



On the canonical formulation of gauge field theories and Poincaré transformations

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Abstract

We address the Hamiltonian formulation of classical gauge field theories while putting forward results some of which are not entirely new, though they do not appear to be well known. We refer in particular to the fact that neither the *canonical* energy momentum vector (P^μ) nor the *gauge invariant* energy momentum vector (P^μ_{inv}) do generate space-time translations of the gauge field by means of the Poisson brackets: In a general gauge, one has to consider the so-called kinematical energy momentum vector and, in a specific gauge (like the radiation gauge in electrodynamics), one has to consider the Dirac brackets rather than the Poisson brackets. Similar arguments apply to rotations and to Lorentz boosts and are of direct relevance to the “nucleon spin crisis” since the spin of the proton involves a contribution which is due to the angular momentum vector of gluons and thereby requires a proper treatment of the latter. We conclude with some comments on the relationships between the different approaches to quantization (canonical quantization based on the classical Hamiltonian formulation, Gupta-Bleuler, path integrals, BRST, covariant canonical approaches).

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1. Introduction

In 1918, Emmy Noether published her famous article on invariant variational problems in which she stated and proved the so-called Noether theorem(s) [1]. Over the years the latter have become a pillar of classical mechanics and field theory, see reference [2] for an historical account and reference [3] for a general discussion and various applications. According to Noether’s first theorem, the invariance of a field theoretic action functional $S[\phi]$ under an m -dimensional Lie group (of global symmetry transformations) implies the existence of m local conservation laws

for any solution ϕ of the equations of motion $\delta S/\delta\phi = 0$. Following Noether's work, Felix Klein raised the question about the application of Noether's results to the free electromagnetic field. In 1921, E. Bessel-Hagen tackled this problem [4] while taking into account the remark made to him by E. Noether that the invariance of the action functional $S[\phi] \equiv \int_{\mathbb{R}^n} d^n x \mathcal{L}(\phi, \partial_\mu \phi, x)$ allows for the addition of a total divergence term $\partial_\mu \Omega^\mu(\phi, x)$ to the Lagrangian density \mathcal{L} ("divergence symmetry"): By starting from the conformal invariance of the free Maxwell equations (discovered in 1910 by H. Bateman and E. Cunningham) and cleverly combining with local gauge invariance (to which Noether's second theorem applies), he could determine fifteen conserved, gauge invariant quantities in four dimensional Minkowski space.

Among the conformal transformations we have the Poincaré transformations and in particular the space-time translations. More specifically, the invariance of the action under translations $x^\nu \rightsquigarrow x^\nu - a^\nu$ in \mathbb{R}^n implies the local conservation law for the canonical EMT (energy-momentum tensor), $\partial_\mu T_{\text{can}}^{\mu\nu} = 0$, and thereby the existence of n conserved "charges" $P^\nu \equiv \int_{\mathbb{R}^{n-1}} d^{n-1}x T_{\text{can}}^{0\nu}$ which are interpreted as the total energy-momentum of the fields. In their pioneering work on the "quantum dynamics of wave fields" of 1929 [5], W. Heisenberg and W. Pauli presented the general Lagrangian and Hamiltonian formulation of classical relativistic field theories as well as the procedure of canonical quantization (based on equal-time commutation relations). It is commonly believed that, within the Hamiltonian formulation of a classical field theory, the Noether charges generate the symmetry transformations of the phase space variables $\varphi \in \{\phi, \pi \equiv \partial\mathcal{L}/\partial\dot{\phi}\}$ by means of the Poisson brackets, e.g. for infinitesimal translations, $\delta_a \varphi(x) \equiv \{\varphi(x), a^\mu P_\mu\} = a^\mu \partial_\mu \varphi(x)$. This is indeed the case for matter fields (scalar or Dirac fields), but, as we will discuss in detail in the present article, it is definitely more subtle for a gauge field (A^μ): This is due to the fact that the gauge invariance of the action functional $S[A]$ implies the presence of constraints for the phase space variables (as was already noted by Heisenberg and Pauli for electrodynamics in their pioneering work).

As was only realized recently [6], the treatment of the constraints appearing in Lagrangian (or Hamiltonian) dynamical systems with local symmetries like electrodynamics or general relativity has been systematically investigated in 1930 upon Pauli's impetus by his assistant Léon Rosenfeld in a seminal work whose goal was the quantization of the Maxwell-Dirac-Einstein field equations [7]. In the sequel, Rosenfeld moved to other subjects and his work fell into oblivion. In the late forties and fifties, P. Bergmann and his collaborators [8] as well as P.A.M. Dirac rediscovered the results found, or at least anticipated, twenty years earlier by L. Rosenfeld and they worked them out further (e.g. Dirac's modification of the Poisson brackets). In particular, Dirac exposed the general approach to constrained Hamiltonian dynamics in his celebrated Yeshiva lectures of 1964 [9] (see [10–13] for more recent introductions). More recently, the quantization of (non-Abelian) gauge field theories has been revolutionized by the discovery of the so-called BRST-symmetry (Becchi, Rouet, Stora 1974 [14], Tyutin 1975 [15]) and its application to the perturbative renormalization of these theories in their Lagrangian formulation. Yet, the Hamiltonian formulation of classical Abelian or non-Abelian gauge field theory and the canonical approach to its quantization continue to represent a basic tool and useful device for exploring various aspects of gauge theories. Thus, it is worthwhile to have a clear view of the action of Poincaré transformations on the phase space variables in the Hamiltonian formulation. To a large extent, these aspects have already been addressed about forty years ago by some of the masters of the subject (A. J. Hanson, T. Regge and C. Teitelboim) in their Roma lectures [10]. The goal of the present article (the impetus for which came in part from our joint work with M. Reboud and M. Schweda [16]) is to give a short pedagogical account of these ideas. We hope that our presentation clarifies some misleading or erroneous statements made in the literature

and will prove to be useful as a complement to the basic textbook treatments of classical gauge theories and their quantization.

Our text is organized as follows. In section 2, we briefly recall the definition and salient features of the Hamiltonian formulation of classical relativistic field theories. As reviewed in section 3, the description of the geometric transformations of matter fields (scalar and Dirac fields) within this setting is unproblematic. In section 4, we introduce the canonical and improved (gauge invariant) current densities and charges following from the Poincaré invariance of the action functional for pure Yang-Mills theories. The Hamiltonian formulation of Abelian and non-Abelian gauge field theories is then dealt with in section 5 and 7, respectively. The quantization procedure(s) for these theories are addressed in section 6 and 8, respectively, while the coupling to matter fields is considered in section 9. The identification of the physical observables of angular momentum (and its decomposition into different contributions) is outlined in section 10 while the concluding remarks gather some remarks on other approaches to classical (gauge) field theories like the multisymplectic or covariant phase space formulations. In order to provide a better understanding of the structure of geometric symmetry transformations in gauge theories and of their relationship with conserved quantities we devote an appendix to a concise and unified derivation of conserved gauge invariant currents associated to the conformal group.

Notation and conventions: We consider the natural system of units ($c \equiv 1 \equiv \hbar$ and $\varepsilon_0 = 1 \equiv \mu_0$ for electrodynamics). Furthermore, we use the standard notation for the coordinates of n -dimensional Minkowski space, i.e. $x = (x^\mu) = (t, x^i) = (t, \vec{x})$ as well as the signature $(+, -, \dots, -)$ for the Minkowski metric $\eta \equiv (\eta_{\mu\nu})$.

2. Hamiltonian formulation of field theory

The Hamiltonian formulation of classical field theory in \mathbb{R}^n is the starting point for its canonical quantization, and we briefly recall [17] here its basics for a given Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$. The *canonical momentum* π_ϕ associated to the field ϕ is defined by $\pi \equiv \pi_\phi \equiv \partial \mathcal{L} / \partial \dot{\phi}$ and the *canonical Hamiltonian density* \mathcal{H} is defined in terms of the fields ϕ and their canonical momenta π by means of a Legendre transformation:

$$\mathcal{H} \equiv \dot{\phi} \pi - \mathcal{L}. \quad (2.1)$$

Here, and in similar expressions to follow, the sum over all fields is implicitly understood (e.g. the sum over ϕ and ϕ^* in the case of a complex scalar field ϕ).

We note that in the simplest situation (which is realized for instance for a free real scalar field), the relation $\pi \equiv \partial \mathcal{L} / \partial \dot{\phi}$ can be solved for $\dot{\phi}$ as a function of π (and possibly ϕ and/or the spatial derivatives $\partial_k \phi$). The Hamiltonian function $H[\phi, \pi] \equiv \int d^{n-1}x \mathcal{H}(\phi, \pi, \partial_k \phi)$ is now to be viewed as a functional of the fields ϕ and π .

For any two functionals F, G of bosonic fields ϕ and π , the *canonical Poisson bracket* is defined at fixed time t by

$$\{F, G\} \equiv \int d^{n-1}x \left(\frac{\delta F}{\delta \phi} \frac{\delta G}{\delta \pi} - \frac{\delta F}{\delta \pi} \frac{\delta G}{\delta \phi} \right). \quad (2.2)$$

This bracket is bilinear and antisymmetric in its arguments, and it satisfies the Jacobi identity as well as the Leibniz product rule for each of its arguments, e.g. for the second argument: $\{F, GH\} = \{F, G\}H + G\{F, H\}$. The general expression (2.2) yields the fundamental bracket of fields for any fixed time t , e.g. for a single real scalar field ϕ :

$$\{\phi(t, \vec{x}), \pi(t, \vec{y})\} = \delta(\vec{x} - \vec{y}). \tag{2.3}$$

For a given Hamiltonian function $H[\phi, \pi] \equiv \int d^{n-1}x \mathcal{H}(\phi, \pi, \partial_k \phi)$, the *time evolution* of a functional $F[\phi, \pi]$ is given by

$$\dot{F} = \{F, H\}.$$

For example

$$\dot{\phi} = \{\phi, H\} = \frac{\delta H}{\delta \pi}, \quad \dot{\pi} = \{\pi, H\} = -\frac{\delta H}{\delta \phi},$$

i.e. the Hamiltonian equations of motion of the field theoretic system described by $H[\phi, \pi]$.

Let us again consider the particular case of a free real scalar field ϕ . The *space-time translations of the phase space variables* $\varphi \equiv (\phi, \pi)$ are then generated by the conserved Noether charges P^μ (which are associated to the translation invariance of the action) and these transformations are described in terms of the canonical Poisson bracket: For $a \equiv (a^\mu) \in \mathbb{R}^n$, we have

$$\delta_a \varphi(x) \equiv \{\varphi(x), a^\mu P_\mu\} = a^\mu \partial_\mu \varphi(x). \tag{2.4}$$

Similarly the transformation laws of the phase space variables under Lorentz transformations are generated by the Noether charges $J^{\rho\sigma} = -J^{\sigma\rho}$ associated to the Lorentz invariance of the action: Denoting the constant symmetry parameters by $\varepsilon_{\rho\sigma} = -\varepsilon_{\sigma\rho}$, we have

$$\delta_\varepsilon \varphi(x) \equiv \{\varphi(x), \varepsilon_{\rho\sigma} J^{\rho\sigma}\} = \varepsilon_{\rho\sigma} (x^\rho \partial^\sigma - x^\sigma \partial^\rho) \varphi(x). \tag{2.5}$$

In quantum field theory, the variables $\varphi(x)$ and the observables $P^\mu, J^{\rho\sigma}$ become operators, the Poisson bracket being replaced by $1/i\hbar$ times the commutator of operators.

The result (2.4) also holds for free spinor fields. This result states that an infinitesimal, global, Lagrangian symmetry transformation of fields like $\delta_a^L \phi \equiv a^\mu \partial_\mu \phi$ coincides with the infinitesimal Hamiltonian (canonical) symmetry transformation $\delta_a^H \phi(x) \equiv \{\phi(x), a^\mu P_\mu\}$ which is given by the Poisson bracket of fields ϕ with the Lagrangian Noether charges P_μ (expressed in terms of phase space variables ϕ, π). This fact has actually been proven quite generally in classical mechanics (see section 7.12.3 of reference [41]) for the case of non singular Lagrangians $L(q^i, \dot{q}^i)$, i.e. for the case where $\det(\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}) \neq 0$. The line of arguments of the latter proof should also carry over to continuous systems, i.e. to non singular Lagrangian field theories as considered above. However, gauge field theories represent singular dynamical systems. More precisely, for pure gauge theories, the gauge invariance leads to a so-called *constraint*: E.g. for the free Maxwell field (A^μ), we have $\pi_0 \equiv \partial \mathcal{L} / \partial \dot{A}^0 = F^{00} = 0$ where $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$ are the components of the Faraday tensor. The space-time translations are then no longer generated by the canonical Noether charge P^μ and by the canonical Poisson brackets as in equation (2.4): In a general gauge, the canonical expression for P^μ has to be extended following Dirac’s treatment of constrained Hamiltonian systems so as to *construct a “kinematical energy-momentum vector” P_{kin}^μ for gauge fields which generates space-time translations of fields*. If the gauge freedom is completely fixed (e.g. by choosing the radiation gauge or the axial gauge), then the Poisson brackets have to be replaced by the so-called Dirac brackets [10]. A similar conclusion holds for the Lorentz transformations generated by the components of the angular momentum: these components are important for instance for the determination of the spin of the nucleon, the latter being made up of the angular momenta of its constituents (quarks and gluons) [18–20].

To conclude, we note that relations (2.1)–(2.5) represent the conventional formulation of Hamiltonian dynamics which is based on Poisson brackets that are defined at equal times. This

formulation is also referred to as the *instant form* formulation of the classical dynamical system. As noted by Dirac in 1949 [21] (see also [10] and references therein), one may replace the hyperplanes “ $x^0 = \text{constant}$ ” of Minkowski space \mathbb{R}^n by a family of hypersurfaces defined by a condition of the form

$$F(x) = \tau = \text{constant}, \tag{2.6}$$

where F denotes a suitably chosen function. For instance, for $F(x) = x^0$, one recovers the conventional constant time hypersurfaces, and for

$$F(x) \equiv \frac{1}{\sqrt{2}}(x^0 + x^{n-1}) \equiv x^+,$$

one obtains the so-called *null-plane* or *light-front formulation* which has received a lot of attention in the context of two-dimensional conformal field theory, of string theories as well as for gauge field theories in general dimension, e.g. see references [22]. In the formulation of field theory based on (2.6), the fields are considered to be functions of “time” τ and of $n - 1$ “spatial” coordinates $\vec{\sigma}$ which are chosen in such a way that $(\tau, \vec{\sigma})$ parametrizes Minkowski space.

Following R. E. Peierls [23], one may also consider the so-called *Peierls bracket* [23,24] which represents a Poisson bracket of fields at different times. We will come back to this bracket in our concluding remarks.

3. Scalar and Dirac fields

The canonical EMT (energy-momentum tensor) $T_{\text{can}}^{\mu\nu}[\phi]$ for a *free real massive scalar field* ϕ in \mathbb{R}^n , whose dynamics is described by the Lagrangian density $\mathcal{L} \equiv \frac{1}{2} [(\partial^\mu \phi)(\partial_\mu \phi) - m^2 \phi^2]$, yields the conserved energy-momentum vector $P^\nu \equiv \int d^{n-1}x T_{\text{can}}^{0\nu}$ with

$$P^0 = \frac{1}{2} \int d^{n-1}x [\pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2] = H, \quad \vec{P} = - \int d^{n-1}x \pi \vec{\nabla}\phi. \tag{3.1}$$

From these expressions and definition (2.2) of the canonical Poisson bracket, one readily infers that (2.4) holds for $\varphi = \phi$ and $\varphi = \pi$ (while taking into account the equation of motion ($\square + m^2)\phi = 0$).

For the *free Dirac field* described by the Lagrangian

$$\mathcal{L}_{\text{real}}(\psi) \equiv i \bar{\psi} \gamma^\mu \overleftrightarrow{\partial}_\mu \psi - m \bar{\psi} \psi \equiv \frac{i}{2} [\bar{\psi} \gamma^\mu \partial_\mu \psi - (\partial_\mu \bar{\psi}) \gamma^\mu \psi] - m \bar{\psi} \psi,$$

or by
$$\mathcal{L}(\psi) \equiv \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi, \tag{3.2}$$

we have

$$P^0 = \int d^{n-1}x \bar{\psi} (-i\gamma^k \partial_k + m) \psi = H, \quad \vec{P} = \int d^{n-1}x \psi^\dagger (-i\vec{\nabla}\psi), \tag{3.3}$$

and relations (2.4) hold for $\varphi = \psi_\alpha$ and $\varphi = \psi_\alpha^* = i\pi_\alpha^*$. Since the latter equation represents a relation between phase space variables, it represents strictly speaking a constraint equation: $\psi_\alpha^* - i\pi_\alpha^* = 0$. However, these constraints for the Dirac field are *second class* and the replacement of the Poisson bracket by the Dirac bracket [25] yields results of the same form as those obtained by ignoring this subtlety.

We note that relations (3.1) and (3.3) also imply that the charges P^μ Poisson-commute, i.e. we have an Abelian algebra of charges:

$$\boxed{\{P^\mu, P^\nu\} = 0} \quad \text{for } \mu, \nu \in \{0, 1, \dots, n-1\}. \quad (3.4)$$

A consistency check for these results consists in combining (2.4) and (3.1) or (3.3) to verify the Jacobi identity $0 = \{\varphi, \{P^\mu, P^\nu\}\} + \text{cyclic permutations of factors}$.

To summarize, both for free scalar fields and free Dirac fields the components P^μ of the canonical energy-momentum vector generate space-time translations by means of the Poisson brackets, cf. equation (2.4). This result generalizes to the case where one has a multiplet of free scalar or free Dirac fields which is invariant under *global* (rigid) gauge transformations. Even more generally, one can consider a globally gauge invariant self-interaction of matter fields, e.g. include a self-interaction potential $V(\phi^\dagger\phi)$ for a multiplet of scalar fields (like the Higgs field) or an invariant Yukawa-type coupling between scalar and spinor fields. However, if gauge fields are involved, things become more subtle as we will discuss in the next section.

4. Lagrangian formulation of pure gauge theories

4.1. General set-up

In the sequel we are interested in pure Abelian gauge theory (free Maxwell theory) and in pure non-Abelian gauge theory (pure YM theory). For concreteness we will consider the *four dimensional* case and in order to avoid redundancies in the presentation, we will present the generalities for the case of a general symmetry group G , Maxwell’s theory corresponding to the particular case $G = U(1)$.

More precisely, as symmetry group we consider a compact, semi-simple matrix Lie group G of dimension n_G and we denote the associated Lie algebra by \mathfrak{g} . The gauge potential is given by a \mathfrak{g} -valued vector field $A_\mu(x) \equiv A_\mu^a(x)T^a$. Here, $(A_\mu^a)_{\mu \in \{0,1,2,3\}}$ is a real-valued vector field in four space-time dimensions for each value of the internal index $a \in \{1, \dots, n_G\}$ and $\{T^a\}_{a \in \{1, \dots, n_G\}}$ is a basis of the Lie algebra \mathfrak{g} . We have

$$[T^a, T^b] = if^{abc}T^c, \quad (4.1)$$

where the real structure constants f^{abc} can be chosen to be totally antisymmetric in the indices for semi-simple Lie algebras, e.g. $su(N)$. Under an infinitesimal gauge variation parametrized by a \mathfrak{g} -valued function $x \mapsto \omega(x) \equiv \omega^a(x)T_a$, the gauge potential transforms with the covariant derivative of ω :

$$\delta A_\mu = D_\mu \omega \equiv \partial_\mu \omega + iq[A_\mu, \omega]. \quad (4.2)$$

Here, the coupling constant q represents the “non-Abelian” or “YM” charge.

The \mathfrak{g} -valued field strength tensor associated to the gauge potential A_μ reads $F_{\mu\nu} \equiv F_{\mu\nu}^a T_a$ with $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + iq[A_\mu, A_\nu]$. As usual, the components of this tensor will be denoted by

$$F^{i0} = E_i, \quad F^{ij} = -\varepsilon^{ijk}B_k \quad (\text{with the normalization } \varepsilon^{123} = 1).$$

In the non-Abelian case, the vector fields $\vec{E} \equiv (E_i)_{i=1,2,3}$ and $\vec{B} \equiv (B_i)_{i=1,2,3}$ represent the chromo-electric and chromo-magnetic fields,¹ respectively.

¹ The notation E_i for $E_{\chi,i}$ is convenient, but it should be kept in mind in this context that i is not a covariant (Lorentz) index since \vec{E} is not the spatial part of a four-vector (and similarly for B_i).

For free Maxwell theory, the internal index takes a single value $a = 1$ and the totally antisymmetric structure constants f^{abc} in (4.1) vanish, as does the commutator term in the field strength $F_{\mu\nu}$ and in the covariant derivative (4.2). In this case, the vectors \vec{E} and \vec{B} represent the electric and magnetic fields, respectively and there is presently no self-interaction of gauge potentials in the action (4.3) below.

4.2. Lagrangian formulation

Dynamics: The dynamics of pure gauge theory is described by the classical action

$$S[A] \equiv -\frac{1}{4} \int d^4x \operatorname{Tr}(F^{\mu\nu} F_{\mu\nu}) = -\frac{1}{4} \int d^4x F^{a\mu\nu} F_{\mu\nu}^a = \frac{1}{2} \int d^4x \operatorname{Tr}(\vec{E}^2 - \vec{B}^2), \quad (4.3)$$

where we absorbed the so-called *index* of the considered Lie algebra representation into the definition of the trace. The functional (4.3) is gauge invariant and its variation yields the YM field equation $0 = D_\nu F^{\nu\mu} = \partial_\nu F^{\nu\mu} + iq[A_\nu, F^{\nu\mu}]$. In terms of the chromo-electric and -magnetic fields, the latter equations read

$$D_i E_i = 0, \quad \varepsilon^{ijk} D_j B_k = D_0 E_i.$$

In the Abelian case, these equations represent Maxwell's equations $\operatorname{div} \vec{E} = 0$ and $\operatorname{curl} \vec{B} = \partial_t \vec{E}$.

Translational invariance: By virtue of Noether's first theorem, the invariance of the action functional $S[A] \equiv \int d^4x \mathcal{L}$ given by (4.3) under *space-time translations*

$$\delta x^\mu = a^\mu, \quad \delta A^\mu = -a^\nu \partial_\nu A^\mu, \quad (4.4)$$

implies the local conservation law $\partial_\mu T_{\text{can}}^{\mu\nu} = 0$ for the solutions of the (Lagrangian) equations of motion $D_\nu F^{\nu\mu} = 0$. Here,

$$T_{\text{can}}^{\mu\nu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\rho^a)} \partial^\nu A_\rho^a - \eta^{\mu\nu} \mathcal{L} = \operatorname{Tr}(-F^{\mu\rho} \partial^\nu A_\rho + \frac{1}{4} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}), \quad (4.5)$$

represents the *canonical EMT*. By using the equations of motion, this tensor may be decomposed into a gauge invariant part $T_{\text{inv}}^{\mu\nu}$ (generally referred to as the *improved EMT* of the gauge field [16]) and a *superpotential term* (i.e. a total derivative $\partial_\rho \chi^{\mu\rho\nu}$ with $\chi^{\mu\rho\nu} = -\chi^{\rho\mu\nu}$):

$$T_{\text{can}}^{\mu\nu} = T_{\text{inv}}^{\mu\nu} + \partial_\rho \operatorname{Tr}(-F^{\mu\rho} A^\nu), \quad \text{with} \quad \boxed{T_{\text{inv}}^{\mu\nu} = \operatorname{Tr}(F^{\mu\rho} F_\rho^\nu + \frac{1}{4} \eta^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma})}. \quad (4.6)$$

Accordingly, the EMT $T_{\text{inv}}^{\mu\nu}$ is also locally conserved for the solutions of the equations of motion and, for these solutions, it yields the same conserved charges as $T_{\text{can}}^{\mu\nu}$ (for fields which vanish sufficiently fast at spatial infinity). More precisely, for the solutions of the equations of motion, the *canonical Noether charges* P^ν and the *gauge invariant charges* P_{inv}^ν are related by

$$P^\nu \equiv \int d^3x T_{\text{can}}^{0\nu} = P_{\text{inv}}^\nu + \int d^3x \partial_j \operatorname{Tr}(-F^{0j} A^\nu), \quad \text{with} \quad P_{\text{inv}}^\nu \equiv \int d^3x T_{\text{inv}}^{0\nu}. \quad (4.7)$$

The explicit expressions have the form

$$\boxed{P^0 = \int d^3x \operatorname{Tr} \left[\frac{1}{2} (\vec{E}^2 + \vec{B}^2) - A^0 (D_i E_i) \right] = H}, \quad \boxed{\vec{P} = \int d^3x \operatorname{Tr} (E_i \vec{\nabla} A^i)}, \quad (4.8)$$

and

$$\boxed{P_{\text{inv}}^0 = \frac{1}{2} \int d^3x \operatorname{Tr} (\vec{E}^2 + \vec{B}^2) = H_{\text{inv}}}, \quad \boxed{\vec{P}_{\text{inv}} \equiv \int d^3x \operatorname{Tr} (\vec{E} \times \vec{B})}. \quad (4.9)$$

Here, $\frac{1}{2} \operatorname{Tr} (\vec{E}^2 + \vec{B}^2)$ represents the total energy density of the fields and $\operatorname{Tr} (\vec{E} \times \vec{B})$ the associated Poynting vector. We note that the result (4.6) as well as the conserved currents associated to the invariance under Lorentz transformations in \mathbb{R}^n and under scale transformations in \mathbb{R}^4 are derived in a concise and unified manner in appendix A.

Lorentz invariance: The invariance of the action functional $S[A] \equiv \int d^4x \mathcal{L}$ under Lorentz transformations

$$\delta x^\mu = 2\varepsilon^{\mu\nu} x^\nu, \quad \delta A^\mu = \varepsilon_{\rho\sigma} [(x^\rho \partial^\sigma - x^\sigma \partial^\rho) A^\mu + \eta^{\mu\rho} A^\sigma - \eta^{\mu\sigma} A^\rho] \quad \text{with } \varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu} \quad (4.10)$$

implies the local conservation law $\partial_\mu M_{\text{can}}^{\mu\rho\sigma} = 0$ for the solutions of the equations of motion. Here, $M_{\text{can}}^{\mu\rho\sigma}$ denotes the *canonical angular momentum tensor* of the gauge field:

$$M_{\text{can}}^{\mu\rho\sigma} \equiv x^\rho T_{\text{can}}^{\mu\sigma} - x^\sigma T_{\text{can}}^{\mu\rho} + \operatorname{Tr} (-F^{\mu\rho} A^\sigma + F^{\mu\sigma} A^\rho). \quad (4.11)$$

For the solutions of the equations of motion, the latter can be decomposed into a gauge invariant part and a superpotential term:

$$M_{\text{can}}^{\mu\rho\sigma} = M_{\text{inv}}^{\mu\rho\sigma} + \partial_\nu \operatorname{Tr} [-F^{\mu\nu} (x^\rho A^\sigma - x^\sigma A^\rho)], \quad \text{with } \boxed{M_{\text{inv}}^{\mu\rho\sigma} = x^\rho T_{\text{inv}}^{\mu\sigma} - x^\sigma T_{\text{inv}}^{\mu\rho}}. \quad (4.12)$$

For the *canonical conserved charges* $J^{\rho\sigma} \equiv \int d^3x M_{\text{can}}^{0\rho\sigma}$, it is convenient to introduce the following notation for the purely spatial parts: $J^i \equiv \frac{1}{2} \varepsilon^{ijk} J^{jk}$ and $\vec{J} \equiv (J^i)_{i=1,2,3}$. We have

$$J^{ij} = \int d^3x \operatorname{Tr} [E_k (x^i \partial^j - x^j \partial^i) A_k + E_i A^j - E_j A^i], \quad (4.13)$$

$$\text{or } \boxed{\vec{J} = \vec{L} + \vec{S}} \quad \text{with } \boxed{\vec{L} = \int d^3x \operatorname{Tr} [E_k (\vec{x} \times \vec{\nabla}) A^k]}, \quad \boxed{\vec{S} = \int d^3x \operatorname{Tr} (\vec{E} \times \vec{A})}, \quad (4.14)$$

and

$$J^{0i} = \int d^3x \operatorname{Tr} [E_k (x^0 \partial^i - x^i \partial^0) A_k - E_i A^0 + x^i \frac{1}{2} (\vec{E}^2 - \vec{B}^2)], \quad (4.15)$$

$$\text{or } J^{0i} = \int d^3x \operatorname{Tr} [x^0 E_k \partial^i A_k - x^i \frac{1}{2} (\vec{E}^2 + \vec{B}^2) + x^i A^0 (D_k E_k)]. \quad (4.16)$$

For the solutions of the Lagrangian equations of motion $D_\nu F^{\nu\mu} = 0$, the canonical charges $J^{\rho\sigma}$ coincide with *gauge invariant charges* $J_{\text{inv}}^{\rho\sigma} \equiv \int d^3x M_{\text{inv}}^{0\rho\sigma}$ which read as follows (in terms of the notation $\vec{J}_{\text{inv}} \equiv (J_{\text{inv}}^i)_{i=1,2,3}$ with $J_{\text{inv}}^i \equiv \frac{1}{2} \varepsilon^{ijk} J_{\text{inv}}^{jk}$):

$$\vec{J}_{\text{inv}} = \int d^3x \text{Tr}[\vec{x} \times (\vec{E} \times \vec{B})], \quad J_{\text{inv}}^{0i} = \int d^3x \text{Tr}[x^0(\vec{E} \times \vec{B})_i - x^i \frac{1}{2}(\vec{E}^2 + \vec{B}^2)]. \quad (4.17)$$

Thus, we have $\vec{J}_{\text{inv}} = \vec{J} = \vec{L} + \vec{S}$ for the solutions of the equations of motion, but \vec{L} and \vec{S} are not gauge invariant. As a matter of fact, it is not possible to decompose the total angular momentum in a gauge invariant manner into orbital and spin parts — see reference [26] for this issue and for the related question of how to attribute a gauge invariant, and thereby physical, meaning to the spin of the photon.

Conserved charges associated to Poincaré invariance: The structure of the generators (4.14)–(4.17) can nicely be exhibited by using the notation

$$H \equiv \int d^3x \mathcal{H}, \quad H_{\text{inv}} \equiv \int d^3x \mathcal{H}_{\text{inv}}, \quad \vec{P} \equiv \int d^3x \vec{\mathcal{P}}, \quad \vec{P}_{\text{inv}} \equiv \int d^3x \vec{\mathcal{P}}_{\text{inv}},$$

and

$$\vec{K} \equiv \left(J^{0i} \right)_{i=1,2,3}, \quad \vec{K}_{\text{inv}} \equiv \left(J_{\text{inv}}^{0i} \right)_{i=1,2,3},$$

in terms of which we have

$$\boxed{\vec{L} = \int d^3x \vec{x} \times \vec{\mathcal{P}}}, \quad \boxed{\vec{K} = \int d^3x (t \vec{\mathcal{P}} - \vec{x} \mathcal{H})}, \quad (4.18)$$

$$\boxed{\vec{J}_{\text{inv}} = \int d^3x \vec{x} \times \vec{\mathcal{P}}_{\text{inv}}}, \quad \boxed{\vec{K}_{\text{inv}} = \int d^3x (t \vec{\mathcal{P}}_{\text{inv}} - \vec{x} \mathcal{H}_{\text{inv}})}. \quad (4.19)$$

Conserved charges associated to conformal invariance: We note that, in four space-time dimensions, pure Abelian or non-Abelian gauge field theory is not only invariant under the Poincaré group, but also under the larger *conformal group* (see appendix A for the details): The generators corresponding to dilatations and special conformal transformations can be treated along the same lines.

Other conserved charges: For completeness, we mention that a free field theory admits an infinite number of conserved current densities [27–29]. In particular, for free Maxwell theory, one can find gauge invariant conserved quantities differing from those considered above. The latter include in particular the so-called *zilch* currents introduced by D. M. Lipkin [30] (see also [27,31]), i.e. the current densities

$$Z_{\nu\rho}^\mu \equiv \tilde{F}^{\mu\lambda} \overleftrightarrow{\partial}_\rho F_{\lambda\nu} + \tilde{F}_{\nu\lambda} \overleftrightarrow{\partial}_\rho F^{\lambda\mu}, \quad \text{where } \tilde{F}^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}.$$

For their potential physical relevance (in particular the relationship with the optical chirality) and for the underlying symmetry transformations, we refer to [32] and references therein. We also mention that related conserved currents involving spatial non-localities can be found and have been discussed for Maxwell theory [33]. The latter result in particular from the *duality rotations* parametrized by $\theta \in \mathbb{R}$,

$$\vec{E}' = \cos\theta \vec{E} + \sin\theta \vec{B}, \quad \vec{B}' = -\sin\theta \vec{E} + \cos\theta \vec{B},$$

which leave the Maxwell equations (and action) invariant [34–38]. Yet, it should be stressed that the conservation laws which hold for a free field theory do in general not carry over to the interacting theory [27].

5. Hamiltonian formulation of free Maxwell theory

The Hamiltonian formulation being more subtle than the Lagrangian one, we first discuss the Abelian theory in this section and then point out the essential modifications which are brought about by the non-Abelian theory in section 7. We again consider the four dimensional case while starting from the action functional (4.3).

5.1. Canonical momenta and Hamiltonian

The momentum which is canonically conjugate to A_μ is defined by $\pi^\mu \equiv \partial\mathcal{L}/\partial\dot{A}_\mu = F^{\mu 0}$, hence $\vec{\pi}$ coincides with the electric field strength \vec{E} :

$$\boxed{\pi^k = F^{k0} = E_k}. \tag{5.1}$$

From $E_k = F^{k0} = \partial^k A^0 - \partial^0 A^k = (-\vec{\nabla}A^0 - \partial_t \vec{A})^k$, it follows that the relation $\pi^k = \partial\mathcal{L}/\partial\dot{A}_k$ can be solved for the derivative \dot{A}_k in terms of $\vec{\pi}$ and $\vec{\nabla}A^0$: $\dot{A} = -\vec{\pi} - \vec{\nabla}A^0$.

From the antisymmetry of the tensor field ($F^{\mu\nu}$) it follows that $\pi^0 = F^{00} = 0$. Henceforth we cannot use the explicit expression for π^0 to express the time derivative \dot{A}^0 in terms of the fields π^μ (and, possibly, A^μ and/or the spatial derivatives of A^μ) as required by the standard Hamiltonian formulation.² Thus, the dynamical system under consideration represents a so-called *constrained Hamiltonian system* [9–13]: We have the relation

$$0 = \phi_1 \equiv \pi^0, \tag{5.2}$$

whose origin can be traced back to the invariance of the Lagrangian under gauge transformations, e.g. see reference [13]. Since the condition (5.2) results directly from the Lagrangian, it is referred to as a *primary constraint*. (As a matter of fact, π^0 depending on the space-time coordinates, this equation actually represents an infinity of constraints, one for each $\vec{x} \in \mathbb{R}^3$.)

The *canonical Hamiltonian function* $H \equiv P^0 \equiv \int d^3x T_{\text{can}}^{00}$ reads

$$H \equiv \int d^3x \mathcal{H} \equiv \int d^3x (\dot{A}_\mu \pi^\mu - \mathcal{L}) = \int d^3x \left[\frac{1}{2} \pi^i \pi^i + \frac{1}{4} F_{ij} F^{ij} - A_0 (\partial_i \pi^i) \right], \tag{5.3}$$

where the form of the last term results from an integration by parts, assuming as usual that fields vanish at infinity. After substituting $F^{ij} = -\varepsilon^{ijk} B_k$ and $\pi^i = E_i$, we recognize the Abelian special case of expression (4.8) for the canonical Hamiltonian of YM theory.

5.2. Canonical Poisson brackets

The fundamental Poisson-commutator for the canonically conjugate pair (A, π) , which holds for any fixed time t , reads

$$\{A_\mu(t, \vec{x}), \pi^\nu(t, \vec{y})\} = \delta_\mu^\nu \delta(\vec{x} - \vec{y}). \tag{5.4}$$

In particular we have

² The fact that the Lagrangian density does not depend on \dot{A}^0 reflects a degeneracy (related to gauge invariance) and means that A^0 does not really represent a dynamical variable.

$$\{A_i(t, \vec{x}), E_k(t, \vec{y})\} = \delta_i^k \delta(\vec{x} - \vec{y}), \tag{5.5}$$

whence $\{E_i(t, \vec{x}), B_j(t, \vec{y})\} = \varepsilon_{ijk} \partial_k^{(x)} \delta(\vec{x} - \vec{y}),$

(where the last relation follows from the first one by virtue of $\vec{B} = \overrightarrow{\text{curl}} \vec{A}$), all other brackets between A^μ, π^ν and B_k vanishing.

A few remarks concerning these brackets are in order. First, we note that relations (5.5) are sufficient for evaluating brackets between functionals which only depend on \vec{A}, \vec{E} and \vec{B} . For instance, one finds that the brackets between the components of the spin angular momentum \vec{S} , as defined in equation (4.14), satisfy the Lie algebra of infinitesimal rotations,

$$\{S^i, S^j\} = \varepsilon^{ijk} S^k, \tag{5.6}$$

and similarly for the components L^i of \vec{L} and J^i of \vec{J} .

Second, we stress that for the proper evaluation of Poisson brackets like $\{A_\mu(t, \vec{x}), J^{0i}\}$, the functional J^{0i} (as given by expression (4.15) or (4.16)) has to be expressed in terms of canonical variables, i.e. $\partial_0 A^k$ has to be rewritten in terms of $\pi^k = E_k$ and $\partial_k A^0$.

Finally, we note that the basic relation (5.4) is not compatible with the constraint $\pi^0 = 0$. In fact, an important point in this context is that *the constraint equations must not be substituted into the Poisson brackets*: they can only be imposed after computing the Poisson brackets and then amount to projecting the result onto the constraint surface. Indeed, generalizing earlier work of P. Bergmann and his collaborators, Dirac devised a general approach to handle constrained Hamiltonian systems [9–13] which we will also follow below.

5.3. Dirac’s method and extended Hamiltonian

For a function F on the phase space $\{(A^\mu, \pi_\mu)\}$ which vanishes on the primary constraint surface defined by relation (5.2), Dirac introduced the notation $F \approx 0$ (“ F vanishes weakly”), hence we can also write $0 \approx \phi_1 \equiv \pi^0$. The first step of Dirac’s procedure consists of including the primary constraint into the Hamiltonian by means of an undetermined Lagrange multiplier field $\lambda^1(x)$ (which comes without associated momentum):

$$H_p \equiv H + \int d^3x \lambda^1(x) \phi_1(x). \tag{5.7}$$

The second step is to impose that the primary constraint is preserved by the time evolution defined by the *primary Hamiltonian* H_p , i.e. a stability condition for this constraint:

$$0 \overset{!}{\approx} \dot{\phi}_1 \approx \{\phi_1, H_p\} = \partial_i \pi^i.$$

Obviously this condition gives rise to the so-called *secondary constraint*

$$0 \approx \phi_2 \equiv \partial_i \pi^i = \text{div } \vec{E}. \tag{5.8}$$

For the free field theory under consideration, the relation $\text{div } \vec{E} = 0$ represents the Maxwell equation describing the Gauss law in vacuum and (5.8) is therefore referred to as *Gauss law constraint*. Its time evolution does not give rise to further constraints since $\dot{\phi}_2 \approx \{\phi_2, H_p\} = -\partial_i \partial_j F^{ij} = 0$. Since $\{\phi_1, \phi_2\} = 0$, the constraints ϕ_1, ϕ_2 are referred to as *first class constraints* (FCC’s).

The Hamiltonian equations of motion determined by H_p are equivalent to the Lagrangian equations of motion of the dynamical system. For the study of symmetries in the Hamiltonian

framework, Dirac considered a generalization of the Lagrangian formalism given by the so-called *extended Hamiltonian*

$$H_E \equiv H + \sum_{j=1}^2 \int d^3x \lambda^j \phi_j = \int d^3x \left[\frac{1}{2} (\vec{E}^2 + \vec{B}^2) + \lambda^1 \pi^0 + (\lambda^2 - A_0) \operatorname{div} \vec{E} \right], \quad (5.9)$$

which is defined on the *extended phase space* $\{(A_\mu, \pi^\mu, \lambda^j)\}$. The Hamiltonian (5.9) involves a linear combination of all of the FCC's ϕ_j (the primary and secondary FCC's being treated on an equal footing) involving Lagrange multipliers λ^1 and λ^2 . Geometrically speaking, the extended Hamiltonian generates all possible flows of the system under consideration, namely the time evolution determined by the canonical Hamiltonian H as well as the flow generated by the FCC's (i.e. vanishing conserved quantities) [39].

The *extended Hamiltonian equations* are given by $\phi_1 = 0 = \phi_2$ and $\dot{F} = \{F, H_E\}$ for $F \in \{A^0, \dots, A^3, \pi_0, \dots, \pi_3\}$: for the extended Hamiltonian (5.9), we thus obtain

$$\begin{aligned} \dot{A}^0 &= \lambda^1, & \dot{\pi}_0 &= \operatorname{div} \vec{E}, \\ \dot{\vec{A}} &= -\vec{\pi} - \overrightarrow{\operatorname{grad}} A^0 + \overrightarrow{\operatorname{grad}} \lambda^2, & \dot{\vec{\pi}} &= \operatorname{curl} \vec{B}. \end{aligned} \quad (5.10)$$

If we substitute the expressions $\pi_0 = 0$ and $\vec{\pi} = \vec{E}$ into these relations, then we recognize the equations for $\dot{\pi}_\mu$ as Maxwell's free field equations $\partial_\mu F^{\mu\nu} = 0$. The equation of motion $\dot{A}^0 = \lambda^1$ determines λ^1 in terms of \dot{A}^0 : we will come back to this relation below. Here we only note that the last term in (5.9) shows that the field A^0 amounts to a Lagrange multiplier combining with the multiplier λ^2 which also represents an arbitrary function.

5.4. Hamiltonian gauge symmetries

By definition, the *Hamiltonian gauge symmetries* (which are parametrized at the infinitesimal level by $\varepsilon^j(x)$) are generated by the FCC's ϕ_1 and ϕ_2 : for a functional F of (A, π) , we have

$$\delta_{(\varepsilon^j)} F = \{F, G_{(\varepsilon^j)}\}, \quad \text{with } G_{(\varepsilon^j)} \equiv \sum_{j=1}^2 \int d^3x \varepsilon^j(x) \phi_j(x). \quad (5.11)$$

By virtue of (5.4), this relation readily leads to $\delta_{(\varepsilon^j)} \pi^\mu = 0$ and

$$\delta_{(\varepsilon^j)} A_0 = \varepsilon^1, \quad \delta_{(\varepsilon^j)} A_i = -\partial_i \varepsilon^2 \quad \text{for } i \in \{1, 2, 3\}. \quad (5.12)$$

One can show on general grounds [12] (and easily check for the case at hand) that the extended equations of motion (5.10) are invariant under these gauge transformations if the *Lagrange multiplier fields* λ^j transform as

$$\delta_{(\varepsilon^j)} \lambda^1 = \dot{\varepsilon}^1, \quad \delta_{(\varepsilon^j)} \lambda^2 = \dot{\varepsilon}^2 + \varepsilon^1. \quad (5.13)$$

For the present dynamical system (for which the Hamiltonian (5.3) is quadratic in the momenta, and the constraints (5.2), (5.8) are linear in the momenta), the Hamiltonian gauge symmetries generated by the FCC's yield the *Lagrangian symmetries* [13], i.e. the infinitesimal gauge transformations $\delta A_\mu = \partial_\mu \varepsilon$. Indeed, according to (5.12), the latter transformation laws are recovered for the choice of parameters $\varepsilon^1 \equiv \partial_0 \varepsilon$, $\varepsilon^2 \equiv -\varepsilon$, which implies (by virtue of (5.13)) $\delta_{(\varepsilon^j)} \lambda^2 = 0$ and $\delta_{(\varepsilon^j)} \lambda^1 = \ddot{\varepsilon}$. This choice of parameters amounts to imposing the following *generalized gauge condition*, i.e. gauge fixing condition for the Hamiltonian gauge symmetries (5.12), (5.13):

$$\text{gauge condition} \quad \lambda^2 = 0. \tag{5.14}$$

In this case, the extended Hamiltonian reduces to the primary Hamiltonian (which is equivalent to the Lagrangian formulation) and the *gauge generator* $G_{(\epsilon^j)}$ reduces to the one of Lagrangian gauge transformations which reads

$$G_\epsilon = \int d^3x \left[(\partial_0 \epsilon) \pi^0 - \epsilon (\partial_i \pi^i) \right] = \int d^3x \pi^\mu \partial_\mu \epsilon. \tag{5.15}$$

Thus, with (5.14) we have

$$\delta_\epsilon A_\mu \equiv \{A_\mu, G_\epsilon\} = \partial_\mu \epsilon, \quad \delta_\epsilon \pi^\mu \equiv \{\pi^\mu, G_\epsilon\} = 0. \tag{5.16}$$

Conclusion: This means that *the Lagrangian gauge transformations are the residual Hamiltonian gauge transformations in the generalized gauge in which the Lagrange multiplier associated to the secondary constraint is put to zero.* The Lagrange multiplier λ^1 associated with the primary FCC $\pi^0 \approx 0$ still remains undetermined. It can be determined by imposing a consistent generalized gauge condition while leaving the Lagrangian gauge freedom unfixed (see next subsection) or it can be determined as a consequence of a complete gauge fixing of the Lagrangian gauge freedom (subsection 5.6).

5.5. *General gauge in the extended Hamiltonian formalism: kinematical energy-momentum of gauge fields*

From equation (5.10) we see that the relation $\{\pi_\mu, H_E\} = \dot{\pi}_\mu$ (i.e. H_E generates time translations of π_μ) does not involve λ^1, λ^2 and therefore holds for any value of the Lagrange multipliers. By contrast, *the relation* $\{A^\mu, H_E\} = \dot{A}^\mu$ entails that λ^1 and λ^2 are determined in terms of the basic fields (recall from equation (5.1) that $\vec{\pi} = \vec{E} = -\overrightarrow{\text{grad}} A^0 - \dot{\vec{A}}$): it amounts to imposing the

$$\text{generalized gauge conditions} \quad \boxed{\lambda^1 = \dot{A}^0, \quad \lambda^2 = 0}. \tag{5.17}$$

(Comparison with (5.14) shows that the gauge condition $\lambda^2 = 0$ is also the one which allows us to recover the Lagrangian gauge transformations from the Hamiltonian ones.)

In summary, if we impose the gauge conditions (5.17), then the extended Hamiltonian (5.9) only depends on the phase space variables A^μ, π_μ (and the derivatives of A^μ) and reads

$$H_{\text{kin}} \equiv H_E|_{\lambda^j \text{ fixed}} = H + \int d^3x \pi^0 \dot{A}^0. \tag{5.18}$$

Here, H can be decomposed into a gauge invariant part $H_{\text{inv}} \equiv \frac{1}{2} \int d^3x (\vec{E}^2 + \vec{B}^2)$ representing the energy of the electromagnetic field, and a remainder term, i.e.

$$H_{\text{kin}} = H_{\text{inv}} + \int d^3x \left[\pi^0 \dot{A}^0 + \vec{\pi} \cdot \overrightarrow{\text{grad}} A^0 \right] = H_{\text{inv}} + G_{\epsilon=A^0}, \tag{5.19}$$

where $G_{\epsilon=A^0}$ is the gauge generator (5.15) with $\epsilon = A^0$. By construction (see eqn. (5.18)), the functional (5.19) generates time translations of fields in the Hamiltonian framework:

$$\{F, H_{\text{kin}}\} = \dot{F} \quad \text{for } F \in \{A^0, \dots, A^3, \pi_0, \dots, \pi_3\}. \tag{5.20}$$

Since $H = \int d^3x T_{\text{can}}^{00} = P_{\text{can}}^0$, this line of arguments can be generalized as follows to construct P_{kin}^v with $\{F, P_{\text{kin}}^v\} = \partial^v F$. Following reference [10], we define extended quantities involving Lagrange multiplier fields $\Lambda_1^{0v}, \Lambda_2^{0v}$ for the FCC's:

$$P_E^v \equiv \int d^3x T_E^{0v} \equiv \int d^3x \left(T_{\text{can}}^{0v} + \Lambda_1^{0v} \pi^0 + \Lambda_2^{0v} \text{div } \vec{\pi} \right). \tag{5.21}$$

We remark that notational consistency with (5.9) requires $\Lambda_1^{00} = \lambda^1$ and $\Lambda_2^{00} = \lambda^2$, and we note that one can introduce more generally [10] the extended EMT within the Hamiltonian formulation by the expression

$$T_E^{\mu\nu} \equiv T_{\text{can}}^{\mu\nu} + \Lambda_1^{\mu\nu} \pi^0 + \Lambda_2^{\mu\nu} \text{div } \vec{\pi}. \tag{5.22}$$

With the

$$\text{extended gauge conditions} \quad \Lambda_1^{0v} = \partial^v A^0, \quad \Lambda_2^{0v} = 0, \tag{5.23}$$

which encompass the condition (5.17) for $v = 0$, we then obtain the so-called **kinematical energy-momentum vector** of the gauge field,

$$P_{\text{kin}}^v \equiv P_E^v |_{\Lambda_j^{0v} \text{ fixed}} = P^v + \int d^3x \pi^0 \partial^v A^0, \tag{5.24}$$

or

$$\boxed{P_{\text{kin}}^v \equiv P_E^v |_{\Lambda_j^{0v} \text{ fixed}} = P_{\text{inv}}^v + \int d^3x \left[\pi^0 \partial^v A^0 - A^v \partial_i \pi^i \right]}. \tag{5.25}$$

Here, the last term corresponds to the last term in equation (5.19) and the gauge invariant contributions are the familiar ones as given in equation (4.9).

The main result of this subsection is the following one. By construction, **the functional (5.25)** only depends on the phase space variables A^μ, π_μ (and the derivatives of A^μ) and it **generates space-time translations of fields without any gauge condition imposed on (A^μ)**:

$$\boxed{\delta_a \varphi(x) \equiv \{\varphi(x), a_\mu P_{\text{kin}}^\mu\} = a_\mu \partial^\mu \varphi(x)} \quad \text{for } \varphi \in \{A^0, \dots, A^3, \pi_0, \dots, \pi_3\}. \tag{5.26}$$

For $\vec{a} = \vec{0}$, we recover expressions (5.18)–(5.20). It can be explicitly checked that we have the Poisson commutator relations

$$\boxed{\{P_{\text{kin}}^\mu, P_{\text{kin}}^v\} = 0}.$$

On the constraint surface we have $\pi^0 = 0$, hence $\text{div } \vec{E} = 0$ by virtue of the extended Hamiltonian equations (5.10). Then T_{can}^{0v} and T_{inv}^{0v} differ by a divergence $\partial_i (E_i A^v)$ so that the charges P^v and P_{inv}^v coincide with each other. However, as emphasized above, the constraints must not be substituted directly in the Poisson brackets and therefore the charges P^v, P_{inv}^v do not generate space-time translations of the gauge field (A^μ) by virtue of the Poisson bracket: in this respect we have to consider the charge P_{kin}^v which involves an additional π^0 -dependent term, see equations (5.25), (5.26).

We note that the decomposition of the canonical Hamiltonian H into a gauge invariant part H_{inv} and a remainder term, as considered in equations (5.18), (5.19), is also encountered in the context of the *Hamiltonian BRST quantization* where it amounts to treating the field A^0 as a Lagrange multiplier in the Hamiltonian formulation [40].

Interpretation and underpinnings of results: The result (5.26) states that the infinitesimal, global, Lagrangian symmetry transformation $\delta_a^L A^\mu \equiv a_\nu \partial^\nu A^\mu$ of the gauge field coincides with the infinitesimal Hamiltonian symmetry transformation $\delta_a^H A^\mu(x) \equiv \{A^\mu(x), a_\nu P_{\text{kin}}^\nu\}$. By virtue of (5.25) and (5.11) we have

$$a_\nu P_{\text{kin}}^\nu = a_\nu P_{\text{inv}}^\nu + G_{(\varepsilon^1 = a \cdot \partial A^0, \varepsilon^2 = -a \cdot A)}, \quad (5.27)$$

where P_{inv}^ν is the gauge invariant Noether charge associated to translation invariance of the Lagrangian action functional and $G_{(\varepsilon^1, \varepsilon^2)}$ represents the generator of Hamiltonian (canonical) gauge transformations with parameters ε^J depending in a specific way on the translation parameters a^μ and on the gauge field.³ This result as well as the underlying construction (5.21)-(5.25) are quite analogous to the derivation of gauge invariant currents in the Lagrangian formulation as presented in appendix A, e.g. compare the decomposition (5.27) with the expression (A.8) of infinitesimal Lagrangian translations of gauge fields, i.e. (for Abelian gauge fields)

$$\delta_a^L A_\mu = \delta_a^{\text{inv}} A_\mu + \delta_{\varepsilon = a \cdot A}^{\text{gauge}} A_\mu \equiv a^\nu F_{\nu\mu} + \partial_\mu (a \cdot A). \quad (5.28)$$

We remark that relation $\delta_a^H A_\mu(x) \equiv \{A_\mu(x), a_\rho P_{\text{kin}}^\rho\}$ implies that

$$\delta_a^H F_{\mu\nu}(x) = \{F_{\mu\nu}(x), a_\rho P_{\text{kin}}^\rho\} = \{F_{\mu\nu}(x), a_\rho P_{\text{inv}}^\rho\} = a^\rho \partial_\rho F_{\mu\nu}(x). \quad (5.29)$$

Here, the second equality follows from the gauge invariance of the field strength tensor $F_{\mu\nu}$ and the last equality can be verified by using (5.5) as well as the equations of motion, i.e. the free Maxwell equations. Thus the gauge invariant (or the canonical) energy-momentum vector generates space-time translations of the field strengths \vec{E} and \vec{B} (by contrast to the case of the gauge-variant gauge field (A_μ) for which one has to combine this generator with the one of a specific gauge transformation).

Summary: As we noted in section 2 for non singular field theories (i.e. in particular in the absence of gauge symmetries), the Lagrangian symmetries are recovered in the Hamiltonian formulation by the canonical Noether charges. For gauge field theories, the Lagrangian translations of the gauge fields (A^μ) can only be recovered by including a specific gauge transformation in the Hamiltonian symmetry variation describing translations.

More precisely, the gauge fixing conditions (5.17) for the Hamiltonian gauge symmetries (i.e. for the symmetries in extended phase space $\{(A_\mu, \pi^\mu, \lambda^J)\}$ generated by the FCC's ϕ_1, ϕ_2) allow us to reduce the Hamiltonian gauge symmetries to the usual Lagrangian gauge symmetries (i.e. to $\delta A_\mu = \partial_\mu \epsilon$) and, together with their extension (5.23), they allow us to generate space-time translations of phase space variables, in particular of the gauge field (A^μ) in a general gauge (i.e. for gauge potentials (A^μ) which are not constrained by any subsidiary condition). *Lorentz transformations* can be handled in a similar way, see subsection 5.7. We note that one may also formulate a so-called *extended Lagrangian formalism* starting from the notion of an extended configuration space and discuss symmetries in this (less familiar) setting [41].

If one is interested in the *quantization* of the theory, then one has to gauge fix the local symmetry $\delta A_\mu = \partial_\mu \epsilon$ which is at the origin of degeneracies, yielding in particular a singular gauge field propagator. To implement this gauge fixing within the Hamiltonian formulation, one chooses other gauge fixing conditions than relations (5.17) for λ^1 and λ^2 : We will treat this point in the

³ We thank one of the anonymous referees for his insightful comments clarifying the nature and derivation of these results.

next subsection where we will also come back once more to space-time translations of the gauge field.

5.6. Complete gauge fixing for (A^μ) and Dirac brackets

5.6.1. Generalities

In order to fix the Hamiltonian gauge symmetries (5.11) generated by the FCC's $0 \approx \phi_1 \equiv \pi^0$ and $0 \approx \phi_2 \equiv \partial_i \pi^i = \text{div } \vec{\pi}$, we completely break the symmetry generated by $G_{(\varepsilon^j)}$ by imposing appropriate gauge fixing conditions (such that we are only left with the two physical degrees of freedom of a massless vector field in four dimensions): for each FCC $\phi_j(A, \pi) \approx 0$, one introduces a so-called **canonical gauge fixing condition for the Hamiltonian gauge symmetry** $G_{(\varepsilon^j)}$,

$$\boxed{f_j(A, \pi) \approx 0} \quad \text{for } j \in \{1, 2\}. \quad (5.30)$$

The admissibility criteria for these two (independent) conditions are the usual ones, i.e. the gauge slice $f_1 = 0 = f_2$ must be reachable by means of a gauge transformation and it should fix the gauge uniquely, i.e. the gauge slice should be transversal to the gauge orbits. Concerning the latter point, we note that the invertibility of the 2×2 matrix $\mathcal{A} \equiv [\{f_j, \phi_{j'}\}]$ is related to the fact that the only gauge transformation which leaves the gauge fixing condition $f_j = 0$ invariant is the identity transformation, i.e. $0 = \delta_{(\varepsilon^{j'})} f_j = \int d^3x \varepsilon^{j'} \{f_j, \phi_{j'}\}$ implies $\varepsilon^{j'} = 0$ for all j' . Moreover, the conditions of invertibility of \mathcal{A} and of stability of the gauge fixing condition (5.30) under time evolution imply that the Lagrange multipliers λ^1, λ^2 appearing in the extended Hamiltonian are determined in a consistent manner:

$$0 \approx \dot{f}_j \approx \{f_j, H_E\} \approx \{f_j, H\} + \{f_j, \phi_{j'}\} \lambda^{j'}, \quad \text{hence } \lambda^j = -\left(\mathcal{A}^{-1}\right)^{jj'} \{f_{j'}, H\}. \quad (5.31)$$

Examples of admissible gauge fixing conditions for the free electromagnetic field are given by the **radiation gauge**

$$\boxed{\begin{array}{ll} \text{FCC 1: } 0 \approx \phi_1 \equiv \pi^0, & \text{Gauge fixing 1: } 0 \approx f_1(A, \pi) \equiv A^0, \\ \text{FCC 2: } 0 \approx \phi_2 \equiv \partial_i \pi^i, & \text{Gauge fixing 2: } 0 \approx f_2(A, \pi) \equiv \text{div } A, \end{array}} \quad (5.32)$$

or by the **special axial gauge**

$$\boxed{\begin{array}{ll} \text{FCC 1: } 0 \approx \phi_1 \equiv \pi^0, & \text{Gauge fixing 1: } 0 \approx f_1(A, \pi) \equiv A^3, \\ \text{FCC 2: } 0 \approx \phi_2 \equiv \partial_i \pi^i, & \text{Gauge fixing 2: } 0 \approx f_2(A, \pi) \equiv \pi^3 + \partial_3 A^0. \end{array}} \quad (5.33)$$

Before considering these particular cases, we recall some generalities on the gauge fixed Hamiltonian theory. First, let us denote the constraints ϕ_j and the corresponding gauge functions f_j collectively by φ_a ,

$$(\varphi_a)_{a=1, \dots, 4} \equiv (\phi_1, \phi_2, f_1, f_2), \quad (5.34)$$

and define the *reduced phase space* Γ_r as the submanifold of phase space $\Gamma \equiv \{(A, \pi)\}$ defined by the relations $\varphi_a \approx 0$:

$$\Gamma_r \equiv \{(A, \pi) \in \Gamma \mid \phi_j(A, \pi) = 0 = f_j(A, \pi) \text{ for } j = 1, 2\}. \quad (5.35)$$

This space may be viewed as the *physical subspace* of phase space for the constrained dynamical system under consideration.

Furthermore, we introduce the 4×4 -matrix X with elements $X_{ab} \equiv \{\varphi_a, \varphi_b\}$. From $\{\phi_j, \phi_{j'}\} \approx 0$ (FCC's) and the invertibility of the matrix $[\{f_j, \phi_{j'}\}]$, it follows that, whatever the value of the bracket $\{f_j, f_{j'}\}$, one has

$$\det X = \det \left[\begin{array}{c|c} \{\phi_j, \phi_{j'}\} & \{\phi_j, f_{j'}\} \\ \hline \{f_j, \phi_{j'}\} & \{f_j, f_{j'}\} \end{array} \right] \approx (\det [\{f_j, \phi_{j'}\}])^2 \neq 0.$$

Thus, the matrix X is invertible on Γ_r :

$$X \approx \left[\begin{array}{c|c} 0 & -\mathcal{A}^t \\ \hline \mathcal{A} & \mathcal{B} \end{array} \right] \implies X^{-1} \approx \left[\begin{array}{c|c} \mathcal{A}^{-1}\mathcal{B}(\mathcal{A}^{-1})^t & \mathcal{A}^{-1} \\ \hline -(\mathcal{A}^{-1})^t & 0 \end{array} \right]. \quad (5.36)$$

We remark that the FCC's ϕ_j supplemented with gauge fixing conditions $f_j \approx 0$ such that $\det [\{f_j, \phi_{j'}\}] \neq 0$ can be viewed as a set of second class constraints. (The fact that we do not have any FCC's anymore reflects the fact that the gauge has been completely fixed.) The quantization of such a *purely second class system* is based on the introduction of the so-called **Dirac bracket**: For any two functions F, G on phase space, one defines this bracket by

$$\boxed{\{F, G\}_D \equiv \{F, G\} - \{F, \varphi_a\} (X^{-1})^{ab} \{\varphi_b, G\}}. \quad (5.37)$$

The Dirac bracket enjoys the same algebraic properties as the Poisson bracket (i.e. bilinearity, antisymmetry, the Jacobi identity, and the derivation property). Moreover, we have

$$\{\varphi_a, F\}_D = 0 \quad \text{for any function } F, \quad (5.38)$$

since

$$\{\varphi_a, F\}_D = \{\varphi_a, F\} - \{\varphi_a, \varphi_b\} (X^{-1})^{bc} \{\varphi_c, F\} = 0$$

by virtue of $\{\varphi_a, \varphi_b\} = X_{ab}$. The result (5.38) means that the second class system (φ_a) can be set to zero before or after the evaluation of the Dirac bracket. Thus, *after the theory has been formulated in terms of Dirac brackets, the constraints and gauge fixing conditions can be used as strong equalities, i.e. as identities expressing some dynamical variables in terms of others*. In particular, these identities can be imposed as *operatorial identities* in quantum theory where the Dirac brackets of the classical theory become commutators multiplied by $1/i\hbar$.

As a matter of fact, the Dirac and Poisson brackets coincide on the physical subspace Γ_r where $\varphi_a = 0$ for all a . *The aim of the Dirac bracket is to eliminate the unphysical (gauge) degrees of freedom in a consistent way so as to formulate the classical theory solely in terms of the physical degrees of freedom using brackets which differ from the standard Poisson brackets*. Concerning the practical determination of the inverse X^{-1} and thus of the Dirac bracket (5.37), we note that one can proceed in an iterative manner by starting with a subset of the set of all constraints [10].

We now come back again to the free Maxwell field. It follows from the strong equalities $\pi^0 = 0$ and $\text{div } \vec{E} = 0$ that the kinematical energy-momentum vector P_{kin}^ν , as defined by (5.24) or equivalently by (5.25), coincides with the canonical expression P^ν (or with the gauge invariant expression P_{inv}^ν) on the physical subspace. Therefore, **P^ν generates space-time translations of the phase-space variables by means of the Dirac bracket:**

$$\boxed{\delta_a \varphi(x) \equiv \{\varphi(x), a_\mu P^\mu\}_D = a_\mu \partial^\mu \varphi(x)} \quad \text{for } \varphi \in \{A^0, \dots, A^3, \pi_0, \dots, \pi_3\}. \quad (5.39)$$

5.6.2. Radiation gauge

We now consider our first example (5.32) of gauge fixing conditions. We note that the essential condition is the *Coulomb gauge* choice $\text{div} \vec{A} \approx 0$: the Lagrangian field equation

$$0 = \partial_\mu F^{\mu 0} = \partial_i F^{i0} = \partial_i (\partial^i A^0 - \partial^0 A^i) \approx -\Delta A^0, \quad (5.40)$$

then implies the condition $A^0 \approx 0$ for an *appropriate choice of boundary condition of fields* at spatial infinity. We mention [25] that one sometimes also considers the **temporal gauge** choice $A^0 \approx 0$ as the basic gauge condition in (5.32): the field equation $0 = \partial_\mu F^{\mu 0}$ then yields $\partial_0 (\partial_i A^i) \approx 0$. Of course $\text{div} \vec{A} \approx 0$ represents a solution of this equation, but in the present context there is not really a convincing argument for concluding that this represents the only solution since the boundary condition concerns the behavior of fields at spatial infinity. We remark that the Coulomb gauge condition $\partial_i A^i = 0$ is manifestly invariant under rotations (and under translations) and for this reason it was strongly advocated by J. Schwinger (and more recently by S. Weinberg [42]) for the quantization of electrodynamics and the treatment of the spin of the photon (construction of states with good quantum numbers for momentum and angular momentum).⁴

For the radiation gauge choice (5.32), the matrix \mathcal{A} with elements $\mathcal{A}_{jj'} \equiv \{f_j, \phi_{j'}\}$ appearing in (5.36) is invertible⁵:

$$\mathcal{A}(\vec{x}, \vec{y}) \approx \begin{bmatrix} \delta(\vec{x} - \vec{y}) & 0 \\ 0 & \Delta \delta(\vec{x} - \vec{y}) \end{bmatrix} \implies \mathcal{A}^{-1}(\vec{x}, \vec{y}) \approx \begin{bmatrix} \delta(\vec{x} - \vec{y}) & 0 \\ 0 & \frac{-1}{4\pi} \frac{1}{|\vec{x} - \vec{y}|} \end{bmatrix}, \quad (5.41)$$

where we used

$$\Delta G(\vec{x} - \vec{y}) = \delta(\vec{x} - \vec{y}) \quad \text{for } G(\vec{x} - \vec{y}) = \frac{-1}{4\pi} \frac{1}{|\vec{x} - \vec{y}|}. \quad (5.42)$$

Thus, the Dirac brackets (5.37) between the variables A_i and π_j take the form

$$\begin{aligned} \{A_i(t, \vec{x}), \pi_j(t, \vec{y})\}_D &\equiv \{A_i(t, \vec{x}), \pi_j(t, \vec{y})\} \\ &\quad - \int_{\mathbb{R}^3} d^3 z \int_{\mathbb{R}^3} d^3 w \{A_i(t, \vec{x}), \varphi_a(t, \vec{z})\} (X^{-1})^{ab}(\vec{z}, \vec{w}) \\ &\quad \times \{\varphi_b(t, \vec{w}), \pi_j(t, \vec{y})\}, \end{aligned}$$

⁴ The tensorial nature of the observable field strength $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ (i.e. the transformation law $F'^{\mu\nu}(x') = \Lambda^\mu_\rho \Lambda^\nu_\sigma F^{\rho\sigma}(x)$ for $x' = \Lambda x + a$) is not affected by assuming that Lorentz transformations of the gauge potential A^μ mix with gauge transformations (i.e. assuming that $A'^\mu(x') = \Lambda^\mu_\nu [A^\nu(x) + \partial^\nu \omega_\Lambda(x)]$ where ω_Λ is a real-valued function associated to the transformation Λ) [18,43,44]. The Coulomb or special axial gauge conditions are not covariant if A^μ is a four vector, but hold in every inertial system if A^μ transforms with Λ and an appropriately chosen function ω_Λ .

⁵ The inverse of $\mathcal{A}(\vec{x}, \vec{y})$ is defined by

$$\sum_{j' \in \mathbb{R}^3} \int_{\mathbb{R}^3} d^3 z \mathcal{A}_{jj'}(\vec{x}, \vec{z}) (\mathcal{A}^{-1})_{j'j''}(\vec{z}, \vec{y}) = \delta_{jj''} \delta(\vec{x} - \vec{y}),$$

and it is supposed that all fields vanish at spatial infinity.

i.e.

$$\boxed{\{A_i(t, \vec{x}), \pi_j(t, \vec{y})\}_D = -\delta_{ij} \delta(\vec{x} - \vec{y}) + \frac{1}{4\pi} \partial_{x^i} \partial_{y^j} \frac{1}{|\vec{x} - \vec{y}|}}, \tag{5.43}$$

or

$$\boxed{\{A_i(t, \vec{x}), \pi_j(t, \vec{y})\}_D = -\delta_{ij}^\perp(\vec{x} - \vec{y})} \quad \text{with } \delta_{ij}^\perp(\vec{x}) \equiv \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \left(\delta_{ij} - \frac{k_i k_j}{\vec{k}^2} \right), \tag{5.44}$$

where we used the fact that the inverse Fourier transform of \vec{k}^{-2} is $-(4\pi|\vec{x}|)^{-1}$. All other fundamental Dirac brackets vanish, in particular

$$\{A_0(t, \vec{x}), \pi^0(t, \vec{y})\}_D = 0. \tag{5.45}$$

Expression δ_{ij}^\perp is known as the *divergenceless* or *transverse delta function* since it satisfies $\sum_{i=1}^3 \partial_i \delta_{ij}^\perp(\vec{x}) = 0$ for $j \in \{1, 2, 3\}$.

Once the Dirac brackets are considered, the constraints and gauge fixing conditions can be used as strong equalities. This explains why the brackets (5.44), (5.45) do not have the form of canonical (Poisson-)commutation relations. From the physical point of view, we are left with the vector field \vec{A} satisfying the wave equation $\square \vec{A} = \vec{0}$ and the transversality condition $\text{div} \vec{A} = 0$, i.e. with the *two physical degrees of freedom corresponding to the two transverse polarizations of the photon field*. The transformation law (5.39) can be explicitly checked with P^μ given by (4.8) with $\text{div} \vec{E} = 0$, i.e. $P^0 = \frac{1}{2} \int d^3x (\vec{E}^2 + \vec{B}^2)$ and $\vec{P} = \int d^3x E_i \vec{\nabla} A^i$ for the canonical as well as for the gauge invariant energy-momentum vectors. In particular, we have $\{A_i(x), P^j\}_D = \partial^j A_i(x)$.

Obviously, expression (5.44) coincides with the commutator which is obtained in quantum field theory by postulating canonical commutation relations for the creation and annihilation operators of the photon field. In fact, *the previous considerations provide a general framework for the heuristic approach to the canonical quantization of the free electromagnetic field (in the radiation gauge) as considered in classic textbooks*, e.g. see reference [43]. In this context we note that the coupling of gauge fields to a source (j^μ) consists of the addition of a current/field coupling $-j^\mu A_\mu$ to the Lagrangian, i.e. the addition of $+j^\mu A_\mu$ to the Hamiltonian density. By contrast to the Lagrangian formulation (where the equation of motion for A^0 in the Coulomb gauge reads $\Delta A^0 = -j^0$), the field A^0 can now be set to zero consistently [40] since A^0 only amounts to a redefinition of the Lagrange multiplier field λ^2 in the extended Hamiltonian (5.9). For a general discussion of the canonical quantization in the Coulomb gauge for matter fields coupled to the Maxwell field, we refer to the textbook of Weinberg [42].

5.6.3. Special axial gauge

Next we come to our second example (5.33) of gauge fixing conditions. The essential condition is the *special axial gauge condition* $A^3 \approx 0$ since the expression for the canonical momentum,

$$\pi^3 = F^{30} = \partial^3 A^0 - \partial^0 A^3 \approx \partial^3 A^0, \tag{5.46}$$

then implies the condition $\pi^3 + \partial_3 A^0 \approx 0$.

The matrix X of Poisson brackets (at fixed time t) of the four constraints (5.33) can easily be determined and its inverse (5.36) presently reads

$$X^{-1} = \begin{bmatrix} 0 & -g & 0 & f \\ g & 0 & f & 0 \\ 0 & f & 0 & 0 \\ f & 0 & 0 & 0 \end{bmatrix} \quad \text{with} \quad \begin{cases} \partial_{x^3} g(x, y) = f(x, y) \\ \partial_{x^3} f(x, y) = (\partial_{x^3})^2 g(x, y) = \delta(\vec{x} - \vec{y}). \end{cases}$$

Thus, g is a Green function of the linear differential operator $(\partial_{x^3})^2$ and, for an *appropriate choice of boundary conditions*, the latter function is given by [10]

$$g(x, y) = g(\vec{x} - \vec{y}) = \frac{1}{2} \delta(x^1 - y^1) \delta(x^2 - y^2) |x^3 - y^3|, \\ \text{hence} \quad f(x, y) = f(\vec{x} - \vec{y}) = \frac{1}{2} \delta(x^1 - y^1) \delta(x^2 - y^2) \text{sgn}(x^3 - y^3). \quad (5.47)$$

From (5.37) it readily follows that the Dirac brackets of the variables A^1, A^2, π^1, π^2 have canonical form, i.e. we have the non-vanishing brackets

$$\{A_i(t, \vec{x}), \pi^j(t, \vec{y})\}_D = \delta_i^j \delta(\vec{x} - \vec{y}) \quad \text{for } i, j \in \{1, 2\}, \quad (5.48)$$

and by construction these variables have vanishing Dirac brackets with all constraints.

Once the Dirac brackets are considered, the fields A^3 and π^0 vanish while $\pi^3 = -\partial_3 A^0$ where A^0 is a functional of the independent fields (A^1, A^2, π^1, π^2) by virtue of the constraint $\partial_i \pi^i = 0$: Indeed, substitution of $\pi^3 = -\partial_3 A^0$ into the relation $\partial_i \pi^i = 0$ yields an inhomogeneous partial differential equation for A^0 ,

$$0 = \partial_1 \pi^1 + \partial_2 \pi^2 + \partial_3 \pi^3 = \partial_1 \pi^1 + \partial_2 \pi^2 - (\partial_3)^2 A^0 \\ = -\partial_1 \dot{A}^1 - \partial_2 \dot{A}^2 - \Delta A^0,$$

hence A^0 can be expressed in terms of $\partial_1 \pi^1 + \partial_2 \pi^2$ (or of $\partial_1 \dot{A}^1 + \partial_2 \dot{A}^2$) by using the Green function considered for $(\partial_{x^3})^2$ (or for Δ). The space-time translations of the field variables are generated by the canonical Noether charges and the Dirac bracket.

5.7. Lorentz transformations

For Lorentz transformations, i.e. for rotations in \mathbb{R}^3 and for boosts, we proceed in analogy to space-time translations.

Case of a general gauge: In this case concerning the extended Hamiltonian formalism, we add the integral of a linear combination of the constraint functions π^0 and $\text{div} \vec{E}$ to the canonical charges $J^{\rho\sigma}$ (cf. eqn. (5.21))

$$J_E^{\rho\sigma} \equiv J^{\rho\sigma} + \int d^3x [\xi_1^{\rho\sigma} \pi^0 + \xi_2^{\rho\sigma} \text{div} \vec{E}], \quad (5.49)$$

and we fix the multipliers $\xi_1^{\rho\sigma}, \xi_2^{\rho\sigma}$ by requiring that the Poisson bracket of A^μ with the functional $J_E^{\rho\sigma}$ reproduces the correct transformation law (4.10) of A^μ . This procedure yields the result

$$\boxed{\delta_\varepsilon \varphi(x) = \{\varphi(x), \varepsilon_{\rho\sigma} J_{\text{kin}}^{\rho\sigma}\}}, \quad (5.50)$$

with (cf. equations (5.24), (5.25))

$$\boxed{J_{\text{kin}}^{ij} \equiv J_E^{ij} \Big|_{\xi_k^{ij} \text{ fixed}} = J^{ij} + \int d^3x \pi^0 (x^i \partial^j - x^j \partial^i) A^0}, \quad (5.51)$$

or

$$J_{\text{kin}}^{ij} = J_{\text{inv}}^{ij} + \int d^3x [\pi^0 (x^i \partial^j - x^j \partial^i) A^0 - (x^i A^j - x^j A^i) \text{div } \vec{E}], \quad (5.52)$$

and

$$\boxed{J_{\text{kin}}^{0i} \equiv J_E^{0i} \Big|_{\xi_k^{0i} \text{ fixed}} = J^{0i} + \int d^3x \pi^0 [(x^0 \partial^i - x^i \partial^0) A^0 + A^i]}, \quad (5.53)$$

or

$$J_{\text{kin}}^{0i} = J_{\text{inv}}^{0i} + \int d^3x \left\{ \pi^0 [(x^0 \partial^i - x^i \partial^0) A^0 + A^i] - (x^0 A^i - x^i A^0) \text{div } \vec{E} \right\}. \quad (5.54)$$

We note that the gauge invariant electromagnetic field strengths \vec{E} and \vec{B} have a vanishing Poisson bracket with the last terms of (5.52) and (5.54), respectively. Thus, their Lorentz transformations are simply generated by the gauge invariant (or canonical) charges $J_{\text{inv}}^{\rho\sigma}$.

Case of the radiation gauge: In this case, it follows from $\text{div } \vec{E} = 0$ that the expressions for the canonical and the gauge invariant angular momentum vectors coincide with each other: by virtue of (4.13) and (4.16), we have the expressions

$$\begin{aligned} J^{ij} &= \int d^3x [E_k (x^i \partial^j - x^j \partial^i) A_k + E_i A^j - E_j A^i], \\ J^{0i} &= \int d^3x [x^0 E_k \partial^i A_k - x^i \frac{1}{2} (\vec{E}^2 + \vec{B}^2)]. \end{aligned} \quad (5.55)$$

One can readily verify that *the rotations of the gauge field are generated by the canonical charges J^{ij} and by the Dirac bracket*, i.e. for infinitesimal rotations with parameters $\varepsilon_{ij} = -\varepsilon_{ji}$, we have

$$\boxed{\delta_\varepsilon A^\mu(x) = \{A^\mu(x), \varepsilon_{ij} J^{ij}\}_D}. \quad (5.56)$$

However, for a boost of A_k generated by J^{0i} , one gets an additional contribution which has the form of a field dependent gauge transformation [10]:

$$\{A_k(x), J^{0i}\}_D = (x^0 \partial^i - x^i \partial^0) A_k - \frac{\partial}{\partial x^k} \int \frac{d^3y}{4\pi |\vec{x} - \vec{y}|} \frac{\partial A_i}{\partial x^0}(x^0, \vec{y}). \quad (5.57)$$

In fact, the latter term ensures the vanishing of the bracket $\{\partial_k A_k(x), J^{0i}\}_D$ which has to hold by virtue of the gauge fixing condition $\text{div } \vec{A} = 0$ (which is not invariant under Lorentz boosts). Quite generally, for an *infinitesimal boost of the gauge field* (A^μ) parametrized by ε_{0i} , we have to include (for the radiation gauge) the field dependent gauge transformation appearing in the previous equation:

$$\begin{aligned} A'^\mu(x) &= A^\mu(x) + \varepsilon_{0i} \left[(x^0 \partial^i - x^i \partial^0) A^\mu + \eta^{\mu 0} A^i - \eta^{\mu i} A^0 \right. \\ &\quad \left. - \frac{\partial}{\partial x^\mu} \int \frac{d^3y}{4\pi |\vec{x} - \vec{y}|} \frac{\partial A_i}{\partial x^0}(x^0, \vec{y}) \right]. \end{aligned} \quad (5.58)$$

The gauge condition $A^0 = 0$ then also holds by virtue of the field equations (which read $\square A^i = 0$ in the radiation gauge).

6. On the quantization of Abelian gauge field theory

The *canonical quantization* (operator quantization) of a classical constrained Hamiltonian system consists of replacing the Dirac brackets by $1/(i\hbar)$ times the commutators of the corresponding operators. In this section, we have another look at the derivation of Dirac brackets and we put the considered approach to quantization into a general context. Concerning the choice of gauges, we note that a given choice will be more or less convenient depending on the problem under consideration. The form or derivation of the Poincaré transformations will also be commented upon for the different formulations.

6.1. Canonical quantization with a complete gauge fixing

Dirac's approach to constrained Hamiltonian systems starts with the primary constraints which result directly from the Lagrangian without any reference to the equations of motion: the stability condition for the primary constraints (i.e. their preservation under the time evolution defined by the primary Hamiltonian H_p) may yield a secondary constraint whose stability may lead to a tertiary constraint and so on. Thus, one has a *chain of constraints* (which stops in practice after a few steps), e.g. we have a total of two constraints for the free Maxwell theory. FCC's correspond to local Hamiltonian symmetries which have to be gauge fixed so as to eliminate the redundant degrees of freedom. To realize the gauge fixing, one can proceed as for the constraints, i.e. one imposes a single gauge fixing condition and then determines the equations which follow from it by imposing its preservation under time evolution, while iterating the procedure for the resulting equation. Thereby one obtains a *chain of gauge fixing conditions* [45]. By proceeding along these lines for free Maxwell theory, we found in equations (5.40) and (5.46) that the Coulomb gauge fixing condition $\text{div}\vec{A} \approx 0$ yields $A^0 \approx 0$ (upon a proper choice of boundary condition of fields at spatial infinity) and that the special axial gauge condition $A^3 \approx 0$ yields $\pi^3 + \partial_3 A^0 \approx 0$. Stability of these "secondary gauge fixing conditions" yields an equation which fixes the undetermined primary Lagrange multiplier λ^1 . Thus, for the Coulomb and special axial gauge fixing conditions, one has as many independent gauge fixing conditions as constraints: the gauge is fixed completely (which implies that the four degrees of freedom of the gauge field (A^μ) are reduced to its two physical degrees of freedom). This type of gauge fixing is referred to as *class I gauge fixing* in the terminology of Burnel [45,46]. The fact that the Lorentz invariance is not realized manifestly in this approach is unpleasant for calculations, but does not raise a problem for the final physical results since the latter can be shown to be Lorentz invariant (eventually with a fair amount of labor, see [47] and references therein). We note that apart from the radiation gauge and the special axial gauge there exist some other interesting complete gauge fixing conditions, in particular the so-called *light-cone* or *light-front gauge* [10].

Eventually, one may also try to *solve explicitly the constraints*, e.g. for the radiation gauge by decomposing the fields into transversal and longitudinal components and then investigating the brackets between the latter: This approach has some advantages, but it involves non-local expressions and does not strictly follow the canonical procedure in that it ignores the conjugate momentum π^0 [10]. We will briefly expand on this approach in equation (10.4) below.

A general issue of the approach of complete gauge fixing is the unavoidable occurrence of non-localities. For instance, the implementation of the radiation gauge results in Dirac brackets involving a non-local term (i.e. the second, derivative term on the right hand side of equation (5.43)). Similarly, for the special axial gauge, the variable A^0 depends on the independent variables (A^1, A^2, π^1, π^2) by means of an integral, i.e. a non-local expression. These *non-*

localities appearing for a complete gauge fixing result from the derivative terms in the gauge fixing conditions ($\text{div}\vec{A} \approx 0$ and $\pi^3 + \partial_3 A^0 \approx 0$, respectively) and can be traced back to the presence of derivatives in the constraint $\partial_i \pi^i \approx 0$. These non-localities in the Hamiltonian formulation of Abelian gauge field theory do not represent an obstacle for investigating the corresponding quantum field theory and for deriving important physical results [42,43]. However, the whole framework is not fully compatible with the axioms of local relativistic field theory.

At this stage, we also mention the alternative *approach to constrained dynamical systems proposed by L. D. Faddeev and R. Jackiw* [40,48,49] which is essentially equivalent to Dirac’s procedure [50]. The idea of this approach is to formulate the theory in canonical form on reduced phase space, i.e. solely in terms of unconstrained variables which describe the physical degrees of freedom. Yet, the determination of the reduced coordinates amounts to solving explicitly the FCC’s and gauge fixing conditions of Dirac’s approach and thereby non-local expressions appear for the basic variables in gauge field theories [40,49].

6.2. Canonical quantization in the Lorenz gauge $\partial_\mu A^\mu = 0$ (Gupta-Bleuler method)

An alternative to complete gauge fixing within the Hamiltonian formulation consists of modifying the initial Lagrangian by adding to it a *Lorentz covariant gauge fixing term*, e.g. involving $(\partial_\mu A^\mu)^2$: this implies that the field π^0 no longer vanishes and that one has a time evolution equation for all components A^μ of the gauge field. The historical realization of this idea (which goes back to W. Heisenberg in 1928) is to consider $\mathcal{L} \rightsquigarrow \mathcal{L} + \mathcal{L}_{\text{fix}}(A)$ with $\mathcal{L}_{\text{fix}}(A) \equiv -\frac{1}{2\xi} (\partial_\mu A^\mu)^2$ where ξ is a real non-zero parameter, a convenient choice being $\xi = 1$ (“Feynman gauge”). The gauge field is now unconstrained and involves four degrees of freedom which describe two transverse polarizations, a longitudinal one and a scalar one. Accordingly, the Hilbert space of states in quantum theory involves more states than just the physical ones. In the classical Lagrangian field theory, we have (for $\xi = 1$) the equation of motion $\square A^\mu = 0$ which implies $\square(\partial_\mu A^\mu) = 0$. Thus, $\partial_\mu A^\mu$ represents a free scalar field which can eventually be put to zero, thus implementing the *Lorenz gauge condition* (L. Lorenz, 1867). However, in the quantum theory, the Lorenz gauge condition cannot be imposed as an operatorial identity $\partial_\mu A^\mu = 0$ since the latter is inconsistent with the canonical commutation relations for A^μ and π^ν . The way out (i.e. the method to reduce the number of degrees of freedom to the physical ones) is based on a proposal by E. Fermi (1929) and consists of imposing a weaker condition on the theory by restricting the full state space \mathcal{H} to the subspace $\mathcal{H}_{\text{phys}}$ of vectors $|\Psi\rangle$ for which the gauge constraint is satisfied in the mean, i.e. $\langle\Psi|\partial_\mu A^\mu|\Psi\rangle = 0$. The successful implementation of this program was put forward in 1950 (in the Feynman gauge) by S. N. Gupta for the free field case and by K. Bleuler for the interaction of the radiation field with matter [25]: Fermi’s condition is realized for states $|\Psi\rangle$ which satisfy the

$$\text{Gupta-Bleuler subsidiary condition: } \quad \partial^\mu A_\mu^{(+)}(x) |\Psi\rangle = 0 \quad \text{for all } x, \quad (6.1)$$

where $A_\mu^{(+)}(x) \equiv \int \frac{d^3k}{(2\pi)^{3/2} \sqrt{2|\vec{k}|}} a_\mu(|\vec{k}|, \vec{k}) e^{-ikx}$ represents the positive frequency part of A^μ .

Thus, *the gauge field is unconstrained*, but the state space is restricted to a subspace. (One also says that the Lorenz condition holds in the mean for certain states.) Since the Poisson brackets for the gauge field (A^μ) and its conjugate momentum (π_ν) have the canonical form, the *Poincaré transformations* of fields are generated by these brackets in the standard manner.

6.3. Canonical quantization by a generalized Gupta-Bleuler procedure

The Gupta-Bleuler approach for the Lorenz gauge described above can also be formulated by introducing a scalar Lagrange multiplier field b and considering the following modification of the gauge invariant Maxwell field Lagrangian (the modification in this general form being due to T. Kibble [51]):

$$\mathcal{L} \rightsquigarrow \mathcal{L} + \mathcal{L}_{\text{fix}}(A, b), \quad \text{with} \quad \mathcal{L}_{\text{fix}}(A, b) \equiv b(\partial_\mu A^\mu) + \frac{\xi}{2} b^2, \quad (6.2)$$

where ξ is an arbitrary (possibly zero) real constant. The equation of motion of b (i.e. $b = -\frac{1}{\xi} \partial_\mu A^\mu$ if $\xi \neq 0$) then states that the scalar field b coincides up to a factor with the field $\partial_\mu A^\mu$. If one substitutes this equation into the Lagrangian $\mathcal{L}_{\text{fix}}(A, b)$ then one recovers the Lagrangian $\mathcal{L}_{\text{fix}}(A) = -\frac{1}{2\xi} (\partial_\mu A^\mu)^2$ which is usually considered in the Gupta-Bleuler approach (with $\xi = 1$). We remark that one may refer to $b(\partial_\mu A^\mu) + \frac{\xi}{2} b^2$ as the first order form of the gauge fixing Lagrangian and to $-\frac{1}{2\xi} (\partial_\mu A^\mu)^2$ as the second order form [51] very much like the first and second order forms of the gravitational Lagrangian.

The b -field formulation [45] which was put forward by N. Nakanishi [52] and B. Lautrup [44] towards 1966 (with some later refinements [53]) represents an elegant generalization of the Gupta-Bleuler approach to the case of generic values of the gauge parameter ξ . In this formulation, the application of ∂_ν to the equation of motion of A_μ (i.e. to $\partial_\mu F^{\mu\nu} = \partial^\nu b$) yields $\square b = 0$, i.e. b represents a free scalar field. The *Gupta-Bleuler subsidiary condition* presently becomes the

$$\text{Nakanishi-Lautrup subsidiary condition:} \quad b^{(+)}(x) |\Psi\rangle = 0 \quad \text{for all } x, \quad (6.3)$$

which ensures that longitudinal and scalar photons do not contribute to physical processes [45]. By virtue of the equation of motion of b , i.e. $b = -\frac{1}{\xi} \partial_\mu A^\mu$, the condition (6.3) is equivalent (for $\xi \neq 0$) to the Gupta-Bleuler condition (6.1). For the application of Dirac's Hamiltonian approach to the modified (gauge fixed) Maxwell Lagrangian (6.2), we refer to the works [45,70].

It turns out that this approach based on the introduction of an auxiliary field can be generalized to much more general *linear gauge fixing conditions* than the Lorenz gauge, in particular to **algebraic non-covariant gauges** and to gauges interpolating between various of these gauges, see [45] and references therein. Here, we only spell out the *gauge fixing Lagrangian* and a few particular cases which are covered by the latter:

$$\mathcal{L}_{\text{fix}}(A, b, b') = -C_{\mu\nu}(\partial^\mu b)A^\nu + \frac{\xi}{2} b^2 + \frac{\xi'}{2} (\partial^\mu b')(\partial_\mu b') + \xi' b C^\mu \partial_\mu b'. \quad (6.4)$$

In this expression, b and b' are two independent real scalar fields, ξ and ξ' two independent real gauge parameters, $C_{\mu\nu}$ is a given constant, not necessarily symmetric, tensor of rank two (with $C_{00} \neq 0$), and C_μ a given constant four-vector. Interesting particular cases are obtained by expressing $C_{\mu\nu}$ and C_μ in terms of the Minkowski metric $\eta_{\mu\nu}$ and some fixed four-vectors n, n^* . The equation of motion of the auxiliary field b yields the

$$\text{gauge fixing condition} \quad 0 = C_{\mu\nu}(\partial^\mu A^\nu) + \xi b + \xi' C^\mu \partial_\mu b'. \quad (6.5)$$

For instance, for $C_{\mu\nu} = \eta_{\mu\nu}$ and $\xi' = 0$, we recover the Lorenz gauge condition discussed above. In this case, the relativistic invariance is manifestly realized. Another interesting particular case is given by the choice $\xi = 0 = \xi'$ and $C_{\mu\nu} = n_\mu n_\nu - \alpha \eta_{\mu\nu}$ where (n_μ) is a fixed four-vector (with $n^2 > 0$) and α a real constant: The gauge fixing condition (6.5) then reads $0 = (n \cdot \partial)(n \cdot$

$A) - \alpha \partial \cdot A$, hence for $\alpha \rightarrow \infty$ we recover the Lorenz gauge and for $\alpha = 1$ we have a condition generalizing the Coulomb gauge choice. Indeed, the latter is realized in the special frame where $n = (1, \vec{0})$ so that $(n \cdot \partial)(n \cdot A) - \partial \cdot A = -\text{div } \vec{A}$. Remarkably, with some amount of labor [45], the Gupta-Bleuler procedure can be applied for the general gauge fixing Lagrangian (6.4). More precisely, the subsidiary condition selecting physical states $|\Psi\rangle$ that we encountered above, i.e. $b^{(+)}(x)|\Psi\rangle = 0$, now has to be supplemented together with the same condition involving the auxiliary field b' .

For $C_{\mu\nu} \neq \eta_{\mu\nu}$ or $C_\mu \neq 0$, *Lorentz invariance* is broken in the classical theory due to the presence of these fixed tensors, hence the total Lagrangian no longer transforms like a scalar under the infinitesimal Lorentz transformations (4.10). It rather transforms as $\delta\mathcal{L} = \varepsilon_{\mu\nu}(x^\mu \partial^\nu - x^\nu \partial^\mu)\mathcal{L} + K^{\mu\nu} - K^{\nu\mu}$ where $K^{\mu\nu}$ is an asymmetric tensor depending on A^μ, b and b' (which vanishes for $C_{\mu\nu} = \eta_{\mu\nu}, C_\mu = 0$). Consequently, the canonical angular momentum tensor, as given in equation (4.12), is not conserved: $\partial_\mu M_{\text{can}}^{\mu\rho\sigma} = K^{\rho\sigma} - K^{\sigma\rho}$. However, in quantum theory, the subsidiary conditions $b^{(+)}(x)|\Psi\rangle = 0 = b'^{(+)}(x)|\Psi\rangle = 0$ imply that the expectation values of the operators b, b' vanish for physical states and thereby the (normally ordered) operator $K^{\mu\nu}$ also does, i.e. $\langle\Psi|:K^{\mu\nu}:|\Psi\rangle = 0$. Henceforth, *Poincaré invariance* holds in the physical sector of the underlying quantum field theory [45].

6.4. BRST quantization and path integral quantization

A powerful generalization of the Gupta-Bleuler approach to Abelian gauge field theory is given by the BRST quantization. The latter also allows to tackle non-Abelian (i.e. non-linear) gauge field theories for which the Gupta-Bleuler procedure no longer works. It can be applied within the Lagrangian or the Hamiltonian formulation of field theory and it allows us to implement a large variety of Lorentz covariant or non-covariant gauge choices, including *non-linear* ones: We will discuss this point further in section 8 (where we also comment on the path integral approach and on the relationships between these different approaches). Here we only note that *the BRST quantization results in the Hamiltonian framework in a characterization of physical states as those which are left invariant by the so-called BRST operator*: For Abelian gauge field theory, the latter condition is nothing else but the Gupta-Bleuler subsidiary condition, see equations (8.3)–(8.7) below. An example for a *non-linear gauge* in electrodynamics is given by the ‘*t Hooft-Veltman gauge*’ [54] i.e. $0 = \partial_\mu A^\mu + \frac{1}{2}\alpha A_\mu A^\mu$, where $\alpha \neq 0$ represents a real dimensionless constant (see references [55–57] for a study of this gauge).

7. Hamiltonian formulation of pure non-Abelian YM theory

In this section, we outline the non-Abelian generalization of the results presented in section 5 concerning the free Maxwell theory in four dimensions. Thus, our starting point is the action functional (4.3).

7.1. Canonical momenta and Hamiltonian

With the notation $\pi^\mu \equiv \partial\mathcal{L}/\partial\dot{A}_\mu = F^{\mu 0}$ and $F^{i0} \equiv E_i, F^{ij} \equiv -\varepsilon^{ijk} B_k$ (chromo-electric and -magnetic fields), the conserved charges $P^v \equiv \int d^3x T_{\text{can}}^{0v}$ following from the local conservation law $\partial_\mu T_{\text{can}}^{\mu\nu} = 0$ have the form (4.8), (4.9):

$$\begin{aligned}
 P^0 \equiv H &= H_{\text{inv}} + \int_{\mathbb{R}^3} d^3x \text{Tr}[-A^0(D_i\pi^i)], & \text{with } H_{\text{inv}} &\equiv \int_{\mathbb{R}^3} d^3x \text{Tr}\left[\frac{1}{2}(\vec{E}^2 + \vec{B}^2)\right], \\
 \vec{P} &= \vec{P}_{\text{inv}} + \int_{\mathbb{R}^3} d^3x \text{Tr}[-\vec{A}(D_j\pi^j)], & \text{with } \vec{P}_{\text{inv}} &\equiv \int_{\mathbb{R}^3} d^3x \text{Tr}(\vec{E} \times \vec{B}). \quad (7.1)
 \end{aligned}$$

One readily finds that the constraints (5.2), (5.8) of the free Abelian gauge theory presently generalize to Lie algebra-valued constraints

$$\boxed{\pi^0 \approx 0, \quad D_i\pi^i \approx 0}, \quad (7.2)$$

which are again of first class.

7.2. General gauge: kinematical energy-momentum of gauge fields

The Abelian gauge theory expressions (5.22)–(5.25) now generalize to

$$T_E^{\mu\nu} \equiv T_{\text{can}}^{\mu\nu} + \text{Tr}[\Lambda_1^{\mu\nu} \pi^0 + \Lambda_2^{\mu\nu} D_i\pi^i], \quad \text{with } \Lambda_1^{0\nu} = \partial^\nu A^0, \quad \Lambda_2^{0\nu} = 0, \quad (7.3)$$

hence the **kinematical energy-momentum vector of gauge fields** is given by

$$\boxed{P_{\text{kin}}^\nu \equiv P_E^\nu|_{\Lambda_j^{0\nu} \text{ fixed}} = P_{\text{inv}}^\nu + \int d^3x \text{Tr} \left[\pi^0 \partial^\nu A^0 - A^\nu D_i \pi^i \right]}, \quad (7.4)$$

where the contributions P_{inv}^ν are the ones specified in equation (7.1).

7.3. Results for the radiation gauge

Let us consider the *Coulomb gauge* fixing condition

$$\boxed{\text{div} \vec{A} \approx 0}, \quad \text{i.e. } \partial_i A_a^i \approx 0 \text{ for } a \in \{1, \dots, n_G\}. \quad (7.5)$$

From the expression of the canonical momentum,

$$\pi^i = F^{i0} = \partial^i A^0 - \partial^0 A^i + iq[A^i, A^0], \quad (7.6)$$

it follows by substitution of the Coulomb gauge condition for \vec{A} that we have a partial differential equation for A^0 :

$$\partial_i \pi^i \approx -\Delta A^0 + iq[A^i, \partial_i A^0],$$

i.e.

$$Q^{ab} A_b^0 \approx -\partial^i \pi_i^a, \quad \text{with } Q^{ab} \equiv \delta^{ab} \Delta - qf^{abd} A_d^i \partial_i. \quad (7.7)$$

Here, the linear differential operator Q^{ab} represents a deformation of the Laplacian operator which is parametrized by the potential \vec{A} . We note that, by virtue of the secondary constraint $D_i\pi^i \approx 0$, the divergence of $\vec{\pi}$ may also be written as a commutator: $\partial_i \pi^i \approx -iq[A_i, \pi^i]$. (As a matter of fact, the latter commutator represents the density of the conserved charge which is associated to the invariance of the action functional under global gauge transformations.)

A solution A^0 of the inhomogeneous differential equation (7.7) is obtained by convoluting an inverse G_{ab} of Q^{ab} (i.e. a Green function of the differential operator Q^{ab}) with the inhomogeneous term $-\partial^i \pi_i^a$ (or equivalently with $iq[A^i, \pi_i]^a$):

$$\boxed{A_a^0(x) + \int d^3y G_{ab}(x, y) \partial_{y^i} \pi_b^i(y) \approx 0} \tag{7.8}$$

More precisely, we consider the \vec{A} -dependent Green function $G_{ab}(x, y)$ defined by

$$Q^{ab} G_{bc}(x, y) = \delta_c^a \delta(\vec{x} - \vec{y}), \tag{7.9}$$

which decays as $1/r$ at spatial infinity. Although one does not have an explicit expression for G_{ab} , relation (7.9) can be solved iteratively and thus G_{ab} can be written [10] as a deformation of the Green function (5.42) of the Abelian theory (which decays as $1/r$), namely as $\delta_{ab}G$ plus an infinite power series in the coupling constant q .

In summary, we have the *constraint equations* (7.2) and the *gauge fixing conditions* (7.5) and (7.8), i.e. a set of relations which reduces to (5.32) for the Abelian theory. Determination of the Dirac brackets leads to a non-linear generalization of the bracket (5.43),

$$\boxed{\{A_i^a(t, \vec{x}), \pi_j^b(t, \vec{y})\}_D = -\delta^{ab} \delta_{ij} \delta(\vec{x} - \vec{y}) - D_i^{ac} \partial_{y^j} G_{cb}(x, y)}, \tag{7.10}$$

where $D_i^{ac} = \delta^{ac} \partial_{x^i} - q f^{abc} A_i^b(x)$ represents the covariant derivative. (By construction, the bracket (7.10) is compatible with the constraints $\partial^i A_i^a = 0$ and $D^j \pi_j^b = 0$.) One also finds non-vanishing Dirac brackets for A_a^0 and A_b^0 , for A_a^0 and A_b^i and for π_i^a and π_j^b . By contrast to the Abelian theory, the Dirac brackets are now highly non-local in the gauge field \vec{A} (so that the quantization becomes an extremely difficult endeavor). If one considers the Dirac brackets rather than the Poisson brackets, all constraints can be imposed as strong equalities and the kinematical energy-momentum four-vector (7.4) then reduces to the expression P_{inv}^ν specified in equation (7.1). The generators of *Lorentz transformations* can be discussed along the same lines.

We note that (7.10) are not the commutators which are generally considered in the literature for the quantization of YM theories in the Coulomb gauge [58–60] where one rather decomposes the Lie algebra-valued canonical momentum $\vec{\pi} \equiv (\pi^i)$ into transverse and longitudinal parts, i.e. $\vec{\pi} = \vec{\pi}_\perp - \vec{\nabla} \Omega$. By virtue of the Coulomb gauge condition $\text{div} \vec{A} = 0$, the constraint equation $0 = D_i \pi^i = \vec{D} \cdot \vec{\pi}$ is then equivalent to the transversality condition $\text{div} \vec{\pi}_\perp = 0$ (as in the Abelian case) supplemented with the condition

$$\vec{\nabla} \cdot \vec{D} \Omega = -\rho, \quad \text{with } \rho \equiv -iq [A_i, \pi_\perp^i].$$

Very much like (7.7) with $\partial_i \pi^i \approx -iq[A_i, \pi^i]$, this relation represents the non-Abelian generalization of the Poisson equation of electrodynamics with ρ being interpreted as the *density of color charges* of the gauge fields. For the components of \vec{A} and $\vec{\pi}_\perp$, one is then led to a commutator having the same form as in the Abelian theory, i.e. expression (5.43). The corresponding expression for the generators of *Poincaré transformations* and the relativistic invariance of the Hamiltonian formulation of YM in the Coulomb gauge are discussed in references [60,61].

7.4. Results for the special axial gauge

For the *special axial gauge* condition $A^3 \approx 0$, equation (7.6) implies $\pi^3 = \partial^3 A^0$, hence we have the *gauge fixing conditions*

$$\boxed{A^3 \approx 0, \quad \pi^3 + \partial_3 A^0 \approx 0}, \tag{7.11}$$

i.e. we have n_G copies of the gauge fixing condition (5.33) discussed for Abelian gauge theory (though the constraint equations (7.2) presently involve the covariant derivative of $\vec{\pi}$). Thus, there are close parallels with the Abelian theory: the inverse X^{-1} of the matrix X of Poisson brackets again involves the Green function g of the operator $(\partial_{x^3})^2$ as given by (5.47), and we again have the four independent variables (A^1, A^2, π^1, π^2) (which are now Lie algebra-valued) and whose Dirac brackets have the canonical form. By virtue of the constraints and gauge fixing conditions, the variables A^3 and π^0 vanish while A^0 and π^3 can be expressed in terms of the independent variables (A^1, A^2, π^1, π^2) by means of the Green function g . The *space-time translations* are once more generated by the canonical Noether charges and the Dirac brackets.

8. On the quantization of non-Abelian gauge field theory

8.1. Canonical quantization with a complete gauge fixing

The remarks made in section 6 concerning the *canonical quantization* of Abelian gauge field theory also hold for the non-Abelian case: interesting physical results can be derived, but this approach lacks manifest Lorentz invariance and it involves non-local terms. The latter problem is presently worsened quite severely due to the complicated (non-polynomial) field dependence of the non-local terms, e.g. the last term in the Dirac brackets (7.10) or in relation (7.8). For instance, a proof of renormalizability of YM-theory in the Coulomb gauge remains an open problem [62]. Nevertheless, various perturbative or non-perturbative aspects or applications can be (and have been) addressed, e.g. see references [63,64]. For instance, the Hamiltonian light-front formulation of Quantum Chromodynamics (i.e. the gauge theory of strong interactions) is considered to be a promising approach to the problem of determining the field theoretic solutions which describe hadrons, e.g. see [65] and references therein to the large number of related works. Canonical quantization in the Coulomb gauge also represents a useful approach to the exploration of confinement in QCD, e.g. see reference [66] for a review.

We note that, for non-Abelian gauge field theory, a gauge fixing can generally not be realized in a global manner in the space of all gauge fields, i.e. the so-called *Gribov problem* [67] which finds its mathematical expression in a theorem of I. M. Singer. The latter theorem as well as any careful study of finite gauge transformations in non-Abelian gauge theories rely on the consideration of a specific asymptotic behavior of gauge fields in order to render the configuration space mathematically precise: this rules out some gauge choices like the axial gauge [24]. Yet, the Gribov problem is related to large gauge transformations, i.e. non-perturbative calculations, see [62,68,69] and references therein for recent reviews.

8.2. On the Gupta-Bleuler approach

The quantization procedure of Gupta and Bleuler cannot be applied in the non-Abelian case since the addition of a term $-\frac{1}{2} \text{Tr}(\partial_\mu A^\mu)^2$ to the gauge invariant YM Lagrangian yields the modified YM equation $0 = D_\mu F^{\mu\nu} + \partial^\nu(\partial \cdot A)$: application of ∂_ν leads to $\square(\partial \cdot A) = -iq\partial_\nu[A_\mu, F^{\mu\nu}]$, i.e. the fields $\partial^\mu A_\mu^a$ are not free fields. This implies [25,70] that one cannot decompose them in a time invariant manner into positive and negative frequency parts so as to impose a subsidiary condition of the form $\partial^\mu A_\mu^{(+)}(x)|\Psi\rangle = 0$ holding for all times. Thus, the subsidiary conditions of Gupta-Bleuler and of Nakanishi-Lautrup are not consistent with time evolution in the non-Abelian case.

8.3. Path integral quantization

Instead of the canonical quantization, we can consider Feynman’s path integral approach to the quantization of gauge field theories. In this framework, one functionally integrates over all gauge fields A^μ , i.e. one has a functional integral of the form $\int \mathcal{D}A e^{\frac{i}{\hbar} S_{\text{inv}}[A]}$. However, in the latter integral, the gauge fields are overcounted since all gauge equivalent fields should only be counted once. The well-known remedy, put forward by Faddeev and Popov (FP), consists of the choice of a gauge fixing slice in the space of all gauge fields and in the introduction of the corresponding FP determinant in the functional integral. By introducing FP ghost and anti-ghost fields, both of these contributions can be rewritten in a local form so that the action in the exponential (over which one integrates in the functional integral) becomes a total action $S_{\text{tot}} \equiv S_{\text{inv}} + S_{\text{fix}} + S_{\text{FP}}$. For a *generalized gauge fixing condition* of the form $f(A) = B$ (where f and B denote given Lie algebra-valued functions and where B does not depend on A), the ghost action depends on the Lie algebra-valued ghost and anti-ghost fields c, \bar{c} and it has the structure $S_{\text{FP}} = \int d^4x \int d^4y \bar{c}^a(x) \mathcal{M}_{ab}(x, y) c^b(y)$ where

$$\mathcal{M}_{ab}(x, y) \equiv \left. \frac{\delta f^a(\omega A(x))}{\delta \omega^b(y)} \right|_{\omega=0} \quad (\text{with } \omega A_\mu \equiv A_\mu + D_\mu \omega).$$

For instance, for the *homogeneous Coulomb gauge* $\text{div} \vec{A} = 0$, we have

$$\mathcal{M}^{ab}(x, y) = \partial_x^i D_{xi}^{ab} \delta(x - y), \quad \text{hence} \quad S_{\text{FP}} = \int d^4x \text{Tr}(\bar{c} \partial^i D_i c), \quad (8.1)$$

and for the *homogeneous special axial gauge* $A^3 = 0$, we have (upon implementation of $A^3 = 0$)

$$\mathcal{M}^{ab}(x, y) = \delta^{ab} \partial_{x3} \delta(x - y), \quad \text{hence} \quad S_{\text{FP}} = \int d^4x \text{Tr}(\bar{c} \partial_3 c). \quad (8.2)$$

The field-dependent derivative $\partial^i D_i$ in the action (8.1) corresponds to the field-dependent term in the Dirac bracket (7.10) and gives rise to ghost loops in the Coulomb gauge, e.g. see reference [71]. The absence of a field-dependent term in the axial gauge FP action (8.2) reflects the absence of such terms in the Dirac brackets of A^1, A^2, π^1, π^2 and implies that the FP-ghosts decouple in the special axial gauge, such gauges being referred to as ‘ghost-free’ or ‘physical gauges’ (see however [72] and references therein for subtleties related to infrared divergences): This is convenient, but these gauges also come along with a number of complications, e.g. see references [63,71,73,74] for a general discussion and assessment.

8.4. BRST quantization

If one considers the Lorenz gauge, then the total action for pure YM theory appearing in the path integral over A^μ has the form

$$S_{\text{tot}} \equiv S_{\text{inv}} + S_{\text{fix}} + S_{\text{FP}} \equiv \int d^4x \text{Tr} \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + b(\partial_\mu A^\mu) + \frac{\xi}{2} b^2 + \bar{c} \partial^\mu D_\mu c \right], \quad (8.3)$$

where the auxiliary field b is a real Lie algebra-valued scalar field and ξ a gauge parameter. By construction this action is not gauge invariant, but it is invariant under the so-called *BRST transformations* (Becchi, Rouet, Stora 1974 [14], Tyutin 1975 [15]). The latter define a global symmetry (parametrized by a constant, anticommuting parameter) and represent a relic of local

gauge symmetry. By virtue of Noether’s first theorem, this invariance of the gauge fixed action functional S_{tot} yields a conserved charge, the so-called BRST charge.

More precisely [25,70,75], in the **Abelian case**, the gauge fixing action (8.3) leads to a *BRST charge* of the form $Q = \int d^3x [F^{0i} \partial_i c - b \dot{c}]$. If one uses the equation of motion $\partial_\mu F^{\mu\nu} = \partial^\nu b$ of the gauge field, then the expression for Q reduces to

$$Q = - \int d^3x b \overleftrightarrow{\partial}_0 c. \tag{8.4}$$

The total action (8.3) is also invariant under the rescaling of ghosts $c \mapsto e^\rho c$, $\bar{c} \mapsto e^{-\rho} \bar{c}$ (with a constant parameter ρ) which leads to a conserved *ghost number charge*

$$Q_c = - \int d^3x \bar{c} \overleftrightarrow{\partial}_0 c. \tag{8.5}$$

In the Hamiltonian (canonical) formulation of quantum theory, one requires that the physical states $|\Psi\rangle$ are invariant under both operators Q and Q_c , i.e. the

$$\mathbf{Kugo-Ojima subsidiary condition:} \quad Q |\Psi\rangle = 0, \quad Q_c |\Psi\rangle = 0. \tag{8.6}$$

For the Fourier components of the fields c, b , this condition implies

$$c^{(+)}(\vec{k}) |\Psi\rangle = 0, \quad b^{(+)}(\vec{k}) |\Psi\rangle = 0 \quad \text{for all } \vec{k}. \tag{8.7}$$

According to the first relation, the states do not involve ghost particles. By virtue of the second relation and the equation of motion of b (i.e. $b = -\frac{1}{\xi} \partial^\mu A_\mu$), these states are annihilated by $k^\mu A_\mu^{(+)}(\vec{k})$, i.e. the Gupta-Bleuler subsidiary condition (6.1) written in momentum space. We note that for the Abelian theory, the last term in (8.3) does not involve a coupling to the gauge field, hence the ghost fields decouple in this case: *For Abelian gauge theory, the BRST approach then amounts to an elegant formulation of the Gupta-Bleuler method in which the BRST symmetry allows us to eliminate the unphysical degrees of freedom.*

For the case of **non-Abelian YM-theories**, where the Gupta-Bleuler method no longer works, the BRST quantization method can be applied straightforwardly. Actually, this approach to quantization can be applied to quite general field theories involving local symmetries and it can be used to implement quite general linear or non-linear gauge fixing conditions, e.g. see [74,76] and references therein for the Lagrangian framework and [12,40] for the Hamiltonian framework. Concerning the EMT which we discussed in the previous sections, we note that it does not only receive contributions from the gauge invariant YM action, but also from the gauge fixing and ghost terms – see expression (8.3) for a Lorentz covariant gauge fixing. However, the latter terms in the action represent a BRST-exact functional, i.e. $S_{\text{fix}} + S_{\text{FP}}$ has the form of a graded commutator of the BRST charge Q with a gauge fixing fermion Φ_{gf} : $S_{\text{fix}} + S_{\text{FP}} = [Q, \Phi_{\text{gf}}]$. This implies that their contribution $T_{\text{gf}}^{\mu\nu}$ to the total EMT is also BRST-exact⁶ and thereby BRST invariant by virtue of the nilpotency of the BRST operator. This ensures that the matrix elements of the operator $T_{\text{gf}}^{\mu\nu}$ between physical states $|\Psi\rangle, |\Psi'\rangle$ vanish [18] due to the subsidiary condition (8.6).

⁶ In this respect, we note that the EMT can equivalently be defined (e.g. see references [16,77]) by coupling the system to an external gravitational field described by a fixed metric tensor field $(g_{\mu\nu}(x))$ that is BRST-invariant: the EMT in Minkowski space is then given by the flat space limit, i.e. $g_{\mu\nu}(x) = \eta_{\mu\nu}$, of the Einstein-Hilbert EMT in curved space as defined by $T^{\mu\nu}[\varphi, \mathbf{g}] \equiv \frac{-2}{\sqrt{|\mathbf{g}|}} \frac{\delta S_M[\varphi, \mathbf{g}]}{\delta g_{\mu\nu}}$ where $\mathbf{g} \equiv (g_{\mu\nu})$ and $g \equiv \det \mathbf{g}$.

9. Matter field interacting with a gauge field

For simplicity, we consider the case of a complex *scalar field* ϕ of charge e in \mathbb{R}^n which is minimally coupled to an *Abelian gauge field* (A^μ). The matter field Lagrangian then reads

$$\mathcal{L}_M(\phi, A) = (D^\mu \phi^*)(D_\mu \phi) - m^2 \phi^* \phi, \quad \text{with } D_\mu \phi \equiv \partial_\mu \phi + ieA_\mu \phi, \quad D_\mu \phi^* \equiv (D_\mu \phi)^*. \tag{9.1}$$

The complete action $S[A, \phi] \equiv S_{\text{gauge}}[A] + S_M[\phi, A]$ now yields the Maxwell equation $\partial_\nu F^{\nu\mu} = j^\mu$ where

$$j^\mu \equiv j^\mu(\phi, A) \equiv ie [\phi^* D^\mu \phi - \phi D^\mu \phi^*], \tag{9.2}$$

represents the matter current.

The EMT for the minimally coupled field ϕ has [16] the expression

$$T_{\text{int}}^{\mu\nu}(\phi, A) \equiv T_{M,\text{can}}^{\mu\nu} - j^\mu A^\nu = (D^\mu \phi^*)(D^\nu \phi) + (D^\mu \phi)(D^\nu \phi^*) - \eta^{\mu\nu} \mathcal{L}_M(\phi, A), \tag{9.3}$$

where $T_{M,\text{can}}^{\mu\nu} \equiv \frac{\partial \mathcal{L}_M}{\partial(\partial_\mu \phi)} \partial^\nu \phi + \frac{\partial \mathcal{L}_M}{\partial(\partial_\mu \phi^*)} \partial^\nu \phi^* - \eta^{\mu\nu} \mathcal{L}_M$ is the canonical EMT of the matter field. The canonically conjugate momenta associated to ϕ^* and ϕ are given by the covariant derivatives of the fields: $\pi \equiv \partial \mathcal{L} / \partial \dot{\phi}^* = D_0 \phi$ and $\pi^* \equiv \partial \mathcal{L} / \partial \dot{\phi} = D_0 \phi^*$. Thus the components $P_{\text{int}}^\nu \equiv \int d^{n-1}x T_{\text{int}}^{0\nu}[\phi, A]$ of the energy-momentum vector take the form

$$\begin{aligned} P_{\text{int}}^0 &= \int d^{n-1}x \left[\pi^* \pi + (\vec{D}\phi^*)(\vec{D}\phi) + m^2 \phi^* \phi \right], \\ P_{\text{int}}^k &= - \int d^{n-1}x \left[\pi^* D_k \phi + \pi D_k \phi^* \right]. \end{aligned} \tag{9.4}$$

By construction these expressions are gauge invariant.

Given the minimal coupling of matter fields, it does not come as a surprise that the charges P_{int}^μ generate gauge covariant translations,

$$\boxed{\{\varphi(x), P_{\text{int}}^\mu\} = (D^\mu \varphi)(x)} \quad \text{for } \varphi \in \{\phi, \phi^*, \pi, \pi^*\}, \tag{9.5}$$

and that they satisfy a non-Abelian algebra involving the field strength tensor of the gauge field:

$$\boxed{\{P_{\text{int}}^\mu, P_{\text{int}}^\nu\} = - \int d^{n-1}x F^{\mu\nu} j^0} \quad \text{with } j^0 = ie(\phi^* \pi - \phi \pi^*). \tag{9.6}$$

The *kinematical energy-momentum vector* P_{kin}^μ of matter which generates ordinary space-time translations is presently defined by

$$\boxed{P_{\text{kin}}^\mu \equiv P_{\text{int}}^\mu + \int d^{n-1}x A^\mu j^0} \tag{9.7}$$

and it satisfies

$$\{\varphi, P_{\text{kin}}^\mu\} = \partial^\mu \varphi, \quad \{P_{\text{kin}}^\mu, P_{\text{kin}}^\nu\} = 0. \tag{9.8}$$

In fact, by comparing the redefinition (9.7) with relation (9.3) we conclude that P_{kin}^μ is nothing else but the canonical energy-momentum vector of the free scalar field,

$$\boxed{P_{\text{kin}}^\mu = P_{\text{can}}^\mu}, \tag{9.9}$$

which was of course to be expected. The four-vectors (P_{int}^μ) and $(P_{\text{kin}}^\mu) = (P_{\text{can}}^\mu)$ of a matter field which is minimally coupled to a gauge field (A^μ) can be compared to the vectors $m\dot{\vec{x}} = \vec{p} - e\vec{A}$ and \vec{p} for a charged particle which is minimally coupled to a vector potential \vec{A} in classical mechanics. The angular momentum for a scalar or Dirac field can also be discussed along the previous lines: for different expressions and aspects, we refer to [18,19].

10. On the observables of (angular) momentum in gauge theories

As in our previous treatment of gauge theories, we again consider the four dimensional case.

From the physical point of view, the components $p^i \equiv T^{0i}$ of the EMT represent the *density* \vec{p} of linear momentum while the components (M^{0jk}) of the angular momentum tensor represent the *density of total angular momentum*, i.e. of orbital angular momentum \vec{l} and of intrinsic (spin) angular momentum \vec{s} :

$$\vec{P} \equiv \int_{\mathbb{R}^3} d^3x \vec{p}, \quad \vec{J} \equiv \int_{\mathbb{R}^3} d^3x \vec{j} \equiv \int_{\mathbb{R}^3} d^3x (\vec{l} + \vec{s}) \equiv \vec{L} + \vec{S}. \tag{10.1}$$

Any two densities differing by a superpotential term, e.g. $p^i \equiv T^{0i}$ and $(p^i)' = (T^{0i})' = T^{0i} + \partial_j \chi^{0ji}$ (where χ^{0ji} decreases fast enough at spatial infinity) yield the same integrals, i.e. charges P^i (and similarly for J^i, L^i and S^i). In quantum field theory, the latter charges become self-adjoint operators which play an important role, e.g. in characterizing the physical states (momentum, spin or helicity). *Two classically equivalent charges may eventually give rise to operators in quantum theory which have quite different properties, e.g. satisfy different commutation relations.* These issues have physical consequences in quantum electrodynamics for instance for the characteristics of laser beams or in quantum chromodynamics for instance for the spin of the nucleon, the latter being made up of the angular momenta of its constituents (quarks and gluons) [18–20]. In view of these physical applications, we briefly summarize here the naturally given classical expressions for the densities of momentum and angular momentum of a gauge field encountered in section 4, as well as some of the expressions put forward in the literature [18–20]. We refer to the latter work as well as to [78–81] for a discussion of problems related to quantization, in particular the issue of gauge transformations of operators.

The *canonical expressions* (4.5), (4.11) for the EMT and angular momentum tensor of a gauge field yield *gauge-dependent densities* for the linear and angular momentum: The latter can be read off from expressions (4.8) and (4.14),

$$\vec{p}_{\text{can}} = \text{Tr}(E_i \vec{\nabla} A^i), \quad \vec{l}_{\text{can}} = \text{Tr}[E_i (\vec{x} \times \vec{\nabla}) A^i], \quad \vec{s}_{\text{can}} = \text{Tr}(\vec{E} \times \vec{A}). \tag{10.2}$$

In section 5 we saw that, within the extended Hamiltonian formalism, the conditions (5.17) only fix the Lagrange multipliers, but leave the gauge field unfixed. The fundamental Poisson brackets for A^μ and its canonically conjugate momentum π_μ (which hold at fixed time t) then imply standard Poisson brackets for the components of the spin momentum $\vec{S}_{\text{can}} \equiv \int_{\mathbb{R}^3} d^3x \vec{s}_{\text{can}}$, i.e. the Poisson algebra relations (5.6). Upon replacing the Poisson bracket by $1/i\hbar$ times the commutator of operators, we obtain the standard commutator algebra of angular momentum in quantum theory for $\vec{S}_{\text{can}}, \vec{L}_{\text{can}}$ and \vec{J}_{can} .

The so-called *improved expressions* for the EMT and angular momentum tensor of a gauge field as defined by expressions (4.6) and (4.12), respectively, give rise to *gauge invariant densities* which can be read off from (4.9) and (4.17):

$$\vec{p}_{\text{inv}} = \text{Tr}(\vec{E} \times \vec{B}), \quad \vec{j}_{\text{inv}} = \text{Tr}[\vec{x} \times (\vec{E} \times \vec{B})]. \tag{10.3}$$

For an Abelian gauge field these are the familiar expressions of classical electrodynamics [82]. In the sequel we focus on this particular case.

Equivalent expressions of physical interest can be obtained by a decomposition of the vector field \vec{A} into its transverse and longitudinal components. In this respect we recall [83] that any vector field \vec{A} which decreases for $|\vec{x}| \rightarrow \infty$ faster than $1/|\vec{x}|$ admits a unique *Helmholtz decomposition*

$$\vec{A} = \vec{A}_{\parallel} + \vec{A}_{\perp} = -\vec{\nabla}V + \vec{\nabla} \times \vec{C}, \quad \text{where } \vec{\nabla} \times \vec{A}_{\parallel} = \vec{0}, \quad \vec{\nabla} \cdot \vec{A}_{\perp} = 0, \tag{10.4}$$

and

$$V(\vec{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} d^3x' \frac{\vec{\nabla}' \cdot \vec{A}(\vec{x}')}{|\vec{x} - \vec{x}'|}, \quad \vec{C}(\vec{x}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} d^3x' \frac{\vec{\nabla}' \times \vec{A}(\vec{x}')}{|\vec{x} - \vec{x}'|}.$$

In Fourier space, the transversality and longitudinality conditions become $\vec{k} \cdot \vec{A}_{\perp}(\vec{k}, t) = 0$ and $\vec{k} \times \vec{A}_{\parallel}(\vec{k}, t) = \vec{0}$. It should be noted that the expressions for \vec{A}_{\parallel} and \vec{A}_{\perp} are non-local in \vec{A} . For a gauge transformation $\vec{A} \rightsquigarrow \vec{A}' = \vec{A} + \vec{\nabla}\alpha$, we have $\vec{A}'_{\perp} = \vec{A}_{\perp}$ and $\vec{A}'_{\parallel} = \vec{A}_{\parallel} + \vec{\nabla}\alpha$, i.e. \vec{A}_{\perp} is gauge invariant. We remark that for the more general case of non-Abelian gauge fields, the geometric structure underlying the decomposition (10.4) is related to the so-called *dressing field method* to construct gauge invariants, see [84] and references therein.

Using the free field equation $\vec{\nabla} \cdot \vec{E} = 0$, one can easily verify that the canonical densities (10.2) are related by a divergence (superpotential term) to the following *densities considered by Chen, Lu, Sun, Wang, and Goldman* [85]:

$$\vec{p}_{\text{Chen}} = E_i \vec{\nabla} A_{\perp}^i, \quad \vec{j}_{\text{Chen}} = \vec{l}_{\text{Chen}} + \vec{s}_{\text{Chen}}, \quad \text{with } \begin{cases} \vec{l}_{\text{Chen}} = E_i (\vec{x} \times \vec{\nabla}) A_{\perp}^i, \\ \vec{s}_{\text{Chen}} = \vec{E} \times \vec{A}_{\perp}. \end{cases} \tag{10.5}$$

Since the densities (10.5) are gauge invariant, one may as well spell them out in a convenient gauge, for instance in the *radiation gauge*, i.e. for gauge potentials (A^μ) satisfying $\vec{\nabla} \cdot \vec{A} = 0$ and $A^0 = 0$. Then we have $\vec{E}_{\parallel} = -\vec{\nabla}A^0 = \vec{0}$, hence $\vec{E} = \vec{E}_{\perp} = -\dot{\vec{A}}$ (with $\vec{\nabla} \cdot \dot{\vec{A}} = 0$), and thereby we obtain the so-called *gauge invariant canonical expressions* [20] which only involve \vec{E}_{\perp} and \vec{A}_{\perp} :

$$\vec{p}_{\text{gic}} = E_{\perp i} \vec{\nabla} A_{\perp}^i, \quad \vec{j}_{\text{gic}} = \vec{l}_{\text{gic}} + \vec{s}_{\text{gic}}, \quad \text{with } \begin{cases} \vec{l}_{\text{gic}} = E_{\perp i} (\vec{x} \times \vec{\nabla}) A_{\perp}^i, \\ \vec{s}_{\text{gic}} = \vec{E}_{\perp} \times \vec{A}_{\perp}. \end{cases} \tag{10.6}$$

(For instance, for plane waves with frequency ω , we have $\vec{A}_{\perp}(t, \vec{x}) = \vec{A}_{\perp 0}(\vec{x}) e^{-i\omega t}$ with $\vec{\nabla} \cdot \vec{A}_{\perp} = 0$, which implies $\vec{E} = \vec{E}_{\perp} = -\dot{\vec{A}}_{\perp} = i\omega \vec{A}_{\perp}$, hence $\vec{A}_{\perp} = \frac{1}{i\omega} \vec{E}$ is a local field.) In the present setting (where $\vec{E}_{\parallel} = \vec{0}$ and $\vec{E} = \vec{E}_{\perp} = -\dot{\vec{A}}$), the Poisson brackets are to be chosen to have the Dirac form (5.44) with $\vec{\pi} = \vec{E} = -\dot{\vec{A}}$. In this case, \vec{L}_{gic} and \vec{S}_{gic} represent physically measurable quantities, but they cannot really be interpreted as the orbital and spin angular momentum of the electromagnetic field due to the fact that they do not satisfy the algebra of angular momenta, e.g. the components of the vectorial operator \vec{S}_{gic} commute with each other — see [19,20] and references therein. For a recent assessment of the physical issues in the absence or presence of matter and in particular the role of boundary terms, we refer to [80].

11. Covariant Hamiltonian approaches

We recall that our starting point for relativistic gauge field theories was the Lagrangian formulation – see section 4. Thereafter, we considered the standard Hamiltonian approach to these theories. Since time derivatives of fields are treated differently from spatial derivatives in the latter approach, Lorentz covariance is not manifest. For this reason, covariant canonical formulations have been sought for which retain as much as possible the advantages of the standard Hamiltonian approach. Several such approaches have attracted a lot of attention during the last decades. We mention the *multisymplectic approach* following ideas put forward, in particular, towards 1970 by the Warsaw school (notably J. Kijowski [86], K. Gawędzki [87] and W. M. Tulczyjew [88]) and independently by the Spanish school [89,90] as well as H. Goldschmidt and S. Sternberg [91]: for this set-up there exist numerous variants, e.g. see reference [92] for a partial overview. Another formulation is the *covariant phase approach* based on the so-called covariant phase space, i.e. the infinite-dimensional space of all solutions of the field equations. For this set-up, one can adopt the view-point of *symplectic geometry* (following again the Warsaw school as well as more recent work of E. Witten [93] and G. Zuckerman [94]) or consider the so-called *Peierls bracket* introduced by R. E. Peierls [23] and thoroughly investigated by B. DeWitt [24]. There exist relationships between all of these approaches as well as the standard Hamiltonian approach that we followed here (e.g. see references [95–98] for some results in this direction), all formulations having their advantages and shortcomings. Since the covariant approaches rely on physical and mathematical concepts that are sensibly different from the ones of the standard Hamiltonian approach, we will not expand further on these issues here and rather defer this discussion (in particular the treatment of symmetries and conserved currents/charges in gauge field theories) to a separate work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Derivation of the gauge invariant currents associated to conformal invariance

In the last paragraph of this appendix, we present a concise and straightforward derivation of the gauge invariant currents associated to the conformal symmetry (EMT, angular momentum tensor, scale current,...) as well as of their conservation laws, this derivation providing also the superpotential terms which relate these currents to the canonical expressions. Our argumentation generalizes the one considered for free Maxwell theory in four dimensions by the author of reference [99]. It relies on the use of the well-known conformal Killing vector fields of Minkowski space-time and on a gauge covariantization procedure, i.e. expressing ordinary derivatives in terms of gauge covariant derivatives. The latter procedure has been repeatedly (re-)discovered in the literature (in particular in the case of translation invariance), one of the earliest (if not the

first) consideration being due to R. Jackiw [100]. Our treatment of conformal invariance of pure YM theories applies in \mathbb{R}^4 and more generally in \mathbb{R}^n for the particular case of Poincaré transformations. Although the considered geometric approach [99] is appealing and works quite well in the case of four dimensions, it should be noted that the description of infinitesimal conformal transformations by Lie derivatives requires some modifications in a space-time of arbitrary dimension: we will first elaborate on this fact while generalizing some results of reference [102].

Conformal group: By definition, conformal transformations in Minkowski space-time (\mathbb{R}^n, η) are transformations $x \rightsquigarrow x'(x)$ which preserve the angles, i.e. the Minkowski metric is preserved under these transformations up to a scale factor: $ds^2 \rightsquigarrow e^\lambda ds^2$ where λ represents a constant real parameter [3]. The associated infinitesimal transformations $x'^\mu(x) \simeq x^\mu + \xi^\mu(x)$ are generated by *conformal Killing vector fields* $\xi \equiv \xi^\mu \partial_\mu$, i.e. solutions of the

$$\text{conformal Killing equation : } \quad \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{2}{n} (\partial_\rho \xi^\rho) \eta_{\mu\nu} = 0. \tag{A.1}$$

The general solution of this equation is given by

$$\xi_\mu = a_\mu + \varepsilon_{\mu\nu} x^\nu + \rho x_\mu + 2(c \cdot x) x_\mu - c_\mu x^2, \tag{A.2}$$

where a_μ, ρ, c_μ and $\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}$ are constant real parameters. For Poincaré transformations we have $\xi_\mu(x) = a_\mu + \varepsilon_{\mu\nu} x^\nu$: this transformation also preserves the lengths and solves the ordinary Killing equation $\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0$ (whence $\partial_\rho \xi^\rho = 0$). The parameter ρ labels *scale transformations (dilatations)* $x' = e^\rho x$ while (c_μ) labels *conformal boosts (special conformal transformations)*.

Transformation laws of fields: The gauge potential (A_μ) and the associated field strength $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + iq [A_\mu, A_\nu]$ introduced in section 4.1 represent relativistic tensor fields and thereby transform with the Lie derivative L_ξ under infinitesimal diffeomorphisms generated by a vector field $\xi \equiv \xi^\mu \partial_\mu$:

$$\delta_\xi A_\mu = (L_\xi A)_\mu = \xi^\nu \partial_\nu A_\mu + (\partial_\mu \xi^\nu) A_\nu, \tag{A.3a}$$

$$\delta_\xi F_{\mu\nu} = (L_\xi F)_{\mu\nu} = \xi^\rho \partial_\rho F_{\mu\nu} + (\partial_\mu \xi^\rho) F_{\rho\nu} + (\partial_\nu \xi^\rho) F_{\mu\rho}. \tag{A.3b}$$

For the particular case of the vector field $\xi[a, \varepsilon] = (a^\mu + \varepsilon^{\mu\nu} x_\nu) \partial_\mu$ describing infinitesimal Poincaré transformations, the variation (A.3a) yields the standard form of infinitesimal translations and Lorentz transformations as considered in equations (4.4) and (4.10), respectively. For the case of scale transformations where $\xi^\mu[\rho] = \rho x^\mu$ in \mathbb{R}^n , the variations (A.3a)-(A.3b) become

$$\delta_{\xi[\rho]} A_\mu = \rho (x^\nu \partial_\nu A_\mu + 1 A_\mu) \tag{A.4a}$$

$$\delta_{\xi[\rho]} F_{\mu\nu} = \rho (x^\nu \partial_\nu F_{\mu\nu} + 2 F_{\mu\nu}). \tag{A.4b}$$

Here, the first equation is the standard form of a scale transformation $\delta_\rho A_\mu$ of a vector field in \mathbb{R}^4 with parameter ρ (and so is the second equation for a tensor field with components $F_{\mu\nu}$). The scale factors in the last terms of the geometric transformations (A.4a)-(A.4b) simply reflect the rank of the covariant tensor fields under consideration. However, for a space-time of generic dimension n , the *scale transformation of a vector field*, i.e.

$$\delta_\rho A_\mu = \rho (x^\nu \partial_\nu A_\mu + d_A(n) A_\mu) \tag{A.5}$$

involves the *scale (canonical) dimension* $d_A(n) \equiv \frac{n-2}{2}$ of this field in \mathbb{R}^n [3]. To recover this transformation law of A_μ for $\xi^\mu[\rho] = \rho x^\mu$ from the geometric transformation law (A.3a), the latter has to be supplemented by an additional term [102]:

$$\delta_{CG} A_\mu \equiv (L_\xi A)_\mu + \frac{n-4}{2n} (\partial_\nu \xi^\nu) A_\mu. \tag{A.6}$$

For Poincaré transformations, i.e. for $\xi^\mu[a, \varepsilon] = a^\mu + \varepsilon^{\mu\nu} x_\nu$, the last term in expression (A.6) vanishes. For dilatations, i.e. for $\xi^\mu[\rho] = \rho x^\mu$, relation (A.6) yields the correct transformation law (A.5) of A_μ . It also does for conformal boosts, i.e. for $\xi^\mu[c] = 2(c \cdot x) x^\mu - c^\mu x^2$. Henceforth, the variation (A.6) (whose last term vanishes for $n = 4$ as well as for Poincaré transformations in \mathbb{R}^n) describes all infinitesimal transformations of A_μ associated to the conformal group (whence the label CG in the variation (A.6)). Expression (A.6) induces the following transformation law of the field strength:

$$\delta_{CG} F_{\mu\nu} = (L_\xi F)_{\mu\nu} + \frac{n-4}{2n} (\partial_\rho \xi^\rho) F_{\mu\nu} + \frac{n-4}{2n} \{ \partial_\mu (\partial_\rho \xi^\rho) A_\nu - \partial_\nu (\partial_\rho \xi^\rho) A_\mu + iq (\partial_\rho \xi^\rho) [A_\mu, A_\nu] \}. \tag{A.7}$$

In the case of an Abelian gauge field theory, this expression reduces to the results given in reference [102], e.g. for dilatations we obtain $\delta_\rho F_{\mu\nu} = \rho (x^\lambda \partial_\lambda F_{\mu\nu} + \frac{n}{2} F_{\mu\nu})$. The A -dependent contributions in (A.7) (which are not gauge covariant) reflect the fact that the field strength does not represent a primary field for $n \neq 4$.

Conformal invariance and associated conservation laws: For $n \neq 4$, the coupling constant q of YM-theory is dimensionful and thereby the YM action is only scale invariant for $n = 4$ (by contrast to the Abelian theory which is scale invariant for all values of n). Accordingly, we can consider the geometric transformation laws (A.3a) -(A.3b) for all conformal Killing vector fields in \mathbb{R}^4 as well as for the conformal Killing vector fields $\xi^\mu[a, \varepsilon]$ which generate Poincaré transformations in \mathbb{R}^n . The derivative of (A_μ) in the transformation law (A.3a) can be expressed in terms of the field strength:

$$\delta_\xi A_\mu = \xi^\nu F_{\nu\mu} + D_\mu(\xi \cdot A), \quad \text{with} \quad \xi \cdot A \equiv \xi^\nu A_\nu. \tag{A.8}$$

Here, the first term reflects a covariantization of $\xi^\nu \partial_\nu A_\mu$ and the second term represents an infinitesimal gauge transformation of A_μ with (field dependent) parameter $\xi \cdot A$. As is well known, (e.g. see reference [101]), such a gauge transformation does not contribute to the Noether charge since it yields a current which vanishes on-shell up to a superpotential term. We note that the coordinate transformations (A.8) of gauge fields are often considered in conjunction with local gauge transformations, in particular in the study of BRST symmetries in flat or curved space-time, e.g. see [103] and references therein.

The transformation law (A.3b) of $F_{\mu\nu}$ induces the following variation of the Lagrangian $\mathcal{L} \equiv -\frac{1}{4} \text{Tr}(F^{\mu\nu} F_{\mu\nu})$ of pure YM theory:

$$\delta_\xi \mathcal{L} = \partial_\mu (\xi^\mu \mathcal{L}) + \frac{1}{2} \text{Tr} [F^{\mu\alpha} F_\alpha{}^\nu (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2} (\partial_\rho \xi^\rho) \eta_{\mu\nu})]. \tag{A.9}$$

Here, the trace term vanishes by virtue of the conformal Killing equation (A.1) with $n = 4$ and it vanishes for all values n if we limit ourselves to Poincaré transformations. Thus, the action

is invariant under the corresponding geometric transformations of the gauge fields. To recover Noether's first theorem, we multiply the equation of motion function $\frac{\delta S}{\delta A_\mu^a} = D_\nu F_a^{\nu\mu}$ with the variation $\delta_\xi A_\mu^a$: substitution of (A.8) and use of the Leibniz rule as well as of the Bianchi identity ($0 = D_\rho F_{\mu\nu} +$ cyclic permutations of the indices) and of $[D_\mu, D_\nu](\xi \cdot A) = iq [F_{\mu\nu}, \xi \cdot A]$ then yields the result

$$0 = \text{Tr} \left(\frac{\delta S}{\delta A_\mu} \delta_\xi A_\mu \right) + \partial_\mu j^\mu - \frac{1}{2} \text{Tr} [F^{\mu\alpha} F_\alpha^\nu (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \frac{1}{2} (\partial_\rho \xi^\rho) \eta_{\mu\nu})], \quad (\text{A.10})$$

with

$$j^\mu \equiv T_{\text{inv}}^{\mu\nu} \xi_\nu + J^\mu, \quad \boxed{T_{\text{inv}}^{\mu\nu} \equiv \text{Tr} (F^{\mu\rho} F_\rho^\nu) - \eta^{\mu\nu} \mathcal{L}}, \quad \boxed{J^\mu \equiv \text{Tr} [-F^{\mu\nu} D_\nu (\xi \cdot A)]}. \quad (\text{A.11})$$

As we noted after eqn. (A.9), the last term in (A.10) vanishes for conformal transformations in \mathbb{R}^4 and it vanishes for Poincaré transformations in \mathbb{R}^n . For the solutions of the equation of motion $0 = \frac{\delta S}{\delta A_\mu} = D_\nu F^{\nu\mu}$, the current density (j^μ) is thus conserved in these cases, (i.e. $\partial_\mu j^\mu \approx 0$) and the contribution J^μ to j^μ writes

$$J^\mu \approx \partial_\nu [\text{Tr} (-F^{\mu\nu} A^\rho) \xi_\rho] \equiv \partial_\nu B^{\mu\nu} \quad \text{with } B^{\mu\nu} = -B^{\nu\mu}. \quad (\text{A.12})$$

This term represents a superpotential term (whose divergence vanishes identically and which does not contribute to the Noether charges). More precisely, for the particular case of *translations*, i.e. for $\xi_\mu = a_\mu$, this superpotential term is the one encountered in eqn. (4.6) and so is the gauge invariant EMT $T_{\text{inv}}^{\mu\nu}$ satisfying the local conservation law $\partial_\mu T_{\text{inv}}^{\mu\nu} \approx 0$: by virtue of equations (A.11)-(A.12) we altogether have

$$\boxed{j^\mu \equiv T_{\text{inv}}^{\mu\nu} a_\nu + \text{Tr} [-F^{\mu\rho} D_\rho (a_\nu A^\nu)] \approx \{T_{\text{inv}}^{\mu\nu} + \partial_\rho [\text{Tr} (-F^{\mu\rho} A^\nu)]\} a_\nu = T_{\text{can}}^{\mu\nu} a_\nu}. \quad (\text{A.13})$$

Thus, we have recovered the canonical EMT and the gauge invariant EMT as well as the superpotential term which relates these conserved currents. In the case of *Lorentz transformations*, i.e. for $\xi_\mu = \varepsilon_{\mu\nu} x^\nu$, we get

$$J^\mu \approx \frac{1}{2} \varepsilon_{\rho\sigma} \partial_\nu \text{Tr} [F^{\mu\nu} (x^\rho A^\sigma - x^\sigma A^\rho)], \quad (\text{A.14})$$

$$j^\mu = T_{\text{inv}}^{\mu\nu} \xi_\nu = -\frac{1}{2} \varepsilon_{\rho\sigma} M_{\text{inv}}^{\mu\rho\sigma}, \quad \text{with } M_{\text{inv}}^{\mu\rho\sigma} \equiv x^\rho T_{\text{inv}}^{\mu\sigma} - x^\sigma T_{\text{inv}}^{\mu\rho},$$

and $\partial_\mu M_{\text{inv}}^{\mu\rho\sigma} \approx 0$. The previous expressions for the angular momentum tensor $M_{\text{inv}}^{\mu\rho\sigma}$ and for the superpotential term J^μ coincide with those in eqn. (4.12). The local conservation law of $M_{\text{inv}}^{\mu\rho\sigma}$ reflects the on-shell symmetry of the EMT $T_{\text{inv}}^{\mu\nu}$ since $\partial_\mu M_{\text{inv}}^{\mu\rho\sigma} \approx T_{\text{inv}}^{\rho\sigma} - T_{\text{inv}}^{\sigma\rho}$. As a matter of fact, the EMT $T_{\text{inv}}^{\mu\nu}$ given by (A.11) is symmetric off-shell. For the *scale transformations*, i.e. for $\xi_\mu = \rho x_\mu$, the current $j^\mu = T_{\text{inv}}^{\mu\nu} \xi_\nu = \rho T_{\text{inv}}^{\mu\nu} x_\nu$ represents the *dilatation current* whose local conservation law expresses the (on-shell) tracelessness of the EMT $T_{\text{inv}}^{\mu\nu}$ in four space-time dimensions:

$$0 \approx \partial_\mu j^\mu \approx \rho T_{\text{inv}\mu}^\mu \quad \text{for } n = 4.$$

The EMT $T_{\text{inv}}^{\mu\nu}$ given by (A.11) is actually traceless off-shell for $n = 4$.

To conclude, we note that a completely analogous covariantization procedure can be applied to minimally coupled matter fields φ : one writes

$$\delta_{\xi}\varphi \equiv \xi^{\nu}\partial_{\nu}\varphi = \xi^{\nu}D_{\nu}\varphi - iq(\xi \cdot A)\varphi,$$

where the last term again describes an infinitesimal (field dependent) gauge transformation.

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