

SIMILARITY RENORMALIZATION GROUP APPROACH TO BOOST INVARIANT HAMILTONIAN DYNAMICS

STANISŁAW D. GŁĄZEK

Institute of Theoretical Physics, Warsaw University
Hoża 69, 00-681 Warsaw

(Received January 6, 1998)

We outline a method of deriving boost invariant dynamics for effective particles in quantum field theory. The method is based on the similarity renormalization group transformation for Hamiltonians in the canonical light-front quantization scheme. The Hamiltonians are defined and calculated using creation and annihilation operators. The renormalization group equations are written for a sequence of unitary transformations which gradually transform the bare canonical creation and annihilation operators of a local theory to the creation and annihilation operators of effective particles in an effective theory with the same dynamical content but a finite range of energy transfers due to form factors in the interaction vertices. The form factors result from the similarity renormalization group flow of effective Hamiltonians. The regularized initial Hamiltonian and the renormalized effective Hamiltonians possess seven kinematical Poincaré symmetries specific to the light-front quantization scheme. Thus, the effective interactions can be used to describe the constituent dynamics in relativistically moving systems including the rest and the infinite momentum frame. Solutions to the general equations for the effective Hamiltonians are illustrated in perturbation theory by second-order calculations of self-energy and two-particle interaction terms in Yukawa theory, QED and QCD. In Yukawa theory, one obtains the generalized Yukawa potential including its full off-energy-shell extension and form factors in the vertices. In QED, the effective Hamiltonian eigenvalue problem converges for small coupling constants to the Schrödinger equation but the typical relativistic ultraviolet singularities at short distances between constituents are regularized by the similarity form factors. In the second-order QCD effective Hamiltonian one obtains a boost invariant logarithmically confining quark-anti-quark potential which may remain uncanceled in the non-abelian dynamics of effective quarks and gluons.

PACS numbers: 11.10. Gh

1. Introduction

This paper describes a boost invariant construction of effective Hamiltonians in quantum field theory. Physical states are assumed to be describable by solutions to the Schrödinger equation with these Hamiltonians.

The Hamiltonians are derived by integrating first-order differential equations of the similarity renormalization group. The initial condition is provided by the regularized canonical light-front Hamiltonian of a local quantum field theory with counterterms. The Hamiltonian acts in the Fock space which is constructed by applying the canonical bare creation operators to a vacuum state. The vacuum is annihilated by the corresponding annihilation operators. Due to the light-front boost symmetry, this representation of states is useful in a relativistic theory and the dynamics in any boosted frame resembles dynamics in the infinite momentum frame.

The similarity renormalization group is defined in terms of running creation and annihilation operators. The running operators interpolate between the bare ones in a local Hamiltonian and effective ones in the effective Hamiltonian. The effective operators are used for construction of the effective basis states in the Fock space. The effective Hamiltonians are calculable term by term using methods of successive approximations and perturbation theory. One has to study the role of different terms in the effective Schrödinger equation. Solutions include bound states.

One reason for the renormalization group to play an important role in the Hamiltonian approach is ultraviolet divergences; the initial expressions for bare Hamiltonians contain divergences of local field theory and the divergences require renormalization. The most prominent example is the canonical Hamiltonian of QED. Old-fashioned tree diagrams of Hamiltonian perturbation theory are finite and closely reproduce experimental data. Renormalization problems appear when one sums over intermediate states and the sum diverges. The divergences correspond to diverging loop integrals in Feynman diagrams. However, the Hamiltonian approach greatly differs from the Lagrangian diagrammatic approaches.

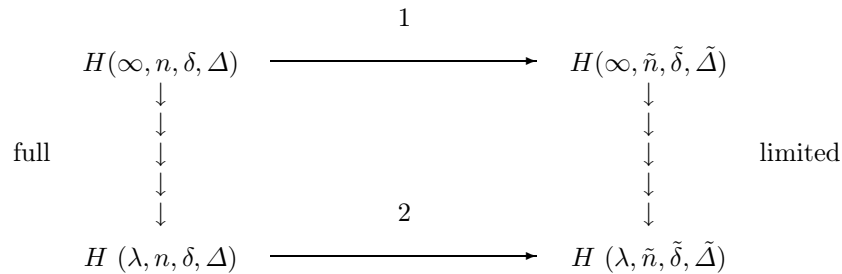
One apparent difference is that the sums over intermediate states involve integrals over a three-dimensional momentum space while the integrals in the Lagrangian calculus are four-dimensional. Although a connection exists for finite integrals which are not sensitive to cutoffs, the connection is broken when the integrals diverge. In the Hamiltonian approach, we introduce the effective Fock space basis and we construct Hamiltonians using three-dimensional regularization and renormalization procedures. In the Lagrangian approach, one directly calculates Green functions using four-dimensional regularization and renormalization techniques. Equivalence of the two approaches in diverging cases remains to be shown, especially when the bound states are taken into account.

Besides removing divergences, the renormalization group is useful in the Hamiltonian approach because it introduces a hierarchy of scales. Phenomena of different scales are dealt with in a certain order. This enables us to solve problems involving many scales. Particle theories contain many, possibly even infinitely many different scales. Using the renormalization group approach, we can start from the Hamiltonian of a basic theory that couples all degrees of freedom of all scales and we can reduce the initial Hamiltonian to an effective renormalized Hamiltonian in which the couplings between degrees of freedom of vastly different scales vanish. Then, the couplings between different scales are further reduced in the renormalization group flow to obtain the effective Hamiltonian matrix which is sufficiently narrow in scale so that its spectrum of eigenstates can be found in practice. The similarity transformation is designed to eliminate all large changes of scale by the effective interactions.

The different scales in the Hamiltonian renormalization group approach are defined by different scales of momentum. The momentum scales are defined using the relative momenta of interacting particles. The definitions will be given in Section 2. Here we need to mention that the effective Hamiltonian of a small width contains interactions which couple particles of similar energies only. The energy changes induced by the Hamiltonian are limited by the running similarity renormalization group cutoff, denoted by λ . The smaller is λ the smaller is the energy width of the effective Hamiltonian. Exact results for physical quantities are independent of λ .

A number of model subspaces of the Fock space need to be considered when one is solving for the spectrum of a field-theoretic effective Hamiltonian because the full space of states is too large for computations. Different physical problems require different model subspaces. Working within a subspace of interest, one should secure that the results for physical quantities are independent of the running cutoff λ . The cutoff independence can appear only in a certain range of cutoffs that corresponds to the physical problem and model subspace under consideration. However, once the cutoff independence in the finite range is achieved, one expects to have solved the theory in this range.

The following diagram will illustrate the situation.



In this diagram, the vertical arrows indicate evolution in the renormalization group parameter λ which limits the relative energy transfers in the interaction terms. It will be explained in detail in next Sections how the limits are imposed. λ ranges from infinity in the initial Hamiltonian to a finite value in an effective Hamiltonian. The Hamiltonians depend on additional parameters n , δ and Δ .

n stands for the cutoff on the *change* of the particle number. It defines the limits on the numbers of creation and annihilation operators that can appear in a single Hamiltonian term. For example, the canonical expressions for light-front Hamiltonians in local field theories of physical interest have the number of creation and annihilation operators in a single term limited to 4, and the particle number cannot change by more than $n = 2$.

δ stands for the infrared cutoff. For example, it may be the lower bound on the longitudinal momentum carried by a particle that appears or disappears in a single interaction.

Δ stands for the ultraviolet cutoff which defines the upper limit on the relative transverse momentum of particles which can appear or disappear in a single interaction.

The left branch of the diagram is marked “full” because it represents the renormalization group flow calculated using the effective creation and annihilation operators with no restriction imposed on the space of states.

The right branch of the figure is marked “limited” because it describes the renormalization group flow in the bare model space which is limited by parameters $\tilde{n}$, $\tilde{\delta}$ and $\tilde{\Delta}$. Imposing the limits is denoted by the arrow marked 1. For example, $\tilde{n}$ can limit the number of bare particles, $\tilde{\delta}$ can limit the bare particle momenta from below and $\tilde{\Delta}$ can limit from above free energies of the states which are taken into account.

The initial $H(\infty, n, \delta, \Delta)$ contains counterterms which are constructed using the condition that physical results have well defined limits when the cutoffs n , δ and Δ are relaxed. The construction of counterterms in perturbation theory will be discussed in detail in the next Sections. Once the counterterms remove the regularization dependence from the effective dynamics the arguments n , δ and Δ in $H(\lambda, n, \delta, \Delta)$ in the lower left corner of the diagram are equivalent to their limiting values, $n = \infty$, $\delta = 0$ and $\Delta = \infty$. Thus, $H(\lambda, n, \delta, \Delta) \equiv H(\lambda)$. One should stress that the infrared regulator δ may still appear in the effective Hamiltonian if there are massless particles in the theory. This is important in gauge theories.

Accuracy of the step denoted by the arrow 1 has to be checked by relaxing the model cutoff parameters $\tilde{n}$, $\tilde{\delta}$ and $\tilde{\Delta}$ and measuring the resulting changes in the spectrum of $H(\lambda, \tilde{n}, \tilde{\delta}, \tilde{\Delta})$. Naturally, these cutoffs may have to be varied in a big range because they are introduced along the arrow 1 for bare particles.

The effective Hamiltonians at the bottom of the diagram, namely, $H(\lambda, n, \delta, \Delta)$ and $H(\lambda, \tilde{n}, \tilde{\delta}, \tilde{\Delta})$ with energy transfers limited by finite λ , are connected by the arrow marked 2. This arrow denotes the procedure of introducing the small space cutoffs $\tilde{n}$, $\tilde{\delta}$ and $\tilde{\Delta}$ which enable us to approximately solve for the spectrum of the effective Hamiltonian $H(\lambda, n, \delta, \Delta)$. This time, however, the final computation cutoffs are introduced at the level of the effective particles, not at the level of the initial bare particles.

The arrow 2 denotes the replacement of the whole effective Hamiltonian matrix by a limited matrix. The procedure of obtaining the small matrix will be discussed below. The spectrum of the small matrix may be very close to the corresponding part of the spectrum of the full matrix because λ is small (this will become clear later). The accuracy of the calculation must be verified by relaxing the cutoffs $\tilde{n}$, $\tilde{\delta}$ and $\tilde{\Delta}$ and observing convergence of results as in the case of the arrow 1 and branch “limited”. But now, it is natural to expect that the cutoffs $\tilde{n}$, $\tilde{\delta}$ and $\tilde{\Delta}$ may have to be varied only in a small range which corresponds to λ . Thus, a finite dynamical problem to solve is defined.

The “full” renormalization group evolution is calculable using the method described in this paper. The “limited” evolution can be calculated using the matrix elements techniques introduced earlier by Głazek and Wilson in Refs. [1] and [2] who drew on the work of Wilson [3,4]. The matrix elements techniques were introduced for application to QCD [5]. Alternatively, one can adopt Wegner’s flow equations for Hamiltonian matrix elements in cases soluble with the energy-independent width λ [6,7]. The present approach can be viewed as a special case of the general similarity renormalization group for Hamiltonians because the Hamiltonians we consider transform by the same unitary transformations as our creation and annihilation operators. However, by having defined the renormalization group transformation for the effective creation and annihilation operators, we remove the need to consider the model space dependence of the renormalization group transformation.

The transformation we describe in this paper is partly similar to the transformation discussed by Melosh [8]. The important difference is that we provide a dynamical theory of the transformation in a form applicable to particles of different kinds. If one restricts attention to QCD, the boost invariant calculus is expected to help in establishing a connection between the constituent quark model, Feynman parton model, and perturbative quantum chromodynamics.

Both ways in the diagram which start from the initial Hamiltonian $H(\infty, n, \delta, \Delta)$ and go through the arrow 1 and the arrows “limited” (called branch 1l) and the arrows “full” and the arrow 2 (called branch f2), lead to a finite $H(\lambda, \tilde{n}, \tilde{\delta}, \tilde{\Delta})$. When calculations of some selected matrix elements of the effective Hamiltonian are done in perturbation theory, both ways of

going through the diagram are equivalent. For only a finite range of particle numbers and momenta can be reached in a limited number of steps of size λ starting from the finite values selected by the external states of the matrix elements in question.

Differences arise when one solves for the spectrum of an effective Hamiltonian and when one attempts to vary the model space parameters $\tilde{n}$, $\tilde{\delta}$ and $\tilde{\Delta}$. In the “full” calculation, one obtains a single effective Hamiltonian which one can solve in successively enlargeable model spaces. In the “limited” calculation, the model space restrictions are imposed at the beginning and they lead to an effective Hamiltonian whose action cannot be considered in a larger model space without repeating the renormalization group calculation in the larger space.

An explicit example of a difference between the two branches 1l and f2 in the diagram above is provided in Ref. [9] which discusses a Tamm–Dancoff (TD) approach analogous to the branch 1l (*cf.* Ref. [10]). In the TD approach, there are restrictions on the particle number which naturally lead to the sector-dependent counterterms as described in Ref. [9], for example, for masses. On the other hand, in the procedure of the branch f2 no such sector dependent counterterms arise. The present paper describes examples of sector-independent mass counterterms.

Proportionality to different powers of the coupling constant helps in estimates of how important are different effective interaction terms and how to choose the model space. Finding the basis which can span a good approximation to the full solution requires trial and error studies. This general feature can be illustrated by the following 2×2 matrix

$$\begin{bmatrix} a + bg^2 & gv \\ gv & c + dg^2 \end{bmatrix}.$$

This matrix is a model of the entire effective Hamiltonian matrix calculated to second order in g including all couplings between all effective Fock sectors as given by the “full” calculation. Thus, we have the Hamiltonian terms order 1, order g and order g^2 . In a perturbative calculation using matrix elements, which is focused on the upper sectors, one would calculate only the terms a and bg^2 .

Assume that a and c are of the same order, and b and d are of the same order, and calculate eigenvalues of the model matrix neglecting terms order g^4 and higher powers of g . For arbitrarily small g , the eigenvalues are given by a and terms quadratic in g . No terms linear in g arise. The quadratic corrections include contributions due to the term gv which couples different sectors. The role of this coupling needs to be estimated. The presence of

dg^2 seems to be irrelevant because it couples to the upper sector through the off-diagonal terms order g . Hence, it seems to contribute only in order g^4 to the eigenvalues.

It is well known that the above analysis is wrong in the case with degenerate diagonal matrix elements no matter how the degeneracy arises. For example, consider the case of a finite g such that $a + bg^2 = c + dg^2$. The eigenvalues are equal $a + bg^2 \pm gv$. They are linear in g instead of being quadratic, for arbitrary v . The lowest eigenvalue eigenstate is a superposition of the upper and the lower sector instead of being dominated by the upper one. In this example, the degeneracy is not visible until the term dg^2 is included in the calculation. As a second example consider the case with degenerate matrix elements c and $a + bg^2$ and d not included. The simple non-degenerate perturbative expansion is again not applicable. But the addition of the term dg^2 can lift the degeneracy and make the simple perturbative expansion work.

Corrections due to interaction terms such as gv may be additionally suppressed for very small λ since the range of v in momentum variables is given by λ . If λ is reduced in the renormalization group flow down to a number on the order of some positive power of g then the resulting interaction can contribute to the eigenvalues in the order implied by g and λ together which is higher than g^2 . In addition, the effective interaction v may contain small factors. For example, in the effective e^+e^- -sector of positronium, the emission of photons is proportional to the velocity of electrons which is order α , on average. The interaction term gv which couples states with an additional photon, plays no role in the eigenvalue in order α^2 if the width λ restricts energy changes to order $\alpha^2 m_{\text{electron}}$ (cf. [11]).

Terms such as dg^2 have been originally discussed in the light-front approach to QCD by Perry [12]. Heavy quarkonia are dominated by the effective $Q\bar{Q}$ sector. Terms such as dg^2 in other Fock sectors may lift up energies of effective gluons due to the non-abelian interactions to a sufficiently high value so that the model Hamiltonian $a + bg^2$ in the $Q\bar{Q}$ sector alone may have eigenstates which approximate the full solution for heavy mesons. The important observation made by Perry [12] in a frame dependent matrix elements approach using coupling coherence is that the terms bg^2 contain a logarithmically confining potential. An analogous boost invariant logarithmic interaction term in the Fock space in our approach will be discussed in Section 3.

The above diagram and the 2×2 matrix model illustrate the structure of our similarity renormalization group approach to the light-front Hamiltonian dynamics in quantum field theory. We summarize the steps here.

The first step is the calculation of the effective Hamiltonian,

$$H(\lambda) = S_{\lambda,n,\delta,\Delta}^\dagger H(\infty, n, \delta, \Delta) S_{\lambda,n,\delta,\Delta} . \quad (1.1)$$

S denotes the similarity transformation. Eq. (1.1) corresponds to the arrows marked full in the diagram.

The second step is to solve the effective Schrödinger equation

$$H(\lambda) |\psi\rangle = E |\psi\rangle . \quad (1.2)$$

$H(\lambda)$ has the same dynamical content and eigenvalues as $H(\infty, n, \delta, \Delta)$. The eigenvalue E is independent of the width λ .

Equation (1.2) greatly differs from the eigenvalue equation for $H(\infty, n, \delta, \Delta)$. The major difference is that the dynamics of $H(\lambda)$ has a limited range on the energy scale and the Hamiltonian does not contain ultraviolet divergences. Therefore, one can attempt to solve the eigenvalue problem scale by scale. Scattering processes are described by the same Hamiltonian. Next Sections will give examples of two fermions scattering in different theories.

Solutions to Eq. (1.2) provide renormalization conditions for the finite parts of counterterms. A general method is necessary for reducing the full eigenvalue problem to a manageable one. This step is marked by the arrow 2 in the diagram. In the case of the 2×2 matrix model, this step corresponds to the calculation of the model space Hamiltonian in the upper-left corner of the matrix. The similarity renormalization scheme guarantees that this step is free from ultraviolet divergences because the width λ is finite.

In the general case, one can apply the well known Bloch [13] technique of calculating model space Hamiltonians. Suppose we want to evaluate a model two-body Hamiltonian knowing $H(\lambda)$ with $\lambda < m$, where m is the effective one-body mass. We can introduce the projection operator P on the effective two-particle sector with a limited center-of-mass energy. We also introduce the operator R which generates the multi-particle and high energy components of the eigenstates from their limited mass two-body part. By assumption, R satisfies the conditions $(1 - P)R = RP = R$ and $PR = R(1 - P) = 0$ and the equation $(P + R - 1)H(\lambda)(P + R) = 0$. Then, the model two-body dynamics is described by the Hamiltonian [4]

$$H_2 = (P + R^\dagger R)^{-1/2} (P + R^\dagger) H(\lambda) (P + R) (P + R^\dagger R)^{-1/2} . \quad (1.3)$$

The same approach can be used for larger model spaces. The model space is characterized by the parameters $\tilde{n}$, $\tilde{\delta}$ and $\tilde{\Delta}$ in the diagram. So, the operation R depends on these parameters. But the resulting spectrum in the range of interest should not depend on the model space boundary when

the width λ is small and the model space contains the dynamically dominant basis states in the selected range of scales. The heuristic Eq. (1.3) can be applied in perturbation theory in the effective interaction even for sizable coupling constants since the effective interaction strength is considerably reduced by the similarity factors.

The scheme outlined above is still prone to the infrared regularization dependence for massless particles. This is particularly important in gauge theories. However, the effective Hamiltonian dynamics is expected to lead to infrared convergent results for gauge invariant quantities. There is also a possibility that new effective interactions are generated from the infrared region and they bring in effects normally associated with a nontrivial vacuum state [5, 14, 15]. We shall make comments on the issue of long distance phenomena in the present approach in Section 2.2 where we describe the range of scales involved in the theory. The reader should refer to [5, 14, 15] and [16] for discussions of the ground state, spontaneous symmetry breaking and zero-modes problems.

The remaining part of this paper is organized as follows. Section 2 describes the Hamiltonian formalism in three subsections. Namely, Section 2.1 introduces the similarity renormalization group equations and describes methods of solution, Section 2.2 describes regularization factors, 2.3 deals with renormalization conditions. Section 3 contains examples of lowest order applications of the formalism. Our derivation of the generalized Yukawa potential is given in Section 3.1. Section 3.2 describes the Schrödinger equation for positronium in QED. Section 3.3 discusses a confining term for constituent quarks in QCD. Section 4 concludes the paper. The list of references is focused on the similarity renormalization group approach to Hamiltonian dynamics in the light-front Fock space. The reader should be aware of this limitation. Examples of other approaches to quantum field theory in the light-front form of dynamics can be found in Ref. [16].

2. Effective Hamiltonians

This Section is divided into three parts. The first part describes our method of calculating effective Hamiltonians in the Fock space. The second part presents our regularization scheme for initial Hamiltonians. The last part discusses renormalization conditions and the effective eigenvalue problem.

2.1. Similarity transformation

We construct a family of effective Hamiltonians in the light-front Fock space. The family is parameterized by a scale parameter λ which ranges

from infinity to a finite value. λ limits energy transfers in the interaction terms.

The Hamiltonians are built of sums of ordered products of creation and annihilation operators. The Hamiltonian labeled by λ is expressed in terms of creation and annihilation operators which correspond to λ . We commonly denote these operators by q_λ . In addition, the creation and annihilation operators carry labels of quantum numbers such as momentum, spin, flavor or color. We will not indicate those numbers in the initial presentation, unless it is necessary.

All Hamiltonians in the family are assumed to be equal. Thus,

$$H_{\lambda_1}(q_{\lambda_1}) = H_{\lambda_2}(q_{\lambda_2}). \quad (2.1)$$

For $\lambda = \infty$, the Hamiltonians H_∞ are expressed in terms of operators creating and annihilating bare particles, q_∞ . Hamiltonians H_∞ can be constructed from the canonical field theoretic expressions for the energy-momentum density tensors.

Unfortunately, expressions for H_∞ in local field theories are divergent. They need to be regularized by introducing a bare ultraviolet cutoff which we shall denote by ε . The ultraviolet cutoff Δ from the previous Section corresponds to Λ^2/ε where Λ is an arbitrary finite constant which carries the necessary dimension of a mass. The limit of removing the bare ultraviolet cutoff will correspond to $\varepsilon \rightarrow 0$.

$H_{\lambda=\infty} = H_\varepsilon$ for all values of ε . For the limit $\varepsilon \rightarrow 0$ to exist the Hamiltonians H_∞ must include a number of additional terms (called counterterms) whose structure will be determined later.

H_∞ may include an infrared regulator, generically denoted by δ . For example, this is required in QED with massless photons and in QCD with massless gluons. $\delta \rightarrow 0$ when the infrared regularization is removed. The parameter δ is indicated explicitly if needed.

Our key assumption is that the particle degrees of freedom for all different scales λ are unitarily equivalent to the bare particle degrees of freedom:

$$q_\lambda = U_\lambda q_\infty U_\lambda^\dagger. \quad (2.2)$$

This assumption says that the quantum numbers of bare and effective particles are the same for all values of λ . The following examples explain the origin of this assumption. (1) Constituent quarks have the same quantum numbers as current quarks. (2) We use the same quantum numbers for photons and electrons independently of the kind of processes we consider in QED or in related effective theories such as the nonrelativistic Schrödinger equation with Coulomb potentials between charges. (3) Pions and nucleons in nuclear physics have the same quantum numbers quite independently of

what kind of interactions, pion-nucleon vertex form factors or other dynamical assumptions one uses.

It follows from Eq. (2.2) that creation and annihilation operators for λ_1 and λ_2 are unitarily equivalent and connected by transformations of the form $U_{\lambda_1} U_{\lambda_2}^\dagger$. The transformations U_{λ_1} or U_{λ_2} will depend on the bare cutoffs but the transformation $U_{\lambda_1} U_{\lambda_2}^\dagger$ for finite λ_1 and λ_2 will have to be finite in the limit $\varepsilon \rightarrow 0$.

The transformation U_λ is defined indirectly through a differential equation of the type used in the similarity renormalization scheme for Hamiltonians of Głazek and Wilson [1, 2]. That scheme was originally developed for application to QCD [5]. Hamiltonians with labels λ_1 and λ_2 are connected by integration of the differential equation from λ_1 to λ_2 . Our guiding principle in writing the differential equation for effective Hamiltonians is that *the resulting interactions between effective particles with considerably different scales of relative momenta are suppressed*.

This principle has its origin in the following examples. (1) Emission and absorption of short wavelength photons are not essential in the formation of atoms. (2) Emission and absorption of hard pions by nucleons is not important in nonrelativistic nuclear physics. (3) Constituent quarks have moderate momenta and their effective dynamics seem to be independent of the very hard gluon emissions. (4) High momentum transfer phenomena are independent of the small momentum transfer effects such as binding. A standard way of achieving this kind of picture in theoretical models is to include form factors in the interaction vertices. The form factors quickly tend to zero when momenta change by more than the size of a specific cutoff parameter.

The cutoff parameter in the form factors sets the scale for allowed changes of momenta. It determines the range or width of the interaction in momentum space. That width is the origin of our scale λ which labels renormalized effective Hamiltonians. Our similarity factors are analogous to the vertex form factors which are commonly used in nonlocal models (see also Ref. [17]). The large momentum transfer dynamics is integrated out through the similarity renormalization group equation.

Boost invariance requires that the individual momenta of effective particles are not restricted because boosts change those unlimitedly. The Hamiltonian width restricts only relative momenta of effective particles. Also, the larger is a relative momentum the larger change is generated by a boost. Therefore, when the free energy of interacting particles in their center-of-mass frame (*i.e.* the free light-front invariant mass) is much larger than λ the immediate change of energy due to interaction is limited by the large energy itself instead of λ . At the same time, this condition takes care of the property of wave mechanics that strong interference occurs between waves

of similar wavelengths within a range of wavelengths on the order of the wavelengths themselves [3].

In our construction, strong dynamical interference effects for states of similar free energies are made not to contribute in the derivation of effective Hamiltonians. For example, the similarity transformation is constructed in such a way that only large energy denominators can appear in the perturbative calculations of effective Hamiltonians and small denominators are excluded [1, 2]. Namely, only large free energy changes are integrated out. In this approach, calculations of strong coherence effects for nearly degenerate states are relegated to a later step of solving for eigenstates of the effective Hamiltonian. That step may be non-perturbative. For example, the Coulomb potential of QED is formally of the first order in α and leads to a variety of bound atomic structures beyond perturbation theory.

Our differential equations require a separation of the changes in creation and annihilation operators from changes in coefficients in front of products of the operators. In order to define this separation we assume that terms with a large number of the operators in a product do not dominate or mediate the effective dynamics of interest. If the latter assumption turns out to be invalid our formalism merely provides a way to approach the resulting problems. The comment due here is that if the dynamics leads to spontaneous symmetry breaking, or condensates, we will have a well defined renormalized Hamiltonian theory to study those phenomena in the desired detail, *cf.* Refs. [5, 14, 15].

The unitary equivalence of creation and annihilation operators at the scale λ and at the infinite scale, *i.e.* those appearing in $H_\infty = H_\varepsilon$, together with the equality of Hamiltonians at all scales imply that

$$H_\lambda(q_\lambda) = U_\lambda H_\lambda(q_\infty) U_\lambda^\dagger = H_\infty(q_\infty). \quad (2.3)$$

We denote $H_\lambda(q_\infty) = \mathcal{H}_\lambda$ and obtain

$$\mathcal{H}_\lambda = U_\lambda^\dagger H_\infty U_\lambda. \quad (2.4)$$

Thus, the effective Hamiltonian H_λ is obtained from the Hamiltonian $\mathcal{H}_\lambda$ by replacing creation and annihilation operators for bare particles by creation and annihilation operators for effective particles with the same quantum numbers. The bare creation and annihilation operators are independent of λ . One calculates λ -dependent coefficients in front of the products of q_∞ in $\mathcal{H}_\lambda$.

The differential equation for $\mathcal{H}_\lambda$ is [2]

$$\frac{d}{d\lambda} \mathcal{H}_\lambda = [\mathcal{H}_\lambda, \mathcal{T}_\lambda], \quad (2.5)$$

where

$$\mathcal{T}_\lambda = U_\lambda^\dagger \frac{d}{d\lambda} U_\lambda. \quad (2.6)$$

$\mathcal{H}_\lambda$ has the following structure

$$\mathcal{H}_\lambda = F_\lambda[\mathcal{G}_\lambda]. \quad (2.7)$$

$F_\lambda[\mathcal{G}_\lambda]$ denotes the similarity form factors in $\mathcal{H}_\lambda$ to be described below. Using the unitary equivalence, we also have

$$H_\lambda(q_\lambda) = F_\lambda[G_\lambda(q_\lambda)], \quad (2.8)$$

where

$$G_\lambda(q_\lambda) = U_\lambda \mathcal{G}_\lambda U_\lambda^\dagger. \quad (2.9)$$

A similar relation holds for $T_\lambda(q_\lambda)$ and $\mathcal{T}_\lambda$ since the latter is expressed in terms of q_∞ .

The operation F_λ acts on the operator $\mathcal{G}_\lambda$ equal to a superposition of terms each of which is an ordered product of creation and annihilation operators. The ordering is arbitrary but needs to be determined. We adopt the order from left to right of creators of fermions, creators of bosons, creators of anti-fermions, annihilators of anti-fermions, annihilators of bosons, annihilators of fermions. At least two operators must appear in a product and at least one creation and one annihilation operator must appear. No product contains only creation or only annihilation operators. This is a special property of light-front Hamiltonians. Hamiltonians in other forms of dynamics do not have this property and lead to the necessity of solving the ground state formation problem before other states can be considered because the pure creation or annihilation terms produce disconnected vacuum dynamics.

The operator $\mathcal{G}_\lambda$ is divided into two parts, $\mathcal{G}_{1\lambda}$ and $\mathcal{G}_{2\lambda}$. $\mathcal{G}_{1\lambda}$ is a superposition of all terms of the form $a^\dagger a$ for a equal q_∞ of any kind. In principle, one could also include in $\mathcal{G}_{1\lambda}$ some chosen terms with a larger number of creation and annihilation operators, *e.g.* terms containing two creation and two annihilation operators. However, plane-wave Fock space basis states are not eigenstates of relevant operators of such type and we limit $\mathcal{G}_{1\lambda}$ to terms $a^\dagger a$ to avoid the difficulty in present calculations.

$\mathcal{G}_{1\lambda}$ becomes the effective free part of G_λ , denoted $G_{1\lambda}$, after q_∞ is replaced by q_λ . The effective free Hamiltonian part $H_{1\lambda}$ is equal to $G_{1\lambda}$ because $G_{1\lambda}$ is not changed by the operation F . Eigenvalues of $G_{1\lambda}$ are called *free energies*.

The remaining part $\mathcal{G}_{2\lambda} = \mathcal{G}_\lambda - \mathcal{G}_{1\lambda}$ gives the interaction part of the effective Hamiltonian H_λ . One replaces q_∞ by q_λ and obtains $G_{2\lambda}$. Then, one applies the operation F_λ which inserts the vertex form factors defined as follows.

Let the momentum labels of all creation operators in a single product in an interaction term be $k_1, k_2, \dots, k_I$ and the momentum labels of all annihilation operators be $k'_1, k'_2, \dots, k'_J$. Each momentum has three components, k^+ ranging from 0 to ∞ and two transverse components $k^\perp = (k^1, k^2)$, both ranging from $-\infty$ to $+\infty$. The z -axis is distinguished by our choice of the light-front. The sum of momentum labels of creation operators, $\sum_{i=1}^I k_i$, equals the sum of momentum labels of annihilation operators, $\sum_{j=1}^J k'_j$. We denote these sums by $P = (P^+, P^\perp)$, (P^+ is positive). Each k^+ or k'^+ is a positive fraction of P^+ ; $x_i = k_i^+/P^+$, $1 > x_i > 0$ and $x'_j = k'_j^+/P^+$, $1 > x'_j > 0$. We have $\sum_{i=1}^I x_i = \sum_{j=1}^J x'_j = 1$. We also define

$$\kappa_n^\perp = k_n^\perp - x_n P^\perp \quad (2.10)$$

for all momenta in the Hamiltonian term. $\sum_{i=1}^I \kappa_i^\perp = \sum_{j=1}^J \kappa'_j{}^\perp = 0$.

These momentum variables appear standard but the way they are used here is not. Namely, P is usually not equal to a total momentum of a physical state. It characterizes the interaction term whose action redistributes P from the set of momenta of the annihilated particles to the set of momenta of the created particles.

Thus, each term in the Hamiltonian is characterized by P and two sets of variables, $X_I = \{(x_i, \kappa_i^\perp)\}_{i=1}^I$ for creation operators and $X'_J = \{(x'_j, \kappa'_j{}^\perp)\}_{j=1}^J$ for annihilation operators. For example, in a product of two creation operators and one annihilation operator we have $x_1 = x$, $x_2 = 1 - x$ and $x'_1 = 1$. Also, $\kappa_1^\perp = -\kappa_2^\perp = \kappa^\perp$ and $\kappa'_1{}^\perp = 0$. P can be arbitrary and the term in question replaces one particle of momentum P by two particles of momenta $xP + \kappa$ and $(1 - x)P - \kappa$ for $+$ and $\perp$ components, $\kappa^+ = 0$. It is convenient to speak of P as a *parent* momentum and about the individual particle momenta as *daughter* momenta. The parent momentum in a Hamiltonian term equals one half of the sum of momenta labeling all creation and annihilation operators in the term. Each daughter particle carries a fraction of the parent momentum. The parent momentum may be carried by one or more particles.

The operation F_λ acting on a product of creation and annihilation operators produces

$$F_\lambda \left[\prod_{i=1}^I a_{k_i}^\dagger \prod_{j=1}^J a_{k'_j} \right] = f_\lambda(X_I, X'_J) \prod_{i=1}^I a_{k_i}^\dagger \prod_{j=1}^J a_{k'_j}. \quad (2.11)$$

The function $f_\lambda(X_I, X'_J)$ is a suitable function which represents our physical intuition about form factors. The arguments of f_λ are invariant with respect to seven kinematical Poincaré transformations of the light-front frame. This

feature results in the boost symmetry of our theory. We impose three conditions on the function f_λ .

The first condition is that f_λ is expressible through the eigenvalues of $G_{1\lambda}$ corresponding to the sets X_I and X'_J so that f_λ equals 1 for small differences between the eigenvalues and quickly goes to zero when the differences become large. This is the basic condition of the similarity renormalization scheme for Hamiltonians [1, 2]. The width of f_λ is set by λ . One can consider functions f_λ which depend on X_I and X'_J in a more general way than through the eigenvalues of the free Hamiltonian but that option will not be investigated here.

The first condition defines the effective nature of the Hamiltonian labeled by λ . Namely, the effective particle states which are separated by the free energy gap which is much larger than λ are not directly coupled by the interactions. In other words, λ limits the free energy changes induced by the effective interaction. Moreover, as a consequence of $f_\lambda \sim 1$ for similar energies, $1 - f_\lambda$ is close to zero for the similar energies and it vanishes proportionally to a power of the energy difference. The higher is the power the smaller is the role of states of similar energies in the calculation of the effective Hamiltonian. This will be explained later. Consequently, the higher is the power the smaller is the role of non-perturbative phenomena due to energy changes below the scale λ in the calculation of the effective Hamiltonian.

Thus, there is a chance for the full Hamiltonian diagonalization process to be divided into two parts: a perturbative calculation of the effective renormalized Hamiltonian and a non-perturbative diagonalization of that effective Hamiltonian. This is our factorization hypothesis in the Hamiltonian approach.

The second condition is that both, $1 - f_\lambda$ and $df_\lambda/d\lambda$, must vanish faster than linearly in the free energy difference. This condition is required to exclude the small energy denominators in perturbation theory and will be explained below. The second condition implies that $1 - f_\lambda$ vanishes as at least second power of the energy difference near zero.

The third condition is defined by saying that multi-particle interactions (especially interactions that change the number of effective particles by many) should not be important in the effective Hamiltonian dynamics which is characterized by changes of energies below the scale λ . This may be possible if f_λ as a function of the daughter variables approximates the shape of one particle irreducible vertices which is characteristic to the theory under consideration. Structure of G_λ depends on the choice of the function f_λ . Some choices will lead to more complicated effective interactions than others. The best choices for the most efficient description of physical phenomena at some scale λ are such that the effective particles interact in a

way that is most easy to understand and which can be parametrized with the least possible number of parameters over the range of scales of physical interest. One can conceive variational estimates for the best choices of f_λ that minimize complexity of the effective Hamiltonians. For example, it is clear that creation of effective particles will be suppressed when the width λ becomes comparable to the effective masses of those particles.

To satisfy the first condition above in a boost invariant way we define a boost invariant gap between free energy eigenvalues for effective particles which is to be compared with the running cutoff parameter λ . The light-front quantization scheme does not explicitly preserve rotational symmetry. Nevertheless, it has been shown that if counterterms provide enough freedom through their finite parts and multi-particle effects are suppressed one can obtain rotationally invariant results [11, 18, 19].

The free energy eigenvalues relevant to a particular Hamiltonian term with daughter variables X_I and X'_J are

$$\sum_{i=1}^I \frac{k_i^{\perp 2} + m_i^2(\lambda)}{k_i^+} = \frac{P^{\perp 2} + \mathcal{M}_I^2}{P^+} \quad (2.12)$$

and

$$\sum_{j=1}^J \frac{k'_j{}^{\perp 2} + m_j^2(\lambda)}{k'_j{}^+} = \frac{P^{\perp 2} + \mathcal{M}_J^2}{P^+}, \quad (2.13)$$

where

$$\mathcal{M}_I^2 = \sum_{i=1}^I \frac{\kappa_i^{\perp 2} + m_i^2(\lambda)}{x_i} \quad (2.14)$$

and

$$\mathcal{M}'^2_J = \sum_{j=1}^J \frac{\kappa'_j{}^{\perp 2} + m_j^2(\lambda)}{x'_j}. \quad (2.15)$$

The individual effective particle masses are allowed to depend on the effective Hamiltonian width parameter λ . We define *the mass difference* for a Hamiltonian term to be

$$\Delta \mathcal{M}^2 = \mathcal{M}'^2_J - \mathcal{M}_I^2, \quad (2.16)$$

and *the mass sum* to be

$$\Sigma \mathcal{M}^2 = \mathcal{M}'^2_J + \mathcal{M}_I^2. \quad (2.17)$$

To be specific, we define details of the function $f_\lambda(X_I, X'_J)$ introducing a parameter z_λ . Following the similarity renormalization scheme [1, 2], z_λ can

be chosen in such a way that z_λ is close to zero for $\Delta\mathcal{M}^2$ small in comparison to λ^2 or $\Sigma\mathcal{M}^2$ and $|z_\lambda|$ is close to 1 for $\Delta\mathcal{M}^2$ large in comparison to λ^2 or comparable to $\Sigma\mathcal{M}^2$. For example,

$$z_\lambda = \frac{\Delta\mathcal{M}^2}{\Sigma\mathcal{M}^2 + \lambda^2}. \quad (2.18)$$

The definition includes $\Sigma\mathcal{M}^2$ to ease estimates in high order perturbation theory, especially in the analysis of overlapping divergences [1]. The new feature here is that the introduction of $\Sigma\mathcal{M}^2$ does not violate the light-front boost invariance and basic cluster decomposition properties. $f_\lambda(X_I, X'_J)$ is defined for the purpose of this article to be a function of $z_\lambda^{2^n}$, $n \geq 1$, which is analytic in the vicinity of the interval $[0, 1]$ on the real axis, equals 1 for $z_\lambda = 0$ and quickly approaches 0 for $z_\lambda \sim 1$;

$$f_\lambda(X_I, X'_J) = f(z_\lambda^{2^n}). \quad (2.19)$$

For example,

$$f(u) = \left[1 + \left(\frac{u(1-u_0)}{u_0(1-u)} \right)^{2^m} \right]^{-1}, \quad (2.20)$$

where $1 > u_0 > 0$ and $m \geq 1$. The larger the exponent m the closer $f(u)$ approaches $\theta(u_0 - u)$ for $0 \leq u \leq 1$. Eq. (2.20) concludes our definition of the operation F_λ .

The smallest possible value of $\Sigma\mathcal{M}^2$ in Eq. (2.18) is $\left[\sum_{i=1}^I m_i(\lambda) \right]^2 + \left[\sum_{j=1}^J m_j(\lambda) \right]^2$. Thus, z_λ is small for small positive λ^2 when $\Delta\mathcal{M}^2$ is small in comparison to particle masses. Therefore, u_0 must be much smaller than 1 to force $\Delta\mathcal{M}^2$ to be small in comparison to $\Sigma\mathcal{M}^2$ when λ^2 is small. One can also force $\Delta\mathcal{M}^2$ to be small in comparison to the particle masses by making λ^2 negative so that it subtracts from $\Sigma\mathcal{M}^2$ its minimal value. Then, the mass difference is compared to the sum of kinetic energies due to the relative motion only. It is also useful to limit the small mass differences by choosing an infinitesimally small u_0 and introducing $\lambda^2 = u_0^{-1/2^n} \tilde{\lambda}^2$. Then, $|\Delta\mathcal{M}^2| \leq \tilde{\lambda}^2$ in the θ -function limit. In this case, the band-diagonal Hamiltonian width becomes independent of the mass sum for as long as the latter is small in comparison to λ^2 .

The infinitesimal transformation $\mathcal{T}_\lambda$ in Eq. (2.5) is defined as follows. Eq. (2.5) is rewritten using Eqs. (2.8) and (2.9), into the form

$$\mathcal{H}' = f'\mathcal{G} + f\mathcal{G}' = [f\mathcal{G}_1, \mathcal{T}] + [f\mathcal{G}_2, \mathcal{T}]. \quad (2.21)$$

The prime denotes differentiation with respect to λ . We have simplified the notation of $F_\lambda[\mathcal{G}_\lambda]$ to $f\mathcal{G}$. Three universal relations $f\mathcal{G}_1 = \mathcal{G}_1$ and $(1-f)\mathcal{G}_1 = f'\mathcal{G}_1 = 0$ are then used without saying.

Equation (2.21) involves two unknowns, $\mathcal{G}$ and $\mathcal{T}$. Additional condition is required to define $\mathcal{T}$. One recalls that if the interaction is absent, *i.e.* when $\mathcal{G}_2 = 0$, then no evolution with λ may appear. Therefore, in the limit of negligible interactions, both $\mathcal{G}' = 0$ and $\mathcal{T} = 0$. $\mathcal{G}'$ should differ from zero if and only if the interactions are important. The first term on the right-hand side is order $\mathcal{T}$ since $\mathcal{G}_1$ contains terms independent of interactions. The second term on the right-hand side is at least of second order in interactions. The first term can be used as a seed for defining $\mathcal{T}$ through a series of powers of the interaction.

We associate the derivative of $\mathcal{G}$ with the second term on the right-hand side. The first term on the right-hand side and a part of the second term which is left after the derivative of $\mathcal{G}$ is defined, together determine $\mathcal{T}$. $\mathcal{T}$ is defined through the commutator $[\mathcal{G}_1, \mathcal{T}]$ using a curly bracket notation. We write

$$A = \{B\}_{\mathcal{G}_1} \quad (2.22)$$

when

$$[A, \mathcal{G}_1] = B. \quad (2.23)$$

Subscripts of such curly brackets are often omitted in later discussion. Suppose B contains a term which involves a product

$$\prod_{i=1}^I a_{k_i}^\dagger \prod_{j=1}^J a_{k'_j}. \quad (2.24)$$

Then, $\{B\}_{\mathcal{G}_{1\lambda}}$ contains the same product (as a part of the same expression) with an additional factor equal

$$\left[\sum_{j=1}^J \frac{k_j'^{\perp 2} + m_j^2(\lambda)}{k_j'^+} - \sum_{i=1}^I \frac{k_i^{\perp 2} + m_i^2(\lambda)}{k_i^+} \right]^{-1}. \quad (2.25)$$

The sums of individual energies satisfy Eqs. (2.12) and (2.13), respectively, and the factor (2.25) equals

$$\left[\frac{\Delta \mathcal{M}^2}{P^+} \right]^{-1}, \quad (2.26)$$

where P^+ is the parent momentum for the product under consideration and the mass difference is defined in Eq. (2.16). All terms in the operator B are multiplied by the corresponding factors.

The factor (2.25) explodes to infinity when the denominator approaches zero. Hence, for the operator A to be well defined, the coefficients of products

of the form (2.24) in the operator B must vanish at least as fast as the energy denominator itself when the denominator approaches zero. Therefore, our definition of $[\mathcal{G}_1, \mathcal{T}]$ is given in terms of an operator which has such property. Eq. (2.21) is split into two equations as follows.

$$f\mathcal{G}' = f[f\mathcal{G}_2, \mathcal{T}], \quad (2.27)$$

$$[\mathcal{T}, \mathcal{G}_1] = (1 - f)[f\mathcal{G}_2, \mathcal{T}] - f'\mathcal{G}. \quad (2.28)$$

Our second condition introduced below Eq. (2.11) on the functions $1 - f_\lambda(X_I, X'_J)$ and $f'_\lambda(X_I, X'_J)$ guarantees that $\mathcal{T}$ is well defined and tends to zero in the region of vanishing energy denominators because the right-hand side of Eq. (2.28) vanishes at least as fast as the first power of the energy differences.

Equation (2.27) is a first order differential equation. One has to provide an initial condition to define a theory. The initial conditions are set in this paper by canonical light-front Hamiltonians plus counterterms. The latter are determined from the condition that the effective Hamiltonians have well defined limits when the bare cutoff is removed. In other words, one has to find the class of initial conditions at $\lambda = \infty$ which imply ε -independent H_λ 's for all finite λ 's when $\varepsilon \rightarrow 0$.

A general iterative procedure for calculating the effective Hamiltonians is analogous to the one from Refs. [1, 2]. However, instead of iterating two coupled equations for H_λ and T_λ we base iteration on an equivalent single equation for H_λ with an explicit solution for T_λ already built in. Simple algebra and substitution of Eq. (2.28) into Eq. (2.27), lead to

$$\frac{d}{d\lambda}\mathcal{G}_\lambda = \left[f_\lambda\mathcal{G}_{2\lambda}, \left\{ \frac{d}{d\lambda}(1 - f_\lambda)\mathcal{G}_{2\lambda} \right\}_{\mathcal{G}_{1\lambda}} \right]. \quad (2.29)$$

Equation (2.29) drives the renormalization group formalism in this paper. Note that the right-hand side is given in terms of a commutator. Therefore, *the effective renormalized Hamiltonians contain only connected interactions*. This is essential for cluster decomposition properties of the effective Hamiltonians [20].

Equation (2.29) is of the form

$$\frac{d}{d\lambda}\mathcal{G}_\lambda = T_\lambda[\mathcal{G}_\lambda]. \quad (2.30)$$

The right hand side contains terms which are bilinear in the effective interaction strength. The initial condition for Eq. (2.29), or (2.30), is given at $\lambda = \infty$: $\mathcal{G}_{\lambda=\infty} = \mathcal{G}_\varepsilon$.

With accuracy to the first order in powers of the interaction strength, $\mathcal{G}_\lambda$ is independent of λ and $\mathcal{G}_\varepsilon$ is equal to the initial regularized Hamiltonian expression one intends to study, denoted by $H_\varepsilon^{(0)}$. In this initial approximation, $\mathcal{H}_\lambda^{(0)} = f_\lambda^{(0)} \mathcal{G}_\lambda^{(0)}$, where $\mathcal{G}_\lambda^{(0)} = H_\varepsilon^{(0)}$ and $f_\lambda^{(0)}$ is the similarity factor calculated using eigenvalues of $\mathcal{G}_{1\lambda}^{(0)}$. $\mathcal{H}_\lambda^{(0)}$ forms our first approximation to the similarity renormalization group trajectory of operators $\mathcal{G}_\lambda$ parametrized by λ .

Eq. (2.30) can then be written in the iterative form for successive approximations to the trajectory $\mathcal{G}_\lambda$. Namely,

$$\frac{d}{d\lambda} \mathcal{G}_\lambda^{(n+1)} = T_\lambda^{(n)}[\mathcal{G}_\lambda^{(n)}]. \quad (2.31)$$

This is an abbreviated notation for

$$\frac{d}{d\lambda} \mathcal{G}_\lambda^{(n+1)} = \left[f_\lambda^{(n)} \mathcal{G}_{2\lambda}^{(n)}, \left\{ \frac{d}{d\lambda} (1 - f_\lambda^{(n)}) \mathcal{G}_{2\lambda}^{(n)} \right\}_{\mathcal{G}_{1\lambda}^{(n)}} \right]. \quad (2.32)$$

$f_\lambda^{(n)}$ denotes a function of $z_\lambda^{(n)}$ expressed through eigenvalues of $\mathcal{G}_{1\lambda}^{(n)}$, such as in Eqs. (2.19) and (2.20). The initial condition is set by $\mathcal{G}_\infty^{(n+1)} = \mathcal{G}_\varepsilon^{(n+1)}$. Thus, the solution is

$$\mathcal{G}_\lambda^{(n+1)} = \mathcal{G}_\varepsilon^{(n+1)} - \int_\lambda^\infty T_s^{(n)}[\mathcal{G}_s^{(n)}]. \quad (2.33)$$

$\mathcal{G}_\infty$ contains the ε -regulated canonical Hamiltonian terms and counterterms. The counterterms remove the part of the integral in Eq. (2.33) which diverges for finite λ when $\varepsilon \rightarrow 0$. Matrix elements of the Hamiltonian of the effective theory are required to have a limit when ε is made very small. The condition that the necessary $\mathcal{G}_\infty$ exists is the Hamiltonian version of renormalizability. It does not require the number of counterterms to be finite, although a finite number has the clear advantage of simplicity.

The part of the integrand in Eq. (2.33) which leads to the divergence is denoted by $\left[T_s^{(n)}[\mathcal{G}_s^{(n)}] \right]_{\text{div}}$, and the remaining part by $\left[T_s^{(n)}[\mathcal{G}_s^{(n)}] \right]_{\text{conv}}$. $\mathcal{G}_\varepsilon^{(n+1)}$ contains the initial regulated Hamiltonian terms and counterterms. The counterterms in $\mathcal{G}_\varepsilon^{(n+1)}$ are discovered from inspection of $F_\lambda^{(n+1)}[G_\lambda^{(n+1)}]$ dependence on ε when $\varepsilon \rightarrow 0$ in the absence of counterterms.

Note that $F_\lambda^{(n+1)}[G_{1\lambda}^{(n+1)}] = G_{1\lambda}^{(n+1)}$ and it is not necessary to know $F_\lambda^{(n+1)}$ to calculate $G_{1\lambda}^{(n+1)}$. One calculates $F_\lambda^{(n+1)}$ after $G_{1\lambda}^{(n+1)}$ is made independent of ε when $\varepsilon \rightarrow 0$.

The diverging dependence on ε when $\varepsilon \rightarrow 0$, is typically of the form ε^{-1} or $\log \varepsilon$ times operator coefficients. The operator coefficients can be found by integrating the diverging part of the integrand from some arbitrary finite value of λ , say λ_0 , to infinity. The divergence originates from the upper limit of the integration and it is independent of λ_0 . The remaining finite part of the integral is sensitive to the lower limit of integration and depends on λ_0 . The counterterm does not depend on λ_0 but it contains an arbitrary finite part which emerges in the following way.

The counterterm subtracts the diverging part of the integral. But subtracting terms with diverging functions of ε times known operators does not tell us what finite parts times the same operators to keep. Thus, one needs to add arbitrary finite parts to the numbers $1/\varepsilon$ and $\log \varepsilon$ in the counterterms. These finite parts are unknown theoretically and have to be fitted to data. In particular, observed symmetries may impose powerful constraints on the finite parts.

The diverging part of the integrand is such that the lower limit of its integration produces the same operator structure as the upper limit but the diverging numbers such as ε^{-1} or $\log \varepsilon$ from the upper limit are replaced by finite numbers at the lower limit. Those finite numbers depend on λ_0 but, once they are replaced by the required unknown finite parts, one obtains a valid expression for the counterterm. The replacement is achieved by adding to the integral the same operators multiplied by the numbers which are equal to the differences between the unknown numbers and the numbers resulting from the lower limit of the integration. Thus, the unknown numbers we need to add to the integral of the diverging part of the integrand from λ_0 to infinity depend on λ_0 . One can write the λ_0 -independent $\mathcal{G}_\varepsilon^{(n+1)}$ as $\mathcal{G}_{\varepsilon \text{ finite}}^{(n+1)}(\lambda_0) + \int_{\lambda_0}^{\infty} [T_s^{(n)}[\mathcal{G}_s^{(n)}]]_{\text{div}}$. The free finite parts of the counterterms are contained in $\mathcal{G}_{\varepsilon \text{ finite}}^{(n+1)}(\lambda_0)$ and one can fit them to data using predictions obtained from effective Hamiltonians at some convenient scales λ .

More than one scale λ may become necessary for accurate determination of the free parameters when their values have to be of considerably different orders of magnitude and require knowledge of physical phenomena at different scales. In the present work a single scale $\lambda = \lambda_0$ is sufficient for practical calculations. The renormalization conditions are set using H_{λ_0} . One may also consider renormalization conditions for parameters in H_{λ_0} which are set using another effective Hamiltonian at some nearby scale $\lambda_1 \neq \lambda_0$. This will be illustrated in the next Section.

The complete recursion including construction of counterterms in Eq. (2.33) is given by

$$\mathcal{G}_\lambda^{(n+1)} = \mathcal{G}_{\varepsilon \text{ finite}}^{(n+1)}(\lambda_0) + \int_{\lambda_0}^{\lambda} ds \left[T_s^{(n)}[\mathcal{G}_s^{(n)}] \right]_{\text{div}} - \int_{\lambda}^{\infty} ds \left[T_s^{(n)}[\mathcal{G}_s^{(n)}] \right]_{\text{conv}}. \quad (2.34)$$

In the limit $n \rightarrow \infty$, if the limit exists, one obtains

$$\mathcal{G}_\lambda = \mathcal{G}_{\varepsilon \text{ finite}}(\lambda_0) + \int_{\lambda_0}^{\lambda} ds [T_s[\mathcal{G}_s]]_{\text{div}} - \int_{\lambda}^{\infty} ds [T_s[\mathcal{G}_s]]_{\text{conv}}. \quad (2.35)$$

H_λ is obtained from Eq. (2.35) through the replacement of q_∞ by q_λ (to obtain G_λ) and action of F_λ on G_λ .

Perturbative calculations of renormalized effective Hamiltonians are based on the observation that the rate of change of $\mathcal{G}_\lambda$ with λ can be expanded in a power series in the effective interaction $\mathcal{G}_{2\lambda}$ at the same running scale λ . This is obtained by repeated application of Eq. (2.29). One rewrites Eq. (2.29) as

$$\frac{d\mathcal{G}_\lambda}{d\lambda} = \left[f_\lambda \mathcal{G}_{2\lambda}, \{ -f'_\lambda \mathcal{G}_{2\lambda} \}_{\mathcal{G}_{1\lambda}} \right] + \left[f_\lambda \mathcal{G}_{2\lambda}, \left\{ (1 - f_\lambda) \frac{d\mathcal{G}_\lambda}{d\lambda} \right\}_{\mathcal{G}_{1\lambda}} \right]. \quad (2.36)$$

Then, one replaces $\mathcal{G}'_\lambda$ in the last term on the right-hand side of Eq. (2.36) by the preceding terms. Two successive substitutions produce an expression for $\mathcal{G}'_\lambda$ with four explicit powers of the effective interactions and the remaining terms are of higher order [note that $(1 - f_\lambda)\mathcal{G}'_{1\lambda} = 0$].

$$\begin{aligned} \frac{d\mathcal{G}_\lambda}{d\lambda} &= [f\mathcal{G}, \{-f'\mathcal{G}\}] + [f\mathcal{G}, \{(1 - f)[f\mathcal{G}, \{-f'\mathcal{G}\}]\}] \\ &\quad + [f\mathcal{G}, \{(1 - f)[f\mathcal{G}, \{(1 - f)[f\mathcal{G}, \{-f'\mathcal{G}\}]\}]\}] + o(\mathcal{G}^5). \end{aligned} \quad (2.37)$$

We have omitted subscripts 2, λ and $\mathcal{G}_{1\lambda}$ on the right-hand side. All the subscripts appear in the same pattern as in Eq. (2.36). Correspondingly, the infinite chain of substitutions produces an expression ordered by explicit powers of the effective interactions, to infinity.

$$\frac{d}{d\lambda} \mathcal{G}_\lambda = \sum_{n=0}^{\infty} \left[f\mathcal{G}, \left(\left\{ (1 - f) \left[f\mathcal{G}, \right. \right. \right. \right. \left. \left. \left. \left\{ -f'\mathcal{G} \right\} \right\} \right)^{(n)} \right]. \quad (2.38)$$

The round bracket raised to the n -th power means n consecutive repetitions of the symbols from within the round bracket. The subscripts are omitted for clarity as in Eq. (2.37).

The above expansion in powers of the effective interactions provides a systematic order by order algorithm for building an expression for the effective Hamiltonian. The energy denominators and functions f_λ are calculated using eigenvalues of $\mathcal{G}_{1\lambda}$. Therefore, in Eqs. (2.37) and (2.38), the derivatives of the function f_λ contain two kinds of contributions: those resulting from differentiating the explicit λ dependence in the arguments z_λ (for example, λ^2 in Eq. (2.18)), and those resulting from differentiating the free energy eigenvalues (for example, $\Delta\mathcal{M}^2$ in Eq. (2.18)). Since the free energy terms include effective masses which depend on the width λ , the derivatives of the effective masses appear in the equations on the right-hand side. Moving them to the left-hand side leads to coupled nonlinear differential equations for the effective Hamiltonians.

The general iterative approach in Eq. (2.34) or the expansion in Eq. (2.38), can be analysed using expansion in the running coupling constants. One can divide $\mathcal{G}_{1\lambda}$ into two parts: one which is independent of the coupling constants and another one which vanishes when the coupling constants are put equal to zero. The parts depending on the coupling constants are moved over to $\mathcal{G}_{2\lambda}$ and treated as an interaction. After $\mathcal{G}_{1\lambda}$ is reduced to the part which is independent of the interactions, the derivatives of f_λ in Eqs. (2.36) to (2.38) do not introduce additional powers of the interaction strength and the series is strictly ordered in powers of the interactions according to their explicit appearance in the formula (2.38). This series then provides the perturbative expansion in terms of the running coupling constants.

The simplest case of the perturbative expansion involves a single coupling constant at a single scale. Firstly, one expands the renormalization group equations into a series of terms ordered by powers of the bare coupling g_0 . Secondly, one evaluates the effective coupling g_1 at the chosen scale λ_1 as a power series in the bare coupling. Thirdly, the latter series is inverted and the bare coupling is expressed as a series in the effective coupling g_1 . Then, one can pursue perturbative calculations in terms of the effective coupling. In particular, one can reduce the Hamiltonian width to $\lambda_2 < \lambda_1$ and calculate g_2 as a series in g_1 . Such steps can be repeated. For example, one can reduce the width in each step by a factor 2 [3,4]. N steps will reduce the width by the factor 2^{-N} . This way one can build the renormalization group flow indicated by the chains of small arrows in the diagram discussed in Section 1. If many coupling constants appear but they can be reduced to functions of a finite set of independent running coupling constants the finite set determines the theory and one speaks of coupling coherence [21].

2.2. Regularization

A canonical bare Hamiltonian obtained from a local field theory is divergent. This Section describes how the ultraviolet singularities in the canonical Hamiltonian are regularized with the bare cutoff ε . We also introduce infrared regularization. Our presentation is ordered as follows. First, we briefly explain connection between the ultraviolet and infrared regularizations in light-front dynamics. Then, we proceed with definitions of the canonical Hamiltonian terms. For that purpose, we have to discuss the fundamental set of scales in the Hamiltonian approach and explain the role of Lagrangian densities for classical fields in the construction. Then, we describe details of the ultraviolet and, subsequently, infrared regularizations.

The ultraviolet and infrared regularizations are connected through masses. The infrared structure is influenced by the masses in the initial Hamiltonian $H_\varepsilon^{(0)}$. $H_\varepsilon^{(0)}$ carries the superscript 0 to indicate that it is the initial Hamiltonian which does not yet include counterterms. If the ultraviolet counterterms change the masses the infrared behavior is changed too.

An initial mass value is generically denoted by $m_\varepsilon^{(0)}$. A light-front energy, p_m^- , of a free particle with a four-momentum $p_m = (p^+, p^\perp, p_m^- = (p^{\perp 2} + m^2)/p^+)$, tends to infinity when $m^2 > 0$ and p^+ tends to zero. But p_m^- may be finite or even approach zero in this limit if $m^2 = 0$ and $p^\perp$ approaches zero too. The limit of small momentum p^+ is always a high-energy limit when $m^2 > 0$. But it ceases to be the high-energy limit for the small transverse momenta if $m^2 \rightarrow 0$. Thus, $m_\varepsilon^{(0)}$ in the initial Hamiltonian is capable of switching from the high-energy regime in the longitudinal direction to the low-energy one when we take the limit $m_\varepsilon^{(0)} \rightarrow 0$. Conversely, introducing masses turns the infrared low-energy regime into the high-energy regime.

We begin the construction of $H_\varepsilon^{(0)}$ with an enumeration of momentum scales. We distinguish scales related to the boundary conditions for fields at spatial infinity, small momentum cutoffs, phenomenological parameters and large momentum cutoffs.

The bare Hamiltonian $H_\varepsilon^{(0)}$ is defined in terms of the operators q_∞ . The bare quantum fields are built from these operators with plane-wave coefficients [20]. The initial basis in the Fock space is built from the vacuum state $|0\rangle$ using $q_\infty^\dagger$. Fermion, anti-fermion and boson creation and annihilation operators are denoted by $b^\dagger$, $d^\dagger$, $a^\dagger$, b , d and a , respectively. For example, $|k\sigma\rangle = b_{k\sigma}^\dagger|0\rangle$ denotes a state of one bare fermion of momentum $k = (k^+, k^\perp)$. Spin z -axis projection, flavor, color or other quantum numbers, are denoted by a common symbol σ . The momentum variable in the subscript is distinguished in order to describe the scales involved in the defi-

inition of the Hamiltonian. The order of scales in momentum space is reverse to the order of scales in position space.

The largest scale in the position space is the quantization volume. In other words, the momenta can be thought of as discrete when necessary. But we insure by our choices of scales that the granulation in momentum space is never noticeable and the quantization volume is effectively infinite for all our purposes. Thus, we universally adopt continuous notation for momentum variables.

Potentially related to the boundary conditions, hypotheses about zero-modes and spontaneously broken symmetry in light-front quantum field theories were recently discussed in Ref. [5] which quotes important earlier literature on the subject. Basically, one may expect that new terms emerge in the effective Hamiltonians and the new terms account for the large scale dynamical effects. Susskind and co-workers have proposed a way to think about the wee parton dynamics in a model [14]. Ref. [15] describes QCD sum rules technique using the notion of vacuum condensates in the light-front scheme. However, the original quantum dynamics of the vacuum formation and spontaneous symmetry breaking are not yet understood and cannot be discussed further here.

The next smaller size in position space is the inverse of the infrared regulator. Two types of the infrared regulator appear. The first is a lower bound, denoted by δ , on the parent $+$ -momentum fraction that can be carried by an operator in an interaction term. The second is a mass parameter μ_δ . μ_δ appears as the mass parameter $m_\varepsilon^{(0)}$ in the initial Hamiltonian $H_\varepsilon^{(0)}$. It is introduced for massless bare particles. μ_δ cuts off the small longitudinal momentum region at a small scale order μ_δ^2/Δ^2 , where Δ^2 is the invariant mass upper bound. When the infrared regularization is being removed, δ or μ_δ are sent to zero but their inverse is always kept negligible in comparison with the quantization volume scale.

The next smaller scale in position space is set by the size of the volume used for preparation of incoming and detection of outgoing particles (including bound states) and the corresponding time scale. Physics is contained within this scale and observables are allowed to depend on this scale since the preparation and detection of states is a part of a physical process.

The order of magnitude of momenta larger than the experimental wave packet widths are characterized in terms of three different scales, (1) masses of particles, (2) the width of the effective Hamiltonian (*i.e.* λ), and (3) the bare cutoff scale ε^{-1} . When solving for the Hamiltonian spectrum, a new scale may emerge dynamically, determined by the effective coupling constants, masses and width of the effective Hamiltonian. Scale invariance at large momenta may be violated through a dimensional transmutation even if all mass scales are negligible in comparison to the momenta and λ .

The width λ ranges from ∞ to convenient finite values. Description of physical phenomena involving energy-momentum transfers of the order of k require λ to be larger than k . It is also useful to use λ not too large in comparison to k in order to avoid too much detail in the dynamics. For example, useful values of λ in nonrelativistic systems are smaller than effective masses. In QED, the convenient λ in the hydrogen description is much larger than the binding energy and much smaller than the electron mass.

The bare cutoff scale ε^{-1} is the largest momentum scale in the theory. The formal limit $\lambda \rightarrow \infty$ is used only to remove λ dependence from the Hamiltonian regulated by ε . In other words, no λ dependence appears in the Hamiltonians with λ larger than the scale implied by ε^{-1} . No physical quantity depends on ε when $\varepsilon \rightarrow 0$. The similarity renormalization scheme for Hamiltonians is built to achieve this goal to all orders in perturbation theory (*cf.* [1, 2]).

We proceed to the explicit construction of simplest terms in the Hamiltonian $H_\varepsilon^{(0)}$. Details of counterterms are not known from the outset. Light-front power counting rules are helpful [5] in determining possible structures of the counterterms but more is required in practice. The similarity renormalization group provides the required details.

All starting Hamiltonians in quantum field theories contain a free part which we denote by $\mathcal{G}_1^{(0)}$. The free part for fermions and bosons has the form

$$\mathcal{G}_1^{(0)} = \sum_{\sigma} \int [k] \left[\frac{k^{\perp 2} + m_\varepsilon^{(0)2}}{k^+} (b_{k\sigma}^\dagger b_{k\sigma} + d_{k\sigma}^\dagger d_{k\sigma}) + \frac{k^{\perp 2} + \mu_\varepsilon^{(0)2}}{k^+} a_{k\sigma}^\dagger a_{k\sigma} \right]. \quad (2.39)$$

We adopt the following conventions. Summation over σ denotes a sum over all quantum numbers except the momentum.

$$\int [k] = \frac{1}{16\pi^3} \int_0^\infty \frac{dk^+}{k^+} \int d^2 k^\perp. \quad (2.40)$$

The creation and annihilation operators in Eq. (2.39) are the bare ones denoted in Section 2.1 by q_∞ . They satisfy standard commutation or anti-commutation relations

$$[a_{k\sigma}, a_{k'\sigma'}^\dagger] = \{b_{k\sigma}, b_{k'\sigma'}^\dagger\} = \{d_{k\sigma}, d_{k'\sigma'}^\dagger\} = 16\pi^3 k^+ \delta^3(k - k') \delta_{\sigma\sigma'} \quad (2.41)$$

with all other commutators or anti-commutators equal zero as dictated by the spin and statistics assignments of Yukawa theory, QED or QCD.

The initial mass parameters $m_\varepsilon^{(0)}$ and $\mu_\varepsilon^{(0)}$ do not include effects of any interactions and are independent of the interaction strength. We may have

to consider limits where the mass parameters are close to zero, in comparison to all other quantities of relevance to physics. For example, $\mu_\varepsilon^{(0)}$ may be the infrared regulator mass denoted by μ_δ . Recall that the subscript ε indicates that the mass parameters stand in the Hamiltonian with $\lambda = \infty$.

The initial Hamiltonian contains an interaction part, $\mathcal{G}_2^{(0)} = H_\varepsilon^{(0)} - \mathcal{G}_1^{(0)}$. For example, electrons may emit photons. One writes the corresponding interaction term in QED as

$$\sum_{\sigma_1 \sigma_2 \sigma'_1} \int [k_1][k_2][k'_1] 16\pi^3 \delta^3(k_1 + k_2 - k'_1) \bar{u}_{m_\varepsilon^{(0)} k_1 \sigma_1} e \not{\epsilon}_{k_2 \sigma_2}^* u_{m_\varepsilon^{(0)} k'_1 \sigma'_1} b_{k_1 \sigma_1}^\dagger a_{k_2 \sigma_2}^\dagger b_{k'_1 \sigma'_1}. \quad (2.42)$$

We use conventions to be specified shortly. The Hamiltonian term (2.42) is contained in the expression

$$h = \int dx^- d^2 x^\perp \left[e \bar{\psi}_{m_\varepsilon^{(0)}}(x) \not{A}(x) \psi_{m_\varepsilon^{(0)}}(x) \right]_{x^+=0}, \quad (2.43)$$

where the fields $\psi_{m_\varepsilon^{(0)}}(x)$ and $A^\nu(x)$ for $x^+ = 0$ are defined by writing

$$\psi_{m_\varepsilon^{(0)}}(x) = \sum_\sigma \int [k] \left[u_{m_\varepsilon^{(0)} k \sigma} b_{k \sigma} e^{-ikx} + v_{m_\varepsilon^{(0)} k \sigma} d_{k \sigma}^\dagger e^{ikx} \right] \quad (2.44)$$

and

$$A^\nu(x) = \sum_\sigma \int [k] \left[\varepsilon_{k \sigma}^\nu a_{k \sigma} e^{-ikx} + \varepsilon_{k \sigma}^{\nu*} a_{k \sigma}^\dagger e^{ikx} \right]. \quad (2.45)$$

Spinors $u_{mk\sigma}$ and $v_{mk\sigma}$ are defined by boosting spinors for fermions at rest, $u_{m\sigma}$ and $v_{m\sigma}$, to the momentum k , as if the fermion mass were m . This is done using the light-front kinematical boost representation for fermions

$$S(m, k) = (mk^+)^{-1/2} [\Lambda_+ k^+ + \Lambda_- (m + \alpha^\perp k^\perp)]. \quad (2.46)$$

Namely, $u_{mk\sigma} = S(m, k)u_{m\sigma}$ and $v_{mk\sigma} = S(m, k)v_{m\sigma}$. Solving constraint equations for the free fermion fields in canonical field theory amounts to using these spinors. The same boost operation defines the polarization vectors for photons which are independent of the photon mass. We have $\varepsilon_{k\sigma} = (\varepsilon_{k\sigma}^+ = 0, \varepsilon_{k\sigma}^- = 2k^\perp \varepsilon_\sigma^\perp / k^+, \varepsilon_{k\sigma}^\perp = \varepsilon_\sigma^\perp)$. The spin label σ denotes the spin projection on the z-axis. We adopt a number of conventions from Ref. [22]. It is useful to work with the above spinors and polarization vectors because they provide insight into the physical interpretation of the calculated matrix elements. For example, the spinors and polarization vectors help in tracing cancelations which result from the current conservation (*e.g.* see Eq. (3.103) *etc.* in the next Section).

Equation (2.43) includes 5 terms in addition to (2.42). The other 5 terms lead to emission of photons by positrons, absorption of photons by electrons or positrons, or to transitions between electron-positron pairs and photons. There is no term leading to creation of an electron-positron pair and a photon, or to annihilation of such three particles. This is the distinguished property of the light-front Hamiltonians: conservation of momentum $k^+ > 0$ excludes a possibility that the three momenta sum up to zero.

Strictly speaking, one has to limit each k^+ from below by a nonzero positive lower bound in order to make sure that the three $+$ -momentum components cannot add up to zero. This lower bound is provided by the inverse of the quantization volume. Our cutoffs and scale hierarchy ensure that this largest of spatial scales in the theory does not need to be invoked in the description of physical phenomena. The regularization procedure cuts off such small momenta long before they have a chance to become relevant. If high-order perturbation theory subsequently leads to effective Hamiltonians which describe universal low momentum components in all physical states the notion of a nontrivial vacuum has to be taken seriously into account for practical computational reasons. *A priori*, we cannot exclude this will happen. But we postpone considerations of such a situation until it becomes necessary in the future work.

The product $\bar{\psi} \mathcal{A} \psi$ denotes a sum of 6 basic interactions. The products of creation and annihilation operators are ordered as indicated at the beginning of this Section. However, Eq. (2.43) requires additional steps before one can assign it a well defined meaning because operators such as (2.42) can easily produce states of infinite norm. One needs to define the individual terms such as (2.42) in order to provide meaning to the whole combination of similar terms in Eq. (2.43)

There are inverse powers of k^+ in Eq. (2.42) and k^+ may be arbitrarily close to 0. For example, when k_3^+ and k_1^+ in (2.42) are similar (and they are allowed to be arbitrarily close to each other no matter what their own size is), the photon momentum $k_2^+ = k_3^+ - k_1^+$ is arbitrarily close to zero. The problem is that the photon momentum appears in the photon polarization vector in the denominator: $\varepsilon_{k_2 \sigma_2}^- = 2k_2^\perp \varepsilon_{\sigma_2}^\perp / k_2^+$. Unless $k_2^\perp$ is close to zero the resulting emission strength approaches ∞ for $k_2^+ \rightarrow 0$. Therefore, even for a very small coupling constant e , the interaction can be arbitrarily strong. This divergence is canceled in special circumstances. For example, in the tree diagrams for the S-matrix elements in QED, the cancelation is a consequence of the presence of more terms in the Hamiltonian and the energy and charge current conservation in physical processes. However, for the off-energy-shell matrix elements of the T-matrix, in loop diagrams, or in bound state equations, such cancelations will not be ensured automatically and could lead to ill-defined expressions.

In particular, one has to keep in mind that in the perturbative calculation of the S-matrix it is possible to apply energy and momentum conservation laws for incoming and outgoing particles on their mass-shells. In contrast, in the bound state calculations, the individual particle momenta cannot simultaneously be on the individual mass-shells and still sum up to the bound state momentum — the bound state dynamics is always off-shell and the on-shell perturbative mechanisms for cancelations cease to be sufficient.

In Eq. (2.42), the inverse powers of the longitudinal momentum also appear in the fermion spinors. These can be a source of divergences too. However, the examples we describe in this article do not lead to problems with infrared fermion divergences and we do not dwell on this subject here.

The spinor matrix elements depend on the transverse momenta of the fermions and the boson polarization vector depends on the transverse momentum of the boson. The strength of the interaction grows when the relative transverse momenta grow and leads to divergences. The divergence problem manifests itself clearly when one attempts to evaluate the ratio of norms of the states $h|k\sigma\rangle$ and $|k\sigma\rangle$. This ratio is certainly not finite and it is not well defined.

One might ask if it is useful to consider the ill-defined Hamiltonian term (2.43). The answer is unambiguous yes because scattering amplitudes calculated using this term in combination with two other terms in second order perturbation theory agree very well with observable scattering of electrons and photons. No loop integration appears in these calculations to indicate the divergence problem.

It is well known that the terms one should put into the light-front Hamiltonian are provided by the formal Lagrangian density for electrodynamics $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\mathcal{D} - m)\psi$. One can rewrite the Lagrangian density into a corresponding light-front Hamiltonian density by using an expression for the energy-momentum tensor density $T^{\mu\nu}$. Integrating T^{+-} over the light-front hyper-plane gives the expression one starts from in building the light-front Hamiltonian for QED.

The initial Hamiltonian $H_\varepsilon^{(0)}$ for QED results from formal operations on fields ψ_+ and $A^\perp$ [23]. One uses the gauge $A^+ = 0$ and solves the constraint equations, substitutes expansions of the form (2.44) and (2.45) into the formal expression for T^{+-} , integrates the density over the light-front hyper-plane and normal-orders all terms. The normal-ordering produces terms that involve numerically divergent momentum integrals. The classical field theory does not tell us what to do with the divergences resulting from the ordering of operators.

To deal with the divergences one has to regularize the Hamiltonian theory from the outset. The naive connection between the classical theory and the quantum theory as given by the quantization rules is broken by the regular-

ization. The regularization turns out to force new terms in the Hamiltonian. To gain control on the regularization effects one has to construct a renormalization theory for Hamiltonians. The diverging terms which result from normal ordering can be safely dropped in the form they appear ill-defined in the canonical approach because the renormalization procedure introduces other terms of the same operator structure to replace them.

The regularization for light-front Hamiltonians which we apply to expressions resulting from field theoretic Lagrangian densities is first described for the term (2.42). In that term, the parent momentum P equals k'_1 . The spinors and polarization vectors conveniently group a number of terms with different momentum dependences into a combination which is invariant under light-front kinematical symmetry transformations. Among those terms there are terms containing masses, terms which in field theory result from derivatives $i\partial^\perp$ or $i\partial^+$ or from inverting the operator $i\partial^+$. All those derivatives are replaced in the term (2.42) by momenta of particles created or destroyed by that term.

We first introduce the daughter momentum variables for the created electron and photon. We have introduced daughter momenta in a similar configuration in Section 2.1 while defining the similarity functions f_λ . Here, we use the daughter momentum variables for the purpose of regularization. The variables are

$$x_1 = k_1^+/k_1'^+ = x, \quad (2.47a)$$

$$x_2 = k_2^+/k_1'^+ = 1 - x, \quad (2.47b)$$

$$x'_1 = k_1'^+/k_1'^+ = 1, \quad (2.47c)$$

$$\kappa_1^\perp = k_1^\perp - x_1 P^\perp = \kappa^\perp, \quad (2.47d)$$

$$\kappa_2^\perp = k_2^\perp - x_2 P^\perp = -\kappa^\perp, \quad (2.47e)$$

$$\kappa_1'^\perp = k_1'^\perp - x'_1 P^\perp = 0. \quad (2.47f)$$

For each creation and annihilation operator in the interaction term (2.42) we define a *daughter energy* variable. Namely,

$$e_1 = \frac{\kappa_1^{\perp 2} + m_\varepsilon^{(0)2}}{x_1} = \frac{\kappa^{\perp 2} + m_\varepsilon^{(0)2}}{x}, \quad (2.48a)$$

$$e_2 = \frac{\kappa_2^{\perp 2} + \mu_\varepsilon^{(0)2}}{x_2} = \frac{\kappa^{\perp 2} + \mu_\varepsilon^{(0)2}}{1 - x}, \quad (2.48b)$$

$$e'_1 = \frac{\kappa_1'^{\perp 2} + m_\varepsilon^{(0)2}}{x'_1} = m_\varepsilon^{(0)2}. \quad (2.48c)$$

For each creation and annihilation operator in the interaction term (2.42) we introduce a factor which is a function, $r(y_i)$, of the variable $y_i = \varepsilon e_i/\Lambda^2$,

where the subscript i denotes the operator in question. In the no cutoff limit, $\varepsilon \rightarrow 0$. Λ is an arbitrary constant with dimension of a mass ($\hbar = c = 1$). All masses and momenta are measured in units of Λ . In this article, we choose $r(y) = (1 + y)^{-1}$. Thus, the term (2.42) is regulated by the factor

$$(1 + \varepsilon e_1/\Lambda^2)^{-1}(1 + \varepsilon e_2/\Lambda^2)^{-1}(1 + \varepsilon e_3/\Lambda^2)^{-1} \quad (2.49)$$

under the integral. The third factor in the above expression can be replaced by 1, since $m_\varepsilon^{(0)}$ is a finite constant and it cannot compensate the smallness of ε . We shall make such replacements wherever the parent momentum is carried by a single creation or annihilation operator.

In the case of terms which contain only 1 creation and 1 annihilation operator, *i.e.* in $\mathcal{G}_1^{(0)}$, no regularization is introduced. Restrictions on the particle momenta in these terms would violate kinematical symmetries of the light-front Hamiltonian dynamics because momenta in these terms are equal to the parent momenta and limiting the parent momenta violates the light-front boost invariance.

In the initial expressions for Hamiltonian densities of Yukawa theory, QED or QCD, only terms with products of up to four fields appear. Therefore, we have only two more situations to consider in addition to the cases such as $\mathcal{G}_1^{(0)}$ and terms of the type (2.42). In the first situation we have three creation operators and one annihilation operator or vice versa, and in the second situation we have two creation and two annihilation operators. Both cases are regularized using the same general rule.

Independently of the number of creation and annihilation operators in a product, the regularization is introduced by multiplying every creation and annihilation operator in the product by a function $r(y)$ such as in the factor (2.49), where $y = \varepsilon e_d/\Lambda^2$ and e_d is the corresponding daughter energy variable. Later, after counterterms are calculated, the same regularization factors are introduced in the counterterms.

An additional step is required in the case of Hamiltonian terms which originate from the products of four fields including inverse powers of $i\partial^+$ acting on a product of two fields. We introduce two kinds of a fifth daughter momentum and two corresponding daughter energy variables, e_{512} and e_{534} . The numbering originates from assigning numbers to the fields in the product according to the schematic notation $\phi_1\phi_2(i\partial^+)^{-n}\phi_3\phi_4$. One of the fifth daughter energy variables is associated with the operators coming from the fields number 1 and 2, and the other one is associated with the operators coming from the fields number 3 and 4. The regularized terms will contain an additional product of functions $r(y_{512})$ and $r(y_{534})$ with the arguments $y_{512} = \varepsilon e_{512}/\Lambda^2$ and $y_{534} = \varepsilon e_{534}/\Lambda^2$.

The auxiliary daughter energy variables e_{512} and e_{534} are calculated as if they represented daughter energy variables for an intermediate particle, a

boson or a fermion, created and annihilated in the vertices which contained the products $\phi_1\phi_2$ and $\phi_3\phi_4$, respectively. Those vertices are treated as if each of them contained three fields instead of two but the field of the intermediate particle was contracted so that the corresponding creation operator and the corresponding annihilation operator are absent in the resulting term. This particular definition of a gedanken intermediate particle does not refer to any particular Fock state and remains valid when the operators q_∞ are replaced with q_λ by the unitary transformation U_λ . The definition was inspired by Refs. [23] and [24] where the correspondence between the intermediate states with backward moving particles with spin in the infinite momentum frame and the light-front seagull interaction terms is extensively described.

Mathematically, the definition of e_{512} and e_{534} is introduced in the following way. Every creation and annihilation operator in the fields ϕ_1, ϕ_2, ϕ_3 and ϕ_4 is assigned a corresponding number s_i , $i=1,2,3,4$. s_i equals $+1$ for a creation operator and s_i equals -1 for an annihilation operator. We define $k_5^+ = |s_3k_3^+ + s_4k_4^+|$ and $s_5 = (-s_3k_3^+ - s_4k_4^+)/k_5^+$. The gedanken particle is thought to be created in the product of fields including $\phi_3\phi_4$ when $s_5 = 1$ and it is thought to be annihilated in that product when $s_5 = -1$. We define the momentum $k_5 = (k_5^+, k_5^\perp)$ by the relation $s_5k_5 = -s_3k_3 - s_4k_4 = s_1k_1 + s_2k_2$. We also introduce two auxiliary parent momenta, $P_{34} = \frac{1}{2}(k_5 + k_3 + k_4)$ and $P_{12} = \frac{1}{2}(k_5 + k_1 + k_2)$. Then, we introduce the daughter momentum and energy variables

$$x_{512} = k_5^+ / P_{12}^+, \quad (50a)$$

$$\kappa_{512}^\perp = k_5^\perp - x_{512} P_{12}^\perp, \quad (50b)$$

$$e_{512} = \frac{\kappa_{512}^{\perp 2} + m_{\varepsilon 5}^{(0)2}}{x_{512}}, \quad (50c)$$

$$x_{534} = k_5^+ / P_{34}^+, \quad (50d)$$

$$\kappa_{534}^\perp = k_5^\perp - x_{534} P_{34}^\perp, \quad (50e)$$

$$e_{534} = \frac{\kappa_{534}^{\perp 2} + m_{\varepsilon 5}^{(0)2}}{x_{534}}, \quad (50f)$$

where $m_{\varepsilon 5}^{(0)}$ equals $m_\varepsilon^{(0)}$ for regularization of the terms involving $(i\partial^+)^{-1}$ and $m_{\varepsilon 5}^{(0)}$ equals $\mu_\varepsilon^{(0)}$ for regularization of the terms involving $(i\partial^+)^{-2}$. This step completes our definition of the ultraviolet regularization of initial Hamiltonians.

We proceed to the definition of the infrared regularization. Inverse powers of $i\partial^+$ for massive particles are already regulated when the ultraviolet regularization is imposed. This was explained above.

For each creation and annihilation operator of an initially massless particle we introduce a factor which limits the daughter momentum fraction x for that operator to be greater than δ . An example of such a factor is given by $(1 + \delta/x)^{-1}$. Note that our definition also implies that the same regularization factor is inserted for the gedanken particles with x_{512} defined in Eq. (2.50a) and x_{534} defined in Eq. (2.50d).

Besides introducing the cutoff δ on the momentum fractions carried by massless particles, we can also introduce for each initially massless particle a finite regularization mass term which is denoted by m_δ . In other words, in the case of the initially massless particles, $m_\varepsilon^{(0)} = m_\delta$. Such finite masses in the daughter energies lead to additional suppression of the infrared longitudinal momentum region. The additional mass terms are introduced through mass counterterms which contain unknown finite parts. Since the finite parts are arbitrary and not known to be zero we introduce the finite mass terms and investigate their role.

2.3. Renormalization conditions

The free finite parts of counterterms are determined by renormalization conditions which result from comparison of theoretical predictions with data. Calculations of observables require solutions to bound state or scattering problems using renormalized Hamiltonians. In principle, one could work with Hamiltonians of any width λ . In practice, one is limited to consider some subspaces in the Fock space. Therefore, the issue of setting renormalization conditions is subtle.

In theories with small coupling constants and without confinement one has an option of defining on-mass-shell renormalization conditions for single particles and scattering states in perturbation theory. It means that one can determine free parameters in the effective Hamiltonians by demanding that single particle eigenstates of an effective Hamiltonian and the S-matrix calculated using this Hamiltonian have the required properties. The key examples to be discussed in detail in the next Section are Yukawa theory (pseudoscalar coupling) and QED.

In theories with confinement one has to choose mass parameters for confined particles and these are not directly observable. We suggest in this case to use similar renormalization conditions in perturbative calculations of effective Hamiltonians as in QED. Details are described in Section 3.3. Besides ultraviolet the perturbative self-interactions of quarks and gluons diverge also in the infrared region where intermediate states have similar energies to the outer states and the effective dynamics is no longer perturbative. Therefore, the perturbatively renormalized mass terms introduce large infrared effects in the effective dynamics where perturbative cancela-

tion mechanisms are no longer valid. These large infrared effects are welcome as a source of confining potentials suggested by Perry [12].

The key question we have to answer in practice is how many effective particles have to be taken into account to solve the effective eigenvalue problem and how many can be included in a doable calculation. A good example of a theoretical problem one can think of is how the momentum or spin of a proton is shared by its constituents. The phenomenology of deep inelastic scattering of leptons and nucleons suggests a considerable number of constituents even at moderate momentum transfers. If the number of effective constituents has to be large one may encounter ambiguities in the determination of free parts of counterterms because observables will be calculable only through complicated procedures. On the other hand, the constituent quark model suggests that the large number of constituents is not needed to explain main features of the spectrum of hadrons. Therefore, one can expect that many states are important in the large width Hamiltonian dynamics but only a few effective particles appear in the small width case. The renormalization conditions set through the small width dynamics will use a small number of constituents but require non-perturbative solutions for the spectrum.

$\mathcal{G}_\lambda$ in Eq. (2.35) contains the unknown finite parts of counterterms in $\mathcal{G}_{\varepsilon \text{ finite}}(\lambda_0)$. The Hamiltonian H_{λ_0} can be used to calculate scattering amplitudes and bound state properties. The most familiar example of QED is largely perturbative as far as renormalization is concerned in order α . One can calculate the physical electron energy defined as the lowest eigenvalue of the effective Hamiltonian for the eigenstates with electron quantum numbers. Thanks to the symmetries of the light-front dynamics the eigenvalue has the form $(p^\perp{}^2 + m_e^2)/p^+$ and m_e has to be equal to the physical electron mass. Note also that the effective mass term for the interacting photons must be different from zero (and growing with λ) in order to obtain massless photon eigenstates. Examples of the renormalization conditions for QED are presented in the next Section. The same procedure in QCD is expected to lead to strong infrared effects because the non-abelian QCD interactions prevent the same cancelation of infrared divergences as in QED.

Hamiltonian belongs to the algebra of Poincaré generators. The Poincaré algebra commutation relations can be studied order by order in perturbation theory to find out constraints the algebra imposes on the counterterms. The general structure of the similarity transformation for creation and annihilation operators allows extension of the Hamiltonian renormalization procedure to the whole algebra. The renormalization group evolution is given by the same Eq. (2.5) for all generators. Renormalization of the Poincaré algebra is not further analysed in this article [25].

3. Examples of application

This Section describes a set of examples of lowest order calculations of renormalized effective Hamiltonians using the scheme from Section 2. We begin by the description of generic rules for calculating the right-hand side of the renormalization group equation (2.29). The rules follow from the commutator structure. Then, we discuss examples from Yukawa theory, QED and QCD.

3.1. Evaluation of commutators

The right-hand sides of Eqs. (2.29) and (2.38) are commutators. This implies that the interactions which appear in the Hamiltonians $H_\lambda(q_\lambda)$ and in the counterterms in H_ε are connected. This Section explains how this result comes about.

The commutators can be evaluated in a number of equivalent ways but some of the ways are more convenient than others. Suppose we are to evaluate

$$\hat{H} = [\hat{A}, \{\hat{B}\}_{\hat{C}}]. \quad (3.1)$$

$\hat{A} = A(X, Y) \prod_{i=1}^{I_A} a_{x_i}^\dagger \prod_{j=1}^{J_A} a_{y_j}$, $\hat{B} = B(V, W) \prod_{k=1}^{I_B} a_{v_k}^\dagger \prod_{l=1}^{J_B} a_{w_l}$ and $\hat{C} = \sum_z E(z) a_z^\dagger a_z$. The right-hand side of Eq. (3.1) equals

$$\begin{aligned} \hat{H} = & A(X, Y) \prod_{i=1}^{I_A} a_{x_i}^\dagger \prod_{j=1}^{J_A} a_{y_j} \frac{B(V, W)}{E_w - E_v} \prod_{k=1}^{I_B} a_{v_k}^\dagger \prod_{l=1}^{J_B} a_{w_l} \\ & - \frac{B(V, W)}{E_w - E_v} \prod_{k=1}^{I_B} a_{v_k}^\dagger \prod_{l=1}^{J_B} a_{w_l} A(X, Y) \prod_{i=1}^{I_A} a_{x_i}^\dagger \prod_{j=1}^{J_A} a_{y_j}, \end{aligned} \quad (3.2)$$

where $E_w = \sum_{l=1}^{J_B} E(w_l)$ and $E_v = \sum_{k=1}^{I_B} E(v_k)$. By commuting $\prod_{j=1}^{J_A} a_{y_j}$ in the first term through $\prod_{k=1}^{I_B} a_{v_k}^\dagger$ one generates the contracted terms with a number of contractions ranging from 1 to the smaller of the numbers J_A and I_B , and a term with $\prod_{j=1}^{J_A} a_{y_j}$ standing to the right of $\prod_{k=1}^{I_B} a_{v_k}^\dagger$. Then, by commuting $\prod_{l=1}^{J_B} a_{w_l}$ in the latter term through $\prod_{i=1}^{I_A} a_{x_i}^\dagger$, one obtains new contracted terms with the number of contractions ranging from 1 to the smaller of the numbers I_A and J_B , and a term equal to the second term in Eq. (3.2) with an opposite sign which thereby is canceled out leaving only connected terms in the result for $\hat{H}$. This result holds despite anti-commutation relations for fermions because interactions contain even numbers of fermion operators.

After the second term in Eq. (3.2) is canceled one is left with a number of partially contracted terms in which annihilation operators may still stand

to the left of creation operators. A number of ordering transpositions need to be done before a generic ordering of operators adopted in the previous Section is achieved. In fact, the process of commuting factors in $\hat{A}$ through factors in $\{\hat{B}\}_{\hat{C}}$ in the first term on the right-hand side of Eq. (3.2) produced above a number of terms with creation operators moved to the right of annihilation operators unnecessarily. These transpositions have to be undone to recover final answers with the adopted ordering. Nevertheless, it is visible that disconnected terms cannot appear and the following rule simplifies the calculations.

The right-hand side of Eq. (3.2) equals the sum of the contracted terms which result from $\hat{A}\{\hat{B}\}_{\hat{C}}$ by moving $\prod_{j=1}^{J_A} a_{y_j}$ through $\prod_{k=1}^{I_B} a_{v_k}^\dagger$ and, the contracted terms which result from $-\{\hat{B}\}_{\hat{C}}\hat{A}$ by moving $\prod_{l=1}^{J_B} a_{w_l}$ through $\prod_{i=1}^{I_A} a_{x_i}^\dagger$. All other terms cancel out.

3.2. Yukawa theory

The standard procedure from Ref. [23] leads from the Lagrangian density $\mathcal{L}_Y = \bar{\psi}(i \not{\partial} - m - g\phi)\psi + \frac{1}{2}(\partial^\mu \phi \partial_\mu \phi - \mu^2 \phi^2)$ to the light-front Hamiltonian expression of the form

$$H_Y = \int dx^- d^2 x^\perp \left[\bar{\psi}_m \gamma^+ \frac{-\partial^{\perp 2} + m^2}{2i\partial^+} \psi_m + \frac{1}{2} \phi (-\partial^{\perp 2} + \mu^2) \phi + g \bar{\psi}_m \psi_m \phi + g^2 \bar{\psi}_m \phi \frac{\gamma^+}{2i\partial^+} \phi \psi_m \right]_{x^+=0}. \quad (3.3)$$

We replace fields $\psi_m(x)$ and $\phi(x)$ for $x^+ = 0$ by the Fourier superpositions of creation and annihilation operators, order the operators in all terms and drop the terms containing divergent integrals which result from the contractions. Then, we introduce the regularization factors.

In the course of calculating effective Hamiltonians we will also add new terms to H_Y due to the presence of the regularization, in accord with the renormalization theory from the previous Section. For example, we will add a small term $\delta m_\varepsilon^2 = m_\varepsilon^2 - m^2$ to m^2 in the first term and $\delta \mu_\varepsilon^2 = \mu_\varepsilon^2 - \mu^2$ to μ^2 in the second term. We will calculate these terms below using the renormalization theory to order g^2 .

In order to consider particles with quantum numbers of nucleons and pions one needs to include the isospin and replace the scalar coupling by $i\gamma_5$ [17]. However, for the purpose of the illustration of the renormalization procedure to second order in the coupling g , we do not have to introduce these explicitly. The additional factors merely lead to somewhat different algebra which can be traced throughout the whole calculation and final

results including isospin and $i\gamma_5$ can be read from the results in the Yukawa theory. In this Section we assume $m > \mu > 0$.

3.2.1. Meson mass squared

The simplest example of a second order expression for a term in an effective Hamiltonian in the Yukawa theory is provided by the meson mass squared. We first describe steps which produce this expression. The number of distinct steps in the procedure is 10: defining the regularized initial Hamiltonian, calculation of the effective Hamiltonian, analysis of the cutoff dependence of finite matrix elements of the calculated terms and extraction of the structure of the divergence, evaluation of the counterterm, isolation of the finite part, calculation of the effective Hamiltonian knowing the structure of the counterterm, solving a physical problem such as an eigenvalue problem or a scattering problem using the effective Hamiltonian, adjusting the finite part of the counterterm to match data (including adjustments for the observed symmetries), and computing the final expression for the effective Hamiltonian with the counterterm finite part determined from the fit to data.

The simplest example is described in full detail of the 10 steps. Such extensive presentation is not provided in later examples where more complicated expressions would require too much space. The first example is discussed in such detail despite the fact that in this case it is easy to predict the answer.

For example, one might propose the structure of the counterterm using, as is usually done, some scattering amplitude instead of the matrix elements of an effective Hamiltonian. Note that one can also impose renormalization conditions using a scattering amplitude which results from a calculation performed without use of the effective Hamiltonian.

However, the systematic approach from Section 2 is the only tool we have for dealing with more complicated cases of light-front Hamiltonians and their eigenvalue equations. In other words, the simplest available case is used to present all the steps in detail because it illustrates the procedure in a familiar setting. When we proceed to more complicated interactions details of the calculation are discussed only where a new feature appears.

Equation (2.36) implies to second order in $\mathcal{G}_{2\lambda}$ that

$$\frac{d}{d\lambda}\mathcal{G}_{1\lambda} = \left[\mathcal{G}_{12\lambda} \frac{\frac{d}{d\lambda}f^2(z_\lambda^2)}{\mathcal{G}_{1\lambda} - E_{1\lambda}} \mathcal{G}_{21\lambda} \right]_{11} + \sum_{p=3}^{\infty} \left[\mathcal{G}_{1p\lambda} \frac{\frac{d}{d\lambda}f^2(z_\lambda^2)}{\mathcal{G}_{1\lambda} - E_{1\lambda}} \mathcal{G}_{p1\lambda} \right]_{11}, \quad (3.4)$$

where the double-digit subscripts refer to the number of creation and annihilation operators (in that order) and the bracket subscript denotes the part which contributes to the rate of change of $\mathcal{G}_{1\lambda}$ with λ . $E_{1\lambda}$ is the eigenvalue of

$\mathcal{G}_{1\lambda}$ which corresponds to the creation and annihilation operators indicated by the subscript 11. The reason for that only one free energy eigenvalue appears in the denominators is that $\mathcal{G}_{1\lambda}$ of Eq. (3.4) is a one-body operator and quantum numbers which label creation and annihilation operators in $\mathcal{G}_{1\lambda}$ are the same, including momentum. Therefore, the free energy eigenvalues are also the same: both are equal to $E_{1\lambda}$. Consequently, all commutators are written on the right-hand side of Eq. (3.4) in the simplified form. The numerator similarity factors reduce to the derivative of f_λ^2 (we have chosen $n = 1$ in Eq. (2.19)). Terms with more than two intermediate particles ($p \geq 3$) are of order g^4 or higher.

Assuming that g in Eq. (3.3) is extremely small, writing $\mathcal{G}_{1\lambda}$ as a series in powers of g and keeping only terms order g^2 , we obtain the following result from Eq. (3.4) for the meson free energy term.

$$\mathcal{G}_{1\text{ meson } \lambda} = \int [k] \frac{k^{\perp 2} + \mu_\lambda^2}{k^+} a_k^\dagger a_k. \quad (3.5)$$

A remarkable feature in this result is that no correction arises to the term $k^{\perp 2}/k^+$ which is protected by the kinematical symmetries; the total transverse momentum does not appear in a boost invariant expression.

The width dependence of μ_λ is determined by the equation

$$\frac{d\mu_\lambda^2}{d\lambda} = g^2 \int [x\kappa] \frac{df^2(z_\lambda^2)}{d\lambda} \frac{[2\mathcal{M}^2 - 8m^2]}{\mathcal{M}^2 - \mu^2} r_\varepsilon(x, \kappa), \quad (3.6)$$

where $\mathcal{M}^2 = (\kappa^2 + m^2)/x(1-x)$. m^2 and μ^2 are the original bare mass squared parameters from Eq. (3.3). They do not include terms order g^2 and higher because such terms would lead to higher order corrections than g^2 for the whole expression. The terms order g^2 and higher are treated as interactions in the perturbative calculation.

In terms of graphs for the effective Hamiltonian calculus, Eq. (3.6) represents the contribution of a fermion loop on a meson line. However, the graphs are not provided in order to avoid confusion with other diagrammatic techniques.

$$\int [x\kappa] = (16\pi^3)^{-1} \int_0^1 \frac{dx}{x(1-x)} \int d^2\kappa^\perp. \quad (3.7)$$

Using Eq. (2.20) in the limit of a θ -function, $f(u) = \theta(u_0 - u)$, one obtains

$$f^2(z_\lambda^2) = \theta \left[\lambda^2 + \frac{1 + \sqrt{u_0}}{\sqrt{u_0}} \mu^2 - \frac{1 - \sqrt{u_0}}{\sqrt{u_0}} \mathcal{M}^2 \right]. \quad (3.8a)$$

For example, for $u_0 = \frac{1}{4}$ one has $f^2(z_\lambda^2) = \theta[\lambda^2 + 3\mu^2 - \mathcal{M}^2]$. Therefore, the derivative of f_λ with respect to λ forces the invariant mass of the fermion-anti-fermion pair, $\mathcal{M}^2$, to be equal $\lambda^2 + 3\mu^2$. The derivative selects the range of energies in the integral where the similarity function changes most rapidly. The regions where the function approaches a constant, *i.e.* 1 near the diagonal and 0 beyond the Hamiltonian width, are strongly suppressed. The region that contributes is the edge of the diagonal proximum [1]. The derivative of f_λ is large and positive in this region and it approaches a δ -function in the limit of Eq. (3.8).

In the limit of an infinitesimally small u_0 , as discussed below Eq. (2.20), one would substitute $\lambda^2 = u_0^{-1/2} \tilde{\lambda}^2$. Then,

$$f^2(z_\lambda^2) = \theta \left[\tilde{\lambda}^2 + \mu^2 - \mathcal{M}^2 \right]. \quad (3.8b)$$

The numerator factor in the square bracket in Eq. (3.6) originates from spinors of the intermediate fermions, $\text{Tr}(\not{p}_m + m)(\not{p}_m - m)$ with $p_m^2 = \bar{p}_m^2 = m^2$. The subscript m indicates that the $-$ component is calculated from the mass-shell condition knowing $+$ and $\perp$ components. $+$ and $\perp$ components of p and $\bar{p}$ are constrained by the light-front spatial momentum conservation law, $p + \bar{p} = k$, where k is the meson momentum. The pseudoscalar interaction with $i\gamma_5$ gives the same result with an additional term $+8m^2$ in the numerator.

According to Eq. (2.49),

$$r_\varepsilon(x, \kappa) = \left[1 + \varepsilon \frac{\mathcal{M}^2}{\Lambda^2} + \left(\varepsilon \frac{\mathcal{M}^2}{\Lambda^2} \right)^2 x(1-x) \right]^{-2}. \quad (3.9)$$

No infrared regularization is required in Yukawa theory with massive particles, $m > 0$ and $\mu > 0$.

If the regularization factors in Eq. (2.49) contain e_i divided by $1 - x_i$ in place of e_i one obtains here

$$r_\varepsilon(x, \kappa) = \left[1 + \varepsilon \left(\frac{\mathcal{M}^2}{\Lambda^2} \right) \right]^{-4} \quad (3.10)$$

instead of Eq. (3.9). The integrand function of x and κ^2 can be reduced to a function of $\mathcal{M}^2$. Such simplifications are helpful in a qualitative analysis of the cutoff dependence.

In the limit of Eq. (3.8a) for $u_0 = \frac{1}{4}$ one obtains

$$\frac{d\mu_\lambda^2}{d\lambda^2} = \frac{3\alpha}{2\pi} \left(1 + \frac{\mu^2}{\lambda^2 + 2\mu^2} \right) \theta(z_0^2) \left(\frac{2}{a} \right)^4 \int_0^{z_0} dz \frac{z^2}{[(1 + 2/a)^2 - z^2]^2}, \quad (3.11)$$

where $a = \varepsilon(\lambda^2 + 3\mu^2)/\Lambda^2$ and $z_0 = \sqrt{1 - 4m^2/(\lambda^2 + 3\mu^2)}$. Note that for $\lambda^2 \leq 4m^2 - 3\mu^2$ the derivative of the effective meson mass equals zero and the mass stays at the width independent value $\mu_{4m^2-3\mu^2}^2$. If one uses Eq. (3.10) instead, the corresponding result is

$$\frac{d\mu_\lambda^2}{d\lambda^2} = \frac{\alpha}{2\pi} \left(1 + \frac{\mu^2}{\lambda^2 + 2\mu^2} \right) \theta(z_0^2) z_0^3 (1+a)^{-4}. \quad (3.12)$$

Eqs. (3.11) and (3.12) are the same for $a \ll 1$ which is the limit of removing the regularization cutoff, $\varepsilon \rightarrow 0$, for a fixed Hamiltonian width λ . In this limit one has

$$\frac{d\mu_\lambda^2}{d\lambda^2} = \frac{\alpha}{2\pi} \left(1 + \frac{\mu^2}{\lambda^2 + 2\mu^2} \right) \left(1 - \frac{4m^2}{\lambda^2 + 3\mu^2} \right)^{3/2} \theta(\lambda^2 + 3\mu^2 - 4m^2). \quad (3.13)$$

If one assumes that the meson mass squared parameter in the effective Hamiltonian has some finite value, $\mu_0^2 = \mu_{\lambda_0}^2$ at some λ_0 such that $\lambda_0^2 \geq 4m^2 - 3\mu^2$ then, the integration of Eq. (3.13) demonstrates that

$$\mu_\lambda^2 = \mu_0^2 + \frac{\alpha}{2\pi} (\lambda^2 - \lambda_0^2) + \frac{\alpha}{2\pi} (\mu^2 - 6m^2) \log \frac{\lambda^2}{\lambda_0^2} + \mu_{\text{conv}}^2(\lambda, \lambda_0). \quad (3.14)$$

$\mu_{\text{conv}}^2(\lambda, \lambda_0)$ denotes the result of integrating the convergent part of the integrand,

$$\mu_{\text{conv}}^2(\lambda, \lambda_0) = \frac{\alpha}{2\pi} \int_{\lambda_0^2}^{\lambda^2} ds \left[\left(1 + \frac{\mu^2}{s + \mu^2} \right) \left(1 - \frac{4m^2}{s + 3\mu^2} \right)^{3/2} - 1 - \frac{\mu^2 - 6m^2}{s} \right]. \quad (3.14a)$$

$\mu_{\text{conv}}^2(\lambda, \lambda_0)$ has a finite (*i.e.* λ -independent) limit for large λ . It contains the terms which vanish for large λ as inverse powers of λ^2 . The dependence of $\mu_{\text{conv}}^2(\lambda, \lambda_0)$ on m and μ is not indicated explicitly because we will not need it in the discussion of counterterms. However, one should keep in mind that the mass parameters determine the value of $\lambda = \sqrt{4m^2 - 3\mu^2}$ where the effective mass stabilizes. We simplify our notation assuming that the effective cutoffs are always above the point of stabilization. Below the stabilization point, the meson mass has the constant value which is independent of λ . We will show later that the constant value is equal to the physical meson mass.

The mass squared term in the effective Hamiltonian with a non-negligible coupling g grows linearly with λ^2 . A logarithmic correction appears with an opposite sign due to the factor z_0^3 in Eq. (3.13), as indicated in Eq. (3.14). However, one cannot make contact in Eq. (3.14) with the initial Hamiltonian

by letting λ grow to infinity because one would obtain an ill-defined result. The factors depending on a in Eqs. (3.11) or (3.12) remove the infinite growth of μ_λ when $\lambda \rightarrow \infty$. Eq. (3.12) is simpler than Eq. (3.11) and illustrates the same idea so we start with Eq. (3.12).

Equation (3.12) can be integrated over λ from infinity to any finite value because the factor $(1+a)^{-4}$ provides convergence for $\lambda^2 > \Lambda^2/\varepsilon$. Nevertheless, the integral diverges as a function of ε when $\varepsilon \rightarrow 0$. The divergence appears as a single number. Therefore, the counterterm is also a number. We add $\mu_\varepsilon^2 - \mu^2$ to μ^2 in the initial Hamiltonian. We also write μ_ε^2 as a series in powers of g , $\mu_\varepsilon^2 = \mu^2 + \delta\mu_\varepsilon^2 + o(g^4)$ so that $\delta\mu_\varepsilon^2 \sim g^2$. Thus,

$$\begin{aligned} \mu_\lambda^2 = & \mu^2 + \delta\mu_\varepsilon^2 - \frac{\alpha}{2\pi} \int_{\lambda^2}^{\infty} ds \left(1 + \frac{\mu^2}{s + 2\mu^2}\right) \left(1 - \frac{4m^2}{s + 3\mu^2}\right)^{3/2} \left(1 + \varepsilon \frac{s + 3\mu^2}{\Lambda^2}\right)^{-4} \\ & + o(g^4). \end{aligned} \quad (3.15)$$

This is an example of Eq. (2.33) in a perturbative application to second order in powers of g in Yukawa theory. The counterterm $\delta\mu_\varepsilon^2$ will be calculated following the steps described below Eq. (2.33).

The diverging part of the integrand equals $\alpha/2\pi [1 + (\mu^2 - 6m^2)/s]$ and the remaining part is convergent. The convergent part of the integrand has the same structure as in $\mu_{\text{conv}}^2(\lambda, \lambda_0)$ but the integral now extends from λ^2 to infinity instead of from λ_0^2 to λ^2 . In the convergent part, one can replace the regulating factor by 1. Simplifications occur in the limit $\varepsilon \rightarrow 0$ and the result of integration in Eq. (3.15) is

$$\begin{aligned} \mu_\lambda^2 = & \mu^2 + \delta\mu_\varepsilon^2 + \frac{\alpha}{2\pi} \left[-\frac{\Lambda^2}{3} \frac{1}{\varepsilon} + (\lambda^2 + 3\mu^2) + (\mu^2 - 6m^2) \left(\log \varepsilon \frac{\lambda^2}{\Lambda^2} + \frac{11}{6} \right) \right] \\ & - \mu_{\text{conv}}^2(\infty, \lambda) + o(g^4), \end{aligned} \quad (3.16)$$

where the square bracket originates from the diverging part.

Following the procedure described below Eq. (2.33), we define the counterterm $\delta\mu_\varepsilon^2$ as the negative of the integral of the diverging integrand for some arbitrarily chosen $\lambda = \lambda_0$ plus an unknown finite piece corresponding to λ_0 and denoted by $\delta\mu_{\varepsilon \text{ finite}}^2(\lambda_0)$. Namely,

$$\begin{aligned} \delta\mu_\varepsilon^2 = & \frac{\alpha}{2\pi} \left[\frac{\Lambda^2}{3} \frac{1}{\varepsilon} + (\mu^2 - 6m^2) \log \frac{1}{\varepsilon} - (\lambda_0^2 + 3\mu^2) - (\mu^2 - 6m^2) \left(\log \frac{\lambda_0^2}{\Lambda^2} + \frac{11}{6} \right) \right] \\ & + \delta\mu_{\varepsilon \text{ finite}}^2(\lambda_0) + o(g^4), \end{aligned} \quad (3.17)$$

where

$$\delta\mu_{\varepsilon}^2{}_{\text{finite}}(\lambda_0) = \frac{\alpha}{2\pi} \left[\frac{A^2}{3} c_1 + (\mu^2 - 6m^2) c_2 + (\lambda_0^2 + 3\mu^2) + (\mu^2 - 6m^2) \left(\log \frac{\lambda_0^2}{A^2} + \frac{11}{6} \right) \right] + o(g^4) \quad (3.17a)$$

with the unknown finite numbers c_1 and c_2 . So, in fact,

$$\delta\mu_{\varepsilon}^2 = \frac{\alpha}{2\pi} \left[\frac{A^2}{3} \left(\frac{1}{\varepsilon} + c_1 \right) + (\mu^2 - 6m^2) \left(\log \frac{1}{\varepsilon} + c_2 \right) \right]. \quad (3.17b)$$

Since the whole expression on the right-hand side of Eq. (3.17) is merely a number, it is not necessary to find c_1 and c_2 or any other part of it separately. One can easily find the number $\delta\mu_{\varepsilon}^2{}_{\text{finite}}(\lambda_0)$ from the knowledge of μ_{λ}^2 at some value of λ . The resulting counterterm will render well defined finite boson mass squared parameter in the effective Hamiltonians in the limit $\varepsilon \rightarrow 0$. Using Eqs. (3.16) and (3.17) one obtains

$$\begin{aligned} \mu_{\lambda}^2 &= \mu^2 + \delta\mu_{\varepsilon}^2{}_{\text{finite}}(\lambda_0) + \frac{\alpha}{2\pi} \left[\lambda^2 - \lambda_0^2 + (\mu^2 - 6m^2) \log \frac{\lambda^2}{\lambda_0^2} \right] \\ &\quad - \mu_{\text{conv}}^2(\infty, \lambda) + o(g^4). \end{aligned} \quad (3.18)$$

Equation (3.18) is an example of Eq. (2.35).

The unknown finite term $\delta\mu_{\varepsilon}^2{}_{\text{finite}}(\lambda_0)$ has to be found by comparison with data, which might include symmetry requirements. We shall discuss an example of a renormalization condition later in this Section.

Let us assume now that at some arbitrarily chosen value of $\lambda = \lambda_1$ the effective meson mass squared required in Eq. (3.5) by a fit to data equals μ_1^2 , *i.e.* $\mu_{\lambda_1}^2 = \mu_1^2$. We can calculate $\delta\mu_{\varepsilon}^2{}_{\text{finite}}(\lambda_0)$ using Eq. (3.18) with $\lambda = \lambda_1$ and $\mu_{\lambda_1}^2$ on the left-hand side replaced by the number μ_1^2 inferred from the experimental data. The result is

$$\begin{aligned} \delta\mu_{\varepsilon}^2{}_{\text{finite}}(\lambda_0) &= \mu_1^2 - \mu^2 - \frac{\alpha}{2\pi} \left[\lambda_1^2 - \lambda_0^2 + (\mu^2 - 6m^2) \log \frac{\lambda_1^2}{\lambda_0^2} \right] \\ &\quad + \mu_{\text{conv}}^2(\infty, \lambda_1) + o(g^4). \end{aligned} \quad (3.19)$$

Note that one has to include the contribution of the convergent terms in the determination of the arbitrary constants when using the value of μ_1^2 . Knowing $\delta\mu_{\varepsilon}^2{}_{\text{finite}}(\lambda_0)$ one can calculate μ_{λ}^2 . Namely,

$$\mu_{\lambda}^2 = \mu_1^2 + \frac{\alpha}{2\pi} \left[\lambda^2 - \lambda_1^2 + (\mu^2 - 6m^2) \log \frac{\lambda^2}{\lambda_1^2} \right] + \mu_{\text{conv}}^2(\lambda, \lambda_1) + o(g^4). \quad (3.20)$$

As expected, Eq. (3.20) is the same as Eq. (3.14) when $\lambda_1 = \lambda_0$ and $\mu_1 = \mu_0$. One can also trace the origin of all the terms from Eq. (3.14); the diverging and converging terms and the counterterm in Eq. (3.15).

We can now analyze Eq. (3.11) analogously to Eq. (3.12) without calculating all integrals explicitly. Integrating both sides of Eq. (3.11), we have

$$\mu_\lambda^2 = \mu_\varepsilon^2 - \int_{\lambda^2}^{\infty} ds \frac{d\mu_s^2}{ds}. \quad (3.21)$$

By demanding that μ_ε^2 removes the diverging (*i.e.* ε -dependent in the limit $\varepsilon \rightarrow 0$) part of the integral, and leaving the finite part of μ_ε^2 free so that at some $\lambda = \lambda_0$ the effective boson mass squared parameter has a desired value μ_0^2 , we obtain

$$\mu_\lambda^2 = \mu_0^2 + \int_{\lambda_0^2}^{\lambda^2} ds \frac{d\mu_s^2}{ds}. \quad (3.22)$$

The integrand is given by the right-hand side of Eq. (3.11) with $\lambda^2 = s$. Since s ranges only from λ_0^2 to λ^2 and both are finite we can take the limit $\varepsilon \rightarrow 0$ under the integral sign and the integrand becomes equal to the right-hand side of Eq. (3.13) with λ^2 replaced with s . Integration over s produces exactly the same answer as given by Eq. (3.14). Thus, we see that the result of Eq. (3.14) is independent of the regularization scheme. It is determined by the initial Hamiltonian of Yukawa theory as given by Eq. (3.3). The only unknown in Eq. (3.14) is the value of μ_0^2 . More precisely, we know that $\mu_0^2 = \mu^2 + \delta\mu_0^2 + o(g^4)$ and the unknown term is $\delta\mu_0^2 \sim g^2$.

Note that the above calculations can be carried out in a different way using the following observation. Equation (3.6) in the lowest order of perturbation theory has a particularly simple structure. Namely, the only dependence on λ comes from the similarity function $f(z_\lambda^2)$ and both sides of the equation are equal to the derivatives with respect to λ . Therefore, one can directly integrate both sides and obtain a compact integral expression for μ_λ^2 for arbitrary functions f .

One should remember that such simplifications do not occur in higher order calculations or beyond simple perturbative expansion when the coupling constant and mass parameters depend on λ themselves. Therefore, we stress that the dominant contribution to the rate of change of μ_λ^2 with λ comes from the edge of the diagonal proximum. This fact remains generally valid and the procedure applied above represents a generic situation despite simplicity of the example. This example does not involve a distinction between the bare coupling g and a renormalized coupling because to order g^2 there is none.

In order to determine $\delta\mu_0^2$ we need to specify a renormalization condition. A natural requirement for μ_0^2 is that the effective Hamiltonian has one boson eigenstates parametrized by momenta p^+ and $p^\perp$ with eigenvalues of the form $p^- = (p^{\perp 2} + \tilde{\mu}^2)/p^+$ where $\tilde{\mu}$ is equal to the physical boson mass. Our approach preserves kinematical symmetries of the light-front frame explicitly and the eigenvalue is guaranteed to appear in that form. Therefore, one can calculate a whole spectrum of eigenvalues for eigenstates of different momenta by calculating the single mass parameter $\tilde{\mu}$.

In order to write the effective eigenvalue equation and find out μ_0^2 which leads to the desired value of $\tilde{\mu}$ (if it is possible), the following steps need to be taken.

One inserts Eq. (3.14) into Eq. (3.5). Then, one replaces the bare operators $a_k^\dagger$ and a_k in the whole $\mathcal{G}_\lambda$ by the effective ones, $a_{\lambda k}^\dagger$ and $a_{\lambda k}$, obtaining G_λ . Next, by applying the operation F_λ , one obtains the effective Hamiltonian with the form factors in the vertices, $H_\lambda = F_\lambda[G_\lambda]$.

The effective eigenvalue equation for bosons is an example of Eq. (1.2). Here, it is sufficient to consider the eigenvalue equation for H_λ in the expansion to second order in g since we are interested in $\delta\mu_0^2$ which is proportional to g^2 . The only terms that contribute are the free energy term including the effective mass squared and the interaction terms which change the particle number by one. The latter equal the canonical interactions with the similarity form factors.

To zeroth order in g , a physical meson state equals a single effective meson state, and $\tilde{\mu}^2 = \mu^2$.

No terms order g arise in $\tilde{\mu}^2$ and the next correction is order g^2 . This correction can be calculated using the operation R and the model Hamiltonian defined in a perturbative expansion from Eq. (1.3). Using expansion into a series of powers of g to second order, one can restrict the model space to the single effective boson sector. The effect of coupling to the fermion-anti-fermion pair states is included in perturbation theory.

Another method is to arbitrarily limit the number of effective Fock sectors and diagonalize the effective Hamiltonian in that limited space. Such procedure could be called the effective Tamm–Dancoff approach (ETD), *cf.* Refs. [9] and [10]. The term of the second order in g in the eigenvalue will determine $\delta\mu_0^2$. One can limit the space of states to one effective boson and effective fermion-anti-fermion pairs. Note that interactions in the ETD are tempered by the similarity factors of width λ which is on the order of particle masses and no ultraviolet divergence exists. This way our ETD approach overcomes the old problem of ultraviolet mass renormalization in the TD approach.

The model calculation using R and the ETD calculation have to agree with each other for small coupling constants in the presence of a finite energy

gap between states with different numbers of particles, which is the case here for $0 < \mu < 2m$. We discuss only second order corrections in g . Therefore, we can focus on a straight-forward perturbation theory anyways. Nevertheless, our simple calculations have two interesting aspects.

The first one is that no coupling renormalization effects arise to order g^2 . Therefore, the expansion in powers of g up to g^2 is equivalent to the expansion in powers of an effective coupling, no matter how the latter is defined. It is important to realize that the expansion in powers of g is understood here to be the expansion in powers of the effective coupling which appears in the effective Hamiltonian of width λ ; $g_\lambda = g$ to order g^2 . It is not meant to be the expansion in the initial field theory coupling constant.

The second aspect is following. The perturbative expansions applied in the effective eigenvalue problem are expansions in the interaction which is suppressed in strength by the similarity vertex form factor of width λ . Therefore, the range in momentum space of the effective interaction terms is infinitely smaller than the momentum range of the analogous interaction in the bare Hamiltonian. In other words, the effective strength of the interactions is greatly reduced and much smaller than the value of g itself would imply if it stood in the initial bare Hamiltonian. Our initial expansion in powers of g can now be understood as a shortcut to the expansion in powers of the effective coupling. The latter expansion may have a considerable range of rapid convergence because the form factors reduce the size of coefficients in the expansion. The effective coefficients are much smaller than in the case of the initial Hamiltonian without form factors.

Thus, the operation R on H_λ expanded in powers of the effective interaction (the coupling constant itself can be sizable), opens new options for studying the effective eigenvalue problem in the whole Fock space using the basis built with the effective creation and annihilation operators corresponding to the width λ . One can estimate contributions of various components by making different choices of the model spaces and solving model dynamical problems numerically. Wave functions are expected to fall off sharply for large momenta and particle numbers if g is not too large. The effective coupling constant needs to be set equal to the right value at λ .

The second order expression in perturbation theory implies

$$\begin{aligned} \frac{p^\perp{}^2 + \tilde{\mu}^2}{p^+} \langle p' | p \rangle &= \frac{p^\perp{}^2 + \mu_\lambda^2}{p^+} \langle p' | p \rangle \\ &- \langle p' | F_\lambda[G_{12\lambda}] \frac{1}{G_{1\lambda} - (p^\perp{}^2 + \mu^2)/p^+} F_\lambda[G_{21\lambda}] | p \rangle. \end{aligned} \quad (3.23)$$

$|p\rangle$ denotes a single effective meson state with momentum p^+ and $p^\perp$, $\langle p' | p \rangle = 16\pi^3 p^+ \delta^3(p' - p)$. The term order g^2 produces

$$\tilde{\mu}^2 = \mu_\lambda^2 - \int [x\kappa]gf(z_\lambda^2) \frac{[2\mathcal{M}^2 - 8m^2]}{\mathcal{M}^2 - \mu^2} gf(z_\lambda^2) + o(g^4), \quad (3.24)$$

where the notation is the same as in Eq. (3.6). Using Eq. (3.8) with $u_0 = \frac{1}{4}$ at $\lambda = \lambda_0$ one obtains

$$\mu_0^2 = \tilde{\mu}^2 + \frac{\alpha}{4\pi} \int_0^1 dx \int_0^\infty d\kappa^2 \frac{[2\mathcal{M}^2 - 8m^2]}{x(1-x)(\mathcal{M}^2 - \mu^2)} \theta(\lambda_0^2 + 3\mu^2 - \mathcal{M}^2) + o(g^4). \quad (3.25)$$

For $\lambda_0 \leq \sqrt{4m^2 - 3\mu^2}$ the effective meson mass parameter equals the physical meson mass, as promised. For fermions with masses order 0.9 GeV, this implies no corrections to a light meson mass such as μ_π for cutoffs smaller than 1.8 GeV. But one has to remember that the correction for the pseudoscalar πN interaction is different from Eq. (3.25), *i.e.* the spin factor has to be enlarged by $8m^2$ (see comments above Eq. (3.9)).

However, the actual measure of the off-shell effects is not given directly by μ_0^2 but by the sum of μ_0^2 and the self-energy resulting from the effective interactions. According to Eqs. (3.22) and (3.24), the sum of both contributions in the physical pion mass itself is actually equal zero to order α .

Using Eqs. (3.21), (3.22) and (3.25), one can express the meson mass squared term in the initial renormalized Hamiltonian in terms of the physical meson mass $\tilde{\mu}$ and the initial mass parameter μ . Namely, $\mu^2 = \tilde{\mu}^2 + o(g^2)$ and

$$\mu_\varepsilon^2 = \tilde{\mu}^2 + g^2 \int [x\kappa] \frac{[2\mathcal{M}^2 - 8m^2]}{\mathcal{M}^2 - \mu^2} r_\varepsilon(x, \kappa) + o(g^4). \quad (3.26)$$

3.2.2. Fermion mass squared

In complete analogy to Eqs. (3.4) to (3.6) one obtains the fermion energy operator,

$$\mathcal{G}_{1 \text{ fermion } \lambda} = \sum_\sigma \int [k] \frac{k^\perp{}^2 + m_\lambda^2}{k^+} (b_{k\sigma}^\dagger b_{k\sigma} + d_{k\sigma}^\dagger d_{k\sigma}). \quad (3.27)$$

Results for fermions and anti-fermions are identical. We have

$$\frac{dm_\lambda^2}{d\lambda} = g^2 \int [x\kappa] \frac{df^2(z_\lambda^2)}{d\lambda} \frac{\bar{u}_{m\sigma k}(\not{p}_m + m)u_{m\sigma k}}{\mathcal{M}^2 - m^2} r_\varepsilon(x, \kappa). \quad (3.28)$$

$\mathcal{M}^2 = (m^2 + \kappa^2)/x + (\mu^2 + \kappa^2)/(1-x)$. The regularization factor of Eq. (2.49) implies

$$r_\varepsilon(x, \kappa) = \left[1 + \frac{\varepsilon}{\Lambda^2} \mathcal{M}^2 + \left(\frac{\varepsilon}{\Lambda^2} \right)^2 \frac{\kappa^2 + m^2}{x} \frac{\kappa^2 + \mu^2}{1-x} \right]^{-2}. \quad (3.29)$$

The spin factor in Eq. (3.28) can be rewritten as

$$\bar{u}_{m\sigma k}(\not{p}_m + m)u_{m\sigma k} = \bar{u}_{m\sigma k}[x \not{k}_m + m + \frac{1}{2}\gamma^+(p_m^- - xk_m^-)]u_{m\sigma k}. \quad (3.30)$$

$\not{k}_m$ between spinors is equivalent to m . The term with γ^+ is typical in light-front calculations. Its part proportional to $k^\perp{}^2/k^+$ cancels out. The term linear in $k^\perp$ does not contribute because it is odd in $\kappa^\perp$ and all other factors including the regularization factor depend only on the modulus of $\kappa^\perp$. The spin factor is thus reduced to

$$\bar{u}_{m\sigma k} \left[(x+1)m + \frac{\gamma^+}{2k^+} \frac{\kappa^2 + (1-x^2)m^2}{x} \right] u_{m\sigma k} = \frac{\kappa^2 + (1+x)^2m^2}{x}. \quad (3.31)$$

Result for a pseudoscalar interaction with $i\gamma_5$ is the same except for the opposite sign in front of x in the numerator. Inclusion of the isospin introduces the number of bosons in the theory in front of the integral in Eq. (3.28).

We observe a similar structure in Eq. (3.28) as in the meson mass dependence on λ^2 in Eq. (3.6). Namely, there are terms diverging linearly and logarithmically and there is a series of convergent terms. We observe that the divergences amount to a number which grows when $\varepsilon \rightarrow 0$ and integrate both sides of Eq. (3.28) to obtain

$$m_\lambda^2 = m_\varepsilon^2 - g^2 \int [x\kappa] [1 - f^2(z_\lambda^2)] \frac{\kappa^2 + (1+x)^2m^2}{x(\mathcal{M}^2 - m^2)} r_\varepsilon(x, \kappa), \quad (3.32)$$

where, according to Eq. (2.18), $z_\lambda = (\mathcal{M}^2 - m^2)/(\mathcal{M}^2 + m^2 + \lambda^2)$. The ε -dependent terms originate from 1 in the bracket, which is independent of λ . The counterterm $\delta m_\varepsilon^2 \sim g^2$ in $m_\varepsilon^2 = m^2 + \delta m_\varepsilon^2 + o(g^4)$ removes the divergence. The finite part of the counterterm is left to be determined by data.

$$m_\lambda^2 = m^2 + \delta m_{\varepsilon \text{ finite}}^2 + g^2 \int [x\kappa] f^2(z_\lambda^2) \frac{\kappa^2 + (1+x)^2m^2}{x(\mathcal{M}^2 - m^2)}. \quad (3.33)$$

The value of $\delta m_{\varepsilon \text{ finite}}^2$ is determined from the value of m_λ^2 required in the effective Hamiltonian H_λ by some physical condition. If we had defined the divergent part by an integral from λ_0 we would have to take into account that $m_{\varepsilon \text{ finite}}^2$ depends on λ_0 to compensate for the λ_0 dependence of the diverging integral. When we define the counterterm to be given by the whole λ -independent part of the integral in Eq. (3.32), plus a finite constant to be determined by data, then $\delta m_{\varepsilon \text{ finite}}^2$ does not depend on λ_0 . Nevertheless, it can be expressed in terms of $m_{\lambda_0}^2$. For example, if the effective fermion mass squared at $\lambda = \lambda_0$ should be $m_{\lambda_0}^2 = m_0^2$ then,

$$m_\lambda^2 = m_0^2 + g^2 \int [x\kappa] [f^2(z_\lambda^2) - f^2(z_{\lambda_0}^2)] \frac{\kappa^2 + (1+x)^2m^2}{x(\mathcal{M}^2 - m^2)}. \quad (3.34)$$

$m_0^2 = m^2 + \delta m_0^2 + o(g^4)$. $\delta m_0^2 \sim g^2$ can be found from a renormalization condition for the physical fermion mass.

A natural condition for fitting m_0^2 is that the effective Hamiltonian at the scale λ_0 has fermionic eigenstates with eigenvalues of the form $p^- = (p^{\perp 2} + \tilde{m}^2)/p^+$, where $\tilde{m}$ denotes the physical fermion mass. In analogy to Eqs. (3.23) and (3.24) one obtains

$$\tilde{m}^2 = m_\lambda^2 - \int [x\kappa] g f(z_\lambda^2) \frac{\kappa^2 + (1+x)^2 m^2}{x(\mathcal{M}^2 - m^2)} g f(z_\lambda^2) + o(g^4). \quad (3.35)$$

So,

$$m_0^2 = \tilde{m}^2 + g^2 \int [x\kappa] f^2(z_{\lambda_0}^2) \frac{\kappa^2 + (1+x)^2 m^2}{x(\mathcal{M}^2 - m^2)} + o(g^4). \quad (3.36)$$

The initial m_ε^2 can be calculated in terms of m^2 , $\tilde{m}^2$, g and ε from Eq. (3.32). The effective fermion mass parameter in the interacting Hamiltonian of width λ is

$$m_\lambda^2 = \tilde{m}^2 + g^2 \int [x\kappa] f^2(z_\lambda^2) \frac{\kappa^2 + (1+x)^2 m^2}{x(\mathcal{M}^2 - m^2)} + o(g^4). \quad (3.37)$$

Analogous equation in the case of nucleons coupled to pions is the same as Eq. (3.37) except for $(1-x)^2$ instead of $(1+x)^2$ in the numerator and the isospin factor 3 in front of the integral. In the limit of a θ -function for the similarity factor f one obtains the result $m^2 = \tilde{m}^2 + o(g^2)$, $\tilde{m} = m_N$, and

$$m_\lambda^2 = m_N^2 + 3g^2 \int [x\kappa] \theta(\lambda^2 + 3m^2 - \mathcal{M}^2) \frac{\kappa^2 + (1-x)^2 m^2}{x(\mathcal{M}^2 - m^2)} + o(g^4). \quad (3.38)$$

For $\lambda^2 = (m + n_\pi \mu_\pi)^2 - 3m^2$, where n_π is a small integer one obtains ($\alpha = g^2/4\pi$)

$$m_\lambda^2 = m_N^2 + m_N^2 \frac{3\alpha}{4\pi} c. \quad (3.39)$$

$c = 4/3(n_\pi \mu_\pi/m_N)^3 + o(\mu_\pi^4)$. The expansion formula for c shows the correction is small for small meson masses. Note that λ^2 must be negative for small n_π , as explained below Eq. (2.20). The exact result for $n_\pi = 3$ gives $c \sim 0.03$ and $n_\pi = 4$ gives $c \sim 0.12$. Even for quite large couplings the effective mass parameter in the Hamiltonian deviates only a little from the physical nucleon mass if the Hamiltonian width allows momentum changes of the order of a few meson masses only. Moreover, the physically relevant off-shell effects are not given directly by the above numbers but by the difference between these and the effects of the interactions present in the effective Hamiltonian. The combined effect is zero for the nucleon mass

itself to order g^2 . Eq. (3.39) suggests that the self-interaction effects can be calculable in perturbation theory. This is encouraging for the program outlined in Ref. [17].

If we used Eq. (2.20) in the θ -function limit with an infinitesimal u_0 and $\lambda^2 = \tilde{\lambda}^2/\sqrt{u_0}$ for $n = 1$ the θ -function under the integral in Eq. (3.38) would be replaced by $\theta(\tilde{\lambda}^2 + m^2 - \mathcal{M}^2)$.

3.2.3. Fermion-fermion interaction

Our next example is the second order calculation of the effective Hamiltonian term which contains products of two creation and two annihilation operators for fermions. The differential equation we need to consider results from Eq. (2.29) for the two-fermion terms;

$$\frac{d}{d\lambda}\mathcal{G}_{22\lambda} = \left[f_{\lambda}\mathcal{G}_{12\lambda} \left\{ \frac{d}{d\lambda}(1 - f_{\lambda})\mathcal{G}_{21\lambda} \right\}_{\mathcal{G}_{1\lambda}} - \left\{ \frac{d}{d\lambda}(1 - f_{\lambda})\mathcal{G}_{12\lambda} \right\}_{\mathcal{G}_{1\lambda}} f_{\lambda}\mathcal{G}_{21\lambda} \right]_{\mathcal{G}_{1\lambda}}^{22}. \quad (3.40)$$

The subscript 22 denotes a term with two creation and two annihilation operators for fermions. 21 denotes a term with one annihilation operator and one creation operator for fermions and one creation operator for bosons. 12 denotes a term which annihilates a fermion and a boson and creates a fermion. For a hermitean Hamiltonian, we have $\mathcal{G}_{12} = \mathcal{G}_{21}^{\dagger}$.

The right-hand side of Eq. (3.40) does not contain disconnected interactions (it never does, *cf.* Section 3.1) and one can isolate the terms with two creation and two annihilation operators for fermions by contracting one creation operator and one annihilation operator for bosons. The only term which contributes is the ordered and regularized third term on the right-hand side of Eq. (3.3). Thus, in Eq. (3.40), we have

$$\begin{aligned} \mathcal{G}_{21\lambda} &= \sum_{\sigma\sigma_f} \int [k_1 k k_2] 16\pi^3 \delta^3(k_1 + k - k_2) g \bar{u}_{mk_1\sigma_f} u_{mk_2\sigma} r \left(\frac{\varepsilon e_f}{\Lambda^2} \right) r \left(\frac{\varepsilon e_b}{\Lambda^2} \right) b_{k_1\sigma_f}^{\dagger} a_k^{\dagger} b_{k_2\sigma} \\ &= \int [P] \frac{1}{P+} \sum_{\sigma\sigma_f} \int [x\kappa] g \bar{u}_{m x P + \kappa \sigma_f} u_{m P \sigma} r \left(\frac{\varepsilon e_f}{\Lambda^2} \right) r \left(\frac{\varepsilon e_b}{\Lambda^2} \right) b_{x P + \kappa \sigma_f}^{\dagger} a_{(1-x)P - \kappa}^{\dagger} b_{P \sigma}, \end{aligned} \quad (3.41)$$

where $e_f = (\kappa^2 + m^2)/x$ and $e_b = (\kappa^2 + \mu^2)/(1 - x)$. This representation illustrates appearance of the parent and daughter momentum variables in the interaction term. The factor $g \bar{u}_{mk_1\sigma_f} u_{mk_2\sigma} r(\varepsilon e_f/\Lambda^2) r(\varepsilon e_b/\Lambda^2)$ will be denoted by g_{21} . The analogous factor in $\mathcal{G}_{12\lambda}$ will be denoted by g_{12} . Similarly,

in the case of the four-fermion interaction term, we have

$$\mathcal{G}_{22\lambda} = \int [P] \frac{1}{P^+} \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \int [x\kappa][y\rho] g_{22\lambda} b_{xP+\kappa\sigma_1}^\dagger b_{(1-x)P-\kappa\sigma_3}^\dagger b_{yP+\rho\sigma_2} b_{(1-y)P-\rho\sigma_4}, \quad (3.42)$$

where $g_{22\lambda}$ is a function of the daughter momentum variables x , $\kappa^\perp$, y , $\rho^\perp$ and the fermion spin projections on the z-axis: σ_1 , σ_2 , σ_3 and σ_4 . Details of the notation will become clear shortly. To order g^2 , only f_λ depends on λ on the right-hand side of Eq. (3.40) and it can be written in terms of the coefficient functions as

$$\frac{d}{d\lambda} g_{22} = [f\{-f'\} - \{-f'\}f] [g_{12}g_{21}]_{22}. \quad (3.43)$$

The subscript λ and arguments of the functions are not indicated, to simplify the formula. Expression in the first bracket is called *the inner similarity factor* for $\mathcal{G}_{22}$. The word “inner” is used to distinguish this factor from the form factor introduced by the operation F_λ when this operation is applied to G_{22} . The latter form factor can be called the *outer* similarity factor because it depends on the incoming and outgoing invariant masses only, independently of the internal structure of the interaction.

Momentum variables in Eq. (3.43) can be expressed by the daughter momentum variables from Eq. (3.42). One needs to express the parent and daughter variables of $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$, and the energy denominators, in terms of x , $\kappa^\perp$, y and $\rho^\perp$ from Eq. (3.42). This is done as follows.

The momentum labels of the fermion annihilation operators in Eq. (3.42) are denoted by k_2 and k_4 and the momentum labels of the fermion creation operators in Eq. (3.42) are denoted by k_1 and k_3 . The numbers assigned to the fermion momenta correspond to the numbers labeling their spin projections on the z-axis in Eq. (3.42). In terms of the origin of the annihilation and creation operators in Eq. (3.42), $\mathcal{G}_{21}$ provides the fermion creation operator with momentum k_1 and the fermion annihilation operator with momentum k_2 . $\mathcal{G}_{12}$ provides the fermion creation operator with momentum k_3 and the fermion annihilation operator with momentum k_4 . There is a change of sign due to the reordering of the fermion operators. The boson operators from $\mathcal{G}_{12}$ and $\mathcal{G}_{21}$ are contracted and provide factors similar to the factors associated with an intermediate particle in the old-fashioned Hamiltonian calculations of the S-matrix.

It is useful to think about the effective Hamiltonian in terms of a scattering amplitude with two vertices but the reader should remember that the formula we are describing is not for an S-matrix matrix element. Therefore, the “scattering” language has a limited meaning. The fine point is that, after evaluation of $\mathcal{G}_\lambda$, one has to replace the bare creation and annihilation

operators by the effective ones in order to obtain G_λ . The term in $\mathcal{G}_\lambda$ is not directly related to any scattering process before the replacement is made. The replacement prevents confusion between the Hamiltonian calculus which uses the bare operators, and the S-matrix calculus which uses the effective Hamiltonian and the corresponding incoming, outgoing and intermediate states of effective particles. The scattering language becomes particularly confusing in theories with gauge symmetry, spontaneous symmetry breaking and confinement. None of these features appear here in the calculation to order g^2 . Therefore, the scattering language is useful in the current example.

The intermediate boson momentum is defined for $+$ and $\perp$ components as $k_5 = k_3 - k_4$ and $k_{5\mu}^- = (k_5^{\perp 2} + \mu^2)/k_5^+$. These four components form the four-momentum of the exchanged boson. It is denoted by $k_{5\mu}$ to indicate the mass which determines the minus component. The same result for $k_{5\mu}$ is obtained by subtracting k_1 from k_2 instead of k_4 from k_3 . It is so because the translational invariance of the Hamiltonian on the light-front implies momentum conservation for the $+$ and $\perp$ components.

Thus, the inner similarity factor in Eq. (3.43) is

$$[f\{-f'\} - \{-f'\}f] = f(z_{12\lambda}^2) \frac{[-f(z_{21\lambda}^2)]'}{\Delta E_{21}} - \frac{[-f(z_{12\lambda}^2)]'}{\Delta E_{12}} f(z_{21\lambda}^2). \quad (3.44)$$

The prime denotes differentiation with respect to λ .

In the case of $\mathcal{G}_1$ in Eq. (3.4), the whole inner similarity factor of the analogous structure was equal to the derivative of $f^2(z_\lambda^2)$ divided by a single denominator. For both functions f in Eq. (3.4) had the same argument z_λ^2 and the two corresponding energy denominators were the same. In Eq. (3.44) we have two different arguments of the similarity functions f and two different energy changes. Namely, $z_{12\lambda}$ and ΔE_{12} for the vertex of $\mathcal{G}_{12}$ with momenta k_{3m} , k_{4m} and $k_{5\mu}$, and $z_{21\lambda}$ and ΔE_{21} for the vertex of $\mathcal{G}_{21}$ with momenta k_{1m} , k_{2m} and $k_{5\mu}$.

The parent momentum for the vertex of $\mathcal{G}_{12}$ is $P_{12} = (k_{5\mu} + k_{3m} + k_{4m})/2$ so that for the $+$ and $\perp$ components we have $P_{12} = k_3$. Similarly, the parent momentum for the vertex of $\mathcal{G}_{21}$ is $P_{21} = (k_{5\mu} + k_{1m} + k_{2m})/2$ so that for the $+$ and $\perp$ components we have $P_{21} = k_2$.

Now, the rules provided by Eqs. (2.12) to (2.20) imply the following formulae for the arguments of the similarity functions f .

$$\Delta \mathcal{M}_{12}^2 = (k_{5\mu} + k_{4m})^2 - k_{3m}^2 = 2(k_{5\mu} + k_{4m} - k_{3m})P_{12}. \quad (3.45)$$

$$\Sigma \mathcal{M}_{12}^2 = \mathcal{M}_{12}^2 + 2m^2. \quad (3.46)$$

$$z_{12\lambda} = \frac{\Delta \mathcal{M}_{12}^2}{\Sigma \mathcal{M}_{12}^2 + \lambda^2}. \quad (3.47)$$

$$\Delta\mathcal{M}_{21}^2 = k_{2m}^2 - (k_{5\mu} + k_{1m})^2 = -2(k_{5\mu} + k_{1m} - k_{2m})P_{21}. \quad (3.48)$$

$$\Sigma\mathcal{M}_{21}^2 = -\mathcal{M}_{21}^2 + 2m^2. \quad (3.49)$$

$$z_{21\lambda} = \frac{\Delta\mathcal{M}_{21}^2}{\Sigma\mathcal{M}_{21}^2 + \lambda^2}. \quad (3.50)$$

Equations (2.12) to (2.16) imply

$$\Delta E_{12} = \frac{\Delta\mathcal{M}_{12}^2}{P_{12}^+} \quad (3.51)$$

and

$$\Delta E_{21} = \frac{\Delta\mathcal{M}_{21}^2}{P_{21}^+}. \quad (3.52)$$

In terms of the familiar parameters x , $\kappa^\perp$, y and $\rho^\perp$ from Eq. (3.42), *i.e.*

$$P = k_1 + k_3 = k_2 + k_4, \quad (3.53)$$

$$x = k_1^+ / P^+, \quad (3.54)$$

$$\kappa^\perp = k_1^\perp - xP^\perp, \quad (3.55)$$

and

$$y = k_2^+ / P^+, \quad (3.56)$$

$$\rho^\perp = k_2^\perp - yP^\perp, \quad (3.57)$$

the mass differences which determine the arguments of the similarity functions read as follows.

$$\Delta\mathcal{M}_{12}^2 = (1-x) \left[\frac{(\kappa^\perp - \rho^\perp)^2 + \mu^2}{y-x} + \frac{\rho^{\perp 2} + m^2}{1-y} - \frac{\kappa^{\perp 2} + m^2}{1-x} \right], \quad (3.58)$$

$$\Delta\mathcal{M}_{21}^2 = y \left[\frac{\rho^{\perp 2} + m^2}{y} - \frac{\kappa^{\perp 2} + m^2}{x} - \frac{(\kappa^\perp - \rho^\perp)^2 + \mu^2}{y-x} \right]. \quad (3.59)$$

The corresponding energy denominators are

$$\Delta E_{12} = \left[\frac{(\kappa^\perp - \rho^\perp)^2 + \mu^2}{y-x} + \frac{\rho^{\perp 2} + m^2}{1-y} - \frac{\kappa^{\perp 2} + m^2}{1-x} \right] / P^+ \quad (3.60)$$

and

$$\Delta E_{21} = \left[\frac{\rho^{\perp 2} + m^2}{y} - \frac{\kappa^{\perp 2} + m^2}{x} - \frac{(\kappa^\perp - \rho^\perp)^2 + \mu^2}{y-x} \right] / P^+. \quad (3.61)$$

In Eqs. (3.40) to (3.61) always $y > x$. Evaluating matrix elements of the effective interaction between states of indistinguishable fermions one obtains

results in which the momentum and spin variables are properly symmetrized (antisymmetrized) as dictated by the statistics.

Evaluation of the second bracket in Eq. (3.43) gives

$$[g_{12}g_{21}]_{22} = -g\bar{u}_{m(1-x)P-\kappa\sigma_3}u_{m(1-y)P-\rho\sigma_4}r(\varepsilon e_4/\Lambda^2)r(\varepsilon e_{12}/\Lambda^2) \\ \times \frac{1}{(y-x)P^+}g\bar{u}_{mxP+\kappa\sigma_1}u_{myP+\rho\sigma_2}r(\varepsilon e_1/\Lambda^2)r(\varepsilon e_{21}/\Lambda^2). \quad (3.62)$$

The arguments of the regularization factors appear in the mass differences. Namely,

$$\Delta\mathcal{M}_{12}^2 = e_4 + e_{12} - m^2 = \frac{\kappa_{12}^{\perp 2} + m^2}{x_{12}} + \frac{\kappa_{12}^{\perp 2} + \mu^2}{1 - x_{12}} - m^2, \quad (3.63a)$$

where

$$x_{12} = \frac{1-y}{1-x}, \quad (3.63b)$$

$$\kappa_{12}^{\perp} = -\rho^{\perp} + x_{12}\kappa^{\perp}, \quad (3.63c)$$

and

$$\Delta\mathcal{M}_{21}^2 = m^2 - e_1 + e_{21} = m^2 - \frac{\kappa_{21}^{\perp 2} + m^2}{x_{21}} - \frac{\kappa_{21}^{\perp 2} + \mu^2}{1 - x_{21}}, \quad (3.64a)$$

where

$$x_{21} = \frac{x}{y}, \quad (3.64b)$$

$$\kappa_{21}^{\perp} = \kappa^{\perp} - x_{21}\rho^{\perp}. \quad (3.64c)$$

Eqs. (3.62) to (3.64) contain daughter energies in the notation introduced already in Eqs. (2.48) and (2.49). Similar subscript notation will be used to label the regularization factors. Equation (3.43) reads, in the abbreviated notation, as follows.

$$\frac{d}{d\lambda}g_{22} = \left[\frac{y}{y-x} \frac{f_{12}f'_{21}}{\Delta\mathcal{M}_{21}^2} - \frac{1-x}{y-x} \frac{f'_{12}f_{21}}{\Delta\mathcal{M}_{12}^2} \right] \theta(y-x)g\bar{u}_3u_4r_4r_{12}g\bar{u}_1u_2r_1r_{21}. \quad (3.65)$$

In the familiar limit of Eq. (3.8) where the similarity function f approaches the θ -function, *i.e.* $f_{12} = \theta[\lambda^2 + 2m^2 - \Delta\mathcal{M}_{12}^2]$ and $f_{21} = \theta[\lambda^2 + 2m^2 + \Delta\mathcal{M}_{21}^2]$, the derivatives of the similarity functions become δ -functions and one can integrate Eq. (3.65) using the relation $\int_a^\infty ds\theta(s-b)\delta(s-c) = \theta(c-b)\theta(c-a)$. The result is

$$g_{22\lambda} = g_{22\varepsilon} + \left[\frac{y\theta_{21-12}\theta_{21-\lambda}}{(y-x)|\Delta\mathcal{M}_{21}^2|} + \frac{(1-x)\theta_{12-\lambda}\theta_{12-21}}{(y-x)|\Delta\mathcal{M}_{12}^2|} \right] \\ \times \theta(y-x)g\bar{u}_3u_4r_4r_{12}g\bar{u}_1u_2r_1r_{21}. \quad (3.66a)$$

The symbols θ with various subscripts denote the following functions:

$$\theta_{12-21} = 1 - \theta_{21-12} = \theta(|\Delta\mathcal{M}_{12}^2| - |\Delta\mathcal{M}_{21}^2|), \quad (3.66b)$$

$$\theta_{12-\lambda} = \theta(|\Delta\mathcal{M}_{12}^2| - \lambda^2 - 2m^2), \quad (3.66c)$$

and

$$\theta_{21-\lambda} = \theta(|\Delta\mathcal{M}_{21}^2| - \lambda^2 - 2m^2). \quad (3.66d)$$

The initial value term of $g_{22\varepsilon}$ at $\lambda = \infty$ is absent in the canonical Hamiltonian. It is equal zero if matrix elements of $H_{22\lambda} = F_\lambda [G_{22\lambda}]$ between finite free energy states have a limit when $\varepsilon \rightarrow 0$. If the limit does not exist due to a diverging ε -dependence a counterterm containing nonzero $g_{22\varepsilon}$ is required to remove the divergence.

If we used Eq. (2.20) in the θ -function limit with an infinitesimal u_0 and $\lambda^2 = \tilde{\lambda}^2/\sqrt{u_0}$ for $n = 1$ then, $\lambda^2 + 2m^2$ in the θ -function arguments above would be replaced by $\tilde{\lambda}^2$. This feature will be used later in the case of nonrelativistic bound states.

The easiest momentum configuration to analyze $g_{22\varepsilon}$ is the one where the sum of free energies for the momenta of creation operators equals the sum of free energies for the momenta of annihilation operators: $(k_{1m} + k_{3m})^2 = (k_{2m} + k_{4m})^2 = \mathcal{M}^2$. We will refer to this configuration as *the energy-diagonal part of the interaction*. In the energy-diagonal part of the interaction, we have $\Delta E_{12} = -\Delta E_{21}$, and

$$\frac{\rho^{\perp 2} + m^2}{1 - y} - \frac{\kappa^{\perp 2} + m^2}{1 - x} = \frac{\kappa^{\perp 2} + m^2}{x} - \frac{\rho^{\perp 2} + m^2}{y}. \quad (3.67)$$

Thus,

$$\frac{|\Delta\mathcal{M}_{12}^2|}{1 - x} = \frac{|\Delta\mathcal{M}_{21}^2|}{y} = \frac{\mu^2 + \vec{q}^2}{y - x}, \quad (3.68)$$

where $\vec{q}^2 = (\kappa^\perp - \rho^\perp)^2 + (y - x)^2 \mathcal{M}^2$. These relations imply $\theta_{12-21} = 1 - \theta_{21-12} = \theta(1 - x - y)$, $\theta_{12-\lambda} = \theta[\mu^2 + \vec{q}^2 - (y - x)(\lambda^2 + 2m^2)/(1 - x)]$ and $\theta_{21-\lambda} = \theta[\mu^2 + \vec{q}^2 - (y - x)(\lambda^2 + 2m^2)/y]$. Therefore, $\theta_{12-\lambda} = 1$ in the same momentum range where $\theta_{21-\lambda} = 1$. Thus, the energy-diagonal part of the fermion-fermion effective interaction order g^2 is

$$\begin{aligned} g_{22\lambda} = g_{22\varepsilon} + & \frac{g\bar{u}_3 u_4 r_4 r_{12} g\bar{u}_1 u_2 r_1 r_{21}}{\mu^2 + \vec{q}^2} \theta(y - x) \\ & \times \theta \left[\mu^2 + \vec{q}^2 - \frac{y - x}{\max(y, 1 - x)} (\lambda^2 + 2m^2) \right]. \end{aligned} \quad (3.69)$$

Equation (3.69) is helpful because it provides insight into the more complicated interaction from Eq. (3.66). When the momentum transfer is sufficiently large and $(y - x)(\lambda^2 + 2m^2)/\max(y, 1 - x)$ is negligible so that the

θ -functions and the regularization functions r_4 , r_{12} , r_1 and r_{21} in Eq. (3.69) equal 1, the second term on the right-hand side of Eq. (3.69) is equal to Feynman's expression for the one boson-exchange scattering amplitude for two fermions. Namely, the numerator factors are standard for the Yukawa interaction and the denominator equals $\mu^2 - (k_{3m} - k_{4m})^2 = \mu^2 - (k_{2m} - k_{1m})^2$ (the necessary antisymmetrization for identical fermions requires evaluation of matrix elements of the Hamiltonian term under consideration). However, there is a difference between the energy diagonal part of the effective Hamiltonian matrix elements and the on-shell Feynman scattering amplitude due to the θ -functions and the regularization factors in Eq. (3.69) where they differ from 1. In other words, our theory is regularized *ab initio* and the resulting amplitudes contain the regularizing factors. The width dependent factor of the effective Hamiltonian does not belong in a physical scattering amplitude and we will explain how it goes away when one evaluates S-matrix elements later in this Section.

The θ -functions force the momentum transfer carried by the intermediate boson, $|\vec{q}|$, to be larger than $[(y - x)(\lambda^2 + 2m^2)/y - \mu^2]^{1/2}$. The size of this number depends on how large λ^2 and the ratios of the longitudinal momenta are. If λ^2 is negative and compensates $2m^2$, the lower bound on the momentum transfer is absent. For larger λ^2 , the ratio of $y - x$ to the parent fermion y has to be smaller than $\mu^2(\lambda^2 + 2m^2)^{-1}$ for the lower bound on the momentum transfer to be absent. Otherwise, the momentum transfer is limited from below. This means that the effective interaction term does not include the long distance part of the Yukawa potential.

The regularization factors in the limit $\varepsilon \rightarrow 0$ converge pointwise to 1. No diverging cutoff dependence is obtained when evaluating matrix elements of $G_{22\lambda}$ between states of finite invariant masses $\mathcal{M}^2$ so the matrix elements of $H_{22\lambda}$ are free from divergences. Therefore, $g_{22\varepsilon} = 0$. We can replace the regularization factors in the limit $\varepsilon \rightarrow 0$ by 1.

We proceed to the analysis of Eq. (3.66). No divergences appear in the finite matrix elements of the effective Hamiltonian of width λ when $\varepsilon \rightarrow 0$. One can see that this is the case using Eqs. (3.63a) and (3.64a). Namely, the arguments of the regularization factors are finite for finite $\Delta\mathcal{M}_{12}^2$ and $\Delta\mathcal{M}_{21}^2$ and they approach 0 when $\varepsilon \rightarrow 0$. One demands that the free invariant masses of the states of fermions used to calculate the matrix elements are finite. The only possibility for $\Delta\mathcal{M}_{12}^2$ or $\Delta\mathcal{M}_{21}^2$ to diverge emerges when x approaches y , *i.e.* when the longitudinal momentum transfer between the fermions approaches zero. In such case, e_{12} and e_{21} approach infinity even for a vanishing transverse momentum transfer because the meson mass squared is greater than zero.

Now, the remaining factors of spinors and energy denominators, the latter multiplied by the boson phase-space factor of $y - x$, are all finite in the limit $x \rightarrow y$. The regularization factors r_{12} and r_{21} deviate from 1 only in

the small region in the momentum space where $|x - y| < \varepsilon \Lambda^2 / \mu^2$ (or in a still smaller region for a nonzero meson transverse momentum). All other factors in the interaction are finite in this region. Therefore, for finite wave packets or bound state wave functions used in the evaluation of the matrix element, this small region produces a contribution which is proportional to ε . Thus, it vanishes in the limit $\varepsilon \rightarrow 0$. Consequently, the matrix elements of $g_{22\varepsilon}$ are equal 0 and the regularization factors can be replaced by 1.

The full result for the effective fermion-fermion interaction in the limit $\varepsilon \rightarrow 0$ is

$$H_{22\lambda} = F_\lambda [G_{22\lambda}] = \int [P] \frac{1}{P_+} \times \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \int [x\kappa][y\rho] g_{22\lambda} f(z_{22\lambda}^2) b_{\lambda x P + \kappa \sigma_1}^\dagger b_{\lambda(1-x)P - \kappa \sigma_3}^\dagger b_{\lambda y P + \rho \sigma_2} b_{\lambda(1-y)P - \rho \sigma_4}, \quad (3.70a)$$

where

$$g_{22\lambda} = \left[\frac{\theta_{12-21}\theta_{21-\lambda}}{\mu^2 - q_{21}^2} + \frac{\theta_{12-\lambda}\theta_{21-12}}{\mu^2 - q_{12}^2} \right] \theta(y-x) g_{\bar{u}_3 u_4} g_{\bar{u}_1 u_2}, \quad (3.70b)$$

$$q_{12} = k_{3m} - k_{4m}, \quad (3.70c)$$

$$q_{21} = k_{2m} - k_{1m}, \quad (3.70d)$$

$$\theta_{12-21} = 1 - \theta_{21-12} = \theta \left[(1-x)(\mu^2 - q_{12}^2) - y(\mu^2 - q_{21}^2) \right], \quad (3.70e)$$

$$\theta_{12-\lambda} = \theta \left[(1-x)(\mu^2 - q_{12}^2) - (y-x)(\lambda^2 + 2m^2) \right], \quad (3.70f)$$

$$\theta_{21-\lambda} = \theta \left[y(\mu^2 - q_{21}^2) - (y-x)(\lambda^2 + 2m^2) \right]. \quad (3.70g)$$

The argument of the outer similarity factor $f(z_{22\lambda}^2)$ that limits the width of the effective interaction in momentum space, is

$$z_{22\lambda} = \frac{\Delta \mathcal{M}_{22}^2}{\Sigma \mathcal{M}_{22}^2 + \lambda^2}. \quad (3.70h)$$

The mass difference, $\Delta \mathcal{M}_{22}^2 = \mathcal{M}_{24}^2 - \mathcal{M}_{13}^2$, and the mass sum, $\Sigma \mathcal{M}_{22}^2 = \mathcal{M}_{24}^2 + \mathcal{M}_{13}^2$, are expressed by the fermion momenta through relations $\mathcal{M}_{24}^2 = (k_{2m} + k_{4m})^2$ and $\mathcal{M}_{13}^2 = (k_{1m} + k_{3m})^2$.

Equations (3.70a) to (3.70h) explain the structure of the fermion-fermion effective interaction order g^2 in terms of the two four-momentum transfers, q_{12} and q_{21} . The transfer q_{12} appears in the vertex where the intermediate boson is annihilated and the transfer q_{21} appears in the vertex where the boson is created. The θ -functions exclusively select which momentum transfer appears in the denominator. The lower bounds on the momentum transfers depend on the ratio of $|x - y|$ to y and $1 - x$ and on the masses and λ^2 .

We can now evaluate matrix elements of the T-matrix between effective two-fermion states using the Hamiltonian of width λ to second order in g ;

$$T(E) = H_{I\lambda} + H_{I\lambda} \frac{1}{E - H_{0\lambda} + i\varepsilon} H_{I\lambda}. \quad (3.71)$$

We have $H_{0\lambda} = G_{1\lambda}$ and $H_{I\lambda} = F_\lambda [G_{22\lambda} + G_{12\lambda} + G_{21\lambda}]$. The first term on the right-hand side of Eq. (3.71) contributes solely through $H_{22\lambda}$. In the second term, only $H_{12\lambda} + H_{21\lambda}$ contributes in $H_{I\lambda}$.

The first term in Eq. (3.71) has its matrix element given by the anti-symmetrization of the right-hand side of Eq. (3.70b). The multiplication by $f(z_{22\lambda}^2)$ does not matter because $f(z_{22\lambda}^2) = 1$ in the energy-diagonal matrix elements and only the energy-diagonal part contributes to the cross section. The energy-diagonal part of $g_{22\lambda}$ is given by

$$g_{22\lambda} = \frac{g\bar{u}_3 u_4 g\bar{u}_1 u_2}{\mu^2 - q^2} \theta(y - x) \theta_\lambda, \quad (3.72)$$

where $\theta_\lambda = \theta [\max(y, 1 - x)(\mu^2 - q^2) - (y - x)(\lambda^2 + 2m^2)]$ and $q = q_{12} = q_{21}$. The antisymmetrization of the right-hand side of Eq. (3.72) produces the contribution of the first term in Eq. (3.71) to the scattering amplitude.

The second term in Eq. (3.71) provides the one-boson exchange amplitude with form factors in the fermion-boson vertices. The form factors are the similarity functions f_λ . The resulting amplitude is given by the anti-symmetrization of Eq. (3.72) with θ_λ replaced by the product of the form factors. In the θ -function limit, the vertex form factors equal $1 - \theta_{21-\lambda}$ and $1 - \theta_{12-\lambda}$. Their product equals $1 - \theta_\lambda$. Thus, the second term provides the same contribution as the first term but the factor θ_λ is replaced by $1 - \theta_\lambda$.

The sum of both terms in Eq. (3.71) produces the matrix element of the scattering matrix on-energy-shell which is independent of λ . Our complete on-shell result in the effective theory is equal to the well known Feynman result for the one boson exchange scattering amplitude.

There is an important property of the second order calculation above which is worth a separate note. When the Hamiltonian width in the mass difference becomes small the effective meson emission can no longer occur. Thus, the effective theory describes fermions interacting by potential forces. The potentials are given by factors $f(z_{22\lambda}^2)g_{22\lambda}$ in $H_{22\lambda}$. The form factors $f(z_{22\lambda}^2)$ are known off-energy shell. $g_{22\lambda}$ contains also the inner similarity factors which force the intermediate boson to form a sufficiently high invariant mass state but if the width is small enough these factors are equal 1. $f(z_{22\lambda}^2)g_{22\lambda}$ is the generalized relativistic potential term that equals Yukawa potential in the nonrelativistic limit. Thus, we have accomplished a derivation of the boost invariant potential term order g^2 in the effective Yukawa theory.

The nonrelativistic Yukawa theory is obtained when the width λ is such that the allowed energy transfers are much smaller than the effective fermion masses. This condition limits only the relative motion of the effective fermions. It does not limit their total momentum which can still be chosen arbitrarily by taking advantage of the boost invariance. The reduction of the fermion dynamics in Yukawa theory with small λ to the Schrödinger equation in second order perturbation theory is further discussed in Ref. [26]. Initial studies of the 4th order similarity in a Yukawa model are given in Ref. [27].

Note that in the energy diagonal part of the effective potential as well as in the on-energy shell scattering amplitude the outer similarity factor equals 1 independently of the size of the momentum transfer. In other words, one cannot see the outer similarity factor in the physical scattering amplitude order g^2 and the only signs of the effective nature of the potential are the form factors in the interaction vertices.

If we use the interaction $g\bar{\psi}i\gamma_5\vec{\tau}\psi\vec{\phi}$ instead of $g\bar{\psi}\psi\phi$ in writing the initial Hamiltonian of Eq. (3.3), the resulting effective potential corresponds to the one-pion exchange between nucleons. Since the formalism is not limited to the nonrelativistic domain of the fermion momenta or to the lowest order perturbation theory, one can investigate this type of potentials in a wide range of applications in meson-baryon and quark-pion physics.

There exists a possibility that the similarity flow of Hamiltonians may lead to growth of coupling constants for small width. The outer similarity factor reduces the strength of the effective interactions when λ decreases. Effective Hamiltonians with small width may have the same bound state eigenvalues as Hamiltonians of similar structure with large widths and small couplings if the effective coupling constants become large for small widths. The range of coupling constants requires investigation in order to establish if the size of coupling constants required in the meson-nucleon phenomenology can be explained this way.

3.2.4. Fermion-anti-fermion interaction

The fermion-anti-fermion interaction order g^2 satisfies a differential equation which is analogous to Eq. (3.40) but more terms appear. The operator subscripts must distinguish fermions and anti-fermions and one has to include terms which result from the annihilation channel. The fermion-anti-fermion effective interaction term is

$$\mathcal{G}_{1\bar{1}1\bar{1}\lambda} = \int [P] \frac{1}{P_+} \sum_{\sigma_1\sigma_2\sigma_3\sigma_4} \int [x\kappa][y\rho] g_{1\bar{1}1\bar{1}\lambda} b_{xP+\kappa\sigma_1}^\dagger d_{(1-x)P-\kappa\sigma_3}^\dagger d_{(1-y)P-\rho\sigma_4} b_{yP+\rho\sigma_2} \quad (3.73)$$

Note the change of order of the spin numbering and momentum assignments in comparison to Eq. (3.42) for fermions. The new order results from the operator ordering including anti-fermions as defined in Section 2.1. Momenta k_1 and k_2 are used for fermion and k_3 and k_4 for anti-fermion operators with even subscripts for annihilation operators and odd subscripts for creation operators.

There are three terms contributing to the derivative of $g_{1\bar{1}\bar{1}1\lambda}$ with respect to λ : one due to the annihilation channel and two due to the exchange of a boson. One of the latter two contributions results from the emission of the boson by the fermion and absorption by the anti-fermion and the other one from the emission by the anti-fermion and absorption by the fermion. In each of the terms there are two similarity functions with different arguments. We have

$$\begin{aligned} \frac{dg_{1\bar{1}\bar{1}1\lambda}}{d\lambda} &= S_1 g_{\bar{u}_1 v_3} g_{\bar{v}_4 u_2} r_{11} r_{13} r_{14} r_{12} \frac{1}{P^+} \\ &- \left\{ S_2 r_{21} r_{2512} r_{24} r_{2534} \frac{\theta(y-x)}{(y-x)P^+} + S_3 r_{32} r_{3512} r_{33} r_{3534} \frac{\theta(x-y)}{(x-y)P^+} \right\} g_{\bar{u}_1 u_2} g_{\bar{v}_4 v_3}. \end{aligned} \quad (3.74)$$

The inner similarity factors are, $i = 1, 2, 3$,

$$S_i = f(z_{i2}^2) \frac{[-f(z_{i2}^2)]'}{\Delta E_{i2}} - \frac{[-f(z_{i1}^2)]'}{\Delta E_{i1}} f(z_{i2}^2). \quad (3.75)$$

Equation (3.75) is similar to Eq. (3.44) (the subscript λ is skipped for clarity). The second subscript of the arguments of the similarity function f denotes the vertex, *i.e.* 1 stands for the vertex where the boson was annihilated and 2 stands for the vertex where the boson was created. In Eq. (3.74), the fermion regularization factors first subscript is the same as the corresponding inner similarity factor subscript (*i.e.* the subscript of S) and the second subscript is the same as the corresponding fermion momentum subscript. The boson regularization factors are distinguished by the subscript 5 following the convention from Eqs. (2.50). Their first subscript is also the same as the corresponding inner similarity factor subscript. Last two subscripts of the boson regularization factors equal subscripts of the fermion momenta from the vertex where the boson regularization factor originated. Arguments of the regularization factors have the same subscripts as the regularization factors themselves, *i.e.* $r_i = r(\varepsilon e_i/\Lambda^2)$. The daughter energies in the arguments are calculated according to the rules given in Eqs. (2.47) to (2.50). We give the results below for completeness. The same arguments will appear in all theories of physical interest.

$$e_{11} = \frac{\kappa^{\perp 2} + m^2}{x}. \quad (3.76a)$$

$$e_{13} = \frac{\kappa^{\perp 2} + m^2}{1 - x}. \quad (3.76b)$$

$$e_{14} = \frac{\rho^{\perp 2} + m^2}{y}. \quad (3.76c)$$

$$e_{12} = \frac{\rho^{\perp 2} + m^2}{1 - y}. \quad (3.76d)$$

$$e_{21} = \frac{\kappa_{212}^{\perp 2} + m^2}{x_{212}}. \quad (3.77a)$$

$$e_{2512} = \frac{\kappa_{212}^{\perp 2} + \mu^2}{1 - x_{212}}. \quad (3.77b)$$

$$x_{212} = \frac{x}{y}. \quad (3.77c)$$

$$\kappa_{212}^{\perp} = \kappa^{\perp} - x_{212}\rho^{\perp}. \quad (3.77d)$$

$$e_{24} = \frac{\kappa_{234}^{\perp 2} + m^2}{x_{234}}. \quad (3.77e)$$

$$e_{2534} = \frac{\kappa_{234}^{\perp 2} + \mu^2}{1 - x_{234}}. \quad (3.77f)$$

$$x_{234} = \frac{1 - y}{1 - x}. \quad (3.77g)$$

$$\kappa_{234}^{\perp} = -\rho^{\perp} + x_{234}\kappa^{\perp}. \quad (3.77h)$$

$$e_{32} = \frac{\kappa_{312}^{\perp 2} + m^2}{x_{312}}. \quad (3.78a)$$

$$e_{3512} = \frac{\kappa_{312}^{\perp 2} + \mu^2}{1 - x_{312}}. \quad (3.78b)$$

$$x_{312} = \frac{y}{x}. \quad (3.78c)$$

$$\kappa_{312}^{\perp} = \rho^{\perp} - x_{312}\kappa^{\perp}. \quad (3.78d)$$

$$e_{33} = \frac{\kappa_{334}^{\perp 2} + m^2}{x_{334}}. \quad (3.78e)$$

$$e_{3534} = \frac{\kappa_{334}^{\perp 2} + \mu^2}{1 - x_{334}}. \quad (3.78f)$$

$$x_{334} = \frac{1-x}{1-y}. \quad (3.78g)$$

$$\kappa_{334}^\perp = -\kappa^\perp + x_{334}\rho^\perp. \quad (3.78h)$$

Arguments of the similarity functions and energy denominators which appear in Eq. (3.75) are calculated according to the rules given in Eqs. (2.12) to (2.19) and (2.24) to (2.26). The results are universal for all one-particle-exchange two-particle interactions and are given below for completeness.

$$\begin{aligned} \Delta\mathcal{M}_{11}^2 &= (k_1 + k_3)_\mu^2 - (k_{1m} + k_{3m})^2 \\ &= \mu^2 - e_{11} - e_{13}. \end{aligned} \quad (3.79a)$$

$$\Sigma\mathcal{M}_{11}^2 = -\Delta\mathcal{M}_{11}^2 + 2\mu^2. \quad (3.79b)$$

$$\Delta E_{11} = \Delta\mathcal{M}_{11}^2/P^+. \quad (3.79c)$$

$$\begin{aligned} \Delta\mathcal{M}_{12}^2 &= (k_{2m} + k_{4m})^2 - (k_2 + k_4)_\mu^2 \\ &= e_{14} + e_{12} - \mu^2. \end{aligned} \quad (3.79d)$$

$$\Sigma\mathcal{M}_{12}^2 = \Delta\mathcal{M}_{12}^2 + 2\mu^2. \quad (3.79e)$$

$$\Delta E_{12} = \Delta\mathcal{M}_{12}^2/P^+. \quad (3.79f)$$

$$\begin{aligned} \Delta\mathcal{M}_{21}^2 &= (k_{2534\mu} + k_{4m})^2 - k_{3m}^2 \\ &= e_{2534} + e_{24} - m^2. \end{aligned} \quad (3.80a)$$

$$\Sigma\mathcal{M}_{21}^2 = \Delta\mathcal{M}_{21}^2 + 2m^2. \quad (3.80b)$$

$$\Delta E_{21} = \Delta\mathcal{M}_{21}^2/(1-x)P^+. \quad (3.80c)$$

$$\begin{aligned} \Delta\mathcal{M}_{22}^2 &= k_{2m}^2 - (k_{2512\mu} + k_{1m})^2 \\ &= m^2 - e_{2512} - e_{21}. \end{aligned} \quad (3.80d)$$

$$\Sigma\mathcal{M}_{22}^2 = -\Delta\mathcal{M}_{22}^2 + 2\mu^2. \quad (3.80e)$$

$$\Delta E_{22} = \Delta\mathcal{M}_{22}^2/yP^+. \quad (3.80f)$$

$$\begin{aligned} \Delta\mathcal{M}_{31}^2 &= (k_{3512\mu} + k_{2m})^2 - k_{1m}^2 \\ &= e_{3512} + e_{32} - m^2. \end{aligned} \quad (3.81a)$$

$$\Sigma\mathcal{M}_{31}^2 = \Delta\mathcal{M}_{31}^2 + 2m^2. \quad (3.81b)$$

$$\Delta E_{31} = \Delta\mathcal{M}_{31}^2/xP^+. \quad (3.81c)$$

$$\begin{aligned} \Delta\mathcal{M}_{32}^2 &= k_{4m}^2 - (k_{3534\mu} + k_{3m})^2 \\ &= m^2 - e_{3534} - e_{33}. \end{aligned} \quad (3.81d)$$

$$\Sigma\mathcal{M}_{32}^2 = -\Delta\mathcal{M}_{32}^2 + 2\mu^2. \quad (3.81e)$$

$$\Delta E_{32} = \Delta\mathcal{M}_{32}^2/(1-y)P^+. \quad (3.81f)$$

In all cases, the arguments of the similarity functions are given by Eq. (2.18), *i.e.* $z_i = \Delta\mathcal{M}_i^2/(\Sigma\mathcal{M}_i^2 + \lambda^2)$ for all subscripts appearing in Eq. (3.75).

The same reasoning is used to integrate Eq. (3.74) as in the case of Eq. (3.65) for the fermion-fermion interaction. For the similarity function f approaching the θ -function with $u_0 = 1/4$ in Eq. (2.20), we have $f(z_i^2) = \theta(\lambda^2 + 2m_i^2 - |\Delta\mathcal{M}_i^2|)$ with $m_i^2 = \mu^2$ in the first, and $m_i^2 = m^2$ in the second and third inner similarity factors in Eq. (3.74).

Integration of Eq. (3.74) gives

$$\begin{aligned} g_{1\bar{1}\bar{1}1\lambda} &= g_{1\bar{1}\bar{1}1\epsilon} + c_1 g_{\bar{u}1} v_3 g_{\bar{v}4} u_2 r_{11} r_{13} r_{14} r_{12} \\ &+ [c_2 r_{21} r_{2512} r_{24} r_{2534} \theta(y-x) + c_3 r_{32} r_{3512} r_{33} r_{3534} \theta(x-y)] g_{\bar{u}1} u_2 g_{\bar{v}4} v_3, \end{aligned} \quad (3.82a)$$

where the coefficients are,

$$c_1 = \frac{\theta_{12-11}\theta_{12-\lambda}}{|\Delta\mathcal{M}_{12}^2|} + \frac{\theta_{11-\lambda}\theta_{11-12}}{|\Delta\mathcal{M}_{11}^2|}, \quad (3.82b)$$

$$c_2 = \frac{y\theta_{22-21}\theta_{22-\lambda}}{(y-x)|\Delta\mathcal{M}_{22}^2|} + \frac{(1-x)\theta_{21-\lambda}\theta_{21-22}}{(y-x)|\Delta\mathcal{M}_{21}^2|}, \quad (3.82c)$$

$$c_3 = \frac{(1-y)\theta_{32-31}\theta_{32-\lambda}}{(x-y)|\Delta\mathcal{M}_{32}^2|} + \frac{x\theta_{31-\lambda}\theta_{31-32}}{(x-y)|\Delta\mathcal{M}_{31}^2|}. \quad (3.82d)$$

The symbols for θ -functions have the following meaning: $\theta_{i-j} = \theta(|\Delta\mathcal{M}_i^2| - |\Delta\mathcal{M}_j^2|)$ and $\theta_{i-\lambda} = \theta(|\Delta\mathcal{M}_i^2| - 2m_i^2 - \lambda^2)$ with m_i^2 equal μ^2 in c_1 and m^2 in c_2 and c_3 .

The next step is the construction of the interaction $F_\lambda[G_{1\bar{1}\bar{1}1\lambda}]$ from $\mathcal{G}_{1\bar{1}\bar{1}1\lambda}$ of Eq. (3.73) using Eqs. (2.8) and (2.9).

Then, one has to find out if matrix elements of $F_\lambda[G_{1\bar{1}\bar{1}1\lambda}]$ between finite free invariant mass states have a limit when $\epsilon \rightarrow 0$. Stated differently, one checks if the existence of the limit requires the initial value of $g_{1\bar{1}\bar{1}1\epsilon}$ to differ from zero to cancel potential divergences in the limit. Following the same steps as in the case of Eqs. (3.66) and (3.70), one can check that no divergences arise. Therefore, $g_{1\bar{1}\bar{1}1\epsilon} = 0$.

The final answer for the effective fermion-anti-fermion interaction is

$$\begin{aligned} H_{1\bar{1}\bar{1}1\lambda} &= F_\lambda[G_{1\bar{1}\bar{1}1\lambda}] = \int [P] \frac{1}{P^+} \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \int [x\kappa][y\rho] g_{1\bar{1}\bar{1}1\lambda} f(z_{1\bar{1}\bar{1}1\lambda}^2) \\ &\times b_{\lambda x P + \kappa \sigma_1}^\dagger d_{\lambda(1-x)P - \kappa \sigma_3}^\dagger d_{\lambda(1-y)P - \rho \sigma_4} b_{\lambda y P + \rho \sigma_2}, \end{aligned} \quad (3.83a)$$

where

$$g_{1\bar{1}\bar{1}1\lambda} = c_1 g_{\bar{u}1} v_3 g_{\bar{v}4} u_2 + [c_2 \theta(y-x) + c_3 \theta(x-y)] g_{\bar{u}1} u_2 g_{\bar{v}4} v_3. \quad (3.83b)$$

In terms of the fermion momenta,

$$c_1 = \frac{\theta(s - 3\mu^2 - \lambda^2)}{s - \mu^2}, \quad (3.83c)$$

with $s = \max(\mathcal{M}_{13}^2, \mathcal{M}_{24}^2)$, and

$$c_2\theta(y - x) + c_3\theta(x - y) = \frac{\theta_{a-b}\theta_{a-\lambda}}{a} + \frac{\theta_{b-a}\theta_{b-\lambda}}{b}, \quad (3.83d)$$

with $a = \mu^2 - q_{12}^2$, $b = \mu^2 - q_{34}^2$, $\theta_{a-b} = 1 - \theta_{b-a} = \theta(m_{xy}a - m_{1-x1-y}b)$, $\theta_{a-\lambda} = \theta[m_{xy}a - |x - y|(2m^2 + \lambda^2)]$, $\theta_{b-\lambda} = \theta[m_{1-x1-y}b - |x - y| \times (2m^2 + \lambda^2)]$, $m_{xy} = \max(x, y)$ and $m_{1-x1-y} = \max(1 - x, 1 - y)$. The argument of the outer similarity factor in Eq. (3.83.a), *i.e.* $z_{1\bar{1}\bar{1}1\lambda}$, is equal to $z_{22\lambda}$ from Eq. (3.70h). Note that Eqs. (3.83a-d) provide the generalization of Eqs. (3.70a-h) to the case of effective interactions of distinguishable fermions.

When λ^2 is reduced below $4m^2 - 3\mu^2$, the internal similarity factor in the annihilation term stays equal 1 independently of the value of λ . The effective interaction term provides the full contribution of the annihilation channel to the fermion-anti-fermion scattering amplitude of order g^2 . The fermion-anti-fermion-boson term in the effective Hamiltonian which could contribute acting twice in the scattering is zero for such low values of λ^2 because the mass gap between the boson and the fermion pair is larger than λ allows.

The internal similarity factor in the exchange term becomes equal 1 independently of λ only when λ^2 becomes smaller than $-2m^2 + 2m\mu + \mu^2$. The lower bound on λ^2 is $-m^2 - (m + \mu)^2$ (see the discussion of Eqs. (2.18) and (2.20)). In the lower bound region, the effective boson emission and absorption vanish and the exchange interaction term provides the full scattering amplitude due to the one-boson exchange. The amplitude is equal to the standard result on-shell where the outer similarity factor equals 1.

If the boson mass is much smaller than the fermion mass the fermion energies for small momenta are quadratic functions of momentum while the boson energy is a linear function of momentum. Therefore, for sufficiently small momenta, the boson energy is large in comparison to the fermion kinetic energies and their changes. Thus, the one-boson-exchange interaction is mediated by a relatively high energy intermediate state. Consequently, it is contained in a potential term in the effective Hamiltonian.

For small λ , the effective Hamiltonian contains potentials which are equal to standard scattering amplitudes in the Born approximation. The potentials differ from the Born amplitudes off-shell in a unique way which is dictated by principles of the Hamiltonian quantum mechanics and the similarity

renormalization group: the outer similarity factor reduces the strength of the interaction off-energy-shell. In the light-front dynamics the off-shellness is measured in terms of the free invariant mass.

Outside the lower bound region for λ the scattering amplitudes obtain also contributions from the effective interactions which change the number of bosons by one in the transition through the intermediate states. Analysis of Eq. (3.71) in application to the fermion-anti-fermion scattering follows the same steps as for the fermion-fermion scattering in the previous Section. The resulting on-shell scattering amplitude is independent of λ . The amplitude is equal to the well known perturbative result in Yukawa theory to order g^2 .

3.3. QED

This Section describes calculations of the effective mass squared term for photons, the effective mass squared term for electrons and the effective interaction between electrons and positrons in QED. The calculated terms are order e^2 .

The initial expression which we use to calculate the renormalized Hamiltonian of QED is obtained from the Lagrangian $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{D}-m)\psi$ by the procedure of evaluating the energy-momentum tensor $T^{\mu\nu}$ and integrating T^{+-} over the light-front [23]. We have

$$H_{\text{QED}} = \int dx^- d^2x^\perp \left[\bar{\psi}_m \gamma^+ \frac{-\partial^{\perp 2} + m^2}{2i\partial^+} \psi_m - \frac{1}{2} A_0^\nu \partial^{\perp 2} A_{0\nu} + e \bar{\psi}_m \not{A}_0 \psi_m + e^2 \bar{\psi}_m \not{A}_0 \frac{\gamma^+}{2i\partial^+} \not{A}_0 \psi_m + \frac{e^2}{2} \bar{\psi}_m \gamma^+ \psi_m \frac{1}{(i\partial^+)^2} \bar{\psi}_m \gamma^+ \psi_m \right]_{x^+=0}, \quad (3.84)$$

where ψ_m is a free fermion field with mass m and A_0^ν is a free massless photon field with $A_0^+ = 0$.

We replace fields $\psi_m(x)$ and $A_0^\nu(x)$ for $x^+ = 0$ by the Fourier superpositions of creation and annihilation operators, we order the operators in all terms and we drop terms containing divergent integrals which result from contractions. This is done in the same way as in the Yukawa theory but more terms need to be considered. Then, we introduce the regularization factors.

The ultraviolet regularization factors are already familiar and the same as in the Yukawa theory. The additional regularization is required due to the infrared singularities. Photons have diverging polarization vectors when their $+$ -momentum approaches 0. The corresponding seagull term, *i.e.* the 5th term in Eq. (3.84), is diverging too. We introduce the infrared regularization factors $(1 + \delta/x)^{-1}$ as described in Section 2.2.

We also introduce a photon mass term $\mu_\varepsilon^2 = \mu_\delta^2$ by adding it to $-\partial^{\perp 2}$ in the second term in Eq. (3.84). A fixed value of μ_δ leads to the conclusion in perturbation theory that the photon eigenstates have masses equal to μ_δ when the charge e approaches 0. Therefore, we will be forced to consider the limit $\mu_\delta \rightarrow 0$ in order to discuss physical photons to order e^2 . Also, the nonzero mass squared term for photons leads to additional divergences when $\delta \rightarrow 0$ and the limit of $\mu_\delta \rightarrow 0$ removes those.

The infrared finiteness of QED suggests that physical results in our approach should be independent of μ_δ when it is sufficiently small. We introduce the photon mass μ_δ and investigate the limit $\mu_\delta \rightarrow 0$. The second order calculations in this paper lead to results which are independent of μ_δ^2 when it tends to zero.

3.3.1. Photon mass squared

The same procedure from Section 2 which led to Eqs. (3.4) and (3.5) in Yukawa theory leads in QED to

$$\mathcal{G}_{1 \text{ photon } \lambda} = \sum_{\sigma} \int [k] \frac{k^{\perp 2} + \mu_{\lambda}^2}{k^+} a_{k\sigma}^{\dagger} a_{k\sigma}. \quad (3.85)$$

No correction arises to the term $k^{\perp 2}/k^+$ because our regularization preserves the kinematical symmetries of light-front dynamics.

A new feature in comparison to the Yukawa theory is the polarization of photons. With the kinematical symmetries explicitly preserved, only terms diagonal in the photon polarization emerge. For example, terms proportional to $k^i \varepsilon_{\sigma_1}^i k^j \varepsilon_{\sigma_2}^j$ with $\sigma_1 \neq \sigma_2$ cannot appear because the regularization and similarity factors do not introduce dependence on the photon momentum. Note that such terms are allowed by the power counting [5].

The net result of the photon self-interaction is an effective photon mass squared term which is independent of the photon momentum but varies with the effective Hamiltonian width λ . One obtains more complicated results for the effective photon free energy if the regularization or similarity factors violate kinematical symmetries of light-front dynamics [5].

The dependence of μ_{λ}^2 on λ is determined to order e^2 by the equation

$$\frac{d\mu_{\lambda}^2}{d\lambda} \delta^{\sigma_1 \sigma_2} = e^2 \int [x\kappa] \frac{df^2(z_{\lambda}^2)}{d\lambda} \frac{\text{Tr } \not{\varepsilon}_{k\sigma_1}^* (\not{k}_{1m} + m) \not{\varepsilon}_{k\sigma_2} (\not{k}_{2m} - m)}{\mathcal{M}^2 - \mu_{\delta}^2} r_{\varepsilon}(x, \kappa) \quad (3.86)$$

and the initial condition at $\lambda = \infty$. However, it is also sufficient to know the effective mass squared at any single value of λ to determine its value at other values of λ using Eq. (3.86). The initial condition at $\lambda = \infty$ is

distinguished only because it provides connection with standard approaches based on the local Lagrangian for electrodynamics.

In Eq. (3.86), $\mathcal{M}^2 = (\kappa^{\perp 2} + m^2)/x(1-x)$, $\Delta\mathcal{M}^2 = \mathcal{M}^2 - \mu_\delta^2$ and $\Sigma\mathcal{M}^2 = \mathcal{M}^2 + \mu_\delta^2$ so that $z_\lambda = (\mathcal{M}^2 - \mu_\delta^2)/(\mathcal{M}^2 + \mu_\delta^2 + \lambda^2)$. In the limit of Eq. (3.8), we have $f^2(z_\lambda^2) = \theta(\lambda^2 + 3\mu_\delta^2 - \mathcal{M}^2)$. In fact, Eq. (3.86) is free from infrared singularities and we could skip the introduction of μ_δ^2 by letting it go to zero at this point. However, the systematic approach defines the Hamiltonian of QED including the infrared regulator mass for photons and we can keep it here for illustration. The regularization factor $r_\varepsilon(x, \kappa)$ in Eq. (3.86) is the same as in Eq. (3.9) in Yukawa theory because only fermion regularization factors enter Eq. (3.86), according to Eqs. (2.47) to (2.49), and these factors are the same in both theories.

Evaluation of the spin factor gives

$$\frac{d\mu_\lambda^2}{d\lambda} = e^2 \int [x\kappa] \frac{df^2(z_\lambda^2)}{d\lambda} \frac{2\mathcal{M}^2 - 4\kappa^{\perp 2}}{\mathcal{M}^2 - \mu_\delta^2} r_\varepsilon(x, \kappa), \quad (3.87)$$

which is the QED analog of Eq. (3.6) from Yukawa theory.

Integration of Eq. (3.87) is carried out through the same steps as in the Yukawa theory. We can use Eq. (3.22) to calculate the effective photon mass squared μ_λ^2 knowing its value μ_0^2 at some value of $\lambda = \lambda_0$.

The value of μ_0^2 is found by requesting that the effective Hamiltonian eigenvalues for photon states contain the physical photon mass $\tilde{\mu}$, expected to be 0. However, solving the eigenvalue equation to second order in the coupling constant e through the same steps as in the case of mesons in Yukawa theory in Eqs. (3.23) to (3.25), leads to the physical photon mass $\tilde{\mu} = \mu_\delta$. μ_δ is small but finite and it is considered, in terms of powers of e , to be of order $e^0 = 1$ when $e \rightarrow 0$.

In the θ -function limit for the similarity function f with $u_0 = 1/4$ one obtains

$$\mu_0^2 = \mu_\delta^2 + \frac{\alpha}{4\pi} \int_0^1 dx \int_0^\infty d\kappa^2 \frac{2\mathcal{M}^2 - 4\kappa^{\perp 2}}{x(1-x)(\mathcal{M}^2 - \mu_\delta^2)} \theta(\lambda_0^2 + 3\mu_\delta^2 - \mathcal{M}^2) + o(e^4), \quad (3.88)$$

Thus, at other values of λ , the effective photon mass squared is

$$\mu_\lambda^2 = \mu_\delta^2 + \frac{\alpha}{4\pi} \int_0^1 dx \int_0^\infty d\kappa^2 \frac{2\mathcal{M}^2 - 4\kappa^{\perp 2}}{x(1-x)(\mathcal{M}^2 - \mu_\delta^2)} \theta(\lambda^2 + 3\mu_\delta^2 - \mathcal{M}^2) + o(e^4). \quad (3.89)$$

This result naturally depends on the infrared regularization parameter μ_δ^2 but no singularity appears when this parameter is set equal to zero. For

$\lambda^2 + 3\mu_\delta^2 \leq 4m^2$, where $4m^2$ is the lowest possible free invariant mass squared for the two intermediate fermions, the photon mass is independent of the Hamiltonian width λ and it equals μ_δ^2 . For larger values of λ^2 , the effective photon mass grows with the width λ so that its larger value compensates effects of the interactions which become active for the larger width. The net result is that the photon eigenstates have eigenvalues with masses squared equal μ_δ^2 independently of λ . Finally, the result favored by experimental data is obtained in the limit $\mu_\delta^2 \rightarrow 0$ at the end of the calculation.

3.3.2. Electron mass squared

Electron and positron self-interactions through emission and reabsorption of transverse photons result in the fermion free energy terms of the form exactly the same in QED as in Eq. (3.27) in Yukawa theory. However, the effective mass of electrons and positrons depends on the width differently than in the case of Yukawa theory. Instead of Eq. (3.28), one obtains now

$$\frac{dm_\lambda^2}{d\lambda} = e^2 \sum_{\tilde{\sigma}} \int [x\kappa] \frac{df^2(z_\lambda^2)}{d\lambda} \frac{\bar{u}_{m\sigma k} \not{\tilde{k}}_{\tilde{\sigma}} (\not{p}_m + m) \not{\tilde{k}}_{\tilde{\sigma}}^* u_{m\sigma k}}{\mathcal{M}^2 - m^2} r_{\varepsilon\delta}(x, \kappa), \quad (3.90)$$

where $\tilde{k} = (k - p)_0$, $p^+ = xk^+$, $p^\perp = xk^\perp + \kappa^\perp$, $\mathcal{M}^2 = (m^2 + \kappa^2)/x + (\mu_\delta^2 + \kappa^2)/(1-x)$, $\Delta\mathcal{M}^2 = \mathcal{M}^2 - m^2$, $\Sigma\mathcal{M}^2 = \mathcal{M}^2 + m^2$ and $z_\lambda = \Delta\mathcal{M}^2/(\Sigma\mathcal{M}^2 + \lambda^2)$. The regularization factor of Eq. (2.49) for the intermediate particles and the infrared regulator for the intermediate photon, as given at the end of Section 2.2, imply

$$r_{\varepsilon\delta}(x, \kappa) = \left[1 + \frac{\varepsilon}{\Lambda^2} \mathcal{M}^2 + \left(\frac{\varepsilon}{\Lambda^2} \right)^2 \frac{\kappa^2 + m^2}{x} \frac{\kappa^2 + \mu_\delta^2}{1-x} \right]^{-2} \left(1 + \frac{\delta}{1-x} \right)^{-2}. \quad (3.91)$$

The sum over photon polarizations in Eq. (3.90) produces the well known expression

$$\sum_{\tilde{\sigma}} \varepsilon_{\tilde{k}\tilde{\sigma}}^\alpha \varepsilon_{\tilde{k}\tilde{\sigma}}^{*\beta} = -g^{\alpha\beta} + \frac{\tilde{k}^\alpha g^{+\beta} + g^{+\alpha} \tilde{k}^\beta}{\tilde{k}^+}, \quad (3.92)$$

and the spin factor in Eq. (3.90) is

$$\begin{aligned} & \bar{u}_{m\sigma k} \gamma_\alpha (\not{p}_m + m) \gamma_\beta u_{m\sigma k} \left[-g^{\alpha\beta} + \frac{\tilde{k}^\alpha g^{+\beta} + g^{+\alpha} \tilde{k}^\beta}{\tilde{k}^+} \right] \\ &= \frac{2}{x} \left[(1-x)^2 m^2 + \kappa^2 \frac{1+x^2}{(1-x)^2} \right]. \end{aligned} \quad (3.93)$$

The new feature of this expression, in comparison to the Yukawa theory, is the divergence for $x \rightarrow 1$, *i.e.* where the photon longitudinal momentum approaches 0.

The rate of change of the electron mass term versus the effective Hamiltonian width in the θ -function limit for the similarity function with $u_0 = 1/4$ is

$$\frac{dm_\lambda^2}{d\lambda^2} = \frac{\alpha}{4\pi} \int_0^1 dx \int_0^\infty du \delta(3m^2 + \lambda^2 - \mathcal{M}^2) \frac{m^2 \frac{2(1-x)^2}{x} + u \frac{2(1+x^2)}{1-x}}{\mathcal{M}^2 - m^2} r_{\varepsilon\delta}(x, \kappa), \quad (3.94a)$$

where $u = \kappa^2/x(1-x)$,

$$r_{\varepsilon\delta}(x, \kappa) = \left[1 + \frac{\varepsilon}{\Lambda^2} \mathcal{M}^2 + \left(\frac{\varepsilon}{\Lambda^2} \right)^2 \left[(1-x)u + \frac{m^2}{x} \right] \left[xu + \frac{\mu_\delta^2}{1-x} \right] \right]^{-2} \left(1 + \frac{\delta}{1-x} \right)^{-2}, \quad (3.94b)$$

and

$$\mathcal{M}^2 = u + \frac{m^2}{x} + \frac{\mu_\delta^2}{1-x}. \quad (3.94c)$$

The divergence structure of the effective electron mass in Eq. (3.94a) is obscured by the fact that the whole effective mass term is merely a number dependent on λ while three cutoff and regularization parameters appear in the integral: ε , δ and μ_δ . The only available condition is that the effective electron mass should have a limit when $\varepsilon \rightarrow 0$. However, this condition has to be satisfied without generating divergences in the physical electron mass (*i.e.* in the electron eigenvalue energy) when we remove the infrared regularization. Since other contributions to the physical electron mass may diverge as δ or μ_δ tend to 0, and only the sum is finite in the limit, one needs to keep track of the infrared structure in defining the ε -independent (*i.e.* ultraviolet finite) part of the counterterm.

The divergences due to $\varepsilon \rightarrow 0$, $\delta \rightarrow 0$ and $\mu_\delta \rightarrow 0$, are not resolved in the single mass constant. Many elements of a complete analysis overlap in producing the final answer and many simplifications are possible. We will proceed in this Section with a simplified analysis. A more extended analysis will be required for other Hamiltonian terms where the outcome of the procedure is not reduced to finding only one number in the effective interaction. For example, in the electron-positron interaction term the external momenta of fermions introduce a whole range of additional parameters. That case will be illustrated in the next Section.

The δ -function under the integral on the right-hand side of Eq. (3.94a) forces $\mathcal{M}^2$ to be equal $\lambda^2 + 3m^2$. The smallest possible value of $\mathcal{M}$ is $m + \mu_\delta$ and the negative value of $\lambda^2 = (m + \mu_\delta)^2 - 3m^2 = -m^2 - (m + \mu_\delta)^2 + 2[(m +$

$\mu_\delta)^2 - m^2]$ is required to reach this lower bound (see the comments about Eq. (2.20)). Below this bound the right-hand side of Eq. (3.94a) vanishes, no emission or absorption of photons by electrons is possible and the effective electron mass stays constant. The smallest possible value of λ^2 allowed by Eq. (2.18) is $-m^2 - (m + \mu_\delta)^2$. The difference between these bounds vanishes when the photon mass goes to zero.

In the next Section about electron-positron interaction we will also consider the case of the infinitesimal u_0 (see the discussion below Eq. (2.20)), which leads to $\delta(\tilde{\lambda}^2 + m^2 - \mathcal{M}^2)$ in Eq. (3.94a), instead of $\delta(3m^2 + \lambda^2 - \mathcal{M}^2)$.

For $\lambda^2 > (m + \mu_\delta)^2 - 3m^2$, the right-hand side of Eq. (3.94a) is positive. Therefore, the effective electron mass squared term grows together with the width of the Hamiltonian. This growth combines with the growing negative contributions of the corresponding effective transverse photon emission and reabsorption so that the physical electron mass is independent of the Hamiltonian width.

For λ^2 close to the lower bound, u must be close to 0 and $x \sim 1 - \mu_\delta/(m + \mu_\delta)$. But $\mathcal{M}^2$ is limited and determined by the value of λ . Quite generally, as long as $\mathcal{M}$ remains limited, u and $\mu_\delta^2/(1 - x)$ are limited. The invariant mass denominator in Eq. (3.94a) equals $\lambda^2 + 2m^2$. The denominator is small only when λ^2 is close to the lower bound. Then, the integration range is small too. The denominator, when expressed in terms of λ , can be pulled out and put in front of the integral. The integration over u sets $u = \tilde{u}(x) = \lambda^2 + 3m^2 - m^2/x - \mu_\delta^2/(1 - x)$ and forces the condition $\tilde{u}(x) > 0$. This condition implies the following limits on the integration over x , provided $\lambda^2 + 3m^2 > (m + \mu_\delta)^2$ since otherwise the integral is 0.

$$x_0 - \Delta x < x < x_0 + \Delta x, \quad (3.95a)$$

$$x_0 = \frac{1}{2} \left(1 + \frac{m^2 - \mu_\delta^2}{\lambda^2 + 3m^2} \right), \quad (3.95b)$$

$$\Delta x = \sqrt{x_0^2 - \frac{m^2}{\lambda^2 + 3m^2}}. \quad (3.95c)$$

Within these limits, $\tilde{u}(x)$ varies from the minimal value of 0 at the lower bound $x_0 - \Delta x$ through a maximum of $\lambda^2 + 3m^2 - (m + \mu_\delta)^2$ at $x = m/(m + \mu_\delta)$ to the minimal value of 0 again at the upper bound $x_0 + \Delta x$. In the case of infinitesimal u_0 , one replaces $\lambda^2 + 2m^2$ in these formulae by $\tilde{\lambda}^2$. Eq. (3.94a) reads

$$\frac{dm_\lambda^2}{d\lambda^2} = \frac{\alpha\theta[3m^2 + \lambda^2 - (m + \mu_\delta)^2]}{4\pi(\lambda^2 + 2m^2)} \times \int_{x_0 - \Delta x}^{x_0 + \Delta x} dx \left[m^2 \frac{2(1-x)^2}{x} + \tilde{u}(x) \frac{2(1-x^2)}{1-x} \right] r_{\varepsilon\delta}(x, \tilde{u}(x)), \quad (96a)$$

where

$$r_{\varepsilon\delta}(x, \tilde{u}(x)) = \left[1 + \frac{\varepsilon}{\Lambda^2}(3m^2 + \lambda^2) + \left(\frac{\varepsilon}{\Lambda^2} \right)^2 \left[(1-x)\tilde{u}(x) + \frac{m^2}{x} \right] \times \left[x\tilde{u}(x) + \frac{\mu_\delta^2}{1-x} \right] \right]^{-2} \left(1 + \frac{\delta}{1-x} \right)^{-2}. \quad (96b)$$

The upper limit of integration over x for μ_δ^2 much smaller than m^2 and $2m^2 + \lambda^2$, equals $1 - \mu_\delta^2/(2m^2 + \lambda^2)$ and approaches 1 when $\mu_\delta \rightarrow 0$. For x close to 1, the factor $(1-x)^{-1}$ in the square bracket of the integrand is large and leads to a logarithmic dependence of the integral on the upper integration limit. The logarithm would become infinite for $\mu_\delta \rightarrow 0$ if δ were equal 0. Therefore, the limit $\mu_\delta \rightarrow 0$ is sensitive to the presence of the regularization factor with $\delta \neq 0$. For the finite μ_δ , the region of $x \rightarrow 1$ is regulated by ε and can be considered an ultraviolet limit. A counterterm to the diverging ε dependence could remove the divergence due to $x \rightarrow 1$. Then, a separate cutoff parameter δ would not be needed but the resulting terms would diverge for $\mu_\delta^2 \rightarrow 0$. For finite μ_δ^2 , the derivative of the electron mass with respect to λ is finite. For $\mu_\delta^2 = 0$, the integrand in the region $x \sim 1$ is regulated solely by the infrared regularization factor $[1 + \delta/(1-x)]^{-2}$ since the upper limit of integration over x is equal 1.

For finite λ^2 all three terms in $\mathcal{M}^2$, *i.e.* $\tilde{u}(x)$, m^2/x and $\mu_\delta^2/(1-x)$ are limited. Therefore, for finite μ_δ , one can take the limit $\varepsilon \rightarrow 0$ in the integrand. The factor $[1 - \delta/(1-x)]^{-2}$ remains and additionally cuts off the integration region at $x \sim 1 - \delta$. The ratio of δ to $\mu_\delta^2/(2m^2 + \lambda^2)$ determines the size of contributions obtained from the upper range of integration over x . For finite fixed values of μ_δ , one can take the limit $\delta \rightarrow 0$ and $\log[(2m^2 + \lambda^2)/\mu_\delta^2]$ appears in the answer.

The right-hand side of Eq. (3.96a) contains terms which behave for large λ as a constant, as λ^{-2} with factors of logarithms of λ and as functions vanishing faster than λ^{-2} . One integrates Eq. (3.96a) over λ from λ_0 to infinity in order to express the effective electron mass squared term at λ_0 , denoted m_0^2 , in terms of the initial m_ε^2 . Clearly, the integration over λ would diverge without the regularization factor which depends on ε . The integration produces terms behaving as ε^{-1} , $\log \varepsilon$ and terms convergent in

the limit $\varepsilon \rightarrow 0$. m_ε^2 in the initial Hamiltonian must be supplied with a counterterm to subtract the diverging ε -dependent terms in the effective Hamiltonians.

In summary, the infrared divergence due to $\mu_\delta \rightarrow 0$ and $\delta \rightarrow 0$ appears in the derivative of m_λ^2 with respect to λ . Therefore, even if one requests that the electron mass term is finite at some value of λ , the effective masses of electrons in the neighboring Hamiltonians with even slightly different widths will diverge when μ_δ and δ approach 0. We have to abandon the requirement that the effective electron mass term at any value of λ remains finite when the infrared regularization is removed. The effective masses diverge in the limit $\mu_\delta \rightarrow 0$ and $\delta \rightarrow 0$. The only condition we can fulfill through the ultraviolet renormalization is that the effective electron masses for finite λ are independent of ε .

Mathematical details of the effective electron mass term calculation are more complicated than in the Yukawa theory because the infrared regularization parameters are present. Otherwise, the calculation is essentially the same and we skip the description here. We only stress that the counterterm and the effective masses of electrons and positrons depend on the infrared cutoffs and they diverge when the cutoffs are being removed.

Thus, the effective electron mass squared term in the limit $\varepsilon \rightarrow 0$ is

$$m_\lambda^2 = m_0^2 + e^2 \int [x\kappa] [f^2(z_\lambda^2) - f^2(z_{\lambda_0}^2)] \\ \times \frac{m^2 \frac{2(1-x)^2}{x} + \frac{\kappa^2}{x(1-x)} \frac{2(1+x^2)}{1-x}}{\mathcal{M}^2 - m^2} \left(1 + \frac{\delta}{1-x}\right)^{-2} + o(e^4). \quad (3.97)$$

The finite term m_0^2 has a limit when $\varepsilon \rightarrow 0$. Its dependence on the infrared regularization is not displayed. m_0^2 is found from a suitable renormalization condition.

The natural condition to be satisfied by m_0^2 is that the effective Hamiltonian of some width λ has the electron eigenstates with eigenvalues equal $(p^\perp{}^2 + \tilde{m}^2)/p^+$, where p denotes the electron momentum and $\tilde{m}$ is the physical electron mass. The eigenvalue equation for electrons can be solved in perturbation theory in the same way as for bosons in Yukawa theory in Eqs. (3.23)–(3.26) and fermions in Eqs. (3.35)–(3.39), or in QED for photons in Eqs. (3.88)–(3.89). One obtains the condition

$$\tilde{m}^2 = m_0^2 - e^2 \int [x\kappa] f^2(z_{\lambda_0}^2) \frac{m^2 \frac{2(1-x)^2}{x} + \frac{\kappa^2}{x(1-x)} \frac{2(1+x^2)}{1-x}}{\mathcal{M}^2 - m^2} \left(1 + \frac{\delta}{1-x}\right)^{-2} + o(e^4), \quad (3.98)$$

and one can calculate m_0 from this condition. Consequently,

$$m_\lambda^2 = \tilde{m}^2 + e^2 \int [x\kappa] f^2(z_\lambda^2) \frac{m^2 \frac{2(1-x)^2}{x} + \frac{\kappa^2}{x(1-x)} \frac{2(1+x^2)}{1-x}}{\mathcal{M}^2 - m^2} \left(1 + \frac{\delta}{1-x}\right)^{-2} + o(e^4), \quad (3.99)$$

and $m^2 = \tilde{m}^2 + o(e^2)$.

The physical electron mass is independent of the infrared regularization because the regularization dependent m_λ^2 and the effective emission and absorption of photons combine to the regularization independent result.

3.3.3. Electron-positron interaction

Calculation of the effective electron-positron interaction to order e^2 is of interest as a way to derive the Coulomb force in quantum electrodynamics — this interaction is responsible for the formation of positronium. Also, effective interactions between quarks and anti-quarks in QCD have a number of similar features and the QED calculation provides an introduction to the QCD case.

Generally speaking, the QED calculation of the effective electron-positron interaction proceeds in the same way as in the case of fermion-anti-fermion interaction in Yukawa theory except for three new elements.

The first is that photons have polarization vectors which enter in the vertex factors and introduce additional dependence on the exchanged photon momentum. This dependence leads to infrared divergences for small longitudinal momenta of exchanged photons.

The second feature is that the infrared divergences require additional regularization factors. We use the parameter δ and the photon mass squared $\mu_\delta \neq 0$. The limits $\delta \rightarrow 0$ and $\mu_\delta \rightarrow 0$ are generally understood as to be taken at the end of a calculation of observables and not in the effective Hamiltonian itself. However, it may also be possible to take the limits in matrix elements of the Hamiltonian between states which do not induce infrared divergences, *i.e.* do not involve small x photons.

The third feature is that the one-photon exchange interaction needs to be combined with the 5th term from Eq. (3.84) to obtain the standard results for the electron-positron scattering in the Born approximation. The 5th term from Eq. (3.84) provides the initial condition for the renormalization group flow of the effective Hamiltonians. In order e^2 this term is only supplied with the outer similarity factor by the operation F_λ . It does not change in the flow beyond this factor because it is order e^2 itself. The initial condition provides a contribution which is needed to obtain the Coulomb potential. This is a different situation than in Yukawa theory where no four-fermion seagull interactions appeared and the one-meson exchange interaction was

sufficient to produce the Yukawa potential in the effective Hamiltonians of small widths.

The effective electron-positron interaction term has the analogous structure as the fermion-anti-fermion interaction in Eq. (3.73), *i.e.*

$$\mathcal{G}_{1\bar{1}\bar{1}1\lambda} = \int [P] \frac{1}{P^+} \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \int [x\kappa][y\rho] g_{1\bar{1}\bar{1}1\lambda} b_{xP+\kappa\sigma_1}^\dagger d_{(1-x)P-\kappa\sigma_3}^\dagger d_{(1-y)P-\rho\sigma_4} b_{yP+\rho\sigma_2}. \quad (3.100)$$

The coefficient function $g_{1\bar{1}\bar{1}1\lambda}$ of order e^2 satisfies the differential equation

$$\begin{aligned} \frac{dg_{1\bar{1}\bar{1}1\lambda}}{d\lambda} &= S_1 \sum_{\sigma_5} e\bar{u}_1 \not{\epsilon}_{k_5\sigma_5} v_3 e\bar{v}_4 \not{\epsilon}_{k_5\sigma_5}^* u_2 r_{11} r_{13} r_{14} r_{12} \frac{1}{P^+} \\ &\quad - S_2 r_{21} r_{2512} r_{24} r_{2534} r_{3/5} r_{2/5} \frac{\theta(y-x)}{(y-x)P^+} \sum_{\sigma_5} e\bar{v}_4 \not{\epsilon}_{k_5\sigma_5} v_3 e\bar{u}_1 \not{\epsilon}_{k_5\sigma_5}^* u_2 \\ &\quad - S_3 r_{32} r_{3512} r_{33} r_{3534} r_{1/5} r_{4/5} \frac{\theta(x-y)}{(x-y)P^+} \sum_{\sigma_5} e\bar{u}_1 \not{\epsilon}_{k_5\sigma_5} u_2 e\bar{v}_4 \not{\epsilon}_{k_5\sigma_5}^* v_3, \end{aligned} \quad (3.101)$$

which is the QED analog of Eq. (3.74) from Yukawa theory. Notation is the same as in Eqs. (3.74) to (3.81) with the exception that μ^2 is replaced by μ_δ^2 . The new elements are the infrared regularization factors of Section 2.2, *i.e.* $r_{3/5} = r(k_3^+ \delta / k_5^+)$, $r_{2/5} = r(k_2^+ \delta / k_5^+)$, $r_{1/5} = r(k_1^+ \delta / k_5^+)$, $r_{4/5} = r(k_4^+ \delta / k_5^+)$, and the photon polarization vectors. The sum over photon polarizations gives

$$\sum_{\sigma_5} \varepsilon_{k_5\sigma_5}^\alpha \varepsilon_{k_5\sigma_5}^{*\beta} = -g^{\alpha\beta} + \frac{k_{50}^\alpha g^{\beta+} + g^{\alpha+} k_{50}^\beta}{k_\beta^+}. \quad (3.102)$$

The terms proportional to the four-vector k_{50} can be rewritten using the Dirac equation for free fermions of mass m . For example, in the second term on the right-hand side of Eq. (3.101) we have

$$\begin{aligned} \bar{u}_1 \not{k}_{50} u_2 &= \bar{u}_1 \left[\not{k}_{2m} - \not{k}_{1m} + \frac{1}{2} \gamma^+ [(k_2 - k_1)_0^- - k_{2m}^- + k_{1m}^-] \right] u_2 \\ &= \bar{u}_1 \gamma^+ u_2 \frac{-(k_{2m} - k_{1m})^2}{2(k_2^+ - k_1^+)}. \end{aligned} \quad (3.103)$$

Using similar relations for all vertex factors involved one obtains

$$\begin{aligned} \frac{dg_{1\bar{1}\bar{1}1\lambda}}{d\lambda} = & S_1 \left[-g^{\mu\nu} - g^{\mu+} g^{\nu+} \frac{s_{13} + s_{24}}{2P+2} \right] e\bar{u}_1 \gamma_\mu v_3 e\bar{v}_4 \gamma_\nu u_2 r_{11} r_{13} r_{14} r_{12} \frac{1}{P+} \\ & - \left\{ S_2 r_{21} r_{2512} r_{24} r_{2534} r_{3/5} r_{2/5} \frac{\theta(y-x)}{(y-x)P+} + S_3 r_{32} r_{3512} r_{33} r_{3534} r_{1/5} r_{4/5} \frac{\theta(x-y)}{(x-y)P+} \right\} \\ & \times \left[-g^{\mu\nu} - g^{\mu+} g^{\nu+} \frac{q_{12}^2 + q_{34}^2}{2(x-y)^2 P+2} \right] e\bar{u}_1 \gamma_\mu u_2 e\bar{v}_4 \gamma_\nu v_3. \end{aligned} \quad (3.104)$$

We use the notation $s_{ij} = (k_i + k_j)^2$ and $q_{ij}^2 = (k_i - k_j)^2$.

Integration of Eq. (3.104) proceeds in the same way as in the case of Eq. (3.74). The initial condition at $\lambda = \infty$ includes the seagull term.

$$\begin{aligned} H^{\text{seagull}} = & \int [P] \frac{1}{P+} \sum_{\sigma_1 \sigma_2 \sigma_3 \sigma_4} \int [x\kappa][y\rho] g_{1\bar{1}\bar{1}1}^{\text{seagull}} \\ & \times b_{xP+\kappa\sigma_1}^\dagger d_{(1-x)P-\kappa\sigma_3}^\dagger d_{(1-y)P-\rho\sigma_4} b_{yP+\rho\sigma_2}, \end{aligned} \quad (105a)$$

where

$$\begin{aligned} g_{1\bar{1}\bar{1}1}^{\text{seagull}} = & - \left[r_{21} r_{2512} r_{24} r_{2534} r_{3/5} r_{2/5} \frac{\theta(y-x)}{(y-x)^2} \right. \\ & + r_{32} r_{3512} r_{33} r_{3534} r_{1/5} r_{4/5} \frac{\theta(x-y)}{(x-y)^2} \left. \right] e\bar{u}_1 \gamma_\mu u_2 e\bar{v}_4 \gamma_\nu v_3 \frac{g^{\mu+} g^{\nu+}}{P+2} \\ & + e\bar{u}_1 \gamma_\mu v_3 e\bar{v}_4 \gamma_\nu u_2 r_{11} r_{13} r_{14} r_{12} \frac{g^{\mu+} g^{\nu+}}{P+2}. \end{aligned} \quad (105b)$$

The result of the integration of Eq. (104) is

$$\begin{aligned} g_{1\bar{1}\bar{1}1\lambda} = & g_{1\bar{1}\bar{1}1\epsilon}^{\text{counterterm}} - c_1 e\bar{u}_1 \gamma^\mu v_3 e\bar{v}_4 \gamma_\mu u_2 r_{11} r_{13} r_{14} r_{12} \\ & + \left[-\frac{1}{2} c_1 (s_{13} + s_{24}) + 1 \right] \frac{e\bar{u}_1 \gamma^+ v_3 e\bar{v}_4 \gamma^+ u_2}{P+2} r_{11} r_{13} r_{14} r_{12} \\ & - \left[c_2 r_{21} r_{2512} r_{24} r_{2534} r_{3/5} r_{2/5} \theta(y-x) + c_3 r_{32} r_{3512} r_{33} r_{3534} r_{1/5} r_{4/5} \theta(x-y) \right] \\ & \times e\bar{u}_1 \gamma^\mu u_2 e\bar{v}_4 \gamma_\mu v_3 - \left\{ \left[\frac{1}{2} c_2 (q_{12}^2 + q_{34}^2) + 1 \right] r_{21} r_{2512} r_{24} r_{2534} r_{3/5} r_{2/5} \theta(y-x) \right. \\ & + \left. \left[\frac{1}{2} c_3 (q_{12}^2 + q_{34}^2) + 1 \right] r_{32} r_{3512} r_{33} r_{3534} r_{1/5} r_{4/5} \theta(x-y) \right\} \frac{e\bar{u}_1 \gamma^+ u_2 e\bar{v}_4 \gamma^+ v_3}{(x-y)^2 P+2}. \end{aligned} \quad (3.106)$$

The coefficients c_1 , c_2 and c_3 are given by universal Eqs. (3.82.b) to (3.82.d) which were derived already in Yukawa theory, with the replacement of the meson mass by the photon mass, $\mu^2 = \mu_\delta^2$.

This result will be now analysed term by term for illustration in the electron-positron scattering in second order perturbation theory. The first term in Eq. (3.71) for the T-matrix calculated to order e^2 has matrix elements equal to the matrix elements of the effective interaction from Eq. (3.106). When evaluating the S-matrix elements, one considers configurations where the free energy of incoming fermions equals the free energy of the outgoing fermions. This configuration selects the energy-diagonal part of the effective interaction: $s_{13} = s_{24} = s$ and $q_{12}^2 = q_{34}^2 = q^2$. Also, the external similarity factor which appears in the effective Hamiltonian as an additional factor to $g_{1\bar{1}1\bar{1}\lambda}$ equals 1. Thus, $g_{1\bar{1}1\bar{1}\lambda}$ in the energy diagonal part can be viewed as the scattering amplitude. It simplifies in the energy diagonal part to

$$\begin{aligned} g_{1\bar{1}1\bar{1}\lambda} &= g_{1\bar{1}1\bar{1}\epsilon}^{\text{counterterm}} - c_1 e \bar{u}_1 \gamma^\mu v_3 e \bar{v}_4 \gamma_\mu u_2 \\ &+ [-c_1 s + 1] \frac{e \bar{u}_1 \gamma^+ v_3 e \bar{v}_4 \gamma^+ u_2}{P+2} \\ &- [c_2 r_{2512} r_{2534} r_{3/5} r_{2/5} \theta(y-x) + c_3 r_{3512} r_{3534} r_{1/5} r_{4/5} \theta(x-y)] e \bar{u}_1 \gamma^\mu u_2 e \bar{v}_4 \gamma_\mu v_3 \\ &- \{ [c_2 q^2 + 1] r_{2512} r_{2534} r_{3/5} r_{2/5} \theta(y-x) \\ &+ [c_3 q^2 + 1] r_{3512} r_{3534} r_{1/5} r_{4/5} \theta(x-y) \} \frac{e \bar{u}_1 \gamma^+ u_2 e \bar{v}_4 \gamma^+ v_3}{(x-y)^2 P+2}. \end{aligned} \quad (3.107)$$

We have removed the regularization factors which equal 1 for fermions with finite free energy.

In the limit $\mu_\delta \rightarrow 0$ and for the cutoff λ close to $-2m^2$, a number of simplifications occur. We display the result for the case where the intermediate photon momentum fraction $|x-y| \gg (x, y, 1-x, 1-y)\delta$, *i.e.* when the exchanged photon momentum is not negligible in comparison to the fermion momenta. In this case, the infrared regularization factors for the photon which are still kept in Eq. (3.107) equal 1 and

$$\begin{aligned} g_{1\bar{1}1\bar{1}\lambda} &= g_{1\bar{1}1\bar{1}\epsilon}^{\text{counterterm}} - \frac{u_1 \gamma^\mu v_3 e \bar{v}_4 \gamma_\mu u_2}{s} \\ &+ \theta[|q^2| - |x-y|(2m^2 + \lambda^2)/\max(x, y, 1-x, 1-y)] \frac{e \bar{u}_1 \gamma^\mu u_2 e \bar{v}_4 \gamma_\mu v_3}{q^2} \\ &- \theta[|x-y|(2m^2 + \lambda^2)/\max(x, y, 1-x, 1-y) - |q^2|] \frac{e \bar{u}_1 \gamma^+ u_2 e \bar{v}_4 \gamma^+ v_3}{(x-y)^2 P+2}. \end{aligned} \quad (3.108)$$

The first term is the potentially necessary counterterm which we have not yet determined. Since the remaining terms are not sensitive to ϵ , the counterterm matrix element is equal zero.

The second term is the well known expression for the electron-positron annihilation channel scattering amplitude in the Born approximation. No

limits on the fermion momenta appear because the invariant mass squared of two fermions is larger than $4m^2$ which is the minimal invariant mass difference in the transition between the two fermions and a one massless photon state. Since we assume the width λ^2 to be small the effective Hamiltonian contains the full amplitude for transition through the intermediate photon state. The third term is equal to the standard second order expression for the electron-positron scattering amplitude via one-photon exchange, except for the θ -function factor which forces the momentum transfer to be sufficiently large.

The meaning of this restriction is visible in the T-matrix. The third term contributes to the e^+e^- -scattering amplitude through the first term in Eq. (3.71). The same contribution would originate from the second term in Eq. (3.71) if we were using the initial Hamiltonian to calculate the T-matrix. In contrast, the effective Hamiltonian with the small width λ limits the effective photon emissions and absorptions to small momentum transfers and, therefore, it is not able to provide this contribution through the second term in Eq. (3.71). This contribution is then contained in the effective Hamiltonian itself and comes in the scattering matrix through the first term in Eq. (3.71).

The fourth term is unusual in the sense that it should not appear in the electron-positron scattering at all. The fourth term distinguishes the z -axis in its structure and diverges when $x \rightarrow y$. The θ -function factor in the fourth term is equal 1 where the θ -function factor of the third term is equal 0. And vice versa, the fourth term θ -function equals 0 where the third term θ -function equals 1.

The need for the fourth term becomes clear when one recalls that the second term in Eq. (3.71) also contributes to the electron-positron scattering amplitude. The relevant contribution comes through the effective one-photon exchange which results from the double action of $H_{I\lambda}$. $H_{I\lambda}$ is given by the operation F_λ applied to the third term of the QED Hamiltonian from Eq. (3.84), (see Eq. (2.11)). The operation F_λ multiplies the photon emission and absorption vertices by the factor f_λ . This factor was set equal to a θ -function in the current example. Each interaction provides one factor of the θ -function. The resulting factor in the second term of Eq. (3.71) is the same in QED as in Yukawa theory in Eq. (3.72), except for the antisymmetrization effect which leads to the θ -function which stands in front of the fourth term in Eq. (3.108).

Now, the second term in Eq. (3.71) contains spin factor which is the same as in the last term in Eq. (3.104). The $g^{\mu\nu}$ part complements the third term in Eq. (3.108) and produces the full well known one-photon exchange scattering amplitude which is free from the θ -function factor. The remaining part provides the term which cancels the odd fourth term in Eq. (3.108).

Thus, the effective Hamiltonian calculated to second order in powers of the charge e contains an odd term and θ -functions which are required to compensate for the odd contributions to the scattering amplitude from the effective Hamiltonian order e acting twice. The above analysis of the energy diagonal part of the second order effective Hamiltonian explains the role of different terms in Eqs. (3.106) to (3.108).

The analysis also suggests that apparently infrared diverging terms in the effective Hamiltonian may mutually compensate their diverging contributions in the scattering amplitude on energy-shell. In the current example, we see the interplay between the second-order seagull term and the double action of the first order emission and absorption of photons. The first order Hamiltonian matrix elements diverge when the photon longitudinal momentum approaches zero. The second order seagull term compensates this divergence in the on-energy-shell T-matrix elements.

The remaining point to make here is that the result of Eq. (3.106) with the counterterm equal 0 leads to the effective light-front Hamiltonian version of the Coulomb force in the limit of small $\lambda^2 + 2m^2 \ll \alpha m^2$. The key elements in deriving this conclusion are the outer similarity factors and the smallness of α . The outline of the derivation is following (*cf.* Refs. [11] and [26]).

If only the small energy transfers are allowed by the outer similarity factor, *i.e.* transfers much smaller than the electron mass, then the wave functions of the lowest mass eigenstates of the effective Hamiltonian are strongly peaked at small relative electron momenta and they fall off very rapidly as functions of the relative momentum. This is not true without the outer similarity factor because the function $g_{1\bar{1}\bar{1}1\lambda}$ alone is too large at the large energy transfers and it would produce singular contributions in the large relative momentum region making the eigenvalue problem sensitive to the ultraviolet regularization cutoffs.

Below the width scale the wave functions fall off as dictated by the eigenvalue equation with small α . Above the width scale the fall off is very fast due to the similarity factor which justifies restriction to momenta much smaller than m and the nonrelativistic approximation for all factors in Eq. (3.106) becomes accurate.

In the nonrelativistic approximation, $q_{12}^2 = q_{34}^2 = q^2$ and Eq. (3.107) applies. Further, the θ -functions in Eq. (3.108) become effectively equal 1 and 0, respectively. The last term is not leading to important contribution despite its divergent longitudinal structure because it is canceled by the effective massless photon exchange as described earlier in this Section.

The dominant contributions are provided by the second and the third terms from Eq. (3.108) which are well known to have the right nonrelativistic structure for predicting positronium properties in the Schrödinger equation. Now, the outer similarity factor becomes irrelevant to the spectrum in the

leading approximation because the coupling constant is very small (*cf.* [11] and [26]). In the dominant region the electron velocity is order α , the non-relativistic approximation to the full dynamics produces wave functions with relative momenta order αm and the outer similarity factor in the effective interaction can be replaced by 1.

We can use the infinitesimal u_0 in Eq. (2.20) and replace $\lambda^2 + 2m^2$ by $\tilde{\lambda}^2$ (see the discussion below Eq. (2.20)). When $\tilde{\lambda}$ is order αm and $x - y$ is order α the momentum transfer $\tilde{q}^2$ is typically order $\alpha^2 m^2$ which is much larger than $2(x - y)\tilde{\lambda}^2$ in Eq. (3.108). Thus, the θ -function is equal 1 and the second term in Eq. (3.108) becomes equal to the standard Coulomb interaction with the well known Breit-Fermi structure of the spin factors. This step completes the derivation of the Coulomb potential. The derivation explains the effective nature of the Schrödinger equation with the Coulomb potential in the light-front Hamiltonian formulation of QED.

3.4. QCD

The main conceptual complication in Hamiltonian calculations in QCD is confinement which is not yet fully understood. To order g^2 the manifestation of the confinement problem is the lack of a well defined initial condition for the renormalization group flow of the effective Hamiltonians. It will require an extended research effort to find the class of acceptable initial conditions. For example, the on-mass-shell renormalization conditions for the quark and gluon mass terms are questionable.

We stress the urgent need for the higher order calculations by describing some details of the second order calculation of the $q\bar{q}$ effective interaction. The calculation is similar to the one in QED above, except for the option for a different treatment of the last term in Eq. (3.108). Perry [12] suggested that the long distance part of this term may remain uncanceled in the effective QCD dynamics because of the gluon non-abelian gauge interactions. If the last term in Eq. (3.108) remained uncanceled it could be claimed to generate confinement in the light-front Hamiltonian approach to QCD [12]. We describe the structure of this term in the present approach since it is different than in Ref. [12]. The differences result from the different definitions of the similarity transformation, boost invariance and not invoking coupling coherence. The coupling coherence arguments are replaced by a plain perturbative renormalization condition for quark mass terms.

3.4.1. Quark and gluon mass terms

Results one obtains from Eq. (2.38) in QCD in second order in $\mathcal{G}_{2\lambda}$ can be illustrated by two equations for the effective masses of quarks and gluons.

The range of widths and coupling constants for which these equations can pertain to physics are not known yet.

We do not write regularization factors in detail. One can write them easily using results of Section 2 and previous examples in Section 3. In the case of effective masses a number of simplifications occur as explained in the previous Sections. Let us consider an infinitesimal u_0 in Eq. (2.20) and simplify notation by replacing $\tilde{\lambda}$ by λ itself. Then, we can write

$$\frac{d}{d\lambda}\mathcal{G}_{1\lambda} = \left[\mathcal{G}_{12\lambda} \frac{df^2(z_\lambda/\lambda^2)/d\lambda}{\mathcal{G}_{1\lambda} - E_{1\lambda}} \mathcal{G}_{21\lambda} \right]_{11}. \quad (3.109)$$

$E_{1\lambda}$ is the eigenvalue of $\mathcal{G}_{1\lambda}$ corresponding to the subscript 11. A set of arguments z_λ is needed. Namely,

$$z_1 = \frac{\kappa^2 + \mu_\lambda^2}{x(1-x)} - \mu_\lambda^2, \quad (3.110)$$

$$z_2 = \frac{\kappa^2 + m_\lambda^2}{x(1-x)} - \mu_\lambda^2, \quad (3.111)$$

$$z_3 = \frac{\kappa^2 + m_\lambda^2}{x} + \frac{\kappa^2 + \mu_\lambda^2}{1-x} - m_\lambda^2. \quad (3.112)$$

m_λ and μ_λ are the effective quark and gluon masses, respectively. Then,

$$\frac{dm_\lambda^2}{d\lambda} = \int [x\kappa] g_{q\lambda}^2 z_3^{-1} \frac{df^2(z_3/\lambda^2)}{d\lambda} [\kappa^2 [2/x + 4/(1-x)^2] + 2m_\lambda^2 (1-x)^2/x] r_{qg\varepsilon}(x, \kappa) \quad (3.113)$$

and

$$\begin{aligned} \frac{d\mu_\lambda^2}{d\lambda} = & 3 \int [x\kappa] g_{g\lambda}^2 z_1^{-1} \frac{df^2(z_1/\lambda^2)}{d\lambda} \kappa^2 [4/x^2 + 2] r_{gg\varepsilon}(x, \kappa) \\ & + \int [x\kappa] g_{q\lambda}^2 z_2^{-1} \frac{df^2(z_2/\lambda^2)}{d\lambda} \left[\frac{\kappa^2 + m_\lambda^2}{x(1-x)} - 2\kappa^2 \right] r_{qq\varepsilon}(x, \kappa). \end{aligned} \quad (3.114)$$

The gluon couples to the quark-anti-quark pairs and pairs of gluons while the quark couples only to the quark-gluon pairs. The number of colors above is equal to 3 and the number of flavors to 1.

These equations are not further studied here for two major reasons. The first one is that we do not know the initial conditions to use for such study. The second is that the equations involve two running couplings which are not known yet. The third and fourth order calculations are required to find them.

The importance of effective mass issue for quarks and gluons is illustrated below by the calculation of the small energy transfer effective forces between quarks and anti-quarks.

3.4.2. Quark-anti-quark effective interaction

The second order results are similar to QED. For heavy quarkonia one can directly look at Eq. (3.108). The only change required is the color SU(3) matrices sandwiched between color vectors of quarks and summed over colors of the exchanged gluons. The counterterm is 0. The second term gives the annihilation channel potential but the color matrix is traceless and this excludes the single gluon annihilation channel from the dynamics of color singlet $Q\bar{Q}$ states. The third term leads to the color Coulomb potential with the well known Breit-Fermi spin factors.

About the last term in Eq. (3.108) it was suggested by Perry [12] that a term of this kind may become a seed for confining interactions if it is not canceled. A cancelation occurs in perturbation theory when one evaluates the model interaction in the $Q\bar{Q}$ sector in the explicit expansion in powers of g to second order using the operation R and Eq. (1.3) in the nonrelativistic approximation for the quark relative momentum, exactly the same way as for electrons in QED with massless photons.

However, there are reasons for effective gluons to become separated by an energy gap through the non-abelian terms which are absent in effective QED. One can assume that the gluon energies in the $Q\bar{Q}g$ sector may be lifted up so that the gluons cannot contribute to the model $Q\bar{Q}$ interaction in the way the abelian massless photons can in the model electron-positron interaction in QED. As a result of this assumption one obtains the last term in Eq. (3.108) acting in the effective $Q\bar{Q}$ sector.

In the nonrelativistic approximation, the Coulomb term and the term in question are (see Eq. (3.108))

$$-\theta \left[\vec{q}^2 - |x - y| 2\tilde{\lambda}^2 \right] \frac{g^2 4m^2}{\vec{q}^2} - \theta \left[|x - y| 2\tilde{\lambda}^2 - \vec{q}^2 \right] \frac{g^2}{(x - y)^2}. \quad (3.115)$$

The regularization factors are the same in both terms and they are not displayed. The only effect of their presence which matters here is that $|x - y|$ is limited from below by about $\delta/2$. The initial infrared regularization parameter δ comes in through the initial condition in the renormalization group flow of the seagull term. The flow is limited in the second order calculation to the dependence of the outer similarity factor on $\tilde{\lambda}$. The factor $1/2$ results from $|x - y|/x_{\text{quark}}$ being limited from below by δ and the quarks having $x_{\text{quark}} \sim 1/2$. In fact, the lower bound on $|x - y|$ is given by $\delta \max(x, 1 - x, y, 1 - y)$. This is different from Ref. [12] where instead of the ratios of the $+$ -momentum fractions a separate frame dependent scale for $+$ -momentum is introduced.

The third component of the exchanged gluon momentum is $q_3 = (x - y)2m$. Thus, we see that the uncanceled singular term is represented by the potential which is analogous to the Coulomb potential except for that the factor

$-1/\vec{q}^2$ is replaced by $-1/q_3^2$ and both terms have mutually excluding and complementary supports in the momentum transfer space.

The seagull θ -function can be rewritten as $\theta[\omega^2 - (|q_3| - \omega)^2 - q^\perp{}^2]$, where $\omega = \tilde{\lambda}^2/2m$. The support of this function is two spheres of radius ω centered at $q^\perp = 0$ and $q_3 = \pm\omega$. The spheres touch each other at the point $q^\perp = q_3 = 0$. In this point, q_3^2 in the denominator produces a singularity.

Let us initially consider both terms in Eq. (3.115) as the actual interaction in the model $Q\bar{Q}$ sector, *i.e.* as if they were not affected by the operation R in Eq. (1.3). The Coulomb term works outside the two spheres in the $\vec{q}$ -space and the singular seagull term works inside.

In the region of the singularity, both $q^\perp$ and q_3 are small in comparison to ω . In this rough analysis one can neglect the outer similarity factor $\theta(\tilde{\lambda}^2 - |k^2 - k'^2|)$ since it is equal 1 when $\vec{q} = \vec{k} - \vec{k}'$ approaches 0. $\vec{k}$ is the relative momentum of the created $Q\bar{Q}$ pair and $\vec{k}'$ is the relative momentum of the annihilated $Q\bar{Q}$ pair. $\omega = (\tilde{\lambda}/m)\tilde{\lambda}/2 \ll \tilde{\lambda}/2$ and the spheres have the radius about $\tilde{\lambda}/m$ times smaller than the outer similarity factor width in the quark relative momenta, *i.e.* the relative size of the spheres in comparison to the outer similarity factor support approaches 0 when $\tilde{\lambda}/m \rightarrow 0$.

Since $q^\perp{}^2$ is order q_3 in the singular region the divergence when $q_3 \rightarrow 0$ is logarithmic. The lower limit of integration over $|q_3|$ for a given x is given by $2m\delta \max(x, 1-x)$. However, we assume $x = 1/2 + o(g^2)$ and we neglect terms of higher order than g^2 in the model $Q\bar{Q}$ Hamiltonian.

The potential resulting from the uncanceled seagull term is given by the following expression (*cf.* [12]),

$$V(\vec{r}) \sim - \int \frac{d^3q}{(2\pi)^3} \exp(i \vec{q} \vec{r}) \frac{\theta(2\omega|q_3| - \vec{q}^2) \theta(|q_3| - 2m\delta)}{q_3^2}. \quad (3.116)$$

The sign $\sim$ means that the diverging dependence on δ is subtracted and the same coefficient stands in front of the integral as in the Coulomb potential term. The argument for the infrared subtraction goes as follows.

If the gluons cannot cancel the last term in Eq. (3.108) they presumably cannot contribute to the model quark self-energies either for the same reason. Because the size of the quark mass in the effective Hamiltonian is unknown one may propose that its value is chosen in the second order calculation in the same way as for nucleons in the Yukawa theory in Eq. (3.37) or electrons in QED in Eq. (3.99). A would-be quark eigenstate has a finite constituent quark mass when the gluons are allowed to contribute in the whole range of momenta from zero up in the eigenvalue equation. This setting is equivalent to the solution Perry proposed for his coupling coherence condition for the quark self-energies [12]. The argument also illustrates the urgency of questions concerning the initial conditions and higher order analysis in the similarity renormalization group flow.

There is nothing wrong with the mass adjustment despite the infrared divergence. We have noticed in the previous Section that the arbitrary finite parts of the ultraviolet counterterms can be infrared divergent. This time, however, the positive and infrared logarithmically divergent part of the effective quark mass term in the model eigenvalue equation for heavy quarkonia remains uncanceled when the transverse gluons with $2\omega|q_3| - \vec{q}^2 > 0$ are declared to be absent from the model dynamics. The uncanceled part of the effective quark mass term stands in the eigenvalue problem. The point is it can now cancel the diverging δ dependence in the seagull term which is not canceled because the gluon exchange below $\tilde{\lambda}$ is missing. The new cancelation between the incomplete masses and the seagull occurs in the colorless states. It is analysed here in the nonrelativistic limit.

We describe the cancelation mechanism in the case of equal masses of quarks. The mechanism is similar but not identical to that in Ref. [12]. The infrared divergent mass squared term comes into the quarkonium eigenvalue equation divided by $x(1-x)$. But in the second order analysis the mass divergence appears only as a logarithmically divergent constant and the x -dependence is of higher order. The same diverging constant with the opposite sign is generated by the seagull term.

The infrared divergent terms and their cancelation are not directly related to the ultraviolet renormalization procedure. They appear in the ultraviolet-finite effective small width Hamiltonian dynamics. Note also that the introduction of the gluon mass μ_δ in the regularization could matter for the lower bound on $|x-y|$ and it could even eliminate the whole contribution when the upper bound of ω meets the lower bound of μ_δ . We assume here $\mu_\delta = 0$.

The divergent part in Eq. (3.116) is independent of $\vec{r}$ and it is easily removed by subtracting 1 from $\exp i\vec{q}\vec{r}$. Evaluation of the integral leads to the answer that for large r the seagull term produces a logarithmic potential of the form

$$V(\vec{r}) \sim \frac{2\omega a(\hat{e}_r)}{\pi} \log r, \quad (3.117)$$

where a is equal 1 for the radial versor $\hat{e}_r$ along the z -axis and it equals 2 when $\vec{r}$ is purely transverse. This potential is confining. It is also boost invariant. But the rotational asymmetry of the potential raises doubts. It suggests that an important piece of physics is missing in the reasoning used to derive it. The obvious sources of questions are the mechanism of blocking the effective gluon emissions and absorptions, role of the operation R , the role of the nonrelativistic approximation, the size of the quark and gluon masses and the strong dependence of the term on the width $\tilde{\lambda}$. The most urgent question is what happens in higher order calculations.

4. Conclusion

We have defined and illustrated on a few perturbative examples a general method of calculating light-front Hamiltonians which can be used for the relativistic description of interacting particles. The starting point in the calculation is a field theoretic expression for the bare Hamiltonian density. This expression is multiply divergent in the physically interesting cases. Therefore, the Hamiltonian theory requires renormalization.

In the renormalization process, one calculates a whole family of effective Hamiltonians as functions of the width parameter λ which determines the range of the effective interactions on the energy scale.

An effective Hamiltonian of a small width λ is much different from the initial bare Hamiltonian. It couples only those states whose masses differ by less than a prescribed amount. Thus, the effective theory contains only near-neighbor interactions on the energy scale. No scale is removed in the calculation but the correlations between dynamics at significantly different energy scales are integrated out. Therefore, in principle, the effective eigenvalue problem can be solved scale by scale using standard techniques for finite matrices which describe dynamics at a single scale.

Our formalism is based on the earlier work on renormalization of Hamiltonians from Refs. [1] and [2] where the Hamiltonians are defined by their matrix elements in a given set of basis states. Wegner has developed similar equations for Hamiltonian matrix elements in solid state physics [6]. The present approach to renormalization of Hamiltonians introduces the following features.

Our similarity transformation is defined in terms of creation and annihilation operators. Consequently, calculations of counterterms in perturbation theory can be performed without knowing details of the specific Fock states which are needed to evaluate the matrix elements. This is useful because a large number of Fock states needs to be considered. The renormalization scheme is free from practical restrictions on the Fock space sectors.

Expressing the effective Hamiltonians in terms of the creation and annihilation operators of effective particles and showing that the effective interactions are connected is a prerequisite to obtain the cluster decomposition property [20]. The effective interactions in our approach do not contain disconnected terms. The number of creation and annihilation operators in a single term is limited in perturbation theory by $2 + n(V - 2)$ where n is the order of a perturbation theory and V is the number of operators in the perturbing term.

The physically motivated assumptions about the model space of effective states included in solving a particular problem are introduced after the effective Hamiltonian is calculated. The interaction terms in the effective

Hamiltonian contain the similarity factors which diminish the dynamical significance of the Fock sectors with numbers of effective particles considerably different from the number of effective particles in the dominant sectors.

The present operator formulation does not introduce spectator dependent interactions, even in the case where we include the sums of the invariant masses for incoming and outgoing particles in the similarity factors. The sums are useful for estimates of cutoff dependence in perturbation theory.

The formalism explicitly preserves kinematical symmetries of the light-front frame. The structure of counterterms is constrained by these symmetries, including boost invariance. Hence, the number of possible terms is greatly limited. Preserving boost invariance is particularly important because it is expected to help in understanding the parton model and constituent quark model in QCD, simultaneously.

It is essential to include the running of the coupling constants in the calculation of the small width dynamics. The examples of second order calculations we described in this article do not include the running coupling constant effects. Inclusion of these effects requires higher order calculations.

Wegner's equation can be adapted to building an operator approach similar to what we described in the present article. The initial equation which replaces our Eq. (2.29) when one uses the Wegner generator of the similarity transformation is

$$\frac{d\mathcal{H}_\lambda}{d\lambda^2} = \frac{-1}{\lambda^4} [[\mathcal{H}_{1\lambda}, \mathcal{H}_{2\lambda}], \mathcal{H}_\lambda] . \quad (4.1)$$

However, there is little flexibility left in the equation so that the widening of the Hamiltonian band is not readily available.

There exists a class of generalized equations for the flow of Hamiltonian matrix elements described in Ref. [7] and already studied in a simple numerical model. These equations allow widening of the effective Hamiltonian matrix at large energies. The generalized equations can also be adapted for the construction of the creation and annihilation operator calculus. Namely,

$$\frac{d\mathcal{H}_\lambda}{d\lambda} = [F\{\mathcal{H}_{2\lambda}\}, \mathcal{H}_\lambda] . \quad (4.2)$$

These equations require detailed definitions of the similarity factors generated by the operation F [7].

In summary, the present formalism for renormalization of Hamiltonians in the light-front Fock space provides a tool for working on a host of theoretical issues in particle dynamics. Second order applications produce boost invariant Yukawa potential, Schrödinger equation for internal bound state dynamics and logarithmically confining quark-anti-quark interaction. However, it remains to be verified if the formalism can lead to quantitative

improvements in our description of particles. Rotational symmetry and infrared singularities in gauge theories require further studies. Most urgent are the calculations of effective Hamiltonians in the third and fourth order perturbation theory.

The author thanks Ken Wilson, Bob Perry, Billy Jones, Tomek Masłowski, Marek Więckowski, Martina Brisudová and Brent Allen for numerous discussions.

REFERENCES

- [1] St.D. Głazek, K.G. Wilson, *Phys. Rev.* **D48**, 5863 (1993).
- [2] St.D. Głazek, K.G. Wilson, *Phys. Rev.* **D49**, 4214 (1994).
- [3] K. Wilson, *Phys. Rev.* **140**, **B445** (1965).
- [4] K.G. Wilson, *Phys. Rev.* **D2**, 1438 (1970).
- [5] K.G. Wilson *et al.*, *Phys. Rev.* **D49**, 6720 (1994); and references therein.
- [6] F. Wegner, *Ann. Phys.* **3**, 77 (1994).
- [7] St.D. Głazek, K.G. Wilson, *Phys. Rev.* **D57**, 3558 (1998).
- [8] H.J. Melosh, *Phys. Rev.* **D9**, 1095 (1974).
- [9] R.J. Perry, A. Harindranath, K.G. Wilson, *Phys. Rev. Lett.* **65**, 2959 (1990).
- [10] K. Harada, A. Okazaki, *Phys. Rev.* **D55**, 6198 (1997).
- [11] B.D. Jones, “Light-front Hamiltonian approach to the bound state problem in QED”, Ph.D. Thesis, The Ohio State University, hep-ph/9709463; B.D. Jones, R.J. Perry, St.D. Głazek, *Phys. Rev.* **D55**, 6561 (1997); B.D. Jones, R.J. Perry, *Phys. Rev.* **D55**, 7715 (1997).
- [12] R.J. Perry, in *Hadron Physics’94; Topics on the Structure and Interaction of Hadron Systems*, Proceedings, Gramado, Brazil, edited by V. Herscovitz *et al.*, World Scientific, Singapore 1995, and revised version hep-th/9411037. See also “A Simple Confinement Mechanism for Light-Front Quantum Chromodynamics”, in *Theory of Hadrons and Light-Front QCD*, Ed. St.D. Głazek, World Scientific, 1995, p. 56, and references therein; M. Brisudová, R.J. Perry, *Phys. Rev.* **D54**, 1831 (1996); M. Brisudová, R.J. Perry, K.G. Wilson, *Phys. Rev. Lett.* **78**, 1227 (1997); M. Brisudová, S. Szpigel, R.J. Perry, “Effects of massive gluons on quarkonia in light-front QCD”, hep-ph/9709479, *Phys. Lett.* **B**, in print.
- [13] C. Bloch, *Nucl. Phys.* **6**, 329 (1958).
- [14] A. Casher, L. Susskind, *Phys. Lett.* **B44**, 171 (1973) and *Phys. Rev.* **D9**, 436 (1974); J. Kogut, L. Susskind, *Phys. Rep.* **C8**, 75 (1973) and references therein. For a recent presentation see also L. Susskind, M. Burkardt, “A Model of Mesons Based on χ SB in the Light-Front Frame”, in *Theory of Hadrons and Light-Front QCD*, Ed. St.D. Głazek, World Scientific, 1995, pp.5–14.
- [15] St.D. Głazek, *Phys. Rev.* **D38**, 3277 (1988).

- [16] M. Burkardt, *Adv. Nucl. Phys.* **23**, 1 (1996), hep-ph/9505259; A. Harindranath “An Introduction to Light-Front Dynamics for Pedestrians”, lectures delivered at the First International School on Light-Front Quantization and Non-Perturbative QCD, May 6–June 2, 1996, International Institute of Theoretical and Applied Physics, Iowa State University, Ames, IA, U.S.A., hep-ph/9612244; S.J. Brodsky, H.C. Pauli, S.S. Pinsky, “Quantum Chromodynamics and other field theories on the light-cone”, SLAC-PUB-7484, hep-ph/9705477; R.J. Perry, “Light-front QCD: a constituent picture of hadrons”, in *Confinement, Duality and Nonperturbative Aspects of QCD*, NATO Advanced Study Institute on Confinement, Duality and Nonperturbative Aspects of QCD, Cambridge, England, June 23–July 4, 1997, Proceedings, ed. by P. van Baal, Plenum Press, 1998. (*NATO ASI Series, Series B, Physics*, 368); hep-th/9710175 and references therein.
- [17] A. Bylev, St.D. Głazek, J. Przeszowski, *Phys. Rev.* **C53**, 3097 (1996) and references therein.
- [18] St. D. Głazek, R.J. Perry, *Phys. Rev.* **D45**, 3734 (1992).
- [19] St. Głazek, A. Harindranath, S. Pinsky, J. Shigemitsu, K. Wilson, *Phys. Rev.* **D47**, 1599 (1993).
- [20] S. Weinberg, *The Quantum Theory of Fields*, Cambridge University Press, New York, 1995.
- [21] R.J. Perry, K.G. Wilson, *Nucl. Phys.* **B403**, 587 (1993); R.J. Perry, *Ann. Phys.* **232**, 116 (1994) and references therein.
- [22] G.P. Lepage, S.J. Brodsky, *Phys. Rev.* **D22**, 2157 (1980).
- [23] S.-J. Chang, R.G. Root, T.-M. Yan, *Phys. Rev.* **D7**, 1133 (1973); S.-J. Chang, T.-M. Yan, *Phys. Rev.* **7**, 1147 (1973); T.-M. Yan, *Phys. Rev.* **7**, 1760 (1973); *Phys. Rev.* **7**, 1780 (1973); and references therein.
- [24] S.J. Brodsky, R. Roskies, R. Suaya, *Phys. Rev.* **D8**, 4574 (1973).
- [25] T. Masłowski, “Construction of the Poincaré algebra in scalar field theories in the weak interaction limit”, M. Sc. diploma work, Warsaw University, June 1997.
- [26] M. Więckowski, “Perturbative reduction of two fermions Hamiltonian dynamics in quantum field theory to the Schrödinger equation”, M. Sc. diploma work, Warsaw University, September 1997.
- [27] T. Masłowski, M. Więckowski, *Phys. Rev.* **D57**, 4976 (1998).