7b^{-2}\right)}+4e^{\frac{2i\pi}{3}\left(7b^2+2b^{-2}\right)}+20e^{\frac{2i\pi}{3}\left(7b^2+3b^{-2}\right)}$ $-20e^{\frac{2i\pi}{3}\left(7b^2+4b^{-2}\right)}-4e^{\frac{2i\pi}{3}\left(7b^2+5b^{-2}\right)}+12e^{\frac{2i\pi}{3}\left(7b^2+6b^{-2}\right)}$ $-4e^{\frac{4i\pi}{3}\left(2b^2+b^{-2}\right)}+15e^{2i\pi\left(2b^2+b^{-2}\right)}+30e^{\frac{2i\pi}{3}\left(3b^2+b^{-2}\right)}$ $\left.-24e^{\frac{2i\pi}{3}\left(4b^2+b^{-2}\right)}+18e^{\frac{2i\pi}{3}\left(6b^2+b^{-2}\right)}-12e^{\frac{2i\pi}{3}\left(7b^2+b^{-2}\right)}\right)$

Table 3.2: The S_b^3 partition function of $T[L(p,1),U(N)]$. In this table p ranges from 1 to 3.

p	$U(2)$	$U(3)$
4	$2-2e^{-\frac{i\pi}{2b^2}}-2e^{-\frac{1}{2}i\pi b^2}-2e^{-\frac{i\pi}{2}\left(b^2+b^{-2}\right)}$	$-2e^{-2i\pi\left(b^2+b^{-2}\right)}\times$ $\left(-3-2e^{\frac{i\pi}{2b^2}}+2e^{\frac{3i\pi}{2b^2}}+3e^{\frac{2i\pi}{b^2}}-2e^{\frac{1}{2}i\pi b^2}+2e^{\frac{3}{2}i\pi b^2}+3e^{2i\pi b^2}+4e^{\frac{i\pi}{2}\left(b^2+b^{-2}\right)}\right.$ $+4e^{\frac{3i\pi}{2}\left(b^2+b^{-2}\right)}-3e^{2i\pi\left(b^2+b^{-2}\right)}+4e^{\frac{i\pi}{2}\left(b^2+3b^{-2}\right)}-6e^{\frac{i\pi}{2}\left(b^2+4b^{-2}\right)}$ $\left.+6e^{\frac{i\pi}{2}\left(3b^2+4b^{-2}\right)}+6e^{\frac{i\pi}{2}\left(4b^2+3b^{-2}\right)}+4e^{\frac{i\pi}{2}\left(3b^2+b^{-2}\right)}-6e^{\frac{i\pi}{2}\left(4b^2+b^{-2}\right)}\right)$
5	$2-2e^{-\frac{2i\pi}{5b^2}}-2e^{-\frac{2}{5}i\pi b^2}+2\cos\frac{4\pi}{5}e^{-\frac{2i\pi}{5}\left(b^2+b^{-2}\right)}$	$6-12e^{-\frac{2i\pi}{5b^2}}+12e^{-\frac{6i\pi}{5b^2}}-6e^{-\frac{8i\pi}{5b^2}}-12e^{-\frac{2}{5}i\pi b^2}$ $+12e^{-\frac{6}{5}i\pi b^2}-6e^{-\frac{8}{5}i\pi b^2}+4\left(\cos\frac{8\pi}{5}+e^{\frac{4i\pi}{5}}\right)e^{-\frac{2i\pi}{5}\left(4b^2+b^{-2}\right)}$ $4\left(\cos\frac{8\pi}{5}+2\cos\frac{4\pi}{5}\right)e^{-\frac{2i\pi}{5}\left(b^2+4b^{-2}\right)}+8\left(\cos\frac{4\pi}{5}+2\cos\frac{2\pi}{5}\right)e^{-\frac{2i\pi}{5}\left(b^2+b^{-2}\right)}$ $+8\left(\cos\frac{12\pi}{5}+2\cos\frac{6\pi}{5}\right)e^{-\frac{6i\pi}{5}\left(b^2+b^{-2}\right)}+2\left(\cos\frac{16\pi}{5}+2\cos\frac{8\pi}{5}\right)$ $\times e^{-\frac{8i\pi}{5}\left(b^2+b^{-2}\right)}-8e^{-\frac{2i\pi}{5}\left(b^2+3b^{-2}\right)}-8e^{-\frac{2i\pi}{5}\left(b^2-3+3b^{-2}\right)}-8e^{-\frac{2i\pi}{5}\left(b^2+3+3b^{-2}\right)}$ $-8e^{-\frac{2i\pi}{5}\left(3b^2+b^{-2}\right)}-4e^{-\frac{2i\pi}{5}\left(3b^2+4b^{-2}\right)}-4e^{-\frac{2i\pi}{5}\left(3b^2-6+4b^{-2}\right)}$ $-8e^{-\frac{2i\pi}{5}\left(3b^2-3+b^{-2}\right)}-8e^{-\frac{2i\pi}{5}\left(3b^2+3+b^{-2}\right)}-4e^{-\frac{2i\pi}{5}\left(3b^2+6+4b^{-2}\right)}$ $-4e^{-\frac{2i\pi}{5}\left(4b^2+3b^{-2}\right)}-4e^{-\frac{2i\pi}{5}\left(4b^2-6+3b^{-2}\right)}-4e^{-\frac{2i\pi}{5}\left(4b^2+6+3b^{-2}\right)}$
6	$2-2e^{-\frac{i\pi}{3b^2}}-2e^{-\frac{1}{3}i\pi b^2}+e^{-\frac{i\pi}{3}\left(b^2+b^{-2}\right)}$	$e^{-\frac{4i\pi}{3}\left(b^2+b^{-2}\right)}\times$ $\left(-12e^{\frac{i\pi}{3b^2}}-6e^{\frac{i\pi}{b^2}}-6e^{\frac{4i\pi}{3b^2}}-12e^{\frac{1}{3}i\pi b^2}-6e^{i\pi b^2}-6e^{\frac{4}{3}i\pi b^2}-8e^{\frac{i\pi}{3}\left(b^2+b^{-2}\right)}\right.$ $+4e^{i\pi\left(b^2+b^{-2}\right)}+6e^{\frac{4i\pi}{3}\left(b^2+b^{-2}\right)}+8e^{\frac{i\pi}{3}\left(b^2+3b^{-2}\right)}+12e^{\frac{i\pi}{3}\left(b^2+4b^{-2}\right)}$ $\left.-12e^{\frac{i\pi}{3}\left(3b^2+4b^{-2}\right)}-12e^{\frac{i\pi}{3}\left(4b^2+3b^{-2}\right)}+8e^{\frac{i\pi}{3}\left(3b^2+b^{-2}\right)}+12e^{\frac{i\pi}{3}\left(4b^2+b^{-2}\right)}-3\right)$

Table 3.3: The S_b^3 partition function of $T[L(p,1),U(N)]$. This table, with p ranging from 4 to 6, is the continuation of the previous table 3.2. Due to the limitation of space, only partition functions for $U(2)$ and $U(3)$ are given.

from the change of framing is added[8]. This agreement is not unexpected because, for $p=1$, $\mathcal{M}_{\text{flat}}$ consists of just a single point and there are no such things as relative phases between contributions from different flat connections. Even for $p=2$, the naive way (3.49) of computing partition function of complex Chern-Simons theory seems to be still valid modulo an overall factor. However, starting from $p=3$, the two sides start to differ significantly. See table 3.4 for a comparison between the S_b^3 partition function of $T[L(p,1)]$ and the "naive" partition function of the complex Chern-Simons theory on $L(p,1)$ for $G=U(2)$.

[8]The complex Chern-Simons theory obtained from the 3d-3d correspondence is naturally in "Seifert framing", as the $T[L(p,1)]$ we used is obtained by reducing M5-brane on the Seifeit S^1 fiber of $L(p,1)$ in Chapter 2. However, the computation in appendix A.1 is in "canonical framing" and differs from Seifert framing by $(3-p)$ units [99].

p	S_b^3 partition function of $T[L(p,1);U(2)]$	“naive” partition function of $GL(2,\mathbb{C})$ Chern-Simons theory
1	$2-2q^{-1}-2\overline{q}^{-1}+2\left(q\overline{q}\right)^{-1}$	$2-2q^{-1}-2\overline{q}^{-1}+2\left(q\overline{q}\right)^{-1}$
2	$2+2q^{-\frac{1}{2}}+2\overline{q}^{-\frac{1}{2}}+2\left(q\overline{q}\right)^{-\frac{1}{2}}$	$2i(2+2q^{-\frac{1}{2}}+2\overline{q}^{-\frac{1}{2}}+2\left(q\overline{q}\right)^{-\frac{1}{2}})$
3	$2+\left(1-\sqrt{3}i\right)q^{-\frac{1}{3}}+\left(1-\sqrt{3}i\right)\overline{q}^{-\frac{1}{3}}+\frac{1}{2}\left(1+\sqrt{3}i\right)\left(q\overline{q}\right)^{-\frac{1}{3}}$	$2+\left(1-3\sqrt{3}i\right)\overline{q}^{\frac{1}{3}}+\left(1-3\sqrt{3}i\right)q^{\frac{1}{3}}+\frac{1}{2}\left(1+3\sqrt{3}i\right)\left(q\overline{q}\right)^{\frac{1}{3}}$
4	$2-2iq^{-\frac{1}{4}}-2i\overline{q}^{-\frac{1}{4}}+2\left(q\overline{q}\right)^{-\frac{1}{4}}$	$8i\left(q\overline{q}\right)^{\frac{1}{2}}\left(1+iq^{\frac{1}{4}}+i\overline{q}^{\frac{1}{4}}+\left(q\overline{q}\right)^{\frac{1}{4}}\right)$
5	$2-2e^{\frac{2\pi i}{5}}q^{-\frac{1}{5}}-2e^{\frac{2\pi i}{5}}\overline{q}^{-\frac{1}{5}}+2\cos\frac{4\pi}{5}e^{\frac{4\pi i}{5}}\left(q\overline{q}\right)^{-\frac{1}{5}}$	$q\overline{q}\left(2-2\left(e^{\frac{3\pi i}{5}}+2e^{\frac{4\pi i}{5}}\right)\overline{q}^{\frac{1}{5}}-2\left(e^{\frac{3\pi i}{5}}+2e^{\frac{4\pi i}{5}}\right)q^{\frac{1}{5}}\right.$ $\left.+\left(1+2e^{\frac{\pi i}{5}}+3e^{\frac{2\pi i}{5}}-4e^{\frac{3\pi i}{5}}-4e^{\frac{4\pi i}{5}}\right)\left(q\overline{q}\right)^{\frac{1}{5}}\right)$
6	$2-\left(1+\sqrt{3}i\right)q^{-\frac{1}{6}}-\left(1+\sqrt{3}i\right)\overline{q}^{-\frac{1}{6}}-\frac{1}{2}\left(1-\sqrt{3}i\right)\left(q\overline{q}\right)^{-\frac{1}{6}}$	$6i\left(q\overline{q}\right)^{\frac{3}{2}}\left(2+(-1+i\sqrt{3})q^{\frac{1}{6}}+(-1+i\sqrt{3})\overline{q}^{\frac{1}{6}}+\frac{1}{2}\left(1+i\sqrt{3}\right)\left(q\overline{q}\right)^{\frac{1}{6}}\right)$

Table 3.4: The comparison between the S_b^3 partition function of $T[L(p,1),U(2)]$ and the “naive” partition function of the $GL(2,\mathbb{C})$ Chern-Simons theory, obtained by putting together two copies of the $U(2)$ Chern-Simons theory using (3.80), on lens space $L(p,1)$ in the “Seifert framing.” Notice that when p increases, the difference between the two columns becomes larger and larger.

4.1 Equivalence between two TQFTs

The complex Chern-Simons theory was studied by embedding it into string theory in Chapter 2, and the starting point is the now familiar configuration of M-theory fivebranes:

$$\begin{array}{lccccc} \text{space-time:} & L(k,1)_b & \times & T^*M_3 & \times & \mathbb{R}^2 \\ & & & \cup & & \\ N \text{ fivebranes:} & L(k,1)_b & \times & M_3 & & \end{array} \tag{4.1}$$

If one reduces along the squashed lens space $L(k,1)_b$, one obtains complex Chern-Simons theory at level k on M_3 [13]. Even in the simple case where M_3 is the product of a Riemann surface Σ with a circle S^1, this system is extremely interesting and can be used to gain a lot of insight into complex Chern-Simons theory. For example, the partition function of the 6d $(2,0)$-theory on this geometry gives the "equivariant Verlinde formula", which can be identified with the dimension of the Hilbert space of the complex Chern-Simons theory at level k on Σ:

$$Z_{\text{M5}}(L(k,1)\times\Sigma\times S^1,\beta) = \dim_\beta \mathcal{H}_{\text{CS}}(\Sigma,k). \tag{4.2}$$

Here β is an "equivariant parameter" associated with a geometric $U(1)_\beta$ action whose precise definition will be reviewed in section 4.2. The left-hand side of (4.2) has been computed in several ways in Chapters 2 and 3, and each gives unique insight into the equivariant Verlinde formula, the complex Chern-Simons theory and the 3d-3d correspondence in general. In this chapter, we will add to the list yet another method of computing the partition of the system of M5-branes by relating it to superconformal indices of class $\mathcal{S}$ theories.

The starting point is the following observation. For $M_3 = \Sigma\times S^1$, the setup (4.1) looks like:

$$\begin{array}{lccccccc} N \text{ fivebranes:} & L(k,1)_b & \times & \Sigma & \times & S^1 & & \\ & & & \cap & & & & \\ \text{space-time:} & L(k,1)_b & \times & T^*\Sigma & \times & S^1 & \times & \mathbb{R}^3 \end{array} \tag{4.3}$$

and it is already very reminiscent of the setting of lens space superconformal indices of class $\mathcal{S}$ theories [17–21]:

$$
\begin{array}{rccccc}
N \text{ fivebranes:} & L(k,1)\times S^1 & \times & \Sigma & & \\
 & & & \cap & & \\
\text{space-time:} & L(k,1)\times S^1 & \times & T^*\Sigma & \times & \mathbb{R}^3 \\
 & \circlearrowright & & \circlearrowright & & \circlearrowright \\
\text{symmetries:} & SO(4)_E & & U(1)_N & & SU(2)_R
\end{array}
\tag{4.4}
$$

In this geometry, one can turn on holonomies of the symmetries along the S^1 circle in a supersymmetric way and introduce three "universal fugacities" (p, q, t). Then the partition function of M5-branes in this geometry is the lens space superconformal index of the 4d $\mathcal{N} = 2$ theory $T[\Sigma]$ of class $\mathcal{S}$:

$$Z_{\mathrm{M5}}(L(k,1)\times S^1\times\Sigma, p, q, t) = \mathcal{I}(T[\Sigma], p, q, t), \tag{4.5}$$

where we have adopted the following convention for the index[1]

$$\mathcal{I}(p,q,t) = \mathrm{Tr}(-1)^F p^{\frac{1}{2}\delta_{1+}} q^{\frac{1}{2}\delta_{1-}} t^{R+r} e^{-\beta''\widetilde{\delta}_{1\dot{-}}}. \tag{4.6}$$

As the left-hand sides of (4.2) and (4.5) are almost identical, it is very tempting to ask whether the equivariant Verlinde formula for a Riemann surface Σ actually equals the index of the theory $T[\Sigma]$. The answer to this question is affirmative, and, as we show in section 4.2, the relation is

$$\boxed{\begin{array}{c}\text{Equivariant Verlinde formula}\\ \text{at level } k \text{ on } \Sigma\end{array}} \quad = \quad \boxed{\begin{array}{c}\text{Coulomb branch index}\\ \text{of } T[\Sigma] \text{ on } L(k,1)\times S^1\end{array}}\,. \tag{4.7}$$

The above relation may not be a complete surprise as the equivariant Verlinde formula for G can be written as an integral over the Hitchin moduli space $\mathcal{M}_H(\Sigma, G)$ [44]:

$$\dim_\beta \mathcal{H}_{\mathrm{CS}}(\Sigma, G_{\mathbb{C}}, k) = \dim_\beta H^0(\mathcal{M}_H, \mathcal{L}^{\otimes k}) = \int_{\mathcal{M}_H} \mathrm{ch}(\mathcal{L}^{\otimes k};\beta)\wedge \mathrm{Td}(\mathcal{M}_H, \beta), \tag{4.8}$$

[1] In the literature there are several other conventions in use. The other two most commonly used universal fugacities are (ρ, σ, τ) which are related to our convention via $p = \sigma\tau, q = \rho\tau, t = \tau^2$, and (t, y, v) with $t = \sigma^{\frac{1}{6}}\rho^{\frac{1}{6}}\tau^{\frac{1}{3}}, y = \sigma^{\frac{1}{2}}\rho^{-\frac{1}{2}}, v = \sigma^{\frac{2}{3}}\rho^{\frac{2}{3}}\tau^{-\frac{2}{3}}$.

and $\mathcal{M}_H(\Sigma, G)$ is precisely the Coulomb branch of the theory $T[\Sigma, G]$ on $S^1 \times \mathbb{R}^3$ [100]. In fact, for small values of k, one can directly check (4.7). For example, the $k = 0$ equivariant Verlinde formula is given by[2]

$$\dim_\beta \mathcal{H}_{\text{CS}}(\Sigma, G_{\mathbb{C}}, k = 0) = \dim_\beta H^0(\mathcal{B}, \mathbb{C}) = \int_{\mathcal{B}} \text{Td}(\mathcal{B}, \beta), \tag{4.9}$$

where $\mathcal{B}$ is the "Hitchin base" [101] or physically the Coulomb branch of $T[\Sigma]$ on $\mathbb{R}^4$. The integral over $\mathcal{B}$ computes the "equivariant character" of the $U(1)_\beta$ Hitchin action and is given by[3]

$$\int_{\mathcal{B}} \text{Td}(\mathcal{B}, \beta) = \left[\frac{1}{\prod_{i=1}^{\text{rank}\, G} (1 - \mathfrak{t}^{d_i})^{h_i}} \right]^{g-1} . \tag{4.10}$$

Here the d_i's are degrees of the fundamental invariants of $\mathfrak{g} = \text{Lie}\, G$, the h_i's are the dimension of the space of d_i-differentials on Σ, and $\mathfrak{t} = e^{-\beta}$ is the exponentiated equivariant parameter. For $k = 1$ and G of type ADE, the equivariant Verlinde formula gives

$$\dim_\beta \mathcal{H}_{\text{CS}}(\Sigma, G_{\mathbb{C}}, k = 1) = \frac{|\mathcal{Z}(G)|^g}{\left[\prod_{i=1}^{\text{rank}\, G} (1 - \mathfrak{t}^{d_i})^{h_i} \right]^{g-1}} , \tag{4.11}$$

where $|\mathcal{Z}(G)|$ is the order of the center of group G. The reader may have already recognized that (4.11) is exactly the Coulomb branch index of $T[\Sigma, G]$ on $L(k = 1, 1) = S^3$ times $|\mathcal{Z}(G)|^g$. As we will explain in great detail later, the $|\mathcal{Z}(G)|^g$ factor can also be interpreted as the summation over 't Hooft fluxes, which are labeled precisely by elements in $\mathcal{Z}(G)$. For $k > 1$, the relation (4.7) becomes even more non-trivial. Even if one sets $\mathfrak{t} = 0$, the identification of Verlinde algebra with the algebra of 't Hooft fluxes is completely novel.

This chapter is organized as follows. In section 4.2, we examine more closely the two fivebranes systems (4.1) and (4.4), and derive the relation (4.7) between the equivariant Verlinde formula and the Coulomb branch index together with various variants of this relation. We find that the groups G and ${}^L G$ that appear on the two sides are related by electric-magnetic or Langlands duality. In section 4.3, after reviewing basic facts and ingredients of the index, we verify our proposals

[2] Here we have used the Serre spectral sequence, coming from the the Hitchin fibration $\mathbf{F} \to \mathcal{M}_H \to \mathcal{B}$, to identify the two cohomology groups.

[3] In this chapter, we use "$\mathfrak{t}$" to denote the (exponentiated) equivariant parameter and reserve "t" for the third "universal fugacity" in the index of 4d $\mathcal{N} = 2$ SCFTs, which is quite a standard and widely adopted convention.

by reproducing the already known $SU(2)$ equivariant Verlinde algebra from the Coulomb branch index of class $\mathcal{S}$ theories on lens space. We will see that after an appropriate normalization, the TQFT algebras on both sides are identical, and so are the partition functions. In section 4.4, we will use the proposed relation (4.7) to derive the $SU(3)$ equivariant Verlinde algebra from the index of $T[\Sigma, SU(3)]$ computed via the Argyres-Seiberg duality. Careful analysis of the results reveals interesting geometry of the Hitchin moduli space $\mathcal{M}_H(\Sigma, SU(3))$.

4.2 Equivariant Verlinde algebra and Coulomb branch index

One obvious difference between the two brane systems (4.1) and (4.4) is that the S^1 factor appears on different sides of the correspondence. From the geometry of (4.1), one would expect that

$$\begin{array}{c}\text{Equivariant Verlinde formula}\\ \text{at level } k \text{ on } \Sigma\end{array} = \begin{array}{c}\text{Partition function of}\\ T[\Sigma\times S^1] \text{ on } L(k,1)\end{array}. \tag{4.12}$$

In particular, there should be no dependence on the size of the S^1, so it is more natural to use "3d variables":

$$t = e^{L\beta-(b+b^{-1})L/r}, \quad p = e^{-bL/r}, \quad q = e^{-b^{-1}L/r}. \tag{4.13}$$

Here, L is the size of the S^1 circle, b is the squashing parameter of $L(k,1)_b$, r measures the size of the Seifert base S^2, and β parametrizes the "canonical mass deformation" of the 3d $\mathcal{N}=4$ theory (in our case $T[\Sigma\times S^1]$) into 3d $\mathcal{N}=2$. The latter is defined as follows on flat space. The 3d $\mathcal{N}=4$ theory has R-symmetry $SU(2)_N\times SU(2)_R$ and we can view it as a 3d $\mathcal{N}=2$ theory with the R-symmetry group being the diagonal subgroup $U(1)_{N+R}\subset U(1)_N\times U(1)_R$ with $U(1)_N$ and $U(1)_R$ being the Cartans of $SU(2)_N$ and $SU(2)_R$ respectively. The difference $U(1)_{N-R} = U(1)_N - U(1)_R$ of the original R-symmetry group is now a flavor symmetry $U(1)_\beta$ and we can weakly gauge it to introduce real masses proportional to β. It is exactly how the "equivariant parameter" in Chapter 2, denoted by the same letter β, is defined[4].

In Chapter 2, it was observed that much could be learned about the brane system (4.1) and the Hilbert space of complex Chern-Simons theory by preserving supersymmetry along lens space $L(k,1)$ in a different way, namely by doing partial topological twist instead of deforming the supersymmetry algebra. Geometrically, this corresponds to combining the last $\mathbb{R}^3$ factor in (4.3) with $L(k,1)$ to

[4]More precisely, the dimensionless combination βL is used. And from now on, we will rename $\beta_{\text{new}} = \beta_{\text{old}}L$ and $r_{\text{new}} = r_{\text{old}}/L$ to make all 3d variables dimensionless.

form $T^*L(k,1)$ regarded as a local Calabi-Yau 3-fold with $L(k,1)_b$ being a special Lagrangian submanifold:

$$
\begin{array}{lccccc}
N \text{ fivebranes:} & L(k,1)_b & \times & \Sigma & \times & S^1 \\
& \cap & & \cap & & \\
\text{space-time:} & T^*L(k,1)_b & \times & T^*\Sigma & \times & S^1 \\
& \circlearrowright & & \circlearrowright & & \\
\text{symmetries:} & U(1)_R & & U(1)_N & & .
\end{array}
\tag{4.14}
$$

In this geometry, $U(1)_N$ acts by rotating the cotangent fiber of Σ, while $U(1)_R$ rotates the cotangent fiber of the Seifert base S^2 of the lens space[5]. This point of view enables one to derive the equivariant Verlinde formula as it is now the partition function of the *supersymmetric* theory $T[L(k,1),\beta]$ on $\Sigma \times S^1$.

Although the geometric setting (4.14) appears to be different from the original one (4.1), there is substantial evidence that they are related. For example, the equivariant Verlinde formula can be defined and computed on both sides and they agree. Also, the modern viewpoint on supersymmetry in curved backgrounds is that the deformed supersymmetry is an extension of topological twisting, see *e.g.* [102]. Therefore, one should expect that the equivariant Verlinde formula at level k could be identified with a particular slice of the four-parameter family of 4d indices (k,p,q,t) (or in 3d variables (k,β,b,r)). And this particular slice should have the property that the index has no dependence on the geometry of $L(k,1)_b$. Since $T[L(k,1)]$ is derived in the limit where $L(k,1)$ shrinks, one should naturally take the $r \to 0$ limit for the superconformal index. In terms of the 4d parameters, that corresponds to

$$p,q,t \to 0. \tag{4.15}$$

This is known as the Coulomb branch limit. In this particular limit, the only combination of (k,p,q,t) independent of b and r that one could possibly construct is

$$\mathrm{t} = \frac{pq}{t} = e^{-\beta}, \tag{4.16}$$

[5]Note, $U(1)_N$ is always an isometry of the system whereas the $U(1)_R$ is only an isometry in certain limits where the metric on $L(k,1)$ is singular (*e.g.* for a small torus fibered over a long interval). However, if we are only interested in questions that have no dependence on the metric on $L(k,1)$, we can always assume the $U(1)_R$ symmetry to exist. For example, $T[L(k,1)]$, or in general $T[M_3]$ for any Seifert manifolds M_3 enjoys an extra flavor symmetry $U(1)_\beta = U(1)_N - U(1)_R$.

and this is precisely the parameter used in the Coulomb branch index. Therefore, one arrives at the following proposal:

$$\boxed{\begin{array}{c}\text{Equivariant Verlinde formula}\\ \text{of } U(N)_k \text{ on } \Sigma\end{array}} \quad = \quad \boxed{\begin{array}{c}\text{Coulomb branch index}\\ \text{of } T[\Sigma, U(N)] \text{ on } L(k,1)\times S^1\end{array}}\,. \tag{4.17}$$

This relation should be more accurately viewed as the natural isomorphism between two TQFT functors

$$Z_{\mathrm{EV}} = Z_{\mathrm{CB}}. \tag{4.18}$$

At the level of partition function on a closed Riemann surface Σ, it is the equality between the equivariant Verlinde formula and the Coulomb index of $T[\Sigma]$

$$Z_{\mathrm{EV}}(\Sigma) = Z_{\mathrm{CB}}(\Sigma). \tag{4.19}$$

Going one dimension lower, we also have an isomorphism between the Hilbert spaces of the two TQFTs on a circle:

$$\mathcal{H}_{\mathrm{EV}} = Z_{\mathrm{EV}}(S^1) = \mathcal{H}_{\mathrm{CB}} = Z_{\mathrm{CB}}(S^1). \tag{4.20}$$

As these underlying vector spaces set the stages for any interesting TQFT algebra, the equality above is the most fundamental and needs to be established first. We now show how one can canonically identify the two seemingly different Hilbert spaces $\mathcal{H}_{\mathrm{EV}}$ and $\mathcal{H}_{\mathrm{CB}}$.

$\mathcal{H}_{\mathrm{EV}}$ vs. $\mathcal{H}_{\mathrm{CB}}$

In the equivariant Verlinde TQFT, operator-state correspondence tells us that states in $\mathcal{H}_{\mathrm{EV}}$ are in one-to-one correspondence with local operators. Since these local operators come from codimension-2 "monodromy defects" [74] (see also [103] in the context of 3d-3d correspondence) in $T[L(k,1)]$ supported on the circle fibers of $\Sigma \times S^1$, they are labeled by

$$\mathbf{a} = \mathrm{diag}\{a_1, a_2, a_3, \ldots, a_N\} \in \mathfrak{u}(N) \tag{4.21}$$

together with a compatible choice of Levi subgroup $\mathfrak{L} \subset U(N)$. In the equivariant Verlinde TQFT, one only needs to consider maximal defects with $\mathfrak{L} = U(1)^N$ as they are enough to span the finite-dimensional $\mathcal{H}_{\mathrm{EV}}$. The set of continuous parameters $\mathbf{a}$ is acted upon by the affine Weyl group W_{aff} and therefore can be chosen to live in the Weyl alcove:

$$1 > a_1 \geq a_2 \geq \ldots \geq a_N \geq 0. \tag{4.22}$$

In the presence of a Chern-Simons term at level k, gauge invariance imposes the following integrality condition

$$e^{2\pi i k\,\mathbf{a}} = \mathbf{1}. \tag{4.23}$$

We can then define

$$\mathbf{h} = k\mathbf{a} \tag{4.24}$$

whose elements are now integers in the range $[0, k)$. The condition (4.23) is also the condition for the adjoint orbit

$$\mathcal{O}_{\mathbf{h}} = \{ghg^{-1} \| g \in U(N)\} \tag{4.25}$$

to be quantizable. Via the Borel-Weil-Bott theorem, quantizing $\mathcal{O}_{\mathbf{h}}$ gives a representation of $U(N)$ labeled by a Young tableau $\vec{h} = (h_1, h_2, \ldots, h_N)$. So, we can also label the states in $\mathcal{H}_{\mathrm{EV}}(S^1)$ by representations of $U(N)$ or, more precisely, integrable representations of the loop group of $U(N)$ at level k. In other words, the Hilbert space of the equivariant Verlinde TQFT is the same as that of the usual Verlinde TQFT (better known as the G/G gauged WZW model). This is, of course, what one expects as the Verlinde algebra corresponds to the $\mathtt{t} = 0$ limit of the equivariant Verlinde algebra, and the effect of $\mathtt{t}$ is to modify the algebra structure without changing $\mathcal{H}_{\mathrm{EV}}$. In particular, the dimension of $\mathcal{H}_{\mathrm{EV}}$ is independent of the value of $\mathtt{t}$.

One could also use the local operators from the dimensional reduction of Wilson loops as the basis for $\mathcal{H}_{\mathrm{EV}}(S^1)$. In pure Chern-Simons theory, the monodromy defects are the same as Wilson loops. In $T[L(k,1), \beta]$ with β turned on, these two types of defects are still linearly related by a transformation matrix, which is no longer diagonal. One of the many reasons that we prefer the maximal monodromy defects is because, under the correspondence, they are mapped to more familiar objects on the Coulomb index side. To see this, we first notice that the following brane system

$$\begin{array}{lccccccccc}
N \text{ fivebranes:} & L(k,1)_b & \times & \Sigma & \times & S^1 & & \\
 & & & \cap & & & & \\
\text{space-time:} & L(k,1)_b & \times & T^*\Sigma & \times & S^1 & \times & \mathbb{R}^3 \\
 & & & \cup & & & & \\
n \times N \text{ “defect” fivebranes:} & L(k,1)_b & \times & T^*|_{p_i}\Sigma & \times & S^1 & &
\end{array} \tag{4.26}$$

gives n maximal monodromy defects at $(p_1, p_2, \ldots, p_n) \in \Sigma$. If one first compactifies the brane system above on Σ, one obtains the 4d $\mathcal{N} = 2$ class $\mathcal{S}$ theory $T[\Sigma_{g,n}]$ on

$L(k,1)_b \times S^1$. This theory has flavor symmetry $U(N)^n$ and one can consider sectors of the theory with non-trivial flavor holonomies $\{\exp[\mathbf{a}_i], i = 1, 2, \ldots, n\}$ of $U(N)^n$ along the Hopf fiber. The $L(k,1)$-Coulomb branch index of $T[\Sigma_{g,n}]$ depends only on $\{\mathbf{a}_i, i = 1, 2, \ldots, n\}$ and therefore states in the Hilbert space $\mathcal{H}_{\mathrm{CB}}$ of the Coulomb branch index TQFT associated to a puncture on Σ are labeled by a $U(N)$ holonomy $\mathbf{a}$. (Notice that, for other types of indices, the states are in general also labeled by a continuous parameter corresponding to the holonomy along the S^1 circle and the 2d TQFT for them is in general infinite-dimensional). As the Hopf fiber is the generator of $\pi_1(L(k,1)) = \mathbb{Z}_k$, one has

$$e^{2\pi i k \mathbf{a}} = \mathrm{Id}. \tag{4.27}$$

This is exactly the same as the condition (4.23). In fact, we have even used the same letter $\mathbf{a}$ in both equations, anticipating the connection between the two. What we have found is the canonical way of identifying the two sets of basis vectors in the two Hilbert spaces

$$\begin{array}{ccc} \mathcal{H}_{\mathrm{EV}}^{\otimes n} & & \mathcal{H}_{\mathrm{CB}}^{\otimes n} \\ \Cup & & \Cup \\ \boxed{\begin{array}{c}\text{Monodromy defects on } \Sigma_{g,n} \times S^1 \\ \text{in } GL(N,\mathbb{C})_k \text{ Chern-Simons theory}\end{array}} & = & \boxed{\begin{array}{c}\text{Flavor holonomy sectors} \\ \text{of } T[\Sigma_{g,n} \times S^1, U(N)] \text{ on } L(k,1)\end{array}} \end{array}. \tag{4.28}$$

And, of course, this relation is expected as both sides are labeled by flat connections of the Chan-Paton bundle associated to the coincident N "defect" M5-branes in (4.26). Using the relation (4.28), henceforth we identify $\mathcal{H}_{\mathrm{EV}}$ and $\mathcal{H}_{\mathrm{CB}}$.

The statement for a general group

The proposed relation (4.7) between the $U(N)$ equivariant Verlinde formula and the Coulomb branch index for $T[\Sigma, U(N)]$ can be generalized to other groups. First, one could consider decoupling the center of mass degree of freedom for all coincident stacks of M5-branes. However, there are at least two different ways of achieving this. Namely, one could get rid of the $\mathfrak{u}(1)$ part of $\mathbf{a}$ by either

1. subtracting the trace part from $\mathbf{a}$:

$$\mathbf{a}_{\mathrm{SU}} = \mathbf{a} - \frac{1}{N}\mathrm{tr}\,\mathbf{a}, \tag{4.29}$$

2. or forcing $\mathbf{a}$ to be traceless by imposing

$$a_N = -\sum_{i}^{N-1} a_i \tag{4.30}$$

to get

$$\mathbf{a}_{\text{PSU}} = \text{diag}(a_1, a_2, \ldots, a_{N-1}, -\sum_{i}^{N-1} a_i). \tag{4.31}$$

Naively, one may expect the two different approaches to be equivalent. However, as we are considering lens space index, the global structure of the group comes into play. Indeed, the integrality condition (4.23) becomes different:

$$e^{2\pi i k \cdot \mathbf{a}_{\text{SU}}} \in \mathbb{Z}_N = \mathcal{Z}(SU(N)) \tag{4.32}$$

while

$$e^{2\pi i k \cdot \mathbf{a}_{\text{PSU}}} = \mathbf{1} = \mathcal{Z}(PSU(N)). \tag{4.33}$$

Here $PSU(N) = SU(N)/\mathbb{Z}_N$ has trivial center but a non-trivial fundamental group. As a consequence of having different integrality conditions, one can get either Verlinde formula for $SU(N)$ or $PSU(N)$. In the first case we obtain

$$\boxed{\begin{array}{c}\text{Equivariant Verlinde formula}\\ \text{of } SU(N)_k \text{ on } \Sigma\end{array}} = \boxed{\begin{array}{c}\text{Coulomb branch index}\\ \text{of } T[\Sigma, PSU(N)] \text{ on } L(k,1)\times S^1\end{array}}. \tag{4.34}$$

The meaning of $T[\Sigma, PSU(N)]$ and the way to compute its Coulomb branch index will be discussed shortly. On the other hand, if one employs the second method to decouple the $U(1)$ factor, one finds a similar relation with the role of $SU(N)$ and $PSU(N)$ reversed:

$$\boxed{\begin{array}{c}\text{Equivariant Verlinde formula}\\ \text{of } PSU(N)_k \text{ on } \Sigma\end{array}} = \boxed{\begin{array}{c}\text{Coulomb branch index}\\ \text{of } T[\Sigma, SU(N)] \text{ on } L(k,1)\times S^1\end{array}}. \tag{4.35}$$

Before delving into these statements, we first give a proposal for a more general compact group[6] G:

$$\boxed{\begin{array}{c}\text{Equivariant Verlinde formula}\\ \text{of } G_k \text{ on } \Sigma\end{array}} = \boxed{\begin{array}{c}\text{Coulomb branch index}\\ \text{of } T[\Sigma, {}^L G] \text{ on } L(k,1)\times S^1\end{array}}, \tag{4.36}$$

where ${}^L G$ is the Langlands dual group of G. As ${}^L U(N) = U(N)$ and ${}^L SU(N) = PSU(N)$, (4.36) is a generalization of all the previous proposals. This general

[6]Currently, the right-hand side is only defined for group of Cartan type ADE. However, it is not inconceivable that $T[\Sigma, G]$ could also be defined for the B and C series.

proposal also gives a geometric/physical interpretation of the Coulomb index of $T[\Sigma, G]$ on $L(k,1)$ by relating it to the quantization of the Hitchin moduli space $\mathcal{M}_H(\Sigma, {}^L G)$. In fact, one can make a even more general conjecture for all 4d $\mathcal{N}=2$ superconformal theories (not necessarily of class $\mathcal{S}$):

$$\boxed{\begin{array}{c} L(k,1) \text{ Coulomb index of a} \\ \text{4d } \mathcal{N}=2 \text{ superconformal theory } \mathcal{T} \end{array}} \stackrel{?}{=} \boxed{\begin{array}{c} \text{Graded dimension of Hilbert space} \\ \text{from quantizing } (\widetilde{\mathcal{M}}_{\mathcal{T}}, k\omega_I) \end{array}}. \tag{4.37}$$

Here, $\widetilde{\mathcal{M}}_{\mathcal{T}}$ is the SYZ mirror [104] of the Coulomb branch $\mathcal{M}_{\mathcal{T}}$ of $\mathcal{T}$ on $\mathbb{R}^3 \times S^1$. Indeed, $\mathcal{M}_{\mathcal{T}}$ has the structure of a torus fibration:

$$\begin{array}{ccc} \mathbf{T}^{2d} & \hookrightarrow & \mathcal{M}_{\mathcal{T}} \\ & & \downarrow \\ & & \mathcal{B} \end{array} . \tag{4.38}$$

Here $\mathcal{B}$ is the d-(complex-)dimensional Coulomb branch of $\mathcal{T}$ on $\mathbb{R}^4$, $\mathbf{T}^{2d}$ is the 2d-torus parametrized by the holomonies of the low energy $U(1)^d$ gauge group along the spatial circle S^1 and the expectation values of d dual photons. One can perform T-duality on $\mathbf{T}^{2d}$ to obtain the mirror manifold[7] $\widetilde{\mathcal{M}}_{\mathcal{T}}$

$$\begin{array}{ccc} \widetilde{\mathbf{T}}^{2d} & \hookrightarrow & \widetilde{\mathcal{M}}_{\mathcal{T}} \\ & & \downarrow \\ & & \mathcal{B} \end{array} . \tag{4.39}$$

The dual torus $\widetilde{\mathbf{T}}^{2d}$ is a Kähler manifold equipped with a Kähler form ω, which extends to ω_I, one of the three Kähler forms $(\omega_I, \omega_J, \omega_K)$ of the hyper-Kähler manifold $\widetilde{\mathcal{M}}_{\mathcal{T}}$. Part of the R-symmetry that corresponds to the $U(1)_N - U(1)_R$ subgroup inside the $SU(2)_R \times U(1)_N$ R-symmetry group of $\mathcal{T}$ becomes a $U(1)_\beta$ symmetry of $\widetilde{\mathcal{M}}_{\mathcal{T}}$.

Quantizing $\widetilde{\mathcal{M}}_{\mathcal{T}}$ with respect to the symplectic form $k\omega_I$ yields a Hilbert space $\mathcal{H}(\mathcal{T}, k)$. Because $\widetilde{\mathcal{M}}_{\mathcal{T}}$ is non-compact, the resulting Hilbert space $\mathcal{H}(\mathcal{T}, k)$ is infinite-dimensional. However, because the fixed point set of $U(1)_\beta$ is compact and is contained in the nilpotent cone (= the fiber of $\widetilde{\mathcal{M}}_{\mathcal{T}}$ at the origin of $\mathcal{B}$), the

[7] In many cases, the mirror manifold $\widetilde{\mathcal{M}}_{\mathcal{T}} = \mathcal{M}_{\mathcal{T}'}$ is also the 3d Coulomb branch of a theory $\mathcal{T}'$ obtained by replacing the gauge group of $\mathcal{T}$ with its Langlands dual. One can easily see that $\mathcal{T}'$ obtained this way always has same 4d Coulomb branch B as $\mathcal{T}$.

following graded dimension is free of any divergences and can be computed with the help of the equivariant index theorem

$$\dim_\beta \mathcal{H}(\mathcal{T},k) = \sum_{m=0}^{\infty} \mathrm{t}^m \dim \mathcal{H}^m(\mathcal{T},k) = \int_{\widetilde{\mathcal{M}}_\mathcal{T}} \mathrm{ch}(\mathcal{L}^{\otimes k},\beta) \wedge \mathrm{Td}(\widetilde{\mathcal{M}}_\mathcal{T},\beta). \quad (4.40)$$

Here $\mathrm{t} = e^{-\beta}$ is identified with the parameter of the Coulomb branch index, $\mathcal{L}$ is a line bundle whose curvature is ω_I, and $\mathcal{H}^m(\mathcal{T},k)$ is the weight-m component of $\mathcal{H}(\mathcal{T},k)$ with respect to $U(1)_\beta$ action.

Now let us give a heuristic argument for why (4.40) computes the Coulomb branch index. The lens space $L(k,1)$ can be viewed as a torus fibered over an interval. Following [22, 24, 25] and [105], one can identify the Coulomb branch index with the partition function of a topological A-model living on a strip, with $\mathcal{M}_\mathcal{T}$ as the target space. The boundary condition at each end of the strip gives a certain brane in $\mathcal{M}_\mathcal{T}$. One can then apply mirror symmetry and turn the system into a B-model with $\widetilde{\mathcal{M}}_\mathcal{T}$ as the target space. Inside $\widetilde{\mathcal{M}}_\mathcal{T}$, there are two branes $\mathfrak{B}_1$ and $\mathfrak{B}_2$ specifying the boundary conditions at the two endpoints of the spatial interval. The partition function for this B-model computes the dimension of the Hom-space between the two branes:

$$Z_{\text{B-model}} = \dim \mathrm{Hom}(\mathfrak{B}_1,\mathfrak{B}_2). \quad (4.41)$$

Now $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are objects in the derived category of coherent sheaves on $\widetilde{\mathcal{M}}_\mathcal{T}$ and the quantity above can be computed using the index theorem. The equivariant version is

$$Z_{\text{B-model},\beta} = \dim_\beta \mathrm{Hom}(\mathfrak{B}_1,\mathfrak{B}_2) = \int_{\widetilde{\mathcal{M}}_\mathcal{T}} \mathrm{ch}(\mathfrak{B}_1^*,\beta) \wedge \mathrm{ch}(\mathfrak{B}_2,\beta) \wedge \mathrm{Td}(\widetilde{\mathcal{M}}_\mathcal{T},\beta). \quad (4.42)$$

We can choose the duality frame such that $\mathfrak{B}_1 = \mathcal{O}$ is the structure sheaf. Then $\mathfrak{B}_2$ is obtained by acting $T^k \in SL(2,\mathbb{Z})$ on $\mathfrak{B}_1$. A simple calculation shows $\mathfrak{B}_2 = \mathcal{L}^{\otimes k}$. So the Coulomb branch index indeed equals (4.40), confirming the proposed relation (4.37).

$SU(N)$ vs. $PSU(N)$

Now let us explain why (4.34) and (4.35) are expected. Both orbits, $\mathcal{O}_{\mathbf{a}_{\mathrm{SU}}}$ and $\mathcal{O}_{\mathbf{a}_{\mathrm{PSU}}}$, are quantizable and give rise to representations of $\mathfrak{su}(N)$. However, as the integrality conditions are different, there is a crucial difference between the two classes of representations that one can obtain from $\mathbf{a}_{\mathrm{SU}}$ and $\mathbf{a}_{\mathrm{PSU}}$. Namely, one can get all representations of $SU(N)_k$ from $\mathcal{O}_{\mathbf{a}_{\mathrm{SU}}}$ but only representations[8] of $PSU(N)_k$ from $\mathcal{O}_{\mathbf{a}_{\mathrm{PSU}}}$. This can be directly verified as follows.

For either $\mathbf{a}_{\mathrm{SU}}$ or $\mathbf{a}_{\mathrm{PSU}}$, quantizing $\mathcal{O}_{\mathbf{a}}$ gives a representation of $SU(N)$ with the highest weight[9]

$$\vec{\mu} = (h_1-h_N, h_2-h_N, \ldots, h_{N-1}-h_N) \equiv k(a_1-a_N, a_2-a_N, \ldots, a_{N-1}-a_N) \pmod N. \tag{4.44}$$

The corresponding Young tableau consists of $N-1$ rows with $h_i - h_N$ boxes in the i-th row. The integrality condition (4.32) simply says that $\vec{\mu}$ is integral. With no other constraints imposed, one can get all representations of $SU(N)$ from $\mathbf{a}_{\mathrm{SU}}$. On the other hand, the condition (4.33) requires the total number of boxes to be a multiple of N,

$$\sum_{i=1}^{N-1} \mu_i = N \cdot \sum_{i=1}^{N-1} a_i \equiv 0 \pmod N, \tag{4.45}$$

restricting us to these representations of $SU(N)$ where the center $\mathbb{Z}_N$ acts trivially. These are precisely the representations of $PSU(N)$.

What we have seen is that in the first way of decoupling $U(1)$, one arrives at the equivariant Verlinde algebra for $SU(N)_k$, while the second option leads to the $PSU(N)_k$ algebra. Then, what happens on the lens space side?

$T[\Sigma, SU(N)]$ vs. $T[\Sigma, PSU(N)]$

In the second approach of removing the center, the flavor $U(N)$-bundles become well-defined $SU(N)$-bundles on $L(k,1)$ and decoupling all the central $U(1)$'s on the lens space side simply means computing the lens space Coulomb branch index of $T[\Sigma, SU(N)]$. So we arrive at the equivalence (4.35) between $PSU(N)_k$ equivariant Verlinde algebra and the algebra of the Coulomb index TQFT for $SU(N)$. On the other hand, in the first way of decoupling the $U(1)$, the integrality condition

$$e^{2\pi i k\cdot \mathbf{a}} = 1 \tag{4.46}$$

[8] In our conventions, representations of $PSU(N)_k$ are those representations of $SU(N)_k$ invariant under the action of the center. There exist different conventions in the literature and one is related to ours by $k' = \lfloor k/N \rfloor$. Strictly speaking, when $N \nmid k$, the 3d Chern-Simons theory is not invariant under large gauge transformation and doesn't exist. Nonetheless, the 2d equivariant Verlinde algebra is still well defined and matches the algebra from the Coulomb index side.

[9] Sometimes it is more convenient to use a different convention for the highest weight

$$\vec{\lambda} = (h_1 - h_2, h_2 - h_3, \ldots, h_{N-1} - h_N) \equiv k \cdot (a_1 - a_2, a_2 - a_3, \ldots, a_{N-1} - a_N) \pmod N. \tag{4.43}$$

is not satisfied for $\mathbf{a}_{\mathrm{SU}}$. And as in (4.32), the right-hand side can be an arbitrary element in the center $\mathbb{Z}_N$ of $SU(N)$. In other words, after using the first method of decoupling the central $U(1)$, the $U(N)$-bundle over $L(k,1)$ becomes a $PSU(N) = SU(N)/\mathbb{Z}_N$-bundle. Another way to see this is by noticing that for $\exp[2\pi i \mathbf{a}] \in \mathcal{Z}(SU(N))$,

$$\mathbf{a}_{\mathrm{SU}} = \mathbf{a} - \frac{1}{N}\mathrm{tr}\,\mathbf{a} = 0. \tag{4.47}$$

This tells us that the $U(1)$ quotient done in this way has collapsed the $\mathbb{Z}_N$ center of $U(N)$, giving us not a well-defined $SU(N)$-bundle but a $PSU(N)$-bundle. Therefore, it is very natural to give the name "$T[\Sigma, PSU(N)]$" to the resulting theory living on $L(k,1) \times S^1$, as the class $\mathcal{S}$ theory $T[\Sigma, G]$ doesn't currently have proper definition in the literature if G is not simply-connected.

For a general group G, the path integral of $T[\Sigma, G]$ on $L(k,1) \times S^1$ is defined as the path integral of $T[\Sigma, \widetilde{G}]$ with summation over all possible 't Hooft fluxes labeled by $\pi_1(G) \subset \mathcal{Z}(\widetilde{G})$. Here $\widetilde{G}$ is the universal cover of G, *i.e.* the simply-connected Lie group with the Lie algebra $\mathfrak{g}$. This amounts to viewing flat connections on a G-bundle as the collection of flat connections on all of the $\widetilde{G}$-bundles, which are related to each other by twisting with a topologically non-trivial line bundle. In physics language, such a topology changing twist for a G-bundle amounts to having a non-trivial 't Hooft flux labeled by an element in $\mathcal{Z}(\widetilde{G})$, or equivalently a surface operator with central monodromy whose Levi subgroup is the entire group [74]. In our geometry, the flux tube lives on a $S^1 \subset L(k,1)$ that has linking number 1 with the Hopf fiber.

When G is a group of adjoint type (*i.e.* $\mathcal{Z}(G)$ is trivial), we will call the index of $T[\Sigma, G]$ defined this way the "full Coulomb branch index" of $T[\Sigma, \widetilde{G}]$, which sums over *all* elements of $\mathcal{Z}(\widetilde{G})$. As it contains the most information about the field theory, it is also the most interesting in the whole family associated to the Lie algebra $\mathfrak{g}$. This is not at all surprising as on the other side of the duality, the $\widetilde{G}$ equivariant Verlinde algebra involves all representations of $\mathfrak{g}$ and is the most interesting one among its cousins.

As for the A_{N-1} series that we will focus on in the rest of this chapter, we will be studying the correspondence (4.34) between the $SU(N)$ equivariant Verlinde algebra and the Coulomb index of $T[\Sigma, PSU(N)]$. But before going any further, we will first address a common concern that the reader may have. Namely, charge quantization appears to be violated in the presence of these non-integral $SU(N)$ holonomies. Shouldn't this suggest that the index is just zero with a non-trivial flux background? Indeed, for a state transforming under the fundamental representation of $SU(N)$, translation along the Hopf fiber of $L(k,1)$ k times gives a non-abelian Aharonov-Bohm phase

$$e^{2\pi i k \mathbf{a}_{\mathrm{SU}}}. \tag{4.48}$$

Since the loop is trivial in $\pi_1(L(k,1))$, one would expect this phase to be trivial. However, in the presence of a non-trivial 't Hooft flux, (4.48) is a non-trivial element in the center of $SU(N)$. Then the partition function with insertion of such an 't Hooft operator is automatically zero. However, this is actually what one must have in order to recover even the usual Verlinde formula in the $\mathfrak{t} = 0$ limit. As we will explain next, what is observed above in the $SU(2)$ case is basically the "selection rule" saying that in the decomposition of a tensor product

$$\text{(half integer spin)} \otimes \text{(integer spin)} \otimes \ldots \otimes \text{(integer spin)} \tag{4.49}$$

there is no representation with integer spins! What we will do next is to use Dirac quantization conditions in $T[\Sigma, PSU(N)]$ to derive the selection rule above and analogous rules for the $SU(N)$ Verlinde algebra.

Verlinde algebra and Dirac quantization

The Verlinde formula associates to a pair of pants a fusion coefficient f_{abc} which tells us how to decompose a tensor product of representations:

$$R_a \otimes R_b = \bigoplus_c f_{ab}{}^c R_c. \tag{4.50}$$

Equivalently, this coefficient gives the dimension of the invariant subspace of three-fold tensor products

$$\dim \mathrm{Inv}(R_a \otimes R_b \otimes R_c) = f_{abc}. \tag{4.51}$$

Here, upper and lower indices are related by the "metric"

$$\eta_{ab} = \dim \mathrm{Inv}(R_a \otimes R_b) = \delta_{a\bar{b}}, \tag{4.52}$$

which is what the TQFT associates to a cylinder.

In the case of $SU(N)$, the fusion coefficients f_{abc} are zero whenever a selection rule is not satisfied. For three representations labeled by the highest weights $\vec{\mu}^{(1)}, \vec{\mu}^{(2)}, \vec{\mu}^{(3)}$ in (4.44) the selection rule is

$$\sum_{i=1}^{N-1} (\mu_i^{(1)} + \mu_i^{(2)} + \mu_i^{(3)}) \equiv 0 \quad (\mathrm{mod}\ N). \tag{4.53}$$

This is equivalent to the condition that $\mathbb{Z}_N$ acts trivially on $R_a \otimes R_b \otimes R_c$. Of course, when this action is non-trivial, it is easy to see that there can't be any invariant subspace.

Our job now is to reproduce this rule on the Coulomb index side via Dirac quantization. We start with the familiar case of $SU(2)$. The theory $T_2 = T[\Sigma_{0,3}, SU(2)]$ consists of eight 4d $\mathcal{N} = 2$ half-hypermultiplets transforming in the tri-fundamental of the $SU(2)_a \times SU(2)_b \times SU(2)_c$ flavor symmetry. The holonomy $(H_a, H_b, H_c) \in U(1)^3$ of this flavor symmetry along the Hopf fiber is given by a triple(m_a, m_b, m_c) with

$$H_I = e^{2\pi i m_I/k}, \quad I = a, b, c. \tag{4.54}$$

The Dirac quantization requires that the Aharonov-Bohm phase associated with a trivial loop must be trivial. So, in the presence of the non-trivial holonomy along the Hopf fiber, a physical state with charge (e_a, e_b, e_c) needs to satisfy

$$H_a^{ke_a} H_b^{ke_b} H_c^{ke_c} = e^{2\pi i \sum_{I=a,b,c} e_I m_I} = 1, \tag{4.55}$$

or, equivalently,

$$\sum_{I=a,b,c} e_I m_I \in \mathbb{Z}. \tag{4.56}$$

When decomposed into representations of $U(1)^3$, the tri-fundamental hypermultiplet splits into eight components:

$$(\mathbf{2}, \mathbf{2}, \mathbf{2}) \rightarrow \bigoplus_{\text{All } \pm} (\pm 1, \pm 1, \pm 1). \tag{4.57}$$

Therefore, one needs to satisfy eight equations

$$\pm m_a \pm m_b \pm m_c \in \mathbb{Z}. \tag{4.58}$$

For individual m_I, the condition is

$$m_I \in \frac{\mathbb{Z}}{2}, \tag{4.59}$$

which is the same as the relaxed integrality condition (4.32) for $SU(2)$. This already suggests that the condition (4.32) is the most general one and there is no need to relax it further. Indeed, m_i is the "spin" of the corresponding $SU(2)$ representation and we know that all allowed values for it are integers and half-integers.

Besides the individual constraint (4.59), there is an additional one:

$$m_a + m_b + m_c \in \mathbb{Z}, \tag{4.60}$$

which is precisely the "selection rule" we mentioned before. Only when this rule is satisfied, could R_{m_c} appear in the decomposition of $R_{m_a} \otimes R_{m_b}$.

We then proceed to the case of $SU(N)$. When $N = 3$ the theory T_3 doesn't have a Lagrangian description but is conjectured to have E_6 global symmetry [106]. And the matter fields transform in the 78-dimensional adjoint representation of E_6 [107–109] which decomposes into $SU(3)^3$ representations as follows

$$\mathbf{78} = (\mathbf{3},\mathbf{3},\mathbf{3}) \oplus (\overline{\mathbf{3}},\overline{\mathbf{3}},\overline{\mathbf{3}}) \oplus (\mathbf{8},\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{1},\mathbf{8}). \tag{4.61}$$

The **8** is the adjoint representation of $\mathfrak{su}(3)$ and, being a representation for both $SU(3)$ and $PSU(3)$, imposes no additional restriction on 't Hooft fluxes. So we only need to understand the quantization condition in the presence of a tri-fundamental matter $(\mathbf{3},\mathbf{3},\mathbf{3})$. A natural question, then, is whether it happens more generally, *i.e.*,

$$\begin{array}{c}\text{Dirac quantization condition}\\ \text{for the } T_N \text{ theory}\end{array} = \begin{array}{c}\text{Dirac quantization condition}\\ \text{for a tri-fundamental matter.}\end{array} \tag{4.62}$$

This imposes on the T_N theory an interesting condition, which is expected to be true as it turns out to give the correct selection rule for $SU(N)$ Verlinde algebra.

Now, we proceed to determine the quantization condition for the tri-fundamental of $SU(N)^3$. We assume the holonomy in $SU(N)^3$ to be

$$(H_a, H_b, H_c), \tag{4.63}$$

where

$$H_I = \exp\left[\frac{2\pi i}{k}\mathrm{diag}\{m_{I1}, m_{I2}, \ldots, m_{IN}\}\right]. \tag{4.64}$$

The tracelessness condition looks like

$$\sum_{j=1}^{N} m_{Ij} = 0 \quad \text{for all } I = a, b, c. \tag{4.65}$$

We now have N^3 constraints given by

$$m_{aj_1} + m_{bj_2} + m_{cj_3} \in \mathbb{Z} \quad \text{for all choices of } j_1, j_2 \text{ and } j_3. \tag{4.66}$$

Using (4.65), one can derive the individual constraint for each $I = a, b, c$:[10]

$$\mathbf{m}_I \equiv \left(\frac{1}{N}, \frac{1}{N}, \frac{1}{N}, \ldots, \frac{1}{N}\right) \cdot \mathbb{Z} \quad (\mathrm{mod}\ \mathbb{Z}). \tag{4.67}$$

[10] In this chapter, bold letters like **m** are used to denote an element in the Cartan subalgebra of $\mathfrak{g}$. They are sometimes viewed as a diagonal matrix and sometimes a multi-component vector. The interpretation should be clear from the context.

This is exactly the same as (4.32). There is only one additional "selection rule" that needs to be satisfied:

$$\sum_{I=a,b,c}\sum_{j=1}^{N-1}(m_{Ij}-m_{IN})\equiv 0 \pmod N, \tag{4.68}$$

which coincides with (4.53). Therefore, we have demonstrated the equivalence between the Dirac quantization condition of the tri-fundamental and the selection rules in the $SU(N)$ Verlinde algebra. Since the argument is independent of the value of $\mathfrak{t}$, the same set of selection rules also applies to the equivariant Verlinde algebra.

Beside pairs of pants, one needs one more ingredient to build a 2d TQFT—the cylinder. It can be used to glue punctures together to build general Riemann surfaces. Each cylinder corresponds to a free 4d $\mathcal{N}=2$ vector multiplet. Since all of its components transform under the adjoint representation, it does not alter the individual constraints (4.67). However, the holonomies associated with the two punctures need to be the inverse of each other as the two flavor symmetries are identified and gauged. So the index of $T[\Sigma_{0,2}, SU(N)]$ gives a diagonal "metric"

$$\eta_{ab}\sim\delta_{a\bar{b}}. \tag{4.69}$$

The proportionality constant is $\mathfrak{t}$ dependent and will be determined in later sections.

We can also derive the the Dirac quantization condition for $T[\Sigma_{g,n}, PSU(N)]$. We use m_{Ij} to label the j-th component of the $U(1)^N$ holonomy associated to the I-th puncture. Then the index or any kind of partition function of $T[\Sigma_{g,n}, SU(N)]$ is zero unless

1. each $\vec{m}_I$ satisfies the individual constraint (4.67), and
2. an additional constraint analogous to (4.68),

$$\sum_{I=1}^{n}\sum_{j=1}^{N-1}(m_{Ij}-m_{IN})\equiv 0 \pmod N, \tag{4.70}$$

   is also satisfied.

To end this section, we will explain how the additional numerical factor in (4.11) in the introduction arises from non-trivial 't Hooft fluxes. For $G=SU(N)$, one has

$$Z_{\mathrm{EV}}(\Sigma, k=1,\mathfrak{t}) = N^g\cdot\left[\frac{1}{\prod_{i=1}^{\mathrm{rank}\,G}(1-\mathfrak{t}^{i+1})^{2i+1}}\right]^{g-1}. \tag{4.71}$$

Here we are only concerned with the first factor N^g which is the $k = 1$ Verlinde formula for $SU(N)$

$$Z_{\mathrm{EV}}(\Sigma, k = 1, \mathfrak{t} = 0) = N^g. \tag{4.72}$$

We now derive this result on the index side.

Consider the twice-punctured torus, obtained by gluing two pairs of pants. Let (a_1, a_2, a_3) and $(b_1, b_2, b_3) \in \mathbb{Z}_N^3$ label the 't Hooft fluxes corresponding to all six punctures. We glue a_2 with b_2, a_3 with b_3 to get $\Sigma_{1,2}$. Then we have the following set of constraints:

$$a_2 b_2 = 1, \;\; a_3 b_3 = 1, \tag{4.73}$$

and

$$a_1 a_2 a_3 = 1, \;\; b_1 b_2 b_3 = 1. \tag{4.74}$$

From these constraints, we can first confirm that

$$a_1 b_1 = 1, \tag{4.75}$$

which is what the selection rule (4.70) predicts. Then there is a free parameter a_2 that can take arbitrary values in $\mathbb{Z}_N$. So in the $\mathfrak{t} = 0$ limit, the Coulomb index TQFT associates to $\Sigma_{1,2}$

$$Z_{\mathrm{CB}}(\Sigma_{1,2}, SU(N), \mathfrak{t} = 0) = N \delta_{a_1, \overline{b_1}}. \tag{4.76}$$

We can now glue $g - 1$ twice-punctured tori to get

$$Z_{\mathrm{CB}}(\Sigma_{g-1,2}, SU(N), \mathfrak{t} = 0) = N^{g-1} \delta_{a_1, \overline{b}_{g-1}}. \tag{4.77}$$

Taking trace of this gives[11]

$$Z_{\mathrm{CB}}(\Sigma_{g,0}, SU(N), \mathfrak{t} = 0) = N^g. \tag{4.78}$$

Combining this with the $\mathfrak{t}$ dependent part of (4.11), we have proved that, for $k = 1$, the equivariant Verlinde formula is the same as the full Coulomb branch index.

We will now move on to cases with more general k to perform stronger checks.

[11] What we have verified is basically that the algebra of $\mathbb{Z}_N$ 't Hooft fluxes gives the $SU(N)$ Verlinde algebra at level $k = 1$, which is isomorphic to the group algebra of $\mathbb{Z}_N$. Another TQFT whose Frobenius algebra is also related to the group algebra of $\mathbb{Z}_N$ is the 2d $\mathbb{Z}_N$ Dijkgraaf-Witten theory [110]. However, the normalizations of the trace operator are different so the partition functions are also different.

4.3 A check of the proposal

In this section, we perform explicit computation of the Coulomb branch index for the theory $T[\Sigma_{g,n}, PSU(2)]$ in the presence of 't Hooft fluxes (or half-integral flavor holonomies). We will see that after taking into account a proper normalization, the full Coulomb branch index nicely reproduces the known $SU(2)$ equivariant Verlinde algebra. First, we introduce the necessary ingredients of 4d $\mathcal{N} = 2$ superconformal index on $S^1 \times L(k, 1)$ for a theory with a Lagrangian description.

The lens space index and its Coulomb branch limit

The lens space index of 4d $\mathcal{N} = 2$ theories is a generalization of the ordinary superconformal index on $S^1 \times S^3$, as $S^3 = L(1, 1)$ [111]. For $k > 1$, $L(k, 1)$ has a nontrivial fundamental group $\mathbb{Z}_k$, and a supersymmetric theory on $L(k, 1)$ tends to have a set of degenerate vacua labeled by holonomies along the Hopf fiber. This feature renders the lens space index a refined tool to study the BPS spectra of the superconformal theory; for instance it can distinguish between theories with gauge groups that have the same Lie algebra but different topologies (*e.g.* $SU(2)$ versus $SO(3)$ [112]). Moreover, as it involves not only continuous fugacities but also discrete holonomies, lens space indices of class $\mathcal{S}$ theories lead to a very large family of interesting and exotic 2d TQFTs [20, 21, 111].

The basic ingredients of the lens space index are indices of free supermultiplets, each of which can be conveniently expressed as a integral over gauge group of the plethystic exponential of the "single-letter index", endowed with gauge and flavor fugacities. This procedure corresponds to constructing all possible gauge invariant multi-trace operators that are short with respect to the superconformal algebra.

In particular, for a gauge vector multiplet the single-letter index is

$$f^V(p, q, t, m, k) = \frac{1}{1 - pq}\left(\frac{p^m}{1 - p^k} + \frac{q^{k-m}}{1 - q^k}\right)\left(pq + \frac{pq}{t} - 1 - t\right) + \delta_{m,0}, \quad (4.79)$$

where m will be related to holonomies of gauge symmetries. For a half-hypermultiplet, one has

$$f^{H/2}(p, q, t, m, k) = \frac{1}{1 - pq}\left(\frac{p^m}{1 - p^k} + \frac{q^{k-m}}{1 - q^k}\right)\left(\sqrt{t} - \frac{pq}{\sqrt{t}}\right). \quad (4.80)$$

In addition, there is also a "zero point energy" contribution for each type of field.

For a vector multiplet and a half hypermultiplet, they are given by

$$\begin{aligned} I^0_V(p,q,t,\mathbf{m},k) &= \prod_{\alpha\in\Delta^+}\left(\frac{pq}{t}\right)^{-[\![\alpha(\mathbf{m})]\!]_k+\frac{1}{k}[\![\alpha(\mathbf{m})]\!]_k^2}, \\ I^0_{H/2}(p,q,t,\mathbf{m},\widetilde{\mathbf{m}},k) &= \prod_{\rho\in\Re}\left(\frac{pq}{t}\right)^{\frac{1}{4}\left([\![\rho(\mathbf{m},\widetilde{\mathbf{m}})]\!]_k-\frac{1}{k}[\![\rho(\mathbf{m},\widetilde{\mathbf{m}})]\!]_k^2\right)}, \end{aligned} \tag{4.81}$$

where $[\![x]\!]_k$ denotes remainder of x divided by k. The boldface letters $\mathbf{m}$ and $\widetilde{\mathbf{m}}$ label holonomies for, respectively, gauge symmetries and flavor symmetries[12]; they are chosen to live in the Weyl alcove and can be viewed as a collection of integers $m_1 \geq m_2 \geq \cdots \geq m_r$.

Now the full index can be written as

$$\begin{aligned} \mathcal{I} &= \sum_{\mathbf{m}} I^0_V(p,q,t,\mathbf{m}) I^0_{H/2}(p,q,t,\mathbf{m},\widetilde{\mathbf{m}}) \int \prod_i \frac{dz_i}{2\pi i z_i}\Delta(z)_{\mathbf{m}} \\ &\times \exp\left(\sum_{n=1}^{+\infty}\sum_{\alpha,\rho}\frac{1}{n}\left[f^V(p^n,q^n,t^n,\alpha(\mathbf{m}))\alpha(z)+f^{H/2}(p^n,q^n,t^n,\rho(\mathbf{m},\widetilde{\mathbf{m}}))\rho(z,F)\right]\right). \end{aligned} \tag{4.82}$$

Here, to avoid clutter, we only include one vector multiplet and one half-hypermultiplet. Of course, in general one should remember to include the entire field contents of the theory. Here, F stands for the continuous flavor fugacities and the z_i's are the gauge fugacities; for $SU(N)$ theories one should impose the condition $z_1z_2\ldots z_N = 1$. The additional summation in the plethystic exponential is over all the weights in the relevant representations. The integration measure is determined by $\mathbf{m}$:

$$\Delta_{\mathbf{m}}(z_i) = \prod_{i,j;m_i=m_j}\left(1-\frac{z_i}{z_j}\right), \tag{4.83}$$

since a nonzero holonomy would break the gauge group into its stabilizer.

In this chapter we are particularly interested in the Coulomb branch limit, *i.e.* (4.15) and (4.16). From the single letter index (4.79) and (4.80) we immediately conclude that $f^{H/2} = 0$ identically, so the hypermultiplets contribute to the index only through the zero point energy. As for f^V, the vector multiplet gives a non-zero contribution $pq/t = \mathfrak{t}$ for each root α that has $\alpha(\mathbf{m}) = 0$. So the zero roots (Cartan generators) always contribute, and non-zero roots can only contribute when

[12] As before, the holonomies are given by $e^{2\pi i\mathbf{m}/k}$.

the gauge symmetry is enhanced from $U(1)^r$, *i.e.* when $\mathbf{m}$ is at the boundary of the Weyl alcove. This closely resembles the behavior of the "metric" of the equivariant Verlinde algebra, as we will see shortly.

More explicitly, for $SU(2)$ theory, the index of a vector multiplet in the Coulomb branch limit is

$$I_V(\mathrm{t}, m, k) = \mathrm{t}^{-[\![2m]\!]_k+\frac{1}{k}[\![2m]\!]_k^2}\left(\frac{1}{1-\mathrm{t}}\right)\left(\frac{1}{1+\mathrm{t}}\right)^{\delta_{[\![2m]\!],0}}, \tag{4.84}$$

while for tri-fundamental hypermultiplet the contribution is

$$I_{H/2}(\mathrm{t}, m_1, m_2, m_3, k) = \prod_{s_i=\pm}(\mathrm{t})^{\frac{1}{4}\sum_{i=1}^{3}\left([\![m_i s_i]\!]_k-\frac{1}{k}[\![m_i s_i]\!]_k^2\right)}, \tag{4.85}$$

where all holonomies take values from $\{0, 1/2, 1, 3/2, \ldots k/2\}$.

Unsurprisingly, this limit fits the name of the "Coulomb branch index." Indeed, in the case of $k = 1$, the index receives only contributions from the Coulomb branch operators, *i.e.* a collection of "Casimir operators" for the theory [19] (*e.g.* $\mathrm{Tr}\phi^2$, $\mathrm{Tr}\phi^3, \ldots, \mathrm{Tr}\phi^N$ for $SU(N)$, where ϕ is the scalar in the $\mathcal{N} = 2$ vector multiplet). We see here that a general lens space index also counts the Coulomb branch operators, but the contribution from each operator is modified according to the background holonomies.

Another interesting feature of the Coulomb branch index is the complete disappearance of continuous fugacities of flavor symmetries. Punctures are now only parametrized by discrete holonomies along the Hopf fiber of $L(k, 1)$. This property ensures that we will obtain a *finite-dimensional* algebra.

Then, to make sure that the algebra defines a TQFT, one needs to check associativity, especially because non-integral holonomies considered here are novel and may cause subtleties. We have checked by explicit computation in t that the structure constant and metric defined by lens space index do satisfy associativity, confirming that the "Coulomb branch index TQFT" is indeed well-defined. In fact, even with all p, q, t turned on, the associativity still holds order by order in the expansion in terms of fugacities.

Equivariant Verlinde algebra from Hitchin moduli space

As explained in greater detail in Chapter 2, the equivariant Verlinde TQFT computes an equivariant integral over $\mathcal{M}_H$, the moduli space of Higgs bundles (4.8). In the case of $SU(2)$, the relevant moduli spaces are simple enough and one can deduce the TQFT algebra from geometry of $\mathcal{M}_H$. For example, one can obtain the fusion coefficients from $\mathcal{M}_H(\Sigma_{0,3}, \alpha_1, \alpha_2, \alpha_3; SU(2))$. Here the α_i's are the ramification data specifying the monodromies of the gauge field [74] and take discrete values in the presence of a level k Chern-Simons term. Since in this case the moduli space is just a point or empty, one can directly evaluate the integral. The result is as follows.

Define $\lambda = 2k\alpha$ whose value is quantized to be $0, 1, \ldots, k$. Let

$$\begin{aligned} d_0 &= \lambda_1 + \lambda_2 + \lambda_3 - 2k, \\ d_1 &= \lambda_1 - \lambda_2 - \lambda_3, \\ d_2 &= \lambda_2 - \lambda_3 - \lambda_1, \\ d_1 &= \lambda_3 - \lambda_1 - \lambda_2, \end{aligned} \tag{4.86}$$

and moreover

$$\Delta\lambda = \max(d_0, d_1, d_2, d_3), \tag{4.87}$$

then[13]

$$f_{\lambda_1\lambda_2\lambda_3} = \begin{cases} 1 & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is even and } \Delta\lambda \le 0, \\ \mathrm{t}^{-\Delta\lambda/2} & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is even and } \Delta\lambda > 0, \\ 0 & \text{if } \lambda_1 + \lambda_2 + \lambda_3 \text{ is odd.} \end{cases} \tag{4.88}$$

On the other hand, the cylinder gives the trace form (or "metric") of the algebra

$$\eta_{\lambda_1\lambda_2} = \{1 - \mathrm{t}^2, 1 - \mathrm{t}, \ldots, 1 - \mathrm{t}, 1 - \mathrm{t}^2\}. \tag{4.89}$$

Via cutting-and-gluing, we can compute the partition function of the TQFT on a general Riemann surface $\Sigma_{g,n}$.

Matching two TQFTs

So far we have introduced two TQFTs: the first one is given by equivariant integration over Hitchin moduli space $\mathcal{M}_H$, the second one is given by the $L(k,1)$ Coulomb branch index of the theory $T[\Sigma, PSU(2)]$. It is easy to see that the underlying vector space of the two TQFTs are the same, confirming in the $SU(2)$ case the more general result we obtained previously:

$$Z_{\mathrm{EV}}(S^1) = Z_{\mathrm{CB}}(S^1). \tag{4.90}$$

[13]In this chapter, we use lower indices (as opposed to upper indices in Chapter 2) for the fusion coefficients defined in this way.

We can freely switch between two different descriptions of the same set of basis vectors, by either viewing them as integrable highest weight representations of $\widehat{su}(2)_k$ or $SU(2)$ holonomies along the Hopf fiber. In this section, we only use highest weights λ as the labels for puncture data, and one can easily translate them into holonomies via $\lambda = 2m$.

Then, one needs to compare the algebraic structure of the two TQFTs and may notice that there are apparent differences. Namely, if one compares $I_{H/2}$ and I_V with f and η in (4.88) and (4.89), there are additional factors coming from the zero point energy in the expressions on the index side. However, one can simply rescale states in the Hilbert space on the Coulomb index side to absorb them.

The scaling required is

$$|\lambda\rangle = \mathrm{t}^{\frac{1}{2}\left([\![\lambda]\!]_k - \frac{1}{k}[\![\lambda]\!]_k^2\right)}|\lambda\rangle'. \tag{4.91}$$

This makes I_V exactly the same as $\eta^{\lambda\mu}$. After rescaling, the index of the half-hypermultiplet becomes

$$I_{H/2} \Rightarrow f'_{\lambda_1\lambda_2\lambda_3} = \mathrm{t}^{-\frac{1}{2}\sum_{i=1}^{3}\left([\![\lambda_i]\!]_k - \frac{1}{k}[\![\lambda_i]\!]_k^2\right)} I_{H/2}(\mathrm{t}, \lambda_1, \lambda_2, \lambda_3, k), \tag{4.92}$$

and this is indeed identical to the fusion coefficient $f_{\lambda\mu\nu}$ of the equivariant Verlinde algebra, which we show as follows. If we define

$$\begin{aligned} g_0 &= m_1 + m_2 + m_3 = \frac{1}{2}\left(\lambda_1 + \lambda_2 + \lambda_3\right), \\ g_1 &= m_1 - m_2 - m_3 = \frac{1}{2}\left(\lambda_1 - \lambda_2 - \lambda_3\right), \\ g_2 &= m_2 - m_1 - m_3 = \frac{1}{2}\left(\lambda_2 - \lambda_1 - \lambda_3\right), \\ g_3 &= m_3 - m_1 - m_2 = \frac{1}{2}\left(\lambda_3 - \lambda_1 - \lambda_3\right), \end{aligned} \tag{4.93}$$

then our pair of pants can be written as

$$\begin{aligned} f'_{\lambda_1\lambda_2\lambda_3} =& \mathrm{t}^{\frac{1}{2k}\left([\![g_0]\!]_k[\![-g_0]\!]_k + [\![g_1]\!]_k[\![-g_1]\!]_k + [\![g_2]\!]_k[\![-g_2]\!]_k + [\![g_2]\!]_k[\![-g_2]\!]_k\right)} \\ & \times \mathrm{t}^{-\frac{1}{2k}\left(\lambda_1(k-\lambda_1) + \lambda_2(k-\lambda_2) + \lambda_3(k-\lambda_3)\right)}. \end{aligned} \tag{4.94}$$

Now we can simplify the above equation further under various assumptions of each g_i. For instance if $0 < g_0 < k$ and $g_i < 0$ for $i = 1, 2, 3$, then

$$f'_{\lambda_1\lambda_2\lambda_3} = 1. \tag{4.95}$$

	$k=1$	$k=2$	$k=3$	$k=4$
$g=2$	$\frac{4}{(1-t^2)^3}$	$\frac{2}{(1-t^2)^3}(5t^2+6t+5)$	$\frac{4}{(1-t^2)^3}(4t^3+9t^2+9t+5)$	$\frac{1}{(1-t^2)^3}\left(16t^4+49t^3 +81t^2+75t+35\right)$
$g=3$	$\frac{8}{(1-t^2)^6}$	$\frac{4}{(1-t^2)^6}\left(9t^4+28t^3 +54t^2+28t+9\right)$	$\frac{8}{(1-t^2)^6}\left(8t^6+54t^5+159t^4 +238t^3+183t^2+72t+15\right)$	$\frac{1}{(1-t^2)^6}\left(64t^8+384t^7+1793t^6 +5250t^5+8823t^4+8828t^3 +5407t^2+1890t+329\right)$
$\forall g$	$2\left(\frac{2}{(1-t^2)^3}\right)^{g-1}$	$\left(\frac{2(1-t)^2}{(1-t^2)^3}\right)^{g-1} +2\left(\frac{2(1+t)^2}{(1-t^2)^3}\right)^{g-1}$	$2\left(\frac{5+9t+9t^2+4t^3-\sqrt{5+4t}(1+5t+t^2)}{(1-t^2)^3}\right)^{g-1}+ 2\left(\frac{5+9t+9t^2+4t^3+\sqrt{5+4t}(1+5t+t^2)}{(1-t^2)^3}\right)^{g-1}$	$\left(\frac{(3+t)(1-t)^2}{(1-t^2)^3}\right)^{g-1}+2\left(\frac{4}{1-t^2}\right)^{g-1} +\left(\frac{4(3+t)(1+t)^3}{(1-t^2)^3}\right)^{g-1}$

Table 4.1: The partition function $Z_{\text{EV}}(T[L(k,1),SU(2)],\mathfrak{t}) = Z_{\text{CB}}(T[\Sigma_g,PSU(2)],\mathfrak{t})$ for genus $g=2,3$ and level $k=1,2,3,4$.

If on the other hand, $g_0 > k$ and $g_i < 0$ for $i=1,2,3$, which means $\max(g_0-k,g_1,g_2,g_3)=g_0-k$, then

$$f'_{\lambda_1\lambda_2\lambda_3} = t^{g_0-k}, \tag{4.96}$$

this is precisely what we obtained by (4.88).

Therefore, we have shown that the building blocks of the two TQFTs are the same. And by the TQFT axioms, we have proven the isomorphism of the two TQFTs. For example, they both give $\mathfrak{t}$-deformation of the $\widehat{su}(2)_k$ representation ring; at level $k=10$ a typical example is

$$|3\rangle\otimes|3\rangle = \frac{1}{1-\mathfrak{t}^2}|0\rangle\oplus\frac{1}{1-\mathfrak{t}}|2\rangle\oplus\frac{1}{1-\mathfrak{t}}|4\rangle\oplus\frac{1}{1-\mathfrak{t}}|6\rangle\oplus\frac{\mathfrak{t}}{1-\mathfrak{t}}|8\rangle\oplus\frac{\mathfrak{t}^2}{1-\mathfrak{t}^2}|10\rangle. \tag{4.97}$$

For closed Riemann surfaces, we list partition functions for several low genera and levels in table 4.1. And this concludes our discussion of the $SU(2)$ case.

4.4 $SU(3)$ equivariant Verlinde algebra from the Argyres-Seiberg duality

In the last section, we have tested the proposal about the equivalence between the equivariant Verlinde algebra and the algebra from the Coulomb index of class $\mathcal{S}$ theories. Then one would ask whether one can do more with such a correspondence and what are its applications. For example, can one use the Coulomb index as a tool to access geometric and topological information about Hitchin moduli spaces? Indeed, the study of the moduli space of Higgs bundles poses many interesting and challenging problems. In particular, doing the equivariant integral directly on $\mathcal{M}_H$ quickly becomes unpractical when one increases the rank of the gauge group. However, our proposal states that the equivariant integral could be computed in a completely different way by looking at the superconformal index of familiar SCFTs! This is exactly what we will do in this section—we will put the correspondence to good use and probe the geometry of $\mathcal{M}_H(\Sigma, SU(3))$ with superconformal indices.

The natural starting point is still a pair of pants or, more precisely, a sphere with three "maximal" punctures (for mathematicians, three punctures with full-flag parabolic structure). The 4d theory $T[\Sigma_{0,3}, SU(3)]$ is known as the T_3 theory [113], which is first identified as an $\mathcal{N} = 2$ strongly coupled rank-1 SCFT with a global E_6 symmetry[14] [106]. In light of the proposed correspondence, one expects that the Coulomb branch index of the T_3 theory equals the fusion coefficients $f_{\lambda_1\lambda_2\lambda_3}$ of the $SU(3)$ equivariant Verlinde algebra.

Argyres-Seiberg duality and Coulomb branch index of T_3 theory

A short review

As the T_3 theory is an isolated SCFT, there is no Lagrangian description, and currently no method of direct computation of its index is known in the literature. However, there is a powerful duality proposed by Argyres and Seiberg [109], that relates a superconformal theory with Lagrangian description at infinite coupling to a weakly coupled gauge theory obtained by gauging an $SU(2)$ subgroup of the E_6 flavor symmetry of the T_3 SCFT.

To be more precise, one starts with an $SU(3)$ theory with six hypermultiplets (call it theory A) in the fundamental representation $3\square \oplus 3\overline{\square}$ of the gauge group. Unlike its $SU(2)$ counterpart, the $SU(3)$ theory has the electric-magnetic duality group $\Gamma^0(2)$, a subgroup of $SL(2, \mathbb{Z})$. As a consequence, the fundamental domain of the gauge coupling τ has a cusp and the theory has an infinite coupling limit. As argued by Argyres and Seiberg through direct analysis of the Seiberg-Witten curve at strong couplings, it was shown that the theory can be naturally identified as another theory B obtained by weakly gauging the E_6 SCFT coupled to an additional hypermultiplet in fundamental representation of $SU(2)$. There is much evidence supporting this duality picture. For instance, the E_6 SCFT has a Coulomb branch operator with dimension 3, which could be identified as the second Casimir operator $\mathrm{Tr}\phi^3$ of the dual $SU(3)$ gauge group. The E_6 theory has a Higgs branch of $\dim_{\mathbb{C}} \mathcal{H} = 22$ parametrized by an operator $\mathbb{X}$ in adjoint representation of E_6 with Joseph relation [107]; after gauging $SU(2)$ subgroup, two complex dimensions are removed, leaving the correct dimension of the Higgs branch for the theory A. Finally, Higgsing this

[14] In the following we will use the name "T_3 theory" and "E_6 SCFT" interchangeably.

$SU(2)$ leaves an $SU(6) \times U(1)$ subgroup of the maximal E_6 group, which is the same as the $U(6) = SU(6) \times U(1)$ flavor symmetry in the A frame.

In [114], the Argyres-Seiberg duality is given a nice geometric interpretation. To obtain theory A, one starts with a 2-sphere with two $SU(3)$ maximal punctures and two $U(1)$ simple punctures, corresponding to global symmetry $SU(3)_a \times SU(3)_b \times U(1)_a \times U(1)_b$, where two $U(1)$ are baryonic symmetry. In this setup, the Argyres-Seiberg duality relates different degeneration limits of this Riemann surface, see figure 4.1 and 4.2.

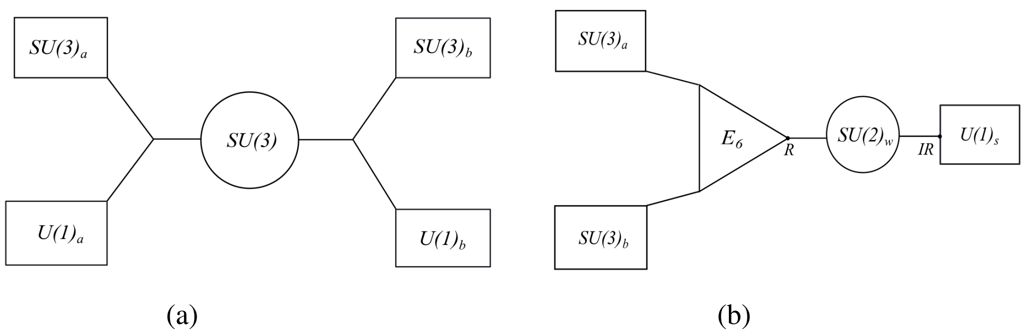


Figure 4.1: Illustration of Argyres-Seiberg duality. (a) The theory A, which is an $SU(3)$ superconformal gauge theory with six hypermultiplets, with the $SU(3)_a \times U(1)_a \times SU(3)_b \times U(1)_b$ subgroup of the global $U(6)$ flavor symmetry. (b) The theory B, obtained by gauging an $SU(2)$ subgroup of the E_6 symmetry of T_3. Note in the geometric realization the cylinder connecting both sides has a regular puncture R on the left and an irregular puncture IR on the right.

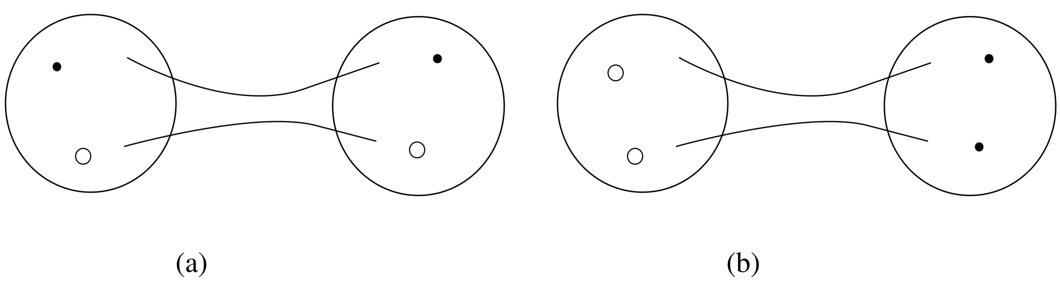


Figure 4.2: Illustration of geometric realization of Argyres-Seiberg duality for T_3 theory. The dots represent simple punctures while circles are maximal punctures. (a) The theory A, which is an $SU(3)$ superconformal gauge theory with six hypermultiplets, is pictured as two spheres connected by a long tube. Each of them has two maximal and one simple punctures. (b) The theory B, which is obtained by gauging an $SU(2)$ subgroup of the flavor symmetry of the theory T_3. This gauge group connects a regular puncture and an irregular puncture.

The Argyres-Seiberg duality gives access to the superconformal index for the E_6 SCFT [108]. The basic idea is to start with the index of theory A and, with the aid of the inversion formula of elliptic beta integrals, one identifies two sets of flavor fugacities and extracts the E_6 SCFT index by integrating over a carefully chosen kernel. It was later realized that the above procedure has a physical interpretation, namely the E_6 SCFT can be obtained by flowing to the IR from an $\mathcal{N} = 1$ theory which has Lagrangian description [115]. The index computation of the $\mathcal{N} = 1$ theory reproduces that of [108], and the authors also compute the Coulomb branch index in the large k limit.

Here we would like to obtain the index for general k. In principle, we could start with the $\mathcal{N} = 1$ theory described in [115] and compute the Coulomb branch index on lens space directly. However, a direct inversion is more intuitive here due to simplicity of the Coulomb branch limit, and can be generalized to arbitrary T_N theories. In the next subsection we outline the general procedure of computing the Coulomb branch index of T_3.

Computation of the index

To obtain a complete basis of the TQFT Hilbert space, we need to turn on all possible flavor holonomies and determine when they correspond to a weight in the Weyl alcove. For the T_3 theory each puncture has $SU(3)$ flavor symmetry, so we can turn on holonomies as $\mathbf{h}^* = (h_1^*, h_2^*, h_3^*)$ for $* = a, b, c$ with constraints $h_1^* + h_2^* + h_3^* = 0$. The Dirac quantization condition tells us that

$$h_i^r + h_j^s + h_k^t \in \mathbb{Z} \tag{4.98}$$

for arbitrary $r, s, t \in \{a, b, c\}$ and $i, j, k = 1, 2, 3$. This means there are only three classes of choices modulo $\mathbb{Z}$, namely

$$\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right), \quad \text{or} \quad \left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right), \quad \text{or} \quad (0, 0, 0) \qquad (\mathrm{mod}\ \mathbb{Z}). \tag{4.99}$$

Furthermore, the three punctures either belong to the same class (for instance, all are $(1/3, 1/3, -2/3) \pmod{\mathbb{Z}}$) or to three distinct classes. Recall that the range of the holonomy variables are also constrained by the level k, so we pick out the Weyl alcove as the following:

$$D(k) = \{(h_1, h_2, h_3) | h_1 \geq h_2, h_1 \geq -2h_2, 2h_1 + h_2 \leq k\}, \tag{4.100}$$

with a pictorial illustration in figure 4.3.

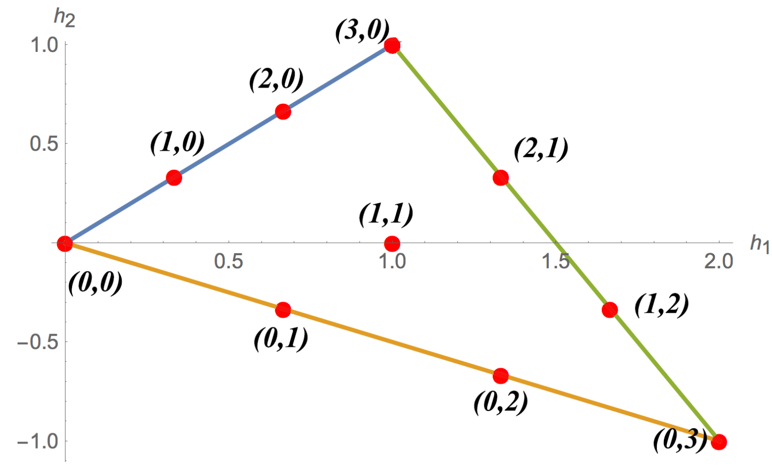


Figure 4.3: The Weyl alcove for the choice of holonomy variables at level $k = 3$. The red markers represent the allowed points. The coordinates beside each point denote the corresponding highest weight representation. The transformation between flavor holonomies and highest weight is given by (4.101).

As we will later identify each holonomy as an integrable highest weight representation for the affine Lie algebra $\widehat{su}(3)_k$, it is more convenient to use the label (λ_1, λ_2) defined as

$$\lambda_1 = h_2 - h_3, \quad \lambda_2 = h_1 - h_2. \tag{4.101}$$

They are integers with $\lambda_1 + \lambda_2 \leq k$ and (λ_1, λ_2) lives on the weight lattice of $su(3)$. The dimension of the representation with the highest weight (λ_1, λ_2) is

$$\dim R_{(\lambda_1,\lambda_2)} = \frac{1}{2}(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_1 + \lambda_2 + 2). \tag{4.102}$$

Next we proceed to compute the index in the Coulomb branch limit. As taking the Coulomb branch limit simplifies the index computation dramatically, one can easily write down the index for theory A[15]:

$$\begin{aligned}&\mathcal{I}_A(\mathrm{t}, \widetilde{\mathbf{m}}_a, \widetilde{\mathbf{m}}_b, n_a, n_b)\\ &\qquad = \sum_{\mathbf{m}} I_{H/2}(\mathrm{t}, \mathbf{m}, \widetilde{\mathbf{m}}_a, n_a) \int \prod_{i=1}^{2} \frac{dz_i}{2\pi i z_i} \Delta(z)_{\mathbf{m}} I_V(\mathrm{t}, z, \mathbf{m}) I_{H/2}(\mathrm{t}, -\mathbf{m}, \widetilde{\mathbf{m}}_b, n_b),\end{aligned} \tag{4.103}$$

[15]In [115] the authors try to compensate for the non-integral holonomies of n_a and n_b by shifting the gauge holonomies $\mathbf{m}$. In contrast, our approach is free from such subtleties because we allow non-integral holonomies for all flavor symmetries as long as the Dirac quantization condition is obeyed.

where $\mathbf{m}_a, \mathbf{m}_b$ and n_a, n_b denote the flavor holonomies for $SU(3)_{a,b}$ and $U(1)_{a,b}$ respectively. It is illustrative to write down what the gauge integrals look like:

$$\begin{aligned} I_V(\mathrm{t}, \mathbf{m}) &= \int \prod_{i=1}^{2} \frac{dz_i}{2\pi i z_i} \Delta(z)_{\mathbf{m}} I_V(\mathrm{t}, z, \mathbf{m}) \\ &= I_V^0(\mathrm{t}, \mathbf{m}) \times \begin{cases} \frac{1}{(1-t^2)(1-t^3)}, & m_1 \equiv m_2 \equiv m_3 \pmod{k}, \\ \frac{1}{(1-t)(1-t^2)}, & m_i \equiv m_j \neq m_k \pmod{k}, \\ \frac{1}{(1-t)^2}, & m_1 \neq m_2 \neq m_3 \pmod{k}. \end{cases} \end{aligned} \tag{4.104}$$

Except the zero point energy $I_V^0(\mathrm{t}, \mathbf{m})$ the rest looks very much alike our “metric" for the $SU(3)$ equivariant Verlinde TQFT. Moreover,

$$I_{H/2}(\mathbf{m}, \widetilde{\mathbf{m}}_a, n_a) = \prod_{\psi \in R_\Phi} \mathrm{t}^{\frac{1}{4}\left([\![\psi(\mathbf{m}, \widetilde{\mathbf{m}}_a, n_a)]\!]_k - \frac{1}{k} [\![\psi(\mathbf{m}, \widetilde{\mathbf{m}}_a, n_a)]\!]_k^2 \right)}, \tag{4.105}$$

where for a half-hypermultiplet in the fundamental representation of $SU(3) \times SU(3)_a$ with positive $U(1)_a$ charge we have

$$\psi_{ij}(\mathbf{m}, \widetilde{\mathbf{m}}_a, n_a) = \mathbf{m}_i + \widetilde{\mathbf{m}}_{a,j} + n_a. \tag{4.106}$$

Now we write down the index for theory B. Take the $SU(3)_a \times SU(3)_b \times SU(3)_c$ maximal subgroup of E_6 and gauge $SU(2)$ subgroup of the $SU(3)_c$ flavor symmetry. This leads to the replacement

$$\{h_{c,1}, h_{c,2}, h_{c,3}\} \to \{w + n_y, n_y - w, -2n_y\}, \tag{4.107}$$

where n_y denotes the fugacity for the remaining $U(1)_y$ symmetry, and n_s is the fugacity for $U(1)_s$ flavor symmetry rotating the single hypermultiplet. We then write down the index of theory B as

$$\mathcal{I}_B(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, n_y, n_s) = \sum_{w} C^{E_6}(\mathbf{h}_a, \mathbf{h}_b, w, n_y) I_V(\mathrm{t}, w) I_{H/2}(-w, n_s), \tag{4.108}$$

where $I_V(\mathrm{t}, w)$ is given by (4.84) with substitution $m \to w$, and $w = 0, 1/2, \ldots, k/2$. Argyres-Seiberg duality tells us that

$$\mathcal{I}_A(\mathrm{t}, \widetilde{\mathbf{m}}_a, \widetilde{\mathbf{m}}_b, n_a, n_b) = \mathcal{I}_B(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, n_y, n_s), \tag{4.109}$$

with the following identification of the holonomy variables:

$$\begin{aligned}\widetilde{\mathbf{m}}_a &= \mathbf{h}_a, \quad \widetilde{\mathbf{m}}_b = \mathbf{h}_b;\\ n_a &= \frac{1}{3}n_s - n_y, \quad n_b = -\frac{1}{3}n_s - n_y.\end{aligned} \tag{4.110}$$

On the right-hand side of the expression (4.108) we can view the summation as a matrix multiplication with w and n_s being the row and column indices respectively. Then we can take the inverse of the matrix $I_{H/2}(-w, n_s)$, $I^{-1}_{H/2}(n_s, w')$, by restricting the range[16] of n_s to be the same as w and multiply it to both sides of (4.108). This moves the summation to the other side of the equation and gives:

$$\boxed{C^{E_6}(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, w, n_y, k) = \sum_{n_s} \frac{1}{I_V(\mathrm{t}, w)} \mathcal{I}_A(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, n_a, n_b, k) I^{-1}_{H/2}(n_s, w)}\,. \tag{4.111}$$

We now regard $C^{E_6}(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, \mathbf{h}_c, k)$ as the fusion coefficient of the 2d equivariant Verlinde algebra, and have checked the associativity. Moreover, let us confirm that the index obtained in this way is symmetric under permutations of the three $SU(3)$ flavor fugacities, and the flavor symmetry group is indeed enhanced to E_6. First of all, we have permutation symmetry for three $SU(3)$ factors at, for instance, level $k = 2$:

$$C^{E_6}\left(\frac{2}{3}, \frac{2}{3}, 0, 0, \frac{4}{3}, -\frac{2}{3}\right) = C^{E_6}\left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}, -\frac{2}{3}, 0, 0\right) = \cdots = C^{E_6}\left(\frac{4}{3}, -\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, 0, 0\right) = \frac{1 + \mathrm{t}^4}{1 - \mathrm{t}^3}. \tag{4.112}$$

To show that the index C^{E_6} is invariant under the full E_6 symmetry, one needs to show that the two $SU(3)$ factors, combined with the $U(1)_y$ symmetry, enhance to an $SU(6)$ symmetry. The five Cartan elements of this $SU(6)$ group can be expressed as the combination of the fluxes [115]:

$$\left(h^a_1 - n_y, h^a_2 - n_y, -h^a_1 - h^a_2 - n_y, h^b_1 + n_y, h^b_2 + n_y\right). \tag{4.113}$$

Then the index should be invariant under the permutation of the five Cartans. Note the computation is almost the same as in [115] except that not all permutations necessarily exist. An allowed permutation should satisfy the charge quantization condition. Restraining ourselves from the illegal permutations, we have verified that the global symmetry is enlarged to E_6.

[16] As long as it satisfies the Dirac quantization condition, we do not have to know what the range of n_s should be. For example, $n_s = 0, 1/2, \ldots, k/2$ is a valid choice.

Finally, at large k our results reproduce these of [115], as can be checked by analyzing the large k limit of the matrix $I^{-1}_{H/2}(n_s, w)$. Indeed, at large k the matrix $I_{H/2}(w, n_s)$ can be simplified as

$$I_{H/2} = \mathrm{t}^{\frac{1}{2}(|w+n_s|+|-w+n_s|)} = \begin{pmatrix} 1 & 0 & \mathrm{t} & 0 & \mathrm{t}^2 & 0 & \dots \\ 0 & \sqrt{\mathrm{t}} & 0 & \mathrm{t}^{\frac{3}{2}} & 0 & \mathrm{t}^{\frac{5}{2}} & \\ \mathrm{t} & 0 & \mathrm{t} & 0 & \mathrm{t}^2 & 0 & \\ 0 & \mathrm{t}^{\frac{3}{2}} & 0 & \mathrm{t}^{\frac{3}{2}} & 0 & \mathrm{t}^{\frac{5}{2}} & \\ \mathrm{t}^2 & 0 & \mathrm{t}^2 & 0 & \mathrm{t}^2 & 0 & \\ 0 & \mathrm{t}^{\frac{5}{2}} & 0 & \mathrm{t}^{\frac{5}{2}} & 0 & \mathrm{t}^{\frac{5}{2}} & \\ \vdots & & & & & & \ddots \end{pmatrix}. \tag{4.114}$$

Upon inversion it gives

$$I^{-1}_{H/2} = \begin{pmatrix} \frac{1}{1-\mathrm{t}} & 0 & -\frac{1}{1-\mathrm{t}} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sqrt{\mathrm{t}}(1-\mathrm{t})} & 0 & -\frac{1}{\sqrt{\mathrm{t}}(1-\mathrm{t})} & 0 & 0 & \\ -\frac{1}{1-\mathrm{t}} & 0 & \frac{1+\mathrm{t}}{\mathrm{t}(1-\mathrm{t})} & 0 & -\frac{1}{\mathrm{t}(1-\mathrm{t})} & 0 & \\ 0 & -\frac{1}{\sqrt{\mathrm{t}}(1-\mathrm{t})} & 0 & \frac{1+\mathrm{t}}{\mathrm{t}^{\frac{3}{2}}(1-\mathrm{t})} & 0 & -\frac{1}{\mathrm{t}^{\frac{3}{2}}(1-\mathrm{t})} & \\ 0 & 0 & -\frac{1}{\mathrm{t}(1-\mathrm{t})} & 0 & \frac{1+\mathrm{t}}{\mathrm{t}^2(1-\mathrm{t})} & 0 & \\ 0 & 0 & 0 & -\frac{1}{\mathrm{t}^{\frac{3}{2}}(1-\mathrm{t})} & 0 & \frac{1+\mathrm{t}}{\mathrm{t}^{\frac{5}{2}}(1-\mathrm{t})} & \\ \vdots & & & & & & \ddots \end{pmatrix}. \tag{4.115}$$

Here w goes from $0, 1/2, 1, 3/2, \cdots$. For a generic value of w only three elements in a single column can contribute to the index[17]. For large k the index of vector multiplet becomes

$$I_V(w) = \mathrm{t}^{-2w}\left(\frac{1}{1-\mathrm{t}}\right), \tag{4.116}$$

[17] By "generic" we mean the first and the second column are not reliable due to our choice of domain for w. It is imaginable that if we take w to be a half integer from $(-\infty, +\infty)$, then such "boundary ambiguity" can be removed. But we refrain from doing this to have weights living in the Weyl alcove.

and we get

$$C^{E_6}(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, w, n_y) = \mathrm{t}^w \Big[(1+\mathrm{t})\mathcal{I}_A(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, n_y, w, k) \\ -\mathrm{t}\ \mathcal{I}_A(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, n_y, w-1, k) - \mathcal{I}_A(\mathrm{t}, \mathbf{h}_a, \mathbf{h}_b, n_y, w+1, k)\Big], \quad (4.117)$$

which exactly agrees with [115].

$SU(3)$ **equivariant Verlinde algebra**

Now with all the basic building blocks of the 2d TQFT at our disposal, we assemble the pieces and see what interesting information could be extracted.

The metric of the TQFT is given by the Coulomb branch index of an $SU(3)$ vector multiplet, with a possible normalization factor. Note the conjugation of representations acts on a highest weight state (λ_1, λ_2) via

$$\overline{(\lambda_1, \lambda_2)} = (\lambda_2, \lambda_1), \quad (4.118)$$

and the metric $\eta^{\lambda\mu}$ is non-vanishing if and only if $\mu = \overline{\lambda}$. Let

$$N(\lambda_1, \lambda_2, k) = \mathrm{t}^{-\frac{1}{k}([\![\lambda_1]\!]_k[\![-\lambda_1]\!]_k + [\![\lambda_2]\!]_k[\![-\lambda_2]\!]_k + [\![\lambda_1+\lambda_2]\!]_k[\![-\lambda_1-\lambda_2]\!]_k)}, \quad (4.119)$$

and we rescale our TQFT states as

$$(\lambda_1, \lambda_2)' = N(\lambda_1, \lambda_2, k)^{-\frac{1}{2}}(\lambda_1, \lambda_2). \quad (4.120)$$

Then the metric η takes a simple form (here we define $\lambda_3 = \lambda_1 + \lambda_2$):

$$\eta^{(\lambda_1,\lambda_2)\overline{(\lambda_1,\lambda_2)}} = \begin{cases} \frac{1}{(1-t^2)(1-t^3)}, & \text{if } [\![\lambda_1]\!]_k = [\![\lambda_2]\!]_k = 0, \\ \frac{1}{(1-t)(1-t^2)}, & \text{if only one } [\![\lambda_i]\!]_k = 0 \text{ for } i = 1,2,3, \\ \frac{1}{(1-t)^2}, & \text{if all } [\![\lambda_i]\!]_k \neq 0. \end{cases} \quad (4.121)$$

Next we find the "pair of pants" $f_{(\lambda_1,\lambda_2)(\mu_1,\mu_2)(\nu_1,\nu_2)}$, from the normalized Coulomb branch index of E_6 SCFT:

$$f_{(\lambda_1,\lambda_2)(\mu_1,\mu_2)(\nu_1,\nu_2)} = \\ (N(\lambda_1,\lambda_2,k)N(\mu_1,\mu_2,k)N(\nu_1,\nu_2,k))^{\frac{1}{2}}\, C^{E_6}(\mathrm{t}, \lambda_1, \lambda_2; \mu_1, \mu_2; \nu_1, \nu_2; k). \quad (4.122)$$

Along with the metric we already have, they define a $\mathfrak{t}$-deformation of the $\widehat{su}(3)_k$ fusion algebra. For instance we could write down at level $k = 3$:

$$
\begin{aligned}
(1,0) \otimes (1,0) = {} & \frac{1+\mathfrak{t}+\mathfrak{t}^3}{(1-\mathfrak{t})(1-\mathfrak{t}^2)(1-\mathfrak{t}^3)}(0,1) \oplus \frac{1+2\mathfrak{t}^2}{(1-\mathfrak{t})(1-\mathfrak{t}^2)(1-\mathfrak{t}^3)}(2,0) \\
& \oplus \frac{\mathfrak{t}(2+\mathfrak{t})}{(1-\mathfrak{t})(1-\mathfrak{t}^2)(1-\mathfrak{t}^3)}(1,2).
\end{aligned}
\tag{4.123}
$$

Using dimensions to denote representations, the above reads

$$
\begin{aligned}
\mathbf{3} \times \mathbf{3} = {} & \frac{1+\mathfrak{t}+\mathfrak{t}^3}{(1-\mathfrak{t})(1-\mathfrak{t}^2)(1-\mathfrak{t}^3)}\bar{\mathbf{3}} + \frac{1+2\mathfrak{t}^2}{(1-\mathfrak{t})(1-\mathfrak{t}^2)(1-\mathfrak{t}^3)}\mathbf{6} \\
& + \frac{\mathfrak{t}(2+\mathfrak{t})}{(1-\mathfrak{t})(1-\mathfrak{t}^2)(1-\mathfrak{t}^3)}\overline{\mathbf{15}}.
\end{aligned}
\tag{4.124}
$$

When $\mathfrak{t} = 0$, it reproduces the fusion rules of the affine $\widehat{su}(3)_k$ algebra, and $f_{\lambda\mu\nu}$ becomes the fusion coefficients $N^{(k)}_{\lambda\mu\nu}$. These fusion coefficients are worked out combinatorically in [116–118].

With pairs of pants and cylinders, one can glue them together to get the partition function on a closed Riemann surface, which gives the $SU(3)$ equivariant Verlinde formula: a $\mathfrak{t}$-deformation of the $SU(3)$ Verlinde formula. For genus $g = 2$, at large k, one can obtain

$$
\begin{aligned}
& \dim_\beta \mathcal{H}_{CS}(\Sigma_{2,0}; SL(3,\mathbb{C}), k) \\
& = \frac{1}{20160}k^8 + \frac{1}{840}k^7 + \frac{7}{480}k^6 + \frac{9}{80}k^5 + \frac{529}{960}k^4 + \frac{133}{80}k^3 + \frac{14789}{5040}k^2 + \frac{572}{210}k + 1 \\
& + \left(\frac{1}{2520}k^8 + \frac{1}{84}k^7 + \frac{17}{120}k^6 + \frac{17}{20}k^5 + \frac{319}{120}k^4 + \frac{15}{4}k^3 + \frac{503}{2520}k^2 - \frac{1937}{420}k - 3\right)\mathfrak{t} \\
& + \left(\frac{1}{560}k^8 + \frac{9}{140}k^7 + \frac{31}{40}k^6 + \frac{39}{10}k^5 + \frac{727}{80}k^4 + \frac{183}{20}k^3 + \frac{369}{140}k^2 - \frac{27}{70}k + 1\right)\mathfrak{t}^2 \\
& + \ldots,
\end{aligned}
\tag{4.125}
$$

and the reader can check that the degree zero piece in $\mathfrak{t}$ is the usual $SU(3)$ Verlinde formula for $g = 2$ [119]:

$$
\begin{aligned}
& \dim \mathcal{H}(\Sigma_{g,0}; SU(3), k) = \\
& \frac{(k+3)^{2g-2}6^{g-1}}{2^{7g-7}} \sum_{\lambda_1,\lambda_2} \left(\sin\frac{\pi(\lambda_1+1)}{k+3} \sin\frac{\pi(\lambda_2+1)}{k+3} \sin\frac{\pi(\lambda_1+\lambda_2+2)}{k+3}\right)^{2-2g},
\end{aligned}
\tag{4.126}
$$

expressed as a polynomial in k.

For a 2d TQFT, the state associated with the "cap" contains interesting information, namely the "cap state" tells us how to close a puncture. Moreover, there are many close cousins of the cap. There is one type which we call the "central cap" that has a defect with central monodromy with the Levi subgroup being the entire gauge group (there is no reduction of the gauge group when we approach the singularity). For $SU(3)$ equivariant Verlinde algebra, besides the "identity-cap" the central cap also includes "ω-cap" and "ω^2-cap," and the corresponding TQFT states are denoted by $|\phi\rangle_1, |\phi\rangle_\omega$ and $|\phi\rangle_{\omega^2}$. One can also insert on the cap a minimal puncture (gauge group only reduces to $SU(2)\times U(1)$ as opposed to $U(1)^3$ for maximal punctures) and the corresponding states can be expressed as linear combinations of the maximal puncture states which we use as the basis vectors of the TQFT Hilbert space.

The cap state can be deduced from f and η written in (4.122) and (4.121), since closing a puncture on a three-punctured sphere gives a cylinder. In algebraic language,

$$f_{\lambda\mu\phi} = \eta_{\lambda\mu}. \tag{4.127}$$

One can easily solve this equation, obtaining

$$|\phi\rangle_1 = |0,0\rangle - \mathrm{t}(1+\mathrm{t})|1,1\rangle + \mathrm{t}^2|0,3\rangle + \mathrm{t}^2|3,0\rangle - \mathrm{t}^3|2,2\rangle. \tag{4.128}$$

For other two remaining caps, by multiplying[18] ω and ω^2 on the above equation (4.128), we obtain

$$\begin{aligned} |\phi\rangle_\omega &= |k,0\rangle - \mathrm{t}(1+\mathrm{t})|k-2,1\rangle + \mathrm{t}^2|k-3,0\rangle + \mathrm{t}^2|k-3,3\rangle - \mathrm{t}^3|k-4,2\rangle, \\ |\phi\rangle_{\omega^2} &= |0,k\rangle - \mathrm{t}(1+\mathrm{t})|1,k-2\rangle + \mathrm{t}^2|0,k-3\rangle + \mathrm{t}^2|3,k-3\rangle - \mathrm{t}^3|2,k-4\rangle. \end{aligned} \tag{4.129}$$

When closing a maximal puncture using $|\phi\rangle_\omega$, we have a "twisted metric" $\eta'_{\lambda\mu}$ which is non-zero if and only if $(\mu_1,\mu_2) = (\lambda_1, k-\lambda_1-\lambda_2)$. When closing a maximal puncture using $|\phi\rangle_{\omega^2}$, we have another twisted metric $\eta''_{\lambda\mu}$ which is non-zero if and only if $(\mu_1,\mu_2) = (k-\lambda_1-\lambda_2, \lambda_2)$. When there are insertions of central monodromies on the Riemann surface, it is easier to incorporate them into twisted metrics instead of using the expansion (4.129).

For minimal punctures, the holonomy is of the form $(u,u,-2u)$, modulo the action of the affine Weyl group, where u takes value $0, 1/3, 2/3, \ldots, k-2/3, k-1/3$.

[18] More precisely, we multiply holonomies with these central elements and translate the new holonomies back to weights.

We can use index computation to expand the corresponding state $|u\rangle_{U(1)}$ in terms of maximal punctures. After scaling by a normalization constant

$$\mathrm{t}^{\frac{1}{2}\left(\llbracket 3u\rrbracket_k-\frac{1}{k}\llbracket 3u\rrbracket_k^2\right)}, \tag{4.130}$$

the decomposition is given by the following:

(1). $\langle 0,0\rangle - \mathrm{t}^2\langle 1,1\rangle$, if $k=u$ or $u=0$;

(2). $\langle 3u,0\rangle - \mathrm{t}\langle 3u-1,2\rangle$, if $k>3u>0$;

(3). $\langle 3u,0\rangle - \mathrm{t}^2\langle 3u-2,1\rangle$, if $k=3u$;

(4). $\langle 2k-3u,3u-k\rangle - \mathrm{t}\langle 2k-3u-1,3u-k-1\rangle$, if $3u/2<k<3u$;

(5). $\langle 0,3u/2\rangle - \mathrm{t}^2\langle 1,3u/2-2\rangle$, if $k=3u/2$;

(6). $\langle 0,3k-3u\rangle - \mathrm{t}\langle 2,3k-3u-1\rangle$, if $u<k<3u/2$.

The above formulae have a natural $\mathbb{Z}_2$-symmetry of the form $C\circ\psi$, where

$$\psi : (u,k) \to (k-u,k), \tag{4.131}$$

and C is the conjugation operator that acts linearly on Hilbert space:

$$C : (\lambda_1,\lambda_2) \to (\lambda_2,\lambda_1), \quad (\lambda_1,\lambda_2)\in\mathcal{H}. \tag{4.132}$$

This $\mathbb{Z}_2$ action sends each state in the above list to itself. Moreover, it is interesting to observe that when $\mathrm{t}=0$, increasing u from 0 to k corresponds to moving along the edges of the Weyl alcove ($c.f.$ figure 4.3) a full cycle. This may not be a surprise because closing a maximal puncture actually implies that one only considers states whose $SU(3)$ holonomy (h_1,h_2,h_3) preserves at least $SU(2)\subset SU(3)$ symmetry, which are precisely the states lying on the edges of the Weyl alcove.

From algebra to geometry

This TQFT structure reveals a lot of interesting geometric properties of moduli spaces of rank 3 Higgs bundles. But as the current dissertation is from a physicist's point of view, we only look at a one example—but arguably the most interesting one—the moduli space $\mathcal{M}_H(\Sigma_{0,3},SU(3))$. In particular this moduli space was studied in [120, 121] and [122] from the point of view of differential equations. Here, from index computation, we can recover some of the results in the mathematical literature and reveal some new features for this moduli space. In particular, we propose the following formula for the fusion coefficient $f_{\lambda\mu\nu}$:

$$f_{(\lambda_1,\lambda_2)(\mu_1,\mu_2)(\nu_1,\nu_2)} = \mathrm{t}^{k\eta_0}\left(\frac{k\mathrm{Vol}(\mathcal{M})+1}{1-\mathrm{t}}+\frac{2\mathrm{t}}{(1-\mathrm{t})^2}\right)+\frac{Q_1(\mathrm{t})}{(1-\mathrm{t}^{-1})(1-\mathrm{t}^2)}+\frac{Q_2(\mathrm{t})}{(1-\mathrm{t}^{-2})(1-\mathrm{t}^3)}. \quad (4.133)$$

This ansatz comes from Atiyah-Bott localization of the equivariant integral done in similar fashion as in Chapter 2. The localization formula enables us to write the fusion coefficient f in (4.122) as a summation over fixed points of the $U(1)_H$ Hitchin action. In (4.133), η_0 is the moment map[19] for the lowest critical manifold $\mathcal{M}$. When the undeformed fusion coefficients $N^{(k)}_{\lambda\mu\nu} \neq 0$, one has

$$k\mathrm{Vol}(\mathcal{M})+1 = N^{(k)}_{\lambda\mu\nu}, \quad \eta_0 = 0. \quad (4.134)$$

Numerical computation shows that $Q_{1,2}(\mathrm{t})$ are individually a sum of three terms of the form

$$Q_1(\mathrm{t}) = \sum_{i=1}^{3}\mathrm{t}^{k\eta_i}, \quad Q_2(\mathrm{t}) = \sum_{j=4}^{6}\mathrm{t}^{k\eta_j}, \quad (4.135)$$

where η_i are interpreted as the moment maps at each of the six higher fixed points of $U(1)_H$.

The moduli space $\mathcal{M}$ of $SU(3)$ flat connections on $\Sigma_{0,3}$ is either empty, a point or $\mathbb{C}\mathbf{P}^1$ depending on the choice of (λ,μ,ν) [123], and when it is empty, the lowest critical manifold of η is a $\mathbb{C}\mathbf{P}^1$ with $\eta_0 > 0$ and we will still use $\mathcal{M}$ to denote it. The fixed loci of $\mathcal{M}_H(\Sigma_{0,3}, SU(3))$ under $U(1)$ action consist of $\mathcal{M}$ and the six additional points, and there are Morse flow lines traveling between them. The downward Morse flow coincides with the nilpotent cone [124]—the singular fiber of the Hitchin fibration, and its geometry is depicted in figure 4.4. The Morse flow carves out six spheres that can be divided into two classes. Intersections of $D_i^{(1)} \bigcap D_i^{(2)}$ are denoted as $P^{(1)}_{1,2,3}$, and at the top of these $D_i^{(2)}$'s there are $P^{(2)}_{1,2,3}$. We also use $P_1,\ldots,P_6$ and $D_1,\ldots,D_6$ sometimes to avoid clutter. The nilpotent cone can be decomposed into

$$\mathcal{N} = \mathcal{M}\cup D_i^{(1)}\cup D_j^{(2)}, \quad (4.136)$$

[19] Recall the $U(1)_H$ Hitchin action is generated by a Hamiltonian, which we call η—not to be confused with the metric, which will make no appearance from now on. η is also the norm squared of the Higgs field.

which gives an affine E_6 singularity (IV* in Kodaira's classification) of the Hitchin fibration. Knowing the singular fiber structure, we can immediately read off the Poincaré polynomial for $\mathcal{M}_H(\Sigma_{0,3}, SU(3))$:

$$\mathcal{P}_{\mathrm{r}} = 1 + 7\mathrm{r}^2, \tag{4.137}$$

which is the same as that given in [121].

To use the Atiyah-Bott localization formula, we also need to understand the normal bundle to the critical manifolds. For the base, the normal bundle is the cotangent bundle with $U(1)_H$ weight 1. Its contribution to the fusion coefficient is given by

$$t^{k\eta_0}\int_{\mathcal{M}} \frac{\mathrm{Td}(\mathbb{C}\mathbf{P}^1)\wedge e^{k\omega}}{1-e^{-\beta+2\omega'}} = \mathrm{t}^{k\eta_0}\left(\frac{k\mathrm{Vol}(\mathcal{M})+1}{1-\mathrm{t}} + \frac{2\mathrm{t}}{(1-\mathrm{t})^2}\right). \tag{4.138}$$

For the higher fixed points, the first class $P^{(1)}$ has normal bundle $\mathbb{C}[-1]\oplus\mathbb{C}[2]$ with respect to $U(1)_H$, which gives a factor

$$\frac{1}{(1-\mathrm{t}^{-1})(1-\mathrm{t}^2)} \tag{4.139}$$

multiplying $e^{k\eta_{1,2,3}}$. For the second class $P^{(2)}$, the normal bundle is $\mathbb{C}[-2]\oplus\mathbb{C}[3]$ and we instead have a factor

$$\frac{1}{(1-\mathrm{t}^{-2})(1-\mathrm{t}^3)}. \tag{4.140}$$

In this thesis, we won't give the analytic expression for the seven moment maps and will leave (4.133) as it is. Instead, we will give a relation between them:

$$\begin{aligned} 2k &= 6(N_{\lambda\mu\nu}^{(k)}-1) + 3k(\eta_1+\eta_2+\eta_3) + k(\eta_4+\eta_5+\eta_6) \\ &= 6k\mathrm{Vol}(\mathcal{M}) + 3k(\eta_1+\eta_2+\eta_3) + k(\eta_4+\eta_5+\eta_6). \end{aligned} \tag{4.141}$$

This is verified numerically and can be explained from geometry. Noticing that the moment maps are related to the volume of the D's:

$$\begin{aligned} &\mathrm{Vol}(D_1) = \eta_1,\ \ \mathrm{V}ol(D_2) = \eta_2,\ \ \mathrm{V}ol(D_3) = \eta_3, \\ &\mathrm{Vol}(D_4) = \frac{\eta_4-\eta_1}{2},\ \ \mathrm{V}ol(D_5) = \frac{\eta_5-\eta_2}{2},\ \ \mathrm{Vol}(D_6) = \frac{\eta_6-\eta_3}{2}. \end{aligned} \tag{4.142}$$

The factor 2 in the second line of (4.142) is related to the fact that $U(1)_H$ rotates the $D^{(2)}$'s twice as fast as it rotates the $D^{(1)}$'s. Then we get the following relation between the volume of the components of $\mathcal{N}$:

$$\mathrm{Vol}(\mathbf{F}) = 6\mathrm{V}ol(\mathcal{M}) + 4\sum_{i=1}^{3}\mathrm{V}ol(D_i) + 2\sum_{i=4}^{6}\mathrm{V}ol(D_j). \tag{4.143}$$

Here F is a generic fiber of the Hitchin fibration and has volume

$$\mathrm{Vol}(\mathbf{F}) = 2. \tag{4.144}$$

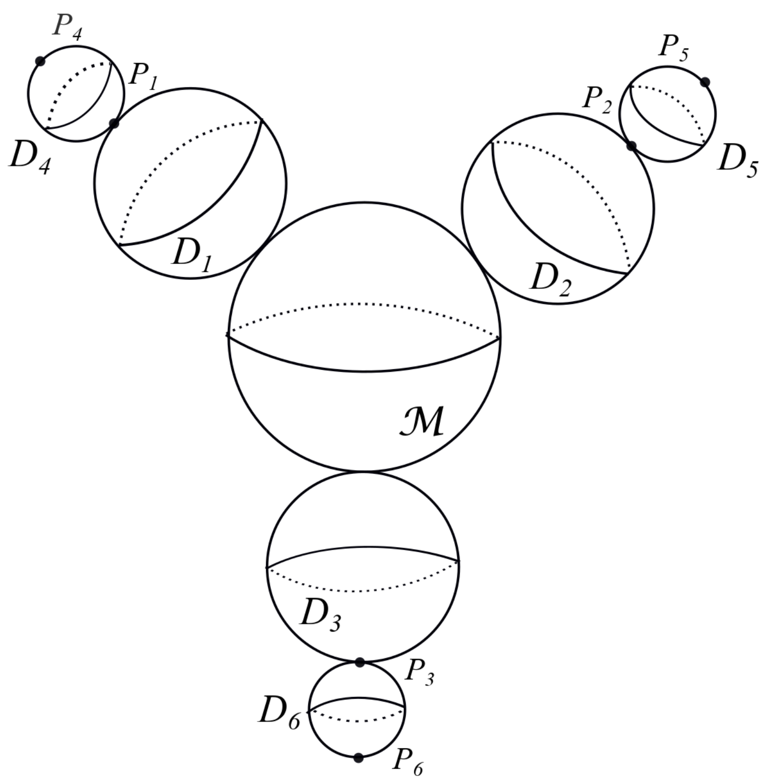


Figure 4.4: The illustration of the nilpotent cone in $\mathcal{M}_H(\Sigma_{0,3}, SU(3))$. Here $\mathcal{M}$ is the base $\mathbb{C}\mathbf{P}^1$, $D_{1,2,3}$ consist of downward Morse flows from $P_{1,2,3}$ to the base, while $D_{4,5,6}$ include the flows from $P_{4,5,6}$ to $P_{1,2,3}$.

The intersection form of different components in the nilpotent cone gives the Cartan matrix of affine E_6. Figure 4.5 is the Dynkin diagram of $\widehat{E}_6$, and coefficients in (4.143) are Dynkin labels on the corresponding node. These numbers tell us the combination of D's and $\mathcal{M}$ that give a null vector $\mathbf{F}$ of $\widehat{E}_6$.

Comments on T_N theories

The above procedure can be generalized to arbitrary rank, for all T_N theories, if we employ the generalized Argyres-Seiberg dualities. There are in fact several ways to generalized Argyres-Seiberg duality [113, 114, 125]. For our purposes, we want no punctures of the T_N to be closed under dualities, so we need the following setup [114].

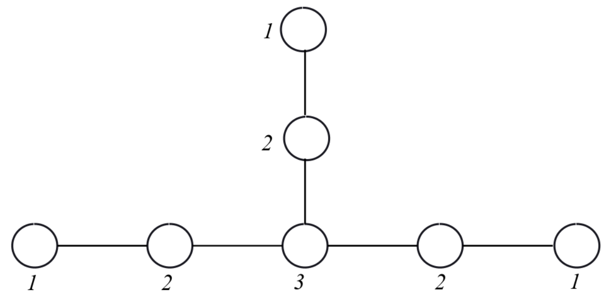


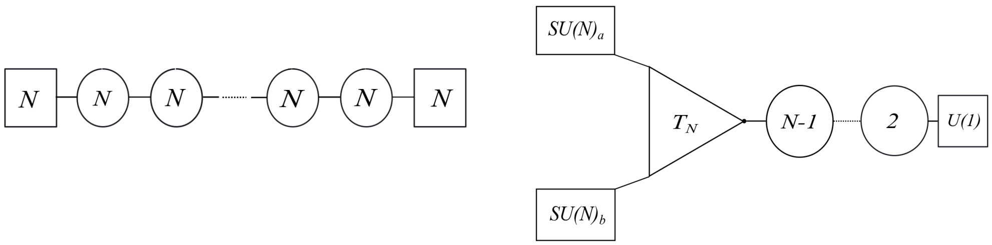
Figure 4.5: The affine $\widehat{E}_6$ extended Dynkin diagram. The Dynkin label gives the multiplicity of each node in the decomposition of the null vector.

We start with a linear quiver gauge theory A' with $N-2$ nodes of $SU(N)$ gauge groups, and at each end of the quiver we associate N hypermultiplets in the fundamental representation of $SU(N)$. One sees immediately that each gauge node is automatically superconformal. Geometrically, we actually start with a punctured Riemann sphere with two full $SU(N)$ punctures and $N-1$ simple punctures. Then, the $N-1$ simple punctures are brought together and a hidden $SU(N-1)$ gauge group becomes very weak. In our original quiver diagram, such a procedure of colliding $N-1$ simple punctures corresponds to attaching a quiver tail of the form $SU(N-1) - SU(N-2) - \cdots - SU(2)$ with a single hypermultiplet attached to the last $SU(2)$ node. See figure 4.6 for the quiver diagrams and figure 4.7 for the geometric realization.

Here we summarize briefly how to obtain the lens space Coulomb index of T_N. Let $\mathcal{I}_A^N$ be the index of the linear quiver theory, which depends on two $SU(N)$ flavor holonomies $\mathbf{h}_a$ and $\mathbf{h}_b$ (here we use the same notation as that of $SU(3)$) and $N-1$ $U(1)$-holonomies n_i where $i = 1, 2, \ldots, N-1$. In the infinite coupling limit, the dual weakly coupled theory B' emerges. One first splits the $SU(N)_c$ subgroup of the full $SU(N)^3$ flavor symmetry group into $SU(N-1) \times U(1)$ and then gauges the $SU(N-1)$ part with the first gauge node in the quiver tail. As in the T_3 case there is a transformation:

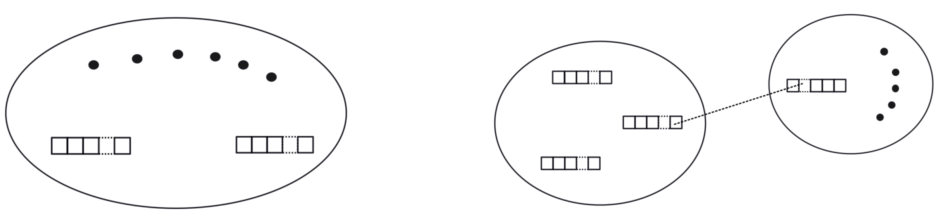
$$\left(h_1^c, h_2^c, \cdots, h_N^c\right) \to \left(w_1, w_2, \cdots w_{N-2}, \widetilde{n}_0\right). \tag{4.145}$$

After the $SU(N-1)$ node, there are $N-2$ more $U(1)$ symmetries, we will call those associated holonomies $\widetilde{n}_j$ with $j = 1, 2, \ldots, N-2$. Again there exists a



(a) (b)

Figure 4.6: Illustration of generalized Argyres-Seiberg duality for the T_N theories. (a) The theory A', which is a linear quiver gauge theory with $N-2$ $SU(N)$ vector multiplets. Between each gauge node there is a bi-fundamental hypermultiplet, and at each end of the quiver there are N fundamental hypermultiplets. In the quiver diagram we omit the $U(1)^{N-1}$ baryonic symmetries. (b) The theory B' is obtained by gauging an $SU(N-1)$ subgroup of the $SU(N)^3$ flavor symmetry of T_N, giving rise to a quiver tail. Again the $U(1)$ symmetries are implicit in the diagram.



(a) (b)

Figure 4.7: Illustration of the geometric realization of generalized Argyres-Seiberg duality for T_N theories. (a) The theory A' is obtained by compactifying 6d $(2,0)$ theory on a Riemann sphere with two maximal $SU(N)$ punctures and $N-1$ simple punctures. (b) The theory B', obtained by colliding $N-1$ simple punctures, is then the theory that arises from gauging a $SU(N-1)$ flavor subgroup of T_N by a quiver tail.

correspondence as in the T_3 case:

$$(n_1, n_2, \ldots, n_{N-1}) \to (\widetilde{n}_0, \widetilde{n}_1, \ldots, \widetilde{n}_{N-2}) \,. \tag{4.146}$$

Then the Coulomb branch index of the theory B' is

$$\begin{aligned} \mathcal{I}_B^N(\mathbf{h}^a, \mathbf{h}^b, \widetilde{n}_0, \widetilde{n}_1, \ldots, \widetilde{n}_{N-2}) = \\ \sum_{\{w_i\}} C^{T_N}(\mathbf{h}^a, \mathbf{h}^b, w_1, w_2, \cdots w_{N-2}, \widetilde{n}_0) \mathcal{I}_T(w_i; \widetilde{n}_1, \ldots, \widetilde{n}_{N-2}), \end{aligned} \tag{4.147}$$

where $\mathcal{I}_T$ is the index of the quiver tail:

$$\begin{aligned} \mathcal{I}_T(w_i; \widetilde{n}_1, \ldots, \widetilde{n}_{N-2}) = \\ \sum_{\{w_i^{(N-2)}\}} \sum_{\{w_i^{(N-3)}\}} \cdots \sum_{\{w_i^{(2)}\}} \mathcal{I}^V_{N-1}(w_i) \mathcal{I}^H_{N-1,N-2}(w_i, w_j^{(N-2)}, \widetilde{n}_1) \mathcal{I}^V_{N-2}(w_i^{(N-2)}) \\ \times \mathcal{I}^H_{N-2,N-3}(w_i^{(N-2)}, w_j^{(N-3)}, \widetilde{n}_2) \mathcal{I}^V_{N-3}(w_i^{(N-3)}) \times \ldots \\ \times \mathcal{I}^V_2(w_i^{(2)}) \mathcal{I}^H_{2,1}(w_i^{(2)}, \widetilde{n}_{N-2}). \end{aligned} \tag{4.148}$$

Now we can view $\mathcal{I}_T$ as a large matrix $\mathfrak{M}_{\{w_i\},\{\widetilde{n}_j\}}$, and in fact it is a square matrix. Although the set $\{\widetilde{n}_j\}$ appears to be bigger, there is an affine Weyl group $\widehat{A}_{N-2}$ acting on it. From the geometric picture, one can directly see the $A_{N-2} = S_{N-2}$ permuting the $N-2$; and the shift $n_i \to n_i + k$, which gives the same holonomy in $U(1)_i$, enlarges the symmetry to that of $\widehat{A}_{N-2}$. After taking quotient by this symmetry, one requires $\{\widetilde{n}_j\}$ to live in the Weyl alcove of $\mathfrak{su}(N-1)$, reducing the cardinality of the set $\{\widetilde{n}_j\}$ to that of $\{w_i\}$. Then one can invert the matrix $\mathfrak{M}_{\{w_i\},\{\widetilde{n}_j\}}$ and obtain the index C^{T_N}, which in turn gives the fusion coefficients and the algebra structure of the $SU(N)$ equivariant TQFT.

The metric of the TQFT coming from the cylinder is also straightforward even in the $SU(N)$ case. It is always diagonal and only depends on the symmetry reserved by the holonomy labeled by the highest weight λ. For instance, if the holonomy is such that $SU(N) \to U(1)^n \times SU(N_1) \times SU(N_2) \times SU(N_l)$, we have

$$\eta^{\lambda\overline{\lambda}} = \frac{1}{(1-\mathrm{t})^n} \prod_{j=1}^{l} \frac{1}{(1-\mathrm{t}^2)(1-\mathrm{t}^3)\ldots(1-\mathrm{t}^{N_j})}. \tag{4.149}$$

This can be generalized to arbitrary group G. If the holonomy given by λ has stabilizer $G' \subset G$, the norm square of λ in the G_k equivariant Verlinde algebra is

$$\eta^{\lambda\overline{\lambda}} = P(BG', \mathrm{t}). \tag{4.150}$$

Here $P(BG', \mathrm{t})$ is the Poincaré polynomial[20] of the infinite-dimensional classifying space of G'. In the "maximal" case of $G' = U(1)^r$, we indeed get

$$P\left(BU(1)^r, \mathrm{t}\right) = P\left(\left(\mathbb{C}\mathbf{P}^\infty\right)^r, \mathrm{t}\right) = \frac{1}{(1-t)^r}. \tag{4.151}$$

[20] More precisely, it is the Poincaré polynomial in the variable $\mathrm{t}^{1/2}$. But as $H^*(BG, \mathbb{C})$ is zero in odd degrees, this Poincaré polynomial is also a series in t with integer powers.

5.1 Going one step further

In previous chapters, we have analyzed in great detail the 3d-3d correspondence for $L(k,1)$ and $\Sigma \times S^1$. The beautiful stories there pose the question: how far are we from the getting the 3d-3d correspondence for *all* Seifert manifolds? In this final chapter, we will sketch the derivation of the theory $T[M_3]$ when M_3 is a general orientable Seifert manifold and briefly discuss ways to test the 3d-3d correspondence in this case, leaving details and computations to [4] and future publications.

In fact, we are already very close to achieving full generality. Recall that a Seifert manifold can be obtained by doing Dehn Surgery along the S^1 fibers of $\Sigma \times S^1$. In fact, this is how a Seifert invariant should be translated into a Seifert manifold:

$$\{b, g; (q_1, p_1), \ldots, (q_n, p_n)\} \rightsquigarrow M_3. \tag{5.1}$$

Here, b and g are positive integers and all the (q, p)'s are pairs of coprime positive integers.

The exact procedure of (5.1) is the following. One starts with a genus-g Riemann surface Σ with $n+1$ marked points $P_0, \ldots, P_n$. One then removes from $\Sigma \times S^1$ all tubular neighborhoods of $P_i \times S^1$ along $n+1$ tori, and glues them back after performing the $SL(2,\mathbb{Z})$ transformation given by

$$\gamma_i = \begin{pmatrix} q_i & -p_i \\ r_i & q_i' \end{pmatrix} \tag{5.2}$$

on the i-th torus. Here q_i' and r_i are integers such that

$$q_i q_i' + p_i r_i = 1. \tag{5.3}$$

In particular, q' is the inverse of q (mod p) and will be denoted q^{-1} when no confusions will be caused. And for $i = 0$, we have $q_0 = 1$, $p_0 = b$. As b can be absorbed into the (q, p)'s, without loss of generality, we will henceforth set $b = 0$.

The above procedure can be directly translated into the language of field theory on the other side of the 3d-3d correspondence. Indeed, Dehn drilling leave us with $\Sigma_{g,n} \times S^1$, whose corresponding SCFT $T[\Sigma_{g,n} \times S^1]$ has a UV Lagrangian description [126] depicted in figure 5.1 (it is the 3d mirror of the other UV description obtained

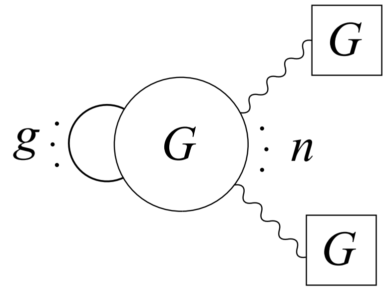


Figure 5.1: The 3d mirror description of $T[\Sigma_{g,n} \times S^1]$. The diagram is in 3d $\mathcal{N} = 4$ notation. A round circle denotes a gauge group, a square box denotes a flavor symmetry and a curly line represents a $T[G]$ theory that has $G \times G$ symmetry. (To avoid clutter, we ingore the distinctions between ${}^LG$ and G.)

from the compactification of a class $\mathcal{S}$ theory on S^1). Then, performing $SL(2,\mathbb{Z})$ transformation on a solid torus creates numerous "duality walls" near a Neumann boundary of a 4d $\mathcal{N} = 4$ super-Yang-Mills theory [127]. Finally, Dehn filling simply means that one has to glue the two pieces together to get $T[M_3]$.

As the 3d theory $T[\Sigma_{g,n}]$ is studied in some detail in [126], our job is to understand the theory that corresponds to the $SL(2,\mathbb{Z})$-transformed solid torus. As we will argue in the next section, this theory is closely related to the "general lens space theory" $T[L(p,q)]$, which has a 3d $\mathcal{N} = 2$ linear super-Chern-Simons quiver description with gauge nodes glued together using the theory $T[G]$—the 3d $\mathcal{N} = 4$ theory living on the S-duality wall of the 4d $\mathcal{N} = 4$ super-Yang-Mills theory.

5.2 The theory $L(p,q)$ and its dualities

When the gauge group is abelian, $T[L(p,q)]$ is a quiver super-Chern-Simons theory derived and studied in [41]. The non-abelian version can be derived in similar fashion and is expected to have all the "Kirby dualities" enjoyed by its abelian counterpart. More precisely, one can view $L(p,q)$ as a torus fibration over an interval with the boundary conditions on the two sides being related by the $SL(2,\mathbb{Z})$ element

$$\gamma = T^{a_1} S T^{a_2} S \dots S T^{a_s}, \tag{5.4}$$

where the a_i's are the coefficients in the continued fraction expansion of p/q

$$\frac{p}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_s}}}. \tag{5.5}$$

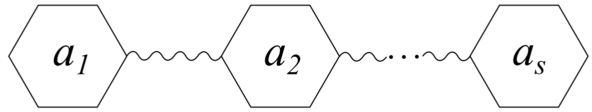


Figure 5.2: The "lens space theory" $T[L(p,q)]$. A curly line denotes a $T[G]$ theory and a hexagonal node with a number a on it represents a 3d $\mathcal{N}=4$ vector multiplet plus a 3d $\mathcal{N}=2$ super-Chern-Simons term at level a (or equivalently a 3d $\mathcal{N}=2$ Chern-Simons term plus an adjoint chiral multiplet). It is easy to see that $T[L(p,q)]$ will indeed become the $T[L(p,1)]$ theory studied in previous chapters once we set $q=1$.

As the 6d $(2,0)$ theory on a torus gives the 4d $\mathcal{N}=4$ SYM, we obtain a description of the theory $T[L(p,q)]$ as the 4d $\mathcal{N}=4$ SYM on an interval with Neumann boundary conditions[1] on the two ends and with S- and T-duality walls inserted in between. Systems like this were studied in [127] and known to be described by a linear quiver with Chern-Simons couplings in the infrared. The quiver for the theory $T[L(p,q)]$ is drawn in figure 5.2. If the 6d (2,0) theory is labeled by the simple Lie algebra $\mathfrak{g}$ and G is the simply-connected group with Lie algebra $\mathfrak{g}$, then $T[L(p,q)]$ consists of s 3d $\mathcal{N}=2$ G super-Chern-Simons theories at levels $a_1, a_2, \ldots, a_s$, with an adjoint chiral multiplet at each node, coupled together by $s-1$ $T[G]$ theories. (Recall that $T[G]$ has flavor symmetry ${}^LG \times {}^L G$ at quantum level and could serve as a "bi-fundamental" that connects two adjacent gauge nodes[2].) When $G=U(1)$, the theory $T[U(1)]$ is an empty theory by itself but still dictates that the gauge fields at the two ends are coupled via

$$-\int A_i \wedge dA_{i+1} + A_{i+1} \wedge dA_i \tag{5.6}$$

and that the coupling between adjoint chirals ϕ_i and ϕ_{i+1} are generated by the superpotential

$$W = \mathrm{Tr}_{\mathfrak{g}}\left(\phi_i \phi_{i+1}\right). \tag{5.7}$$

So our general proposal for $T[L(p,q),G]$ reduces[3] to the abelian quiver super-Chern-Simons theory in [41] when $G=U(1)$. For general non-abelian G, as one cannot make manifest the *full* flavor symmetry of $T[G]$ at the classical level, the description of $T[L(p,q)]$ given above is non-Lagrangian. However, there is no problem in

[1] We choose to work in this duality frame as it is the simplest in many ways.

[2] As G is simply-connected, one can use it to gauge any ${}^LG$ flavor symmetry, and we won't try to distinguish them and will simply use G in all figures for ${}^LG$ symmetries to avoid clutter.

[3] More precisely, there is an additional free chiral multiplet $\phi=\phi_1$ left over after integrating out the rest of the s ϕ_i's using the superpotential couplings. This free chiral will show up as an additional constant in indices or partition functions and the inclusion of this free field is important to match results from the two sides of the 3d-3d correspondence.

verifying this proposal via computation of indices and partition functions, in which the symmetries are broken to the Cartan by real masses and FI parameters. In the literature, localization techniques have already been applied to the $T[G]$ theory started in [128], and in [129] in the context of the 3d-3d correspondence.

As there are infinitely many ways to expand p/q into continued fractions, the description of $T[L(p,q)]$ is far from unique and there are actually infinitely many different descriptions all related by dualities generated by 3d Kirby moves [130]. In fact, even by looking at figure 5.2, it is apparent that there are four descriptions associated with the same set of coefficients $(a_1, \ldots, a_s)$ because:

1. one can read the quiver from right to left[4] and replace the sequence $(a_1, a_2, \ldots, a_s)$ with the reverse sequence $(a_s, a_{s-1}, \ldots, a_1)$,
2. one can invert all the signs of a_i with a change of spacetime orientation,
3. and one can combine the two above to get yet another sequence $(-a_s, -a_{s-1}, \ldots, -a_1)$.

These symmetries are the field theory incarnations of the homeomorphisms between the four equivalent lens spaces $L(p, \pm q^{\pm 1})$. For example, for $(a_1 = 3, a_2 = 5, a_3 = 8)$,

$$[3,5,8] = \frac{109}{39} \tag{5.8}$$

$$[8,5,3] = \frac{109}{14} \tag{5.9}$$

and indeed 14 is the inverse of 39 in $\mathbb{F}_{109}$ as

$$39 \times 14 = 546 \equiv 1 \pmod{109}. \tag{5.10}$$

In general, if

$$[a_1, a_2, \ldots, a_s] = \frac{p}{q} \tag{5.11}$$

then

$$[a_s, a_{s-1}, \ldots, a_1] = \frac{p}{q'} \tag{5.12}$$

with

$$qq' \equiv 1 \pmod{k}. \tag{5.13}$$

[4]This procedure is actually less trivial than it seems to be, as the theory $T[G]$ is, *a priori*, "directed". For example, only one of the two ${}^{L}G$ flavor symmetries can be realized at the classical level, and the R-symmetry acts differently on the Coulomb and Higgs branches.

This interesting relation is proven quite recently in [131][5].

The 3d Kirby moves are generated by two basic moves called “blowing-up/down” and “handle slide”. In general, they will change the sequence $(a_1, \ldots, a_s)$ or even produce a Kirby diagram that cannot be directly translated to a quiver theory. These dualities was already proposed and studied in [41] and the generalization to the non-abelian case is straightforward. We will end this section by putting this infinite family of dualities into context and show that they don’t live on an isolated island but should be rather viewed as a far-reaching generalization of dualities that we already knew.

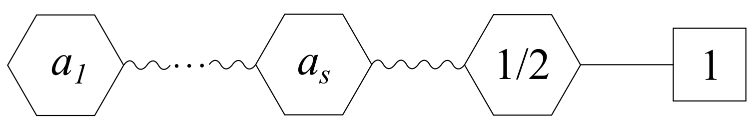
In fact, a very special limit of the dualities generated by blowing-up/down,

$$(a_1, \ldots, a_s) \leftrightarrow (a_1, \ldots, a_s + 1, 1), \tag{5.14}$$

with $s = 0$ actually yields the “duality appetizer” [78] for $G = SU(2)$ and its generalizations in [79] for $U(N)$! As written in (5.14), it may be unclear how to define the field theories on the two sides for $s = 0$ and, in fact, by “taking the $s = 0$ limit”, we mean the following. For $G = U(N)$, the duality between field theories

$$T_{\text{left}} \leftrightarrow T_{\text{right}} \tag{5.15}$$

associated to the move (5.14) can be derived by starting with a “big theory” T_{big}



.

Then one tunes the FI parameter ζ of the last gauge node as Kapustin and Willett did in [79] and as we did in Chapter 2 and 3. In the two different limits of $\zeta \ll 0$ and $\zeta \gg 0$, T_{big} will become either T_{left} or T_{right}. One can play the same trick for $s = 0$, and rediscover the duality:

$$\boxed{\begin{array}{c}U(N)_1 \text{ super-Chern-Simons theory}\\ + \text{ adjoint chiral multiplet}\end{array}} \xleftrightarrow{\text{duality}} \boxed{\begin{array}{c}N \text{ free chiral}\\ \text{multiplets}\end{array}}. \tag{5.16}$$

5.3 Coupling lens space theories to the star-shaped quiver

It is very easy to go from $T[L(p, q)]$ to the theory $T[\gamma(D^2 \times S^1)]$ associated with the solid tori transformed by $\gamma \in SL(2, \mathbb{Z})$. The gauge node at the leftmost end of the

[5] Notice that we are always using the “negative” continued fraction expansion instead of the standard expansion. The relation is $[a_1, a_2, \ldots, a_s]_{\text{here}} = [a_1, -a_2, \ldots, (-)^{s-1} a_s]_{\text{Standard}}$.

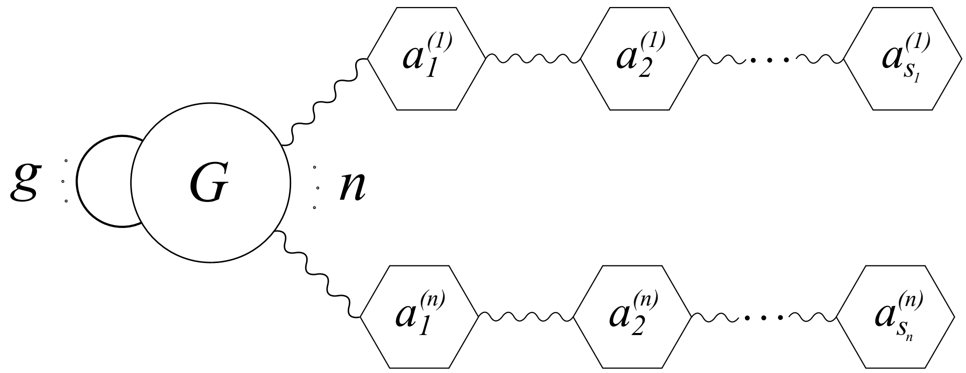


Figure 5.3: The "Seifert theory". In the figure, $(a_1^{(i)}, a_2^{(i)}, \ldots, a_{s_i}^{(i)})$ are the coefficients in the continued fraction expansion of p_i/q_i.

linear quiver in figure 5.2 becomes a flavor node with a "background Chern-Simons term" at level a_1, which will shift the Chern-Simons level for this node once it is gauged. Making the a_1 node special breaks the symmetry of

$$T[L(p,q)] \leftrightarrow T[L(p,q^{-1})] \tag{5.17}$$

generated by reading the sequence $(a_1, a_2, \ldots, a_s)$ in reverse order. The absence of this symmetry is actually a desired feature as changing a pair (q_i, p_i) in the Seifert invariant of an M_3 in this way

$$(q_i, p_i) \rightsquigarrow (q_i^{-1}, p_i) \tag{5.18}$$

generally gives rise to a completely different Seifert manifold.

Then, by gluing the $T[\Sigma_{g,n}]$ theory and n copies of the $T[\gamma_i(D^2 \times S^1)]$ theories, we obtain the "Seifert theory" depicted in figure 5.3. As an example, we have the theory labeled by the Poincaré fake sphere depicted in figure 5.4.

The Seifert theory $T[M_3]$ could have different flavor symmetries for different M_3, but will always have a $U(1)_\beta$ flavor symmetry. From the field theory perspective, this symmetry acts in the following way. Consider removing all Chern-Simons couplings, then the system has 3d $\mathcal{N} = 4$ supersymmetry with the R-symmetry group being $SU(2)_N \times SU(2)_R$. And, as in previous chapters, we can form the $U(1)_{N+R}$ and $U(1)_\beta = U(1)_{N-R}$. Once the Chern-Simons terms are added back, the $SU(2)_N \times SU(2)_R$ will be broken to $U(1)_{N+R} \times U(1)_\beta$. While the first factor can be identified with the R-symmetry of the 3d $\mathcal{N} = 2$ supersymmetry, $U(1)_\beta$ will become a flavor symmetry of $T[M_3]$. From the brane picture, this symmetry is an isometry of the brane system only in a certain limit where the metric of M_3 becomes singular and M_3 becomes a torus fibered over a "graph".

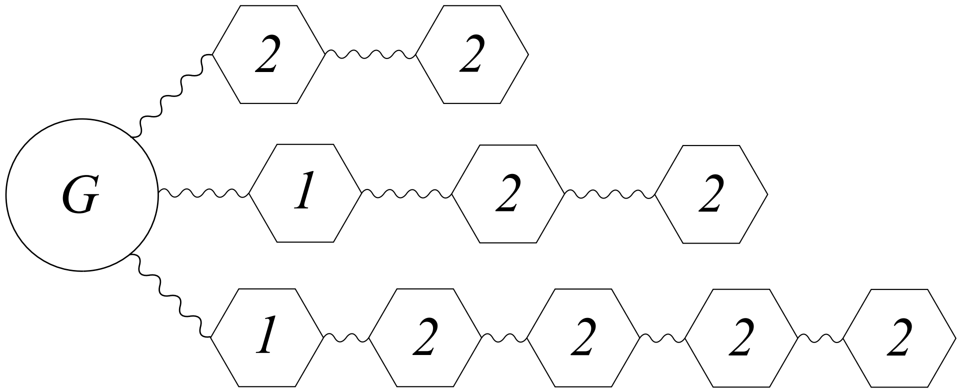


Figure 5.4: The theory corresponding to the Poincaré fake sphere. It has three exceptional Seifert fibers labeled by $(2, 3)$, $(3, 1)$ and $(5, 1)$.

The Seifert theory enjoys an even larger group of dualities, as one can now use Kirby moves to create interactions between different quiver tails associated with different marked points on Σ. Most of the new dualities are highly non-trivial. The simplest is perhaps the class generated by

$$(q_i, p_i), (q_j, p_j) \rightsquigarrow (q_i, p_i - q_i), (q_j, p_j + q_j) \tag{5.19}$$

for a choice of i and j with $i \neq j$, which only modifies the gauge nodes connected to the central node.

Testing the new "Kirby dualities" would provide a very non-trivial check on the proposal we have offered for the Seifert theory $T[M_3]$. One could also perform different tests using SUSY-protected quantities such as indices, partition functions on S^3_b, on lens spaces $L(k, 1)_b$ and on even more general lens spaces $L(k, l)_b$ [4], and compare the results with partition functions of complex Chern-Simons theory at various levels on M_3 (see *e.g.* [132] for the newest development in the study of complex Chern-Simons theory on a Seifert manifold). To get more information and stronger checks, one can turn on the fugacity t associated with the $U(1)_\beta$ flavor symmetry. Then the partition function of $T[M_3]$ on $L(k, 1)$ will give a generalization of the equivariant Velinde formula on an arbitrary Seifert manifold, enabling one to extract the information about the modular $SL(2, \mathbb{Z})$ action on the equivariant Verlinde algebra.

A.1 Complex Chern-Simons theory on lens spaces

Lens space $L(p,q)$ can be obtained by gluing two solid tori $S^1 \times D^2$ along their boundary T^2's using an element in $\mathrm{MCG}(T^2) = SL(2,\mathbb{Z})$:

$$\begin{pmatrix} -q & * \\ p & * \end{pmatrix}\begin{pmatrix} m \\ l \end{pmatrix} = \begin{pmatrix} m' \\ l' \end{pmatrix}. \tag{A.1}$$

Here (m,l) and (m',l') are meridian and longitude circles of the two copies of $T^2 = \partial(S^1 \times D^2)$. So the meridian m' of one torus is mapped to $-qm + pl$ of the other torus. As for l, we do not need to track what it is mapped into as the choice only affects the framing of $L(p,q)$. A canonical choice of an $SL(2,\mathbb{Z})$ element in (A.1) is given by

$$ST^{a_1}ST^{a_2}S\ldots T^{a_s}S, \tag{A.2}$$

where $(a_1, a_2, \ldots, a_s)$ are coefficients in continued fraction expansion of p/q. For $q = 1$, the element that gives $L(p,1)$ is

$$ST^pS. \tag{A.3}$$

As $SL(2,\mathbb{Z})$ naturally acts on the Hilbert space $\mathcal{H}^{\mathrm{CS}}(T^2;G)$ of the Chern-Simons theory on the two-torus, one has

$$Z_{\mathrm{CS}}(L(p,q);G) = \langle 0|\mathcal{S}\mathcal{T}^{a_1}\mathcal{S}\mathcal{T}^{a_2}\mathcal{S}\ldots\mathcal{T}^{a_s}\mathcal{S}|0\rangle. \tag{A.4}$$

Here $|0\rangle \in \mathcal{H}$ is the state associated to the solid torus while $\mathcal{S}$ and $\mathcal{T}$ give the action of $S,T \in SL(2,\mathbb{Z})$ on $\mathcal{H}$. When G is compact, $\mathcal{S}$ and $\mathcal{T}$ are known from the study of the 2D WZW model and affine Lie algebra [133] and can be directly used to evaluate (A.4). Partition functions of Chern-Simons theory on lens spaces were first obtained precisely in this manner in [134] for $SU(2)$ and in [135, 136] for higher rank gauge groups. Define $\widehat{k} = k + \check{h}$, then the partition function of the G Chern-Simons theory on $L(p,q)$ is given by

$$\begin{aligned} Z(L(p,q),\widehat{k}) = &\frac{1}{(\widehat{k}|p|)^{N/2}} \exp\left(\frac{i\pi}{\widehat{k}} s(q,p)|\rho|^2\right) \\ &\times \sum_{w\in W} \det(w) \exp\left(-\frac{2\pi i}{p\widehat{k}}\langle\rho, w(\rho)\rangle\right) \\ &\times \sum_{m\in Y^\vee/pY^\vee} \exp\left(i\pi\frac{q}{p}\widehat{k}|m|^2\right)\exp\left(2\pi i\frac{1}{p}\langle m, q\rho - w(\rho)\rangle\right). \end{aligned} \tag{A.5}$$

Here $s(q,p)$ is the Dedekind sum:

$$s(q,p) = \frac{1}{4p}\sum_{n=1}^{p-1}\cot\left(\frac{\pi n}{p}\right)\cot\left(\frac{\pi q n}{p}\right), \tag{A.6}$$

ρ the Weyl vector of the Lie algebra $\mathfrak{g}$, W the Weyl group, $Y^\vee$ the coroot lattice, N the rank of the gauge group, and the inner product, $\langle\cdot,\cdot\rangle$, is taken with respect to the standard Killing form of $\mathfrak{g}$.

Now we start computing the partition function of complex Chern-Simons theory using (3.50) for $G_\mathbb{C} = GL(N,\mathbb{C})$. The first step is to separate (A.5) into contributions from different flat connections. As discussed in section 3.3, the moduli space $\mathcal{M}_{\text{flat}}$ of $U(N)$ flat connections of $L(p,q)$ — whose foundamental group is $\mathbb{Z}_p$ — consists of discrete points. Each point can be labelled by $(a_1, a_2, \ldots, a_N)$, where the a_j's are the p-th roots of unity. For convenience we use a different set of labels, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathfrak{g}^*$, with the α_j's being integers between 0 and $p-1$ that satisfy

$$e^{2\pi i \alpha_j/p} = a_j. \tag{A.7}$$

Then (A.5) can be rewritten as [137]:

$$\begin{aligned} Z(L(p,q),\widehat{k}) &= \frac{1}{N!}\sum_{\alpha} Z_\alpha(L(p,q),\widehat{k}), \\ Z_\alpha(L(p,q),\widehat{k}) &= \frac{1}{(\widehat{k}|p|)^{l/2}}\exp\left(\frac{i\pi}{\widehat{k}}N(N^2-1)s(q,p)\right)\exp\left(i\pi\frac{q}{p}\widehat{k}|\alpha|^2\right) \\ &\quad \sum_{w,\widetilde{w}\in S_N}\det(w)\exp\left(-\frac{2\pi i}{p\widehat{k}}\langle\rho, w(\rho)\rangle\right)\exp\left(2\pi i\frac{1}{p}\langle\widetilde{w}(\alpha), q\rho - w(\rho)\rangle\right). \end{aligned} \tag{A.8}$$

The set $\{\alpha\}$ is redundant for labeling flat connections in $\mathcal{M}_{\text{flat}}$ because the Weyl group $\mathcal{W} = S_N \subset U(N)$ acts on $\{\alpha\}$ by permuting the α_j's. We will use $\widetilde{\alpha}$ to denote equivalence classes of α under Weyl group action and each $\widetilde{\alpha}$ corresponds to one flat connection modulo gauge transformations. A canonical representative of $\widetilde{\alpha}$ is given by $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ with $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_N$. Using $\widetilde{\alpha}$, (A.5) can be written as

$$Z(L(p,q),\widehat{k}) = \sum_{\widetilde{\alpha}}\frac{1}{|\mathcal{W}_{\widetilde{\alpha}}|}Z_{\widetilde{\alpha}}(L(p,q),\widehat{k}), \tag{A.9}$$

where $\mathcal{W}_{\widetilde{\alpha}} \subset \mathcal{W}$ is the stabilizer subgroup of $\widetilde{\alpha} \in \mathfrak{g}^*$.

Using the naive way (3.49) of computing the partition function of complex Chern-Simons theory when $\mathcal{M}_{\text{flat}}$ is zero-dimensional, one has

$$Z(G_{\mathbb{C}};\tau,\overline{\tau}) = \frac{1}{N!}\sum_{\alpha} Z_{\alpha}\left(G;\frac{\tau}{2}-\check{h}\right) Z_{\alpha}\left(G;\frac{\overline{\tau}}{2}-\check{h}\right). \tag{A.10}$$

Notice that using $\widetilde{\alpha}$ labels, this is

$$Z(G_{\mathbb{C}};\tau,\overline{\tau}) = \sum_{\widetilde{\alpha}} \frac{1}{|\mathcal{W}_{\widetilde{\alpha}}|} Z_{\widetilde{\alpha}}\left(G;\frac{\tau}{2}-\check{h}\right) Z_{\widetilde{\alpha}}\left(G;\frac{\overline{\tau}}{2}-\check{h}\right), \tag{A.11}$$

and the $\frac{1}{|\mathcal{W}_{\widetilde{\alpha}}|}$ factor should not be squared. This is because $G_{\mathbb{C}}$ and G have the same Weyl group $\mathcal{W}$ and in complex Chern-Simons theory $\mathcal{W}$ acts simultaneously on $\mathcal{A}$ and $\overline{\mathcal{A}}$.

(A.11), together with (A.8), is the equation we use to compute the partition function of the complex Chern-Simons theory. In the making of the table 3.1, we have dropped a universal factor

$$\left(\frac{4}{\tau\overline{\tau}}\right)^{N/2} \propto (\ln q)^{N}. \tag{A.12}$$

This matches the factor that is also omitted on the supersymmetric index side.

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