Symmetries and mode stability of gravitational instantons

Gustav Nilsson

Univ.-Diss.
zur Erlangung des akademischen Grades
„doctor rerum naturalium“ (Dr. rer. nat.)
in der Wissenschaftsdisziplin Mathematik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät
Institut für Mathematik
der Universität Potsdam

Ort und Tag der Disputation: Potsdam, der 13. Februar 2025

Supervisors:	Prof. Dr. Lars Andersson
	Prof. Dr. Mattias Dahl
Reviewers:	Prof. Dr. Lars Andersson
	Prof. Dr. Olivier Biquard
	Prof. Dr. Hans-Joachim Hein
Mentor:	Prof. Dr. Jan Metzger

Published online on the
Publication Server of the University of Potsdam:
https://doi.org/10.25932/publishup-67859
https://nbn-resolving.org/urn:nbn:de:kobv:517-opus4-678594

Abstract

We study two aspects of gravitational instantons: mode stability of gravitational instantons, and T^2-symmetric (toric) gravitational instantons.

We study Ricci-flat perturbations of gravitational instantons of Petrov type D. Analogously to the Lorentzian case, the Weyl curvature scalars of extreme spin weight satisfy a Riemannian version of the separable Teukolsky equation. As a step toward infinitesimal rigidity of the type D Kerr and Taub-bolt families of instantons, we prove mode stability, that is, that the Teukolsky equation admits no solutions compatible with regularity and asymptotic (local) flatness.

For an asymptotically locally Euclidean (ALE) or ALF gravitational instanton (M, g) with toric symmetry, we express the signature of (M, g) directly in terms of its rod structure. Applying Hitchin–Thorpe-type inequalities for Ricci-flat ALE/ALF manifolds, we formulate, as a step toward a classification of toric ALE/ALF instantons, necessary conditions that the rod structures of such spaces must satisfy. Finally, we apply these results to the study of rod structures with three turning points.

Zusammenfassung

Wir untersuchen zwei Aspekte von Gravitationsinstantonen: die Modenstabilität von Gravitationsinstantonen sowie T^2-symmetrische (torische) Gravitationsinstantonen.

Wir betrachten Ricci-flache Störungen von Gravitationsinstantonen vom Petrov-Typ D. Analog zum lorentzschen Fall erfüllen die Weyl-Krümmungsskalare extremen Spin-Gewichts eine riemannsche Version der separierbaren Teukolsky-Gleichung. Als Schritt in Richtung infinitesimaler Starrheit der Typ-D-Familien der Kerr- und Taub-Bolt-Instantonen zeigen wir die Modenstabilität, das heißt, dass die Teukolsky-Gleichung keine Lösungen zulässt, die sowohl Regularität als auch asymptotische (lokale) Flachheit erfüllen.

Für einen asymptotisch lokal euklidischen (ALE) oder ALF-Gravitationsinstanton (M, g) mit torischer Symmetrie drücken wir die Signatur von (M, g) direkt in Abhängigkeit von seiner Rod-Struktur aus. Als Schritt in Richtung einer Klassifikation torischer ALE-/ALF-Instantonen formulieren wir unter Anwendung von Hitchin–Thorpe-artigen Ungleichungen notwendige Bedingungen an die Rod-Strukturen von Ricci-flachen ALE-/ALF-Mannigfaltigkeiten. Abschließend wenden wir diese Ergebnisse auf die Untersuchung von Rod-Strukturen mit drei Wendepunkten an.

Acknowledgments

First, I would like to thank my supervisor Lars Andersson for guidance, inspiration, patience, and support in many ways throughout my doctoral studies. We have had many invaluable discussions, and Lars has introduced me to numerous interesting areas of mathematics. I would also like to give a special thanks to my co-supervisor Mattias Dahl for interesting discussions and support. Thanks should be given to Steffen Aksteiner for assistance with computations using the computer algebra package xAct for Mathematica™, and for helpful guidance in understanding the Newman–Penrose formalism. For the latter, I would also like to give special thanks to Bernardo Araneda. I would also like to thank Oliver Petersen, Klaus Kröncke, Eric Ling, Walter Simon, and Hans-Joachim Hein for helpful comments and discussions. This research was supported by the IMPRS for Mathematical and Physical Aspects of Gravitation, Cosmology and Quantum Field Theory.

This thesis is based on the following papers:

[33] Gustav Nilsson. „Mode stability for gravitational instantons of type D“. in: *Classical and Quantum Gravity* 41.8 (Mar. 2024), p. 085004. DOI: 10.1088/1361-6382/ad296f. URL: https://dx.doi.org/10.1088/1361-6382/ad296f.

[34] Gustav Nilsson. „Topology of toric gravitational instantons“. In: *Differential Geometry and its Applications* 96 (2024), p. 102171. ISSN: 0926-2245. DOI: 10.1016/j.difgeo.2024.102171. URL: https://www.sciencedirect.com/science/article/pii/S0926224524000640.

[2] Steffen Aksteiner et al. *Gravitational instantons with S^1 symmetry*. 2023. arXiv: 2306.14567 [math.DG]. URL: https://arxiv.org/abs/2306.14567. Submitted to *Journal für die reine und angewandte Mathematik*

Contents

Chapter 1

Introduction

A gravitational instanton[1] is a complete and non-compact Ricci-flat Riemannian four-manifold with quadratic curvature decay.[2] Gravitational instantons are the vacuum solutions to the Riemannian analog of the Einstein equations. They were originally studied in the context of quantum gravity, as an analogue of the instantons of Yang–Mills theory. [23]

The assumption of quadratic curvature decay implies that the volume growth of a gravitational instanton must be either quartic (ALE), cubic (AF/ALF), quadratic (ALG), or linear (ALH). An ALE instanton has a metric that is asymptotic to $\mathbb{R}^4/\Gamma$, where Γ is a finite subgroup of SO(4) which acts freely on S^3. On the other hand, an ALF instanton is asymptotic to a circle bundle over either $\mathbb{R}^3$ or $\mathbb{R}^3/\mathbb{Z}_2$, and is then said to be either ALF-A_k or ALF-D_k, respectively, here, k is an integer defined in terms of the topology of the circle bundle. A notable special case is ALF-A_{-1}, in which the circle bundle is trivial, and this case is often denoted by AF. This thesis primarily explores ALE and ALF-A_k/AF instantons.

Apart from the trivial example given by $\mathbb{R}^4$ with the flat metric, an example of an ALE instanton is the hyper-Kähler Eguchi–Hanson instanton, whose topology is the total space of the cotangent bundle of the two-sphere. Some notable examples of ALF-A_k instantons are the hyper-Kähler Taub–NUT instanton with topology $\mathbb{R}^4$, the (non-hyper-Kähler) Taub-bolt instanton with topology $\mathbb{C}P^2 \setminus \{*\}$, the Riemannian Kerr instanton with topology $\mathbb{R}^2 \times S^2$, and the Chen–Teo instanton with topology $\mathbb{C}^2 \setminus S^1$, of which the latter two are AF. A trivial example of an AF instanton is also given by the standard (flat) metric on $\mathbb{R}^3 \times S^1$. These examples are discussed in further detail in Section 2.5

Apart from these examples, there are some general results about gravitational instantons, many of which hold under various symmetry assumptions such as the existence of a $U(1)$ or $U(1) \times U(1)$ isometry group, see e.g. [2, 34, 8, 1]. A well-known conjecture in the study of compact Ricci-flat manifolds is the Besse conjecture [7], which states that all compact Ricci-flat manifolds have special holonomy. This is a wide-open conjecture; there are no known examples of compact Ricci-flat four-manifolds with generic holonomy, that is, holonomy group $SO(4)$. In contrast, the analogue statement for the non-compact case is false, since there are known examples of gravitational instantons with generic holonomy. However, all currently known examples of gravitational instantons are Hermitian, which is a slightly weaker condition than that of having special holonomy. Therefore, it is natural to conjecture that all gravitational instantons are Hermitian, see [2] where this conjecture is stated for the ALF case. A first step toward such a

[1] We shall sometimes refer to gravitational instantons as simply *instantons*, when the context is clear.

[2] Some authors include further requirements in the definition, of which one of the most common is to require a gravitational instanton to be hyper-Kähler. We do not assume the latter a priori.

result is given by proving *rigidity*, that is, for various known examples of gravitational instantons, showing that there are no other Ricci-flat metrics close to that metric.

In the recent paper [9], it was shown that rigidity holds for the Riemannian Kerr and Taub-bolt families. However, *infinitesimal rigidity*, that is, that Ricci-flat linear perturbations of these instantons decaying sufficiently fast at infinity must be perturbations within the respective family, is still open. Due to the fact that rigidity holds and that the relevant families are smooth manifolds, infinitesimal rigidity is equivalent to *integrability*, that is, that any such Ricci-flat linear perturbation integrates to a curve of Ricci-flat metrics.

It was shown in [42], that for perturbations of the Lorentzian Kerr metric whose frequency lies in the upper half plane, and satisfying certain boundary conditions, the perturbations of the Weyl scalars of extreme spin weight vanish identically. This result is known as *mode stability*. Furthermore, mode stability for frequencies on the real axis was shown in [5]. It was shown in [41], see also [4], that perturbations of the Lorentzian Kerr metric whose Weyl scalars of extreme spin weight vanish identically must be perturbations within the Kerr family, modulo gauge.

As we show in Chapter 3, the Riemannian analog of mode stability holds in the ALF type D case.[3]. The only ALF instantons of type D are the Riemannian Kerr and the Taub-bolt metrics, and these results suggest that infinitesimal rigidity in the previously described sense holds for these instantons.

The examples mentioned so far are all *toric*, that is, admit an effective action of the torus $T^2 = U(1) \times U(1)$ by isometries. Toric gravitational instantons were studied in the paper [21]. In that paper, the author introduced the formalism of so-called *rod structures* (which we define in detail in Section 2.4.1), which are combinatorial objects associated with toric gravitational instantons, containing information about the T^2-action. The rod structure formalism was later used in the papers [25, 14].

Given a toric gravitational instanton (M, g), the rod structure consists of a sequence of 2-vectors $(v_0, \ldots, v_n)$ with integer entries, where the integer n is called the *number of turning points* of the rod structure. The definition of the rod structure is given only in terms of the underlying topology and its interactions with the T^2-action, and does not involve the metric g.

Much effort has been devoted to constructing new examples of toric gravitational instantons. One example of such a method is the *soliton method* [6, 12], which was used to construct the Chen–Teo instanton. Another method is given in the more recent paper [28]. Despite much progress, it is in general a difficult problem to determine whether a rod structure can be realized as the rod structure of a toric gravitational instanton.

The rod structure of M can be shown to fully determine its topology. In particular, the Euler characteristic is well-known to equal the number of turning points. In Chapter 4 we determine the signature of M in terms of the rod structure. We do so by directly reconstructing M as a manifold with a T^2-action from its rod structure. In terms of this explicit reconstruction, we determine a basis for $H_2(M)$,[4] and then compute the intersection form in terms of this basis. As a result, we obtain Theorem 5. This theorem is then used together with Hitchin–Thorpe-type inequalities for ALE or ALF instantons to give necessary conditions on a rod structure that occurs as the rod structure of an actual ALE or ALF gravitational instanton. The necessary conditions that we obtain as a result are inequalities where the quantities involved can be described solely in terms of the rod structure, and as a consequence lead to restrictions on the possible rod structures.

The particular case of rod structures with three turning points was discussed in [14], where the question was raised whether topological or other constraints could perhaps rule out certain rod structures. We address this question with Theorem 9 and Theorem 4.1, which provide some restrictions on the possible rod structures in this case.

[3]For the definition of type D, see Section 2.3

[4]Here $H_k(M)$ refers to the k:th **singular** homology group (with **integral** coefficients).

Another class of interest is that of S^1-symmetric gravitational instantons. Although at first sight, the requirement of S^1-symmetry appears less restrictive than that of T^2-symmetry, there exist definitions of the former which do not subsume latter. Such a definition is given in the paper [2], where an S^1 *instanton of ALF type* is defined as an ALF instanton admitting an effective action of $S^1 = U(1)$ by isometries, along with the further requirement that the Killing field generating this action is bounded. Under this definition, the Kerr family of ALF instantons is a family for which all members of the family are toric, but for which only a small subset of its members are ALF S^1 instantons. In [2], uniqueness results are given, showing that every S^1 instanton of ALF-A_k type with the same topology as the Kerr family must in fact be a member of the Kerr family, and an analogous result is also provided for the Taub-bolt family. Although the paper does not show the analogous result for the Chen–Teo family of instantons, another main result of [2] is that an S^1-instanton of ALF-A_k type with the same topology as that of the Chen–Teo family must belong to the family of Hermitian metrics, of which the Chen–Teo family is a subfamily.

By definition, an S^1 instanton of type ALF-A_k is an ALF-A_k instanton admitting a periodic Killing field of bounded norm. It is known that ALF-A_k instantons which are Kähler must belong to the multi-Taub–NUT family. For the remaining case of non-Kähler ALF-A_k instantons, the paper [2] conjectures that if the periodicity requirement in the definition of ALF S^1 instanton is replaced by the requirement of being non-Kähler and Hermitian, then any such instanton belongs to one of the Kerr, Chen–Teo, Taub-bolt and Taub–NUT families, where in the latter case, the orientation is the opposite orientation of that for which the instanton is Kähler. A much more general conjecture is also given, stating that the former conjecture is still true after removing the requirement of S^1-symmetry. The main results of [2] imply in particular that a specific subclass of the ALF instantons are all Hermitian, and it is conjectured in [2] that this extends to the class of all ALF instantons.

The proofs in [2] rely on the G-signature theorem, along with a divergence identity related to the Israel–Robinson identities used in the proofs of black hole uniqueness.

Chapter 2

Background

2.1 Gravitational instantons

In this section, we introduce the concept of a gravitational instanton, which is the main object of study of this thesis. We then introduce the various possibilities for the asymptotic behavior of gravitational instantons.

Definition 1. A *gravitational instanton* is a Riemannian 4-manifold (M, g) which is non-compact, simply connected, complete, and Ricci-flat, satisfying the following.

1. There exists a non-increasing function $K : [0, \infty) \to [0, \infty)$ such that

$$\int_1^\infty \frac{K(s)}{s}\, ds < \infty, \tag{2.1}$$

   along with a point $p \in M$ such that

$$|\mathrm{Rm}_q|_g \leq \frac{K(d(p,q))}{d(p,q)^2} \tag{2.2}$$

   for all $q \in M$, where $d(\cdot,\cdot)$ is the Riemannian distance function.

2. There exists a constant $\kappa > 0$, a function $\epsilon : (0, \infty) \to (0, \infty)$ satisfying $\lim_{r\to 0} \epsilon(r) = 0$, along with a compact set $K \subseteq M$ such that for any $r > 0$ and for any geodesic loop γ whose length is less than κr, and which is based at a point in $M \setminus K$, the holonomy of γ rotates any vector by at most the angle $\epsilon(r)$.

Remark 1. We do not require (M, g) to be hyper-Kähler (see Definition 4), an assumption which is sometimes included in the definition of gravitational instanton.

From the conditions 1 and 2 in Definition 1, it follows (see [11]) that (M, g) has must have one out of several special geometries near infinity. The different cases are divided into ALE, ALF, ALG, and ALH depending on the volume growth, and the possible subcases [11] are shown in Table 2.1. Here, we say that the *boundary at infinity* of M is N if there exists a compact set $K \subseteq M$ such that $\overline{M \setminus K}$ is homeomorphic to $[0, \infty) \times N$, such that the homeomorphism can be chosen to map ∂K onto $\{0\} \times N$. Although the methods used in the proof of the following lemma are standard, we write down a detailed proof here since the details will be needed to obtain a slightly stronger version of the result in later sections.

Table 2.1: Possible asymptotic geometries of gravitational instantons.

	Boundary at infinity
ALE[1]	S^3/Γ
ALF-A_{-1}[2]	$S^2 \times S^1$
ALF-A_k, $k \neq -1$	$L(\lvert k+1\rvert, -\operatorname{sgn}(k+1))$
ALF-D_2[3]	$(S^2 \times S^1)/\mathbb{Z}_2$
ALF-D_k, $k \neq 2$	$S^3/D_{4\lvert k-2\rvert}$
ALG	T^2-bundle over S^1
ALH-splitting	$\{\pm 1\} \times T^3$
ALH-non-splitting	T^3

Lemma 1. *The boundary at infinity is well-defined up to h-cobordism. In other words, if N_1 and N_2 are closed 3-manifolds, and if they are both the boundary at infinity of M, then there is an h-cobordism between N_1 and N_2.*

Proof. For $i = 1, 2$, let $K_i \subseteq M$ be compact and let $\Phi_i : \overline{M \setminus K_i} \to [0, \infty) \times N_i$ be a homeomorphism such that $\Phi_i(\partial K_i) = \{0\} \times N_i$. Without loss of generality, assume that $K_1 \subseteq K_2$ (otherwise, replace K_2 by $K_2 \cup \Phi_2^{-1}([0, R] \times N_2)$, for large enough R).

Now pick R_1 such that $\overline{K_2 \setminus K_1} \subseteq \Phi_1^{-1}([0, R_1])$, and then pick R_2 such that $\Phi_1^{-1}([0, R_1] \times N_1) \subseteq \Phi_2^{-1}([0, R_2] \times N_2)$. Then $W := \overline{\Phi_2^{-1}([0, R_2]) \setminus K_1}$ is a 4-manifold with boundary. The boundary ∂W is the disjoint union of $\Phi_1^{-1}(\{0\} \times N_1) \cong N_1$ and $\Phi_2^{-1}(\{R_2\} \times N_2) \cong N_2$, so that W is a cobordism between N_1 and N_2.

Using the fact that $[0, R_2] \times N_2$ deformation retracts onto $\{0\} \times N_2$, we can use the map Φ_2 to define a deformation retraction from W to $\overline{K_2 \setminus K_1}$, and similarly, the fact that $[0, R_1] \times N_1$ deformation retracts onto $\{0\} \times N_1$, implies that $\Phi_1^{-1}([0, R_1] \times N_1)$ deformation retracts onto $\Phi_1^{-1}(\{0\} \times N_1)$. Since $\overline{K_2 \setminus K_1} \subseteq \Phi_1^{-1}([0, R_1] \times N_1)$, this implies that W deformation retracts onto $\Phi_1^{-1}(\{0\} \times N_1)$. In a similar fashion, one sees that W deformation retracts onto $\Phi_2^{-1}(\{R_2\} \times N_2)$, from which it follows that W is a h-cobordism. □

2.2 NP formalism

The Newman–Penrose (NP) formalism [32], commonly used in general relativity, can be adapted to a Riemannian signature (see [3, 19]). Let $(l, \bar{l}, m, \overline{m})$ be a tetrad of vector fields with complex coefficients[4] in which the metric takes the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.3}$$

When viewed as first order differential operators, we denote the vector fields $l, \bar{l}, m, \overline{m}$ by $D, \Delta, \delta, -\tilde{\delta}$, respectively. With respect to the tetrad $(l, \bar{l}, m, \overline{m})$, the Levi-Civita connection

[1]Here, Γ is any finite subgroup of SO(4) which acts freely on S^3.
[2]ALF-A_{-1} is also called AF.
[3]Here, the action of $\mathbb{Z}_2$ on $S^2 \times S^1$ is given by $(x, y, z, \theta) \mapsto (-x, -y, -z, -\theta)$.
[4]Formally expressed, this means that they are sections of the complexified tangent bundle of M.

is represented by 24 *spin coefficients*, denoted by Greek letters and defined to be the coefficients in the right-hand sides of the equations

$$\frac{1}{2}(\bar{l}^a\nabla_b l_a - m^a\nabla_b\overline{m}_a) = \gamma l_b + \epsilon\bar{l}_b - \alpha m_b + \beta\overline{m}_b, \tag{2.4}$$

$$\bar{l}^a\nabla_b\overline{m}_a = -\nu l_b - \pi\bar{l}_b + \lambda m_b - \mu\overline{m}_b, \tag{2.5}$$

$$m^a\nabla_b l_a = \tau l_b + \kappa\bar{l}_b - \rho m_b + \sigma\overline{m}_b, \tag{2.6}$$

$$\frac{1}{2}(\bar{l}^a\nabla_b l_a + \overline{m}^a\nabla_b m_a) = \tilde{\gamma} l_b + \tilde{\epsilon}\bar{l}_b - \tilde{\beta} m_b + \tilde{\alpha}\overline{m}_b, \tag{2.7}$$

$$\bar{l}^a\nabla_b m_a = \tilde{\nu} l_b + \tilde{\pi}\bar{l}_b - \tilde{\mu} m_b + \tilde{\lambda}\overline{m}_b, \tag{2.8}$$

$$\overline{m}^a\nabla_b l_a = -\tilde{\tau} l_b - \tilde{\kappa}\bar{l}_b + \tilde{\sigma} m_b - \tilde{\rho}\overline{m}_b. \tag{2.9}$$

From the fact that all inner products of the tetrad vectors are constant, it can also be seen that

$$\overline{\alpha} = \beta, \qquad \overline{\gamma} = -\epsilon, \qquad \overline{\kappa} = \nu, \qquad \overline{\lambda} = -\sigma, \qquad \overline{\mu} = -\rho, \qquad \overline{\pi} = \tau, \tag{2.10}$$

$$\overline{\tilde{\alpha}} = \tilde{\beta}, \qquad \overline{\tilde{\gamma}} = -\tilde{\epsilon}, \qquad \overline{\tilde{\kappa}} = \tilde{\nu}, \qquad \overline{\tilde{\lambda}} = -\tilde{\sigma}, \qquad \overline{\tilde{\mu}} = -\tilde{\rho}, \qquad \overline{\tilde{\pi}} = \tilde{\tau}. \tag{2.11}$$

The Levi-Civita connection is thus represented by $6+6$ independent complex scalars.

We also have the Weyl scalars:

$$\begin{aligned} &\Psi_0 = -W(l,m,l,m), \quad \Psi_1 = -W(l,\bar{l},l,m), \quad \Psi_2 = W(l,m,\bar{l},\overline{m}),\\ &\Psi_3 = W(l,\bar{l},\bar{l},\overline{m}), \quad \Psi_4 = -W(\bar{l},\overline{m},\bar{l},\overline{m}), \end{aligned} \tag{2.12}$$

$$\begin{aligned} &\tilde{\Psi}_0 = -W(l,\overline{m},l,\overline{m}), \quad \tilde{\Psi}_1 = -W(l,\bar{l},l,\overline{m}), \quad \tilde{\Psi}_2 = W(l,\overline{m},\bar{l},m),\\ &\tilde{\Psi}_3 = W(l,\bar{l},\bar{l},m), \quad \tilde{\Psi}_4 = -W(\bar{l},m,\bar{l},m), \end{aligned} \tag{2.13}$$

where W denotes the Weyl curvature tensor.

From the definitions, one sees immediately that $\overline{\Psi}_k = \Psi_{4-k}$ and $\overline{\tilde{\Psi}}_k = \tilde{\Psi}_{4-k}$, so that the Weyl tensor is determined by the $2+2$ complex scalars $\Psi_0, \Psi_1, \tilde{\Psi}_0, \tilde{\Psi}_1$ and the $1+1$ real scalars $\Psi_2, \tilde{\Psi}_2$. The Weyl scalars of *extreme spin weight* are defined to be $\Psi_0, \Psi_4, \tilde{\Psi}_0$ and $\tilde{\Psi}_4$.

Expressing the condition of Ricci-flatness in terms of these quantities, one arrives at 20 independent equations, and the differential Bianchi identity gives rise to an additional 8 independent equations. These are written out fully in Appendix A

A tetrad $(l, \bar{l}, m, \overline{m})$ is said[5] to be *adapted* if $\Psi_0 = \Psi_1 = \tilde{\Psi}_0 = \tilde{\Psi}_1 = 0$ and $\Psi_2, \tilde{\Psi}_2 \neq 0$. It can be shown that a Ricci-flat four-manifold admits an adapted tetrad if and only if it has type D. A proof of this fact in the Lorentzian case can be found in [36, Chapter 7].

2.3 Petrov classification

Let (M,g) be a Ricci-flat Riemannian 4-manifold, and let W be its Weyl tensor. Since W is antisymmetric in the first two indices, as well as the last two indices, it can be viewed as a bilinear form $\Lambda^2(TM) \otimes \Lambda^2(TM) \to \mathbb{R}$, which is symmetric by the symmetry of the Weyl tensor under interchange of the first and second pair of vectors. This can then be identified with a homomorphism $\Lambda^2(T^*M) \to \Lambda^2(TM)$. Using the metric to identify TM with T^*M, the latter can be identified with a endomorphism $\mathcal{W} : \Lambda^2(T^*M) \to \Lambda^2(T^*M)$, commonly referred to as the *Weyl operator*. Evidently, $\mathcal{W}$ is self-adjoint, and it follows directly from the Ricci-flatness of M that $\mathcal{W}$ is trace-free.

[5]The corresponding notion in the Lorentzian setting is that of a *principal* tetrad.

The Hodge star $\star : \Lambda^2(T^*M) \to \Lambda^2(T^*M)$ satisfies $\star^2 = \mathrm{id}$, and it therefore induces a splitting $\Lambda^2(T^*M) = \Lambda^2_+(T^*M) \oplus \Lambda^2_-(T^*M)$, where $\Lambda^2_\pm(T^M)$ is the bundle of 2-forms ω satisfying $\star\omega = \pm\omega$. The bundles $\Lambda^2_+(T^*M)$ and $\Lambda^2_-(T^*M)$ are the bundles of *self-dual* and *anti-self-dual* 2-forms, respectively.

For any 2-form ω, we have $\mathcal{W}(\star\omega) = \star\mathcal{W}(\omega)$, and furthermore, the endomorphism $\omega \mapsto \star\mathcal{W}(\omega)$ is trace-free.[39] In particular, if ω is (anti-)self-dual, then $\mathcal{W}(\omega)$ has the same property, which means that $\mathcal{W}$ restricts to endomorphisms $\mathcal{W}_\pm : \Lambda^2_\pm(T^*M) \to \Lambda^2_\pm(T^*M)$, and these are trace-free.

Now fix a point $p \in M$. Since the endomorphisms $\mathcal{W}^\pm_p : \Lambda^2_\pm(T^*_pM) \to \Lambda^2_\pm(T^*_pM)$ are self-adjoint, the spectral theorem implies that they are diagonalizable, and since the spaces $\Lambda^2_\pm(T^*_pM)$ are 3-dimensional, each of the two endomorphisms $\mathcal{W}^\pm_p$ has three eigenvalues. If these are all distinct, the corresponding Weyl operator is said to be of *type I* at p. If there is an eigenvalue with multiplicity two, the operator is instead said to be of *type D*. In the case of an eigenvalue with multiplicity three, the trace-freeness implies that the operator vanishes identically, and is said to be of *type O*. The endomorphism field $\mathcal{W}^\pm : \Lambda^2_\pm(T^*M) \to \Lambda^2_\pm(T^*M)$ is said to have a certain type if its restriction to every point has that type. If both $\mathcal{W}^+$ and $\mathcal{W}^-$ have the same type, then the Riemannian manifold (M, g) is said to have that type.

The following lemma will be central to the mode stability results in this thesis.

Lemma 2. *If (M, g) is a Ricci-flat Riemannian 4-manifold of type D, then there exists an adapted tetrad.*

Proof. See [20, Corollary 5.8]. □

2.4 T^2 symmetry

In this section, we provide the definition of a toric gravitational instanton, along with the definition of its associated rod structure.

Definition 2. A *toric gravitational instanton* is a simply connected gravitational instanton (M, g) together with an effective action of the torus $T^2 = U(1) \times U(1)$ by isometries. We assume that the fixed point set[6] is finite and non-empty, and that there are no points whose isotropy group is non-trivial and finite.

Remark 2. The requirement that there are no points with non-trivial and finite isotropy group is equivalent to requiring M to be *locally standard*, see [43].

2.4.1 Rod Structure

Lemma 3. *Let (M, g) be a toric gravitational instanton. Then the orbit space $\widehat{M} = M/T^2$ is a 2-dimensional (smooth) manifold with corners (see [29, p. 415]). Furthermore, $\widehat{M}$ is homeomorphic to the closed upper half plane $\{z + i\rho \mid z \in \mathbb{R}, \rho \geq 0\}$, and under this homeomorphism, the region where $\rho > 0$ corresponds to orbits with trivial isotropy group. There exist points $z = z_1, \ldots, z_n$ on the axis $\rho = 0$, corresponding to the fixed points, dividing this axis into segments $z_i < z < z_{i+1}$ (where we take $z_0 = -\infty$ and $z_{n+1} = \infty$). For each such segment, the points in the corresponding orbits all have the same isotropy group. This isotropy group is a circle subgroup of the torus, given as*

$$T^2(v_i^1, v_i^2) := \{(e^{iv_i^1\theta}, e^{iv_i^2\theta}) \mid \theta \in \mathbb{R}\} \tag{2.14}$$

[6]That is, the set of points in M whose isotropy group is the entire group T^2.

for some pair $v_i = (v_i^1, v_i^2)$. The vectors $(v_0, \dots, v_n)$ can be chosen in such a way that each vector is a pair of coprime integers, and such that

$$\det\begin{pmatrix} v_{i-1} & v_i \end{pmatrix} = 1 \tag{2.15}$$

holds for all i.

Proof. Since M has no points with a non-trivial finite isotropy group, the fact that $\widehat{M}$ is a 2-manifold with corners follows from the non-compact analog of [24, Theorem 1]. From the discussion following the theorem, it is also clear that the interior of $\widehat{M}$ corresponds to points with trivial isotropy and that the corners correspond to fixed points. It can also be seen from this discussion that for a boundary segment, the isotropy group is constant on this segment, that it is of the form $T^2(v)$ for some pair v of coprime integers, and that if $T^2(v)$ and $T^2(w)$ are the isotropy groups corresponding to two segments meeting in a corner, then $\det\begin{pmatrix} v & w \end{pmatrix} = \pm 1$. It remains to show that $\widehat{M}$ is homeomorphic to the closed upper half plane, for once this is done, the sequence of vectors representing the isotropy groups associated with the boundary can be chosen to satisfy (2.15), using the fact that $T^2(v) = T^2(-v)$ and successively flipping the signs of the vectors appropriately.

To prove that $\widehat{M}$ is homeomorphic to the closed upper half plane, first note that the orbit space $\widehat{M}$ is the quotient of a non-compact manifold by a compact Lie group and is therefore non-compact. We claim that $\widehat{M}$ is simply connected. To see this, take any loop $\widehat{\gamma}$ in $\widehat{M}$. Since M has a fixed point, we can assume that the basepoint of $\widehat{\gamma}$ is the image of such a point. We can lift $\widehat{\gamma}$ to a curve γ in M, and because the fiber of the basepoint of $\widehat{\gamma}$ consists of a single point, γ must also be a loop. Using the assumption that M is simply connected, we thus see that γ, and therefore $\widehat{\gamma}$ is null-homotopic.

Of the asymptotic geometries in Table 2.1, all except ALH-splitting have only one end. Furthermore, M cannot be ALH-splitting since by [10, Theorem 3.7], M would then be homeomorphic to $\mathbb{R} \times T^3$, which contradicts the assumption that M is simply connected. Thus, M has only one end, and one can see from this that $\widehat{M}$ has only one end. By the classification of simply connected surfaces with boundary, $\widehat{M}$ is homeomorphic to the closed upper half plane. □

Definition 3. The sequence $(v_0, \dots, v_n)$, where the vectors $v_i = (v_i^1, v_i^2)$ are as in Lemma 3, is called the *rod structure* of M.

Remark 3. Some papers (e.g., [28]) define the rod structure also to include the lengths $z_i - z_{i-1}$ of the boundary segments. Although these lengths are not well-defined in the context of Lemma 3, since the choice of coordinates is not canonical, they are well-defined if one requires that the homeomorphism from $\widehat{M}$ to the closed upper half plane be given by a canonical set of coordinates, the *Weyl–Papapetrou coordinates.*

Lemma 4. *Let (M, g) be a toric gravitational instanton with rod structure $(v_0, \dots, v_n)$. Then the boundary at infinity of M is the lens space[7] $L(p, q)$, where $p := |\det\begin{pmatrix} v_0 & v_n \end{pmatrix}|$ and $q :=$ $\operatorname{sgn}(\det\begin{pmatrix} v_0 & v_n \end{pmatrix}) \cdot \det\begin{pmatrix} v_1 & v_n \end{pmatrix}$.*

Proof. Fix an identification of the orbit space $\widehat{M}$ with the closed upper half plane $\{z + i\rho \mid \rho \geq 0\}$ as in Lemma 3; such an identification can be chosen to be a diffeomorphism away from the corners. With $v_0, \dots, v_n$ and $z_1, \dots, z_n$ as in Lemma 3, set $R = \max(|z_1|, \dots, |z_n|)$, and consider the subset $\widehat{A} = \{z + i\rho \mid \rho \geq 0, z^2 + \rho^2 > R^2\} \subseteq \widehat{M}$.

Without loss of generality, we may assume that $v_0 = (0, 1)$ and $v_1 = (-1, 0)$; otherwise, we can modify the T^2-action on M by composing it with an appropriate isomorphism $T^2 \to T^2$.

[7] Here, we adopt the convention that $L(0, 1) = S^2 \times S^1$.

Then $v_n = \pm(p, -q)$, so that the isotropy groups associated with the boundary segments of $\widehat{A}$ are $T^2(0,1)$ and $T^2(p,-q)$.

In the case where $v_0 = v_n$, that is, $p = 0$, the set $\widehat{A}$ can be realized as the orbit space of $(R, \infty) \times S^2 \times S^1$. Here, the latter is endowed with the T^2-action given by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (r, t, z_1, z_2) = (r, t, e^{i(\theta_2} z_1, e^{i\theta_1} z_2), \tag{2.16}$$

where we are identifying S^2 as a subset of $\mathbb{R} \times \mathbb{C}$. By [43, Theorem 1.1], this means that there exists a diffeomorphism $\overline{M \setminus K} \to [R+1, \infty) \times S^2 \times S^1$ which maps ∂K onto $\{R+1\} \times S^2 \times S^1$, where K is the preimage of the compact set $\widehat{K} = \{z + i\rho \mid \rho \geq 0, z^2 + \rho^2 \leq (R+1)^2\}$ under the quotient map $M \to \widehat{M}$. Thus, $S^2 \times S^1$ is the boundary at infinity of M.

Now consider the case where $v_0 \neq v_n$, that is, $p \neq 0$. Define a T^2-action on $(R, \infty) \times L(p,q)$ by

$$(e^{i\theta_1}, e^{i\theta_2}) \cdot (r, z_1, z_2) = (r, e^{i\theta_1/p} z_1, e^{-i(\theta_2 + q\theta_1/p)} z_2). \tag{2.17}$$

Then points of the form $(r, z_1, 0)$ have isotropy group $T^2(0,1)$, points of the form $(r, 0, z_2)$ have isotropy group $T^2(p,-q)$, and all other points have trivial isotropy group, and in particular, the orbit space of $(R, \infty) \times L(p,q)$ can be realized as $\widehat{A}$. As in the previous case, the conclusion now follows directly from [43, Theorem 1.1]. $\square$

Theorem 1. *Let (M, g) be a toric gravitational instanton. Then (M, g) is either ALE with a cyclic group Γ, or ALF-A_k.*

Proof. By Lemma 4, we can rule out any asymptotic geometry for which the boundary at infinity is not homotopy equivalent to $S^2 \times S^1$ or to a lens space. Looking at Table 2.1, this immediately rules out ALH-splitting since it is not even path-connected in that case. Using the fact that $\pi_1(L(p,q))$ is cyclic, we can also rule out ALH-non-splitting, along with any boundary of the form S^3/Γ where Γ is not cyclic, since $\pi_1(S^3/\Gamma) = \Gamma$. Thus, M is neither ALF-D_k with $k \neq 2$, ALE with non-cyclic group Γ, nor ALH-non-splitting.

It remains to rule out the cases ALF-D_2 and ALG. For ALF-D_2, note that $(S^2 \times S^1)/\mathbb{Z}_2$ is homeomorphic to the quotient $(S^2 \times \mathbb{R})/G$, where G is the subgroup

$$\left\{ \begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix} \,\middle|\, b = \pm 1, a \in \mathbb{Z} \right\} \subseteq \mathrm{GL}(2, \mathbb{Z}), \tag{2.18}$$

acting on $S^2 \times \mathbb{R}$ by

$$\left(\begin{pmatrix} 1 & a \\ 0 & b \end{pmatrix}, (x_1, x_2, x_3, y) \right) \mapsto (bx_1, bx_2, bx_3, by + a). \tag{2.19}$$

The action of G on $S^2 \times \mathbb{R}$ is easily seen to be a covering space action (see [22, p. 72]), and since $S^2 \times \mathbb{R}$ is simply connected, G is the fundamental group of $(S^2 \times S^1)/\mathbb{Z}_2$. However, G is not cyclic since it is infinite and contains the element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, whose order is 2.

Finally, to rule out ALG, we analyze the fundamental group of a fiber bundle $E \to S^1$ with fiber T^2. By [22, Theorem 4.41, Proposition 4.48], there is an exact sequence of homotopy groups

$$\cdots \longrightarrow \pi_2(S^1) \longrightarrow \pi_1(T^2) \longrightarrow \pi_1(E) \longrightarrow \cdots. \tag{2.20}$$

But the homotopy group $\pi_2(S^1)$ is trivial, since any continuous map $S^2 \to S^1$ lifts to the universal cover $\mathbb{R}$, which is contractible. Since also $\pi_1(T^2) = \mathbb{Z}^2$, this means that there is an injective group homomorphism $\mathbb{Z}^2 \to \pi_1(E)$, which cannot happen if $\pi_1(E)$ is cyclic. $\square$

Remark 4. Consider $M = \mathbb{R}^2 \times S^1 \times S^1$, with a T^2-action where the first circle factor acts by rotating $\mathbb{R}^2$ around the origin, and the second circle factor acts by rotating one of the circle factors. Then the quotient space $\widehat{M} = M/T^2$ can be identified with $S^1 \times [0, \infty) \cong \mathbb{R}^2 \setminus B_1(0)$. Here, the inner circle represents points fixed by the subgroup corresponding to the first factor, while the exterior points correspond to points with a trivial isotropy group. This gives an analog of a rod structure for an ALH space, where there is one rod joined to itself at both ends and no turning points.

2.5 Examples

In this section, we consider some known examples of toric gravitational instantons. Table 2.2 lists their topology and topological invariants, along with their isometry groups, asymptotic geometries, and whether they are hyper-Kähler (see Definition 4) or not. Their rod structures are shown in Figure 2.1. Further details about these instantons can be found in [14, 13, 18].

2.5.1 Euclidean Space

A canonical example is Euclidean space $\mathbb{R}^4$ with the flat metric. Identifying $\mathbb{R}^4$ with $\mathbb{C}^2$ and introducing coordinates (r, θ, ϕ, ψ) by

$$(z_1, z_2) = \left(r \sin\left(\frac{\theta}{2}\right) e^{i\phi}, r \cos\left(\frac{\theta}{2}\right) e^{i\psi} \right), \tag{2.21}$$

where $r > 0$ and $0 < \theta < \pi$, the flat metric takes the form

$$g = dr^2 + \frac{r^2}{4}\, d\theta^2 + r^2 \sin^2\left(\frac{\theta}{2}\right) d\phi^2 + r^2 \cos^2\left(\frac{\theta}{2}\right) d\psi^2 \,. \tag{2.22}$$

The Killing fields

$$\frac{\partial}{\partial \psi} \qquad \text{and} \qquad \frac{\partial}{\partial \phi} \tag{2.23}$$

are 2π-periodic, and thus generate an isometric T^2-action. Under this action, points with $\theta = 0$ and $r > 0$ have isotropy group $T^2(0, 1)$, and points with $\theta = \pi$ and $r > 0$ have isotropy group $T^2(1, 0)$. The point where $r = 0$ (that is, the origin) has isotropy group T^2, that is, is fixed under the whole action, and all other points have a trivial isotropy group.

2.5.2 Riemannian Kerr and Schwarzschild Spaces

Let $M > 0$ and $a \in \mathbb{R}$, introduce the shorthands $\Sigma_{\mathrm{K}} = r^2 - a^2 \cos^2\theta$ and $\Delta_{\mathrm{K}} = r^2 - 2Mr - a^2$, and let $r_\pm := M \pm \sqrt{M^2 + a^2}$. In the *Boyer–Lindquist coordinates*,[8] the *Riemannian Kerr metric* is given by

$$g = \frac{\Sigma_{\mathrm{K}}}{\Delta_{\mathrm{K}}}\, dr^2 + \Sigma_{\mathrm{K}}\, d\theta^2 + \frac{\Delta_{\mathrm{K}}}{\Sigma_{\mathrm{K}}} (d\tau - a \sin^2\theta\, d\phi)^2 + \frac{\sin^2\theta}{\Sigma_{\mathrm{K}}} ((r^2 - a^2)\, d\phi + a\, d\tau)^2, \tag{2.24}$$

where $r > r_+$ and $0 < \theta < \pi$. Like its Lorentzian counterpart, it is a Ricci-flat metric.

[8] These are analogous to the coordinates with the same name used for the Lorentzian Kerr metric in general relativity

Table 2.2: Some known toric gravitational instantons.

	Topology	Euler characteristic	Signature
Euclidean space and Taub–NUT	$\mathbb{R}^4$	1	0
Kerr and Schwarzschild	$\mathbb{R}^2 \times S^2$	2	0
Taub-bolt	$\mathbb{C}P^2 \setminus \{*\}$	2	1
Eguchi–Hanson	T^*S^2	2	1
Chen–Teo	$\mathbb{C}P^2 \setminus S^1$	3	1

	Isometry group	Asymptotic geometry	Hyper-Kähler?
Euclidean space	$\mathbb{R}^4 \rtimes \mathrm{O}(4)$	ALE with trivial group	Yes
Schwarzschild	$\mathrm{O}(2) \times \mathrm{O}(3)$	ALF-A_{-1}	No
Kerr	$\mathrm{O}(2) \times \mathrm{O}(2)$	ALF-A_{-1}	No
Taub–NUT	$(\mathrm{U}(1) \times \mathrm{SU}(2))/\mathbb{Z}_2$	ALF-A_0	Yes
Taub-bolt	$(\mathrm{U}(1) \times \mathrm{SU}(2))/\mathbb{Z}_2$	ALF-A_0	No
Eguchi–Hanson	$(\mathrm{U}(1) \times \mathrm{SU}(2))/\mathbb{Z}_2$	ALE with group $\mathbb{Z}_2$	Yes
Chen–Teo	T^2	ALF-A_{-1}	No

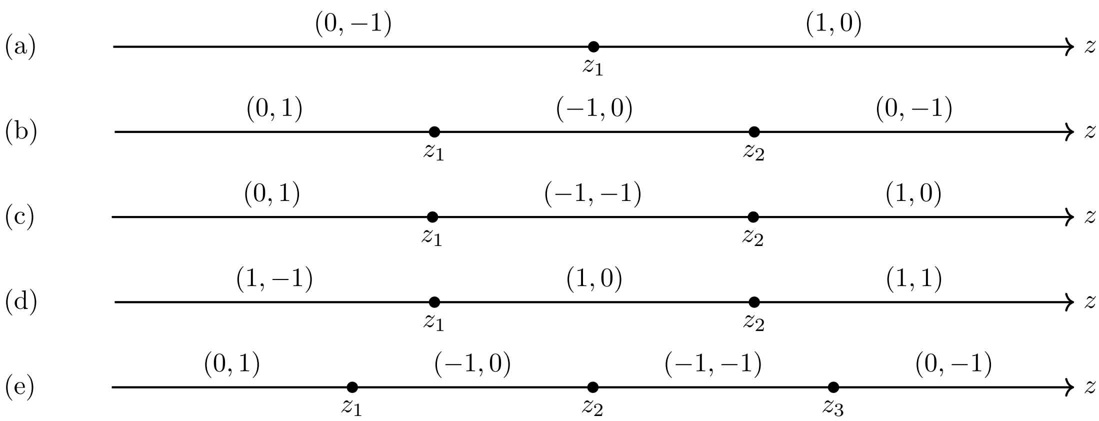


Figure 2.1: Rod structures of (a) $\mathbb{R}^4$ and Taub–NUT; (b) Schwarzschild and Kerr; (c) Taub-bolt; (d) Eguchi–Hanson; (e) Chen–Teo.

Defining new coordinates $(\tilde{t}, \tilde{r}, \theta, \tilde{\phi})$ by

$$\begin{cases} r &= M + \sqrt{M^2 + a^2}\cosh \tilde{r}, \\ t &= \frac{1}{\kappa}\tilde{t}, \\ \phi &= \tilde{\phi} - \frac{\Omega}{\kappa}\tilde{t}, \end{cases} \tag{2.25}$$

where $\kappa = \frac{\sqrt{M^2+a^2}}{2Mr_+}$ and $\Omega = \frac{a}{2Mr_+}$, we see that r is a smooth function of $\tilde{r}^2$, and (2.24) gives

$$g = \Sigma_{\mathrm{K}}(d\tilde{r}^2 + d\theta^2 + (\tilde{r}^2 + O(\tilde{r}^4))\, d\tilde{t}^2 + (\sin^2\theta + O(\sin^4\theta))\, d\tilde{\phi}^2). \tag{2.26}$$

Letting $(\tilde{r}, \tilde{t})$ be polar coordinates on $\mathbb{R}^2$ and letting $(\theta, \tilde{\phi})$ be spherical coordinates on S^2, it follows that g can be made regular at the coordinate singularity $r = r_+$, provided that we identify $\tilde{t}$ and $\tilde{\phi}$ with period 2π independently.[9] Consequently, g is a complete Ricci-flat metric on $\mathbb{R}^2 \times S^2$.

The Killing fields

$$\frac{2Mr_+}{\sqrt{M^2+a^2}}\frac{\partial}{\partial\tau} - \frac{a}{\sqrt{M^2+a^2}}\frac{\partial}{\partial\phi} \qquad \text{and} \qquad \frac{\partial}{\partial\phi} \tag{2.27}$$

are then 2π-periodic, and correspond to rotation of the $\mathbb{R}^2$ factor around its origin, and rotation of the S^2 factor around its polar axis, respectively. Together, they generate an isometric T^2-action. Under this action, points in the set $(\mathbb{R}^2 \setminus \{0\}) \times \{N, S\}$ have isotropy group $T^2(0,1)$, where N and S are the north and south pole of S^2, respectively. Points in the set $\{0\} \times (S^2 \setminus \{N, S\})$ have isotropy group $T^2(1,0)$, and the points $\{0\} \times \{N\}$ and $\{0\} \times \{S\}$ are fixed by all of T^2. All other points have a trivial isotropy group.

In the special case $a = 0$, the metric is often referred to as the *Riemannian Schwarzschild metric*, and in this case it reduces to

$$g = \left(1 - \frac{2M}{r}\right) d\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.28}$$

2.5.3 Taub–NUT and Taub-bolt spaces

Let $M, N > 0$, and define $\Delta_{\mathrm{T}} = r^2 - 2Mr + N^2$, $\Sigma_{\mathrm{T}} = r^2 - N^2$ and $r_\pm = M \pm \sqrt{M^2 - N^2}$. The general *Taub–NUT* family of metrics is given by

$$g = \frac{\Sigma_{\mathrm{T}}}{\Delta_{\mathrm{T}}}\, dr^2 + 4N^2\frac{\Delta_{\mathrm{T}}}{\Sigma_{\mathrm{T}}}(d\psi + \cos\theta\, d\phi)^2 + \Sigma_{\mathrm{T}} a(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{2.29}$$

When $M = N$, we obtain the so-called *self-dual Taub–NUT metric*. In this case, we can introduce new coordinates by

$$\begin{cases} r &= \tilde{r}^2 + N, \\ \psi &= \tilde{\psi} + \tilde{\phi}, \\ \phi &= \tilde{\psi} - \tilde{\phi}, \end{cases} \tag{2.30}$$

in which the metric takes the form

$$\begin{aligned} g = 4(\tilde{r}^2 + 2N)\bigg(d\tilde{r}^2 &+ \frac{\tilde{r}^2}{4}\, d\theta^2 + \tilde{r}^2 \sin^2\left(\frac{\theta}{2}\right) d\tilde{\phi}^2 + \tilde{r}^2\cos^2\left(\frac{\theta}{2}\right) d\tilde{\psi}^2\bigg) \\ &- \frac{4(\tilde{r}^2 + 4N)}{\tilde{r}^2 + 2N}\left(\tilde{r}^2\sin^2\left(\frac{\theta}{2}\right) d\tilde{\phi} + \tilde{r}^2\cos^2\left(\frac{\theta}{2}\right) d\tilde{\psi}\right)^2. \end{aligned} \tag{2.31}$$

[9] This is equivalent to performing the identifications $(t, \phi) \sim (t + \frac{2\pi}{\kappa}, \phi - \frac{2\pi\Omega}{\kappa}) \sim (t, \phi + 2\pi)$.

A comparison with the coordinate expressions in Section 2.5.1 shows that g can be made regular at the coordinate singularities $r = 0$, $\theta = 0$ and $\theta = \pi$ by independently giving each of the coordinates ψ and ϕ the period 2π. The metric g is thus a complete Ricci-flat metric on $\mathbb{R}^4$. The 2π-periodic Killing fields

$$\frac{\partial}{\partial\tilde{\psi}} = \frac{\partial}{\partial\psi} + \frac{\partial}{\partial\phi} \qquad \text{and} \qquad \frac{\partial}{\partial\tilde{\phi}} = \frac{\partial}{\partial\psi} - \frac{\partial}{\partial\phi} \tag{2.32}$$

generate an isometric T^2-action, and when described in terms of the coordinates r and θ, the sets of points with specific isotropy groups are the same as those for Euclidean space with the T^2-action defined in Section 2.5.1.

Another metric of interest, the *Taub-bolt metric*, arises by letting $M = \frac{5}{4}N$, and we shall now briefly account for the regularity of this metric. Introducing the coordinate system $(\tilde{t}, \tilde{r}, \tilde{\theta}, \phi)$ by

$$\begin{cases} r &= \frac{N}{4}(5 + 3\cosh\tilde{r}), \\ t &= 2\tilde{t} - \phi, \\ \theta &= 2\arctan\left(\frac{\tilde{\theta}}{2}\right), \end{cases} \tag{2.33}$$

we see that r is smooth as a function of $\tilde{r}^2$, and that

$$g = \Sigma(d\tilde{r}^2 + (\tilde{r}^2 + O(\tilde{r}^4))\, d\tilde{t}^2 + (1 + O(\tilde{\theta}^2))\, d\tilde{\theta}^2 + (\tilde{\theta}^2 + O(\tilde{\theta}^4))\, d\phi^2 + O(\tilde{r}^2\tilde{\theta}^2)\, d\tilde{t}d\phi). \tag{2.34}$$

Viewing $(\tilde{\theta}, \phi)$ as polar coordinates on $\mathbb{R}^2 \times \{y\} \subseteq \mathbb{R}^4$, and viewing $(\tilde{r}, \tilde{t})$ as polar coordinates on $\{x\} \times \mathbb{R}^2 \subseteq \mathbb{R}^4$, it follows that g extends to a smooth metric on $\mathbb{R}^4 \cong \mathbb{C}^2$, provided that we identify $\tilde{t}$ and ϕ with period 2π independently. This is equivalent to making the identifications $(t, \phi) \sim (t + 4\pi, \phi) \sim (t + 2\pi, \phi + 2\pi)$.

We can also introduce another coordinate system $(\hat{t}, \tilde{r}, \hat{\theta}, \phi)$ by

$$\begin{cases} t &= 2\hat{t} + \phi, \\ \theta &= 2\operatorname{arccot}\left(\frac{\hat{\theta}}{2}\right), \end{cases} \tag{2.35}$$

so that

$$g = \Sigma(d\tilde{r}^2 + (\tilde{r}^2 + O(\tilde{r}^4))\, d\hat{t}^2 + (1 + O(\hat{\theta}^2))\, d\hat{\theta}^2 + (\hat{\theta}^2 + O(\hat{\theta}^4))\, d\phi^2 + O(\tilde{r}^2\hat{\theta}^2)\, d\hat{t}d\phi). \tag{2.36}$$

In the same way as for the previous coordinate system, this shows that g extends to a smooth metric on another copy of $\mathbb{C}^2$. Note that the identifications made to ensure regularity in the coordinate system $(\tilde{t}, \tilde{r}, \tilde{\theta}, \phi)$ also ensure regularity in the coordinate system $(\hat{t}, \tilde{r}, \hat{\theta}, \phi)$. When defined on the union of these copies of $\mathbb{C}^2$, this metric is complete.

Computing the transition map between the two coordinate systems, we see that they are related by $(\hat{t}, \tilde{r}, \hat{\theta}, \phi) = (\tilde{t} - \phi, \tilde{r}, \frac{4}{\tilde{\theta}}, \phi)$. In other words, the two copies of $\mathbb{C}^2$ are glued together according to the map

$$\begin{aligned} (\mathbb{C} \setminus \{0\}) \times \mathbb{C} &\to (\mathbb{C} \setminus \{0\}) \times \mathbb{C}, \\ (z_1, z_2) &\mapsto \left(\frac{4}{\overline{z_1}}, z_2 \cdot \frac{|z_1|}{z_1}\right). \end{aligned}$$

Topologically, this is the same thing as gluing two such copies along the map $(z_1, z_2) \mapsto (\frac{1}{z_1}, z_2 \cdot \frac{|z_1|}{z_1})$, or equivalently, gluing together two copies of $\overline{D}^2 \times \mathbb{C}$ along the map

$$\begin{aligned} S^1 \times \mathbb{C} &\to S^1 \times \mathbb{C}, \\ (z_1, z_2) &\mapsto \left(\frac{1}{z_1}, \frac{z_2}{z_1}\right). \end{aligned} \tag{2.37}$$

We now claim that the manifold is diffeomorphic to $\mathbb{C}P^2$ with a point removed. To see this, consider two of the projective coordinate charts for $\mathbb{C}P^2$, (U_0, ϕ_0) and (U_1, ϕ_1), where

$$U_i = \{[Z_0 : Z_1 : Z_2] \in \mathbb{C}P^2 \mid Z_i \neq 0\}, \tag{2.38}$$

and

$$\begin{aligned}
\phi_0 : U_0 &\to \mathbb{C}^2, \\
[Z_0 : Z_1 : Z_2] &\mapsto \left(\frac{Z_1}{Z_0}, \frac{Z_2}{Z_0}\right), \\
\phi_1 : U_1 &\to \mathbb{C}^2, \\
[Z_0 : Z_1 : Z_2] &\mapsto \left(\frac{Z_0}{Z_1}, \frac{Z_2}{Z_1}\right).
\end{aligned}$$

Since $U_0 \cup U_1 = \mathbb{C}P^2 \setminus \{[0:0:1]\}$, it follows that the latter is topologically equivalent to two copies of $\mathbb{C}^2$, glued together along the transition map

$$\begin{aligned}
(\mathbb{C} \setminus \{0\}) \times \mathbb{C} &\to (\mathbb{C} \setminus \{0\}) \times \mathbb{C}, \\
(z_1, z_2) &\mapsto \left(\frac{1}{z_1}, \frac{z_2}{z_1}\right).
\end{aligned}$$

Again, topologically this is the same thing as gluing together two copies of $\overline{D}^2 \times \mathbb{C}$ along the map (2.37). This shows that the manifold is *homeomorphic* to $\mathbb{C}P^2$ minus a point. To show that these are *diffeomorphic*, we can replace the closed disk with an open disk of radius slightly larger than 1, gluing the two spaces together along a thin open strip around S^1. The gluing map will then be isotopic to the corresponding transition map in $\mathbb{C}P^2$.

2.5.4 Eguchi–Hanson Space

Let $a > 0$. The *Eguchi–Hanson metric* is given by

$$g = \left(1 - \frac{a^4}{r^4}\right)\frac{r^2}{4}(d\psi + \cos\theta\, d\phi)^2 + \left(1 - \frac{a^4}{r^4}\right)^{-1} dr^2 + \frac{r^2}{4}(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{2.39}$$

where $r > a$ and $0 < \theta < \pi$. Letting $r = (\tilde{r}^2 + a^4)^{1/4}$ and $\psi = \tilde{\psi} + \phi$, we have

$$g = \frac{1}{4r^2}(d\tilde{r}^2 + \tilde{r}^2\, d\tilde{\psi}^2) + \frac{r^2}{4}(d\theta^2 + \sin^2\theta\, d\phi^2) + O(\sin^4\theta)\, d\phi^2 + O(\tilde{r}^2\sin^2\theta)\, d\tilde{\psi}\, d\phi\,. \tag{2.40}$$

This shows that g can be made regular at the coordinate singularities $\theta = 0$ and $r = a$ by independently giving each of the coordinates ψ and ϕ periodicity 2π.

Keeping the coordinate $\tilde{r}$, but introducing a coordinate $\hat{\psi}$ by $\psi = \hat{\psi} - \phi$, (2.40) still holds, but with $\hat{\psi}$ in place of $\tilde{\psi}$. Therefore, g can also be made regular at $\theta = \pi$, with the same periodicities mentioned above. The metric g is a complete Ricci-flat metric on T^*S^2, the cotangent bundle of the 2-sphere (see [16]).

The 2π-periodic Killing fields $\frac{\partial}{\partial\psi}$, and $\frac{\partial}{\partial\phi}$ generate an isometric T^2-action, under which points with $\theta = 0$ and $r > a$ have isotropy group $T^2(-1,1)$, points with $\theta = \pi$ and $r > a$ have isotropy group $T^2(1,1)$, and points with $r = a$ and $0 < \theta < \pi$ have isotropy group $T^2(1,0)$. The points $\theta = 0$, $r = a$ and $\theta = \pi$, $r = a$ are fixed under the full T^2-action, and all other points have a trivial isotropy group.

2.5.5 Chen–Teo Space

For parameters $\varkappa \in (0,\infty)$ and $\lambda \in (-1,1)$, let $\gamma = \sqrt{2-\lambda^2}$ and introduce the quantities

$$\kappa_E = \frac{2(\gamma-\lambda^2)}{\varkappa^2(1-\lambda^2)^2(1+\gamma)^2} \tag{2.41}$$

and

$$\Omega_{1E} = -\frac{2(\gamma-\lambda^2)}{\varkappa^2(\gamma-\lambda)^2(1+\gamma)^2}. \tag{2.42}$$

Defining the one-form

$$\begin{aligned}\Omega = \frac{\varkappa^2(1-\lambda^2)(1+\gamma)(x+\lambda)(y+\lambda)}{2(2+\gamma)(x-y)F(x,y)}\Big(& \\ 2(x+1)(y-1)(\lambda(3+\gamma)(2(x+\lambda)(y+\lambda)+(1+\gamma)(\gamma x-\gamma y-2))+3(1-\lambda^2)(x+y)) & \\ +(1+\lambda)^3(\gamma-\lambda+2)^2(x^2-1)-(1-\lambda)^3(\gamma+\lambda+2)^2(y^2-1)\Big)\, d\phi\,, & \end{aligned} \tag{2.43}$$

along with the functions $G(x) = (1-x^2)(x+\lambda)$,

$$\begin{aligned}H(x,y) = \frac{(\lambda+\gamma)(x+\lambda y)+(\lambda-\gamma)(y-\lambda x)+2xy+2\gamma\lambda^2}{4(2+\gamma)}\Big(& \\ (2+\gamma)(x+\lambda)(y+\lambda)(\lambda(1+\gamma)(x+y)+(\lambda^2-\gamma)(x-y)+4+2\gamma\lambda^2-2xy) & \\ +(1-\lambda^2)((\gamma+\lambda+2)(x+\lambda)(x+\lambda\gamma)+(\gamma-\lambda+2)(y+\lambda)(y+\lambda\gamma))\Big), & \end{aligned} \tag{2.44}$$

and

$$F(x,y) = (x+\lambda)(y+\lambda)((1+\lambda\gamma)^2(xy+\lambda x+\lambda y+1)-2\lambda\gamma(1-\lambda)(x-1)(y-1)-(x^2-1)(y^2-1)), \tag{2.45}$$

we define a metric by

$$g = \frac{F(x,y)}{H(x,y)}(d\psi+\Omega)^2 - \frac{\varkappa^4 H(x,y)}{(x-y)^3}\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} + \frac{4G(x)G(y)}{(x-y)F(x,y)}\, d\phi^2\right), \tag{2.46}$$

where $x < -1$ and $y > 1$. We refer to this metric as the *Chen–Teo metric.*

With the identifications $(\psi,\phi) \sim (\psi + 2\pi/\kappa_E, \phi + \Omega_{1E}\phi/\kappa_E) \sim (\psi, \phi + 2\pi)$, the Killing fields

$$\frac{1}{\kappa_E}\frac{\partial}{\partial\psi} + \frac{\Omega_{1E}}{\kappa_E}\frac{\partial}{\partial\phi} \qquad \text{and} \qquad \frac{\partial}{\partial\phi} \tag{2.47}$$

become 2π-periodic, and generate an isometric T^2-action. Introducing appropriate radial and angular coordinates and performing a Taylor expansion, reasoning as in the previous sections, one can show that g can be made regular at points where $x \in \{-\infty, -1\}$ or $y \in \{1, \infty\}$, with the exception of those points where $(x,y) = (-\infty, \infty)$. This makes it into a complete Ricci-flat metric on $\mathbb{C}P^2 \setminus S^1$.

Under the T^2-action, points where $x = -\infty$ and $y > 1$ have isotropy group $T^2(0,1)$, and the same is true for points where $y = \infty$ and $x < -1$. Points where $y = 1$ and $-\infty < x < -1$ have isotropy group $T^2(1,0)$, while points where $x = -1$ and $1 < y < \infty$ have isotropy group $T^2(1,1)$. Points with $(x,y) \in \{(-\infty,1), (-1,1), (-1,\infty)\}$ are fixed under the whole action.

Chapter 3

Mode stability

3.1 The perturbation equations

When referring to a *perturbation* $\dot{g}$ of a metric g, we are referring to a linear perturbation of g, that is, a symmetric 2-tensor $\dot{g}$. When g is Ricci-flat, we say that $\dot{g}$ is a *Ricci-flat perturbation* if it is Ricci-flat to first order, that is, if $\dot{g} \in \ker((D\,\mathrm{Ric})_g)$. In general, for a quantity depending on the metric g, we denote by a dot the derivative of a quantity in the perturbation direction $\dot{g}$. In this notation, $\dot{g}$ is a Ricci-flat perturbation if and only if $\dot{\mathrm{Ric}} = 0$.

Ricci-flat perturbations of the Lorentzian Kerr metric have been studied extensively, and in [40], Teukolsky derived a well-known equation for the perturbation of the Weyl scalars of extreme spin weight, for such perturbations of the metric. The following theorem gives a Riemannian analog of that perturbation equation.

Theorem 2. *Consider a Ricci-flat perturbation $\dot{g}$ of a Ricci-flat type D metric g. For an adapted tetrad, the perturbation $\dot{\Psi}_0$ satisfies the equation*

$$((D - 3\epsilon + \tilde{\epsilon} - \tilde{\rho} - 4\rho)(\Delta - 4\gamma + \mu) - (\delta - \tilde{\alpha} - 3\beta + \tilde{\pi} - 4\tau)(\tilde{\delta} - 4\alpha + \pi) - 3\Psi_2)\dot{\Psi}_0 = 0, \quad (3.1)$$

and the perturbation $\dot{\tilde{\Psi}}_0$ satisfies the equation

$$((D - 3\tilde{\epsilon} + \epsilon - \rho - 4\tilde{\rho})(\Delta - 4\tilde{\gamma} + \tilde{\mu}) - (\tilde{\delta} - \alpha - 3\tilde{\beta} + \pi - 4\tilde{\tau})(\delta - 4\tilde{\alpha} + \tilde{\pi}) - 3\tilde{\Psi}_2)\dot{\tilde{\Psi}}_0 = 0. \quad (3.2)$$

Proof. Since we have an adapted tetrad, $\Psi_0 = \Psi_1 = \tilde{\Psi}_0 = \tilde{\Psi}_1 = 0$, and using (A.15) and (A.17) along with their tilded versions, we also have $\kappa = \tilde{\kappa} = \sigma = \tilde{\sigma} = 0$. The linearized versions of (A.17) and (A.15) become

$$(\Delta - 4\gamma + \mu)\dot{\Psi}_0 = (\delta - 4\tau - 2\beta)\dot{\Psi}_1 + 3\dot{\sigma}\Psi_2 \quad (3.3)$$

and

$$(\tilde{\delta} - 4\alpha + \pi)\dot{\Psi}_0 = (D - 4\rho - 2\epsilon)\dot{\Psi}_1 + 3\dot{\kappa}\Psi_2 \quad (3.4)$$

respectively. Operating on (3.3) with D and on (3.4) with δ, subtracting the resulting equations and using the commutation relation (A.2), we get

$$(D(\Delta - 4\gamma + \mu) - \delta(\tilde{\delta} - 4\alpha + \pi))\dot{\Psi}_0 = ([D, \delta] - 4D\tau - 2D\beta + 4\delta\rho + 2\delta\epsilon)\dot{\Psi}_1 + (3D\dot{\sigma} - 3\delta\dot{\kappa})\Psi_2 \tag{3.5}$$

$$\begin{aligned} &= (-(\tilde{\alpha} + 3\beta - \tilde{\pi} + 4\tau)D + (3\epsilon - \tilde{\epsilon} + \tilde{\rho} + 4\rho))\delta)\dot{\Psi}_1 \\ &\quad - \dot{\Psi}_1(D(4\tau + 2\beta) + \delta(4\rho + 2\epsilon)) + (3D\dot{\sigma} - 3\delta\dot{\kappa})\Psi_2. \end{aligned} \tag{3.6}$$

We proceed by eliminating the first term on the right-hand side, obtaining

$$((D - 3\epsilon + \tilde{\epsilon} - \tilde{\rho} - 4\rho)(\Delta - 4\gamma + \mu) - (\delta - \tilde{\alpha} - 3\beta + \tilde{\pi} - 4\tau)(\tilde{\delta} - 4\alpha + \pi))\dot{\Psi}_0 = A_1 + A_2, \tag{3.7}$$

where

$$\begin{aligned} A_1 = ((-3\epsilon + \tilde{\epsilon} - \tilde{\rho} - 4\rho)(-4\tau - 2\beta) - (-\tilde{\alpha} - 3\beta + \tilde{\pi} - 4\tau)(-4\rho - 2\epsilon) \\ - \dot{\Psi}_1(D(4\tau + 2\beta) + \delta(4\rho + 2\epsilon)) \end{aligned} \tag{3.8}$$

and

$$A_2 = 3((D - 3\epsilon + \tilde{\epsilon} - \tilde{\rho} - 4\rho)\dot{\sigma} - (\delta - \tilde{\alpha} - 3\beta + \tilde{\pi} - 4\tau)\dot{\kappa})\Psi_2. \tag{3.9}$$

By using (A.6), (A.11) and (A.14), we see that $A_1 = 0$. For an adapted tetrad, (A.16) and (A.18) become

$$D\Psi_2 = 3\rho\Psi_2, \qquad \delta\Psi_2 = 3\tau\Psi_2. \tag{3.10}$$

Therefore, by the Leibniz rule,

$$A_2 = 3\dot{\sigma}D\Psi_2 - 3\dot{\kappa}\delta\Psi_2 + 3\Psi_2((D - 3\epsilon + \tilde{\epsilon} - \tilde{\rho} - 4\rho)\dot{\sigma} - (\delta - \tilde{\alpha} - 3\beta + \tilde{\pi} - 4\tau)\dot{\kappa})) \tag{3.11}$$

$$= 3\Psi_2((D - 3\epsilon + \tilde{\epsilon} - \tilde{\rho} - \rho)\dot{\sigma} - (\delta - \tilde{\alpha} - 3\beta + \tilde{\pi} - \tau)\dot{\kappa}) \tag{3.12}$$

$$= 3\dot{\Psi}_0\Psi_2, \tag{3.13}$$

where we used the linearization of (A.10) in the last step, thus verifying (3.1). The proof of (3.2) is similar, referring to the tilded versions of the NP equations instead. □

3.2 Kerr

3.2.1 The Separated Perturbation Equations in Coordinates

We shall be interested in a particular choice of complex null tetrad $(l, \bar{l}, m, \overline{m})$, called the *Carter tetrad*, defined by

$$l = \frac{1}{\sqrt{2\Delta\Sigma}}\left((r^2 - a^2)\frac{\partial}{\partial t} - a\frac{\partial}{\partial\phi}\right) + i\sqrt{\frac{\Delta}{2\Sigma}}\frac{\partial}{\partial r}, \tag{3.14}$$

$$m = \frac{1}{\sqrt{2\Sigma}}\frac{\partial}{\partial\theta} - \frac{i}{\sqrt{2\Sigma}}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + a\sin\theta\frac{\partial}{\partial t}\right). \tag{3.15}$$

Note that $|l|_g = |m|_g = 1$. The spin coefficients for the Carter tetrad are given explicitly in Section B.1. For this tetrad, we have

$$\Psi_2 = \frac{M}{(r - a\cos\theta)^3}, \qquad \tilde{\Psi}_2 = \frac{M}{(r + a\cos\theta)^3}, \tag{3.16}$$

and all other Weyl scalars vanish. In particular, this is an adapted tetrad.

We shall now analyze the perturbation equations in the Carter tetrad. The relevant properties of the equations are given in the following four lemmas.

Lemma 5. *For the Carter tetrad, the perturbation equation* (3.1) *is equivalent to the equation*[1] $\mathbf{L}\Phi = 0$*, where* $\Phi = \Psi_2^{-2/3}\dot{\Psi}_0$ *and*

$$\mathbf{L} = \frac{\partial}{\partial r}\Delta_{\mathrm{K}}\frac{\partial}{\partial r} + \frac{1}{\Delta_{\mathrm{K}}}\left((r^2-a^2)\frac{\partial}{\partial t} - a\frac{\partial}{\partial \phi} + 2i(r-M)\right)^2 + 8i(r + a\cos\theta)\frac{\partial}{\partial t}$$
$$+ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(a\sin^2\theta\frac{\partial}{\partial t} + \frac{\partial}{\partial\phi} - 2i\cos\theta\right)^2. \quad (3.17)$$

Furthermore, if Φ *is a solution to this equation coming from a perturbation of the metric, then we can write*

$$\Phi(t,r,\theta,\phi) = \sum_{m,\omega,\Lambda} e^{i(m\phi-\omega t)}R_{m,\omega,\Lambda}(r)S_{m,\omega,\Lambda}(\theta), \quad (3.18)$$

where m *runs over* $\mathbb{Z}$*,* ω *runs over* $\Omega + \kappa\mathbb{Z}$ *and for each choice of* m, ω, Λ*, the function* $R = R_{m,\omega,\Lambda}$ *solves the equation* $\mathbf{R}R = 0$*. The function* $S = S_{m,\omega,\Lambda}$ *is the unique solution to the boundary value problem* $\mathbf{S}S = 0$*,* $S'(0) = S'(\pi) = 0$*, where*

$$\mathbf{R} = \frac{d}{dr}\Delta_{\mathrm{K}}\frac{d}{dr} + U(r), \quad (3.19)$$

$$U(r) = -\frac{((r^2-a^2)\omega + am + 2(r-M))^2}{\Delta_{\mathrm{K}}} + 8r\omega - \Lambda \quad (3.20)$$

and

$$\mathbf{S} = \frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + V(\cos\theta), \quad (3.21)$$

$$V(x) = 8a\omega x - \frac{1}{1-x^2}\left(a\omega(1-x^2) - m + 2x\right)^2 + \Lambda. \quad (3.22)$$

Here, S *is normalized with respect to the* L^2 *product with measure* $\sin\theta\, d\theta$*, and the separation constant* Λ *runs over the (countable set of) values for which such an* S *exists.*

The same statement holds for the perturbation equation (3.2)*, if* Φ *is replaced by* $\tilde{\Phi}$*, where* $\tilde{\Phi} = \tilde{\Psi}_2^{-2/3}\dot{\tilde{\Psi}}_0$*.*

Proof. The fact that (3.1) is equivalent to $\mathbf{L}\Phi = 0$ follows from a direct computation, using the expressions for the spin coefficients in Section B.1.

Now note that the boundary value problem $\mathbf{S}S = 0$, $S'(0) = S'(\pi) = 0$ is a Sturm-Liouville problem. Thus, there exists an orthonormal L^2 basis of functions $\{S_{m,\omega,\Lambda}\}_\Lambda$ solving it, and furthermore, we can perform a Fourier series decompositions in the coordinates (t,ϕ). From these considerations, we can write (3.18), where

$$R_{m,\omega,\Lambda}(r) = \frac{\kappa}{4\pi^2}\int_0^{2\pi/\kappa}\int_0^{2\pi}\int_0^{\pi} e^{-i(m\phi-\omega t)}\Phi(t,r,\theta,\phi)S_{m,\omega,\Lambda}(\theta)\sin\theta\, d\theta\, d\phi\, dt. \quad (3.23)$$

The fact that $R = R_{m,\omega,\Lambda}$ satisfies $\mathbf{R}R = 0$ now follows directly from (3.23), along with the fact that $\mathbf{L}\Phi = 0$.

For the statement involving $\tilde{\Phi}$, the proof is entirely analogous. ☐

[1]Note that the equation $\mathbf{L}\Phi = 0$ coincides with the corresponding equation for $s = -2$ in the Lorentzian case (see [42, 5]), up to the replacements $t \mapsto it$, $a \mapsto -ia$ and .

Lemma 6. *The equation* $\mathbf{R}R = 0$ *is an ordinary differential equation in a complex variable* r, *which has regular singular points at* $r = r_\pm$. *The point* $r = \infty$ *is an irregular singular point of rank* 1, *except when* $\omega = 0$, *in which case it is a regular singular point. Thus, the equation* $\mathbf{R}R = 0$ *is a confluent Heun equation (see [38, Section 3]) when* $\omega \neq 0$, *and a hypergeometric equation (see [38, Section 2]) when* $\omega = 0$. *The characteristic exponents at* $r = r_+$ *are*

$$\pm\left(1 + \frac{2Mr_+ + am}{r_+ - r_-}\right), \tag{3.24}$$

and those at $r = r_-$ *are*

$$\pm\left(-1 + \frac{2Mr_- + am}{r_+ - r_-}\right). \tag{3.25}$$

When $\omega = 0$, *we have* $\Lambda \geq 0$, *and the characteristic exponents at* $r = \infty$ *are*

$$-\frac{3}{2} \pm i\sqrt{\frac{7}{2} + \Lambda}. \tag{3.26}$$

When $\omega \neq 0$, *the equation* $\mathbf{R}R = 0$ *admits normal solutions (see [17, Section 3.2]), near* $r = \infty$, *of the asymptotic form*

$$R \sim e^{\pm r\omega} r^{-1\pm 2(M\omega - 1)}. \tag{3.27}$$

Proof. The fact that $r = r_\pm$ are regular singular points follows directly from the fact that $\Delta_\mathrm{K} = (r - r_+)(r - r_-)$, the statement about the type and rank of the singular point at $r = \infty$ follows directly from the discussion in [17, Section 3.1], and the expressions for the characteristic exponents can be seen from the discussion in [38, Section 1.1.3]. Substituting $R = y/\sqrt{\Delta_\mathrm{K}}$ transforms the equation $\mathbf{R}R = 0$ into

$$\frac{d^2y}{dr^2} + qy = 0, \tag{3.28}$$

where

$$q(r) = \frac{U(r)}{\Delta_\mathrm{K}} + \left(\frac{r_+ - r_-}{2\Delta_\mathrm{K}}\right)^2 = -\omega^2 - \frac{4\omega(M\omega - 1)}{r} + O(r^{-2}). \tag{3.29}$$

Following [17, Section 3.2], the equation $\mathbf{R}R = 0$ therefore has normal solutions of the asymptotic form

$$R \sim e^{\pm r\omega} r^{-1\pm 2(M\omega - 1)}. \tag{3.30}$$

□

Lemma 7. *For a solution to the equation* $\mathbf{R}R = 0$ *coming from a (globally smooth) perturbation of the Kerr metric, the corresponding characteristic exponent at* $r = r_+$ *is*

$$\left|1 + \frac{2Mr_+ + am}{r_+ - r_-}\right|. \tag{3.31}$$

Proof. Since the set corresponding to $r = r_+$ is compact, and by assumption, the perturbation $\dot{W}$ of the Weyl tensor is continuous, $\dot{W}$ has bounded norm in a neighborhood of this set. Consequently, since l and m have norm 1, it follows that $\dot{\Psi}_0 = -\dot{W}(l, m, l, m)$ is bounded near $r = r_+$. Since $\Psi_2^{-2/3} = O(r^2)$, it follows that Φ, and therefore R, is bounded near $r = r_+$. The statement now follows immediately. □

We say that a perturbation $\dot{g}$ of the Kerr metric is *asymptotically flat* (AF) if $|\dot{g}|_g = O(r^{-1})$, with corresponding decay on derivatives, that is, $|\nabla^k \dot{g}|_g = O(r^{-1-k})$ for every positive integer k. Here, r is the radial coordinate of Kerr defined earlier, and the norm and covariant derivative are taken with respect to g.

Lemma 8. *Let R be a solution to the equation $\mathbf{R}R = 0$ coming from an asymptotically flat perturbation of the Kerr metric. When $\omega = 0$, none of the characteristic exponents at $r = \infty$ are compatible with the asymptotic flatness assumption. When $\omega \neq 0$, exactly one of the asymptotic normal solutions is compatible with this assumption, namely*

$$R \sim e^{-r|\omega|} r^{-1-2(M\omega-1)\,\mathrm{sgn}(\omega)}. \tag{3.32}$$

Proof. By the assumption of asymptotic flatness, we have $W = O(r^{-3})$, which means that Φ, and therefore R, decays as r^{-1} as $r \to \infty$. In particular, we must have $\lim_{r\to\infty} R(r) = 0$, and the result now follows immediately. $\square$

3.2.2 Mode Stability

Equipped with the lemmas of the previous subsection, we are now in a position to prove the following theorem.

Theorem 3. *For Ricci-flat AF perturbations of the Riemannian Kerr metric, the perturbed Weyl scalars $\dot{\Psi}_0, \dot{\tilde{\Psi}}_0$ vanish identically.*

Proof. For $r > r_+$ and $-1 < x < 1$, note that

$$\begin{aligned} U(r) + V(x) = &- \frac{16M(r+ax)}{(r-ax)^2} \\ &- \frac{(a^2x(mx-2) + 2a(x^2-1)(M(r\omega-1)+r) + r(m-2x)(2M-r))^2}{(1-x^2)(\Delta_\mathrm{K} + (1-x^2)a^2)\Delta_\mathrm{K}} \\ &- \frac{(2M(ax+3r) + (r-ax)(r+ax)(-ax\omega + r\omega - 2))^2}{(r-ax)^2(\Delta_\mathrm{K} + (1-x^2)a^2)} \\ &< 0. \end{aligned} \tag{3.33}$$

Here, the strict negativity follows from that of the first term, which holds because $r_+ > |a|$. By an integration by parts, we have

$$U(r) \le U(r) + \int_0^\pi \left(\frac{dS}{d\theta}\right)^2 \sin\theta\, d\theta = U(r) + \int_0^\pi \left(\mathbf{S}S - \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dS}{d\theta}\right)\right) S \sin\theta\, d\theta \tag{3.34}$$

$$= U(r) + \int_0^\pi V(\cos\theta) S^2 \sin\theta\, d\theta = \int_0^\pi (U(r) + V(\cos\theta)) S^2 \sin\theta\, d\theta < 0, \tag{3.35}$$

where the last equality follows from (3.33) and the normalization of S. Multiplying (3.19) by $\overline{R}$ and integrating, the first term being integrated by parts, we get

$$0 = \left[\Delta_\mathrm{K} \frac{dR}{dr}\overline{R}\right]_{r=r_+}^{r=\infty} - \int_{r_+}^\infty \left(\Delta_\mathrm{K}\left|\frac{dR}{dr}\right|^2 - U|R|^2\right) dr. \tag{3.36}$$

We claim that the first term vanishes; to see this, we consider the endpoints separately. Near $r = \infty$, we have $\Delta_\mathrm{K} \sim r^2$, while R and its derivative decays exponentially. Thus, the term in square brackets decays exponentially, and in particular, it tends to zero as $r \to \infty$. Near $r = r_+$, we know that $\overline{R}$ is bounded. The characteristic exponent corresponding to R is either positive, in which case $\frac{dR}{dr} = o(r^{-1})$, or R is analytic in a neighborhood of $r = r_+$, in which case $\frac{dR}{dr}$ is bounded. In either case, the product $\Delta_{\mathrm{K}} \frac{dR}{dr}$ tends to zero as $r \to r_+$, and from this, it immediately follows that the term in square brackets tends to zero.

We have thus shown that

$$\int_{r_+}^{\infty} \left(\Delta_{\mathrm{K}} \left| \frac{dR}{dr} \right|^2 - U|R|^2 \right) dr = 0. \tag{3.37}$$

Since $U < 0$, the terms in the integrand are both non-negative and must therefore vanish. We conclude that R vanishes identically. □

3.3 Taub-Bolt

3.3.1 The Separated Perturbation Equations in Coordinates

As for Kerr, we are interested in a particular choice of complex null tetrad $(l, \bar{l}, m, \overline{m})$, in this case given by

$$l = \frac{1}{\sqrt{2\Sigma}} \left(\frac{1}{\sin\theta} \left(\cos\theta \frac{\partial}{\partial t} - \frac{\partial}{\partial \phi} \right) + i \frac{\partial}{\partial \theta} \right), \tag{3.38}$$

$$m = \sqrt{\frac{\Delta_{\mathrm{T}}}{2\Sigma}} \frac{\partial}{\partial r} + \frac{i\sqrt{\Sigma/2\Delta_{\mathrm{T}}}}{2N} \frac{\partial}{\partial t}, \tag{3.39}$$

satisfying $|l|_g = |m|_g = 1$.

The spin coefficients for this tetrad are given explicitly in Appendix B.2 for completeness. For this tetrad, we have

$$\Psi_2 = \frac{N}{4(r-N)^3}, \qquad \tilde{\Psi}_2 = \frac{9N}{4(r+N)^3}, \tag{3.40}$$

and the rest of the Weyl scalars vanish. Thus, this is an adapted tetrad, and we see that the Taub-bolt metric is of type D.

The following four lemmas give the relevant properties of the perturbation equations (3.1) (3.2) for our analysis. The proofs are entirely analogous to those in Section 3.2.1 and are therefore omitted.

Lemma 9. *For the tetrad given in* (3.38) *and* (3.39)*, the perturbation equation* (3.1) *is equivalent to the equation* $\mathbf{L}\Phi = 0$*, where* $\Phi = \Psi_2^{-2/3}\dot{\Psi}_0$ *and*

$$\mathbf{L} = \frac{\partial}{\partial r}\Delta_{\mathrm{T}}\frac{\partial}{\partial r} - \frac{4N(r+N)}{(r-N)^2} + \frac{\Sigma^2}{4N^2\Delta_{\mathrm{T}}}\left(\frac{\partial}{\partial t} - i\frac{N(4r^2 - 11Nr + 3N^2)}{\Sigma(r-N)}\right)^2$$
$$+ \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\cos\theta\frac{\partial}{\partial t} - \frac{\partial}{\partial\phi} - 2i\cos\theta\right)^2. \tag{3.41}$$

Furthermore, if Φ *is a solution to this equation coming from a perturbation of the metric, then we can write*

$$\Phi(t, r, \theta, \phi) = \sum_{m,\omega,\Lambda} e^{i(m\phi - \omega t)} R_{m,\omega,\Lambda}(r) S_{m,\omega,\Lambda}(\theta), \tag{3.42}$$

where m runs over $\frac{1}{2}\mathbb{Z}$, and ω runs over $m+\mathbb{Z}$, and for each choice of m,ω,Λ, the function $R = R_{m,\omega,\Lambda}$ solves the equation $\mathbf{R}R = 0$, and the function $S = S_{m,\omega,\Lambda}$ solves the boundary value problem $\mathbf{S}S = 0$, $S'(0) = S'(\pi) = 0$, where

$$\mathbf{R} = \frac{d}{dr}\Delta_{\mathrm{T}}\frac{d}{dr} + U(r), \tag{3.43}$$

$$U(r) = -\frac{4N(r+N)}{(r-N)^2} - \frac{\Sigma^2}{4N^2\Delta_{\mathrm{T}}}\left(\omega + \frac{N(4r^2-11Nr+3N^2)}{\Sigma(r-N)}\right)^2 - \Lambda \tag{3.44}$$

and

$$\mathbf{S} = \frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + V(\cos\theta), \tag{3.45}$$

$$V(x) = -\frac{((\omega+2)x+m)^2}{1-x^2} + \Lambda. \tag{3.46}$$

The separation constant Λ runs over the (countable set of) values for which such an S exists, all of which are non-negative.

The same statement holds if Φ is replaced by $\tilde{\Phi} = \tilde{\Psi}_2^{-2/3}\dot{\tilde{\Psi}}_0$, the operator $\mathbf{L}$ is replaced by $\tilde{\mathbf{L}}$, and the operator $\mathbf{R}$ replaced by $\tilde{\mathbf{R}}$, defined in the same way but using a potential $\tilde{U}$ in place of U. Here,

$$\begin{aligned}\tilde{\mathbf{L}} = \frac{\partial}{\partial r}\Delta_{\mathrm{T}}\frac{\partial}{\partial r} - \frac{36N(r-N)}{(r+N)^2} + \frac{\Sigma^2}{4N^2\Delta_{\mathrm{T}}}\left(\frac{\partial}{\partial t} + i\frac{N(4r^2-19Nr+13N^2)}{\Sigma(r+N)}\right)^2 \\ + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\cos\theta\frac{\partial}{\partial t} - \frac{\partial}{\partial\phi} - 2i\cos\theta\right)^2\end{aligned} \tag{3.47}$$

and

$$\tilde{U}(r) = -\frac{36N(r-N)}{(r+N)^2} - \frac{\Sigma^2}{4N^2\Delta_{\mathrm{T}}}\left(\omega - \frac{N(4r^2-19Nr+13N^2)}{\Sigma(r+N)}\right)^2 - \Lambda. \tag{3.48}$$

Lemma 10. *The equation $\mathbf{R}R = 0$ is an ordinary differential equation in a complex variable r, which has regular singular points at $r = 2N$ and $r = N/2$. The point $r = \infty$ is an irregular singular point of rank 1, except when $\omega = 0$, in which case it is a regular singular point. Thus, the equation $\mathbf{R}R = 0$ is a confluent Heun equation (see [38, Section 3]) when $\omega \neq 0$, and a hypergeometric equation (see [38, Section 2]) when $\omega = 0$. The characteristic exponents at $r = 2N$ are*

$$\pm(\omega - 1), \tag{3.49}$$

and those at $r = N/2$ are

$$\pm\left(\frac{\omega}{4} - 1\right). \tag{3.50}$$

When $\omega = 0$, the characteristic exponents at $r = \infty$ are

$$-\frac{3}{2} \pm i\sqrt{\frac{7}{2} + \Lambda}. \tag{3.51}$$

When $\omega \neq 0$, the equation $\mathbf{R}R = 0$ admits normal solutions (see [17, Section 3.2]), near $r = \infty$, of the asymptotic form

$$R \sim e^{\pm r\omega/2N} r^{-1\pm(5\omega/4-2)}. \tag{3.52}$$

Lemma 11. *For a solution to the equation $\mathbf{R}R = 0$ coming from a (globally smooth) perturbation of the Taub-bolt metric, the corresponding characteristic exponent at $r = 2N$ is $|\omega - 1|$.*

A perturbation $\dot{g}$ of the Taub-bolt metric is said to be *asymptotically locally flat* (ALF) if it decays as $O(r^{-1})$, with corresponding decay on derivatives, just like the definition of AF perturbations of the Kerr metric given in Section 3.2.1.

Lemma 12. *Let R be a solution to the equation $\mathbf{R}R = 0$ coming from an asymptotically locally flat perturbation of the Taub-bolt metric. When $\omega = 0$, none of the characteristic exponents at $r = \infty$ are compatible with the assumption of asymptotic local flatness. When $\omega \neq 0$, exactly one of the asymptotic normal solutions is compatible with this assumption, namely*

$$R \sim e^{-r|\omega|/2N} r^{-1-(5\omega/4-2)\,\mathrm{sgn}(\omega)}. \tag{3.53}$$

Corresponding lemmas regarding the asymptotics of the equation $\tilde{\mathbf{R}}R = 0$ also hold. We omit them since they are entirely analogous.

3.3.2 Mode Stability

Theorem 4. *For Ricci-flat ALF perturbations of the Taub-bolt metric, the perturbed Weyl scalars $\dot{\Psi}_0, \dot{\tilde{\Psi}}_0$ vanish identically.*

Proof. In this case, we directly see that $U(r) < 0$, and integrating the equation $\mathbf{R}R = 0$ by parts like in the proof of Theorem 3, we see that R vanishes identically. The case involving the equation $\tilde{\mathbf{R}}R = 0$ is entirely analogous. $\square$

Chapter 4

T^2 symmetry

4.1 Topological invariants in terms of the rod structure

In this section, we explicitly reconstruct M as a smooth manifold with a T^2-action directly from its rod structure. We then calculate the intersection form for the reconstructed model, resulting in the following theorem.

Theorem 5. *Let (M,g) be a toric gravitational instanton with rod structure $(v_0,\dots,v_n)$. Then $H_2(M)\cong\mathbb{Z}^{n-1}$, and $H_2(M)$ admits a basis*[1] *in which the intersection form on M is represented by the matrix*

$$\begin{pmatrix} d_1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & d_2 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & d_3 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & d_{n-3} & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & d_{n-2} & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & d_{n-1} \end{pmatrix}, \tag{4.1}$$

where $d_i=-\det\begin{pmatrix} v_{i-1} & v_{i+1}\end{pmatrix}$. In particular, the signature $\tau(M)$ is the signature of this matrix.

Proof. Let $M_1,\dots,M_n$ denote distinct copies of $\mathbb{R}^4\cong\mathbb{C}^2$, and for each i, define a T^2-action on M_i by

$$(e^{i\theta_1},e^{i\theta_2})\cdot(z_1,z_2)=(e^{i(a_{11}\theta_1+a_{12}\theta_2)}z_1,e^{i(a_{21}\theta_1+a_{22}\theta_2)}z_2), \tag{4.2}$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}=\begin{pmatrix} v_{i-1} & v_i\end{pmatrix}^{-1}. \tag{4.3}$$

Now let $M_i^+=\{(z_1,z_2)\in M_i \mid |z_1|>|z_2|^2+1\}$ and $M_i^-=\{(z_1,z_2)\in M_i \mid |z_2|>|z_1|^2+1\}$. Since

$$\begin{pmatrix} v_i & v_{i+1}\end{pmatrix}^{-1}v_i=\begin{pmatrix}1\\0\end{pmatrix}, \tag{4.4}$$

and since $\det\Big(\begin{pmatrix} v_i & v_{i+1}\end{pmatrix}^{-1}\begin{pmatrix} v_{i-1} & v_i\end{pmatrix}\Big)=1$, we can write

$$\begin{pmatrix} v_i & v_{i+1}\end{pmatrix}^{-1}\begin{pmatrix} v_{i-1} & v_i\end{pmatrix}=\begin{pmatrix} -d_i & 1 \\ -1 & 0\end{pmatrix} \tag{4.5}$$

[1] More precisely, a basis in the sense of $\mathbb{Z}$-modules.

for some $d_i \in \mathbb{Z}$, and since $v_{i-1} = -d_i v_i - v_{i+1}$, we have $\det\begin{pmatrix} v_{i-1} & v_{i+1} \end{pmatrix} = -d_i \det\begin{pmatrix} v_i & v_{i-1} \end{pmatrix} = -d_i$. We have diffeomorphisms $F_i : M_i^+ \to M_{i+1}^-$ given by

$$F_i(z_1, z_2) = \left(\left(\frac{z_1}{|z_1|} \right)^{-d_i} z_2, \left(\frac{z_1}{|z_1|} \right)^{-1} \left(|z_2|^2 + 1 + \frac{1}{|z_1| - |z_2|^2 - 1} \right) \right), \tag{4.6}$$

and gluing the sets M_i together along these diffeomorphisms, we obtain a manifold M'. It can be checked that the diffeomorphisms are equivariant with respect to the T^2-actions defined on the sets M_i and that they are orientation-preserving, where each M_i is given the standard orientation of $\mathbb{R}^4$. This makes M' into an oriented manifold with a T^2-action. By construction, we can map the orbit space M'/T^2 diffeomorphically onto the orbit space M/T^2 in such a way that the isotropy groups match, and by [43, Theorem 1.1], this implies that M and M' are equivariantly diffeomorphic. For simplicity, we therefore identify M with M'.

For $1 \leq i \leq n-1$, define the set

$$R_i = \{(z_1, z_2) \in M_i \mid z_2 = 0\} \cup \{(z_1, z_2) \in M_i \mid z_1 = 0\}. \tag{4.7}$$

This is an embedded 2-sphere in M, and in particular it is a closed orientable submanifold of M. We give R_i the orientation for which the standard coordinate vector fields $(\partial/\partial x_1, \partial/\partial x_2)$ for M_i form a positively oriented frame for R_i. Then the fundamental classes $[R_1], \ldots, [R_{n-1}]$ are elements of $H_2(M)$, and we claim that they form a basis.

To this end, assume that $n > 1$ (otherwise, there is nothing to prove). For $1 \leq i \leq n-1$, we have the Mayer–Vietoris sequence

$$H_2(M_i) \oplus H_2(M_{i+1}) \to H_2(M_i \cup M_{i+1}) \to H_1(M_i \cap M_{i+1}) \to H_1(M_i) \oplus H_1(M_{i+1}). \tag{4.8}$$

Since $M_i \cong M_{i+1} \cong \mathbb{R}^4$, the first and last terms vanish, which implies that the middle map is an isomorphism. We can decompose R_i as the union of two closed disks $D_i \subseteq M_i$ and $D_{i+1} \subseteq M_{i+1}$, joined along their common boundary circle $D_i \cap D_{i+1} \subseteq M_i \cap M_{i+1}$. The middle map can be described explicitly (see, e.g., [22, p. 150]), and it maps $[R_i]$ to $[D_i \cap D_{i+1}]$, where $D_i \cap D_{i+1}$ has been oriented appropriately. But the intersection $M_i \cap M_{i+1}$ deformation retracts to $D_i \cap D_{i+1}$, which shows that $\{[D_i \cap D_{i+1}]\}$ is a basis for $H_1(M_i \cap M_{i+1})$. Hence, it follows that $\{[R_i]\}$ is a basis for $H_2(M_i \cup M_{i+1})$.

Now assume that $2 \leq i \leq n-1$ and put $A = M_1 \cup \cdots \cup M_i$ and $B = M_i \cup M_{i+1}$, and consider the Mayer–Vietoris sequence

$$H_2(A \cap B) \to H_2(A) \oplus H_2(B) \to H_2(A \cup B) \to H_1(A \cap B). \tag{4.9}$$

The first and last terms vanish because $A \cap B = M_i \cong \mathbb{R}^4$, which shows that the middle map is an isomorphism. From the previous paragraph we know that $\{[R_1]\}$ is a basis for $H_2(M_1 \cup M_2)$, and by inducting on i we can conclude that $\{[R_1], \ldots, [R_{n-1}]\}$ is a basis for $H_2(M)$.

What remains is to compute the intersection form with respect to this basis. For $|i-j| > 1$, the submanifolds R_i and R_j are disjoint, and thus the intersection product $[R_i] \cdot [R_j]$ vanishes. Two adjacent submanifolds R_i and R_{i+1} intersect in exactly one point, and they do so transversely, and from this, it follows that $[R_i] \cdot [R_{i+1}] = \pm 1$. Flipping orientations of the submanifolds R_i appropriately, we ensure that $[R_i] \cdot [R_{i+1}] = 1$ for all i, noting that flipping orientation does not affect the self-intersection numbers $[R_i] \cdot [R_i]$

To calculate the self-intersection numbers $[R_i] \cdot [R_i]$, we have to perturb R_i into another submanifold representative of the same homology class, which intersects R_i transversely. We first consider the case where $d_i \geq 0$. In this case, we take a smooth function $f : \mathbb{C} \to \mathbb{C}$ such that

$f(z) = z^{d_i} - 3^{-d_i}$ when $|z| \leq 2/3$, and $f(z) = (z/|z|)^{d_i}$ when $|z| \geq 1$, and such that $f(z) \neq 0$ when $2/3 \leq |z| \leq 1$. Now define a submanifold by

$$R_i' = \{(z_1, z_2) \in M_i \mid z_2 = f(z_1)\} \cup \{(z_1, z_2) \in M_{i+1} \mid z_1 = 1\}. \tag{4.10}$$

The inclusion map $R_i' \to M$ is homotopic to an embedding whose image is R_i. This can be seen by considering the homotopy $R_i' \times [0,1] \to M$ which in M_i is given by $((z_1, z_2), t) \mapsto (z_1, (1-t)z_2)$, and in M_{i+1} is given by $((z_1, z_2), t) \mapsto ((1-t)z_1, z_2)$. Endowing R_i' with the appropriate orientation, this shows that R_i' is homologous to R_i.

Since f has exactly d_i roots, and since it is holomorphic with non-vanishing derivative near these roots, it follows that R_i and R_i' intersect transversely in d_i points and that they intersect positively at each such point (with respect to the orientation of M). In other words, $[R_i] \cdot [R_i] = d_i$. The case where $d_i < 0$ is exactly the same, except that we let $f(z) = \overline{z}^{-d_i} - 3^{d_i}$ for $|z| \leq 2/3$. In that case, the function f is antiholomorphic (instead of holomorphic) in that region so that R_i and R_i' intersect negatively at each root. □

4.2 Hitchin–Thorpe Inequality

The following definition is needed to state the results in this section.

Definition 4. A Riemannian manifold (M, g) is said to be *hyper-Kähler* if it admits three almost complex structures I, J and K such that

- (M, g) is a Kähler manifold with respect to I, J and K separately.
- $I^2 = J^2 = K^2 = -1$.

In its original form, the Hitchin–Thorpe inequality is a statement about compact Einstein 4-manifolds. For such manifolds M, the inequality states that

$$2\chi(M) \geq 3|\tau(M)|, \tag{4.11}$$

and it further states that equality occurs if and only if the universal cover of M is hyper-Kähler. As a special case, this inequality applies when M is Ricci-flat. In general, the inequality does not hold without modification in the non-compact case. For Ricci-flat ALE or ALF manifolds, there are, however, variants of the inequality that involve extra boundary terms. In the case of ALE manifolds, for instance, we have the following variant of the inequality, whose proof can be found in [31, Theorem 4.2].

Theorem 6 (Hitchin–Thorpe Inequality for Ricci-Flat ALE Manifolds)**.** *Let (M, g) be an oriented Ricci-flat ALE manifold with group Γ. Then*

$$2\left(\chi(M) - \frac{1}{|\Gamma|}\right) \geq 3|\tau(M) + \eta_S(S^3/\Gamma)|. \tag{4.12}$$

Equality holds if and only if the universal cover of M is hyper-Kähler.

The number $\eta_S(S^3/\Gamma)$ occurring in Theorem 6 is the so-called *eta-invariant of the signature operator of the space form S^3/Γ*, which is a spectral invariant of this space form. Although we will not describe the eta-invariant in detail, we mention that the eta-invariant is an oriented isometry invariant of the space form S^3/Γ, and that reversing the orientation of this space form flips the sign of the eta-invariant.

The orientation of S^3/Γ that defines the eta-invariant is the boundary orientation on $S^3/\Gamma \cong \{R\} \times S^3/\Gamma$ when viewed as the boundary of $\{x \in \mathbb{R}^4/\Gamma \mid |x| \leq R\}$, where the latter is oriented consistently with the asymptotic diffeomorphism, giving rise to an induced orientation on $L(p,q)$. Replacing the group Γ by its conjugate by an orientation-reversing element of $\mathrm{O}(n)$ if necessary, we may assume that the orientation is induced by the standard orientation on $\mathbb{R}^4$.

Now let M be a toric ALE (with group Γ) instanton, with rod structure $(v_0, \ldots, v_n)$, where we assume that $\det \begin{pmatrix} v_{i-1} & v_i \end{pmatrix} = 1$. Then Theorem 6 applies, and since M is simply connected, the statement about its universal cover applies directly to M. From Lemma 4, it follows that the boundary at infinity of M is $L(p,q)$, where $p = |\det \begin{pmatrix} v_0 & v_n \end{pmatrix}|$ and $q = \operatorname{sgn}(\det \begin{pmatrix} v_0 & v_n \end{pmatrix}) \cdot \det \begin{pmatrix} v_1 & v_n \end{pmatrix}$, and Lemma 1 implies that S^3/Γ and $L(p,q)$ are h-cobordant. In particular, this implies that Γ is a cyclic group of order p, and by the classification of spherical 3-manifolds (see e.g., [35]), $S^3/\Gamma = L(p,q')$ for some q'.

Let M be equipped with the orientation defined in the proof of Theorem 5. By following the proofs of Lemmas 4 and 4 carefully, one verifies that the h-cobordism is an *oriented* h-cobordism, by which we mean that the cobordism W admits an orientation for which the induced boundary orientation on ∂W is that of $(-L(p,q)) \sqcup L(p,q')$. By [15, Theorem 1], this implies that $L(p,q)$ and $L(p,q')$ are orientation-preservingly homeomorphic. The classification of lens spaces then implies that $L(p,q)$ and $L(p,q')$ are orientation-preservingly isometric so that $\eta_S(L(p,q)) = \eta_S(L(p,q'))$. We now cite the following formula for the eta-invariants of lens spaces, whose proof can be found in [26, Theorem 4].

Theorem 7. *For $p, q \in \mathbb{Z}$ with $0 \leq q < p$ and $\gcd(p,q) = 1$, the eta-invariant of the signature operator of $L(p,q)$ is*

$$\eta_S(L(p,q)) = \frac{1}{3p}(p-1)(2pq - 3p - q + 3) - \frac{2}{p} \sum_{k=1}^{q-1} \left\lfloor \frac{kp}{q} \right\rfloor^2. \tag{4.13}$$

By the discussion in the previous paragraph, the eta-invariant which occurs in the inequality (4.12) for the toric ALE instanton M is given by the expression on the right in (4.13), where p and q are given as in the previous paragraph, except that q is reduced[2] modulo p in order to satisfy $0 \leq q < p$.

It is also well known (and can be seen directly by considering the construction in the proof of Theorem 5) that the Euler characteristic is given by $\chi(M) = n$, and taken together, this shows that we can express all of the quantities occurring in Theorem 6 directly in terms of the rod structure. Theorem 6 can thus be interpreted as a necessary condition that rod structures of toric ALE instantons have to satisfy.

As mentioned, there are also variants of the Hitchin–Thorpe inequality for ALF manifolds. In the case of ALF-A_k, we have the following result, whose proof can be found in [11, Theorem 6.12].

Theorem 8 (Hitchin–Thorpe Inequality for Ricci-Flat ALF-A_k Manifolds)**.** *Let (M,g) be an oriented Ricci-flat manifold which is ALF-A_k for some integer k. Then*

$$2\chi(M) \geq 3\left|\tau(M) - \frac{e}{3} + \operatorname{sgn}(e)\right|, \tag{4.14}$$

where $e = -k-1$. Equality holds if and only if the universal cover of M is hyper-Kähler.

Here, the asymptotic diffeomorphism $\overline{M \setminus K} \to (R, \infty) \times L(|e|, \operatorname{sgn} e)$ to be orientation-preserving. For $p \neq 1, 2$, there is no orientation-preserving diffeomorphism $L(p,1) \to L(p,-1)$,

[2] Note that reduction of q modulo p does not change the lens space $L(p,q)$.

which shows that purely topological arguments uniquely determine the sign of e in this case. The cases $|e| = 1$ and $|e| = 2$ are slightly more subtle and require further consideration of how the asymptotic diffeomorphism relates to the metrics of the spaces, but in this paper, we will not consider these aspects.

4.3 Rod structures With Three Turning Points

Consider a rod structure (v_0, v_1, v_2, v_3) with three turning points, satisfying $v_0 = (0,1)$ and $v_1 = (-1,0)$; any such rod structure can be written as in Figure 4.1, for some $a, b \in \mathbb{Z}$. For a toric gravitational instanton (M,g) with this rod structure, we have $\chi(M) = 3$, and the signature $\tau(M)$ is just the difference between the number of positive and negative roots, respectively, of the polynomial $\lambda^2 - (a+b)\lambda + ab - 1$. Assume now, in addition, that (M,g) is ALE. As we shall see, this additional assumption restricts the possible pairs (a,b). An obvious restriction is, of course, that $p := \det\begin{pmatrix} v_0 & v_3 \end{pmatrix} = |1 - ab|$ is non-zero, for otherwise, the boundary at infinity would be $S^2 \times S^1$, which is incompatible with ALE geometry. Moreover, we must have $p > 1$, since if $p = 1$ then M is AE, so the positive mass theorem (see [37]) implies that M is homeomorphic to $\mathbb{R}^4$, which contradicts the fact that $\chi(M) = 3$.

Now let q be the unique integer in the range $0 \leq q < p$ which satisfies $q \equiv b\,\mathrm{sgn}(1-ab) \pmod{p}$. By Theorem 6, we have

$$2\left(3 - \frac{1}{p}\right) \geq 3|\tau(M) + \eta_S(L(p,q))|, \tag{4.15}$$

where $\eta_S(L(p,q))$ is given by the right hand side of (4.13).

The requirement that $p > 1$, along with the inequality (4.15), can be viewed as restrictions on the values of a and b, and for any specific values of a and b, it is straightforward to check whether or not they. In other words, if for some $a, b \in \mathbb{Z}$ one of these requirements does *not* hold, there cannot exist any toric ALE instanton with the corresponding rod structure. By systematically checking all pairs (a,b) with $|a|, |b| \leq 17$, one arrives at Figure 4.2a. The pairs $(a,b) = \pm(2,2)$ (marked with red) are the pairs for which exact equality holds in (4.15). They correspond to the unique hyper-Kähler instanton on the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_3$ (see [27]), equipped with two distinct T^2-actions.

Consider, now, the same question for ALF: Given $a, b \in \mathbb{Z}$, does there exist a toric ALF instanton (M,g) with the rod structure in Figure 4.2? Such a manifold M is necessarily of the type ALF-A_k With the same definitions of p and q as before, and with $e = -k-1$, we have $|e| = p$, and either $q = 1$ or $q = p-1$ holds. Furthermore, Theorem 8 implies that

$$6 \geq 3\left|\tau(M) - \frac{e}{3} + \mathrm{sgn}(e)\right|. \tag{4.16}$$

When $p \neq 1, 2$, the sign of e is uniquely determined, and is given by

$$e = \begin{cases} -p, & q = 1, \\ p, & q = p-1, \end{cases} \tag{4.17}$$

while in the cases $p = 1$ and $p = 2$, the sign e is ambiguous. Nonetheless, in the latter cases, (4.16) must hold with either $e = p$ *or* $e = -p$.

If equality holds in (4.16), then M is hyper-Kähler, and by the classification of hyper-Kähler ALF instantons, M is a triple-Taub–NUT instanton (see [30]). The triple-Taub–NUT instanton has Euler characteristic equal to p, which implies that $p = 3$, and it has an even intersection form, that is, $d_1 = a$ and $d_2 = b$ are even.

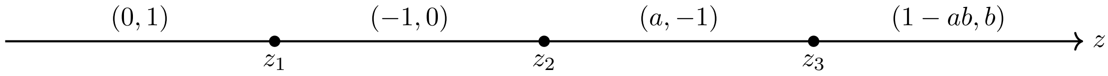
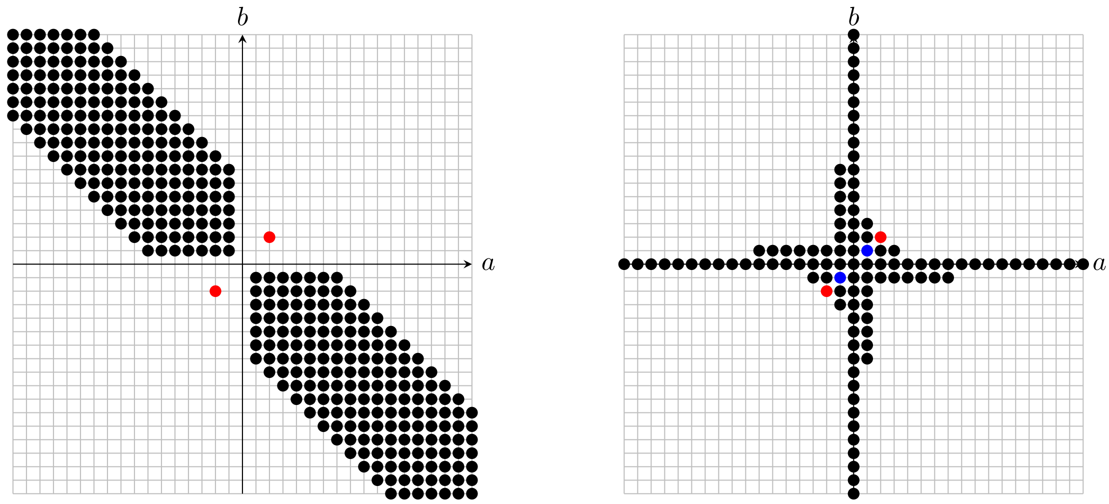


Figure 4.1: A rod structure with three turning points.



(a) ALE

(b) ALF

Figure 4.2: The remaining possibilities for (a,b).

Again, the preceding statements restrict the possible values of a and b. Using the same method to compute the signature $\tau(M)$ as before, systematically checking all pairs (a,b) with $|a|,|b| \leq 17$, one arrives at Figure 4.2b. As for the ALE case, the pairs $(a,b) = \pm(2,2)$ (marked with red) are the pairs for which exact equality holds in (4.16). They correspond to two distinct T^2-actions on the triple-Taub–NUT instanton. Furthermore, the pairs given by $(a,b) = \pm(1,1)$ (marked with blue) correspond to two distinct T^2-actions on the Chen–Teo instanton.

Theorem 9. *Let (M,g) be a toric ALE instanton with the rod structure shown in Figure 4.1. Then either $(a,b) = \pm(2,2)$, or $ab < 0$ and $|a+b| \leq 6$ (as depicted[3] in Figure 4.2a). If instead M is ALF, then either $a = 0$, $b = 0$, or (a,b) is one of the pairs in Figure 4.2b.*

Proof. For the ALF statement, (4.16) implies that

$$6 \geq |e| - 3|\tau(M) + \operatorname{sgn}(e)| \geq |e| - 9, \tag{4.18}$$

so that

$$|ab| \leq 1 + |1 - ab| = 1 + |e| \leq 16. \tag{4.19}$$

Thus, either $|a|,|b| \leq 16$, in which case (a,b) is one of the points in Figure 4.2b, or either a or b vanishes.

We now prove the ALE statement. Since the transformation $(a,b) \mapsto (-a,-b)$ corresponds to $(p,q) \mapsto (p, p-q)$, and since the latter corresponds to an orientation reversal of the lens space $L(p,q)$, it follows that $(a,b) \mapsto (-a,-b)$ amounts to a sign reversal of $\eta_S(L(p,q))$. The

[3] The meaning of the different colors will be explained in Section 4.3.

transformation $(a, b) \mapsto (-a, -b)$ also leads to a sign reversal of $\tau(M)$, and thus preserves the inequality (4.15). Since the conclusion of the theorem is also preserved under this transformation, we may assume without loss of generality that $a \leq 0$. Moreover, since $p > 1$, we must in fact have $a < 0$ and $b \neq 0$.

First assume that $b > 0$; then $p = 1 - ab$, and since $0 < b < 1 - ab$, we have $q = b$. For every integer k with $1 \leq k \leq q - 1$, we have

$$\left\lfloor \frac{kp}{q} \right\rfloor = \left\lfloor \frac{k}{b} - ka \right\rfloor = -ka, \tag{4.20}$$

and by (4.13), this implies that

$$\eta_S(L(p, q)) = -\frac{ab(a+b)}{3(1-ab)}. \tag{4.21}$$

Since the constant coefficient of the polynomial $\lambda^2 - (a+b)\lambda + ab - 1$ is negative, the roots have different signs, which means that $\tau(M) = 0$, and (4.15) now implies that

$$6 - \frac{2}{1-ab} \geq \frac{|ab(a+b)|}{1-ab}, \tag{4.22}$$

or equivalently,

$$(|a+b| - 6)|ab| \leq 4. \tag{4.23}$$

This means that either $|a + b| \leq 6$, in which case we are done, or $|ab| \leq 4$. In the latter case,

$$(a+b)^2 = a^2 + b^2 + 2ab \leq \frac{16}{b^2} + \frac{16}{a^2} + 8 \leq 40, \tag{4.24}$$

which also implies that $|a + b| \leq 6$.

Consider now the case $a, b < -1$, for which $p = ab - 1$ and $q = -b$. For $1 \leq k \leq q - 1$ we have

$$\left\lfloor \frac{kp}{q} \right\rfloor = \left\lfloor \frac{k}{b} - ka \right\rfloor = -ka - 1, \tag{4.25}$$

so (4.13) implies that

$$\eta_S(L(p, q)) = \frac{a^2 b + ab^2}{3(ab-1)} + 2. \tag{4.26}$$

One readily verifies that $\tau(M) = -2$, and thus (4.15) implies that

$$6 - \frac{2}{ab-1} \geq \frac{|a^2 b + ab^2|}{ab-1}, \tag{4.27}$$

or equivalently,

$$6 - \frac{8}{ab} \geq |a+b|. \tag{4.28}$$

In particular, $|a + b| < 6$, so that (a, b) is either $(-2, -2)$, $(-2, -3)$ or $(-3, -2)$. As the latter two do not satisfy (4.28), we must have $(a, b) = (-2, -2)$.

In the case $a = -1$, $b < -1$, we have $\tau = -2$, $p = -b - 1$ and $q = 1$, and

$$\eta_S(L(p, q)) = \frac{(b+2)(b+3)}{3(b+1)}, \tag{4.29}$$

so (4.15) implies that

$$-6b - 8 \geq |(1-b)b|, \tag{4.30}$$

which is a contradiction. The case $a < -1$, $b = -1$ is almost the same, the only difference being that a occurs in place of b in the expressions for p and $\eta_S(L(p, q))$. $\square$

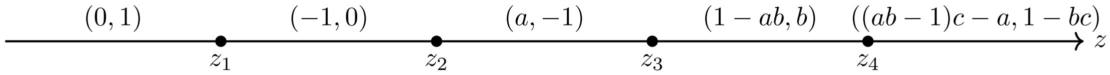


Figure 4.3: A general rod structure with four turning points.

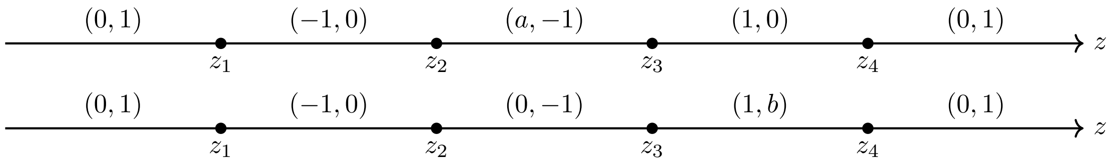


Figure 4.4: Two families of AF rod structures with four turning points.

Remark 5. One can also look at rod structure with four turning points. As in the case with three turning points, we can assume that $v_0 = (0,1), v_1 = (-1,0)$. Such a rod structure will have the form shown in Figure 4.3, with three integer parameters a, b, c. Restricting attention to the AF case, we are left with two families of rod structures, shown in Figure 4.4, along with four exceptional rod structures, shown in Figure 4.5. For AF rod structures with four turning points, it then turns out that the Hitchin–Thorpe inequality is always satisfied with strict inequality. Thus, we cannot rule out any of these rod structures.

Since the inequalities are strict, we can at least conclude that a simply connected AF toric gravitational instanton with four turning points cannot be hyper-Kähler. However, this is already known: by the classification of ALF-A_k hyper-Kähler gravitational instantons, any AF hyper-Kähler gravitational instanton must be a product of $\mathbb{R}^3$ with a circle, which is not simply connected.

In other words, the Hitchin–Thorpe inequality gives no information about AF rod structures with four turning points.

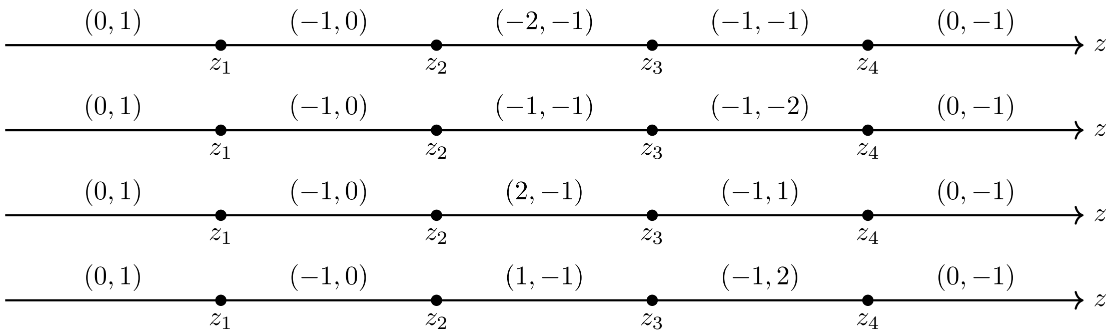


Figure 4.5: Four exceptional AF rod structures with four turning points.

Appendix A

Newman–Penrose Equations

Given an equation expressed in terms of the spin coefficients, Weyl scalars and tetrad derivative operators, we can apply the *tilde operation*, given by formally replacing any such quantity x by the tilded quantity $\tilde{x}$. Here, we adopt the convention that $\tilde{\tilde{x}} = x$, $\tilde{D} = D$ and $\tilde{\Delta} = \Delta$. The result is a new, a priori independent equation, the *tilded version* of the original equation.

We have the Newman–Penrose commutation relations, given by the four equations

$$
\begin{aligned}
[D, \Delta]\psi &= -(\gamma + \tilde{\gamma})D\psi - (\epsilon + \tilde{\epsilon})\Delta\psi + (\pi + \tilde{\tau})\delta\psi + (\tilde{\pi} + \tau)\tilde{\delta}\psi, && \text{(A.1)}\\
[D, \delta]\psi &= -(\tilde{\alpha} + \beta - \tilde{\pi})D\psi - \kappa\Delta\psi + (\epsilon - \tilde{\epsilon} + \tilde{\rho})\delta\psi + \sigma\tilde{\delta}\psi, && \text{(A.2)}\\
[\Delta, \delta]\psi &= \tilde{\nu}D\psi + (\tilde{\alpha} + \beta - \tau)\Delta\psi + (\gamma - \tilde{\gamma} - \mu)\delta\psi - \tilde{\lambda}\tilde{\delta}\psi, && \text{(A.3)}\\
[\delta, \tilde{\delta}]\psi &= (\mu - \tilde{\mu})D\psi + (\rho - \tilde{\rho})\Delta\psi + (-\alpha + \tilde{\beta})\delta\psi + (\tilde{\alpha} - \beta)\tilde{\delta}\psi, && \text{(A.4)}
\end{aligned}
$$

together with their tilded versions.[1]

In terms of the spin coefficients, the vacuum Einstein equations become

$$
\begin{aligned}
D\alpha - \tilde{\delta}\epsilon &= -\tilde{\beta}\epsilon - \gamma\tilde{\kappa} - \kappa\lambda + \pi(\epsilon + \rho) + \alpha(-2\epsilon + \tilde{\epsilon} + \rho) + \beta\tilde{\sigma}, && \text{(A.5)}\\
D\beta - \delta\epsilon &= \Psi_1 - \tilde{\alpha}\epsilon - \kappa(\gamma + \mu) + \epsilon\tilde{\pi} + \beta(-\tilde{\epsilon} + \tilde{\rho}) + (\alpha + \pi)\sigma, && \text{(A.6)}\\
D\gamma - \Delta\epsilon &= \Psi_2 - \tilde{\gamma}\epsilon - \gamma(2\epsilon + \tilde{\epsilon}) - \kappa\nu + \pi(\beta + \tau) + \alpha(\tilde{\pi} + \tau) + \beta\tilde{\tau}, && \text{(A.7)}\\
D\lambda - \tilde{\delta}\pi &= -\tilde{\kappa}\nu + \pi(\alpha - \tilde{\beta} + \pi) + \lambda(-3\epsilon + \tilde{\epsilon} + \rho) + \mu\tilde{\sigma}, && \text{(A.8)}\\
D\rho - \tilde{\delta}\kappa &= \kappa(-3\alpha - \tilde{\beta} + \pi) + \rho(\epsilon + \tilde{\epsilon} + \rho) + \sigma\tilde{\sigma} - \tilde{\kappa}\tau, && \text{(A.9)}\\
D\sigma - \delta\kappa &= \Psi_0 + (3\epsilon - \tilde{\epsilon} + \rho + \tilde{\rho})\sigma - \kappa(\tilde{\alpha} + 3\beta - \tilde{\pi} + \tau), && \text{(A.10)}\\
D\tau - \Delta\kappa &= \Psi_1 - (3\gamma + \tilde{\gamma})\kappa + \tilde{\pi}\rho + (\epsilon - \tilde{\epsilon} + \rho)\tau + \sigma(\pi + \tilde{\tau}), && \text{(A.11)}\\
\Delta\rho - \tilde{\delta}\tau &= -\Psi_2 + \kappa\nu + (\gamma + \tilde{\gamma} - \tilde{\mu})\rho - \lambda\sigma - \tau(\alpha - \tilde{\beta} + \tilde{\tau}), && \text{(A.12)}\\
\delta\alpha - \tilde{\delta}\beta &= -\Psi_2 + \alpha(\tilde{\alpha} - 2\beta) + \beta\tilde{\beta} + \epsilon(\mu - \tilde{\mu}) + (\gamma + \mu)\rho - \gamma\tilde{\rho} - \lambda\sigma, && \text{(A.13)}\\
\delta\rho - \tilde{\delta}\sigma &= -\Psi_1 + \kappa(\mu - \tilde{\mu}) + (\tilde{\alpha} + \beta)\rho + (-3\alpha + \tilde{\beta})\sigma + (\rho - \tilde{\rho})\tau. && \text{(A.14)}
\end{aligned}
$$

together with their tilded versions.

[1] For the first and last equation, applying the tilde operation results in the same equations, up to sign. For the second and third equations, it results in two new, independent equations.

Finally, we have the Bianchi identities, given by the equations

$$\tilde{\delta}\Psi_0 - D\Psi_1 = (4\alpha - \pi)\Psi_0 - 2(\epsilon + 2\rho)\Psi_1 + 3\kappa\Psi_2, \tag{A.15}$$

$$\tilde{\delta}\Psi_1 - D\Psi_2 = \lambda\Psi_0 + 2(\alpha - \pi)\Psi_1 - 3\rho\Psi_2 + 2\kappa\Psi_3, \tag{A.16}$$

$$\Delta\Psi_0 - \delta\Psi_1 = 4\gamma\Psi_0 - \mu\Psi_0 - 2(\beta + 2\tau)\Psi_1 + 3\sigma\Psi_2, \tag{A.17}$$

$$\Delta\Psi_1 - \delta\Psi_2 = \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3, \tag{A.18}$$

together with their tilded versions.

Appendix B

Spin coefficients

B.1 Kerr

$$\alpha = \beta = \frac{r\cos\theta - a}{(r - a\cos\theta)2\sqrt{2\Sigma}\sin\theta}, \tag{B.1}$$

$$\gamma = \epsilon = \frac{i(-\Delta/(r - a\cos\theta) + r - M)}{2\sqrt{2\Delta\Sigma}}, \tag{B.2}$$

$$\mu = \rho = -\frac{i\sqrt{\Delta/2\Sigma}}{r - a\cos\theta}, \tag{B.3}$$

$$\pi = \tau = -\frac{a\sin\theta}{(r - a\cos\theta)\sqrt{2\Sigma}}, \tag{B.4}$$

$$\tilde{\alpha} = \tilde{\beta} = -\frac{r\cos\theta + a}{(r + a\cos\theta)2\sqrt{2\Sigma}\sin\theta}, \tag{B.5}$$

$$\tilde{\gamma} = \tilde{\epsilon} = \frac{i(-\Delta/(r + a\cos\theta) + r - M)}{2\sqrt{2\Delta\Sigma}}, \tag{B.6}$$

$$\tilde{\mu} = \tilde{\rho} = -\frac{i\sqrt{\Delta/2\Sigma}}{r + a\cos\theta}, \tag{B.7}$$

$$\tilde{\pi} = \tilde{\tau} = -\frac{a\sin\theta}{(r + a\cos\theta)\sqrt{2\Sigma}}, \tag{B.8}$$

$$\kappa = \lambda = \nu = \sigma = \tilde{\kappa} = \tilde{\lambda} = \tilde{\nu} = \tilde{\sigma} = 0. \tag{B.9}$$

B.2 Taub-bolt

$$\alpha = \beta = \frac{N(r+N)^2}{8\Sigma\sqrt{2\Delta\Sigma}}, \tag{B.10}$$

$$\gamma = \epsilon = \tilde{\gamma} = \tilde{\epsilon} = \frac{i\cot\theta}{2\sqrt{2\Sigma}}, \tag{B.11}$$

$$\pi = \tau = -\frac{(r+N)\sqrt{\Delta/2\Sigma}}{\Sigma}, \tag{B.12}$$

$$\tilde{\alpha} = \tilde{\beta} = -\frac{9N(r-N)}{8(r+N)\sqrt{\Delta\Sigma}}, \tag{B.13}$$

$$\tilde{\pi} = \tilde{\tau} = \frac{\sqrt{\Delta/2\Sigma}}{r+N}, \tag{B.14}$$

$$\kappa = \lambda = \mu = \nu = \rho = \sigma = \tilde{\kappa} = \tilde{\lambda} = \tilde{\mu} = \tilde{\nu} = \tilde{\rho} = \tilde{\sigma} = 0. \tag{B.15}$$

Bibliography

[1] Steffen Aksteiner, Lars Andersson, and Thomas Bäckdahl. „New identities for linearized gravity on the Kerr spacetime“. In: *Phys. Rev. D* 99.4 (2019), pp. 044043, 19. ISSN: 2470-0010. DOI: 10.1103/physrevd.99.044043. URL: https://doi.org/10.1103/physrevd.99.044043.

[2] Steffen Aksteiner et al. *Gravitational instantons with S^1 symmetry*. 2023. arXiv: 2306.14567 [math.DG]. URL: https://arxiv.org/abs/2306.14567.

[3] A. N. Aliev and Y. Nutku. „Gravitational instantons admit hyper-Kähler structure“. In: *Classical Quantum Gravity* 16.1 (1999), pp. 189–210. ISSN: 0264-9381. DOI: 10.1088/0264-9381/16/1/013. URL: https://doi.org/10.1088/0264-9381/16/1/013.

[4] Lars Andersson, Dietrich Häfner, and Bernard F. Whiting. *Mode analysis for the linearized Einstein equations on the Kerr metric : the large $\mathfrak{a}$ case*. 2022. arXiv: 2207.12952 [math.AP].

[5] Lars Andersson et al. „Mode stability on the real axis“. In: *Journal of Mathematical Physics* 58.7 (July 2017). DOI: 10.1063/1.4991656. URL: https://doi.org/10.1063%2F1.4991656.

[6] V. Belinski and E. Verdaguer. *Gravitational solitons*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, 2001, pp. xiv+258. ISBN: 0-521-80586-4. DOI: 10.1017/CBO9780511535253. URL: https://doi.org/10.1017/CBO9780511535253.

[7] Arthur L. Besse. *Einstein manifolds*. Vol. 10. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1987, pp. xii+510. ISBN: 3-540-15279-2. DOI: 10.1007/978-3-540-74311-8. URL: https://doi.org/10.1007/978-3-540-74311-8.

[8] Olivier Biquard and Paul Gauduchon. „On Toric Hermitian ALF Gravitational Instantons“. In: *Communications in Mathematical Physics* 399.1 (Nov. 2022), pp. 389–422. DOI: 10.1007/s00220-022-04562-z. URL: https://doi.org/10.1007%2Fs00220-022-04562-z.

[9] Olivier Biquard, Paul Gauduchon, and Claude LeBrun. *Gravitational Instantons, Weyl Curvature, and Conformally Kaehler Geometry*. 2023. arXiv: 2310.14387 [math.DG].

[10] Gao Chen and Xiuxiong Chen. „Gravitational instantons with faster than quadratic curvature decay. I“. In: *Acta Mathematica* 227.2 (2021), pp. 263–307. DOI: 10.4310/acta.2021.v227.n2.a2. URL: https://doi.org/10.4310%2Facta.2021.v227.n2.a2.

[11] Xiuxiong Chen and Yu Li. „On the geometry of asymptotically flat manifolds“. In: *Geometry & Topology* 25.5 (Sept. 2021), pp. 2469–2572. DOI: 10.2140/gt.2021.25.2469. URL: https://doi.org/10.2140%2Fgt.2021.25.2469.

[12] Yu Chen. „Gravitational multisoliton solutions on flat space“. In: *Physical Review D* 93.4 (Feb. 2016). ISSN: 2470-0029. DOI: 10.1103/physrevd.93.044021. URL: http://dx.doi.org/10.1103/PhysRevD.93.044021.

[13] Yu Chen and Edward Teo. „A new AF gravitational instanton“. In: *Phys. Lett. B* 703.3 (2011), pp. 359–362. ISSN: 0370-2693. DOI: `10.1016/j.physletb.2011.07.076`. URL: `https://doi.org/10.1016/j.physletb.2011.07.076`.

[14] Yu Chen and Edward Teo. „Rod-structure classification of gravitational instantons with $U(1)\times U(1)$ isometry“. In: *Nuclear Phys. B* 838.1-2 (2010), pp. 207–237. ISSN: 0550-3213. DOI: `10.1016/j.nuclphysb.2010.05.017`. URL: `https://doi.org/10.1016/j.nuclphysb.2010.05.017`.

[15] Margaret Doig and Stephan Wehrli. *A combinatorial proof of the homology cobordism classification of lens spaces.* 2015. arXiv: `1505.06970 [math.GT]`.

[16] Tohru Eguchi and Andrew J. Hanson. „Self-dual solutions to Euclidean gravity“. In: *Ann. Physics* 120.1 (1979), pp. 82–106. ISSN: 0003-4916,1096-035X. DOI: `10.1016/0003-4916(79)90282-3`. URL: `https://doi.org/10.1016/0003-4916(79)90282-3`.

[17] A. Erdélyi. *Asymptotic expansions.* Dover Publications, Inc., New York, 1956, pp. vi+108.

[18] G. W. Gibbons and S. W. Hawking. „Classification of gravitational instanton symmetries“. In: *Comm. Math. Phys.* 66.3 (1979), pp. 291–310. ISSN: 0010-3616. URL: `http://projecteuclid.org/euclid.cmp/1103905051`.

[19] E. Goldblatt. „A Newman-Penrose formalism for gravitational instantons“. In: *Gen. Relativity Gravitation* 26.10 (1994), pp. 979–997. ISSN: 0001-7701. DOI: `10.1007/BF02106666`. URL: `https://doi.org/10.1007/BF02106666`.

[20] A. Rod Gover, C. Denson Hill, and Paweł Nurowski. „Sharp version of the Goldberg-Sachs theorem“. In: *Ann. Mat. Pura Appl. (4)* 190.2 (2011), pp. 295–340. ISSN: 0373-3114,1618-1891. DOI: `10.1007/s10231-010-0151-4`. URL: `https://doi.org/10.1007/s10231-010-0151-4`.

[21] Troels Harmark. „Stationary and axisymmetric solutions of higher-dimensional general relativity“. In: *Phys. Rev. D (3)* 70.12 (2004), pp. 124002, 25. ISSN: 0556-2821. DOI: `10.1103/PhysRevD.70.124002`. URL: `https://doi.org/10.1103/PhysRevD.70.124002`.

[22] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002, pp. xii+544. ISBN: 0-521-79160-X.

[23] S. W. Hawking. „Gravitational instantons“. In: *Phys. Lett. A* 60.2 (1977), pp. 81–83. ISSN: 0375-9601.

[24] Stefan Hollands and Stoytcho Yazadjiev. „A Uniqueness Theorem for Stationary Kaluza-Klein Black Holes“. In: *Communications in Mathematical Physics* 302.3 (Jan. 2011), pp. 631–674. ISSN: 1432-0916. DOI: `10.1007/s00220-010-1176-7`. URL: `http://dx.doi.org/10.1007/s00220-010-1176-7`.

[25] Stefan Hollands and Stoytcho Yazadjiev. „Uniqueness theorem for 5-dimensional black holes with two axial Killing fields“. In: *Comm. Math. Phys.* 283.3 (2008), pp. 749–768. ISSN: 0010-3616. DOI: `10.1007/s00220-008-0516-3`. URL: `https://doi.org/10.1007/s00220-008-0516-3`.

[26] Kiyoshi Katase. „On the value of Dedekind sums and eta-invariants for 3-dimensional lens spaces“. In: *Tokyo J. Math.* 10.2 (1987), pp. 327–347. ISSN: 0387-3870. DOI: `10.3836/tjm/1270134517`. URL: `https://doi.org/10.3836/tjm/1270134517`.

[27] P. B. Kronheimer. „A Torelli-type theorem for gravitational instantons“. In: *J. Differential Geom.* 29.3 (1989), pp. 685–697. ISSN: 0022-040X,1945-743X. URL: `http://projecteuclid.org/euclid.jdg/1214443067`.

[28] Hari K. Kunduri and James Lucietti. „Existence and uniqueness of asymptotically flat toric gravitational instantons“. In: *Lett. Math. Phys.* 111.5 (2021), Paper No. 133, 40. ISSN: 0377-9017,1573-0530. DOI: 10.1007/s11005-021-01475-1. URL: https://doi.org/10.1007/s11005-021-01475-1.

[29] John M. Lee. *Introduction to smooth manifolds.* Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708. ISBN: 978-1-4419-9981-8.

[30] Vincent Minerbe. „Rigidity for multi-Taub-NUT metrics“. In: *J. Reine Angew. Math.* 656 (2011), pp. 47–58. ISSN: 0075-4102. DOI: 10.1515/CRELLE.2011.042. URL: https://doi.org/10.1515/CRELLE.2011.042.

[31] Hiraku Nakajima. „Self-duality of ALE Ricci-flat 4-manifolds and positive mass theorem“. In: *Recent topics in differential and analytic geometry.* Vol. 18. Adv. Stud. Pure Math. Academic Press, Boston, MA, 1990, pp. 385–396. DOI: 10.2969/aspm/01810385. URL: https://doi.org/10.2969/aspm/01810385.

[32] Ezra Newman and Roger Penrose. „An approach to gravitational radiation by a method of spin coefficients“. In: *J. Mathematical Phys.* 3 (1962), pp. 566–578. ISSN: 0022-2488. DOI: 10.1063/1.1724257. URL: https://doi.org/10.1063/1.1724257.

[33] Gustav Nilsson. „Mode stability for gravitational instantons of type D“. In: *Classical and Quantum Gravity* 41.8 (Mar. 2024), p. 085004. DOI: 10.1088/1361-6382/ad296f. URL: https://dx.doi.org/10.1088/1361-6382/ad296f.

[34] Gustav Nilsson. „Topology of toric gravitational instantons“. In: *Differential Geometry and its Applications* 96 (2024), p. 102171. ISSN: 0926-2245. DOI: 10.1016/j.difgeo.2024.102171. URL: https://www.sciencedirect.com/science/article/pii/S0926224524000640.

[35] Peter Orlik. *Seifert manifolds.* Vol. Vol. 291. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1972, pp. viii+155.

[36] Roger Penrose and Wolfgang Rindler. *Spinors and space-time. Vol. 2.* Cambridge Monographs on Mathematical Physics. Spinor and twistor methods in space-time geometry. Cambridge University Press, Cambridge, 1986, pp. x+501. ISBN: 0-521-25267-9. DOI: 10.1017/CBO9780511524486. URL: https://doi.org/10.1017/CBO9780511524486.

[37] Richard M. Schoen. „Variational theory for the total scalar curvature functional for Riemannian metrics and related topics“. In: *Topics in calculus of variations (Montecatini Terme, 1987).* Vol. 1365. Lecture Notes in Math. Springer, Berlin, 1989, pp. 120–154. ISBN: 3-540-50727-2. DOI: 10.1007/BFb0089180. URL: https://doi.org/10.1007/BFb0089180.

[38] Sergei Yu. Slavyanov and Wolfgang Lay. *Special functions.* Oxford Mathematical Monographs. A unified theory based on singularities, With a foreword by Alfred Seeger, Oxford Science Publications. Oxford University Press, Oxford, 2000, pp. xvi+293. ISBN: 0-19-850573-6.

[39] D. C. Spencer and S. Iyanaga, eds. *Global analysis. Papers in honor of K. Kodaira.* University of Tokyo Press, Tokyo; Princeton University Press, Princeton, NJ, 1969, v+414 pp. (1 plate).

[40] Saul A. Teukolsky. „Perturbations of a Rotating Black Hole. I. Fundamental Equations for Gravitational, Electromagnetic, and Neutrino-Field Perturbations“. In: *Astrophysical Journal* 185 (Oct. 1973), pp. 635–648. DOI: 10.1086/152444.

[41] Robert M. Wald. „On perturbations of a Kerr black hole“. In: *Journal of Mathematical Physics* 14.10 (Nov. 2003), pp. 1453–1461. ISSN: 0022-2488. DOI: 10.1063/1.1666203. eprint: https://pubs.aip.org/aip/jmp/article-pdf/14/10/1453/8146257/1453_1_online.pdf. URL: https://doi.org/10.1063/1.1666203.

[42] Bernard F. Whiting. „Mode stability of the Kerr black hole“. In: *J. Math. Phys.* 30.6 (1989), pp. 1301–1305. ISSN: 0022-2488,1089-7658. DOI: 10.1063/1.528308. URL: https://doi.org/10.1063/1.528308.

[43] Michael Wiemeler. „Smooth classification of locally standard T^k-manifolds“. In: *Osaka J. Math.* 59.3 (2022), pp. 549–557. ISSN: 0030-6126.