Topological aspects of the gapless and the gapped

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Publication date
2024

Document Version
Final published version

TOPOLOGICAL ASPECTS

OF THE GAPLESS AND THE GAPPED

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Topological aspects of the gapless and the gapped

This work has been accomplished at the Institute for Theoretical Physics (ITFA) of the University of Amsterdam (UvA). The research was supported by the Spinoza prize awarded to Erik Verlinde.

ISBN: 978-94-93391-27-7

Cover: Stathis Vitouladitis

Topological aspects of the gapless and the gapped

Academisch Proefschrift

ter verkrijging van de graad van doctor

aan de Universiteit van Amsterdam

op gezag van de Rector Magnificus

prof. dr. ir. P.P.C.C. Verbeek

ten overstaan van een door het College voor Promoties

ingestelde commissie,

in het openbaar te verdedigen in de Aula der Universiteit

op dinsdag 20 september 2024, te 14:00 uur

door

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geboren te Marousi

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Publications

This thesis is based on the following publications:

[1] J. R. Fliss and S. Vitouladitis. *Entanglement in BF theory I: Essential topological entanglement* (2023). arXiv: 2306.06158 [hep-th]

Presented in chapter 2.

[2] J. R. Fliss and S. Vitouladitis. *Entanglement in BF theory II: Edge-modes* (2023). arXiv: 2310.18391 [hep-th]

Presented in chapter 3.

[3] D. M. Hofman and S. Vitouladitis. *Generalised symmetries and state-operator correspondence for nonlocal operators* (2024). arXiv: 2406.02662 [hep-th]

Presented in chapter 4.

Other publications by the author:

[4] D. Anninos, D. M. Hofman and S. Vitouladitis. *One-dimensional quantum gravity and the Schwarzian theory*. J. High Energy Phys. 03 (2022), 121. doi: 10.1007/JHEP03(2022)121. arXiv: 2112.03793 [hep-th]

Contribution of the author to the publications:

In all publications, the author list follows an alphabetical order. The author contributed in all conceptual discussions for every publication. Additionally, the author's specific contributions to each publication are outlined below:

[1] The author performed all path integral computations in section 2, as well as all algebraic topology computations throughout the paper. Most of the other computations, except those in subsections 3.2, 3.4, and appendix C, were carried out jointly by both authors of [1]. The writing of sections 2, 3, and 4 was also done jointly, while the author also wrote appendices A and B. Lastly, the author verified the remaining computations.

[2] The author performed all computations and wrote a first draft of sections 2, 3 and 4, and appendices A and B.

[3] The author performed all computations and wrote a first draft of the entire paper.

Contents

1 Introduction

Theoretical physics is concerned with a mathematical description of the physical world. Given the sheer size and complexity of the world we inhabit, it is only natural that the mathematical descriptions of different physical phenomena will vary wildly. And indeed, if one were to chart the space of physical phenomena and their descriptions, they would find a vast landscape, with many dark corners, deep rabbit holes of highly complex mathematics, and surprising, enigmatic portals between seemingly unrelated regions. Yet, the ultimate dream of most theoretical physicists is a single unified description of everything. So, among all pieces of theoretical physics, some of the most valuable ones are those who reveal organising principles underlying the map — one may call them meta-theoretical physics. And the main meta, the crucial lesson that emerges upon reconciling centuries of theoretical physics is that nature is organised by *symmetry* and by *scale*. This is still not capturing the entirety of the map, but it makes is far less intimidating. What's more, it makes the areas that persist this organisation, even more intriguing.

Perhaps the most tangible example of how symmetry organises physical phenomena is the microscopic structure of a solid versus that of a liquid or a gas. In solids, the molecules are tightly packed, forming a crystal structure.[1] In contrast, liquids and gases have a much more fluid structure. Translating (pushing) individual molecules, or clumps of molecules, by arbitrary amounts, or rotating them by arbitrary angles, leaves the fluid invariant, as there is no fixed structure to disrupt. This is not true for solids. In solids molecules, or clumps thereof can only be translated and rotated by specific discrete amounts, due to the periodic arrangement of the crystal lattice. In other words, solids break translational and rotational symmetries that liquids and gases do not.

Another paradigmatic example occurs in ferromagnetic materials, such as iron, nickel,

[1] There are also amorphous solids, but the conclusion is identical.

or cobalt. Heating the material causes it to lose its magnetic properties, and cooling it restores them. This phenomenon is known as a ferromagnetic to paramagnetic phase transition. In that example the spins comprising the material, have a natural order in the ferromagnetic phase — they all align in the same direction, pointing towards the attractive magnet. In contrast, in the paramagnetic phase, this order is lost, and the material is invariant upon flipping the spins. This spin-flip symmetry distinguishes the two phases. More quantitatively, there is an *order parameter*, the local magnetisation, which distinguishes the two phases: it is zero in the paramagnetic phase and non-zero in the ferromagnetic phase.

Both examples are instances of *spontaneous symmetry breaking* and serve as a guide to the general principle. The general principle, first put forward by Lev Landau [5] and now a celebrated paradigm in theoretical physics, is that physical phenomena should be labelled by their symmetries and how they realise them — specifically, whether they break them or not. Furthermore, exactly at the critical point between the symmetry-broken and symmetry-preserving phases, the degrees of freedom are the fluctuations of the order parameter. This is a powerful classification, with applications spanning a whole range of phenomena from particle physics, to cosmology.

How nature is organised by scale is, to some degree, common knowledge. Particles make atoms, atoms make molecules, molecules make larger structures. The chain continues and goes from humans to galaxies, to cosmological scales, bit by bit. Physics at each length scale emerges from the smaller one and affects the immediately larger one. A key insight, here, is the immediate succession of scales. The intricacies and complexities of physics at far smaller scales are washed away.

This last fact makes the overwhelming majority of theories describing natural phenomena *effective*, rather than fundamental. A famous example is Fermi's theory of the weak interaction, which explains the beta decay of a neutron, by positing the interaction of four fermions at a single point. This theory works remarkably well for this process, but it predicts that the probability of such an interaction grows unboundedly as the energies of the participating particles increase. It can, therefore, not be correct at higher energy, or equivalently smaller length scales.[2] It has to be superseeded by a theory, valid at higher energies, that reduces to Fermi's theory, at low energies. Indeed, we now know that this superseeding theory is the standard model of particle physics. This too, despite describing elementary particles, is not a fundamental description, but rather an effective one.

[2] An aside on scales and measurement units. It is common in high-energy physics — and employed throughout this thesis — to use *natural units*, where the speed of light, the reduced Planck constant, and the Boltzmann constant are all set to one: $c = \hbar = k_{\mathrm{B}} = 1$. With these conventions energy, mass, momentum, and temperature are measured in the same units, which are, moreover, the inverse of length, and time. So short length scales are the same as large energy scales, and vice versa.

The mathematical framework underlying this discussion is the notion of *renormalisation group*, originally developed by Kenneth Wilson [6–8], which serves as a cornerstone of modern theoretical physics. The renormalisation group provides a systematic way to move between different scales. Funnily, despite the name one can only move towards longer scales; it is, in fact, a semigroup. Nonetheless, the main qualitative virtue of the renormalisation group is that it gradates the space of descriptions of physical phenomena according to their validity range and provides a criterion to assess whether a description is an effective one, or a rare, fundamental one, valid across all scales.

Most interesting are the endpoints of the renormalisation group: the infinitely small, playfully yet commonly known as the ultraviolet (UV), and the infinitely large, the infrared (IR). At these endpoints — or *fixed points*, as they are more properly termed — there is no sense of scale, as all dimensionful parameters are either exactly zero or exactly infinite. Life at the endpoints is typically simpler. The UV endpoint, despite involving infinitely small, infinitely high-energy physics, simplifies because everything else has completely disappeared. Conversely, the IR endpoint strips away all the details and retains only the essential, scale-invariant information. In this sense, theories valid at intermediate points can be viewed as *flows* between a UV fixed point and an IR fixed point. A UV fixed point contains all information of the theories that flow out of it, while an IR fixed point collects all universal features, of every theory, effective of fundamental, that flows into it.

Furthermore, out of these two organising principles, emerges another valuable feature of the physics map: its linguistics. Remarkably, there seems to be a universal grammar, a framework that describes accurately a plethora of phenomena. This is known as *quantum field theory* (QFT), a framework historically developed to describe relativistic quantum mechanics by postulating that particles are merely excitations of underlying quantum fields. It has been extremely successful in understanding elementary particles and their interactions, and indeed, its most notable application is the development of the standard model of particle physics, which encompasses all known elementary particles and three of the four fundamental forces of nature. However, its utility extends far beyond the realm of particle physics. A well-appreciated, yet astounding fact is that the language of QFT is exactly the right language to describe classical and quantum statistical mechanics.[3] Combined with the fact that most systems in nature have so many degrees of freedom that a statistical description is far more powerful and useful than an exact one, this gives QFT immense power. It describes high-energy particle physics through quantum fields and low-energy statistical regimes through statistical fields.

Moreover, connecting with the renormalisation group, its interesting endpoints

[3]Though time needs to be a bit funny.

are described, in this language by special kinds of QFTs, known as *conformal* and *topological field theories* (CFTs and TQFTs, respectively). While the essence of these theories will be expounded later in the chapter, their spirit can be captured, briefly, by stating that these are theories enjoying more symmetries than generic QFTs. Specifically, CFTs are invariant under rescaling the space they are defined on, while TQFTs are invariant to any smooth deformation in their surroundings. This makes them perfect candidates for renormalisation fixed points as they are both insensitive to scales. This thesis is devoted to the study of such theories, and their universal aspects, exploring their properties and applications.

However, while the perspective taken above is suitable for describing many physical phenomena, it honestly fails to capture the entirety of theoretical physics in its current state. There are outliers, either experimentally observed or theoretically proposed, that break one or both of the organising principles or speak a different language than QFT. The most important and elusive phenomenon is, arguably, *quantum gravity*. Gravity seems to be highly adept at evading the most sophisticated and intricate attempts to describe it quantum mechanically. Despite significant progress and the exploration of promising avenues — most notably, string theory — there is still no comprehensive understanding of quantum gravity in our universe. Although this thesis does not touch upon quantum gravity, much of the underlying research has been and continues to be motivated by this fundamental question.

1.1 Symmetry and topology

Symmetry[4] serves as an organising principle for the mathematical description of the physical world. Besides the arguments mentioned above, an important point is that classical conservation laws, such as conservation of charge, momentum, and angular momentum, are associated with symmetries via Noether's theorem. Recent advancements across various fields such as high-energy physics, condensed-matter physics, quantum information theory, and mathematics have induced a transformative generalisation of the concept of symmetries. This broadeding of our understading has led to notions such as higher-form symmetries, non-invertible symmetries, subsystem symmetries and others, occuring both in quantum field theory (QFT) as well as in microscopic, lattice, models. These generalised symmetries, have given back to the fields they owe their existence to, finding applications in a wide array of quantum systems, such as the Ising model, topological phases of matter (cf. section 1.2), fractons, gauge theory, and string and M-theory.

Ideas relating to generalised symmertries will be crucial and themes relating to them

[4]Throughout this thesis, symmetry will *always* refer to global symmetry. "Gauge symmetry" will always be referred to as gauge redundancy or gauge invariance.

will be recurrent throughout this thesis. The scope of this section is to review some of the recent advancements in this field. Given the quantum-field-theoretic language of the later chapters of this thesis, the discussion of symmetries and their generalisations will be given within the context of relativistic QFT, with departures from that context mentioned explicitly, whenever necessary.

1.1.1 Ordinary symmetries

We begin the discussion by recalling the usual presentation of symmetries in QFT. We will start with a d-dimensional QFT with a U(1) symmetry. Associated with this symmetry there is a current, $J_\mu(x)$, (with $x = (t, \boldsymbol{x})$), that is conserved on-shell:

$$\partial^\mu J_\mu(x) = 0. \tag{1.1.1}$$

One then defines a charge operator:

$$Q(t) := \int_{\Sigma_t} \mathrm{d}^{d-1}\boldsymbol{x} \;\; J_0(t, \boldsymbol{x}), \tag{1.1.2}$$

where the integral is taken over a spatial slice, Σ_t, assumed closed, or equipped with appropriate boundary conditions. As a consequence of (1.1.1), $Q(t)$ is actually independent of t:

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = 0. \tag{1.1.3}$$

In a quantum theory, according to Wigner's theorem, the symmetry is implemented by unitary operators, acting on the Hilbert space. These are simply the exponentials of the charge:

$$U\left(\mathrm{e}^{\mathrm{i}\alpha}, t\right) := \exp(\mathrm{i}\,\alpha\, Q(t)), \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}, \tag{1.1.4}$$

where $U\left(\mathrm{e}^{\mathrm{i}\alpha}, t\right)$ is again, independent of t, due to (1.1.1). This operator implements a rotation by an angle, α. One slight, for now merely notational, change one can make to the above story is to write it in a coordinate-free way. That is, instead of talking about the vector $J_\mu(x)$, one can construct a one-form, $J_{[1]} \in \Omega^1(X)$, which in the previous coordinate system is expressed as:

$$J_{[1]} = J_\mu(x)\,\mathrm{d}x^\mu\,. \tag{1.1.5}$$

Noether's theorem is now expressed as the condition that the one-form $J_{[1]}$ be coclosed, i.e.

$$\mathrm{d} \star J_{[1]} = 0, \tag{1.1.6}$$

where $\star : \Omega^p(X) \to \Omega^{d-p}(X)$ is the Hodge-star operator. We can then integrate $\star J_{[1]}$ on any (closed or equipped with boundary conditions) codimension-one manifold, Σ_{d-1}, to obtain a charge operator:

$$Q[\Sigma_{d-1}] := \int_{\Sigma_{d-1}} \star J_{[1]}. \tag{1.1.7}$$

As a consequence of (1.1.6), the charge operator is insensitive to smooth deformations of Σ_{d-1}. Indeed:

$$Q[\partial Y_d] = \int_{\partial Y_d} \star J_{[1]} = \int_{Y_d} \mathrm{d} \star J_{[1]} = 0, \tag{1.1.8}$$

implying that $Q[\Sigma_{d-1}] = Q[\Sigma'_{d-1}]$, if, Σ_{d-1} and Σ'_{d-1} differ by a boundary. As before, the charges are represented as topological operators acting on the Hilbert space, as

$$U(\mathrm{e}^{\mathrm{i}\alpha}, \Sigma_{d-1}) := \exp(\mathrm{i}\alpha\, Q[\Sigma_{d-1}]), \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}. \tag{1.1.9}$$

While for the most part of the thesis we will be in Euclidean signature, it is worth mentioning that in Lorentzian signature, the operators $U(\mathrm{e}^{\mathrm{i}\alpha}, \Sigma_{d-1})$ play a double role, depending on the nature of Σ_{d-1}. If Σ_{d-1} is a fixed-time slice, such operators are genuine operators acting on the HIlbert space. On the other hand, if Σ_{d-1} is extended in the time direction, it is a topological defect, that modifes the quantisation, giving rise to a twisted Hilbert space. We will refer to $U(\mathrm{e}^{\mathrm{i}\alpha}, \Sigma_{d-1})$, as the symmetry operators.

Since we are in a quantum theory, the continuity equation, (1.1.6), is modified in the presence of a local operator, $\mathcal{O}(x)$ to a Ward identity:

$$\big(\mathrm{d} \star J_{[1]}\big)(y)\, \mathcal{O}(x) = q\, \delta_{[d]}(x-y)\, \mathcal{O}(x), \tag{1.1.10}$$

where $\delta_{[d]}(x-y)$ is a d-form delta function and $q \in \mathbb{Z}$ is the charge of $\mathcal{O}(x)$ under the symmetry. Equation (1.1.10) and all the equations that follow, should be regarded as operator equations, i.e. holding in arbitrary correlation functions, with insertions away from y. Exponentiating the Ward identity we see that the symmetry operators, inserted along a $(d-1)$-dimensional sphere, act on local operators by linking with the point x, as depicted in figure 1.1:

$$\begin{aligned} U\left(\mathrm{e}^{\mathrm{i}\alpha}, \mathbb{S}^{d-1}\right) \cdot \mathcal{O}(x) &= U\left(\mathrm{e}^{\mathrm{i}\alpha}, \partial\mathbb{B}^{d}\right) \cdot \mathcal{O}(x) \\ &= \exp\left(\mathrm{i}q\alpha \int_{\mathbb{B}^d} \delta_{[d]}(x-y)\right)\mathcal{O}(x) = \mathrm{e}^{\mathrm{i}q\alpha}\mathcal{O}(x). \end{aligned} \tag{1.1.11}$$

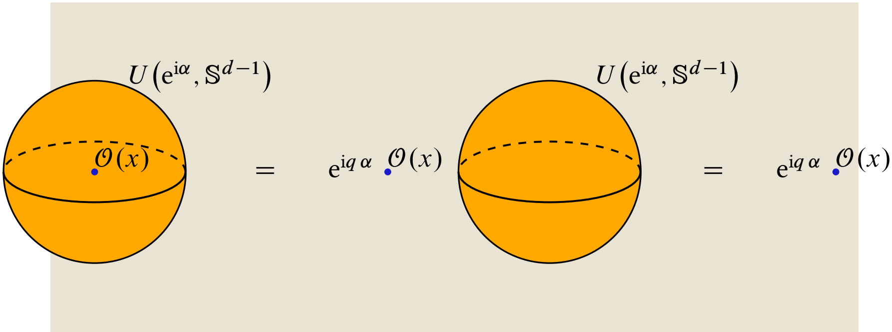

Figure 1.1: *Action of symmetry operator on a local operator. On the left-hand-side, the symmetry operator is inserted on an $\mathbb{S}^{d-1}$ that links with x. In the middle the symmetry operator is deformed and unlinked, resulting in the charged operator picking a representation of the group element. Finally, on the right-hand-side, the topological operator is shrunk to nonexistence.*

From (1.1.11) we see that acting with $U\left(\mathrm{e}^{\mathrm{i}\beta}, \mathbb{S}^{d-1}\right)$, after acting with $U\left(\mathrm{e}^{\mathrm{i}\alpha}, \mathbb{S}^{d-1}\right)$ is equivalent to acting with $\mathrm{e}^{\mathrm{i}\beta} \cdot \mathrm{e}^{\mathrm{i}\alpha} = \mathrm{e}^{\mathrm{i}(\beta+\alpha)}$. In other words, the set of operators $U\left(\mathrm{e}^{\mathrm{i}\alpha}, \mathbb{S}^{d-1}\right)$, together with their composition — usually called their *fusion rules* — implement the action of the symmetry group; in this case, U(1).

More generally, associated with any symmetry group, G, which can be abelian or non-abelian, continuous or discrete, there exists a set of unitary topological operators, supported on codimension-one manifolds, and labeled by group elements: $\{U(g, \Sigma_{d-1}),\ g \in G\}$. These operators fuse according to the group multiplication of G:

$$U(g, \Sigma_{d-1}) \otimes U\left(g', \Sigma_{d-1}\right) = U\left(g \cdot g', \Sigma_{d-1}\right). \tag{1.1.12}$$

The operator $U(\mathbf{1}, \Sigma_{d-1})$, where $\mathbf{1}$ is the identity element of G, acts as the identity operator on the Hilbert space. As a consequence, the operator $U(g^{-1}, \Sigma_{d-1})$ is the inverse of $U(g, \Sigma_{d-1})$. This is to be contrasted with subsection 1.1.3, where not every symmetry operator has an inverse. Local operators transform in representations of the symmetry group:

$$U\left(g, \mathbb{S}^{d-1}\right) \cdot \mathcal{O}^i(x) = \rho(g)^i{}_j \cdot \mathcal{O}^j(x), \tag{1.1.13}$$

where i labels the multiplet that the operator transforms in, in terms of a representation $\rho(g)^i{}_j$ of the group element $g \in G$. The upshot of this definition is that it does not rely on the existence of a conserved current, putting therefore discrete and continuous symmetries on the same footing.

Before moving on with generalisations of symmetries, let us comment on how one probes ordinary symmetries. In case the symmetry is continuous, one can couple the conserved current to background gauge fields, $A_{[1]} = A_\mu(x)\,\mathrm{d}x^\mu$, for the symmetry, by inserting in the path-integral an operator of the form[5]

$$\exp\left(\mathrm{i}\int_X \mathrm{d}^d x\,\sqrt{|g|}\,g^{\mu\nu}(x)A_\mu(x)J_\nu(x)\right) = \exp\left(\mathrm{i}\int_X A_{[1]}\wedge\star J_{[1]}\right). \tag{1.1.14}$$

Note that A is *not* a dynamical gauge field. It is often natural — though not the most generic option — to consider flat connections, $\left[A_{[1]}\right]\in \mathrm{H}^1(X;G)$. These, one can Poincaré-dualise[6] $A_{[1]}$ to a $(d-1)$-cycle, $\left[\widehat{A}_{d-1}\right]\in \mathrm{H}_{d-1}(X;G)$, so (1.1.14) is equivalent to inserting a symmetry operator supported on $\widehat{A}_{d-1}$:

$$U\left(1,\widehat{A}_{d-1}\right) = \exp\left(\mathrm{i}\int_{\widehat{A}_{d-1}}\star J\right). \tag{1.1.15}$$

Computing the partition function in the presence of the background gauge field,

$$\mathcal{Z}[A_{[1]}] := \left\langle U\left(1,\widehat{A}_{d-1}\right)\right\rangle, \tag{1.1.16}$$

allows one to study correlation functions of the currents, as well as to study the anomaly structure of the theory. The latter presentation, (1.1.15), is in fact also suitable for addressing the same quations also for discrete symmetries.

If the symmetry is not anomalous, the procedure outlined above serves as the first step to gauging the symmetry. To gauge the symmetry one needs to, then, insert a sufficiently fine network of (1.1.15) [9–12]. In the case of continuous symmetries, this is usually supplemented with adding kinetic terms for the gauge fields, although some times it suffices to consider flat gauge fields [13, 14]. The "sufficiently fine" network, becomes, in this case, a fully-fledged path integral over the gauge connections, producing at the end the gauged theory. For a discrete symmetry, gauge fields, even when made dynamical, remain, necessarily, flat. The network of topological operators, supported on Poincaré duals of gauge fields implements, in this case the sum over flat gauge bundles of G.

1.1.2 Higher-form symmetries

In their seminal work [15], Gaiotto, Kapustin, Seiberg and Willett formalised the first important generalisation, dubbed *higher-form symmetries*, that sparked the recent symmetry era. This generalisation built upon a collection of related ideas stemming mostly from non-perturbative quantum field theory and string theory [16–34]. The main idea, expressed again in terms of a conserved U(1) current, is to replace the

[5] In some cases, specifically for shift symmetries (in some frame), one needs to also include quadratic terms of $A_{[1]}$, such as $\int A_{[1]}\wedge\star A_{[1]}$, in order to preserve background gauge invariance.

[6] Or Poincaré–Lefschetz in cases when spacetime has a boundary.

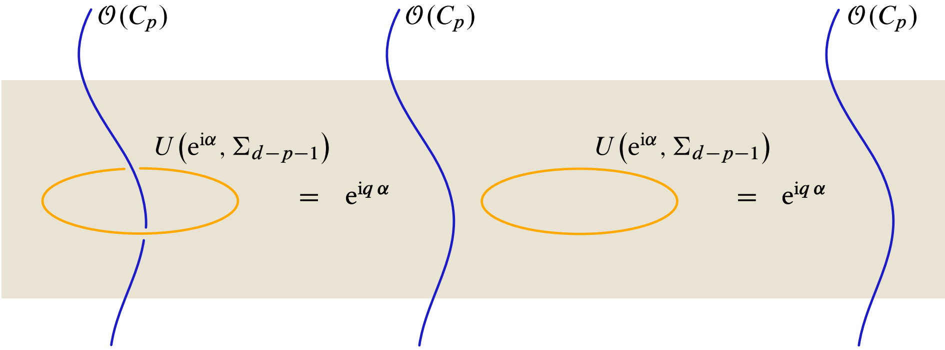

Figure 1.2: *Action of a p-form symmetry operator on a p-dimensional operator. On the left-hand-side, the symmetry operator is inserted on an $\mathbb{S}^{d-p-1}$ that links with the p-dimensional operator. In the middle the symmetry operator is deformed and unlinked, resulting in the charged operator picking a representation of the group element. Finally, on the right-hand-side, the topological operator is shrunk to nonexistence.*

one-form current, $J_{[1]}$, by a $(p+1)$-form one, $J_{[p+1]}$, with the conservarion equation being, still, a coclosedness condition:

$$d \star J_{[p+1]} = 0. \tag{1.1.17}$$

From here on, one can follow the same procedure as before. The first step is to construct charge operators by integrating on a codimension-$(p+1)$ manifold:

$$Q[\Sigma_{d-p-1}] := \int_{\Sigma_{d-p-1}} \star J_{[p+1]}, \tag{1.1.18}$$

and exponentiate to obtain topological, symmetry operators:

$$U(e^{i\alpha}, \Sigma_{d-p-1}) := \exp(i\alpha\, Q[\Sigma_{d-p-1}]), \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}. \tag{1.1.19}$$

The main difference with the case of subsection 1.1.1, is that now the symmetry operators cannot link with local operators. They can link, however, with p-dimensional extended operators (see figure 1.2). They act, in a similar manner as before:

$$U(e^{i\alpha}, \Sigma_{d-p-1}) \cdot \mathcal{O}(C_p) = e^{iq\,\alpha\,\mathrm{Link}(\Sigma_{d-p-1}, C_p)}\mathcal{O}(C_p), \tag{1.1.20}$$

where now the charged operator, is supported on a p-dimensional manifold C_p. The action of the topological operator picks up the charge, $q \in \mathbb{Z}$ and the linking number between Σ_{d-p-1} and C_p.

Similarly as before, the notion of a p-form symmetry group, $G^{[p]}$ — continuous or discrete — in a quantum field theory, can be formalised by the existence of a set

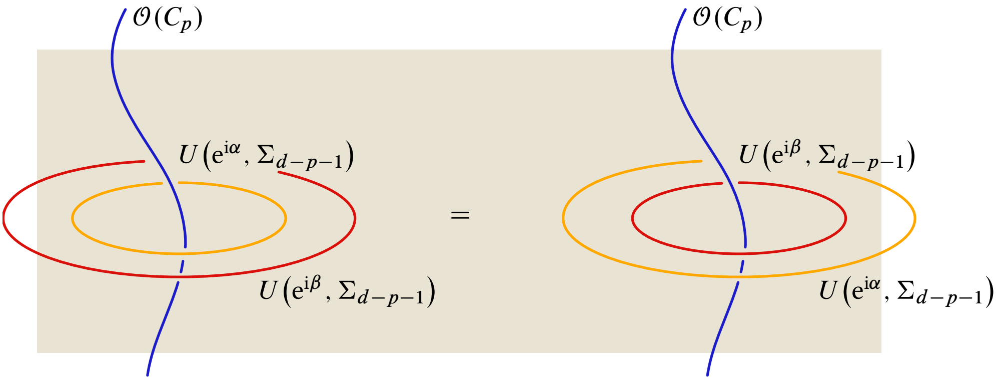

Figure 1.3: *Higher-form symmetries are abelian. On the left-hand-side, the operator $U(e^{i\beta}, \Sigma_{d-p-1})$ acts after $U(e^{i\alpha}, \Sigma_{d-p-1})$. On the right-hand-side, the topological operators were deformed, at no cost, so that the order of their action was reversed.*

of codimension-$(p+1)$ topological operators, $\{U(g, \Sigma_{d-p-1}),\ g \in G^{[p]}\}$, labelled by group elements. The case $p = 0$ corresponds to the ordinary global symmetries (henceforth referred to as *zero-form symmetries*), reviewed in subsection 1.1.1. One key difference between the cases $p > 0$ and $p = 0$ is that, unlike zero-form symmetries, higher-form symmetries are necessarily abelian. This is most easily illustrated with a picture, see figure 1.3. Nonetheless, the general story is similar. The topological operators fuse according to the group multiplication:

$$U\big(g, \Sigma_{d-p-1}\big) \otimes U\big(g', \Sigma_{d-p-1}\big) = U\big(g \cdot g', \Sigma_{d-p-1}\big), \tag{1.1.21}$$

and p-dimensional operators transform in representations of the symmetry group:

$$U\big(g, \Sigma_{d-p-1}\big) \cdot \mathcal{O}(x) = \rho(g) \cdot \mathcal{O}(x). \tag{1.1.22}$$

Here, whenever $p > 0$, $\rho(g)$ is simply a phase, since higher-form symmetries are abelian.

Before moving on, let us give a couple of examples of higher-form symmetries. These examples will, on the one hand, illustrate the abstract discussion above, while on the other hand, they will serve as an entrée to the later chapters of this thesis, where slight alterations of these examples will be prevalent.

The prototypical example is that of Maxwell theory, in d-dimensions. This is a theory of a free photon, i.e. a free one-form, $a \in \Omega^1(X)$, with field-strength $f \in \Omega^2(X)$. The equations of motion are given by

$$\mathrm{d} \star f = 0, \tag{1.1.23}$$

which we immediately recognise as the conservation equation for the two-form current, f. This is a one-form $\mathrm{U}(1)^{[1]}$ symmetry. The symmetry operators associated with it are:

$$U\left(\mathrm{e}^{\mathrm{i}\alpha}, \Sigma_{d-2}\right) := \exp\left(\mathrm{i}\alpha \int_{\Sigma_{d-2}} \star f\right), \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}. \tag{1.1.24}$$

The charged objects under this symmetry are Wilson lines:

$$\mathrm{W}_{q_\mathrm{e}}(C_1) := \exp\left(2\pi\mathrm{i}\, q_\mathrm{e} \int_{C_1} a\right), \tag{1.1.25}$$

where C_1 is either an infinitely extending line, or a closed loop. To measure the charge, one has to link the Wilson line with the one-form symmetry operators. This is Gauss's law in action. The charge, $q_\mathrm{e} \in \mathbb{Z}$, of the Wilson line, is precisely its electric charge. For that reason, this symmetry is called *electric* one-form symmetry, often denoted as $\mathrm{U}(1)^{[1]}_{(\mathrm{e})}$.

Maxwell theory also has another higher-form symmetry, stemming from the Bianchi identity,

$$\mathrm{d}f = 0. \tag{1.1.26}$$

This is a magnetic $(d-3)$-form symmetry, $\mathrm{U}(1)^{[d-3]}_{(\mathrm{m})}$. Its conserved current is $\star f \in \Omega^{d-2}(X)$ and the symmetry operators are given by

$$U\left(\mathrm{e}^{\mathrm{i}\alpha}, \Sigma_2\right) := \exp\left(\mathrm{i}\alpha \int_{\Sigma_2} f\right), \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}. \tag{1.1.27}$$

What is charged under the magnetic one-form symmetry is 't Hooft operators. One way to see it, is within the electric presentation of the theory, where 't Hooft operators are defined by excising a $(d-3)$-dimensional manifold, C_{d-3}, from spacetime and imposing boundary conditions on a manifold Σ_2, linking C_{p-3}, such that

$$\int_{\Sigma_2} f = q_\mathrm{m} \in \mathbb{Z}. \tag{1.1.28}$$

This implies immediately that 't Hooft loops carry charge under the magnetic $(d-3)$-form symmetry; they carry magnetic charge. A different, yet equivalent, way to see this is in a magnetic presentation, where one exchanges the electric gauge potential, a, for a magnetic one, $\check{a} \in \Omega^{d-3}(X)$, whose field-strength is $\check{f} = \star f \in \Omega^{d-2}(X)$. In this presentation, the 't Hooft operators are simply the Wilson operators of the "magnetic photon,"

$$\text{'tH}_{q_\mathrm{m}}(C_{d-3}) := \exp\left(2\pi\mathrm{i}\,\alpha\, q_\mathrm{m} \int_{C_{d-3}} \check{a}\right), \tag{1.1.29}$$

which can be seen immediately to carry charge $q_{\rm m} \in \mathbb{Z}$, under the symmetry generators (1.1.27).

In four-dimensions, both the electric and the magnetic symmetries are one-form symmetries. One may, therefore, have "dyonic" line-operators charged under both of them. This is in fact the starting point of chapter 4, uncovering a richer structure underpinning conformal field theories with continuous higher-form symmetries.

A second, paradigmatic, example of higher-form symmetries in quantum field theory takes place in Chern–Simons theory. The relevant example in this case is abelian Chern–Simons theory. This is a three-dimensional topological gauge theory, with action

$$S_{\rm CS}[a] = \frac{k}{4\pi}\int_X a \wedge {\rm d}a\,. \tag{1.1.30}$$

Here, $a \in \Omega^1(X)$ is a one-form U(1) connection, normalised as $\int a \in 2\pi\mathbb{Z}$, along any one-cycle, and $k \in \mathbb{Z}$ is called the level of the action. Integrality of the level is required to guarantee gauge invariance. The theory, as written above, is defined on a torsion-free manifold; this is sufficient for our purposes. Chern–Simons theories on manifolds with torsion are treated, for example in [35, 36], and (briefly, but in a more general framework) in appendix A.1. The equation of motion for a is simply flatness condition, $k\,{\rm d}a = 0$, and the observables are given by Wilson loops:

$$\mathrm{W}_n[C_1] := \exp\left({\rm i}\,n\int_{C_1} a\right). \tag{1.1.31}$$

with n being an integer. These operators have a slightly peculiar property. They are both the symmetry operators and the charged objects. Let us explain this. They are naturally symmetry operators, as they are topological. Since they are supported on lines, i.e. $(d-2)$-dimensional manifolds, in $d = 3$, they furnish a one-form symmetry. Moreover, from the canonical commutation relations that follow from (1.1.30), one can show that Wilson loops obey an algebra of the form:

$$\mathrm{W}_n[C_1] \otimes \mathrm{W}_m\left[\tilde{C}_1\right] = \exp\left(-\frac{2\pi\,{\rm i}\,n\,m}{k}\mathrm{Link}\left(C_1, \tilde{C}_1\right)\right)\mathrm{W}_m\left[\tilde{C}_1\right] \otimes \mathrm{W}_n[C_1]. \tag{1.1.32}$$

This algebra has three important consequences. Firstly it shows that the fundamental Wilson loop,

$$\mathrm{W}[C_1] := \exp\left({\rm i}\int_{C_1} a\right), \tag{1.1.33}$$

carries charge n under the one-form symmetry described above. Secondly, it is clear that when n is an integer multiple of k, the corresponding Wilson loop is the identity operator. Therefore, n should be regarded only up to shifts by k. In other words, the one-form symmetry is a finite, $\mathbb{Z}_k^{[1]}$, symmetry. Finally, the interesting property that the symmetry generators are charged, despite the abelian nature of the symmetry, indicates that the one-form symmetry is anomalous.

These features, are exhibited in higher-dimensional generalisations of this theory, involving two gauge-fields, of different form-degrees, known as BF theories. BF theories, and their properties will occupy a large part of this thesis. In particular, chapters 2 and 3 are devoted in a systematic study of those theories.

Before moving on to other generalisations of symmetries, it is worth mentioning a few salient features of higher-form symmetries.

Higher-form symmetries can break spontaneously. A spontaneously broken continuous higher-form symmetry gives rise to a gapless Nambu–Goldstone mode, which is in this case a higher-form gauge field [15, 22, 37, 38]. Indeed, the free photon described above, can be seen, in $d \geqslant 4$, as the Goldstone mode of the spontaneously broken $\mathrm{U}(1)^{[1]}_{(e)}$.[7] Spontaneously broken p-form symmetries imply deconfined p-dimensional operators, while, when unbroken, they imply confinement of the charged extended operators. This can be — and is in the case of one-form symmetries [42] — utilised, to characterise phases of matter with higher-form symmetries in the spirit of an extended Landau–Ginzburg theory.

Much like zero-form symmetries, higher-form symmetries can be coupled to background gauge-fields, which are now given by $(p+1)$-form gauge fields, in the same fashion as (1.1.14) and (1.1.15). In the absense of 't Hooft anomalies, higher-form symmetries, both discrete and continuous, can be, subsequently, gauged following the same procedure outlined in subsection 1.1.1. In the case of gauging discrete p-form symmetries, the gauged theory enjoys a $(d-p-2)$-form global symmetry, given by the Pontryagin dual of the original symmetry [15, 43]. Furthermore (discrete) p-form symmetries can also be gauged on a submanifold of higher-codimension, producing new topological defects, which are typically non-invertible [44–48]. This is a novel property of higher-form symmetries, that is not present for zero-form symmetries.

1.1.3 Non-invertible symmetries

In this section we will discuss a different interesting generalisation of symmetries, which will be useful for a part of the main body of this thesis. Such symmetries are, by now, mostly known as *non-invertible symmetries*, although in some parts of the

[7] More generally, a continuous p-form symmetry can break spontaneously in $d \geqslant p+2$ dimensions, while a discrete p-form symmetry can break spontaneously in $d \geqslant p+1$ dimensions [15, 38]. This is a higher-form version of the Coleman–Mermin–Wagner–Hohenberg theorem. [39–41]

phase space they are referred to as *categorical symmetries*.[8] The main characteristic of such symmetries is that, while they are still described by topological operators, their fusion algebra has no inverse. Here, we will not attempt a mathematically rigorous introduction to the various concepts, nor will we follow a historically precise timeline of the developments of this field. Rather, we will present the main ideas, that are relevant for this thesis, and outline their physical applications. For a more complete introduction to non-invertible symmetries we refer the reader to [50, 51].

Perhaps the simplest physical model exhibiting non-invertible symmetries is is a two-dimensional Ising model[9] at the critical temperature, also known as the two-dimensional Ising CFT. The main idea dates back to Fröhlich, Fuchs, Runkel, and Schweigert [52] who reinterpret the classic Kramers–Wannier duality of the Ising model in terms of topological operators. In the Ising model, correlation functions of spins, at temperature inverse temperature β, are equal to those of disorder operators at inverse temperature $\beta_{\text{dual}} = -\frac{1}{2}\log\tanh\beta$, implying a high/low-temperature duality. At the critical temperature, $\beta_{\text{crit}} = -\frac{1}{2}\log\tanh\beta_{\text{crit}}$, this duality gets upgraded to a symmetry. The operator that performs this transformation is a topological operator that has, nonetheless, a curious property. To describe this property we need to first describe the fields that parttake in the critical Ising model. At critical temperature, the model is described by a two-dimensional conformal field theory (CFT), of central charge $c = \frac{1}{2}$. This is a rational CFT, whose local primary operators are the identity operator, a spin operator, $\sigma(x)$, and a thermal operator $\varepsilon(x)$. Correspondingly it has three topological line operators. The identity line, $\mathbf{1}$, a spin-changing line, η, implementing the usual $\mathbb{Z}_2$ symmetry of the Ising CFT, and the aforementioned Kramers–Wannier defect, $\mathcal{D}$, whose action we will now describe (and is illustrated in figure 1.4). Transporting $\mathcal{D}$ past $\varepsilon(x)$, it flips its sign into $-\varepsilon(x)$. Passing it through $\sigma(x)$, however, it deletes the operator, creating in its place a disorder operator, $\mu(x)$, connected to $\mathcal{D}$ by an η line. $\mu(x)$ is a non-genuine operator, of the same scaling weights as $\sigma(x)$. Therefore, upon encircling $\sigma(x)$ by the Kramers–Wannier defect, it produces a "tadpole" diagram: a closed line of $\mathcal{D}$ attached to an η line. Such a correlation function vanishes![10] In other words, the Kramers–Wannier operator, is a non-invertible operator: its action on the state produced by $\sigma(x)$ is zero.

Translated to fusion rules, what we described above can be summarised concisely to the following algebra of topological operators:

$$\eta \otimes \eta = \mathbf{1}, \qquad \eta \otimes \mathcal{D} = \mathcal{D} \otimes \eta = \mathcal{D}, \qquad \mathcal{D} \otimes \mathcal{D} = \mathbf{1} \oplus \eta. \tag{1.1.34}$$

[8] Note, however, that in a small corner of phase space [47, 49] the name categorical symmetry is reserved for a slightly different concept.

[9] Here, and everywhere in this thesis we are using high-energy physics conventions for dimensions. So "two-dimensional" refers to two *spacetime* dimensions.

[10] This can be seen by noting that this diagram should produce a state of conformal weights $(h, \bar{h}) = (0, 0)$, in the $\mathbb{Z}_2$-twisted Hilbert space. But such a state does not exist.

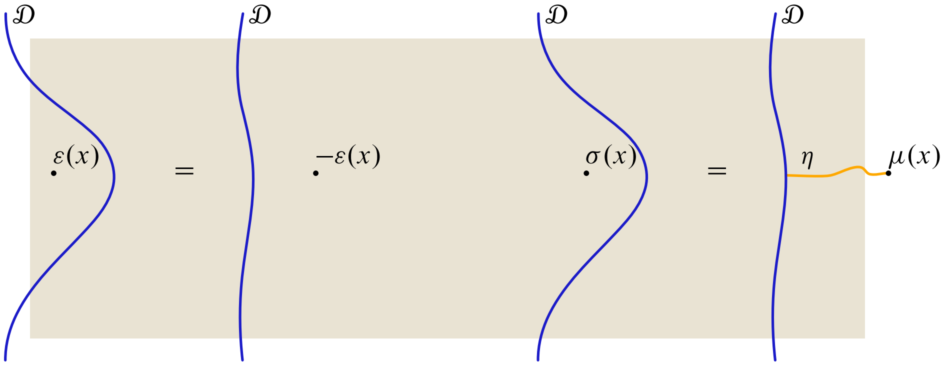

Figure 1.4: *The action of the Kramers–Wannier defect. Left: $\mathcal{D}$ flips the sign of the thermal operator. Right: $\mathcal{D}$ replaces $\sigma(x)$ by a non-genuine operator, $\mu(x)$, connected to a $\mathbb{Z}_2$ line.*

By the above algebra, and specifically by the last equation, we can see that the operator $\mathcal{D}$ has no inverse. In other words, these topological operators do not generate a group, but a fusion category, whose fusion ring is given by (1.1.34). This category is known as $(\mathbb{Z}_2, +)$ Tambara–Yamagami category [53], or simply as *Ising fusion category*.

The story described above is actually quite generic for two-dimensional rational CFTs [54–56]. Such CFTs admit a set of topological lines, the Verlinde lines, [57, 58], whose fusion ring is non-invertible and given by the Verlinde formula [57]. More generally, two-dimensional, not necessarily conformal QFTs also enjoy non-invertible topological lines [9, 12, 52, 59–63]. Such lines are interpreted as generalisations of conventional symmetries, for the reason that, much like ordinary symmetries, they are preserved under the renormalisation group flow, constrain correlation functions, and, when anomalous, give contstraints on the low-energy phases of quantum field theories [60–63]. The mathematics of such symmetries is, generally, better understood in cases when there are finitely many (simple) topological operators, in which case the symmetry is described by a *fusion category*. There are a lot of examples of continuous non-invertible symmetries [59, 61, 64–66], but their mathematical description is less clearly understood.

Non-invertible symmetries are also present in higher-dimensional QFTs. In fact, in higher-dimensions there is also an interplay between higher-form, and non-invertible symmetries, leading to topological operators, supported on higher-codimensional manifolds, with a non-invertible fusion ring. We discuss here a few examples, with many more to be found in the original papers (see for example [67] for a long list of references).

Three-dimensional TQFTs are a prime example. The idea is parallel to the discussion of abelian Chern–Simons theory in subsection 1.1.2, with the added generalisation that the TQFTs can now be taken to be non-abelian. The natural operators of such theories are topological, and because of the non-abelian nature of the TQFT, they fuse non-invertibly. See e.g. [68–74], for examples with this perspecitve.

Another perspective involves gauging non-abelian invertible symmetries. Gauging a non-abelian, finite, p-form symmetry, $G^{[p]}$, in a d-dimensional QFT results in a non-invertible $(d-p-2)$-form non-invertible global symmetry, whose fusion ring is given by there representation ring, $\mathrm{Rep}\big(G^{[p]}\big)$, of $G^{[p]}$ [12, 75–77]. The corresponding topological operators are the Wilson operators of $G^{[p]}$.

Of relevance to this thesis, and particularly chapter 4, is the case of continuous non-invertible symmetries, which is best illustrated by a concrete example. Consider a compact scalar, $\phi \sim \phi + 2\pi$, in generic dimensions. This theory has a global U(1) shift symmetry, $\phi \longmapsto \phi + c$, $c \in \mathbb{R}/2\pi\mathbb{Z}$ and a $\mathbb{Z}_2$ symmetry $\phi \longmapsto -\phi$. Gauging the $\mathbb{Z}_2$ symmetry breaks the shift symmetry because the U(1) charge is odd under $\mathbb{Z}_2$. In other words, if

$$U\big(\mathrm{e}^{\mathrm{i}\alpha}, \Sigma_{d-1}\big) = \exp(\mathrm{i}\alpha Q[\Sigma_{d-1}]), \qquad \alpha \in \mathbb{R}/2\pi\mathbb{Z}, \tag{1.1.35}$$

is the U(1) generator, under a $\mathbb{Z}_2$ gauge transformation it maps to $U(\mathrm{e}^{-\mathrm{i}\alpha}, \Sigma_{d-1})$. However, there is a continuous family of topological operators, that survives the gauging, given by:

$$\mathcal{D}_\alpha(\Sigma_{d-1}) := U\big(\mathrm{e}^{\mathrm{i}\alpha}, \Sigma_{d-1}\big) \oplus U\big(\mathrm{e}^{-\mathrm{i}\alpha}, \Sigma_{d-1}\big) = 2\cos(\alpha\, Q[\Sigma_{d-1}]), \tag{1.1.36}$$

with $\alpha \in \mathbb{R}/\pi\mathbb{Z}$. These operators fuse, however, in a non-invertible fashion [63, 65, 78]:

$$\mathcal{D}_\alpha(\Sigma_{d-1}) \otimes \mathcal{D}_\beta(\Sigma_{d-1}) = \mathcal{D}_{\alpha+\beta}(\Sigma_{d-1}) \oplus \mathcal{D}_{\alpha-\beta}(\Sigma_{d-1}). \tag{1.1.37}$$

This story has a gauge-theory analogue, where the compact scalar is replaced by a gauge field, in which case, the $\mathbb{Z}_2$ symmetry is interpreted as charge-conjugation. The corresponding $\mathbb{Z}_2$-gauged theory, with its also has a continuous family of topological operators fusing similarly to (1.1.37) [76, 79–82].

Finally, a striking appearance of non-invertible symmetries is in (four-dimensional) quantum electrodynamics (QED) and quantum chromodynamics (QCD) [83, 84]. Let us sketch, in brief, the idea in QED. Classically, QED has an axial U(1) symmetry rotating the participating Dirac fermions by a phase $\exp\big(\mathrm{i}\frac{\alpha}{2}\gamma_5\big)$. However, this symmetry suffers from the Adler–Bell–Jackiw (ABJ) anomaly [85, 86], implying that the symmetry is broken, at the quantum level, by instanton effects. However, what was shown in [83, 84] is that the symmetry can be restored by coupling the non-topological operator

$$U_\alpha(\Sigma_3) := \exp\bigg(\mathrm{i}\,\alpha \int_{\Sigma_3} \star J_{\text{axial}}\bigg), \tag{1.1.38}$$

where J_{axial} is the anomalously conserved axial current, for the value $\alpha = \frac{2\pi}{N}$, to a three-dimensional fractional quantum Hall state with filling fraction $\nu = \frac{1}{N}$. This constructs a genuine topological operator in the theory. In this case, the response field, A, of the quantum Hall state, is identified with the dynamical photon of QED, while the dynamical field a of the quantum Hall state is introduced, and integrated out, only on the submanifold Σ_3 of spacetime. More precisely the topological operator produced by this construction is:

$$\mathcal{D}_{\frac{1}{N}}(\Sigma_3) := \int_{\mathscr{C}[\Sigma_3]} \frac{\mathrm{D}a}{\mathrm{vol}(\mathscr{G})} \exp\left(\int_{\Sigma_3} \left(\frac{2\pi \mathrm{i}}{N} \star J_{\text{axial}} + \frac{\mathrm{i}N}{4\pi} a \wedge \mathrm{d}a + \frac{\mathrm{i}}{2\pi} a \wedge \mathrm{d}A\right)\right), \tag{1.1.39}$$

where $\mathscr{C}[\Sigma_3]$ denotes the space of U(1) connections on Σ_3 modulo gauge transformations. This operator is, however, non-invertible. This can be explicitly checked by noting that $\mathcal{D}_{\frac{1}{N}} \otimes \mathcal{D}^{\dagger}_{\frac{1}{N}} \neq \mathbf{1}$. A similar procedure can be performed to obtain $\mathcal{D}_{\alpha}(\Sigma_3)$, for all $\alpha \in 2\pi\mathbb{Q}/\mathbb{Z}$, by stacking with the effective TQFT for fractional quantum Hall state at filling $\nu \in \mathbb{Q}/\mathbb{Z}$ (this is known as a minimal TQFT [87]). The same construction works for QCD, in the massless quark limit, where, now, the axial symmetry rotates the quark doublet. In this limit, it is suggested that spontaneous breaking of this non-invertible axial symmetry explains the dominant decay channel of the neutral pion into two photons. In this spirit, the neutral pion may be viewed as a Nambu–Goldstone boson for this spontaneous symmetry breaking [83, 88, 89].

1.1.4 Symmetry topological field theory

In this section we will briefly present an approach aimed at decoupling the symmetry properties of a given quantum field theory from the dynamics. The idea, based on [90, 91] is to encode the symmetry operators of a given quantum field theory in a one-higher-dimensional TQFT. In the TQFT all operators are, by definition, topological, therefore one may think of them as symmetries of something. Another point-of-view, is to generalise the spirit of anomaly inflow. In an 't Hooft anomalous theory one can capture the anomaly in a one-higher-dimensional invertible TQFT; that is, the low-energy description of an SPT phase (cf. section 1.2). Here, one captures the symmetry by coupling to a topological order in one-higher-dimension.

Concretely, consider a d-dimensional quantum field theory, with a symmetry, $\mathscr{C}$, on a manifold X. This symmetry can be grouplike, or non-invertible, zero-, or higher-form. Typically, the construction works best for discrete symmetries, however proposals for continuous symmetries appeared recently in the literature [14, 92–94]. One then constructs a $(d+1)$-dimensional topological field theory, the "symmetry topological field theory" (symTFT($\mathscr{C}$)), with the following properties.

1. It admits topological boundary conditions, $|\text{top}\rangle$. These are sometimes referred

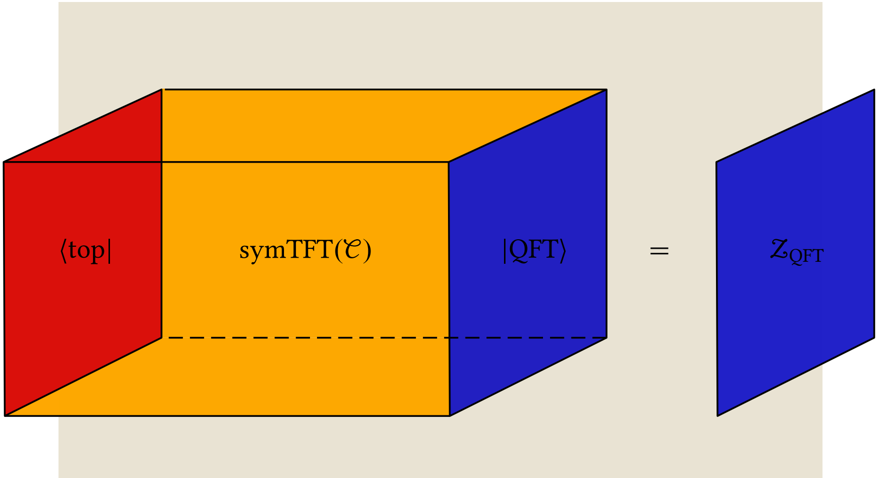

Figure 1.5: *The symmetry TFT setup. Evaluating the symmetry TFT on an interval with topological Dirichlet boundary conditions on the one side and dynamical boundary conditions on the other side produces the QFT partition function.*

to as gapped boundary conditions.

2. It admits the original d-dimensional QFT as a boundary condition, i.e. the starting QFT is compatible with the variational principle of the symTFT($\mathscr{C}$). We will denote this boundary condition as $|\text{QFT}\rangle$.

3. The partition function of the symTFT on $X \times [0, 1]$, with boundary conditions $\langle\text{top}|$ at 0 and $|\text{QFT}\rangle$ at 1 gives the partition function of a global variant of the original QFT. The modifier "global variant" refers to different topological manipulations of the QFT, i.e. stacking with SPTs or discrete gaugings. Exactly which global variant is produced depends on the form of topological boundary conditions one chooses. Topological Dirichlet boundary conditions produce the original QFT. This is illustated in figure 1.5.

As an example, let us assume that the QFT has a finite, grouplike symmetry, G. In this case, the dynamical boundary condition, $|\text{QFT}\rangle$ is

$$|\text{QFT}\rangle = \sum_{\mathcal{A}} Z_{\text{QFT}}[\mathcal{A}]\,|\mathcal{A}\rangle\,, \tag{1.1.40}$$

where $Z_{\text{QFT}}[\mathcal{A}]$ denotes the partition function of the QFT in the presence of background G-gauge fields, in the sense of (1.1.16). An option for the topological boundary condition is Dirichlet boundary condition, which can be represented as

$$|\text{D}[\mathcal{A}]\rangle = \sum_{\mathcal{A}'} \delta\left[\mathcal{A}' - \mathcal{A}\right]\left|\mathcal{A}'\right\rangle. \tag{1.1.41}$$

The sandwich of figure 1.5 produces:

$$\langle \mathrm{D}[\mathcal{A}]|\mathrm{QFT}\rangle = \mathcal{Z}_{\mathrm{QFT}}[\mathcal{A}]. \tag{1.1.42}$$

Other choices of boundary conditions produce the QFT with some subgroup of G gauged. For example, if the group is non-anomalous, there exists a Neumann boundary condition, given by

$$|\mathrm{N}[\mathcal{B}]\rangle = \sum_{\mathcal{A}'} \exp\left(\mathrm{i}\int \mathcal{A}' \cup \mathcal{B}\right)|\mathcal{A}'\rangle\,, \tag{1.1.43}$$

where $\cup$ is the cup product of cocycles and $\mathcal{B}$ is a background gauge field of the Pontryagin dual group, $\widehat{G}$, of G. This produces the partition function of QFT, with G gauged:

$$\langle \mathrm{N}[\mathcal{B}]|\mathrm{QFT}\rangle = \sum_{\mathcal{A}} \exp\left(\mathrm{i}\int \mathcal{A} \cup \mathcal{B}\right)\mathcal{Z}_{\mathrm{QFT}}[\mathcal{A}] = \mathcal{Z}_{\mathrm{QFT}/G}[\mathcal{B}]. \tag{1.1.44}$$

To further illustrate the idea, in an example that will appear prominently in this thesis, let us take the global symmetry to be a p-form $\mathbb{Z}_k^{[p]}$ symmetry. The corresponding symTFT is a $\mathbb{Z}_k$ higher Dijkgraaf–Witten theory [95], or written in terms of U(1) gauge fields [34], it is a level-k, abelian, p-form BF theory. Its action is:

$$S_{\mathrm{BF}}[A, B] := \frac{k}{2\pi}\int_X B_{[d-p-1]} \wedge \mathrm{d}A_{[p]}\,, \tag{1.1.45}$$

where $A_{[p]}$ is a p-form gauge field and $B_{[d-p-1]}$ is a $(d-p-1)$-form gauge field. The observables, are a generalisation of the Chern–Simons Wilson loops (1.1.31), namely Wilson operators for the A and the B fields:

$$\mathrm{W}_n\big[C_p\big] := \exp\left(\mathrm{i}\,n\int_{C_p} A_{[p]}\right) \quad \text{and} \quad \mathrm{V}_m\big[C_p\big] := \exp\left(\mathrm{i}\,m\int_{C_{d-p-1}} B_{[d-p-1]}\right). \tag{1.1.46}$$

The Dirichlet and Neumann boundary conditions are, in this case given by the eigenstates of $\mathrm{W}_n\big[C_p\big]$ and $\mathrm{V}_m\big[C_{d-p-1}\big]$, respectively:

$$\mathrm{W}_n\big[C_p\big]\,|\mathrm{D}[\mathcal{A}]\rangle = \exp\left(\mathrm{i}\,n\int_{C_p}\mathcal{A}\right)|\mathrm{D}[\mathcal{A}]\rangle\,, \tag{1.1.47}$$

$$\mathrm{V}_m\big[C_{d-p-1}\big]\,|\mathrm{N}[\mathcal{B}]\rangle = \exp\left(\mathrm{i}\,m\int_{C_{d-p-1}}\mathcal{B}\right)|\mathrm{N}[\mathcal{B}]\rangle\,. \tag{1.1.48}$$

The important feature in this discussion, is that, in order to study symmetry-related questions, we did not need to specify the dynamical quantum field theory. For example, this is the symmetry TFT for $SU(N)$ Yang–Mills theory, as well as for various pure $\mathcal{N} = 2$ super-Yang–Mills theories with ADE gauge algebra [96].

More generally, the study of symmetry TFTs has become, in recent years, a very active topic of research, seeing applications in various fields, see [14, 89, 92, 93, 96–100] for a sample of the recent literature.

1.1.5 Honorable mentions

There are many more important developments concerning symmetries and their generalisations, which, albeit exciting, are not directly connected to this thesis. In this short section we briefly comment on two of them.

An obvious generalisation concerns the interplay of grouplike symmetries acting on objects of different dimensions, or in other words, the mixing of higher-form symmetries of different degree, into a structure that is not simply a direct product. This structure is known as *higher-group symmetry*. For an incomplete list of references, see [31–33, 75, 95, 96, 100–116].

A different generalisation concerns what is known as *subsystem symmetries*. This departs, slightly, from the point of view we advocated so far, namely that symmetry = topological operators, in the sense that their symmetry generators are not entirely topological, but are nevertheless, conserved in time. They appear in a large class of condensed matter systems, such as fracton models see e.g. [117–123], for an incomplete list. The most surprising feature of these models, which can be traced back to subsystem symmetry is the sensitivity of the low-energy observables, to the high-energy details. For example, such systems have exponentially large ground state degeneracy [124, 125], excitations with restricted mobility, and exhibit UV/IR mixing [120, 125–128], preventing a conventional effective field theory description at low energies.

1.2 The gapless and the gapped

We have, thus far, explained the first half of the title of this thesis. The purpose of this section is to explain the second half.

Let us begin with a quantum system, described by some microscopic Hamiltonian. Given systems described by different Hamiltonians, we can examine whether or not they occupy the same quantum phase of matter by looking into their ground states. One of two things can happen: (1) there is a discretuum of such states,[11] or (2) there is a continuum of such states. The first case is associated with the notion of a *gapped*

[11] Typically there are finitely many states, but we are being generous and allow at most countably infinite states.

phase of matter, while the second one with a *gapless phase*. In what follows we will give a more precise definition of gapped and gapless phases of matter, explain their origin, relevance and universality, and give examples of such phases.

1.2.1 Gapped phases

Let us first describe a gapped phase. A gapped phase can be defined as an equivalence class of gapped, local Hamiltonians. To be precise, let us consider Hamiltonians $H(\lambda_1)$ and $H(\lambda_2)$, with λ_i being a collection of parameters. Let us further denote by

$$\mathrm{GS}_1 := \left\{ |0_{i(1)}\rangle \right\}_{i=1}^{N_1} \quad \text{and} \quad \mathrm{GS}_2 := \left\{ |0_{i(2)}\rangle \right\}_{i=1}^{N_2} \tag{1.2.1}$$

the collection of ground states of $H(\lambda_1)$ and $H(\lambda_2)$, respectively, where N_i denotes the number of degenerate ground states of the Hamiltonian $H(\lambda_i)$. We say that $|0_{i(1)}\rangle$ is equivalent to $|0_{j(2)}\rangle$, denoted by $|0_{i(1)}\rangle \sim |0_{j(2)}\rangle$, if the two states are related by adiabatic evolution for a finite time, $t \in [0, T]$, together with inclusion or removal of product states. This is equivalent to saying that there is a path in parameter space, i.e. $\lambda(t)$, such that $\lambda(0) = \lambda_1$ and $\lambda(T) = \lambda_2$, connecting the two Hamiltonians without closing the gap. We further consider the two Hamiltonians equivalent, $H(\lambda_1) \sim H(\lambda_2)$, if for all $|0_{i(1)}\rangle \in \mathrm{GS}_1$ there exists an $|0_{j(2)}\rangle \in \mathrm{GS}_2$, such that $|0_{i(1)}\rangle \sim |0_{j(2)}\rangle$. This allows us to define a *gapped phase* as the equivalence class of gapped Hamiltonians, with respect to the above equivalence relation:

$$[H(\lambda_1)] := \{H(\lambda) \mid H(\lambda) \sim H(\lambda_1)\}. \tag{1.2.2}$$

This is illustrated in figure 1.6. Note, however, that this is a coarse grained classification of gapped phases, not accounting for phases that can be distinguished by other means besides gap closing. We will come back to rectify that in the following paragraphs.

This definition of a gapped phase, directly implies that Hamiltonians with different ground state degeneracy cannot be in the same phase. And indeed, this principle lies behind the Landau paradigm for the classification of phases: phases can be distinguished by their spontaneous symmetry breaking (SSB) patterns. If a phase of matter breaks spontaneously a discrete zero-form symmetry group, G, it will have $|G|$ ground states. It can therefore be distinguished from the trivial symmetry-preserving phase, with a unique, product ground state.

It seems, however, that the above characterisation distinguishes phases further, beyond the Landau paradigm, as it is not only the number of ground states that it measures, but also other, finer, yet robust, characteristics. The word robust here, refers to the fact that these are characteristics that are not affected by small perturbations of the system, as these would be accounted for in the adiabatic evolution that connects

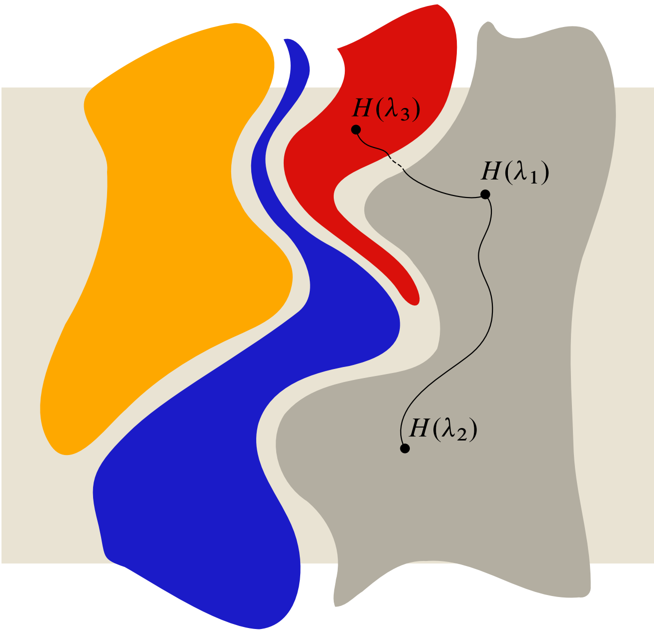

Figure 1.6: *A crude illustration of the quantum phases of matter. The coloured islands indicate different phases of matter, while the light rivers are gap-closing paths. The Hamiltonians $H(\lambda_1)$ and $H(\lambda_2)$ lie in the same phase, but $H(\lambda_3)$ does not.*

the states. These characteristics indicate, therefore, something topological, something that can only change under drastic changes of the physical system. In accordance with the logic of section 1.1, symmetries are intimately connected to topological properties of quantum systems and vice versa. It is therefore conceivable, that the Landau paradigm can be modified to capture this finer distinction of phases of matter, by enlarging the definition of symmetry and symmetry-breaking. Such a programme is subject to intense recent research [42, 129–132].

Let us return to the classification of gapped phases and consider, for now, phases that do not break any ordinary symmetry. Moreover, let us momentarily choose $[H(\lambda_1)]$ to be the trivial phase, i.e. with a unique gapped vacuum in a product state. The above definition guarantees, that, if the ground state of $[H(\lambda_2)]$ is highly entangled, it cannot be deformed to the trivial state through a finite-time adiabatic evolution. This is in fact easier to state in the language of [133], where it is shown that the action of finite adiabatic evolution on states is equivalent to the action of a constant-depth quantum circuit. What the circuit does, is it attempts to disentangle pairs of states, one step at a time. But being a constant-depth circuit, it can only disentangle finitely many such states. This leads to the definition of short- and long-range entangled states. Short-range entangled states are those that can be transformed into a product state by a constant-depth quantum circuit. Long-range entangled are those that cannot. The phases induced by the above ground states, are called, consequently, phases with short- and long-range entanglement respectively. Phases with long-range entanglement are also known as (intrinsically) *topologically ordered phases* [134] (phases with short-range entanglement are sometimes known as non-intrinsically topologically ordered). As is evident by the above criterion, topologically ordered phases are characterised by their patterns of long-range entanglement. This will be a recurrent theme in this thesis, occupying chapters 2 and 3.

We can add an extra layer of fine-graining by supplementing the entanglement classification with symmetry constraints. This gives rise to symmetry-protected and symmetry-enriched topological phases. Namely, given a symmetry, $\mathcal{C}$,[12] of the quantum system, represented by operators, $\mathcal{D}$ acting on the Hilbert space, that commute with our starting local Hamiltonian,

$$[H(\lambda_1), \mathcal{D}] = 0, \tag{1.2.3}$$

we define a symmetry-enriched topologically ordered phase (SET phase) as

$$[H(\lambda_1)]_{\mathcal{C}} := \big\{ H(\lambda) \,\big|\, H(\lambda) \sim H(\lambda_1) \quad \text{and} \quad [H(\lambda), \mathcal{D}] = 0,\ {}^{\forall}\mathcal{D} \in \mathcal{C} \big\}. \tag{1.2.4}$$

In the case where $[H(\lambda_1)]$ is the trivial phase, $[H(\lambda_1)]_{\mathcal{C}}$ is known as symmetry-protected topological phase (SPT phase).

1.2.2 Gapless phases

So far, the discussion was focussed on gapped phases, with gaplessness arising only at boundaries between different phases. From the point-of-view that we have taken up until now, gapless phases are a bit more tricky to define. They are more natural in the quantum field theory point-of-view, that we will connect to in the forthcoming paragraphs. Nevertheless, let us attempt a heuristic definition before changing language. Ground states of gapped phases, have exponentially decaying correlations — this statement was already anticipated by Haldane [135], and finally proven by Hastings and Koma [136]. On the other hand, correlations of ground states of gapless Hamiltonians, generically decay as a power-law with distance. In other words, the correlation length in such phases diverges. Often, this behaviour indicates that the system is at a critical point, undergoing a (second order) phase transition. Such points in the space of theories are described by conformal field theory. We will return to that point in the forthcoming paragraphs and in chapter 4. Nevertheless, it is worth mentioning that from the point-of-view of phases of matter, gapless points are not only present at criticality, but can sometimes occupy stable phases of their

[12] $\mathcal{C}$ can, in general have a group-like, or categorical structure, cf. section 1.1

own, such as gapless quantum spin liquids [137–139], non-Fermi liquids and strange metals [140, 141], as well as gapless symmetry protected phases [142, 143].

1.2.3 Field theory description

At large distances, or equivalently, low energies, many of the quantum systems described above, exhibit universal properties, that can be understood without reference to the original system. In other words, a lot of interesting questions can be addressed without having access to the microscopic, "ultraviolet" (UV) description of the system. Such, low-energy effective Hamiltonians, are often well-approximated by local quantum field theories.[13,14] We are therefore only given access to a quantum field theory, describing our physical system up until some energy scale, Λ, beyond which, anything could happen. We are not interested in that now. What we are interested in is actually the opposite regime, what happens at the "infrared" (IR), i.e. at extremely low energies. To that end, we run the renormalisation group flow, to obtain the low-energy effective action. That is, assuming that we started with an action principle, with some action $S_\Lambda[\phi]$, describing the physics of the degrees of freedom ϕ whose energy/momentum, is at most Λ, the renormalisation group equation tells us, that the effective action at a lower energy, $\Lambda' < \Lambda$ is given by[15]

$$S_{\Lambda'}[\phi_{\text{light}}] = -\log\left[\int_{\mathcal{C}[\Lambda',\Lambda]} \mathrm{D}\phi_{\text{heavy}}\ \exp\big(-S_\Lambda[\phi_{\text{light}} + \phi_{\text{heavy}}]\big)\right]. \tag{1.2.5}$$

In doing so, we have split the degrees of freedom into light and heavy, with energy below Λ', and between Λ' and Λ respectively, and integrated out the heavy fields. This leaves us with an effective theory of the light modes. We can iterate this procedure, going lower and lower in energy scales, until we reach a fixed point of the renormalisation group. In this framework the million dollar question is what the description of the IR phase of a generic QFT is.[16]

The first type of IR fixed points, are gapped fixed points. As alluded to above, these are fixed points with vanishing correlation length. Let us begin the discussion by asking the simplest possible question: What fixed point does the trivially gapped phase flow to? The answer is, naturally, that it flows to a trivial QFT. Namely, a QFT whose operator content is only the identity operator. The next-up question, in terms of complexity, concerns the fixed-points of SPT phases. These are given by invertible topological quantum field theories (iTQFT) [145]. Such theories, are

[13]There are, however, systems that evade such a description, such as fractonic phases, mentioned in subsection 1.1.5.

[14]We will restrict to unitary QFTs, explicitly mentioning the lack of unitarity if needed.

[15]A word on notation. We are be in Euclidean signature and $\mathcal{C}[\Lambda', \Lambda]$ denotes a functional space of fields with energy between Λ' and Λ.

[16]Quite literally. There is a million dollar bounty on answering this question for Yang–Mills theory [144].

mild generalisations of the trivial quantum field theory, coupled to background gauge-fields, for the protecting symmetry, and account for the aforementioned finer, symmetry-protected, classification of trivially gapped phases. Invertible TQFTs in d dimensions — and by extension d-dimensional symmetry-protected topological phases — protected by a finite group, G, are classified by cobordisms [146–149]. In equations:

$$\{[H(\lambda)]_G\} \cong \hom\left(\Omega_d^G, \mathrm{U}(1)\right), \tag{1.2.6}$$

where the left-hand-side denotes the set of isomorphism classes of d-dimensional SPT phases, and on the right-hand-side, Ω_d^G denotes the bordism groups of d-dimensional manifolds with G-structure.

A more challenging question is describing the fixed point that corresponds to a long-range entangled gapped phase. The mantra that follows this question is that topologically ordered systems flow in the IR to a topological field theory [145, 150]. A more precise statement [44, 151], further attempting a classification, is that topological orders flow in the IR to *fully-extended TQFTs*.[17] A more complete classification, further accounting for symmetry-enriched phases, protected by potentially non-invertible finite symmetries is given in [152], in terms of autoequivalences of the protecting category.

The fixed points we have discussed so far exhaust the infrared fate of gapped phases of matter. We are left to discuss the fate of gapless points. As we alluded to, above, for such phases, the correlation length diverges. Therefore, the effective quantum field theory description of the system, the system flows to a scale invariant theory.[18] Oftentimes, scale invariance, combined with Poincaré invariance, and unitarity, enhance to conformal invariance. This is famously the case for all two-dimensional theories, [155, 156], while there are also supporting arguments for four-dimensional theories [157–160]. Therefore, the infrared behaviour of a large class of gapless phases of matter can be described using the techniques of conformal field theory.

Many of the phases we discussed above present apparent exceptions to the Landau paradigm. For instance, topologically ordered gapped phases have no local order parameter. Even worse, there is no symmetry understanding of stable gapless phases. However, a recent perspective advocated in [42, 161, 162] suggests that while the traditional Landau paradigm cannot account for all phases of matter, a modified

[17] In [151], the TQFT is further supplemented with the word anomalous. The word anomalous indicates, there, that a topological order should be thought of up to stacking with a trivially gapped phase, since a trivially gapped phase corresponds to the trivial QFT in the IR. In our discussion this is taken into account by allowing the addition/removal of product states in the definition of the equivalence class defining the phase.

[18] However, note that when the dispersion is not linear the effective field theory is not relativistic [153]. Fixed points of this sort are known as Lifshitz fixed points [154].

Landau paradigm encompassing all generalised notions of symmetry can. Perchance. Indeed, by including higher-form symmetries, one can understand in a unified way deconfined states of gauge theory [15, 37, 42, 163]. Additionally, incorporating non-invertible symmetries, provides a more complete picture of two-dimensional phases of matter [132]. Importantly, this perspective opens a window towards understanding gravity as a phase of matter [129].

1.3 Outline

The rest of this thesis is organised as follows. Chapters 2 and 3 are devoted to the study of the *gapped*. In particular, they are concerned with topologically ordered phases in higher-dimensions and their entanglement structure, focussing on their low-energy topological field theory description. Then, in chapter 4, the focus shifts to the *gapless*, where the interplay between conformal field theory and generalised symmetries is examined and novel results about the structure of CFTs are presented.

In detail, chapter 2 presents a systematic study of the entanglement structure of abelian topological order described by p-form BF theory in arbitrary dimensions. This is done directly in the low-energy topological quantum field theory by considering the algebra of topological surface operators. Two relevant notions of subregion operator algebras are defined, which are related by a form of electric-magnetic duality. It is subsequently shown that with each subregion algebra, there is an associated entaglement entropy, termed *essential topological entanglement* (ETE). This refines a well-known concept in $(2+1)$-dimensional topological orders: the topological entanglement entropy. ETE is intrinsic to the theory, inherently finite, positive, and sensitive to more intricate topological features of the state and the entangling region.

Then, in chapter 3 an alternative perspective is explored. Remaining within the setup of p-form abelian BF theory, the entanglement entropy arising from edge-modes is considered. This is done on arbitrary spatial topology and across arbitrary entangling surfaces. The entropy contains a series of descending area laws plus universal corrections proportional to the Betti numbers of the entangling surface. The calculation comes in two flavours: firstly, through an induced edge-mode theory, appearing on the regulated entangling surface in a replica path integral, and secondly through a more rigorous definition of the entanglement entropy through an extended Hilbert space. Along the way several key results are presented, that are of their own merit. The edge-mode theory is given by a novel combination of $(p-1)$-form and $(d-p-2)$-form Maxwell theories linked by a chirality condition, in what is termed *chiral mixed Maxwell theory.* The thermal partition function of this theory is explicitly evaluated. Additionally, it is shown that the extended Hilbert space is organised into representations of an infinite-dimensional, centrally extended current algebra which naturally generalises 2d Kac–Moody algebras to arbitrary dimension and topology.

Lastly, the two approaches are connected, showing that the thermal partition function of the chiral mixed Maxwell theory is precisely an extended representation character of the current algebra, establishing an exact correspondence of the edge-mode theory and the entanglement spectrum.

Coming to chapter 4, the setup switches to conformal field theory, but the theme of current algebras persists. The main result of this chapter is a one-to-one correspondence between line operators and states in four-dimensional CFTs with continuous 1-form symmetries. Such CFTs enjoy an infinite dimensional current algebra, closely related to the algebras of chapter 3. The representation theory of this current algebra is constructed, and the space of states on an arbitrary closed spatial slice is described in detail. Then, the spectrum on $\mathbb{S}^2 \times \mathbb{S}^1$ is rederived via a path integral on $\mathbb{B}^3 \times \mathbb{S}^1$ with insertions of line operators. This leads to a direct and explicit correspondence between the line operators of the theory and the states on $\mathbb{S}^2 \times \mathbb{S}^1$. An interesting conclusion is that the vacuum state is not prepared by the empty path integral, but by a squeezing operator. Additionally, some of the above results are generalised in two directions. Firstly, universal current algebras are constructed in $(2p + 2)$-dimensional CFTs, with continuous p-form symmetry, and secondly non-invertible generalisations are provided.

Chapter 5 summarises the salient points and the results of the thesis, and discusses interesting future directions and generalisations.

Technical details are collected in the various appendices. Specifically, appendix A contains a careful treatment of BF theory in differential cohomology, mathematical proofs, and details on the decomposition of density matrices pertaining to the body of chapter 2. Appendix B contains a comparison of p-form Maxwell theory and chiral mixed Maxwell theory, and spectral properties of the Hodge Laplacian. Appendix C expands further on the spectrum of the Hodge Laplacian, in particular in situations relevant for chapter 4, discusses the current algebras of the body of the thesis in more general situations, and provides details on the radial evolution, which is central to the state-operator correspondence.

2 Essential topological entanglement

2.1 Introduction

Quantum entanglement is an invaluable framework for modern theoretical physics. This framework has led to profound insights into quantum information theory, quantum field theory, and quantum gravity.[1] Yet some of the most profound applications can be found in the theory of quantum phases of matter. In particular, in (2+1) dimensional gapped systems the presence of topological order cannot be diagnosed by any local order parameter. Entanglement is a non-local phenomenon. It stands to reason that long-range entanglement can provide a clean signature of topological order in (2+1) dimensions: the celebrated "topological entanglement entropy" (TEE) [165, 166].

Low-energy effective field theories are potent tools for exploring TEE in manifestly universal manner. These are topological quantum field theories (TQFTs), the prototypical example being Chern-Simons theory in (2+1) dimensions. Topological order in higher-dimensions is expected to be richer: already the discovery of (3+1) dimensional systems displaying "fracton topological order" [167–169] has broadened our understanding of gapped phases. Yet even the traditional classification of TQFTs can involve a large set of non-Gaussian interactions which induce richer forms of operator statistics [170, 171]. It remains a broad open question as to what universal entanglement signatures diagnose and distinguish topological order in higher dimensions. Here we take modest steps towards understanding this question, focusing on abelian topological orders described by abelian BF theory. This focus buys us some muscle: we will be able to make broad statements about abelian topological order in arbitrary dimensions and quantised on (almost) arbitrary manifolds.[2] We will use this muscle to address two conceptually puzzling aspects of the traditional treatements

[1] See [164] and references therein for an (obviously) non-exhaustive summary.
[2] We do restrict to torsion-free manifolds as well.

of TEE.

The first conceptual puzzle we want to address is the area law. Traditional computations of TEE involve an area law stemming from short-distance correlations at the UV scale and to which the TEE appears as a subleading, scale-independent, correction. Heuristically the scale-independence of this subleading correction is a signal of its universality (however there are subtleties applying this argument to lattice and tensor network models [172, 173]). It is initially surprising that a TQFT, which has a finite dimensional Hilbert space when quantised on a compact surface, can support a divergent entanglement entropy. However, the area law arises from an explicit addition of UV degrees of freedom when calculating TEE. These either come in the form of an embedding into a microscopic model (e.g. a lattice gauge theory [174–176], a "coupled wires" model [177], or a tensor network model) or in the form of "edge-modes" living on an entangling surface [178–181]. These UV degrees of freedom play an important role in calculating entanglement entropy: TQFTs are quantum gauge theories which have a well-known obstruction to factorising the Hilbert space into local subregions [174, 182–185]. In this context the UV degrees of freedom provide an arena, the "extended Hilbert space," in which the Hilbert space can be factorised and the entanglement entropy defined. Here we ask if there is another manner for defining entanglement entropy that (i) bypasses invoking UV degrees of freedom, (ii) is strictly topological, and (iii) is commensurate with a finite dimensional Hilbert space in the IR.

There is indeed an alternative for dealing with this obstruction. In a seminal paper, Casini, Huerta, and Rosabal [183] illustrated how operator algebras provide a natural definition of entanglement in gauge theories. The lack of Hilbert space factorisation manifests itself as a non-trivial centre in the algebra of operators associated to a region. Algebraic definitions of entanglement in gauge theories and their relation to the extended Hilbert space have been largely explored in the context of lattice gauge theories [186, 187]. However the algebraic approach to entanglement is, in principle, valid even in the continuum. For TQFTs it provides an intrinsically IR avenue for defining entanglement entropy. I.e. a definition that utilises only the ground states and operators available at low-energies, without involving UV degrees of freedom, and is strictly finite. Despite the hotbed of research in entanglement entropy in topological phases and quantum gauge theories, this aspect of topological entanglement has been left relatively unexplored.

The second conceptual puzzle we want to address is "semi-locality" of the traditional TEE which in (2+1) dimensions involves topological aspects (Betti numbers) intrinsic to the entangling surface; this relation is argued to hold in higher dimensions [188]. The ground states of topological field theories display extreme long-range entanglement.[3] It is perhaps then surprising that TEE does not "sense" farther than the entangling surface itself and is insensitive, say, to how the entangling surface is topologically embedded into the Cauchy slice defining the Hilbert space.

We will address these two conceptual puzzles in this chapter. Namely, we consider the operator algebra, $\mathfrak{A}[\Sigma]$, acting on a Cauchy slice, Σ, that is directly available in abelian BF theory. The operators generating this algebra are higher-form Wilson surface operators. Due to the topological nature of the field theory, these surface operators are invariant under deformations and so are naturally associated with homology cycles of Σ. Clearly this algebra, and any subalgebra, is inherently topological and defined directly in the IR. However, such operators are not wont to be localised to spatial subregions; as a result, there are potentially large ambiguities in ascribing a subalgebra, $\mathfrak{A}[R]$, to region, R. We describe two natural choices that can roughly be stated as "the set of operators that *can* act entirely in R" and "the set of operators that *must* act, at least partially, in R." We name these two algebras the *topological magnetic algebra* and the *topological electric algebra*, respectively, for reasons that will become clear in due time. They are related by a form of *subregion electric-magnetic duality* which we will make precise below.

We utilise these two notions of subregion algebra to assign an entanglement entropy to ground states in the theory. This entanglement entropy is by nature (i) topological, and (ii) finite and commensurate with a finite dimensional Hilbert space. To distinguish it from the traditional TEE appearing as the subleading correction to an area law, we coin[4] this entropy *essential topological entanglement*, $\mathcal{E}$. It comes in two forms, $\mathcal{E}_{\text{mag}}$ and $\mathcal{E}_{\text{elec}}$, and are related by the subregion electric-magnetic duality mentioned above.

Owing to the power of topological field theory, we will be able to evaluate $\mathcal{E}$ in arbitrary dimensions, on arbitrary surfaces, and associated to arbitrary regions. This allows to us to show that $\mathcal{E}$ is indeed sensitive to more intricate and long-range forms of topology than that of ∂R alone: in both forms it depends on topological aspects of ∂R, Σ, and how ∂R is embedded into Σ. This is, again, innate to the operator algebra definition. Operators in $\mathfrak{A}[R]$ must, foremost, be operators in $\mathfrak{A}[\Sigma]$. It is clear then that cycles of ∂R that embed to trivial cycles of Σ cannot contribute to $\mathcal{E}$.

We pause to mention that similar notions to our definition of $\mathcal{E}$ have appeared in the context of lattice gauge theories by examining the algebras of "ribbon operators" which are also naturally topological operator algebras [190], as well as work utilising similar ideas to discuss the "area operator" in tensor network models of holographic entanglement [191]. However the focus on these quoted works is on (2+1) dimensional

[3]Illustrated, for instance, in "multi-boundary" set-ups in Chern-Simons theory [189].

[4]Competing nomenclatures: intrinsic, core, and boneless topological entanglement.

non-abelian models on spaces and subregions with simple topology. Our focus on BF theory allows us to work directly in the continuum and deftly incorporate spaces and entangling regions of arbitrary topology, albeit at the expense of working in an abelian model. Because of this it is hard to make a direct comparison between these works and ours at this time. We will comment on this further in section 2.4.

2.2 Quantisation of BF theory

We begin by introducing the p-form abelian BF theory, on a d-dimensional, torsion-free manifold X, with action

$$S_{\mathrm{BF}}[A,B] := \frac{\mathbb{K}^{\mathsf{IJ}}}{2\pi}\int_X B_{\mathsf{I}} \wedge \mathrm{d}A_{\mathsf{J}}\,. \tag{2.2.1}$$

In the above $A_{\mathsf{I}} \in \Omega^p(X)$ and $B_{\mathsf{I}} \in \Omega^{d-p-1}(X)$ are vectors of p- and $(d-p-1)$-form gauge fields respectively. We will take $p \neq 0, d-1$.[5] We have also allowed a possible square, integer, non-degenerate — but not necessarily symmetric — $\mathbb{K}$-matrix of rank κ. For notational simplicity we will drop the indices, unless it is necessary. In appendix A.1 we provide a more careful treatment of BF theory, allowing for manifolds with torsion. The action (2.2.1) possesses a gauge redundancy of the form

$$\delta A = \mathrm{d}\alpha \quad \text{and} \quad \delta B = \mathrm{d}\beta\,, \tag{2.2.2}$$

where $\alpha \in \Omega^p(X)$ and $\beta \in \Omega^{d-p-2}(X)$.

Let us suppose that X possesses a boundary and discuss the quantisation of the theory on ∂X. Much of this procedure follows that of [192] and [171], however we provide these details for completeness. We begin with the classical symplectic structure. The variation of the action takes the form

$$\delta S_{\mathrm{BF}}[A,B] = \int_X (\delta B \wedge \textsc{eom}[A] + \delta A \wedge \textsc{eom}[B]) + \int_{\partial X} \boldsymbol{\vartheta}[A,B;\delta A,\delta B], \tag{2.2.3}$$

where the classical equations of motion are flatness conditions:

$$\textsc{eom}[A] = \frac{\mathbb{K}}{2\pi}\,\mathrm{d}A = 0, \qquad \textsc{eom}[B] = (-1)^{(d-p)(p+1)}\frac{\mathbb{K}^\top}{2\pi}\,\mathrm{d}B = 0. \tag{2.2.4}$$

The boundary term defines the symplectic potential, $\boldsymbol{\vartheta}$:

$$\int_{\partial X} \boldsymbol{\vartheta}[A,B;\delta A,\delta B] := (-1)^{d-p-1}\frac{\mathbb{K}}{2\pi}\int_{\partial X} B \wedge \delta A, \tag{2.2.5}$$

[5] We expect much of what follows to morally hold true in these special cases, however some technical details of our proofs would need to be altered.

where the pullback along the embedding map, $\iota_\partial : \partial X \hookrightarrow X$ is implicitly understood above. We see this theory is already in canonical, or Darboux, form, $\boldsymbol{\vartheta} = \mathbf{p} \wedge \delta\mathbf{q}$, with

$$\mathbf{q} = A \quad \text{and} \quad \mathbf{p} = (-1)^{d-p-1} \frac{\mathbb{K}}{2\pi} B, \tag{2.2.6}$$

which is consistent with fixing A as a boundary condition. We can switch the role of $(B, A) \sim (\mathbf{p}, \mathbf{q})$ to $(B, A) \sim (\mathbf{q}, \mathbf{p})$ by the inclusion of the boundary action

$$S_\partial^{\text{alt.}}[A, B] = (-1)^{d-p} \frac{\mathbb{K}}{2\pi} \int_{\partial X} B \wedge A, \tag{2.2.7}$$

but we will work in the former quantisation scheme. The symplectic form on ∂X, given by the variation of $\int_{\partial X} \boldsymbol{\vartheta}$, is

$$\boldsymbol{\Omega}_{\partial X} = (-1)^{d-p-1} \frac{\mathbb{K}}{2\pi} \int_{\partial X} \delta B \wedge \delta A. \tag{2.2.8}$$

This symplectic form is degenerate due to gauge variations. We will take care of this soon below.

We will quantise the BF theory on a $(d-1)$-dimensional manifold, Σ, by performing the path-integral on $X = \mathbb{R} \times \Sigma$. Here, $\mathbb{R}$ is coordinatised by t and the Cauchy slice at time t is represented by $\{t\} \times \Sigma$. The path-integral measure is formally given by

$$\mathrm{d}\mu\,(A, B) = \frac{\mathrm{D}A\ \mathrm{D}B}{\mathrm{vol}(\mathscr{G}_p)\,\mathrm{vol}(\mathscr{G}_{d-p-1})}\ \mathrm{e}^{\mathrm{i}S_{\mathrm{BF}}[A,B]}, \tag{2.2.9}$$

where $\mathscr{G}_p$ and $\mathscr{G}_{d-p-1}$ are the gauge groups for the redundancies (2.2.2). On top of it, it includes a sum over non-trivial bundles. For a full definition of the measure and the gauge groups, we refer the reader to appendix A.1. Currently, we consider the case where $\partial\Sigma = \varnothing$ (we will revisit the case with boundaries in chapter 3). Additionally, let $\iota_\Sigma : \Sigma \hookrightarrow X$ be the embedding of Σ into X.

We can express A and B as $A = A_0 + a$ and $B = B_0 + b$ respectively, where

$$\begin{aligned} \iota_\Sigma^* A_0 = 0 \qquad &\Longleftrightarrow \qquad \iota_\Sigma^* A = \iota_\Sigma^* a, \\ \iota_\Sigma^* B_0 = 0 \qquad &\Longleftrightarrow \qquad \iota_\Sigma^* B = \iota_\Sigma^* b. \end{aligned}$$

To make the decomposition clearer we can use coordinates $\{x_m\}_{m=1}^{d-1}$ for Σ which gives us:

$$\begin{aligned} A = (A_0)_{m_1\cdots m_{p-1}}\, \mathrm{d}t \wedge \mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_{p-1}} \\ + a_{m_1\cdots m_p}\, \mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_p}\,, \quad \text{and} \end{aligned} \tag{2.2.10}$$

$$\begin{aligned} B = (B_0)_{m_1\cdots m_{d-p-2}}\, \mathrm{d}t \wedge \mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_{d-p-2}} \\ + b_{m_1\cdots m_{d-p-1}}\, \mathrm{d}x^{m_1} \wedge \cdots \wedge \mathrm{d}x^{m_{d-p-1}}\,. \end{aligned} \tag{2.2.11}$$

In these coordinates, let us also write $\mathrm{d} = \mathrm{d}t \wedge \partial_t + \mathrm{d}x^m \wedge \partial_m =: \mathrm{d}_{\mathbb{R}} + \mathbf{d}$. Integrating (2.2.1) by parts and utilising the fact that $\mathrm{d}_{\mathbb{R}} A_0 = 0$ and $\mathrm{d}_{\mathbb{R}} B_0 = 0$ (since they involve $\mathrm{d}t \wedge \mathrm{d}t \wedge \cdots = 0$), we arrive at

$$S_{\mathrm{BF}}[A_0 + a, B_0 + b] = \frac{\mathbb{K}}{2\pi} \int_X \Big((-1)^{d-p}\, \mathbf{d}b \wedge A_0 + B_0 \wedge \mathbf{d}a + b \wedge \mathrm{d}a \Big). \tag{2.2.12}$$

It is easy to see that A_0 and B_0 act as Lagrange multipliers enforcing the Σ-flatness of a and b:

$$\mathbf{d}a = \mathbf{d}b = 0. \tag{2.2.13}$$

We will refer to (2.2.13) as the "Gauss law" constraints. Using the property (A.1.6) of the path-integral measure we can write

$$\mathrm{d}\mu(A, B) = \mathrm{d}\mu(A_0, B_0)\ \mathrm{d}\mu(a, b)\ \mathrm{e}^{\mathrm{i}S_{\mathrm{BF}}[A_0, b]}\ \mathrm{e}^{\mathrm{i}S_{\mathrm{BF}}[a, B_0]}, \tag{2.2.14}$$

and performing the integrals over A_0 and B_0 we get $\mathrm{d}\mu(a, b)\, \delta[\mathbf{d}a]\delta[\mathbf{d}b]$. The delta-functions force a and b to be closed under $\mathbf{d}$; Hodge decomposition implies, then, that

$$a = \mathbf{d}\psi + \theta, \quad \text{and} \quad b = \mathbf{d}\chi + \phi, \tag{2.2.15}$$

for some $\psi \in \Omega^{p-1}(X)$, $\theta \in \iota_\Sigma^* \mathrm{H}^p(\Sigma)$, and $\chi \in \Omega^{d-p-2}(X)$, $\phi \in \iota_\Sigma^* \mathrm{H}^p(\Sigma)$. This results into the path-integral measure:

$$\begin{aligned} \mathrm{d}\mu(a, b)\ \delta[\mathbf{d}a]\delta[\mathbf{d}b] &= \mathrm{d}\mu(\chi, \phi)\ \mathrm{d}\mu(\phi, \psi)\ \mathrm{e}^{\mathrm{i}S_{\mathrm{BF}}[\mathbf{d}\chi, \theta]}\ \mathrm{e}^{\mathrm{i}S_{\mathrm{BF}}[\phi, \mathbf{d}\psi]} \\ &= \frac{\mathrm{D}\psi\, \mathrm{D}\chi\, \mathrm{D}\phi\, \mathrm{D}\theta}{\mathrm{vol}(\mathcal{G}_p)\, \mathrm{vol}(\mathcal{G}_{d-p-1})}\ \exp\!\left(\frac{\mathrm{i}\mathbb{K}}{2\pi} \int_X \phi \wedge \mathrm{d}_{\mathbb{R}} \theta \right). \end{aligned} \tag{2.2.16}$$

The integral over ψ and χ over the volumes of the gauge groups yields the Ray–Singer torsion of the manifold [193, 194],[6] and so we simply get

$$\mathcal{Z}_{\mathrm{BF}}[X] = \left(\int \mathrm{D}\phi\, \mathrm{D}\theta\ \exp\!\left(\frac{\mathrm{i}\mathbb{K}}{2\pi} \int_X \phi \wedge \mathrm{d}_{\mathbb{R}} \theta \right) \right) \mathrm{T}_{\mathrm{RS}}[X]^{(-1)^{p-1}}, \tag{2.2.17}$$

where $\mathrm{T}_{\mathrm{RS}}[X]$ is the Ray–Singer torsion:

$$\mathrm{T}_{\mathrm{RS}}[X] := \prod_{k=0}^{d} \left(\frac{(\det{}' \triangle_k)^k}{\det \mathbb{G}_k} \right)^{\frac{1}{2}(-1)^{k+1}}, \tag{2.2.18}$$

with $\triangle_k$ being the Laplacian on the space of k-forms on X, and $\mathbb{G}_k$ the metric in the space of harmonic k-forms, defined as follows. Let $\left\{ \tau_{\mathrm{i}}^{(k)} \right\}_{\mathrm{i}=1}^{\mathrm{b}_k(\Sigma)}$ be the topological basis of harmonic k-forms, with $\mathrm{b}_k(\Sigma)$ the k^{th} Betti number of Σ. This basis is defined

[6] for a modern exposition see also [195]

such that given a basis of k-cycles $\left\{\eta^{\mathrm{i}}_{(k)} \in \mathrm{H}_k(\Sigma)\right\}_{\mathrm{i}=1}^{\mathrm{b}_k(\Sigma)}$, there is a unique harmonic representative, $\tau^{(k)}_{\mathrm{i}}$, of each cohomology class in $\mathrm{H}^k(\Sigma)$, such that

$$\int_{\eta^{\mathrm{i}}_{(k)}} \tau^{(k)}_{\mathrm{j}} = \delta^{\mathrm{i}}_{\mathrm{j}}. \tag{2.2.19}$$

It is in terms of this basis that the matrices $\mathbb{G}_k$ above are defined. Explicitly:

$$[\mathbb{G}_k]_{\mathrm{ij}} := \int_\Sigma \tau^{(k)}_{\mathrm{i}} \wedge \star\tau^{(k)}_{\mathrm{j}}, \tag{2.2.20}$$

where $\star$ is the Hodge-star on Σ. Before moving on let us make a quick digression to mention that the inverse of $\mathbb{G}_k$ is the linking matrix

$$[\mathbb{G}_k]_{\mathrm{ij}}[\mathbb{L}_k]^{\mathrm{jk}} = \delta^{\mathrm{k}}_{\mathrm{i}}, \tag{2.2.21}$$

which can be alternatively be defined as an oriented intersection number in the following way. Let us pick a basis of k-cycles $\{\eta^{\mathrm{i}}\}_{\mathrm{i}=1}^{\mathrm{b}_k(\Sigma)}$ of $\mathrm{H}_k(\Sigma)$ and a basis of $(d-k-1)$-cycles $\{\sigma^{\mathrm{i}}\}_{\mathrm{i}=1}^{\mathrm{b}_k(\Sigma)}$ of $\mathrm{H}_{d-k-1}(\Sigma)$. The transversal intersection of η^{j} and σ^{i} in Σ is a zero-dimensional manifold (that is, a collection of points) and $[\mathbb{L}_k]^{\mathrm{ij}}$ counts the number of points signed by their orientation:

$$[\mathbb{L}_k]^{\mathrm{ij}} \equiv \mathbb{L}_k\left(\eta^{\mathrm{i}}, \sigma^{\mathrm{j}}\right) := \int_{\eta^{\mathrm{i}}\cap\sigma^{\mathrm{j}}} 1. \tag{2.2.22}$$

Let us focus on the remaining path-integral in (2.2.17), which is the quantum mechanics for the large-gauge degrees of freedom, ϕ and θ. We can expand ϕ and θ in terms of the basis $\{\tau_{\mathrm{i}}\}_{\mathrm{i}=1}^{\mathrm{b}_p(\Sigma)}$. Namely

$$\phi(t,x) = \phi^{\mathrm{i}}(t)\,\tau_{\mathrm{i}} \quad \text{and} \quad \theta(t,x) = \theta^{\mathrm{j}}(t)\star\tau_{\mathrm{j}}. \tag{2.2.23}$$

Note that (2.2.19) with (2.2.15) implies

$$\theta^{\mathrm{i}} = \int_{\eta^{\mathrm{i}}} a, \qquad \phi^{\mathrm{i}} = \int_{\sigma^{\mathrm{i}}} b, \tag{2.2.24}$$

in terms of our original field variables. Since $\phi^{\mathrm{i}}(t)$ and $\theta^{\mathrm{j}}(t)$ are circle-valued functions on $\mathbb{R}$ they are identified with

$$\phi^{\mathrm{i}}(t) \sim \phi^{\mathrm{i}}(t) + 2\pi \quad \text{and} \quad \theta^{\mathrm{j}}(t) \sim \theta^{\mathrm{j}}(t) + 2\pi. \tag{2.2.25}$$

All in all the action reduces to

$$S^{\mathrm{eff}}_{\mathrm{BF}}[\phi,\theta] = \frac{\mathbb{K}}{2\pi}[\mathbb{G}_p]_{\mathrm{ij}} \int_{\mathbb{R}} \phi^{\mathrm{i}} \wedge \mathrm{d}_{\mathbb{R}}\theta^{\mathrm{j}}. \tag{2.2.26}$$

This is a simple quantum mechanical system whose symplectic form reads (reinstating the I, J indices)

$$\boldsymbol{\Omega}_\Sigma = \frac{(-1)^{d-p-1}}{2\pi}\mathbb{K}^{\mathsf{IJ}}[\mathbb{G}_p]_{\mathsf{ij}}\,\delta\phi^{\mathsf{i}}_{\mathsf{I}}\wedge\delta\theta^{\mathsf{j}}_{\mathsf{J}}\,. \tag{2.2.27}$$

which is the restriction of (2.2.8) to $\phi^{\mathsf{i}}_{\mathsf{I}}$ and $\theta^{\mathsf{j}}_{\mathsf{J}}$.

Given the our interpretation of $A = \mathbf{q}$ coming from the symplectic potential, (2.2.5), we will identify $\theta^{\mathsf{j}}_{\mathsf{J}}$ as "positions" and $\phi^{\mathsf{i}}_{\mathsf{I}}$ as "momenta".[7] Passing from Poisson brackets to commutators, promoting ϕ and θ to operators, we arrive at

$$\left[\hat{\phi}^{\mathsf{i}}_{\mathsf{I}},\hat{\theta}^{\mathsf{j}}_{\mathsf{J}}\right] = 2\pi\mathrm{i}\,(-1)^{d-p-1}\big[\mathbb{K}^{\perp}\big]_{\mathsf{IJ}}[\mathbb{L}_p]^{\mathsf{ij}}, \tag{2.2.28}$$

where $\mathbb{K}^{\perp} := \left(\mathbb{K}^{\top}\right)^{-1}$ is the inverse transpose. Hereafter we will drop the index p and the square brackets from $\mathbb{L}_p$, and $\mathbb{G}_p$ for conciseness.

Since these operators are U(1)-valued, we should exponentiate them to construct gauge-invariant Wilson surface operators:

$$\hat{\mathrm{W}}^{\boldsymbol{w}_{\mathsf{j}}}_{\eta^{\mathsf{j}}} := \exp\!\Big(w^{\mathsf{J}}_{\mathsf{j}}\,\hat{\theta}^{\mathsf{j}}_{\mathsf{J}}\Big) = \exp\!\left(\int_{\eta^{\mathsf{j}}} w^{\mathsf{J}}_{\mathsf{j}}\,a_{\mathsf{J}}\right), \tag{2.2.29}$$

$$\hat{\mathrm{V}}^{\boldsymbol{v}_{\mathsf{i}}}_{\sigma^{\mathsf{i}}} := \exp\!\Big(v^{\mathsf{I}}_{\mathsf{i}}\,\hat{\phi}^{\mathsf{i}}_{\mathsf{I}}\Big) = \exp\!\left(\int_{\sigma^{\mathsf{i}}} v^{\mathsf{I}}_{\mathsf{i}}\,b_{\mathsf{I}}\right), \tag{2.2.30}$$

where $\left\{w^{\mathsf{J}}_{\mathsf{j}}\right\}^{\mathsf{J}\in\{1,\dots,\kappa\}}_{\mathsf{j}\in\{1,\dots,\mathrm{b}_p(\Sigma)\}}$ and $\left\{v^{\mathsf{I}}_{\mathsf{i}}\right\}^{\mathsf{I}\in\{1,\dots,\kappa\}}_{\mathsf{i}\in\{1,\dots,\mathrm{b}_p(\Sigma)\}}$ are are $\mathrm{b}_p(\Sigma)\times\kappa$ collections of integers. These surface operators are defined with respect to fixed bases of homology p- and $(d-p-1)$-cycles, $\{\eta^{\mathsf{j}}\}^{\mathrm{b}_p(\Sigma)}_{\mathsf{j}=1}$ and $\{\sigma^{\mathsf{i}}\}^{\mathrm{b}_p(\Sigma)}_{\mathsf{i}=1}$, respectively; however it is easy to verify that they are homotopy invariants when acting on gauge-invariant states due to the Gauss-law constraints, (2.2.13), and thus well defined on homology classes.

2.2.1 The algebra of Wilson surface operators

The Wilson surface operators constructed above satisfy a "clock algebra" which can be easily found using the canonical commutation relations (2.2.28):

$$\hat{\mathrm{V}}^{\boldsymbol{v}_i}_{\sigma^{\mathsf{i}}}\,\hat{\mathrm{W}}^{\boldsymbol{w}_j}_{\eta^{\mathsf{j}}} = \mathrm{e}^{2\pi\mathrm{i}\,(-1)^{d-p-1}\,w^{\mathsf{J}}_{\mathsf{j}}v^{\mathsf{I}}_{\mathsf{i}}(\mathbb{K}^{\perp})_{\mathsf{IJ}}\mathbb{L}^{\mathsf{ij}}}\,\hat{\mathrm{W}}^{\boldsymbol{w}_{\mathsf{j}}}_{\eta^{\mathsf{j}}}\,\hat{\mathrm{V}}^{\boldsymbol{v}_{\mathsf{j}}}_{\sigma^{\mathsf{i}}}. \tag{2.2.31}$$

The $\hat{\mathrm{W}}$'s commute amongst each other as do the $\hat{\mathrm{V}}$'s. From the above algebra we can clearly see that $\boldsymbol{w}_{\mathsf{j}}$ and $\boldsymbol{v}_{\mathsf{i}}$ give the same algebra as $\boldsymbol{w}_{\mathsf{j}} + m\cdot\mathbb{K}$ and $\boldsymbol{v}_{\mathsf{i}} + \mathbb{K}\cdot m'$, for arbitrary $m, m' \in \mathbb{Z}^{\kappa}$. The entire algebra is, then, generated by operators labelled by charges in the lattices

$$\boldsymbol{w}_{\mathsf{j}} \in \Lambda_A := \mathbb{Z}^{\kappa}\big/\operatorname{im}\mathbb{K}^{\top} \quad\text{and}\quad \boldsymbol{v}_{\mathsf{i}} \in \Lambda_B := \mathbb{Z}^{\kappa}\big/\operatorname{im}\mathbb{K}. \tag{2.2.32}$$

[7] More correctly handling the index placement, the momenta are $\mathbf{p}^{\mathsf{J}}_{\mathsf{j}} = (-1)^{d-p-1}\frac{\mathbb{K}^{\mathsf{IJ}}}{2\pi}[\mathbb{G}_p]_{\mathsf{ij}}\phi^{\mathsf{i}}_{\mathsf{I}}$.

It will be notationally useful to collect $w_{\mathrm{j}}^{\mathrm{J}}$ and $v_{\mathrm{i}}^{\mathrm{I}}$ as the components of a $\mathrm{b}_p(\Sigma) \times \kappa$-dimensional integer vectors denoted as $\mathfrak{w}$ and $\mathfrak{v}$, respectively. Also for notational convenience, we will define an inner product on these vector spaces as

$$\Gamma(\mathfrak{v}, \mathfrak{w}) := 2\pi(-1)^{d-p-1}\mathfrak{v} \cdot \left(\mathbb{K}^{\perp} \otimes \mathbb{L}\right) \cdot \mathfrak{w} = 2\pi(-1)^{d-p-1} w_{\mathrm{j}}^{\mathrm{J}}\, v_{\mathrm{i}}^{\mathrm{I}} (\mathbb{K}^{\perp})_{\mathrm{IJ}} \mathbb{L}^{\mathrm{ij}}. \quad (2.2.33)$$

Then (2.2.31) can be written succinctly as

$$\hat{\mathrm{V}}^{\mathfrak{v}} \hat{\mathrm{W}}^{\mathfrak{w}} = \mathrm{e}^{\mathrm{i}\Gamma(\mathfrak{v},\mathfrak{w})} \hat{\mathrm{W}}^{\mathfrak{w}} \hat{\mathrm{V}}^{\mathfrak{v}}, \quad (2.2.34)$$

with

$$\hat{\mathrm{V}}^{\mathfrak{v}} = \prod_{i=1}^{\mathrm{b}_p(\Sigma)} \hat{\mathrm{V}}_{\sigma^i}^{\boldsymbol{v}_i}, \qquad \hat{\mathrm{W}}^{\mathfrak{w}} = \prod_{i=1}^{\mathrm{b}_p(\Sigma)} \hat{\mathrm{W}}_{\eta^j}^{\boldsymbol{w}_j}. \quad (2.2.35)$$

In this notation, the $\hat{\mathrm{W}}$'s and $\hat{\mathrm{V}}$'s satisfy an abelian fusion algebra

$$\hat{\mathrm{W}}^{\mathfrak{w}} \hat{\mathrm{W}}^{\mathfrak{w}'} = \hat{\mathrm{W}}^{\mathfrak{w}+\mathfrak{w}'} \quad \text{and} \quad \hat{\mathrm{V}}^{\mathfrak{v}} \hat{\mathrm{V}}^{\mathfrak{v}'} = \hat{\mathrm{V}}^{\mathfrak{v}+\mathfrak{v}'}, \quad (2.2.36)$$

where it is understood that the sums are taken in the respective lattices, $(\Lambda_A)^{\mathrm{b}_p(\Sigma)}$ and $(\Lambda_B)^{\mathrm{b}_p(\Sigma)}$.

Constructing states

To construct the states on $\mathscr{H}_\Sigma$ we pick a maximal set of commuting operators and use the space of their eigenvectors. For that we can use either $\left\{\hat{\phi}^{\mathrm{j}}\right\}_{\mathrm{j}=1}^{\mathrm{b}_p(\Sigma)}$ or $\left\{\hat{\theta}^{\mathrm{i}}\right\}_{\mathrm{i}=1}^{\mathrm{b}_p(\Sigma)}$. We will first use the basis given by $\left\{\hat{\theta}^{\mathrm{i}}\right\}$ eigenvectors which is morally consistent with fixing a as a boundary condition. To construct the states systematically, we will first define a fiducial state $|0\rangle$ annihilated by all $\left\{\hat{\theta}^{\mathrm{i}}\right\}$. This is an eigenstate of $\hat{\mathrm{W}}_{\eta^{\mathrm{j}}}^{\boldsymbol{w}_{\mathrm{j}}}$ with all eigenvalues one:

$$\hat{\mathrm{W}}^{\mathfrak{w}} \left|0\right\rangle = \left|0\right\rangle. \quad (2.2.37)$$

We will call this state the "p-surface operator condensate," or "the condensate" when the context is clear. We then use $\hat{\mathrm{V}}_{\sigma^{\mathrm{i}}}^{\boldsymbol{v}_{\mathrm{i}}}$ as raising operators. A general ground state will then be given by

$$\left|\mathfrak{v}\right\rangle := \hat{\mathrm{V}}^{\mathfrak{v}} \left|0\right\rangle = \prod_{\mathrm{i}=1}^{\mathrm{b}_p(\Sigma)} \hat{\mathrm{V}}_{\sigma^{\mathrm{i}}}^{\boldsymbol{v}_{\mathrm{i}}} \left|0\right\rangle, \quad (2.2.38)$$

for any integer vector, $\mathfrak{v} \in (\Lambda_B)^{\mathrm{b}_p(\Sigma)}$. These are indeed eigenstates of $\hat{\mathrm{W}}^{\mathfrak{w}}$ with eigenvalue

$$\hat{\mathrm{W}}^{\mathfrak{w}} \left|\mathfrak{v}\right\rangle = \mathrm{e}^{\mathrm{i}\Gamma(\mathfrak{v},\mathfrak{w})} \left|\mathfrak{v}\right\rangle. \quad (2.2.39)$$

Additionally, since

$$\langle 0|\mathfrak{v}\rangle = \langle 0|\hat{\mathrm{W}}^{\mathfrak{w}}|\mathfrak{v}\rangle = \mathrm{e}^{\mathrm{i}\Gamma(\mathfrak{v},\mathfrak{w})} \langle 0|\mathfrak{v}\rangle, \quad (2.2.40)$$

for any $\mathfrak{w} \in (\Lambda_A)^{\mathrm{b}_p(\Sigma)}$ and since $\mathbb{K}$ and $\mathbb{L}$ (and thus Γ) are non-degenerate,

$$\langle 0|\mathfrak{v}\rangle = \boldsymbol{\delta}_{\mathfrak{v}} := \prod_{\mathrm{i}=1}^{\mathrm{b}_p(\Sigma)} \delta^{(\Lambda_B)}_{\boldsymbol{v}_\mathrm{i}}, \quad \text{with} \quad \delta^{(\Lambda_B)}_{\boldsymbol{v}} = \begin{cases} 1 & \boldsymbol{v} = 0 \bmod \operatorname{im}\mathbb{K}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.41}$$

This fact, coupled with $(\hat{\mathrm{V}}^{\mathfrak{v}})^\dagger = \hat{\mathrm{V}}^{-\mathfrak{v}}$, and the fusion algebra, (2.2.36), implies the full-orthonormality of $\mathcal{H}_\Sigma = \operatorname{span}\{|\mathfrak{v}\rangle\}$. The dimension of the Hilbert space is

$$\dim \mathcal{H}_\Sigma = \dim_{\mathbb{Z}} (\Lambda_B)^{\mathrm{b}_p(\Sigma)} = \dim_{\mathbb{Z}} (\Lambda_A)^{\mathrm{b}_p(\Sigma)} = |\det \mathbb{K}|^{\mathrm{b}_p(\Sigma)}. \tag{2.2.42}$$

We note that the quantisation can be repeated in a wholly similar procedure by using $\hat{\mathrm{W}}^{\mathfrak{w}}$ as ladder operators (this builds a Hilbert space of eigenvectors of $\hat{\mathrm{V}}^{\mathfrak{v}}$) to arrive a Hilbert space of the same dimension. We will call the isomorphism of the Hilbert spaces built on p- and $(d-p-1)$-surface operator condensates, *electric-magnetic duality* in this context.[8] Below we will describe how this duality can be refined to a notion of *subregion electric-magnetic duality.*

2.3 Subregion algebras and essential topological entanglement

We now move to the main act of this chapter: how to associate subregion entanglement entropy to this theory after we have "integrated out" all of the local degrees of freedom. We will do so in the algebraic approach. We briefy remind the reader of the broad features of this approach.

Starting with a region, $R \subset \Sigma$, one associates a subalgebra, $\mathfrak{A}[R] \subset \mathfrak{A}[\Sigma]$, of the operators which act naturally on R. The commutant of $\mathfrak{A}[R]$ is then associated to the complement of R: $\mathfrak{A}[R^\mathrm{c}] = (\mathfrak{A}[R])^\mathrm{c}$. Given a state,[9] ρ, one can reduce it to $\mathfrak{A}[R]$: i.e. ρ_R is the unique Hermitian and trace-normalised element of the subregion algebra, $\mathfrak{A}[R]$, reproducing the expectation values of all $\mathcal{O}_R \in \mathfrak{A}[R]$. The von Neumann entropy of this reduced density matrix then provides an algebraic definition of the entanglement entropy of ρ reduced to R:

$$\mathcal{S}_{\mathfrak{A}[R]}[\rho] = \mathcal{S}_{\mathrm{vN}}[\rho_R] := -\operatorname{Tr}(\rho_R \log \rho_R). \tag{2.3.1}$$

This situation is complicated in theories with gauge invariance. The non-local manner in which gauge constraints are applied to states manifests itself in a non-trivial centre in the subregion algebra: $\mathfrak{Z}[R] = \mathfrak{A}[R] \cap \mathfrak{A}[R^\mathrm{c}]$. Operators generating $\mathfrak{Z}[R]$ can be

[8]This duality is simply a statement that the Hilbert space built on the p-surface operator condensate is of equal dimension to the Hilbert space built on the $(d-p-1)$-surface operator condensate. They are automatically isomorphic. This is a simple consequence of Hodge duality on Σ.

[9]We will generally call density matrices "states" regardless of their purity.

simultaneously diagonalised. The state, ρ, and subsequenty, the reduced state, ρ_R, can be decomposed with respect to the eigenspaces of the operators generating $\mathfrak{Z}[R]$:

$$\rho_R = \bigoplus_\alpha \lambda_{(\alpha)} \rho_R^{(\alpha)}, \tag{2.3.2}$$

where α labels the eigenspaces. The $\rho_R^{(\alpha)}$ can be individually trace-normalised and so $\sum_\alpha \lambda_{(\alpha)} = 1$. This leads to a refinement of (2.3.1) where the algebraic entanglement entropy naturally splits into the weighted sum of von Neumann entropies of reduced density matrix projected to fixed eigenspaces plus the Shannon entropy of the probability distribution given by $\{\lambda_{(\alpha)}\}$:

$$\mathcal{S}_{\mathfrak{A}[R]}[\rho] = \sum_\alpha \lambda_{(\alpha)} \mathcal{S}_{\text{vN}} \left[\rho_R^{(\alpha)}\right] - \sum_\alpha \lambda_{(\alpha)} \log \lambda_{(\alpha)}. \tag{2.3.3}$$

We will make these broad features explicit in what follows and show that the Shannon contribution takes a universal, topological, form. Before doing so we will first need to define a notion of a subregion algebra, $\mathfrak{A}[R]$. Given the topological nature of the operators in $\mathfrak{A}[\Sigma]$, we will take care to define it in a manifestly topological manner below.[10]

2.3.1 Topological subregion algebras

We begin by regarding Σ as the union of two, otherwise disjoint, submanifolds $\Sigma = R \sqcup_{\partial R} R^c$ sharing a common boundary, ∂R. We will assign an operator algebra, $\mathfrak{A}[R]$, to R and $\mathfrak{A}[R^c]$ is then defined as the commutant of $\mathfrak{A}[R]$. There is some ambiguity in this assignment; in what follows we will assign this in a "natural" way. Since our operators in this theory are only defined up to homotopy, however, there may be multiple "natural" ways to associate an algebra to R. Different choices of subregion algebra may result in different centres and different definitions of the entanglement entropy.

Let us introduce the following notations. Suppose that M is a submanifold of Σ and let $i^M : M \hookrightarrow \Sigma$ be the embedding map. We can use i^M to push-forward homology groups: $i^M_\bullet : \mathrm{H}_\bullet(M) \to \mathrm{H}_\bullet(\Sigma)$.[11] In what follows, given $\alpha \in \mathrm{H}_k(\Sigma)$ and a map as above, we will denote $\alpha \,\tilde{\in}\, M$, iff $\alpha \in \operatorname{im} i^M_k$. In words, $\alpha \,\tilde{\in}\, M$ says that α is continuously deformable within Σ to a cycle completely contained in M. Similarly, we denote $\alpha \,\tilde{\notin}\, M$, for an $\alpha \in \mathrm{H}_k(\Sigma)$, iff $\alpha \in \operatorname{coker} i^M_k$. In words, $\alpha \,\tilde{\notin}\, M$ is not

[10] As a benefit to these definitions applied to $\mathfrak{A}[\Sigma]$ and its subalgebras: these are all Type I von Neumann algebras acting on finite dimensional spaces. As such there is subtlety in defining traces, reduced density matrices, and von Neumann entropies.

[11] For notational convenience we will avoid indexing the push-forward with an asterisk or a hash, as is common in the mathematical literature and we will index it solely by the rank of the homology groups it is connecting.

continuously deformable within Σ to a cycle completely contained in M. We will alternate between the im / coker and $\tilde{\in}/\tilde{\notin}$ notation freely. To state the results of the following sections up-front, there are two natural algebras associated to R:

1. Topological magnetic algebra

$$\mathfrak{A}_{\text{mag}}[R] := \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_\mathrm{i}}_{\eta^\mathrm{i}}, \hat{\mathrm{V}}^{\boldsymbol{v}_\mathrm{j}}_{\sigma^\mathrm{j}} \;\middle|\; \eta^\mathrm{i} \in \operatorname{im} i^R_p,\ \sigma^\mathrm{j} \in \operatorname{im} i^R_{d-p-1}\right\} \tag{2.3.4}$$

$$\equiv \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_\mathrm{i}}_{\eta^\mathrm{i}}, \hat{\mathrm{V}}^{\boldsymbol{v}_\mathrm{j}}_{\sigma^\mathrm{j}} \;\middle|\; \eta^\mathrm{i}, \sigma^\mathrm{j} \,\tilde{\in}\, R\right\}, \tag{2.3.5}$$

where $\mathfrak{U}\{\cdot\}$ denotes the universal enveloping algebra. This algebra consists of all surface operators deformable to being completely contained in R.

2. Topological electric algebra

$$\mathfrak{A}_{\text{elec}}[R] := \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_\mathrm{i}}_{\eta^\mathrm{i}}, \hat{\mathrm{V}}^{\boldsymbol{v}_\mathrm{j}}_{\sigma^\mathrm{j}} \;\middle|\; \eta^\mathrm{i} \in \operatorname{coker} i^{R^\mathrm{c}}_p,\ \sigma^\mathrm{j} \in \operatorname{coker} i^{R^\mathrm{c}}_{d-p-1}\right\} \tag{2.3.6}$$

$$\equiv \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_\mathrm{i}}_{\eta^\mathrm{i}}, \hat{\mathrm{V}}^{\boldsymbol{v}_\mathrm{j}}_{\sigma^\mathrm{j}} \;\middle|\; \eta^\mathrm{i}, \sigma^\mathrm{j} \,\tilde{\notin}\, R^\mathrm{c}\right\}. \tag{2.3.7}$$

This algebra consists of all surface operators that are not deformable to being completely contained in R^c.

As we will soon explain both of these algebras have non-trivial centres, $\mathfrak{Z}[R]$, which we name the *topological magnectic centre* and *topological electric centre*, respectively. We can seek centreless operator algebras by either systematically removing operators in $\mathfrak{Z}[R]$ from $\mathfrak{A}[R]$ or by systematically adding operators to $\mathfrak{A}[R]$ that do not commute with operators in $\mathfrak{Z}[R]$. Doing so here results in two centreless algebras that we will call the *austere algebra* (in the case of reduction) and the *greedy algebra* (in the case of extension). There is a cost to being centreless: these two algebras have more tenuous relationships to their underlying subregion. Additionally we will show that ground states have trivial entanglement with respect to these algebras.

These different choices of subregion algebras are illustrated in figure 2.1.

The topological magnetic algebra

Let us begin the discussion with the topological magentic algebra

$$\mathfrak{A}_{\text{mag}}[R] := \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_\mathrm{i}}_{\eta^\mathrm{i}}, \hat{\mathrm{V}}^{\boldsymbol{v}_\mathrm{j}}_{\sigma^\mathrm{j}} \;\middle|\; \eta^\mathrm{i}, \sigma^\mathrm{j} \,\tilde{\in}\, R\right\}, \tag{2.3.8}$$

The algebra associated with R^c is the commutant of $\mathfrak{A}_{\text{mag}}[R]$. It is clear that the operators that can commute with $\mathfrak{A}_{\text{mag}}[R]$ are precisely those that cannot link homology cycles in R. This is equivalent to the following.

$$\left(\mathfrak{A}_{\text{mag}}[R]\right)^\mathrm{c} = \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_\mathrm{i}}_{\eta^\mathrm{i}}, \hat{\mathrm{V}}^{\boldsymbol{v}_\mathrm{j}}_{\sigma^\mathrm{j}} \;\middle|\; \eta^\mathrm{i}, \sigma^\mathrm{j} \,\tilde{\in}\, R^\mathrm{c}\right\} \equiv \mathfrak{A}_{\text{mag}}[R^\mathrm{c}]. \tag{2.3.9}$$

The proof of this claim is given in appendix A.2.

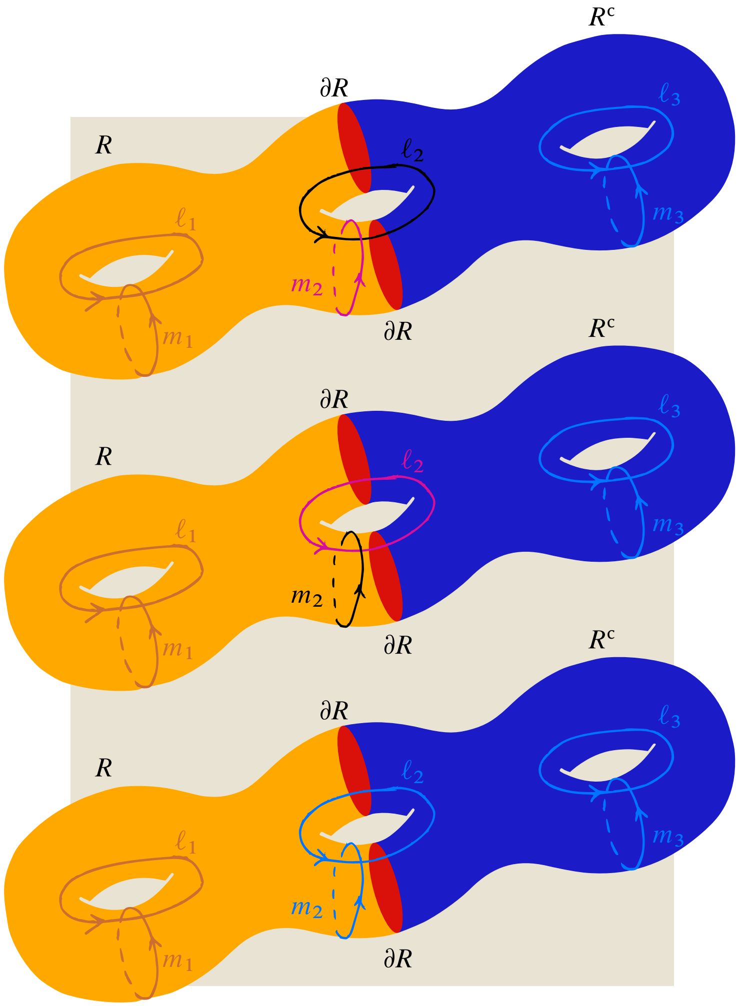

Figure 2.1: *The algebra on a genus 3 surface, generated by a basis of operators along longitude and meridian cycles, $\{\ell_i, m_i\}_{i=1,2,3}$. To the region, R, depicted in yellow, we associate an algebra $\mathfrak{A}[R]$ generated by cycles depicted in orange and purple. The commutant, $\mathfrak{A}[R^c]$, is generated by cycles depicted in blue and purple. The centre, $\mathfrak{Z}[R]$, is generated by cycles depicted in purple. Cycles generated neither in $\mathfrak{A}[R]$ nor $\mathfrak{A}[R^c]$ are depicted in* ***black****. (Top) The topological magnetic algebra, $\mathfrak{A}_{mag}[R]$. (Middle) The topological electric algebra, $\mathfrak{A}_{elec}[R]$. (Bottom) The austere algebra, $\mathfrak{A}_{aus}[R]$, and the greedy algebra, $\mathfrak{A}_{greedy}[R^c]$.*

As alluded to above, this algebra has a centre, $\mathfrak{Z}_{\text{mag}}$, which is generated by surface operators lying within the entangling surface, ∂R, itself:

$$\mathfrak{Z}_{\text{mag}}[R] := \mathfrak{A}_{\text{mag}}[R] \cap \mathfrak{A}_{\text{mag}}[R^c] = \mathfrak{U}\left\{ \hat{\mathrm{W}}^{\boldsymbol{w}_{\mathrm{i}}}_{\eta^{\mathrm{i}}}, \hat{\mathrm{V}}^{\boldsymbol{v}_{\mathrm{j}}}_{\sigma^{\mathrm{j}}} \;\middle|\; \eta^{\mathrm{i}}, \sigma^{\mathrm{j}} \mathrel{\tilde{\in}} \partial R \right\}. \tag{2.3.10}$$

The heuristic argument is simple: $\mathfrak{Z}_{\text{mag}}[R] \subset \mathfrak{A}_{\text{mag}}[R]$ since any cycle, $\eta^{\mathsf{i}} \,\tilde{\in}\, \partial R$ or $\sigma^{\mathsf{j}} \,\tilde{\in}\, \partial R$, can be deformed "slightly inward" along the flow of an inward-pointing normal vector to be contained completely in R. Similarly by flowing in the other direction, "slightly outward," any cycle can be contained completely in R^{c} and so $\mathfrak{Z}_{\text{mag}}[R] \subset \mathfrak{A}_{\text{mag}}[R^{\mathrm{c}}]$. Perhaps one might question if operators in $\mathfrak{Z}_{\text{mag}}$ actually commute with themselves (as required for $\mathfrak{Z}_{\text{mag}}$ to be a centre). A potential puzzle arises because p-cycles and $(d-p-1)$-cycles have no notion of intersection numbers as defined intrinsically on ∂R: the intersection of a p-cycle and a $(d-p-1)$-cycle on a $d-2$ dimension manifold is not a collection of points, but instead itself a 1-dimensional manifold. The key here is the algebra $\mathfrak{Z}_{\text{mag}}$ is not defined intrinsically on ∂R but instead up to homotopy in $\Sigma \supset \partial R$. It is then clear that all $(d-p-1)$-cycles, $\sigma^{\mathsf{j}} \,\tilde{\in}\, \partial R$, can be deformed to have zero linking number with $\eta^{\mathsf{i}} \,\tilde{\in}\, \partial R$ by evolving them slightly along an outward pointing normal vector.

We call $\mathfrak{Z}_{\text{mag}}$ the *topological magnetic centre* because of its similarity to the magnetic centre of 2+1 dimensional lattice gauge theories [183, 190], generated by line and/or ribbon operators wrapping ∂R. Here the interplay of the dimensionality, d, with the degrees of the gauge fields, p and $d-p-1$, allow for a richer flavour of magnetic centre, generated by topological operators of different dimension. Additionally, the topological magnetic centre defined here is sensitive to the bulk topology while the magnetic centre appearing in [183] is only sensitive to the intrinsic topology of ∂R. Namely, an operator can only appear in $\mathfrak{Z}_{\text{mag}}$ if its defining cycle is also non-trivial as a cycle on Σ.

In the interest of counting how many basis operators generate $\mathfrak{Z}_{\text{mag}}$, it will be useful to formalise the above as follows. The dimension of the magnetic centre will be the sum of the number of p-cycle surface operators, $\hat{\mathrm{W}}^{\boldsymbol{w}}_{\eta}$, and $(d-p-1)$-cycle surface operators, $\hat{\mathrm{V}}^{\boldsymbol{v}}_{\sigma}$, spanning $\mathfrak{Z}_{\text{mag}}$:

$$\left|\mathfrak{Z}_{\text{mag}}\right| = |\det \mathbb{K}|^{\mathrm{h}^{p}_{\text{mag}} + \mathrm{h}^{d-p-1}_{\text{mag}}}, \tag{2.3.11}$$

where $\mathrm{h}^{k}_{\text{mag}}$ is defined in the following way. A surface operator, $\hat{\mathrm{W}}^{\boldsymbol{w}}_{\eta}$, in the magnetic centre must be supported on a p-cycle that lies in the intersection of the figures of the push-forward maps i^{R}_{p} and $i^{R^{\mathrm{c}}}_{p}$. Using the push-out square,

$$\begin{array}{ccc} \partial R & \xrightarrow{\;j^{\partial R}\;} & R \\ {\scriptstyle j^{\partial R^{\mathrm{c}}}}\big\downarrow & & \big\downarrow{\scriptstyle i^{R}} \\ R^{\mathrm{c}} & \xrightarrow[\;i^{R^{\mathrm{c}}}\;]{} & \Sigma, \end{array}$$

we see that the corresponding p-cycle then has to lie in the image of push-forward map $\left(i^{R} \circ j^{\partial R}\right)_{p} \cong \left(i^{R^{\mathrm{c}}} \circ j^{\partial R^{\mathrm{c}}}\right)_{p}$. Utilising the associated long-exact sequence we prove in appendix A.2

$$\mathrm{h}^{p}_{\mathrm{mag}} = \sum_{n=0}^{p-1} (-1)^{p-1-n} \mathrm{b}_n(\partial R) + \sum_{n=0}^{p} (-1)^{p-n} \left(\mathrm{b}_n(\Sigma) - \dim \mathrm{H}_n(\Sigma, \partial R)\right), \tag{2.3.12}$$

and similarly for $\mathrm{h}^{d-p-1}_{\mathrm{mag}}$ via the replacement $p \to (d-p-1)$. We remind the reader that $\mathrm{b}_n(\cdot)$ is the n^{th} Betti number. Let us point out two broad features of (2.3.12). Firstly we have the alternating sum of Betti numbers intrinsic to ∂R; as we will show later this will give contributions to the entropy analogous to those found in [188]. Secondly, however, we find an interesting dependence on bulk topology relative to how ∂R is embedded in Σ. Although perhaps initially surprising, we can easily argue why we expect this dependence on the bulk topology to show up: $\mathfrak{A}_{\mathrm{mag}}$ is defined with respect to homotopy equivalence within Σ. If $\mathrm{h}^{p}_{\mathrm{mag}}$ only detected intrinsic topology of ∂R it could easily[12] count more operators in $\mathfrak{Z}_{\mathrm{mag}}[R]$ than actually exist in $\mathfrak{A}_{\mathrm{mag}}[R]$, or even in $\mathfrak{A}[\Sigma]$! These additional bulk terms are then crucial for ensuring that this counting makes sense.

Summing $\mathrm{h}^{p}_{\mathrm{mag}}$ and $\mathrm{h}^{d-p-1}_{\mathrm{mag}}$ the total dimension of $\mathfrak{Z}_{\mathrm{mag}}$ can be simplified utilising the long exact sequence (see appendix A.2) to

$$\begin{aligned} \log\left|\mathfrak{Z}_{\mathrm{mag}}\right| = \Bigg[& 2 \sum_{n=0}^{p-1} (-1)^{p-1-n} \mathrm{b}_n(\partial R) + \big(\mathrm{b}_p(\Sigma) - \dim \mathrm{H}_p(\Sigma, \partial R)\big) \\ & + (-1)^{d-p-1} \big(\dim \mathrm{H}_{d-1}(\Sigma, \partial R) - \dim \mathrm{H}_0(\Sigma, \partial R)\big) \Bigg] \log|\det \mathbb{K}|. \end{aligned} \tag{2.3.13}$$

Above, all of the bulk dependence has been isolated to dimensions of p^{th} absolute and relative homologies, plus the additional, p-independent term: a potential mismatch between the bottom and top relative homologies.

The topological electric algebra

In contrast with the topological magnetic algebra, whose centre is generated by operators "wrapping" the entangling surface, we will pick the topological electric algebra such to be such that its centre is generated by operators "piercing" the entangling surface. Specifically, for a region, R, we define

$$\mathfrak{A}_{\mathrm{elec}}[R] := \mathfrak{U}\left\{ \hat{\mathrm{W}}^{\boldsymbol{w}_{\mathrm{i}}}_{\eta^{\mathrm{i}}}, \hat{\mathrm{V}}^{\boldsymbol{v}^{\mathrm{j}}}_{\sigma^{\mathrm{j}}} \;\middle|\; \eta^{\mathrm{i}}, \sigma^{\mathrm{j}} \not\precsim R^{\mathrm{c}} \right\}. \tag{2.3.14}$$

In words, $\mathfrak{A}_{\mathrm{elec}}[R]$ is generated by operators that cannot be deformed to being contained completely in R^{c}. The algebra associated to R^{c}, the commutant of $\mathfrak{A}_{\mathrm{elec}}[R]$, is

[12] Easy examples are cooked up when Σ is topologically trivial, e.g. a $(d-1)$-sphere, $\mathbb{S}^{d-1}$.

generated by all operators that do not link with any cycle that cannot be deformed to be contained in R^{c}. We claim that this is, in fact, generated by operators that cannot be deformed to be contained in R:

$$(\mathfrak{A}_{\text{elec}}[R])^{\mathrm{c}} = \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_{\mathrm{i}}}_{\eta^{\mathrm{i}}}, \hat{\mathrm{V}}^{\boldsymbol{v}_{\mathrm{j}}}_{\sigma^{\mathrm{j}}} \;\middle|\; \eta^{\mathrm{i}}, \sigma^{\mathrm{j}} \not\precsim R\right\} = \mathfrak{A}_{\text{elec}}[R^{\mathrm{c}}]. \tag{2.3.15}$$

The proof of this claim is given in appendix A.2.

In this case, the centre is then given by cycles of Σ, that cannot be deformed to be contained completely in R nor R^{c}:

$$\begin{aligned}\mathfrak{Z}_{\text{elec}}[R] &= \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_{\mathrm{i}}}_{\eta^{\mathrm{i}}}, \hat{\mathrm{V}}^{\boldsymbol{v}_{\mathrm{j}}}_{\sigma^{\mathrm{j}}} \;\middle|\; \eta^{\mathrm{i}}, \sigma^{\mathrm{j}} \not\precsim R \text{ and } \not\precsim R^{\mathrm{c}}\right\} \\ &= \mathfrak{U}\Bigg\{\hat{\mathrm{W}}^{\boldsymbol{w}_{\mathrm{i}}}_{\eta^{i}}, \hat{\mathrm{V}}^{\boldsymbol{v}_{\mathrm{j}}}_{\sigma^{\mathrm{j}}} \Bigg| \eta^{\mathrm{i}} \in \operatorname{coker} i_p^R \cap \operatorname{coker} i_p^{R^{\mathrm{c}}} \text{ and} \\ &\qquad\qquad \sigma^{\mathrm{j}} \in \operatorname{coker} i^R_{d-p-1} \cap \operatorname{coker} i^{R^{\mathrm{c}}}_{d-p-1}\Bigg\}. \end{aligned} \tag{2.3.16}$$

$\mathfrak{Z}_{\text{elec}}[R]$ are topological surface operators that *must* cross ∂R non-trivially. We name this centre the *topological electric centre* on account of its similarity to the electric centre of lattice gauge theories generated by link operators emanating transversely from the entangling surface [183]. However let us caution that this is a somewhat shallow comparison: the electric centre typically discussed in lattice gauge theories is microscopic, being given by operators acting on all links intersecting ∂R in a UV lattice realisation of a topological phase.[13] Our topological electric centre is an extreme coarse-graining of this, generated by a handful of topological surface operators that are only defined up to homotopy.

With respect to counting the number of surface operators generating $\mathfrak{Z}_{\text{elec}}$,

$$|\mathfrak{Z}_{\text{elec}}| = |\det \mathbb{K}|^{\mathrm{h}^{p}_{\text{elec}} + \mathrm{h}^{d-p-1}_{\text{elec}}}, \tag{2.3.17}$$

we show in appendix appendix A.2 that

$$\mathrm{h}^{p}_{\text{elec}} = \mathrm{h}^{d-p-1}_{\text{mag}}, \qquad \mathrm{h}^{d-p-1}_{\text{elec}} = \mathrm{h}^{p}_{\text{mag}}. \tag{2.3.18}$$

The heuristic argument for this follows: a p-cycle surface operator in $\mathfrak{Z}_{\text{elec}}$ by definition can't be deformable to either the interiors of R or R^{c} and so must wrap a basis $(d-p-1)$-cycle intrinsic to ∂R. This cycle is precisely where one would put a $(d-p-1)$-cycle surface operator lying in $\mathfrak{Z}_{\text{mag}}$. Consequently

$$|\mathfrak{Z}_{\text{elec}}| = |\mathfrak{Z}_{\text{mag}}|. \tag{2.3.19}$$

[13] As emphasised in [186, 187], the electric centre of lattice gauge theories shares many features with extending the Hilbert space with edge-mode degrees of freedom. We discuss the extended Hilbert space of BF theory in chapter 3.

Subregion electric-magnetic duality

The equivalence of the counting $|\mathfrak{Z}_{\mathrm{elec}}| = |\mathfrak{Z}_{\mathrm{mag}}|$ is a particular instance of a refinement of the electric-magnetic duality described in section 2.2 applied to the region R and its operator algebras. More specifically, in appendix A.2 we prove that the intersection pairing, $\mathbb{L}$, induces a one-to-one correspondence between

$$\operatorname{coker} i_p^{R^c} \leftrightarrow \operatorname{im} i_{d-p-1}^{R} \quad \text{as well as} \quad \operatorname{coker} i_{d-p-1}^{R^c} \leftrightarrow \operatorname{im} i_p^{R}. \tag{2.3.20}$$

This then implies a one-to-one correspondence between operators generating $\mathfrak{A}_{\mathrm{elec}}[R]$ and $\mathfrak{A}_{\mathrm{mag}}[R]$. We refer to this correspondence as *subregion electric-magnetic duality.*

2.3.2 Decomposing the Hilbert space

The existence of a centre prohibits the tensor factorisation of the global Hilbert space, $\mathscr{H}_\Sigma$, into Hilbert spaces corresponding to R and R^c in the following way. We will illustrate this first using the topological magnetic algebra. Currently we are organising $\mathscr{H}_\Sigma$ by the eigenvectors of $\hat{\mathrm{W}}^{\mathfrak{w}}$, $|\mathfrak{v}\rangle$, which are created by acting $\hat{\mathrm{V}}^{\mathfrak{v}}$ on the condensate. Given this, we ask: "Can we partition $|\mathfrak{v}\rangle$ into the eigenvalues of $\hat{\mathrm{W}}^{\mathfrak{w}} \in \mathfrak{A}_{\mathrm{mag}}[R]$ and the eigenvalues of $\hat{\mathrm{W}}^{\mathfrak{w}} \in \mathfrak{A}_{\mathrm{mag}}[R^c]$?"

$$|\mathfrak{v}\rangle \overset{?}{=} \left|\left\{\mathfrak{v}^{R}\right\}, \left\{\mathfrak{v}^{R^c}\right\}\right\rangle. \tag{2.3.21}$$

One obvious obstruction to the above is the possible existence of $\hat{\mathrm{W}}^{\mathfrak{w}} \in \mathfrak{Z}_{\mathrm{mag}}[R] = \mathfrak{A}_{\mathrm{mag}}[R] \cap \mathfrak{A}_{\mathrm{mag}}[R^c]$, whose eigenvalues are overcounted in the above partition. A more subtle obstruction to the above comes from $\hat{V}^{\mathfrak{v}} \in \mathfrak{Z}_{\mathrm{mag}}$, which, being deformable to either inside R or R^c, make it ambiguous if their action should shift the $\{\mathfrak{v}^{R}\}$ or the $\{\mathfrak{v}^{R^c}\}$ sets of eigenvalues. To that end let us define the set $\left\{\mathfrak{v}^{\overline{R}}\right\}$ as the eigenvalues of $\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_{\mathrm{i}}}_{\eta^{\mathrm{i}}} \,\middle|\, \eta^{\mathrm{i}} \,\tilde{\in}\, R \text{ and } \eta^{\mathrm{i}} \,\tilde{\not\in}\, R^c\right\}$ and similarly $\left\{\mathfrak{v}^{\overline{R^c}}\right\}$ the eigenvalues of $\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_{\mathrm{i}}}_{\eta^{\mathrm{i}}} \,\middle|\, \eta^{\mathrm{i}} \,\tilde{\in}\, R^c \text{ and } \eta^{\mathrm{i}} \,\tilde{\not\in}\, R\right\}$. We can label the eigenvalues of $\hat{\mathrm{W}}^{\mathfrak{w}} \in \mathfrak{Z}_{\mathrm{mag}}[R]$ as $\{\mathfrak{v}_{\perp}^{\partial R}\}$. The "perpendicular" notation here denotes that because they are measured by $\hat{\mathrm{W}}^{\mathfrak{w}}$ operators "living in ∂R" they are created by the action of $\hat{\mathrm{V}}^{\mathfrak{v}}$ operators which cross the entangling surface transversally. These $\hat{\mathrm{V}}^{\mathfrak{v}}$ operators do not belong in either $\mathfrak{A}_{\mathrm{mag}}[R]$ or $\mathfrak{A}_{\mathrm{mag}}[R^c]$ (in fact they are $(d-p-1)$-cycle surface operators generating $\mathfrak{Z}_{\mathrm{elec}}$). Lastly we will denote by $\left\{\mathfrak{v}_{\parallel}^{\partial R}\right\}$ the labels for states created by the action of $\hat{\mathrm{V}}^{\mathfrak{v}} \in \mathfrak{Z}_{\mathrm{mag}}[R]$ on the condensate. These states are eigenvectors of $\hat{\mathrm{W}}^{\mathfrak{w}}$ which also cross the entangling surface transversally (which belong neither in $\mathfrak{A}_{\mathrm{mag}}[R]$ nor $\mathfrak{A}_{\mathrm{mag}}[R^c]$).

Thus a general state can be partitioned unambiguously as

$$|\mathfrak{v}\rangle = \left|\left\{\mathfrak{v}^{\overline{R}}\right\}, \left\{\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{v}_{\parallel}^{\partial R}\right\}, \left\{\mathfrak{v}^{\overline{R^c}}\right\}\right\rangle, \tag{2.3.22}$$

which have natural action by operators in the centre. I.e for all $\hat{\mathrm{W}}^{\mathfrak{w}} \in \mathfrak{Z}_{\mathrm{mag}}[R]$:

$$\hat{\mathrm{W}}^{\mathfrak{w}} \left|\left\{\mathfrak{v}^{\overline{R}}\right\}, \left\{\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{v}_{\parallel}^{\partial R}\right\}, \left\{\mathfrak{v}^{\overline{R^c}}\right\}\right\rangle = \mathrm{e}^{\mathrm{i}\Gamma\left(\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{w}\right)} \left|\left\{\mathfrak{v}^{\overline{R}}\right\}, \left\{\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{v}_{\parallel}^{\partial R}\right\}, \left\{\mathfrak{v}^{\overline{R^c}}\right\}\right\rangle, \tag{2.3.23}$$

while for all $\hat{V}^{\mathfrak{v}'} \in \mathfrak{Z}_{\mathrm{mag}}[R]$,

$$\hat{\mathrm{V}}^{\mathfrak{v}'} \left|\left\{\mathfrak{v}^{\overline{R}}\right\}, \left\{\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{v}_{\parallel}^{\partial R}\right\}, \left\{\mathfrak{v}^{\overline{R^c}}\right\}\right\rangle = \left|\left\{\mathfrak{v}^{\overline{R}}\right\}, \left\{\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{v}_{\parallel}^{\partial R} + \mathfrak{v}'\right\}, \left\{\mathfrak{v}^{\overline{R^c}}\right\}\right\rangle. \tag{2.3.24}$$

Under this partitioning the Hilbert space decomposes as

$$\mathcal{H}_{\Sigma} = \bigoplus_{\{\mathfrak{v}^{\partial R}\}} \mathcal{H}_{\Sigma}^{\left(\mathfrak{v}^{\partial R}\right)}, \tag{2.3.25}$$

where we've used a short-hand, $\{\mathfrak{v}^{\partial R}\} := \left\{\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{v}_{\parallel}^{\partial R}\right\}$. Each block in this decomposition admits a tensor product on $\overline{R}$ and $\overline{R^c}$:

$$\mathcal{H}_{\Sigma}^{\left(\mathfrak{v}^{\partial R}\right)} = \mathcal{H}_{\overline{R}}^{\left(\mathfrak{v}^{\partial R}\right)} \otimes \mathcal{H}_{\overline{R^c}}^{\left(\mathfrak{v}^{\partial R}\right)}. \tag{2.3.26}$$

This partitioning is useful to illustrate the obstruction of a global tensor-product decomposition of $\mathcal{H}_{\Sigma}$, however it is not computationally useful since the action of $\hat{\mathrm{V}}^{\mathfrak{v}} \in \mathfrak{Z}_{\mathrm{mag}}[R]$ moves between different blocks of (2.3.25). It is more helpful to instead diagonalise their action. Their eigenvectors are associated with $\hat{\mathrm{W}}^{\mathfrak{w}}$ operators crossing the entangling surface transversally and which $\hat{\mathrm{V}}^{\mathfrak{v}} \in \mathfrak{Z}_{\mathrm{mag}}[R]$ link non-trivially: these are the p-cycle surface operators generating $\mathfrak{Z}_{\mathrm{elec}}$. Because of this we will label the eigenvectors with the set $\{\mathfrak{w}_{\perp}^{\partial R}\}$ which amounts to a change of basis on the $\left\{\mathfrak{v}_{\parallel}^{\partial R}\right\}$ part of the state. Thus we partition our ground state as

$$|\mathfrak{v}\rangle \to \left|\left\{\mathfrak{v}^{\overline{R}}\right\}, \left\{\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{w}_{\perp}^{\partial R}\right\}, \left\{\mathfrak{v}^{\overline{R^c}}\right\}\right\rangle, \tag{2.3.27}$$

such that for all $\hat{\mathrm{V}}^{\mathfrak{v}} \in \mathfrak{Z}_{\mathrm{mag}}[R]$

$$\hat{\mathrm{V}}^{\mathfrak{v}} \left|\left\{\mathfrak{v}^{\overline{R}}\right\}, \left\{\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{w}_{\perp}^{\partial R}\right\}, \left\{\mathfrak{v}^{\overline{R^c}}\right\}\right\rangle = \mathrm{e}^{\mathrm{i}\Gamma\left(\mathfrak{v}, \mathfrak{w}_{\perp}^{\partial R}\right)} \left|\left\{\mathfrak{v}^{\overline{R}}\right\}, \left\{\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{w}_{\perp}^{\partial R}\right\}, \left\{\mathfrak{v}^{\overline{R^c}}\right\}\right\rangle. \tag{2.3.28}$$

With respect to this partitioning, we can decompose the global Hilbert space with respect to the eigenvalues of operators spanning $\mathfrak{Z}_{\mathrm{mag}}[R]$:

$$\mathcal{H}_{\Sigma} = \bigoplus_{\{\mathfrak{q}_{\perp}^{\partial R}\}} \mathcal{H}_{\Sigma}^{\left(\mathfrak{q}_{\perp}^{\partial R}\right)}, \tag{2.3.29}$$

where we now use the short-hand, $\{\mathfrak{q}_{\perp}^{\partial R}\} := \{\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{w}_{\perp}^{\partial R}\}$. Again, each block admits a tensor product on $\overline{R}$ and $\overline{R^c}$:

$$\mathcal{H}_{\Sigma}^{\left(\mathfrak{q}_{\perp}^{\partial R}\right)} = \mathcal{H}_{\overline{R}}^{\left(\mathfrak{q}_{\perp}^{\partial R}\right)} \otimes \mathcal{H}_{\overline{R^c}}^{\left(\mathfrak{q}_{\perp}^{\partial R}\right)}. \tag{2.3.30}$$

In this set-up it is easy to see from our state partitions, (2.3.27), that all $\mathcal{H}_{\overline{R}}^{(\mathfrak{q}_\perp^{\partial R})} \otimes \mathcal{H}_{\overline{R^c}}^{(\mathfrak{q}_\perp^{\partial R})}$ blocks are isomorphic. To make the following discussion notationally cleaner will often drop the $(\mathfrak{q}_\perp^{\partial R})$ superscripts from the tensor factors $\mathcal{H}_{\overline{R}}$ and $\mathcal{H}_{\overline{R^c}}$.

For the topological electric algebra, we can decompose the Hilbert space based on the eigenvalues of the operators generating the centre in a similar way. Again, the partitioning of cycles into those that can and cannot be deformed being contained in ∂R provides a useful partition of surface operator charges. It is precisely the operators contained in ∂R (i.e. those in $\mathfrak{Z}_{\text{mag}}[R]$) that can link non-trivally with operators in $\mathfrak{Z}_{\text{elec}}[R]$ and so their charges, $\left\{\mathfrak{v}_\parallel^{\partial R}, \mathfrak{w}_\parallel^{\partial R}\right\}$ are the eigenvalues of the basis generating $\mathfrak{Z}_{\text{elec}}[R]$. A generic state of $\mathcal{H}_\Sigma$ then can be partitioned as

$$|\mathfrak{v}\rangle \rightarrow \left|\left\{\mathfrak{v}^{\overline{R}}\right\}, \left\{\mathfrak{v}_\parallel^{\partial R}, \mathfrak{w}_\parallel^{\partial R}\right\}, \left\{\mathfrak{v}^{\overline{R^c}}\right\}\right\rangle, \tag{2.3.31}$$

corresponding to a Hilbert space decomposition

$$\mathcal{H}_\Sigma = \bigoplus_{\left\{\mathfrak{q}_\parallel^{\partial R}\right\}} \mathcal{H}_\Sigma^{\left(\mathfrak{q}_\parallel^{\partial R}\right)}, \tag{2.3.32}$$

where $\left\{\mathfrak{q}_\parallel^{\partial R}\right\} := \left\{\mathfrak{v}_\parallel^{\partial R}, \mathfrak{w}_\parallel^{\partial R}\right\}$ is a short-hand notation for the central eigenvalues. Each block of (2.3.32) admits a tensor product on $\overline{R}$ and $\overline{R^c}$:

$$\mathcal{H}_\Sigma^{\left(\mathfrak{q}_\parallel^{\partial R}\right)} = \mathcal{H}_{\overline{R}}^{\left(\mathfrak{q}_\parallel^{\partial R}\right)} \otimes \mathcal{H}_{\overline{R^c}}^{\left(\mathfrak{q}_\parallel^{\partial R}\right)}. \tag{2.3.33}$$

Again, since these blocks are all isomorphic, in what follows we will omit the $(\mathfrak{q}_\parallel^{\partial R})$ superscripts from the tensor factors unless clarity demands it.

2.3.3 Algebras with trivial centre

As has been mentioned at several points above, one hallmark of gauge theories is the presence of centres in algebras assigned to subregions. While this feature is generic, the assignment of an algebra, $\mathfrak{A}[R]$, to a region, R, is ultimately a choice. As such one may ask if there is enough ambiguity to assign a centreless algebra to R. To be clear, of course one can always assign, by fiat, a centreless algebra to R; for two (extreme) instances:

$$\mathfrak{A}[R] = \text{span}_{\mathbb{C}}\{1\}, \qquad \text{or} \qquad \mathfrak{A}[R] = \mathfrak{A}[\Sigma], \tag{2.3.34}$$

where $\mathfrak{A}[\Sigma]$ is the full algebra of operators[14] on $\mathcal{H}_\Sigma$. However these assignments tell us absolutely no physics about the region in which we are interested! Barring

[14]We assume that $\mathfrak{A}[\Sigma]$ has no centre. This is true for the algebra of surface operators in this section. However in lattice gauge theories, it is not typically true: there are an extensive number of Gauss operators that generate a global centre [186]. Gauge-invariant states in the BF theory (treated as an IR EFT of such a lattice gauge theory) live in a fixed eigenvalue sector of this "microscopic centre."

such "mad-man assignments," centreless subalgebras associated to a subregion R are highly non-generic in gauge theories, yet they may still exist. We can attempt to build such an subalgebra by starting from a generic (centreful) algebra, $\mathfrak{A}[R]$, and either (i) systematically excluding operators from $\mathfrak{Z}[R]$ [183] (a process we will call *reduction*), or (ii) systematically adding to $\mathfrak{A}[R]$ operators from $\mathfrak{A}[\Sigma]$ that do not commute with $\mathfrak{Z}[R]$ (what we will call *extension*). In the course of both of these processes, the resulting $\mathfrak{A}[R]$ will have a more tenuous relationship to its associated region R. However to maintain at least some degree of association between $\mathfrak{A}[R]$ and R we will focus on reductions and extensions that are minimal to ensuring $\mathfrak{A}[R]$ is centreless.

Starting then from $\mathfrak{A}_{\text{mag}}[R]$, minimal reduction and minimal extension result in two centreless algebras associated to R, respectively:

The austere algebra

For the process of reduction, we remove from $\mathfrak{A}_{\text{mag}}[R]$ the surface operators homotopic to ∂R. Since these are also the operators deformable to being contained in R^{c}, this equivalent to

$$\mathfrak{A}_{\text{aus}}[R] := \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_{\text{i}}}_{\eta^{\text{i}}}, \hat{\mathrm{V}}^{\boldsymbol{v}^{\text{j}}}_{\sigma^{\text{j}}} \;\middle|\; \eta^{\text{i}},\, \sigma^{\text{j}} \,\tilde{\in}\, R \text{ and } \eta^{\text{i}}, \sigma^{\text{j}} \,\tilde{\notin}\, R^{\text{c}}\right\} = \mathfrak{A}_{\text{mag}}[R] \cap \mathfrak{A}_{\text{elec}}[R], \quad (2.3.35)$$

i.e. $\mathfrak{A}_{\text{aus}}[R]$ is generated only by surface operators that, homotopically, *must* be completely in R.

The greedy algebra

For extension, we now add to $\mathfrak{A}_{\text{mag}}[R]$ the minimal basis of surface operators that "pierce" ∂R. These link with operators on ∂R and so prohibit them from forming a centre:

$$\mathfrak{A}_{\text{greedy}}[R] := \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_{\text{i}}}_{\eta^{\text{i}}}, \hat{\mathrm{V}}^{\boldsymbol{v}_{\text{j}}}_{\sigma^{\text{j}}} \;\middle|\; \eta^{\text{i}},\, \sigma^{\text{j}} \,\tilde{\in}\, R \text{ or } \eta^{\text{i}}, \sigma^{\text{j}} \,\tilde{\notin}\, R^{\text{c}}\right\} = \mathfrak{A}_{\text{mag}}[R] \cup \mathfrak{A}_{\text{elec}}[R], \quad (2.3.36)$$

i.e. $\mathfrak{A}_{\text{greedy}}[R]$ is generated by all surface operators that are either homotopically in R, or topologically must have "a leg in R".[15]

It is clear the same algebras can be constructed from the reduction and extension, respectively, of $\mathfrak{A}_{\text{elec}}[R]$, as well. It is also clear that

$$(\mathfrak{A}_{\text{aus}}[R])^{\text{c}} = \mathfrak{A}_{\text{greedy}}[R^{\text{c}}], \qquad \left(\mathfrak{A}_{\text{greedy}}[R]\right)^{\text{c}} = \mathfrak{A}_{\text{aus}}[R^{\text{c}}]. \qquad (2.3.37)$$

Because these two subalgebras have trivial centre, they should correspond to honest tensor factorisations of $\mathcal{H}_\Sigma$. This is indeed the case. Returning to the partition of a

[15] to be precise $\mathfrak{A}_{\text{greedy}}[R]$ should be the smallest algebra containing the two. That is the generated algebra, $\mathfrak{A}_{\text{mag}}[R] \vee \mathfrak{A}_{\text{elec}}[R] := \left(\mathfrak{A}_{\text{mag}}[R] \cup \mathfrak{A}_{\text{elec}}[R]\right)^{\text{cc}}$. But for finite-dimensional operator algebras, as is our case, it coincides with (2.3.36), since $\mathfrak{A}^{\text{cc}} = \mathfrak{A}$ is automatically guaranteed. See also [196].

general state, (2.3.22), for the austere algebra, we simply group the charges associated with central surface operators with R^{c}

$$|\mathfrak{v}\rangle := \left|\left\{\mathfrak{v}^{\overline{R}}\right\}, \left\{\mathfrak{v}^{R^{c}+}\right\}\right\rangle, \qquad \left\{\mathfrak{v}^{R^{c}+}\right\} := \left\{\mathfrak{v}_{\perp}^{\partial R}, \mathfrak{v}_{\parallel}^{\partial R}, \mathfrak{v}^{\overline{R^{c}}}\right\}, \tag{2.3.38}$$

while for the greedy algebra, we group them in with R:

$$|\mathfrak{v}\rangle := \left|\left\{\mathfrak{v}^{R+}\right\}, \left\{\mathfrak{v}^{\overline{R^{c}}}\right\}\right\rangle, \qquad \left\{\mathfrak{v}^{R+}\right\} := \left\{\mathfrak{v}^{\overline{R}}, \mathfrak{v}_{\perp}^{\partial R}, \mathfrak{v}_{\parallel}^{\partial R}\right\}. \tag{2.3.39}$$

Correspondingly the Hilbert space decomposes as

$$\mathcal{H}_{\Sigma} = \mathcal{H}_{\overline{R}} \otimes \mathcal{H}_{R^{c}+}, \quad \text{or} \quad \mathcal{H}_{\Sigma} = \mathcal{H}_{R+} \otimes \mathcal{H}_{\overline{R^{c}}}, \tag{2.3.40}$$

under the action of $\mathfrak{A}_{\text{aus}}[R]$ or $\mathfrak{A}_{\text{greedy}}[R]$, respectively. There is a cost for being centreless: our centreless algebras, $\mathfrak{A}_{\text{aus}}[R]$ and $\mathfrak{A}_{\text{greedy}}[R]$, have looser relationships to their associated region. Correspondingly, Hilbert space decompositions such as (2.3.40) may possess less information about a region than decompositions possessing a centre. Additionally, even though we have arrived at $\mathfrak{A}_{\text{aus(greedy)}}[R]$ through minimal reduction (extension), owing to the topological nature of the theory and its operator content, the dissociation from R may still be drastic indeed. For example, as illustrated in figure 2.2, there are situations (namely when either $\mathfrak{A}[R] = \mathfrak{Z}$ or $\mathfrak{A}[R^{c}] = \mathfrak{Z}$) where minimal extension or subtraction can result in everything-or-nothing centreless algebras

$$\mathfrak{A}_{\text{aus}}[R] = \text{span}_{\mathbb{C}}\{1\}, \qquad \text{or} \qquad \mathfrak{A}_{\text{greedy}}[R] = \mathfrak{A}[\Sigma], \tag{2.3.41}$$

even when R, $\mathfrak{A}_{\text{mag}}[R]$, and $\mathfrak{A}_{\text{elec}}[R]$ are non-trivial. In such situations, insofar as the entanglement entropy is concerned, the cost of being centreless is then very heavy: it is zero for *all* pure states.

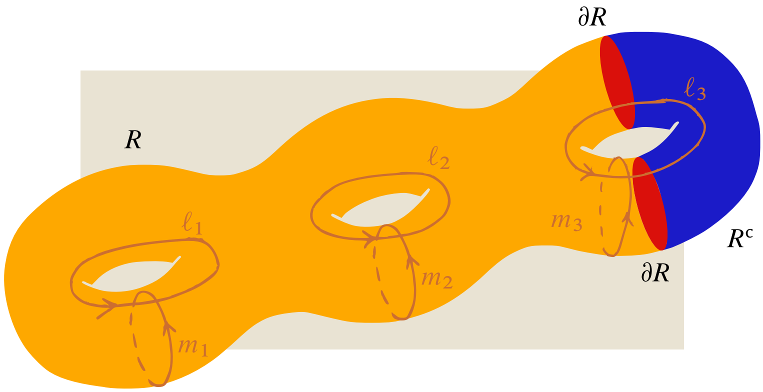

Figure 2.2: *An example of "everything-or-nothing" centreless algebras:* $\mathfrak{A}_{greedy}[R] = \mathfrak{A}[\Sigma]$, $\mathfrak{A}_{aus}[R^{c}] = \text{span}_{\mathbb{C}}\{1\}$.

2.3.4 Decomposing a state and the reduced density matrix

Now let us consider a generic state, ρ, on $\mathscr{H}_\Sigma$. With respect to either Hilbert space decomposition, (2.3.29) or (2.3.32),

$$\mathscr{H}_\Sigma = \bigoplus_{\{\mathfrak{q}\}} \mathscr{H}_\Sigma^{(\mathfrak{q})} = \bigoplus_{\{\mathfrak{q}\}} \mathscr{H}_{\overline{R}}^{(\mathfrak{q})} \otimes \mathscr{H}_{\overline{R^c}}^{(\mathfrak{q})}, \qquad \mathfrak{q} \in \left\{\mathfrak{q}_\perp^{\partial R}, \mathfrak{q}_\parallel^{\partial R}\right\}, \tag{2.3.42}$$

we can write

$$\rho = \bigoplus_{\{\mathfrak{q}\}} \lambda_{(\mathfrak{q})} \rho^{(\mathfrak{q})}, \qquad \mathfrak{q} \in \left\{\mathfrak{q}_\perp^{\partial R}, \mathfrak{q}_\parallel^{\partial R}\right\}, \tag{2.3.43}$$

which corresponds to diagonalising ρ as an operator with respect to the respective centre, $\mathfrak{Z}[R]$. The coefficients, $\lambda_{(\mathfrak{q})}$, are chosen such that $\rho^{(\mathfrak{q})}$ are normalised states on $\mathscr{H}_\Sigma^{(\mathfrak{q})}$:

$$\mathrm{Tr}_{\mathscr{H}_\Sigma^{(\mathfrak{q})}} \rho^{(\mathfrak{q})} = 1, \tag{2.3.44}$$

which requires

$$\sum_{\{\mathfrak{q}\}} \lambda_{(\mathfrak{q})} = 1. \tag{2.3.45}$$

Within each block $\mathscr{H}_\Sigma^{(\mathfrak{q})}$ we can construct the reduced density matrix with respect to the tensor product, (2.3.30) or (2.3.33), by tracing out $\mathscr{H}_{\overline{R^c}}$:

$$\rho_{\overline{R}}^{(\mathfrak{q})} = \mathrm{Tr}_{\mathscr{H}_{\overline{R^c}}} \rho^{(\mathfrak{q})}. \tag{2.3.46}$$

This results in a reduced density matrix on R which follows from the sum decomposition, (2.3.43):

$$\rho_R = \bigoplus_{\{\mathfrak{q}\}} \lambda_{(\mathfrak{q})} \left(\rho_{\overline{R}}^{(\mathfrak{q})} \otimes \tau_{\mathscr{H}_{\overline{R^c}}}\right). \tag{2.3.47}$$

where $\tau_{\mathscr{H}}$ is the trace-normalised identity matrix on a Hilbert space:

$$\tau_{\mathscr{H}} := \frac{\hat{1}_{\mathscr{H}}}{\dim \mathscr{H}}. \tag{2.3.48}$$

Alternatively, we can define this reduced density matrix as the unique unit-trace, Hermitian operator in $\mathfrak{A}[R]$ which reproduces expectation values of all other operators in $\mathfrak{A}[R]$:

$$\rho_R \in \mathfrak{A}[R], \quad \text{such that} \quad \mathrm{Tr}_{\mathscr{H}_\Sigma}\left(\rho_R \mathcal{O}_R\right) = \mathrm{Tr}_{\mathscr{H}_\Sigma}\left(\rho \mathcal{O}_R\right) \quad \forall\, \mathcal{O}_R \in \mathfrak{A}[R]. \tag{2.3.49}$$

Given the decomposition (2.3.47), we assign an entanglement entropy to ρ and $\mathfrak{A}[R]$ via

$$S_{\mathfrak{A}[R]}[\rho] := \sum_{\{\mathfrak{q}\}} \lambda_{(\mathfrak{q})} S_{\mathrm{vN}}\left[\rho_{\overline{R}}^{(\mathfrak{q})}\right] - \sum_{\{\mathfrak{q}\}} \lambda_{(\mathfrak{q})} \log \lambda_{(\mathfrak{q})}, \tag{2.3.50}$$

where

$$\mathcal{S}_{\mathrm{vN}}\left[\rho^{(\mathfrak{q})}_{\overline{R}}\right] := -\operatorname{Tr}_{\mathcal{H}_{\overline{R}}}\left(\rho^{(\mathfrak{q})}_{\overline{R}}\log\rho^{(\mathfrak{q})}_{\overline{R}}\right) \tag{2.3.51}$$

is the von Neumann entropy of the reduced density matrix in a fixed block, (2.3.46). The second term of (2.3.50) is a Shannon entropy of the probability of measuring the set $\{\mathfrak{q}\}$ of eigenvalues in the state ρ. We emphasise that, while ostensibly classical in nature, this Shannon entropy is present even for pure quantum states and is a generic feature of entanglement entropies associated to centreful algebras. It will play an important role in the following section. For a centreless algebra (such as $\mathfrak{A}_{\mathrm{aus}}[R]$ or $\mathfrak{A}_{\mathrm{greedy}}[R]$), the entanglement entropy, $\mathcal{S}_{\mathfrak{A}[R]}[\rho]$, can be defined in the usual way as the von Neumann entropy of ρ reduced on the corresponding tensor-factorisation.

2.3.5 Essential topological entanglement

We now illustrate the entanglement entropy associated to various choices of subregion algebra for a ground state, $|\psi\rangle = |\mathfrak{v}_\star\rangle$, for some fixed $\mathfrak{v}_\star$. As discussed in section 2.2, this state is prepared by the action of $\hat{\mathrm{V}}^{\mathfrak{v}_\star}$ acting on condensate. In the context of the BF path-integral, these states are natural: they are prepared by path-integral on the interior of a manifold having Σ as its boundary and with $\hat{\mathrm{V}}$ Wilson surface operators inserted. In $(2+1)$ dimensional topological phases, such states play a key role in the standard treatment of entanglement entropy (say via the replica trick, lattice regularisation, or an extended Hilbert space), yielding the celebrated "topological entanglement entropy" discussed in section 2.1. The primary upshot of this section is to show that in our setup such states provide a new smoking gun topological signature in the algebraic entropy, which we name the "essential topological entanglement." There will be two varieties: one associated to the topological magnetic algebra, and one associated to the topological electric algebra:

$$\mathcal{E}_{\mathrm{mag/elec}} := \mathcal{S}_{\mathfrak{A}_{\mathrm{mag/elec}}[R]}[|\mathfrak{v}_\star\rangle\langle\mathfrak{v}_\star|]. \tag{2.3.52}$$

We will further see that these two essential topological entanglements are related by a electric-magnetic duality.

Let us begin the discussion with the topological magnetic algebra in mind. We will illustrate the machinery in this instance; the consideration of the other subregion algebras will be wholly clear afterward. We will consider the global pure state

$$\rho = \sum_{\mathfrak{v}_1,\mathfrak{v}_2} \psi_{\mathfrak{v}_1}\psi^*_{\mathfrak{v}_2}\, |\mathfrak{v}_1\rangle\langle\mathfrak{v}_2|\,. \tag{2.3.53}$$

and reduce it down on $\mathfrak{A}_{\mathrm{mag}}[R]$ for some region R. We can express the reduced density matrix of the state, (2.3.53), as element of $\mathfrak{A}_{\mathrm{mag}}[R]$ via

$$\rho_R = \mathcal{N}^{-1}\sum_{\mathfrak{w}^R}\sum_{\mathfrak{v}^R}\sum_{\mathfrak{v}_1}\psi_{\mathfrak{v}_1}\psi^*_{\mathfrak{v}_1-\mathfrak{v}^R}\mathrm{e}^{-\mathrm{i}\Gamma(\mathfrak{v}_1,\mathfrak{w}^R)}\,\hat{\mathrm{W}}^{\mathfrak{w}^R}\,\hat{\mathrm{V}}^{\mathfrak{v}^R}. \tag{2.3.54}$$

where $\mathcal{N} = \dim \mathcal{H}_\Sigma = |\det \mathbb{K}|^{\mathrm{b}_p(\Sigma)}$ ensures that ρ_R is trace normalised over $\mathcal{H}_\Sigma$. See appendix A.3 for details on this decomposition. It will be useful to split this normalisation into

$$\mathcal{N} = \mathcal{N}_{\overline{R}}\, \mathcal{N}_{\partial R}\, \mathcal{N}_{\overline{R^c}} = |\det \mathbb{K}|^{\mathrm{h}_{\overline{R}}}\, |\det \mathbb{K}|^{\mathrm{h}_{\partial R}}\, |\det \mathbb{K}|^{\mathrm{h}_{\overline{R^c}}}, \tag{2.3.55}$$

where $\mathrm{h}_{\overline{R}}$ is the number of independent $(d-p-1)$-cycles deformable to R but not to R^c (i.e. the number of $(d-p-1)$-cycles spanning the austere algebra for R). And vice-versa for $\mathrm{h}_{\overline{R^c}}$. Thus $\mathcal{N}_{\overline{R}}$ and $\mathcal{N}_{\overline{R^c}}$ are simply the dimensions of $\mathcal{H}_{\overline{R}}$ and $\mathcal{H}_{\overline{R^c}}$, respectively, in the decomposition (2.3.30). Above, $\mathrm{h}_{\partial R}$ is the number of $(d-p-1)$-cycles piercing ∂R, $\mathrm{h}^{d-p-1}_{\mathrm{elec}}$, (counting the independent $\{\mathfrak{v}^{\partial R}_\perp\}$ in the decomposition (2.3.27)) plus $\mathrm{h}^{d-p-1}_{\mathrm{mag}}$, the number of $(d-p-1)$-cycles deformable to ∂R. As discussed above in subsection 2.3.2, the latter of these is equal to $\mathrm{h}^{p}_{\mathrm{elec}}$, and counts the independent $\{\mathfrak{w}^{\partial R}_\perp\}$ in (2.3.27). Thus

$$\mathrm{h}^{\partial R} = \mathrm{h}^{p}_{\mathrm{elec}} + \mathrm{h}^{d-p-1}_{\mathrm{elec}} = \mathrm{h}^{p}_{\mathrm{mag}} + \mathrm{h}^{d-p-1}_{\mathrm{mag}}. \tag{2.3.56}$$

In writing (2.3.54), some of the $\hat{\mathrm{W}}$'s and $\hat{\mathrm{V}}$'s belong to the centre, $\mathfrak{Z}_{\mathrm{mag}}[R]$. It will be useful to separate them off as

$$\begin{aligned} \hat{\mathrm{W}}^{\mathfrak{w}^R} &= \prod_{\eta^{\mathrm{j}} \tilde{\in} R, \tilde{\notin} R^c} \hat{\mathrm{W}}^{\boldsymbol{w}^R_{\mathrm{j}}}_{\eta^{\mathrm{j}}} \prod_{\eta^{\mathrm{j}} \tilde{\in} R, \tilde{\in} R^c} \hat{\mathrm{W}}^{\boldsymbol{w}^R_{\mathrm{j}}}_{\eta^{\mathrm{j}}} \equiv \hat{\mathrm{W}}^{\mathfrak{w}^{\overline{R}}} \hat{\mathrm{W}}^{\mathfrak{w}^{\partial R}_{\parallel}} \\ \hat{\mathrm{V}}^{\mathfrak{v}^R} &= \prod_{\sigma^{\mathrm{i}} \tilde{\in} R, \tilde{\notin} R^c} \hat{\mathrm{V}}^{\boldsymbol{v}^R_{\mathrm{i}}}_{\sigma^{\mathrm{i}}} \prod_{\sigma^{\mathrm{i}} \tilde{\in} R, \tilde{\in} R^c} \hat{\mathrm{V}}^{\boldsymbol{v}^R_{\mathrm{i}}}_{\sigma^{\mathrm{i}}} \equiv \hat{\mathrm{V}}^{\mathfrak{v}^{\overline{R}}} \hat{\mathrm{V}}^{\mathfrak{v}^{\partial R}_{\parallel}}. \end{aligned} \tag{2.3.57}$$

Diagonalising these central elements, we can write them in terms of their eigenvalues as

$$\hat{\mathrm{W}}^{\mathfrak{w}^{\partial R}_{\parallel}} = \bigoplus_{\mathfrak{v}^{\partial R}_{\perp}} \mathrm{e}^{\mathrm{i}\Gamma\left(\mathfrak{v}^{\partial R}_{\perp}, \mathfrak{w}^{\partial R}_{\parallel}\right)}, \qquad \hat{\mathrm{V}}^{\mathfrak{v}^{\partial R}_{\parallel}} = \bigoplus_{\mathfrak{w}^{\partial R}_{\perp}} \mathrm{e}^{\mathrm{i}\Gamma\left(\mathfrak{v}^{\partial R}_{\parallel}, \mathfrak{w}^{\partial R}_{\perp}\right)}. \tag{2.3.58}$$

This leads to a reduced density matrix in block diagonal form, as in (2.3.47):

$$\rho_R = \bigoplus_{\{\mathfrak{v}^{\partial R}_{\perp}, \mathfrak{w}^{\partial R}_{\perp}\}} \lambda_{(\mathfrak{v}^{\partial R}_{\perp}, \mathfrak{w}^{\partial R}_{\perp})} \left(\rho_{\overline{R}}^{(\mathfrak{v}^{\partial R}_{\perp}, \mathfrak{w}^{\partial R}_{\perp})} \otimes \tau_{\mathcal{H}_{\overline{R^c}}} \right), \tag{2.3.59}$$

with

$$\begin{aligned} \lambda_{(\mathfrak{v}^{\partial R}_{\perp}, \mathfrak{w}^{\partial R}_{\perp})} &= \mathcal{N}_{\partial R}^{-1} \sum_{\mathfrak{w}^{\partial R}_{\parallel}} \sum_{\mathfrak{v}^{\partial R}_{\parallel}} \sum_{\mathfrak{v}_1} \psi^*_{\mathfrak{v}_1 - \mathfrak{v}^{\partial R}_{\parallel}} \psi_{\mathfrak{v}_1} \mathrm{e}^{-\mathrm{i}\Gamma\left(\mathfrak{v}_1 - \mathfrak{v}^{\partial R}_{\perp}, \mathfrak{w}^{\partial R}_{\parallel}\right) + \mathrm{i}\Gamma\left(\mathfrak{v}^{\partial R}_{\parallel}, \mathfrak{w}^{\partial R}_{\perp}\right)} \\ \rho_{\overline{R}}^{(\mathfrak{v}^{\partial R}_{\perp}, \mathfrak{w}^{\partial R}_{\perp})} &= \mathcal{N}_{\overline{R}}^{-1} \sum_{\mathfrak{w}^{\overline{R}}} \sum_{\mathfrak{v}^{\overline{R}}} \sum_{\mathfrak{v}_1} \psi^*_{\mathfrak{v}_1 - \mathfrak{v}^{\overline{R}}} \psi_{\mathfrak{v}_1} \mathrm{e}^{-\mathrm{i}\Gamma\left(\mathfrak{v}_1, \mathfrak{w}^{\overline{R}}\right)} \hat{\mathrm{W}}^{\mathfrak{w}^{\overline{R}}} \hat{\mathrm{V}}^{\mathfrak{v}^{\overline{R}}}, \end{aligned} \tag{2.3.60}$$

and $\tau_{\mathcal{H}_{\overline{R^c}}}$ is the trace-normalised unit operator of $\mathcal{H}_{\overline{R^c}}$ with respect to the decomposition (2.3.30). We can easily verify that

$$\sum_{\mathfrak{v}_\perp^{\partial R}} \sum_{\mathfrak{w}_\perp^{\partial R}} \lambda_{(\mathfrak{v}_\perp^{\partial R}, \mathfrak{w}_\perp^{\partial R})} = \sum_{\mathfrak{v}_1} \psi_{\mathfrak{v}_1}^* \psi_{\mathfrak{v}_1} = 1, \tag{2.3.61}$$

because the sums over $\mathfrak{w}_\perp^{\partial R}$ and $\mathfrak{v}_\perp^{\partial R}$ enforce delta functions on $\mathfrak{v}_\parallel^{\partial R}$ and $\mathfrak{w}_\parallel^{\partial R}$, respectively.

We now will take our pure state to be a ground state, $|\psi\rangle = |\mathfrak{v}_\star\rangle$ for fixed $\mathfrak{v}_\star$, which sets $\psi_{\mathfrak{v}} = \boldsymbol{\delta}_{\mathfrak{v}-\mathfrak{v}_\star}$ (defined in (2.2.41)). With respect to the decomposition (2.3.30) $|\mathfrak{v}_\star\rangle$ projected onto each central eigenspace remains a product state on $\mathcal{H}_{\overline{R}} \otimes \mathcal{H}_{\overline{R^c}}$ and so $\rho_{\overline{R}}^{(\mathfrak{v}_\perp^{\partial R}, \mathfrak{w}_\perp^{\partial R})}$ is pure on $\mathcal{H}_{\overline{R}}$:

$$\rho_{\overline{R}}^{(\mathfrak{v}_\perp^{\partial R}, \mathfrak{w}_\perp^{\partial R})} = \mathcal{N}_{\overline{R}}^{-1} \sum_{\mathfrak{w}^{\overline{R}}} \mathrm{e}^{-\mathrm{i}\Gamma\left(\mathfrak{v}_\star, \mathfrak{w}^{\overline{R}}\right)} \hat{\mathrm{W}}^{\mathfrak{w}^{\overline{R}}} = \left|\mathfrak{v}_\star^{\overline{R}}\middle\rangle\!\middle\langle\mathfrak{v}_\star^{\overline{R}}\right|. \tag{2.3.62}$$

As such its von Neumann entropy vanishes, $\mathcal{S}_{\mathrm{vN}}\left[\rho_{\overline{R}}^{(\mathfrak{q}_\perp^{\partial R})}\right] = 0$. The entanglement entropy of $|\mathfrak{v}_\star\rangle$ then comes entirely from the Shannon entropy of the distribution, $\left\{\lambda_{(\mathfrak{q}_\perp^{\partial R})}\right\}$, which take the form

$$\lambda_{(\mathfrak{v}_\perp^{\partial R}, \mathfrak{w}_\perp^{\partial R})} = \mathcal{N}_{\partial R}^{-1} \sum_{\mathfrak{w}_\parallel^{\partial R}} \mathrm{e}^{-\mathrm{i}\Gamma\left(\mathfrak{v}_\star - \mathfrak{v}_\perp^{\partial R}, \mathfrak{w}_\parallel^{\partial R}\right)} = \boldsymbol{\delta}_{\mathfrak{v}_{\star\perp}^{\partial R} - \mathfrak{v}_\perp^{\partial R}} |\det \mathbb{K}|^{-\mathrm{h}_{\mathrm{mag}}^{d-p-1}} ; \tag{2.3.63}$$

that is, their support is isolated to the $\mathfrak{v}_\perp^{\partial R}$ sector determined by the original ground state, and is maximally mixed on the $\mathfrak{w}_\perp^{\partial R}$ eigenvalues. The algebraic entanglement entropy associated to $\mathfrak{A}_{\mathrm{mag}}[R]$ of $|\mathfrak{v}_\star\rangle$ is then determines the essential topological entanglement as

$$\mathcal{E}_{\mathrm{mag}} = -\sum_{\{\mathfrak{v}_\perp^{\partial R}, \mathfrak{w}_\perp^{\partial R}\}} \lambda_{(\mathfrak{v}_\perp^{\partial R}, \mathfrak{w}_\perp^{\partial R})} \log \lambda_{(\mathfrak{v}_\perp^{\partial R}, \mathfrak{w}_\perp^{\partial R})} = \mathrm{h}_{\mathrm{mag}}^{d-p-1} \log |\det \mathbb{K}|, \tag{2.3.64}$$

where we remind the reader

$$\begin{aligned} \mathrm{h}_{\mathrm{mag}}^{d-p-1} = & \sum_{n=0}^{d-p-2} (-1)^{d-p-2-n} \mathrm{b}_n(\partial R) \\ & + \sum_{n=0}^{d-p-1} (-1)^{d-p-1-n} \left(\mathrm{b}_n(\Sigma) - \dim \mathrm{H}_n(\Sigma, \partial R)\right). \end{aligned} \tag{2.3.65}$$

Let us dissect this result. We make note of several features of (2.3.64).

- $\mathcal{E}_{\text{mag}}$ probes non-local operator statistics through $|\det \mathbb{K}|$ which can be regarded as a "total quantum dimension" in this theory.

- $\mathcal{E}_{\text{mag}}$ is independent of the ground state, $|\mathfrak{v}_\star\rangle$, in which it is evaluated. This is in keeping with this being an abelian topological phase. Since operator fusion is unique in $\mathfrak{A}[\Sigma]$, the "quantum dimension," $\mathcal{D}_{\mathfrak{v}_\star}$, of the $\hat{\mathrm{V}}^{\mathfrak{v}_\star}$ building $|\mathfrak{v}_\star\rangle$ from the condensate is unity and so any possible contribution going as $\log \mathcal{D}_{\mathfrak{v}_\star}$ will vanish.

- $\mathcal{E}_{\text{mag}}$ probes topological features intrinsic to ∂R: the alternating sum of Betti numbers on ∂R. This contribution mimics a proposed (negative) correction to the area law in higher-dimensional topological order described by membrane-net models [188]. Here we find this term contributes positively and appears without an area law.

- $\mathcal{E}_{\text{mag}}$ possesses additional terms that depend both on the topology of Σ itself, as well as the relative homologies of Σ and ∂R. Thus, the essential topological entanglement is sensitive to more than the topology of ∂R itself, but also how ∂R is embedded into Σ. As we argued in subsection 2.3.1, this has to be the case since the operators counted by $\mathcal{E}_{\text{mag}}$ have to descend from non-trivial topological operators on Σ.

- $\mathcal{E}_{\text{mag}}$ comes entirely from the Shannon contribution to $\mathcal{S}_{\mathfrak{A}_{\text{mag}}}[|\mathfrak{v}_\star\rangle\langle\mathfrak{v}_\star|]$, while the contribution from the sum of von Neumann entropies exactly vanish. It was argued that this latter contribution corresponds to the distillable entanglement in gauge theories [185, 197]. That $\mathcal{E}_{\text{mag}}$ entirely enters through the Shannon term is in keeping with its essential non-locality: it cannot be distilled into Bell pairs by local operations.

We can repeat this same exercise for the entanglement entropy associated to $\mathfrak{A}_{\text{elec}}[R]$. Running through this process one finds for a pure state the reduced density matrix decomposes in a way wholly similar to (2.3.59):

$$\rho_R = \bigoplus_{\left\{\mathfrak{v}_\parallel^{\partial R},\mathfrak{w}_\parallel^{\partial R}\right\}} \lambda_{(\mathfrak{v}_\parallel^{\partial R},\mathfrak{w}_\parallel^{\partial R})}\left(\rho_{\overline{R}}^{(\mathfrak{v}_\parallel^{\partial R},\mathfrak{w}_\parallel^{\partial R})}\otimes \tau_{\mathcal{H}_{\overline{R^c}}}\right), \tag{2.3.66}$$

with

$$\begin{aligned}
\lambda_{(\mathfrak{v}_\parallel^{\partial R},\mathfrak{w}_\parallel^{\partial R})} &= \mathcal{N}_{\partial R}^{-1}\sum_{\mathfrak{w}_\perp^{\partial R}}\sum_{\mathfrak{v}_\perp^{\partial R}}\sum_{\mathfrak{v}_1}\psi^*_{\mathfrak{v}_1-\mathfrak{v}_\perp^{\partial R}}\psi_{\mathfrak{v}_1}\,\mathrm{e}^{-\mathrm{i}\Gamma\left(\mathfrak{v}_1-\mathfrak{v}_\parallel^{\partial R},\mathfrak{w}_\perp^{\partial R}\right)+\mathrm{i}\Gamma\left(\mathfrak{v}_\perp^{\partial R},\mathfrak{w}_\parallel^{\partial R}\right)} \\
\rho_R^{(\mathfrak{v}_\parallel^{\partial R},\mathfrak{w}_\parallel^{\partial R})} &= \mathcal{N}_{\overline{R}}^{-1}\sum_{\mathfrak{w}^{\overline{R}}}\sum_{\mathfrak{v}^{\overline{R}}}\sum_{\mathfrak{v}_1}\psi^*_{\mathfrak{v}_1-\mathfrak{v}^{\overline{R}}}\psi_{\mathfrak{v}_1}\,\mathrm{e}^{-\mathrm{i}\Gamma\left(\mathfrak{v}_1,\mathfrak{w}^{\overline{R}}\right)}\hat{\mathrm{W}}^{\mathfrak{w}^{\overline{R}}}\,\hat{\mathrm{V}}^{\mathfrak{v}^{\overline{R}}}.
\end{aligned} \tag{2.3.67}$$

Again, for a fixed ground state, $|\psi\rangle = |\mathfrak{v}_\star\rangle$, the reduced density matrix projected to a fixed block of (2.3.32) is pure and the electric entanglement entropy comes entirely from the Shannon entropy of $\left\{\lambda_{(\mathfrak{q}_\parallel^{\partial R})}\right\}$. This distribution again is isolated onto a specific block of $\mathfrak{v}_\parallel^{\partial R}$ eigenvalues determined by the ground state and are maximally mixed onto the $\mathfrak{w}_\parallel^{\partial R}$ eigenvalues:

$$\lambda_{(\mathfrak{v}_\parallel^{\partial R},\mathfrak{w}_\parallel^{\partial R})} = \mathcal{N}_{\partial R}^{-1} \sum_{\mathfrak{w}_\perp^{\partial R}} \mathrm{e}^{-\mathrm{i}\Gamma\left(\mathfrak{v}_\star - \mathfrak{v}_\parallel^{\partial R}, \mathfrak{w}_\perp^{\partial R}\right)} = \delta_{\mathfrak{v}_{\star\parallel}^{\partial R} - \mathfrak{v}_\parallel^{\partial R}} |\det \mathbb{K}|^{-\mathrm{h}_{\mathrm{elec}}^{d-p-1}}. \tag{2.3.68}$$

$\mathcal{E}_{\mathrm{elec}}$, being given by the algebraic entanglement entropy associated to $\mathfrak{A}_{\mathrm{elec}}[R]$ of a ground state, is then

$$\mathcal{E}_{\mathrm{elec}} = \mathrm{h}_{\mathrm{elec}}^{d-p-1} \log |\det \mathbb{K}|, \tag{2.3.69}$$

where we remind the reader

$$\mathrm{h}_{\mathrm{elec}}^{d-p-1} = \mathrm{h}_{\mathrm{mag}}^{p} = \sum_{n=0}^{p-1} (-1)^{p-1-n} \mathrm{b}_n(\partial R) + \sum_{n=0}^{p} (-1)^{p-n} \left(\mathrm{b}_n(\Sigma) - \dim \mathrm{H}_n(\Sigma, \partial R)\right). \tag{2.3.70}$$

Again we see many familiar features of this essential topological entanglement: the dependence of $\log|\det \mathbb{K}|$ with a coefficient displaying topological dependence of ∂R as well as Σ and how ∂R is embedded into Σ. Comparing with (2.3.64) we also notice

$$\mathcal{E}_{\mathrm{mag}} \xmapsto{p \to (d-p-1)} \mathcal{E}_{\mathrm{elec}}, \tag{2.3.71}$$

which ultimately stems from the electric-magnetic duality discussed in section 2.2.

Lastly for sake of completeness, we consider the centreless algebras, $\mathfrak{A}_{\mathrm{aus}}$ and $\mathfrak{A}_{\mathrm{greedy}}$. Since the ground states are already product states on the tensor factorisations defined by either $\mathfrak{A}_{\mathrm{aus}}[R]$ or $\mathfrak{A}_{\mathrm{greedy}}[R]$, these two algebras yield zero entanglement entropy:

$$\mathcal{S}_{\mathfrak{A}_{\mathrm{aus}}[R]}\left[|\mathfrak{v}_\star\rangle\langle\mathfrak{v}_\star|\right] = \mathcal{S}_{\mathfrak{A}_{\mathrm{greedy}}[R]}\left[|\mathfrak{v}_\star\rangle\langle\mathfrak{v}_\star|\right] = 0. \tag{2.3.72}$$

2.4 Discussion

In this chapter we considered the algebraic approach to entanglement entropy applied to the algebra of surface operators in abelian BF theories. On a technical level the algebraic approach allowed us to address issues of Hilbert space factorisation while respecting the non-local and topological nature of the gauge-invariant observables available in the low-energy theory. On a conceptual level this investigation was predicated on finding a suitable definition of topological entanglement that (i) can be defined intrinsically in the IR TQFT (i.e. without the need to embed into a microscopic model, or to extend the Hilbert space with "edge modes") and (ii) is sensitive to longer-range and intricate forms of topology than that of the entangling surface itself. To that end we defined two non-trivial algebras that can be assigned to a subregion: the topological magnetic and electric algebras. To each of these algebras we associated an algebraic entanglement entropy which we coin the "essential topological entanglement." Our essential topological entanglement is manifestly finite, positive, and displays a more intricate and long-range features than the topology of the entangling surface itself: namely, how the entangling surface is embedded into the Cauchy slice.

Let us comment on some open questions and open directions implied by this research below.

Comparing with traditional TEE

The essential topological entanglement shares familiar features with traditional TEE: e.g. the log dependence on the total quantum dimension, $|\det \mathbb{K}|$, and the appearance of the alternating sum of $\mathrm{b}_k(\partial R)$ which has been argued to be the coefficient of the $\log |\det \mathbb{K}|$ in higher dimensions [188]. However the additional dependence of $\mathcal{E}$ on bulk topology makes it clear that $\mathcal{E}$ truly a different object than the TEE. We have emphasised above and will emphasise again that this has to be the case. A simple example to keep in mind when comparing the two concepts is when the region is a D-ball: $R = \mathbb{B}^D$. There is simply no non-trivial operator one can assign to either $\mathfrak{A}_{\text{mag}}$ or $\mathfrak{A}_{\text{elec}}$: physically the BF theory has integrated out all local degrees of freedom and there is no probe that can distinguish R from the empty set. It is easy to see that $\mathcal{E}_{\text{mag/elec}} = 0$ in this case. However the TEE proposed by [188] will generically be non-zero: this is because $\partial \mathbb{B}^D = \mathbb{S}^{D-1}$ can support a top and bottom homology group. Again this difference stems from the fact that the TEE arises from the long-range correlations amongst UV degrees of freedom localised to ∂R while $\mathcal{E}$ arises from the long-range correlations of long-range operators delocalised on Σ.

With that difference stated, we can still speculate on the form of the traditional TEE in BF theory. As discussed above, accessing this TEE is contingent on adding in UV degrees of freedom. However we can easily do this by extending the Hilbert space using the methods in [198, 199] or by regulating a replica path-integral with "edge modes." We will return to this question in chapter 3, where we will show that the inheritance of gauge transformations on an entangling surface is an infinite dimensional algebra that completely organises the entanglement spectrum of an edge-mode theory living on ∂R. This algebra is a direct analogue to the Kac-Moody algebras arising on the boundaries of Chern-Simons theories and provides a natural procedure for constructing the extended Hilbert space. The computation of the entanglement entropy of a subregion is entirely controlled by this algebra and leads to area and sub-area laws plus a constant correction depending on Betti numbers of ∂R.

Accessing essential topological entanglement on the lattice

Our definition of the essential topological entanglement is strongly motivated by the IR effective TQFT described a topological phase. In practice, however, it is much more useful to work directly with spin lattice models or with tensor network constructions of ground states. How does one define $\mathcal{E}$ in these settings?

A natural starting place are the operator algebras defined in lattice gauge theories defined on general graphs. One can then look for a projected set of gauge invariant operators that are both homotopy invariant as well as independent to refining or coarse graining the graph. Such algebras were precisely considered in [190] where an algebra of "ribbon operators" were used to define algebraic entanglement entropies in lattice gauge theories that are graph-independent, topological, and finite dimensional. These features resonate strongly with our definition of essential topological entanglement. In that paper all entangling regions have trivial topology and so the contribution to the entanglement entropy comes entirely from surface operators terminating on (non-abelian) quasi-particle punctures in the region. In this work we have not considered punctured states; additionally it is unlikely they will contribute to $\mathcal{E}$ because of the abelian fusion of surface operators. This makes a direct comparison to difficult. It would be interesting to extend the methods of [190] to more interesting topologies to investigate if our notions of algebraic entanglement coincide.

More broadly, it is fair to ask if essential topological entanglement, either defined in a TQFT or in a lattice gauge theory, affords any practical advantages over traditional TEE. A well known use for TEE is to diagnose whether a tensor product ansatz for a gapped Hamiltonian truly captures topological order [200]. With this regard we do not expect $\mathcal{E}$ to provide any significant advantages. However essential topological entanglement likely displays conceptual advantages in models where manifest background independence and diffeomorphism invariance are desired, such as loop quantum gravity [190] or tensor network models of quantum gravity [191].

Probing essential topological entanglement in general states

In this chapter we have defined and evaluated the essential topological entanglement in fixed ground states. The similar calculation for a generic pure or mixed state follows a wholly similar calculation, however the coefficients of the superposition pollute the topological aspects of the entanglement. This also occurs in similar calculations of the TEE using generic pure states; see e.g. [201] for example calculations. One benefit of the algebraic approach to entanglement is that it is clear how the structure of the subregion algebras and their central elements lead to $\mathcal{E}$; it would be useful if this structure could be utilised to isolate $\mathcal{E}$ cleanly in arbitrary states. One such structure is that $\mathfrak{A}_{\text{mag}}$ and $\mathfrak{A}_{\text{elec}}$ appear as *complementary operator algebras* [202]. Viewing $\mathfrak{A}_{\text{greedy}} = \mathfrak{A}_{\text{elec}} \vee \mathfrak{A}_{\text{mag}}$, there is a related structure of complementary conditional expectations, E and E',

$$\begin{array}{ccc} \mathfrak{A}_{\text{greedy}}[R] & \xrightarrow{E} & \mathfrak{A}_{\text{mag}}[R] \\ \updownarrow{\scriptstyle c} & & \updownarrow{\scriptstyle c} \\ \mathfrak{A}_{\text{aus}}[R^{c}] & \xleftarrow{E'} & \mathfrak{A}_{\text{mag}}[R^{c}], \end{array} \tag{2.4.1}$$

(and similarly for $\mathfrak{A}_{\text{elec}}$). One can then try to use entropic certainty relations to place strict bounds on the relative entropies in terms of the index of $[\mathfrak{A}_{\text{mag}} \colon \mathfrak{A}_{\text{greedy}}]$ [202]. As of yet we have been unable to utilise this technology to constrain the entanglement entropy of generic pure states: it is likely possible to construct pure states whose algebraic entropy saturates $\log \dim \mathcal{H}_{\Sigma}$ and so washes out the more intricate features of $\mathcal{E}$. Regardless, this avenue and the related avenue of the topological uncertainty principle [203] are worth exploring further.

Applications beyond BF theory: fractons

As mentioned above, much of this and the next chapters is motivated by the question of topological order in higher dimensions. While our focus has been on standard abelian topological orders, we hope some of our ideas translate to $(3+1)$ gapped fracton phases. This translation is most easily facilitated through the "foliated field theory" framework to describe Type I, or foliated, fracton order [204]. Fracton phases, foliated phases included, have interesting forms of UV/IR mixing that make the distinction between different UV scales (e.g. the energy cutoff, the momentum cutoff, and the lattice scale) subtle and important. Essential topological entanglement eschews at least some of this subtlety: it does not rely on a UV embedding, but instead utilises only the structure of symmetry operators (which may still rely on a lattice scale for foliated fracton phases). It would be very interesting if essential topological entanglement can provide a more natural way to extract universal features of foliated fracton phases directly in the continuum.

Essential topological entanglement in generic theories

Although we have focused on essential topological entanglement in abelian BF theory, the extension to other TQFTs is conceptually straightforward. However, essential topological entanglement might prove to be a useful concept in generic (non-topological) quantum field theories exhibiting generalised global symmetries. This follows from the realisation that symmetries, and more broadly generalised symmetries, are synonymous with topological operators (of various codimensions, invertible or non-invertible) [12, 15].[16] One can then define topological operator

[16] There is a multitude of results stemming from this realisation, see e.g. [67] for a more complete list.

algebras in a generic quantum field theory, by restricting to the algebras of symmetry operators. More formally, the sandwich approach [90, 91] to global symmetries, provides an avenue to delineate these operators from the rest of the theory: all symmetry operators live in the one-higher-dimensional SymTFT [205], which is a topological field theory in its own right. Therefore the essential topological entanglement applied to this SymTFT is a potential probe of an indistillable, symmetry-induced, entanglement of the original theory.

Along these lines, we can lastly speculate on consequences for gravity. It is strongly believed that quantum gravity has no global symmetries (see e.g. [206–208]), although there are exceptions in low-dimensional models excluding black holes [209]. It is tempting to phrase this condition in the language of entanglement and conjecture that *quantum gravity has no essential topological entanglement,* which might be a weaker, but more universal condition on quantum gravity.

3 Edge modes

3.1 Introduction

As alluded to in chapter 1, long-range entanglement is a notion with potent conceptual and practical utility in characterising quantum phases of matter. In gapped ground states, short-range correlations manifest in an 'area law' entanglement coming from degrees of freedom localised and straddling the boundary of the region of the interest (within a correlation length), the so-called "entangling surface." Important, however, are potential long-range corrections to this area law. This is exemplified in (2+1)-dimensional gapped systems where a constant negative correction to the area law can arise from non-local features of the ground state, which constrain the short-ranged correlations at the entangling surface. This is the celebrated "topological entanglement entropy" and is a smoking gun of (2+1)d topological order [165, 166].

This story is mirrored beautifully in topological quantum field theory (TQFT), which provides IR effective field theories of topological order: when restricted to a region of spacetime, TQFTs are host to a robust spectrum of "edge modes" localised to the boundary of the region [178–181]. These edge modes are the inheritance of bulk gauge transformations, which are broken by the existence of a boundary. When "gluing" a region to its complement to form a complete state, the edge modes on either side of the common boundary are maximally entangled up to global constraints [199, 201], giving a divergent area-law entanglement entropy with universal constant corrections. TQFT also provides powerful avenues to corroborate these results, such as through the replica trick and surgery [199, 210].

By this point, the role of edge modes in topological entanglement entropy is extremely well understood in (2+1) dimensions; while generally believed to extend, their role is still relatively unexplored in general dimensions. In this part we make progress in this direction, focussing on the abelian topological phases described by p-form BF theories in d dimensions:

$$S_{\mathrm{BF}} := \frac{\mathbb{K}^{\mathsf{IJ}}}{2\pi}\int B_{\mathsf{I}} \wedge \mathrm{d}A_{\mathsf{J}}. \tag{3.1.1}$$

Above A_{J} is a vector of p-forms and B_{I} is a vector of $(d-p-1)$-forms and $\mathbb{K}$ is a rank κ symmetric matrix[1] of integer entries. These are theories whose ground states are p-form membrane condensates. In chapter 2, we explained how such theories display an extreme long-range form of entanglement, what we name "essential topological entanglement" (ETE), which must be present in the strict IR limit of a TQFT [1]. The ETE is entirely finite: absent from it are all contributions from UV degrees of freedom and their subsequent area law. However, it is still important to know the UV contributions to entanglement entropy which provide benchmarks for simulating topological order with a given UV model[2] (such as through tensor networks or matrix product states). In this direction, the potential contributions to topological entanglement entropy in p-form condensates were deduced by Grover, Turner, and Vishwanath (GTV) [188] by focussing on lattice gauge theories as a particular UV realisation. Despite various example calculations,[3] to date there has not been a comprehensive computation performed in the corresponding continuum TQFT for generic entangling surfaces and any dimension. In this chapter, we fill this gap.

We will explain how the edge mode spectrum leads to area (plus subleading area) law entanglement entropy with constant pieces that are sensitive to topological features of the entangling surface. To be concrete, we find

$$\begin{aligned} S_{\mathrm{EE}} = &\sum_{k=1}^{\lfloor \frac{d-1}{2} \rfloor} \mathrm{C}_k^{(p-1)} \left(\frac{\ell}{\varepsilon}\right)^{d-2k} + \frac{\kappa}{2}\left(\mathcal{I}^{(p-1)}_{\frac{d-2}{2}} + \mathcal{I}^{(d-p-2)}_{\frac{d-2}{2}}\right)\delta_{d,\mathrm{even}} \log\left(\frac{\ell}{\varepsilon}\right) \\ &- \frac{1}{2}(\mathrm{b}_p + \mathrm{b}_{d-p-1}) \log|\det \mathbb{K}|. \end{aligned} \tag{3.1.2}$$

where ε is a short-distance regulator, $\mathrm{C}_k^{(p-1)}$ are non-universal dimensionless numbers, that we compute, $\mathcal{I}^{(k)}_{\frac{d-2}{2}}$ is the $\left(\frac{d-2}{2}\right)$-th heat kernel coefficient for a particular spectral zeta function, ℓ is a characteristic length scale, and b_n denotes the n-th Betti number of the entangling surface. This is the first main result of this chapter. Our result differs from the GTV result; in the discussion, section 3.5, we discuss to what extent the two results can be reconciled through local terms to the entangling surface. We arrive at this result through two independent, yet conceptually complementary, avenues.

[1] In principle, this action doesn't require $\mathbb{K}$ to be symmetric, however only the symmetric part will participate in the edge mode entanglement and this saves us introducing extra notation.

[2] A real-world material being a pertinent example!

[3] See e.g. [211–216]. See [217] for a very general lattice gauge theory calculation.

(I) Firstly we utilise the replica trick and perform path integral on a replica manifold, regulated by excising a small region (of circumference ε) about the entangling surface. The resulting edge mode theory on this regulated entangling surface is a novel mix of $(p-1)$-form and $(d-p-2)$-form Maxwell theories, tied together by a "chirality" condition. We coin this theory "chiral mixed Maxwell theory." The existence of such theories as well as their thermal partition function, which we explicitly compute, is a second main result of this chapter. This edge mode theory contributes an entropy given by (3.1.2).

(II) Secondly, and more rigorously, we address the subtle issues of gauge-invariance in defining entanglement entropy by moving to an "extended Hilbert space" (EHS) [174, 182, 184, 185, 218]. We show that the EHS is organised by an infinite-dimensional current algebra, akin to the Kac–Moody algebras that arise at the edge of (2+1) dimensional phases. This algebra has also appeared in the context of 4d abelian Maxwell theory [37], where it fixes the spectrum of the theory and can, for instance, be used to establish a state-operator correspondence for non-local operators [3], as we will see in chapter 4. However, to our knowledge, the existence of these algebras for general p-forms and in general dimensions has not been explored. Here we will elucidate them in detail, construct their Verma modules, and compute their representation characters. We regard this as a third main result of this chapter. Unlike in (2+1)d, these algebras are not necessarily conformal. Regardless, they completely fix the computation of (3.1.2) which arises from the high-temperature limit of the representation character of this algebra (i.e., a regulated count of the representation dimension).

Along the way, we explain the connection between these two approaches: the chiral mixed Maxwell theory appearing in item (I) has a spectrum that is completely fixed by the infinite-dimensional algebras of approach item (II). Correspondingly, their partition function is given exactly by a representation character, which is the ultimate source of the match in (3.1.2). This is a precise analogue of "edge spectrum = bulk entanglement spectrum" promoted to higher dimensions. We regard this as a final major result of this chapter.

An organising summary of the chapter is as follows. In section 3.2 we introduce the BF theory and perform its path integral on manifolds with and without boundary. While there are many results for the BF path integral on closed manifolds, we will be very careful keeping track of factors of $\mathbb{K}$, which, to our knowledge, had not been fully nailed down in the previous literature. We will use these results in section 3.3 to evaluate the replica integral and describe how the computation ultimately results from the partition function of the chiral mixed Maxwell theory. In section 3.4 we shift gears and describe the more systematic definition of the entanglement entropy through the extended Hilbert space and describe features of the resulting current algebra that organises it. We then use the representation characters of this algebra to compute the entanglement entropy, finding a match with section 3.3. In the discussion, section 3.5, we will put our results in context with the known results of GTV, as well as the ETE of this theory, and possible future extensions of our computation.

3.1.1 Notation

For the reader's ease, we lay out here some basic notation that will be used in what follows.

We will analyse theories on torsion-free manifolds of spacetime dimension d, which we shall collectively denote as X. These manifolds may have a boundary, which we will embed into X using the map $i_\partial : \partial X \hookrightarrow X$. Theories will be quantised on manifolds of dimension $D \equiv d - 1$, which we will refer to as Σ, often calling it the "Cauchy slice" without reference to any causal structure of the TQFT. We will consider a subregion R, which is the closure of a D-dimensional embedded open submanifold of Σ. The interior of R is $\overline{R} := R \setminus \partial R$, and its complement is R^{c}, the closure of $\Sigma \setminus R$. Note that $R \cap R^{\mathrm{c}} = \partial R$. We will denote the space of forms of degree p as $\Omega^p(\,\cdot\,)$. Unless stated otherwise, these forms will be real valued. Cohomology groups will be denoted with their degree placed upstairs, $\mathrm{H}^p(\,\cdot\,)$, while homology groups will be denoted with their degree placed downstairs, $\mathrm{H}_p(\,\cdot\,)$, and these groups are always defined with integer coefficients, unless stated otherwise. For compact, boundary-less manifolds, we notate the dimensions of the groups by Betti numbers, $\mathrm{b}_p(\,\cdot\,)$. For (co)homology groups on manifolds with boundary or for relative homology groups, we will always explicitly write the dimension.

3.2 The BF path integral

In this section, we describe the path integral quantisation of multi-component, abelian, p-form BF theory, on a d-dimensional, torsion-free manifold, X, with a potentially nonempty boundary, ∂X. We will start by reminding the reader of the procedure in the case where X is a closed manifold, presenting general results with the added benefit of a careful accounting of the level matrix. We then move on to modify the situation in the presence of boundaries.

3.2.1 On a closed manifold

Let us set the stage by starting with a closed manifold. We consider BF theory defined by the action

$$S_{\mathrm{BF}}[A, B] := \frac{\mathbb{K}^{\mathsf{IJ}}}{2\pi} \int_X B_{\mathsf{I}} \wedge \mathrm{d}A_{\mathsf{J}}. \tag{3.2.1}$$

In the above, $A_{\mathrm{I}} \in \Omega^{p}(X)$ and $B_{\mathrm{J}} \in \Omega^{d-p-1}(X)$ are vectors of p- and $(d-p-1)$-form gauge fields respectively. The level matrix, $\mathbb{K}$ is a symmetric, integer, and non-degenerate matrix of rank κ. In what follows, we will drop the indices, wherever not necessary, to simplify the notation. Note that, as emphasised above, this definition makes sense whenever X is torsion-free. We will comment on the case of torsion manifolds at the end of this subsection. For a more general and precise definition of BF theory, we refer the reader to [1, Appendix A].

The equations of motion arising from varying the action (3.2.1) are flatness conditions:

$$\textsc{eom}[A] = \frac{\mathbb{K}}{2\pi}\,\mathrm{d}A = 0, \qquad \textsc{eom}[B] = (-1)^{(d-p)(p+1)}\frac{\mathbb{K}}{2\pi}\,\mathrm{d}B = 0. \tag{3.2.2}$$

Moreover, the action (3.2.1) possesses a gauge redundancy of the form

$$\delta A = \alpha \quad \text{and} \quad \delta B = \beta, \tag{3.2.3}$$

where $\alpha \in \Omega^{p}_{\mathrm{cl}}(X)$ and $\beta \in \Omega^{d-p-1}_{\mathrm{cl}}(X)$, are closed p- and $(d-p-1)$-forms respectively. Note that there are two types of gauge shifts. Shifts by harmonic forms correspond to large gauge transformations, in the sense that they are not continuously connected to the identity, and shifts by exact forms correspond to the usual infinitesimal gauge transformations. To properly quantise the theory in the path integral formalism, we must therefore divide by the volume of the gauge groups. However, note that there is a tower of reducibility of the gauge parameters. For example, splitting α off as $\alpha = \alpha_0 + \mathrm{d}\alpha_1$, with $\alpha_0 \in \mathrm{Harm}^{p}(X)$ and $\alpha_1 \in \Omega^{p-1}(X)$, the parameter α_1 generates the same gauge transformation as $\alpha_1 + \tilde{\alpha}_1$, with $\tilde{\alpha}_1 \in \Omega^{p-1}_{\mathrm{cl}}(X)$. This tower continues until one reaches a zero-form gauge redundancy. This redundancy is encoded in the BF path integral in the form of the volume of the gauge group:

$$\mathcal{Z}_{\mathrm{BF}}[X] = \int \frac{\mathrm{D}A\ \mathrm{D}B}{\mathrm{vol}(\mathscr{G})}\, \mathrm{e}^{\mathrm{i}S_{\mathrm{BF}}[A,B]}, \tag{3.2.4}$$

where, $\mathscr{G}$ is the total gauge group, $\mathscr{G} = \mathscr{G}_p \times \mathscr{G}_{d-p-1}$, with

$$\mathscr{G}_k := \Omega^{k}_{\mathrm{cl}}(X)/\mathscr{G}_{k-1}, \tag{3.2.5}$$

and $\mathscr{G}_0 = \mathrm{Harm}^0(X)$. In what follows we will be careless with overall numerical coefficients, such as factors of 2π in the partition function, as they can be absorbed in an overall normalisation of the path integral measure. However, we will pay extra attention to factors of $\mathbb{K}$, as they play a crucial role in the universal terms of the entanglement entropy.

One commonly employed method to properly quantise such higher-gauge theories is to include ghosts and ghosts-for-ghosts, and so on, until the gauge transformations are completely resolved [193]. An alternative approach, due to [194], which was shown to be equivalent, at the level of determinants, to that of Blau and Thompson [193] (which is in turn also equivalent to Schwarz's method of resolvents [219]), is to Hodge-decompose the fields as

$$\begin{aligned} A &= A_0 + \mathrm{d}A_\perp + \mathrm{d}^\dagger A_\parallel, \\ B &= B_0 + \mathrm{d}B_\perp + \mathrm{d}^\dagger B_\parallel. \end{aligned} \tag{3.2.6}$$

with $A_0 \in \mathrm{Harm}^p(X)$, $A_\perp \in \Omega^{p-1}(X)$, $A_\parallel \in \Omega^{p+1}(X)$, $B_0 \in \mathrm{Harm}^{d-p-1}(X)$, $B_\perp \in \Omega^{d-p-2}(X)$, and $B_\parallel \in \Omega^{d-p}(X)$ and perform the path integral directly. In what follows, we will show that, with a bit of care, this method also correctly reproduces the harmonic correction to the partition function[4] and we will obtain the level dependence of the partition function. With the decomposition (3.2.6), taking into account the Jacobians of the transformation, the path integral measure decomposes as

$$\mathrm{D}A = \mathrm{D}A_0\ \mathrm{D}A_\perp\,\mathrm{D}A_\parallel\ \left(\mathrm{det}'_{\Omega^{p-1}(X)}(\mathrm{d}^\dagger\mathrm{d})\right)^{1/2}\left(\mathrm{det}'_{\Omega^{p+1}(X)}(\mathrm{d}\mathrm{d}^\dagger)\right)^{1/2}, \tag{3.2.7}$$

and similarly for $\mathrm{D}B$, while the action takes the form

$$S_{\mathrm{BF}}[A,B] = \frac{\mathbb{K}}{2\pi}\left\langle \mathrm{d}^\dagger B_\parallel, \star \mathrm{d}\mathrm{d}^\dagger A_\parallel\right\rangle, \tag{3.2.8}$$

where $\langle \bullet, \circ\rangle$ is the Hodge inner product, $\int_X \bullet \wedge \star\circ$. Following [220] we will normalise the path integral measure as

$$\mathrm{D}A = \prod_{x\in X} \mathbb{E}\,\mathrm{d}A_x \qquad \mathrm{D}B = \prod_{x\in X} \mathbb{E}\,\mathrm{d}B_x\,, \tag{3.2.9}$$

where $\mathbb{E}$ is a frame matrix, satisfying $\mathbb{E}^2 = \mathbb{K}$. This has the effect of removing the $\mathbb{K}$ dependence from the functional determinants, arising upon integrating over $A_\parallel$ and $B_\parallel$. Note that this is not the only choice. Normalising the modes as $\sim \mathbb{K}^\alpha\,\mathrm{d}A_x$ and $\sim \mathbb{K}^{1-\alpha}\,\mathrm{d}B_x$, for some real α, still takes care of the $\mathbb{K}$-dependence of the functional determinants, but as will become evident below, it rescales the partition function as

$$\mathcal{Z}_{\mathrm{BF}}[X] \longmapsto |\det \mathbb{K}|^{\alpha\,\chi(X)}\mathcal{Z}_{\mathrm{BF}}[X], \tag{3.2.10}$$

where $\chi(X)$ is the Euler characteristic of X. This ambiguity is on the one hand, well documented and understood [221] and on the other hand harmless if one is interested in topological entanglement entropy. The reason is, ultimately, that it can be seen to arise from (or equivalently can be absorbed into) a locally integrated contribution [188]. We will return to this point in the discussion, section 3.5.

[4] This corresponds to the refinement of the Ray–Singer torsion as an element of the determinant line bundle, $\det \mathrm{H}^\bullet$, of $\mathrm{H}^\bullet(X)$.

Ignoring this ambiguity, for the reasons mentioned above, the integral over $A_\parallel$ and $B_\parallel$ with the measure (3.2.9) produces

$$\mathcal{Z}^{\parallel}_{\mathrm{BF}}[X] = \left(\det{}'_{\Omega^p(X)}(\mathrm{d}^\dagger \mathrm{d})\right)^{-\frac{3}{2}}. \tag{3.2.11}$$

Integrating over $A_\perp$ and $B_\perp$, modulo small gauge transformations, i.e. the part of (3.2.5) generated by $\mathrm{d}\Omega^{k-1}(X)$, gives an alternating product of determinants, which can be combined with (3.2.11) to give the analytic torsion[5] [194]:

$$\mathcal{Z}^{\perp}_{\mathrm{BF}}[X]\mathcal{Z}^{\parallel}_{\mathrm{BF}}[X] = \prod_{k=0}^{d}\left(\det{}'_{\Omega^k(X)}\triangle_n\right)^{\frac{k}{2}(-1)^{p+k}} =: \mathrm{T_A}[X]^{(-1)^{p-1}}. \tag{3.2.12}$$

As it stands, $\mathrm{T_A}[X]$ depends on the metric, whenever not all cohomology groups are trivial. So this cannot be the final expression of the partition function; it is the zero-modes that will provide the fix, as is usually the case. Let us see what their contribution is. The integral over A_0 and B_0, modulo large gauge transformations, with the measure (3.2.9) gives [195, 220]:

$$\mathcal{Z}^{0}_{\mathrm{BF}}[X] = \left(\prod_{k=0}^{p}\left|\det\left(\mathbb{K}\otimes\mathbb{G}_k\right)\right|^{\frac{1}{2}(-1)^{p-k}}\right)\left(\prod_{\ell=0}^{d-p-1}\left|\det\left(\mathbb{K}\otimes\mathbb{G}_\ell\right)\right|^{\frac{1}{2}(-1)^{d-p-1-\ell}}\right), \tag{3.2.13}$$

where $\mathbb{G}_k$ is the metric on the moduli space of harmonic k-forms, i.e. the Gram matrix of the topological basis of harmonic k-forms. More explicitly, the topological basis, $\left\{\tau^{(k)}_{\mathrm{i}}\right\}_{\mathrm{i}=1}^{\mathrm{b}_k(X)}$, is defined by the relation

$$\int_{C^{\mathrm{i}}_{(k)}}\tau^{(k)}_{\mathrm{j}} = \delta^{\mathrm{i}}_{\mathrm{j}}, \tag{3.2.14}$$

for a fixed basis, $\left\{C^{\mathrm{j}}_{(k)}\right\}_{\mathrm{i}=1}^{\mathrm{b}_k(X)}$, of k-cycles. Then $\mathbb{G}_k$ is defined as

$$\left[\mathbb{G}_k\right]_{\mathrm{ij}} := \left\langle\tau^{(k)}_{\mathrm{i}}, \tau^{(k)}_{\mathrm{j}}\right\rangle. \tag{3.2.15}$$

Poincaré duality implies that (up to a unimodular matrix, which can be set to one by a suitable choice of basis of the k-cycles) $\mathbb{G}_{d-k} = \mathbb{G}_k{}^{-1}$. Utilising this, we get:

$$\mathcal{Z}^{0}_{\mathrm{BF}}[X] = |\det\mathbb{K}|^{\mathrm{h}_p(X)}\mathrm{T}_{\mathrm{H}^\bullet}[X]^{(-1)^{p-1}}, \tag{3.2.16}$$

[5] Actually we get this expression for odd-dimensional manifolds. For even-dimensional manifolds we get a different power of the (determinant expression of the) analytic torsion. However, on even-dimensional manifolds the analytic torsion is unity, so we can write (3.2.12) for all dimensions.

where

$$\begin{aligned}\mathrm{h}_p(X) &:= \frac{1}{2}(-1)^p \sum_{k=0}^{p}(-1)^k \mathrm{b}_k(X) + \frac{1}{2}(-1)^{d-p-1} \sum_{k=0}^{d-p-1} (-1)^k \mathrm{b}_k(X) \\ &\stackrel{\chi}{=} (-1)^p \sum_{k=0}^{p}(-1)^k \mathrm{b}_k(X),\end{aligned} \tag{3.2.17}$$

where the last equality is modulo removing factors of the Euler characteristic, which can be done by adding a local counterterm, as we alluded to above. Moreover, we have defined

$$\mathrm{T}_{\mathrm{H}^\bullet}[X] := \prod_{k=0}^{d} (\det \mathbb{G}_k)^{\frac{1}{2}(-1)^{k+1}}. \tag{3.2.18}$$

$\mathrm{T}_{\mathrm{H}^\bullet}[X]$ is precisely the cohomological correction to the analytic torsion, necessary to form the metric-independent Ray–Singer torsion:

$$\mathrm{T}_{\mathrm{RS}}[X] = \mathrm{T}_{\mathrm{H}^\bullet}[X]\mathrm{T}_{\mathrm{A}}[X]. \tag{3.2.19}$$

Putting everything together, the partition function for multi-component, p-form BF theory on a compact, closed manifold X, reads:

$$\mathcal{Z}_{\mathrm{BF}}[X] = |\det \mathbb{K}|^{\mathrm{h}_p(X)}\, \mathrm{T}_{\mathrm{RS}}[X]^{(-1)^{p-1}}. \tag{3.2.20}$$

This form of the partition function has a number of desirable features which we list below.

- It is topological. Of course, this was anticipated since the very first appearance of the theory, in [219]. However, to obtain it in the path integral formalism requires some care, regardless of the method one chooses. The method of resolutions [219] was unable to reproduce it, in cases with non-trivial cohomology groups. In the BRST formulation, one needs to append the action by a BRST-closed, but not necessarily BRST-exact, quantum action [193, 195]. In the Batalin–Vilkovisky (BV) [222] formulation of the theory [223] one needs to pay special attention to the residual superfields, furnishing the fibre of the BV integral. Finally, in the direct method that we employed one must be carfeul about the zero-modes appearing in the tower of gauge-for-gauge-for-…-gauge volumes; note, for example, that in [194] the zero-mode piece is not correctly reproduced and hence the formulas there depend implicitly on a choice of metric on X.

- On a three-dimensional manifold, with $p = 1$, (3.2.20) is the square of the partition function of abelian Chern–Simons theory. More explicitly, on $\mathbb{S}^3$ we obtain:

$$\mathcal{Z}_{\mathrm{BF}}\big[\mathbb{S}^3\big] = |\det \mathbb{K}|^{-1}, \tag{3.2.21}$$

which is to be compared with $\mathcal{Z}_{\mathrm{CS}}[\mathbb{S}^3] = |\det \mathbb{K}|^{-1/2}$ in [198]. For a more general case, in $d = 3$ with $p = 1$, allowing also for torsion, see also [224].

- On $X = \mathbb{S}^1 \times \Sigma$, $\mathcal{Z}_{\mathrm{BF}}[\mathbb{S}^1 \times \Sigma]$ simply counts the number of ground states on Σ. Equivalently, it measures the dimension of the Hilbert space on Σ. This is given by $\dim \mathcal{H}_\Sigma = |\det \mathbb{K}|^{\mathrm{b}_p(\Sigma)}$ [1]. From (3.2.20), making use of the Künneth formula: $\mathrm{b}_k(X) = \mathrm{b}_k(\Sigma) + \mathrm{b}_{k-1}(\Sigma)$, if $k \geqslant 1$, while $\mathrm{b}_0(X) = \mathrm{b}_0(\Sigma)$, we correctly find
$$\mathcal{Z}_{\mathrm{BF}}\big[\mathbb{S}^1 \times \Sigma\big] = |\det \mathbb{K}|^{\mathrm{b}_p(\Sigma)}. \tag{3.2.22}$$

Lastly, let us mention that if we allow X to have torsion, in which case we should take into account the differential cohomology definition of BF theory [1, Appendix A], the expression for the partition function is modified as follows. Let $\mathrm{H}_p(X;\mathbb{Z}) = \mathbb{Z}^{\mathrm{b}_p(X)} \oplus \mathrm{T}_p(X)$, where $\mathrm{T}_p(X) = \mathbb{Z}_{p_1(X)} \oplus \cdots \oplus \mathbb{Z}_{p_N(X)}$ is the torsion part. Then, we can obtain from [225], upon a slight modification to account for the multi-component case and to scale away numerical coefficients, that

$$\mathcal{Z}_{\mathrm{BF}}[X] = |\det \mathbb{K}|^{\mathrm{h}_p(X)} \left|\hom\!\big(\mathrm{T}_p(X), \mathbb{Z}^\kappa / \operatorname{im} \mathbb{K}\big)\right| \mathrm{T}_{\mathrm{RS}}[X]^{(-1)^{p-1}}. \tag{3.2.23}$$

Note that Poincaré duality and the universal coefficient theorem give $\mathrm{T}_p(X) \cong \mathrm{T}_{d-p-1}(X)$, making this expression invariant upon exchanging A and B. At this stage, the inclusion of torsion seems like a mathematical curiosity. Indeed, we will not discuss further manifolds with torsion here. However, this result may be of relevance for a future utilisation of "surgery,"[6] see the discussion, section 3.5.

3.2.2 On a manifold with boundary

Consider now the case where X has a boundary, $\partial X \neq \varnothing$. As explained in [1], the action (3.2.1), gives rise to a boundary symplectic form, which as it stands is consistent with fixing A as a boundary condition. In general, the action is supplemented by a boundary term, S_∂. This leads to a modified variational problem, specified by symplectic potential on the boundary, $\boldsymbol{\Theta}_{\partial X}$. Explicitly, the variation of the total action $S_{\text{full}} = S_{\mathrm{BF}} + S_\partial$ reads:

$$\delta S_{\text{full}} = \langle \text{EOM}[A], \delta B\rangle_X + \langle \text{EOM}[B], \delta A\rangle_X + \boldsymbol{\Theta}_{\partial X}, \tag{3.2.24}$$

[6]In this method one needs to compute the partition function of BF theory on a branched cyclic cover of X, ramified over the entangling surface. Although the topology of the replica manifold is completely encoded in the topology of the entangling surface and the original manifold, the replica manifold can have non-trivial torsion, even if the original manifold does not [226].

where $\mathbf{\Theta}_{\partial X}$ can be put in Darboux form:

$$\mathbf{\Theta}_{\partial X} = \int_{\partial X} \mathbf{p} \wedge \star_\partial \delta \mathbf{q}\,, \tag{3.2.25}$$

with $\mathbf{q}$ and $\mathbf{p}$ being functions of the boundary values of the fields, $A_\partial := i_\partial^* A$, $B_\partial := i_\partial^* B$. Having the symplectic potential in the form of (3.2.25) we have two options to proceed, in order to have a well-defined variational problem: fix $\mathbf{p} \overset{!}{=} 0$ or $\delta\mathbf{q} \overset{!}{=} 0$. Without loss of generality, we will proceed with the Dirichlet approach, $\mathbf{p} \overset{!}{=} 0$. The other boundary condition can be attained by a boundary symplectomorphism and can be easily shown to give equivalent results. Note that the symplectic potential is, as it stands, degenerate due to boundary gauge transformations. We will take care of that below when we discuss the boundary path integral measure.

The object one would like to study is the partition function of the combined system:

$$\mathcal{Z}_{\mathrm{BF+edge}}[X, \partial X] := \int_{\mathcal{C}_\mathbf{p}} \frac{\mathrm{D}A\,\mathrm{D}B}{\mathrm{vol}(\mathcal{G})}\, \mathrm{e}^{-S^{\mathrm{E}}_{\mathrm{full}}[A,B]}, \tag{3.2.26}$$

where $\mathcal{C}_\mathbf{p} := \left\{A \in \Omega^p(X),\ B \in \Omega^{d-p-1}(X) \;\middle|\; \mathbf{p} = 0\right\}$ and $S^{\mathrm{E}}_{\mathrm{full}}[A, B]$ is the Euclideanised partition function.[7] Note also that here the gauge group $\mathcal{G}$ consists of gauge transformations that vanish on the boundary. The next step is to decompose the fields A and B as follows:

$$A = \widetilde{A}_\partial + \hat{A}, \tag{3.2.27}$$
$$B = \widetilde{B}_\partial + \hat{B}, \tag{3.2.28}$$

where $\hat{A}$ and $\hat{B}$ are off-shell gauge fields with Dirichlet boundary conditions, $i_\partial^*\,\hat{A} = 0 = i_\partial^*\,\hat{B}$. The boundary conditions are absorbed completely by $\widetilde{A}_\partial$ and $\widetilde{B}_\partial$. The latter are, in the spirit of [223], potentially discontinuous extensions of the boundary fields, A_∂ and B_∂ and are chosen to be on-shell, i.e. flat. This has the effect of disentangling the bulk, from the boundary contribution:

$$S_{\mathrm{full}}[A, B] = S_{\mathrm{BF}}\left[\hat{A}, \hat{B}\right] + S_\partial[A_\partial, B_\partial]. \tag{3.2.29}$$

Similarly, the measure has a piece coming from the hatted fields and giving only bulk contributions, and one coming from the tilded fields, giving all the edge contributions.

The hatted fields are very easy to deal with, they only contribute in the bulk, and they give the full bulk contribution. The computation is completely analogous to that of

[7] note that the bulk piece is not affected by the Wick rotation, since it is a one-derivative action. However, the boundary terms can change in the usual way.

subsection 3.2.1. The only difference is that we make use of the Hodge decomposition theorem for Dirichlet forms on a manifold with boundary [227]:

$$\Omega^\bullet_{\mathrm{D}}(X) = \mathrm{H}^\bullet(X,\partial X) \oplus \mathrm{d}\Omega^{\bullet-1}_{\mathrm{D}}(X) \oplus \left(\mathrm{d}^\dagger\Omega^{\bullet+1}_{\mathrm{N}} \cap \Omega^\bullet_{\mathrm{D}}(X)\right), \tag{3.2.30}$$

where

$$\Omega^\bullet_{\mathrm{D}}(X) := \left\{\omega \in \Omega^\bullet(X) \;\middle|\; i^*_\partial \omega = 0\right\}, \tag{3.2.31}$$
$$\Omega^\bullet_{\mathrm{N}}(X) := \left\{\omega \in \Omega^\bullet(X) \;\middle|\; i^*_\partial \star \omega = 0\right\}, \tag{3.2.32}$$

denote the spaces of Dirichlet and Neumann forms respectively. Altogether we have that

$$\mathcal{Z}_{\text{bulk}}[X] = \int \frac{\mathrm{D}\hat{A}\,\mathrm{D}\hat{B}}{\mathrm{vol}(\mathcal{G})}\,\mathrm{e}^{\mathrm{i}S_{\mathrm{BF}}[\hat{A},\hat{B}]} = |\det \mathbb{K}|^{\mathrm{h}_p(X,\partial X)} \mathrm{T}_{\mathrm{RS}}[X,\partial X]^{(-1)^{p-1}}, \tag{3.2.33}$$

where $\mathrm{h}_p(X,\partial X)$ is the relative version of $\mathrm{h}_p(X)$, appearing in (3.2.20), i.e.:

$$\begin{aligned}\mathrm{h}_p(X,\partial X) &:= \frac{1}{2}(-1)^p \sum_{k=0}^{p} (-1)^k \dim \mathrm{H}^k(X,\partial X) \\ &\quad + \frac{1}{2}(-1)^{d-p-1} \sum_{k=0}^{d-p-1} (-1)^k \dim \mathrm{H}^k(X,\partial X) \\ &\overset{\chi}{=} (-1)^p \sum_{k=0}^{p} (-1)^k \dim \mathrm{H}^k(X,\partial X).\end{aligned} \tag{3.2.34}$$

The contribution of the tilded fields is a little more subtle. We will calculate their contribution, with a specific choice of S_∂, in detail in section 3.3, but let us already give a rough sketch of the computation for a general boundary action. First of all, it is clear from the above discussion that they only contribute on the boundary. Secondly, the flatness constraint becomes a Bianchi identity for the boundary gauge fields. In particular, A_∂ and B_∂ are curvatures of $(p-1)$– and $(d-p-1)$-form gauge fields, a and b, respectively. All in all, the contribution of $\widetilde{A}_\partial$ and $\widetilde{B}_\partial$, including the boundary condition $\mathbf{p} = 0$, in the measure is

$$\sum_{\text{instantons}} \int \frac{\mathrm{D}a\,\mathrm{D}b}{\mathrm{vol}(\mathcal{G}^\partial)}\,\delta[\mathbf{p}], \tag{3.2.35}$$

where the sum over instantons is a sum over all topologically non-trivial configurations, i.e. over cohomology p- and $(d-p-1)$-classes, respectively, of ∂X, and $\mathrm{vol}(\mathcal{G}^\partial)$ indicates the volume of the boundary gauge transformations. Since a and b appear only through their curvatures, there is a redundancy upon shifting them by flat gauge fields. Relatedly, dividing by this volume takes care of the redundancy in the symplectic potential, $\boldsymbol{\Theta}_{\partial X}$.

Putting everything together, we find that the partition function, (3.2.26), takes the form

$$\mathcal{Z}_{\mathrm{BF+edge}}[X,\partial X] = \mathcal{Z}_{\mathrm{bulk}}[X]\mathcal{Z}_{\mathrm{edge}}[\partial X], \tag{3.2.36}$$

with $\mathcal{Z}_{\mathrm{bulk}}[X]$ as in (3.2.33) and $\mathcal{Z}_{\mathrm{edge}}[\partial X]$ is the edge mode partition function, computed with the measure (3.2.35).

3.3 Entanglement from the replica path integral

In this section, we will calculate the entanglement entropy using path integral techniques via the replica trick. We begin with a brief primer on replica path integrals and how they are calculated. The basic ingredient is the path integral representation of a state. We begin with a Cauchy slice, Σ, and a Hilbert space of states associated to it, $\mathcal{H}_\Sigma$. We can imagine generating a state in $\mathcal{H}_\Sigma$ by finding a d-dimensional manifold, X_-, whose boundary is Σ. According to the standard rules of topological field theory, the path integral with specified boundary conditions produces the wavefunction of a state in $\mathcal{H}_\Sigma$. That is,

$$\Psi_{X_-}[\varphi] := \int_{\mathcal{C}[X_-;\varphi]} \mathrm{D}\Phi \ \mathrm{e}^{\mathrm{i}S[\Phi]}, \tag{3.3.1}$$

is the wavefunction associated with the state

$$|\Psi_{X_-}\rangle = \int_{\mathcal{C}[\Sigma]} \mathrm{D}\varphi \ \Psi_{X_-}[\varphi]\,|\varphi\rangle \in \mathcal{H}_\Sigma, \tag{3.3.2}$$

in the Hilbert space $\mathcal{H}_\Sigma$ assigned to Σ. In the above, we have schematically indicated all fields by Φ and $\mathcal{C}[X_-;\varphi]$ is an appropriate functional space (including quotienting out gauge redundancies) over X_-, with boundary conditions $i^*_{\partial X_-}\Phi = \varphi$. $\mathcal{C}[\Sigma]$ is, similarly, an appropriate functional space over Σ. The dual Hilbert space, $\mathcal{H}^\vee_\Sigma$ is canonically isomorphic with the Hilbert space associated to the orientation reversal, $\overline{\Sigma}$, of Σ. Hence, the norm of the state $|\Psi_{X_-}\rangle$, is given by conjugating Ψ_{X_-} and integrating over the boundary conditions. At the level of the path integral, this has the action of gluing X_- onto X_+, its time-reversal about Σ,

$$\|\Psi_{X_-}\|^2 = \int_{\mathcal{C}[X]} \mathrm{D}\Phi \ \mathrm{e}^{\mathrm{i}S[\Phi]} = \mathcal{Z}[X], \tag{3.3.3}$$

yielding the partition function on $X := X_- \cup_\Sigma X_+$. Similarly, the pure density matrix $\rho := |\Psi_{X_-}\rangle\langle\Psi_{X_-}| \in \mathcal{H}_\Sigma \otimes \mathcal{H}^\vee_\Sigma$ has the path integral expression

$$\rho[\varphi_-,\varphi_+] := \langle\varphi_-|\rho|\varphi_+\rangle = \Psi_{X_-}[\varphi_-]\Psi^*_{X_-}[\varphi_+] = \int_{\mathcal{C}[X_{\Sigma\text{-cut}};\varphi_-,\varphi_+]} \mathrm{D}\Phi \ \mathrm{e}^{\mathrm{i}S[\Phi]}, \tag{3.3.4}$$

where $X_{\Sigma\text{-cut}} := X_- \sqcup X_+ \cong X \setminus (\Sigma \times [0,1])$, which has as its boundary two copies of Σ with opposite orientation, $\Sigma_\pm$, on which we impose boundary conditions $\varphi_\pm$, respectively. This density matrix could possibly be unnormalised.

We now choose a subregion $R \subset \Sigma$ and imagine demarcating

$$\varphi_\pm = \Theta_R\, \varphi_\pm + \Theta_{R^c}\, \varphi_\pm =: \varphi_{R,\pm} + \varphi_{R^c,\pm},$$

where Θ_R is the characteristic function of R (taking values 1 inside R and 0 everywhere else) and $\Theta_{R^c} = 1 - \Theta_R$ is the characteristic function for the complement region. The reduced density matrix, ρ_R, is described, at least at a formal level, by identifying $\varphi_{R^c,+} = \varphi_{R^c,-}$ and integrating over their values:

$$\begin{aligned} \rho_R[\varphi_{R,-}, \varphi_{R,+}] &= \int_{\mathcal{C}[R^c]} \mathrm{D}\varphi_{R^c}\, \rho[\varphi_{R,-} + \varphi_{R^c}; \overline{\varphi}_{R,+} + \varphi_{R^c}] \\ &= \int_{\mathcal{C}[X_{R\text{-cut}}; \varphi_{R,-}, \varphi_{R,+}]} \mathrm{D}\Phi\; \mathrm{e}^{\mathrm{i}S[\phi]}, \end{aligned} \tag{3.3.5}$$

resulting in a path integral on $X_{R\text{-cut}} := X_- \cup_{R^c} X_+$, possessing a cut along R, and boundary conditions $\varphi_\pm$, imposed on either side of the cut. This density matrix is unnormalised: its trace is simply $\|\Psi_{X_-}\|^2 = \mathcal{Z}[X]$, which arises from identifying $\varphi_{R,+}$ with $\varphi_{R,-}$ and then integrating, effectively "gluing" the cut closed.

We now aim to calculate the von Neumann entropy of ρ_R. We will make use of the replica trick. First we compute the n-th Rényi entropy as

$$\mathcal{S}_n := \frac{1}{1-n} \log \frac{\mathrm{Tr}\left(\rho_R^n\right)}{(\mathrm{Tr}\, \rho_R)^n}, \tag{3.3.6}$$

where the denominator arises to normalise ρ_R, if it was not already normalised. The von Neumann entropy is then given as the limit:

$$\mathcal{S}_{\mathrm{EE}} = \lim_{n \to 1} \mathcal{S}_n. \tag{3.3.7}$$

The denominator of (3.3.6) is simply given by $\mathcal{Z}[X]^n$, so we now make sense of the numerator. It arises from copying $X_{R\text{-cut}}$ n times; the trace identifies the boundary conditions, $\varphi^{(i)}_{R,+} = \varphi^{(i+1)}_{R,-}$ (where i indexes the replicas mod n) and their subsequent integration glues the replicated manifold into an n-fold branched cover over ∂R, which we denote X_n. This gives a path integral expression of the entanglement entropy as

$$\mathcal{S}_{\mathrm{EE}} = \lim_{n \to 1} \frac{1}{1-n} \log \frac{\mathcal{Z}[X_n]}{\mathcal{Z}[X]^n} = -\frac{\partial}{\partial n} \left(\log \frac{\mathcal{Z}[X_n]}{\mathcal{Z}[X]^n} \right) \bigg|_{n=1}. \tag{3.3.8}$$

We are now tasked with evaluating the path integrals in question taking care with the codimension-2 surface fixed by the $\mathbb{Z}_n$ replica symmetry — the entangling surface.

We will address this by regulating the entangling surface and looking at the edge theory that arises there. To be precise, we will literally excise a tubular neighborhood around the entangling surface: in terms of the state on Σ, this has the interpretation of putting a small buffer region between R and R^{c}. This will serve as a UV regulator. Specifically, we replace the bulk replica manifold X_n with $\mathcal{X}_{n,\varepsilon} := X_n \setminus \left(\mathbb{D}^2_{n\varepsilon} \times \partial R\right)$ and keep in mind the limit as $\varepsilon \to 0$. The interpretation of ε as a regulator on the unreplicated state mandates that the circumference of the disk scales as $n\varepsilon$. We will perform a similar excision of the original manifold: $X \rightsquigarrow \mathcal{X}_{1,\varepsilon} := X \setminus \left(\mathbb{D}^2_{\varepsilon} \times \partial R\right)$. This yields

$$\mathcal{S}_{\mathrm{EE}} = \lim_{\varepsilon\to 0}\lim_{n\to 1} \frac{1}{1-n} \log \frac{\mathcal{Z}_{\mathrm{BF+edge}}[\mathcal{X}_{n,\varepsilon}]}{\mathcal{Z}_{\mathrm{BF+edge}}[\mathcal{X}_{1,\varepsilon}]^n}. \tag{3.3.9}$$

Importantly, these are path integrals on manifolds with boundary, $\partial\mathcal{X}_{n,\varepsilon} = \mathbb{S}^1_{n\varepsilon} \times \partial R$. As we have discussed in section 3.2, we need to supplement these path integrals with boundary conditions and boundary terms on $\mathcal{X}_{n,\varepsilon}$ enforcing those boundary conditions through the variational principle. As we have shown in subsection 3.2.2, the BF path integral on this regulated, replica geometry naturally splits into a product of "bulk" and "edge" terms

$$\mathcal{Z}_{\mathrm{BF+edge}}[\mathcal{X}_{n,\varepsilon}, \partial\mathcal{X}_{n,\varepsilon}] = \mathcal{Z}_{\mathrm{bulk}}[\mathcal{X}_{n,\varepsilon}]\; \mathcal{Z}_{\mathrm{edge}}[\partial\mathcal{X}_{n,\varepsilon}]. \tag{3.3.10}$$

We now insert the relevant $\mathcal{Z}_{\mathrm{BF+edge}}[\mathcal{X}_{n,\varepsilon}, \partial\mathcal{X}_{n,\varepsilon}]$ into the ratio (3.3.9). In this chapter, we will focus on the contribution of the edge modes to the entanglement entropy. It is precisely these terms that we expect to contribute an area law. In the discussion, section 3.5, we will comment on possible bulk contributions. Since the above appears as a product, we can neatly isolate the contribution of the edge modes as:

$$\mathcal{S}_{\mathrm{EE}} = \lim_{\varepsilon\to 0}\lim_{n\to 1} \frac{1}{1-n} \log \frac{\mathcal{Z}_{\mathrm{edge}}[\partial\mathcal{X}_{n,\varepsilon}]}{\mathcal{Z}_{\mathrm{edge}}[\partial\mathcal{X}_{1,\varepsilon}]^n}. \tag{3.3.11}$$

3.3.1 The edge mode theory

We now consider the boundary action. The bare BF action, (3.2.1), is consistent with either fixing A or annihilating B as a boundary condition. However, the subsequent integration over the remaining fluctuating boundary degrees of freedom is now unbounded: we must supplement this theory with a boundary Hamiltonian.[8] We will focus on quadratic boundary actions. In general dimensions these quadratic actions will be dimensionful and general lessons of effective field theory guide us to write the most relevant one. Without loss of generality we can assume that this comes from the quadratic action for the p-forms and write the following boundary action:

$$S_\partial[A_\partial, B_\partial] = \frac{\mathbb{K}}{4\pi}\mu^\Delta \int_{\partial\mathcal{X}_{n,\varepsilon}} A_\partial \wedge \star A_\partial = \frac{\mu^\Delta}{4\pi}\left\langle A_\partial, \mathbb{K}\, A_\partial\right\rangle, \tag{3.3.12}$$

[8] The need to supplement an edge Hamiltonian as a regulator is a theme that will be echoed in section 3.4.

where μ is an arbitrary energy scale and $\Delta = d - 1 - 2p$, as required by dimensional analysis. With this action, the boundary symplectic form (taking also into account the boundary terms from the variation of the bulk action) becomes:

$$\mathbf{\Theta}_{\partial \mathcal{X}_{n,\varepsilon}} = \int_{\partial \mathcal{X}_{n,\varepsilon}} \left(B_\partial + \mu^\Delta (-1)^{(d-p-1)(p+1)} \star A_\partial \right) \wedge \delta A_\partial \,. \tag{3.3.13}$$

As such, the variational principle, with the above boundary action, (3.3.12), is compatible with demanding

$$B_\partial + \mu^\Delta (-1)^{(d-p-1)(p+1)} \star A_\partial \overset{!}{=} 0, \tag{3.3.14}$$

as boundary condition. We will call (3.3.14) a "generalised chiral boundary condition."

This boundary action, (3.3.12), and the corresponding boundary condition, (3.3.14), have been known for a long time to appear on the boundary of BF theories, [228, 229]. Ignoring global issues, the flatness condition that the bulk path integral imposes on A_∂ makes this is a simple theory of a free $(p-1)$-form gauge field, also known as a singleton mode in the string theory literature. Naturally (Hodge-dually), it is also a theory of a free $(d-p-2)$-form. However, on a non-trivial topology, the boundary condition (3.3.14) imposes a constraint on the fluxes. On a manifold of the form $\mathbb{S}^1_\varepsilon \times \Sigma$ — as is our case here — both fields have fluxes obeying Dirac quantisation around purely spatial cycles, but fractional charges, when wrapping the thermal circle. This is entirely reminiscent of the story of the two-dimensional chiral boson on a torus, which can be seen to arise at the edge of a Chern–Simons theory, which is *not* modular invariant: upon demanding periodicity around the spatial circle, one cannot retain periodicity around the thermal circle. It also reflects the story of self-dual gauge fields in higher dimensions [24, 25].[9] In our case, the theory becomes a "chiral half" of a $(p-1)$-form and a $(d-p-2)$-form Maxwell theory, which we call, relatively unimaginatively, *chiral mixed Maxwell theory*. The role of chiral fields as edge modes of BF theories was recently reemphasised in [230, 231]. Note, however, that there, as well as in all other instances where chiral gauge fields make an appearance (see e.g. [24, 25, 232–240] for an incomplete list) it concerns k-form fields in $2k+2$ dimensions, where one can construct a genuinely chiral combination of the gauge fields. This is, to our knowledge, the first time a chiral pure gauge theory is constructed in generic dimensions.[10]

Let us be more concrete. First, we Hodge-decompose A_∂ as $A_\mathrm{h} + \mathrm{d}a$, where A_h is a harmonic p-form and a is the (globally-well defined) $(p-1)$-form gauge field. We do

[9] Of course our approach is much less rigorous than the original story. A rigorous analysis would require a revisit from the lens of differential cohomology. We take this as an opportunity to stress its importance for a solid understanding of gauge theories.

[10] and as a bonus, arbitrary topology

the same for B_∂. Now, if we denote by $\mathrm{d}\tau$ the volume element of the thermal circle, we can split the harmonic parts of the two fields as

$$A_\mathrm{h} = \mathcal{A} + \mathrm{d}\tau \wedge \widetilde{\mathcal{A}}, \tag{3.3.15}$$

$$B_\mathrm{h} = \mathcal{B} + \mathrm{d}\tau \wedge \widetilde{\mathcal{B}}, \tag{3.3.16}$$

where $\mathcal{A} \in \mathrm{Harm}^p(\partial R)$, $\widetilde{\mathcal{A}} \in \mathrm{Harm}^{p-1}(\partial R)$, $\mathcal{B} \in \mathrm{Harm}^{d-p-1}(\partial R)$, and $\widetilde{\mathcal{B}} \in \mathrm{Harm}^{d-p-2}(\partial R)$. Then the boundary condition (3.3.14) relates $\widetilde{\mathcal{A}}$ to $\mathcal{B}$ (and similarly $\widetilde{\mathcal{B}}$ to $\mathcal{A}$) as follows:

$$\mathcal{B} = \mu^\Delta (-1)^{(d-p)p} \star_{\partial R} \widetilde{\mathcal{A}}. \tag{3.3.17}$$

And the boundary action becomes

$$S_\partial[\mathcal{A}, \mathcal{B}] = \frac{\mu^\Delta}{4\pi} \langle \mathcal{A}, \mathbb{K}\, \mathcal{A} \rangle + \frac{\mu^{-\Delta}}{4\pi} \langle \mathcal{B}, \mathbb{K}\, \mathcal{B} \rangle + \frac{\mu^\Delta}{4\pi} \langle \mathrm{d}a\, , \mathbb{K}\ \mathrm{d}a \rangle\, . \tag{3.3.18}$$

We take this as the definition of the action of chiral mixed Maxwell theory.

Let us proceed with quantising this theory, by performing the path integral on $\partial \mathcal{X}_{n,\varepsilon}$. For notational simplicity we will perform the path integral on $\partial \mathcal{X}_{1,\varepsilon}$ and we will rescale the radius in the final formulas to obtain the partition function on $\partial \mathcal{X}_{n,\varepsilon}$. First, observe that writing the action in the form (3.3.18) is equivalent to performing the path integral over $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$ over the delta function that enforces the boundary conditions. This leaves us with

$$\mathcal{Z}_\mathrm{edge}[\partial \mathcal{X}_{1,\varepsilon}] = \mathcal{Z}_\mathrm{inst}[\partial \mathcal{X}_{1,\varepsilon}]\, \mathcal{Z}_\mathrm{osc}[\partial \mathcal{X}_{1,\varepsilon}], \tag{3.3.19}$$

where

$$\mathcal{Z}_\mathrm{inst}[\partial \mathcal{X}_{1,\varepsilon}] := \int \mathrm{D}\mathcal{A}\, \mathrm{D}\mathcal{B}\, \exp\!\left(-\frac{\mu^\Delta}{4\pi} \langle \mathcal{A}, \mathbb{K}\, \mathcal{A} \rangle + \frac{\mu^{-\Delta}}{4\pi} \langle \mathcal{B}, \mathbb{K}\, \mathcal{B} \rangle \right), \tag{3.3.20}$$

$$\mathcal{Z}_\mathrm{osc}[\partial \mathcal{X}_{1,\varepsilon}] := \int \frac{\mathrm{D}a}{\mathrm{vol}\!\left(\mathcal{G}^\partial_{p-1}\right)} \exp\!\left(-\frac{\mu^\Delta}{4\pi} \langle \mathrm{d}a\, , \mathbb{K}\ \mathrm{d}a \rangle \right). \tag{3.3.21}$$

The instanton integral is over the space of harmonic p- and $(d-p-1)$-forms on ∂R and we will evaluate it shortly. The oscillator piece is an integral over a $(p-1)$-form gauge fields modulo their gauge transformations. This is a regular set of $(p-1)$-form Maxwell oscillators.

We first deal with the oscillators. We can expand in modes of the Hodge Laplacian and integrate over those, separating the zero-modes. We must do so for the tower of ghosts that follow from the reducible gauge invariances of the $(p-1)$-form gauge fields. We show in appendix B.1, that the oscillators contribute as

$$\mathcal{Z}_\mathrm{osc}[\partial \mathcal{X}_{1,\varepsilon}] = \left(\eta^{(p-1)}_{\partial R}[q] \right)^{-\kappa} \left(\eta^{(d-p-2)}_{\partial R}[q] \right)^{-\kappa}. \tag{3.3.22}$$

In the above, q denotes the nome, $q := \mathrm{e}^{-\varepsilon\mu}$, and we have defined an analogue of the Dedekind eta function,

$$\eta_Y^{(k)}[q] := q^{-\frac{1}{2}E_0}\left(\prod_{\mathsf{n}\in\mathcal{N}_k^\perp}\sum_{N_\mathsf{n}=0}^{\infty} q^{N_\mathsf{n}\sqrt{\lambda_\mathsf{n}}}\right)^{-1/2}, \tag{3.3.23}$$

associated to the k-form Hodge Laplacian on a closed, compact manifold Y. In the above, $\mathcal{N}_k^\perp$ denotes the index-set of the non-zero spectrum of the k-form Laplacian acting on coclosed forms, λ_n are the corresponding eigenvalues, measured in units of μ, and E_0 is the (potentially divergent) zero-point energy (also in units of μ), $E_0 := \sum_{\mathsf{n}\in\mathcal{N}_k^\perp}\lambda_\mathsf{n}$. Moreover, the above is, strictly speaking, defined only up to a phase. This will not concern us; the existence or choice of a complex structure of Y is insignificant for us, since for our purposes it suffices to only consider real nomes. The definition (3.3.23) is also equivalent to

$$\eta_Y^{(k)}[q] = \left(\prod_{\mathsf{n}_k\in\mathcal{N}_k^\perp}\sinh\left(\frac{1}{2}\varepsilon\mu\sqrt{\lambda_{\mathsf{n}_k}}\right)\right)^{1/2} = \left(\prod_{\mathsf{n}_k\in\mathcal{N}_k^\perp} q^{-\frac{1}{2}\sqrt{\lambda_{\mathsf{n}_k}}}\left(1-q^{\sqrt{\lambda_{\mathsf{n}_k}}}\right)\right)^{1/2}, \tag{3.3.24}$$

Finally it is easy to see that if we take $Y = \mathbb{S}^1$, our generalised eta function reduces to the usual Dedekind eta function: $\eta_{\mathbb{S}^1}^{(0)}[q] = \eta[q]$. Moreover, since the non-zero spectra of the Laplacians acting on transversal $(p-1)$- and $(d-p-2)$-forms coincide, the eta functions associated with those are equal, we write them separately, however for reasons that will become apparent shortly.

Moving on to the instantons, we can further expand $\mathcal{A}$ and $\mathcal{B}$ in terms of the topological basis of harmonic forms on ∂R:

$$\mathcal{A}_\mathrm{h} = \mathcal{A}_\mathrm{i}\tau_\mathrm{i}^{(p)} \quad \text{and} \quad \mathcal{B}_\mathrm{h} = \mathcal{B}_\mathrm{i}\tau_\mathrm{i}^{(d-p-1)}. \tag{3.3.25}$$

Enter Dirac quantisation. Magnetic fluxes along p- and $(d-p-1)$-cycles on ∂R are quantised as

$$\int_\eta \mathcal{A} \in 2\pi\mathbb{Z}^\kappa, \quad \forall\eta\in\mathrm{H}_p(\partial R) \quad \text{and} \quad \int_\gamma \mathcal{B}\in 2\pi\mathbb{Z}^\kappa, \quad \forall\gamma\in\mathrm{H}_{d-p-1}(\partial R). \tag{3.3.26}$$

This means that $\mathcal{A}_\mathrm{i} = 2\pi n_\mathrm{i}$ and $\mathcal{B}_\mathrm{i} = 2\pi m_\mathrm{i}$, where $n_\mathrm{i}, m_\mathrm{i}\in\mathbb{Z}^\kappa$. As such, the action (3.3.18) becomes:

$$S_\partial[\mathcal{A},\mathcal{B}] = \pi\varepsilon\mu\Big(\boldsymbol{n}\cdot\left(\mathbb{K}\otimes\widetilde{\mathbb{G}}_p^{\partial R}\right)\cdot\boldsymbol{n} + \boldsymbol{m}\cdot\left(\mathbb{K}\otimes\widetilde{\mathbb{G}}_{d-p-1}^{\partial R}\right)\cdot\boldsymbol{m}\Big). \tag{3.3.27}$$

In the above, we have combined n_i and m_i into vectors $\boldsymbol{n}\in\mathbb{Z}^{\kappa\,\mathrm{b}_p(\partial R)}$ and $\boldsymbol{m}\in\mathbb{Z}^{\kappa\,\mathrm{b}_{d-p-1}(\partial R)}$. Moreover, $\widetilde{\mathbb{G}}_k^{\partial R}$ is the dimensionless version of $\mathbb{G}_k^{\partial R}$, the Gram matrix of harmonic k-forms on ∂R. This comes from the fact that $\mathbb{G}_k$ has dimension $[\mathbb{G}_k] = \mu^{1+(2k-1-d)}$, so $\widetilde{\mathbb{G}}_p := \mu^{\Delta-1}\mathbb{G}_p$ and $\widetilde{\mathbb{G}}_{d-p-1} := \mu^{1+\Delta}\mathbb{G}_{d-p-1}$ are dimensionless. Therefore, the instanton contribution reads

$$\mathcal{Z}_{\text{inst}}[\partial\mathcal{X}_{1,\varepsilon}] = \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{\kappa b_p(\partial R)} \\ \boldsymbol{m}\in\mathbb{Z}^{\kappa b_{d-p-1}(\partial R)}}} \exp\Big[-\pi\varepsilon\mu\Big(\boldsymbol{n}\cdot\Big(\mathbb{K}\otimes\widetilde{\mathbb{G}}_p^{\partial R}\Big)\cdot\boldsymbol{n} + \boldsymbol{m}\cdot\Big(\mathbb{K}\otimes\widetilde{\mathbb{G}}_{d-p-1}^{\partial R}\Big)\cdot\boldsymbol{m}\Big)\Big]. \tag{3.3.28}$$

To arrive at the final expression for the edge mode partition function, let us also introduce a Siegel-type Theta function, denoted by $\Theta[q;\mathbb{O}_N]$:

$$\Theta[q;\mathbb{O}_N] := \sum_{\boldsymbol{r}\in\mathbb{Z}^N} q^{\pi\boldsymbol{r}\cdot\mathbb{O}_N\cdot\boldsymbol{r}}. \tag{3.3.29}$$

If $\mathbb{O}_N$ is a real, $N\times N$ matrix, when $q\in\mathbb{R}_{\geqslant 0}$, it truly becomes a Siegel Theta function and inherits all the modular properties of those. Expressing the instanton contribution, (3.3.20), in terms of the Theta functions, (3.3.29), and combining with the oscillator contribution, (3.3.22), gives the following simple form for the edge mode partition function:

$$\mathcal{Z}_{\text{edge}}[\partial\mathcal{X}_{1,\varepsilon}] = \frac{\Theta\Big[q;\mathbb{K}\otimes\widetilde{\mathbb{G}}_p\Big]}{\Big(\eta_{\partial R}^{(p-1)}[q]\Big)^{\kappa}}\;\frac{\Theta\Big[q;\mathbb{K}\otimes\widetilde{\mathbb{G}}_{d-p-1}\Big]}{\Big(\eta_{\partial R}^{(d-p-2)}[q]\Big)^{\kappa}}. \tag{3.3.30}$$

We briefly re-emphasise the chiral nature of this result and comment on its importance. To get a feel for it, note first that if $\partial R = \mathbb{S}^1$, with $p = 1$, this becomes exactly the partition function of two (multi-component) chiral bosons. This resonates entirely with the usual story. In $d = 3$ BF theory can be written as a sum of two Chern–Simons theories, each of which supports a chiral scalar on its boundary. The partition function of this chiral boson is *not* modular invariant.[11] In $d = 5$ dimensions, a smoking gun signal that the edge theory is chiral is the fact that it is not S-duality invariant, as can be seen by comparing to the regular Maxwell partition function (cf. appendix B.1). In [229], S-duality of the edge mode theory was crucial in proving an obstruction to symmetry-preserving regulators of edge-states of BF theory.[12] Our analysis provides evidence that this may not be the case; i.e. the fact that the UV lattice models regularising Maxwell theory break S-duality, while correct, does not imply that these models are symmetry breaking from the point of view of edge states of BF theory. Moving away from $d = 5$, for generic dimensions, one can take a half of our partition function as the definition of an abelian chiral gauge theory, but note

[11]This fact, $\mathcal{Z}_{\text{chiral boson}}(\beta) \neq \mathcal{Z}_{\text{chiral boson}}(1/\beta)$, is actually crucial for extracting the correct entanglement entropy of bulk Chern-Simons theory [201]!

[12]This was, in turn interpreted as a gauge theory version of the Nielsen–Ninomiya theorem [241, 242].

that only this combination is well-defined, at least from a path integral perspective. A systematic understanding of this chiral theory deserves further research.

In the next section, we will move on to extract the entanglement entropy. To do that, we will need to compare the edge mode partition function on $\partial\mathcal{X}_{n,\varepsilon}$ to that on $\partial\mathcal{X}_{1,\varepsilon}$. The $\partial\mathcal{X}_{n,\varepsilon}$ partition function can be easily obtained by rescaling the radius of the $\mathbb{S}^1$ to $n\varepsilon$. This results in

$$\mathcal{Z}_{\text{edge}}[\partial\mathcal{X}_{n,\varepsilon}] = \frac{\Theta\left[q^n; \mathbb{K}\otimes\widetilde{\mathbb{G}}_p\right]}{\left(\eta_{\partial R}^{(p-1)}[q^n]\right)^{\kappa}} \frac{\Theta\left[q^n; \mathbb{K}\otimes\widetilde{\mathbb{G}}_{d-p-1}\right]}{\left(\eta_{\partial R}^{(d-p-2)}[q^n]\right)^{\kappa}}. \tag{3.3.31}$$

3.3.2 Extracting the entanglement entropy

We can now simply insert (3.3.30) and (3.3.31) into (3.3.11) to extract the entanglement entropy. We have two kinds of terms to deal with; the Theta functions and the eta functions. Let us deal with them separately.

We treat first the instantons, i.e. the Theta functions, for it will be easier to extract their high-temperature behavior. Fortunately, Siegel Theta functions are very well behaved and obey a sort of modularity equation, which can be easily proved using the Poisson summation formula. Explicitly, they obey the following transformation equation [243]:

$$\Theta[\mathrm{e}^{-\varepsilon\mu}; \mathbb{O}_N] = (\varepsilon\mu)^{-\frac{N}{2}} |\det\mathbb{O}_N|^{-\frac{1}{2}} \Theta\left[\mathrm{e}^{-\frac{1}{\varepsilon\mu}}; \mathbb{O}_N^{-1}\right]. \tag{3.3.32}$$

The high-temperature behavior is then straightforward to obtain:

$$\Theta[\mathrm{e}^{-\varepsilon\mu}; \mathbb{O}_N] \overset{\varepsilon\to 0}{\sim} (\varepsilon\mu)^{-\frac{N}{2}} |\det\mathbb{O}_N|^{-\frac{1}{2}}. \tag{3.3.33}$$

With that, the total contribution of the Theta functions to the entanglement entropy is

$$\begin{aligned} \mathcal{S}_{\text{EE}}\Big|_{\Theta} &= \lim_{\varepsilon\to 0}\lim_{n\to 1}\frac{1}{1-n}\log\frac{\Theta\left[q^n; \mathbb{K}\otimes\widetilde{\mathbb{G}}_p^{\partial R}\right]\Theta\left[q^n; \mathbb{K}\otimes\widetilde{\mathbb{G}}_{d-p-1}^{\partial R}\right]}{\left(\Theta\left[q; \mathbb{K}\otimes\widetilde{\mathbb{G}}_p^{\partial R}\right]\Theta\left[q; \mathbb{K}\otimes\widetilde{\mathbb{G}}_{d-p-1}^{\partial R}\right]\right)^n} \\ &= \frac{1}{2}\left(\mathrm{b}_p(\partial R) + \mathrm{b}_{d-p-1}(\partial R)\right)\left[\kappa(1-\log(\varepsilon\mu)) - \log|\det\mathbb{K}|\right]. \end{aligned} \tag{3.3.34}$$

As we shall see shortly, this term will be partially cancelled by the contribution of the eta functions, ultimately resulting in the topological piece of the entanglement entropy.

Let us get to it then. To evaluate the contribution of the eta functions, we will use the following trick [244, 245]. In words, we will Taylor expand $\log(\eta_{\partial R})^{-1}$ and on each term perform an inverse Mellin tranform.[13] Written as a Mellin integral, the sum over the eigenvalues, $\{\lambda_{\mathrm{n}}\}$ is easily performed to yield the spectral zeta function,

$$\zeta^{(k)}_{\partial R}(s) := \sum_{\mathrm{n}\in\mathcal{N}_k^{\perp}} \lambda_{\mathrm{n}}^{-s}. \tag{3.3.35}$$

In equations:

$$\begin{aligned} -\sum_{\mathrm{n}\in\mathcal{N}_k^{\perp}} \log\Big(1-\mathrm{e}^{-\varepsilon\mu\sqrt{\lambda_{\mathrm{n}}}}\Big)^{-1} &\overset{\text{Taylor}}{=\!=} \sum_{\mathrm{n}\in\mathcal{N}_k^{\perp}} \sum_{m=1}^{\infty} \frac{1}{m}\mathrm{e}^{-\varepsilon\mu\, m\sqrt{\lambda_{\mathrm{n}}}} \\ &\overset{\text{Mellin}^{-1}}{=\!=} \sum_{\mathrm{n}\in\mathcal{N}_k^{\perp}} \sum_{m=1}^{\infty} \frac{1}{m}\int_{c+\mathrm{i}\mathbb{R}} \frac{\mathrm{d}u}{2\pi\mathrm{i}}\Gamma(u)\Big(\varepsilon\mu\, m\sqrt{\lambda_{\mathrm{n}}}\Big)^{-u} \\ &\overset{\text{resum}}{=\!=} \int_{c+\mathrm{i}\mathbb{R}} \frac{\mathrm{d}u}{2\pi\mathrm{i}}\, \zeta(u+1)\, \zeta^{(p-1)}_{\partial R}\Big(\frac{u}{2}\Big)\, \Gamma(u)\, (\varepsilon\mu)^{-u}, \end{aligned} \tag{3.3.36}$$

where $\zeta(u)$ is the ordinary Riemann zeta function. In the above integral $c > 0$ is a positive number lying to the right of the rightmost pole of the integrand. To extract the leading terms as $\varepsilon \to 0$, we then imagine pushing the integration contour to $-1 < c < 0$, so that the integrand vanishes as $\varepsilon \to 0$. The price we pay is that, as we sweep to the left, we pick up the residues from all of the poles of the integrand.

$$\lim_{\varepsilon\to 0} \log\Big(\eta^{(k)}_{\partial R}(q)\Big)^{-1} = \frac{1}{2}\sum_{u_*\geqslant 0} \operatorname*{Res}_{u=u_*}\, \zeta(u+1)\, \zeta^{(k)}_{\partial R}\Big(\frac{u}{2}\Big)\, \Gamma(u)\, (\varepsilon\mu)^{-u}. \tag{3.3.37}$$

In short, evaluating the descendant contribution to $\mathcal{S}_{\mathrm{EE}}$ reduces to analyzing the pole structure of the above expression. In appendix B.2 we carefully analyse the poles of $\zeta^{(k)}_{\partial R}$ which all lie at $u > 0$. Inside (3.3.37) there is a double pole at $u = 0$ from $\Gamma(u)\zeta(u+1)$, with residue

$$\begin{aligned} \mathrm{Res}^{(k)}_0 &= \frac{1}{2}\zeta^{(k)\prime}_{\partial R}(0) - \zeta^{(k)}_{\partial R}(0)\log(\varepsilon\mu) \\ &= \Big(-\mathcal{I}^{(k)}_{\frac{d-2}{2}}\,\delta_{d,\mathrm{even}} + \mathrm{b}_k(\partial R)\Big)\log(\varepsilon\mu). \end{aligned} \tag{3.3.38}$$

In the above, $\mathcal{I}^{(k)}_{\frac{d-2}{2}}$ is the $\left(\frac{d-2}{2}\right)$-th heat kernel coefficient for the spectral zeta function arising from the value of $\zeta^{(k)}_{\partial R}(0)$ and $\mathrm{b}_k(\partial R)$, the k-th Betti number enumerates the number of zero-modes of the k-form Laplacian, again reviewed in appendix B.2.

[13] note that the piece containing the zero-point-energy, E_0, in (3.3.23) or (3.3.24) obviously does not contribute to the entropy so we will focus on the rest

We see then that this term, evaluated for the eta functions associated to the $(p-1)$- and the $(d-p-2)$-form Laplacians, contributes to the entropy as:

$$\mathcal{S}_{\rm EE}\Big|_{\rm Res_0} = \frac{\kappa}{2}\Bigg[\Big(\mathcal{I}^{(p-1)}_{\frac{d-2}{2}} + \mathcal{I}^{(d-p-2)}_{\frac{d-2}{2}}\Big)\delta_{d,\rm even} - \big(\mathrm{b}_{p-1}(\partial R) + \mathrm{b}_{d-p-2}(\partial R)\big)\Bigg](1-\log(\varepsilon\mu)). \tag{3.3.39}$$

We see a happy cancellation between (part of) this term and (part of) the contribution of the Thetas, (3.3.34), owing to Poincaré duality $\mathrm{b}_{p-1}(\partial R) = \mathrm{b}_{d-p-1}(\partial R)$ and $\mathrm{b}_{d-p-2}(\partial R) = \mathrm{b}_p(\partial R)$.

The residues away from zero come from the poles of $\zeta^{(p-1)}_{\partial R}(u/2)$.[14] These lie at $d-2k$, for integer $k \geqslant 1$[15] (with $k \leqslant d/2-1$ when d is even) and give in total

$$\mathrm{Res}_{2d-k} = (4\pi)^{1-\frac{d}{2}}\zeta(d-2k+1)\frac{\Gamma(d-2k)}{\Gamma\left(\frac{d}{2}-k\right)}\mathcal{I}^{(p-1)}_{k-1}(\varepsilon\mu)^{2k-d}, \tag{3.3.40}$$

where, again, $\mathcal{I}^{(p-1)}_{k-1}$ are heat kernel coefficients for the spectral zeta function. They can be written as integrals of local geometrical (but not topological) data of the entangling surface. These residues contribute area law (and subleading) terms to the entropy:

$$\mathcal{S}_{\rm EE}\Big|_{\mathrm{Res}_{d-2k}} = \mathrm{I}^{(p-1)}_k(\varepsilon\mu)^{2k-d}, \tag{3.3.41}$$

where

$$\mathrm{I}^{(p-1)}_k := \kappa(4\pi)^{1-\frac{d}{2}}\zeta(d-2k+1)\frac{\Gamma(d-2k)}{\Gamma\left(\frac{d}{2}-k\right)}(d-2k+1)\,\mathcal{I}^{(p-1)}_{k-1}. \tag{3.3.42}$$

As expected, the area-law terms are non-universal, depend on the geometry of the entangling surface (through the heat-kernel coefficients, $\mathcal{I}^{(p-1)}_{k-1}$) and the regulator, ε.

Assembling the oscillator contributions and adding the contribution of the instantons, we arrive at the main result for the entanglement entropy:

$$\mathcal{S}_{\rm EE} = \sum_{k=1}^{\lfloor\frac{d-1}{2}\rfloor}\mathrm{C}^{(p-1)}_k\left(\frac{\ell}{\varepsilon}\right)^{d-2k} + \frac{\kappa}{2}\Big(\mathcal{I}^{(p-1)}_{\frac{d-2}{2}} + \mathcal{I}^{(d-p-2)}_{\frac{d-2}{2}}\Big)\delta_{d,\rm even}\log\left(\frac{\ell}{\varepsilon}\right) - \frac{1}{2}\big(\mathrm{b}_p(\partial R) + \mathrm{b}_{d-p-1}(\partial R)\big)\log|\det\mathbb{K}|, \tag{3.3.43}$$

[14]Recall that apart from the zero-modes, the eta functions for $(p-1)$- and for $(d-p-2)$-forms are identical.

[15]Note that there is a shift $k \to k-1$ compared with appendix B.2.

where $\mathrm{C}_k := \mathrm{I}_k \mathrm{e}^{2k-d}$ and we traded the energy scale for a length scale $\ell \equiv \mu^{-1}\mathrm{e}$ as is more common in the presentation of entanglement entropies. In the above formula, we have ignored terms with $k > \left\lfloor \frac{d-1}{2} \right\rfloor$ (which are present only when d is odd), since they give vanishing contributions in the limit $\varepsilon \to 0$.

On extracting universal features:

Some comments are in order regarding our main result, (3.3.43). In odd dimensions we have a universal subleading correction given by

$$\mathcal{S}_{\mathrm{TEE}} = -\frac{1}{2}\big(\mathrm{b}_p(\partial R) + \mathrm{b}_{d-p-1}(\partial R)\big) \log|\det \mathbb{K}|. \tag{3.3.44}$$

In even dimensions, however, the log term

$$\frac{\kappa}{2}\left(\mathcal{I}^{(p-1)}_{\frac{d-2}{2}} + \mathcal{I}^{(d-p-2)}_{\frac{d-2}{2}}\right) \log\left(\frac{\ell}{\varepsilon}\right), \tag{3.3.45}$$

spoils the universality of the $\log|\det \mathbb{K}|$ as rescalings of the cutoff result in constant shifts of $\mathcal{S}_{\mathrm{EE}}$. Under general arguments, there is nothing that prohibits log contributions to the entanglement entropy in even dimensions and, indeed, here we find such a contribution. The coefficient of this log is a potentially universal piece of data, but in fact it depends on the geometry of the entangling surface through its heat kernel coefficient. On top of this, throughout the chapter we had been neglectful of terms proportional to the Euler characteristic, $\chi(\partial R)$. The main reason is that, while topological, they ultimately arise (or can be absorbed into) ambiguities in the edge theory path integral measure and so are non-universal. This non-universality is reflected in the entropy: the Euler characteristic can be recast as the integral of a local quantity, through the generalised Gauss–Bonnet theorem. Nevertheless, it was explained in [188] that a generalisation of the Kitaev–Preskill / Levin–Wen protocol [165, 166] is possible in higher dimensions with an explicit construction in $d = 4$. By this, it is meant that one can decompose the entangling surface, ∂R, into pieces, ∂R_i of specific geometry. Then one can consider a linear combination of entanglement entropies across ∂R_i, such that, when surgered together, the locally integrated contributions to the entropy cancel out. The upshot is that we expect, through such a construction, one can extract in all dimensions the topological entanglement entropy:

$$\mathcal{S}_{\mathrm{TEE}} = -\frac{1}{2}\big(\mathrm{b}_p(\partial R) + \mathrm{b}_{d-p-1}(\partial R)\big) \log|\det \mathbb{K}|, \tag{3.3.46}$$

although we do not do so here.

As an example, consider four-dimensional BF theory and take $p = 1$ or $p = 2$. The two cases are interchangeable, as they are mapped to each other by interchanging the roles of the A and B fields in the original action (3.2.1). Generally, it holds that

$\mathcal{I}_1^{(0)} = \frac{1}{6}\chi(\partial R)$ [246]. Taking the entangling surface to be a torus, $\partial R = \mathbb{T}^2$, for $p = 1$ we have $\mathcal{I}_1^{(0)} = 0$. One can explicitly show that the same holds for $p = 2$, as expected, by analyzing the spectrum of the transverse Laplacian acting on one-forms. Therefore, the topological entanglement entropy for the four-dimensional case with an entangling surface with the topology of a torus is given by

$$\mathcal{S}_{\text{TEE}} = -2\log|\det\mathbb{K}|. \tag{3.3.47}$$

Note that in four dimensions the subleading part of the entropy is always topological (although possibly not universal), since $\mathcal{I}_{\frac{d-2}{2}}^{(p-1)}$ is given by the Euler characteristic. In contrast, in higher dimensions, this is not the case. In six dimensions, for example,

$$\mathcal{I}_2^{(0)} = \frac{1}{180}\int_{\partial R} \mathrm{d}\,\mathrm{vol}_{\partial R}\ \left(10\mathcal{R}^2 - \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + 2\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\right),$$

which depends sensitively on the geometry of the entangling surface, and therefore one truly needs a suitable Kitaev–Preskill / Levin–Wen-like subtraction scheme to extract the universal pieces of data.

3.4 The extended Hilbert space and a current algebra

Above we have presented a replica path integral calculation of the entanglement entropy. We have focussed on the entropy that comes entirely from an "edge mode theory," which we have coined as "chiral mixed Maxwell," in the high-temperature limit; this essentially counts a regulated Hilbert space dimension of this edge theory. In this section, we provide a more honest accounting of the entanglement entropy, accounting for the subtleties of gauge invariance. This will provide an alternative view of the role of these edge modes as providing an arena, the "extended Hilbert space" (EHS), by which the entanglement entropy can be precisely defined. We will see that a physical state is embedded as a maximally entangled state in this EHS and thus its entanglement entropy is naturally interpreted as a dimension of the EHS, which we will regulate. The upshot of this section is that we will reproduce the topological entanglement entropy from a more rigorous starting point, while also providing a, perhaps, intuitive view on the need for the edge theory and why it contributes to the entanglement entropy.

Along the way we will show that the EHS is completely organised by an infinite-dimensional spectrum-generating current algebra, entirely analogous to the Kac-Moody algebras appearing at the edge of three-dimensional topological phases. In general dimensions this algebra is non-conformal. Regardless, we will show that it is powerful enough to fix the entire computation of the entanglement entropy. We will also show that this algebra, remarkably, generates the entire spectrum of the edge mode theory from section 3.3. The fact that the spectrum of the chiral Maxwell theory is entirely fixed by a (potentially non-conformal) current algebra echoes the results of [3, 37] for 1-form Maxwell theories in four-dimensions. Here we present the generic story which we regard as a major result of this chapter.

3.4.1 Gauge-(in)variance, entanglement, and the extended Hilbert space

The typical starting point of the entanglement entropy is the supposition of a Hilbert space factorisation between a region, R, and its complement, R^{c}:

$$\mathcal{H} = \mathcal{H}_R \otimes \mathcal{H}_{R^{\mathrm{c}}}. \tag{3.4.1}$$

This supposition typically fails in quantum field theory due to an infinite number of correlated short-distance modes,[16] however, one can imagine regulating this computation by a short-distance regulator, ε, (say by putting the system on a lattice, or utilising a mutual information regulator [248]). However, even in a regulated scenario, (3.4.1), fails for quantum gauge theories due to the global nature of gauge constraints that physical states must satisfy [174, 182, 184, 185, 218, 248, 249]. It is, by now, well understood how to properly define entanglement entropy in gauge theories. One method is to utilise an algebraic definition of the von Neumann entropy applied to the reduced density matrix realised directly as a gauge-invariant operator [248]. We have explored that aspect of entanglement entropy in this family of theories in chapter 2. Here we focus on the alternative formulation, which goes by the name of the "extended Hilbert space" [174, 182, 184, 185, 218].

In short, while the physical Hilbert space, $\mathcal{H}$, cannot be realised as a tensor product, we can embed $\mathcal{H}$ (let us call the embedding $\mathcal{I}$) into an EHS, $\mathcal{H}_{\mathrm{ext}}$, which admits a tensor product.

$$\mathcal{H} \overset{\mathcal{I}}{\hookrightarrow} \mathcal{H}_{\mathrm{ext}} := \mathcal{H}_R \otimes \mathcal{H}_{R^{\mathrm{c}}}. \tag{3.4.2}$$

By definition, $\mathcal{H}_{\mathrm{ext}}$ is furnished with unphysical, gauge-variant states. These are states in either $\mathcal{H}_R$ or $\mathcal{H}_{R^{\mathrm{c}}}$ which carry the action of gauge transformations acting on the entangling surface, ∂R. That is to say, it is furnished with edge mode states living on ∂R. Given a physical state $|\psi\rangle \in \mathcal{H}$ it embeds to a state $|\widetilde{\psi}\rangle \in \mathcal{H}_{\mathrm{ext}}$ which we can then tensor-factorise and reduce upon R:

$$\widetilde{\rho}_R := \mathrm{Tr}_{\mathcal{H}_{R^{\mathrm{c}}}} |\widetilde{\psi}\rangle\langle\widetilde{\psi}|, \tag{3.4.3}$$

and the entanglement entropy can defined in the typical way:

$$\mathcal{S}_{\mathrm{EE}} := -\mathrm{Tr}_{\mathcal{H}_R}(\widetilde{\rho}_R \log \widetilde{\rho}_R). \tag{3.4.4}$$

In practice, this can be computed utilising the replica trick we reviewed above. The important step in this process, however, is identifying the embedding map, $\mathcal{I}$, and thus

[16] More modernly, and more precisely, it fails due to the type-III nature of the von Neumann algebra associated to a subregion; see e.g. [247] and references therein.

the appropriate embedded state $|\widetilde{\psi}\rangle \in \mathcal{H}_{\text{ext}}$. Since $\mathcal{H}_{\text{ext}}$ is furnished (by definition) by gauge-variant states, this embedding is determined by demanding gauge-invariance by hand: i.e. if a gauge transformation labeled by α with support on ∂R acts on $\mathcal{H}_R$ and $\mathcal{H}_{R^c}$ with charge operators $Q_R[\alpha]$ and $Q_{R^c}[\alpha]$, respectively, then we will demand

$$(Q_R[\alpha] \otimes 1 + 1 \otimes Q_{R^c}[\alpha])\, |\widetilde{\psi}\rangle \stackrel{!}{=} 0. \tag{3.4.5}$$

We will refer to (3.4.5) as the "quantum gluing condition." Because α can be arbitrarily local to ∂R, this induces correlations to $|\widetilde{\psi}\rangle$ that are local to ∂R and maximally entangled. As a result $\widetilde{\rho}_R$ will be maximally mixed amongst an infinite number of edge modes that are local to ∂R. Regulating this maximal mixture will result in a (divergent) area law (with possibly subleading area laws) to the entanglement entropy, but there may also exist universal corrections. We will make these ideas concrete in this theory below. To begin, we will first identify a suitable $\mathcal{H}_R$ with which to build the extended Hilbert space.

3.4.2 A current algebra

We will construct $\mathcal{H}_R$ by quantising the theory on $X = \mathbb{R}\times R$. Many of the details here follow the opening sections of [1] for quantising BF theory, however, now allowing for the existence of a boundary $\partial R \neq \varnothing$. We refer the reader there for a more detailed accounting of the procedure and notation and point out only the necessary features special to the current situation here. We will often use the embedding map of the boundary $i_{\partial R} : \partial R \hookrightarrow R$, and $i_{\partial X} : \partial X \hookrightarrow X$, as well as the embedding map of R into X: $i_R : R \hookrightarrow X$.

We split $A = A_0 + a$ and $B = B_0 + b$ with A_0 and B_0 with "a leg" along $\mathbb{R}$ and a and b along R. Correspondingly the action (3.2.1) splits,

$$S_{\text{BF}}[A_0 + a, B_0 + b] = \frac{\mathbb{K}}{2\pi} \int_X \Big((-1)^{d-p} \mathbf{d} b \wedge A_0 + B_0 \wedge \mathbf{d} a + b \wedge da \Big), \tag{3.4.6}$$

once we impose boundary conditions $i^*_{\partial X} A_0 = i^*_{\partial X} B_0 = 0$. Above $\mathbf{d}$ should be regarded as the exterior derivative along R. A_0 and B_0 impose flatness of a and b on R:

$$\mathbf{d} a = \mathbf{d} b = 0, \tag{3.4.7}$$

which we will refer to as the Gauss law constraints. The residual gauge transformations, preserving the boundary conditions act on a and b as

$$\delta b = \mathbf{d}\beta\,, \qquad i^*_{\partial R} \mathrm{d}_{\mathbb{R}} \beta = 0 \tag{3.4.8}$$

$$\delta a = \mathbf{d}\alpha\,, \qquad i^*_{\partial R} \mathrm{d}_{\mathbb{R}} \alpha = 0. \tag{3.4.9}$$

Recall that higher-form gauge theories come with a tower of lower (secondary, tertiary, quaternary, etc.) gauge-invariances. These lower invariances indicate some redundancy in what arises as a global symmetries when R has a boundary. Later on, we will impose restrictions on the gauge parameters that survive all the way to the boundary, to take care of this.

The charges associated to (3.4.8) can be built from the symplectic form

$$\Omega_R = (-1)^{d-p-1}\frac{\mathbb{K}}{2\pi}\int_R \delta b \wedge \delta a. \tag{3.4.10}$$

Above, we view δb and δa as variational one-forms. We define *variational vectors*, $\boldsymbol{v}_\alpha$ and $\boldsymbol{v}_\beta$ such that their interior product gives (3.4.8):

$$\boldsymbol{v}_\alpha \lrcorner\, \delta a = \mathbf{d}\alpha \quad \text{and} \quad \boldsymbol{v}_\beta \lrcorner\, \delta b = \mathbf{d}\beta. \tag{3.4.11}$$

The charges, given by

$$\delta Q[\alpha] := \boldsymbol{v}_\alpha \lrcorner\, \Omega_R \quad \text{and} \quad \delta Q[\beta] := \boldsymbol{v}_\beta \lrcorner\, \Omega_R, \tag{3.4.12}$$

can then be found as

$$\begin{aligned} Q[\alpha] &:= (-1)^{d-p}\frac{\mathbb{K}}{2\pi}\int_R b\wedge \mathbf{d}\alpha = -\frac{\mathbb{K}}{2\pi}\int_{\partial R} b\wedge\alpha, \\ Q[\beta] &:= (-1)^{d-p-1}\frac{\mathbb{K}}{2\pi}\int_R \mathbf{d}\beta\wedge a = (-1)^{d-p-1}\frac{\mathbb{K}}{2\pi}\int_{\partial R}\beta\wedge a \end{aligned} \tag{3.4.13}$$

up to a total variation.[17] The second equality in each line emphasises that the charges localise to ∂R upon imposing the Gauss law constraint, (3.4.7). When these charges do not vanish identically, then they are genuine, global, symmetries of the system, i.e. they act on and transform states. As alluded to above, there is still a need to fix an additional gauge redundancy: if $i^*_{\partial R}\alpha = \mathbf{d}\gamma$ for some $\gamma \in \Omega^{p-2}(\partial R)$ then $Q[\alpha]$ is identically zero by pulling the Gauss constraint on b back to ∂R (a similar argument follows for β). We fix this by imposing

$$\mathrm{d}^\dagger i^*_{\partial R}\alpha = \mathrm{d}^\dagger i^*_{\partial R}\beta = 0. \tag{3.4.14}$$

This constraint fixes completely all the lower-invariances. The algebra of the charges is also given by the symplectic form as

$$\{Q[\alpha], Q[\beta]\}_R = \Omega_R(\boldsymbol{v}_\beta, \boldsymbol{v}_\alpha) = (-1)^{d-p-1}\frac{\mathbb{K}}{2\pi}\int_R \mathbf{d}\beta\wedge\mathbf{d}\alpha. \tag{3.4.15}$$

The canonical quantisation of the charges, $\{\bullet, \circ\} \to -\mathrm{i}[\bullet, \circ]$, then yields a centrally extended algebra

$$\left[\hat{Q}[\alpha], \hat{Q}[\beta]\right]_R = \mathrm{i}(-1)^{d-p-1}\frac{\mathbb{K}}{2\pi}\int_R \mathbf{d}\beta\wedge\mathbf{d}\alpha = \mathrm{i}\frac{\mathbb{K}}{2\pi}\int_{\partial R}\mathbf{d}\beta\wedge\alpha. \tag{3.4.16}$$

[17]This total variation ambiguity doesn't affect the algebra of the charges, of course.

3.4.3 Mode expansion of the current algebra

We want to turn the algebra (3.4.16) into a countable current algebra. For that we want to decompose our various forms into modes of a common basis and find their mode algebra.

In the following, we restrict ourselves within ∂R and drop the bold differential. It is understood that everything below has been pulled back to ∂R. We will keep in mind our restriction that the forms α and β are coclosed (though not necessarily co-exact) in ∂R. The set of eigen-k-forms of the transversal Laplacian, $\mathrm{d}^\dagger \mathrm{d}\big|_{\Omega^k}$, provides a basis for the coclosed k-forms. Poincaré duality and the Hodge decomposition tell us that the non-zero spectrum of $\mathrm{d}^\dagger\mathrm{d}$ acting on k-forms on a D-dimensional compact manifold is the same as its spectrum acting on $(D-k-1)$-forms, since they are both related to the spectrum of $\mathrm{d}\mathrm{d}^\dagger$ on $(D-k)$-forms.

So, as far as the non-zero-modes are of concern, in our case, we need the bases provided by the $(p-1)$-forms and $(d-p-2)$-forms. Since the non-zero spectra of $\mathrm{d}^\dagger\mathrm{d}$ on the $(d-2)$-dimensional manifold ∂R on these two spaces coincide, these forms will be labeled by the same index set $\{\mathsf{n} \in \mathcal{N}_\perp^*\}$. Namely, we have two orthonormal bases

$$\{\varphi_\mathsf{n} \in \Omega^{p-1}(\partial R)\}_{\mathsf{n}\in\mathcal{N}_\perp^*} \quad \text{and} \quad \left\{\chi_\mathsf{n} \in \Omega^{d-p-2}(\partial R)\right\}_{\mathsf{n}\in\mathcal{N}_\perp^*}, \tag{3.4.17}$$
$$\langle \varphi_\mathsf{n}, \varphi_\mathsf{m}\rangle = \delta_{\mathsf{nm}} = \langle \chi_\mathsf{n}, \chi_\mathsf{m}\rangle\,,$$

with

$$\begin{aligned} \mathrm{d}^\dagger\mathrm{d}\varphi_\mathsf{n} &= \lambda_\mathsf{n}\varphi_\mathsf{n}, \\ \mathrm{d}^\dagger\mathrm{d}\chi_\mathsf{n} &= \lambda_\mathsf{n}\chi_\mathsf{n}, \end{aligned} \qquad \lambda_\mathsf{n} \neq 0 \;\forall \mathsf{n} \in \mathcal{N}_\perp^*. \tag{3.4.18}$$

Now, coming to the zero-modes, these are simply the harmonic $(p-1)$- and $(d-p-2)$-forms on ∂R. The natural bases to expand those are the topological bases

$$\left\{\tau_\mathsf{i}^{(p-1)}\right\}_{\mathsf{i}=1}^{\mathrm{b}_{p-1}(\partial R)} \quad \text{and} \quad \left\{\tau_\mathsf{i}^{(d-p-2)}\right\}_{\mathsf{i}=1}^{\mathrm{b}_{d-p-2}(\partial R)}, \tag{3.4.19}$$

satisfying

$$\int_{\eta^\mathsf{j}} \tau_\mathsf{i}^{(p-1)} = \delta_\mathsf{i}^\mathsf{j} = \int_{\sigma^\mathsf{j}} \tau_\mathsf{i}^{(d-p-2)}, \tag{3.4.20}$$

for all $(p-1)$- and $(d-p-2)$-cycles η^j and σ^j respectively.

We can then expand:

$$\alpha = \sum_{\mathrm{n}\in\mathcal{N}_\perp^*} \alpha^{\mathrm{n}}\varphi_{\mathrm{n}} + \sum_{\mathrm{i}=1}^{\mathrm{b}_{p-1}} \alpha^{\mathrm{i}}\tau_{\mathrm{i}}^{(p-1)}, \tag{3.4.21}$$

$$\beta = \sum_{\mathrm{n}\in\mathcal{N}_\perp^*} \beta^{\mathrm{n}}\chi_{\mathrm{n}} + \sum_{\mathrm{i}=1}^{\mathrm{b}_{d-p-2}} \beta^{\mathrm{i}}\tau_{\mathrm{j}}^{(d-p-2)}. \tag{3.4.22}$$

Similarly, we can expand $\hat{a}$ and $\hat{b}$ as

$$\begin{aligned}\hat{a} &= (-1)^{d-p-1}2\pi\big[\mathbb{K}^{-1}\big]\left(\sum_{\mathrm{n}\in\mathcal{N}_\perp^*} \hat{a}_{\mathrm{n}} \star \chi_{\mathrm{n}} + \sum_{\mathrm{i}=1}^{\mathrm{b}_p} \hat{a}_{0\mathrm{i}} \star \tau_{\mathrm{i}}^{(d-p-2)}\right),\\ \hat{b} &= -2\pi\big[\mathbb{K}^{-1}\big]\left(\sum_{\mathrm{n}\in\mathcal{N}_\perp^*} \hat{b}_{\mathrm{n}} \star \varphi_{\mathrm{n}} + \sum_{\mathrm{i}=1}^{\mathrm{b}_{d-p-1}} \hat{b}_{0\mathrm{i}} \star \tau_{\mathrm{i}}^{(p-1)}\right).\end{aligned} \tag{3.4.23}$$

so that the charges become

$$\hat{Q}[\alpha] = \sum_{\mathrm{n}\in\mathcal{N}_\perp^*} \hat{b}_{\mathrm{n}}\alpha^{\mathrm{n}} + \sum_{\mathrm{i,j}=1}^{\mathrm{b}_{p-1}} \hat{b}_{0\mathrm{i}}\Big[\mathbb{G}_{p-1}^{\partial R}\Big]_{\mathrm{ij}}\alpha^{\mathrm{i}}, \tag{3.4.24}$$

$$\hat{Q}[\beta] = \sum_{\mathrm{n}\in\mathcal{N}_\perp^*} \hat{a}_{\mathrm{n}}\beta^{\mathrm{n}} + \sum_{\mathrm{i,j}=1}^{\mathrm{b}_{d-p-2}} \hat{a}_{0\mathrm{i}}\Big[\mathbb{G}_{d-p-2}^{\partial R}\Big]_{\mathrm{ij}}\beta^{\mathrm{i}}, \tag{3.4.25}$$

where we note again that by Poincaré duality it holds that $\mathbb{G}_k^{\partial R} = \big[\mathbb{G}_{d-2-k}^{\partial R}\big]^{-1}$, up to an invertible matrix which can be set to unity by a choice of basis.

Equivalently, we can invert (3.4.23) to write the individual modes as

$$\hat{a}_{\mathrm{n}} = (-1)^{d-p-1}\frac{\mathbb{K}}{2\pi}\int_{\partial R} \chi_{\mathrm{n}} \wedge \hat{a}, \tag{3.4.26}$$

$$\hat{a}_{0\mathrm{i}} = (-1)^{d-p-1}\frac{\mathbb{K}}{2\pi}\sum_{\mathrm{j}=1}^{\mathrm{b}_p}\Big[\mathbb{G}_p^{\partial R}\Big]_{\mathrm{ij}}\int_{\partial R} \tau_{\mathrm{j}}^{(d-p-2)} \wedge \hat{a}, \tag{3.4.27}$$

$$\hat{b}_{\mathrm{n}} = -\frac{\mathbb{K}}{2\pi}\int_{\partial R} \varphi_{\mathrm{n}}\hat{b}, \tag{3.4.28}$$

$$\hat{b}_{0\mathrm{i}} = -\frac{\mathbb{K}}{2\pi}\sum_{\mathrm{j}=1}^{\mathrm{b}_{d-p-1}^{\partial R}}\Big[\mathbb{G}_{d-p-1}^{\partial R}\Big]_{\mathrm{ij}}\int_{\partial R} \tau_{\mathrm{j}}^{(p-1)} \wedge \hat{b}. \tag{3.4.29}$$

With this mode expansion, the current algebra (3.4.16) is written in modes as

$$\left[\hat{b}_{\mathrm{n}}, \hat{a}_{\mathrm{m}}\right] = \frac{\mathrm{i}\,\mathbb{K}}{2\pi}\, C_{\mathrm{nm}}, \qquad \mathrm{n}, \mathrm{m} \in \mathcal{N}_{\perp}^{*}, \quad \text{where} \tag{3.4.30}$$

$$C_{\mathrm{nm}} := \int_{\partial R} \mathrm{d}\chi_{\mathrm{m}} \wedge \varphi_{\mathrm{n}}. \tag{3.4.31}$$

Of course, the zero-modes have trivial commutators with all the other modes as well as amongst themselves. The final step to pin down the algebra completely is to analyse the matrix C_{nm}. Observe that the two bases of the non-zero sectors are related by

$$\mathrm{d}\chi_{\mathrm{n}} = \sum_{\mathrm{m}\in\mathcal{N}_{\perp}^{*}} C_{\mathrm{mn}} \star \varphi_{\mathrm{m}}. \tag{3.4.32}$$

Let us momentarily assume that there is no degeneracy of eigenvalues, i.e. $\lambda_{\mathrm{n}} = \lambda_{\mathrm{m}} \Rightarrow \mathrm{n} = \mathrm{m}$ and act on (3.4.32) with $\mathrm{d}\mathrm{d}^{\dagger}$. We get that for all $\mathrm{n} \in \mathcal{N}_{\perp}^{*}$ we must have

$$\sum_{\mathrm{m}\in\mathcal{N}_{\perp}^{*}} C_{\mathrm{mn}}(\lambda_{\mathrm{n}} - \lambda_{\mathrm{m}}) = 0. \tag{3.4.33}$$

This can only be achieved if C_{mn} is diagonal: $C_{\mathrm{mn}} = C_{\mathrm{m}}\delta_{\mathrm{mn}}$. Now consider the inner-product $\langle \mathrm{d}\chi_{\mathrm{n}}\,, \mathrm{d}\chi_{\mathrm{m}}\rangle$. We have

$$\begin{aligned} \langle \mathrm{d}\chi_{\mathrm{n}}\,, \mathrm{d}\chi_{\mathrm{m}}\rangle &= \left\langle \chi_{\mathrm{n}}, \mathrm{d}^{\dagger}\mathrm{d}\chi_{\mathrm{m}}\right\rangle = \lambda_{\mathrm{m}}\delta_{\mathrm{nm}} \\ &\parallel \\ C_{\mathrm{n}}C_{\mathrm{m}}\,\langle \star\varphi_{\mathrm{n}}, \star\varphi_{\mathrm{m}}\rangle &= C_{\mathrm{n}}^{2}\delta_{\mathrm{nm}}. \end{aligned} \tag{3.4.34}$$

This tells us that $|C_{\mathrm{n}}| = \sqrt{\lambda_{\mathrm{n}}}$. We can arbitrarily choose the sign of the square root. Different signs will correspond to different choices of raising and lowering operators. We choose the following convention, which will make the choice of raising and lowering operators consistent across all d and p: $C_{\mathrm{n}} = \sqrt{\lambda_{\mathrm{n}}}$.

Returning to the general case and allowing for degeneracy of the eigenvalues, we note that the two bases only mix elements of the same eigenvalue, giving thus C_{mn} in a block-diagonal form. Then we can reshuffle the basis within a subspace of a fixed eigenvalue to fully diagonalise C_{mn} and hence the result is unchanged. Altogether we can write the final expression for the current algebra:

$$\boxed{\left[\hat{b}_{\mathrm{n}}, \hat{a}_{\mathrm{m}}\right] = \frac{\mathrm{i}\,\mathbb{K}}{2\pi}\, \sqrt{\lambda_{\mathrm{n}}}\delta_{\mathrm{nm}}.} \tag{3.4.35}$$

We note the similarity to the Kac–Moody algebras appearing in 2d, however it is important to emphasise that the modes here are not labeled, necessarily, by integers, but instead by the countable set of eigenfuctions $\mathcal{N}_{\perp}^{\star}$.

3.4.4 Verma modules, characters, and extended characters

The current algebra, (3.4.35), allows us to build $\mathcal{H}_R$ as a direct sum of Verma modules corresponding to integrable representations of this algebra. It will be useful to supplement this algebra with a positive-definite Hamiltonian. Recall that since the bulk theory is first order in derivatives, its Hamiltonian is identically zero and so this is an ingredient we will add in by hand. The Hamiltonian we will supplement will be the natural generalisation of the "Sugawara Hamiltonian" in 2d Kac–Moody algebras.

Let us define ladder operators of the non-zero-modes as

$$\hat{K}_{\mathsf{n}} := \frac{1}{\sqrt{2}}\left(\mu^{\Delta/2}\hat{a}_{\mathsf{n}} + \mathrm{i}\mu^{-\Delta/2}\hat{b}_{\mathsf{n}}\right) \quad \text{and} \quad \hat{J}_{\mathsf{n}} := \frac{1}{\sqrt{2}}\left(\mu^{\Delta/2}\hat{a}_{\mathsf{n}} - \mathrm{i}\mu^{-\Delta/2}\hat{b}_{\mathsf{n}}\right). \tag{3.4.36}$$

where μ is an energy scale and $\Delta = d - 1 - 2p$. We then write

$$\hat{H} = \hat{H}_{\text{zero}} + \hat{H}_{\text{osc}} \tag{3.4.37}$$

with

$$\hat{H}_{\text{zero}} = \mu^{\Delta}\pi\mathbb{K}^{-1}\sum_{\mathsf{i,j}=0}^{\mathsf{b}_p}\hat{a}_{0\mathsf{i}}\left[\mathbb{G}_p^{\partial R}\right]^{-1}_{\mathsf{ij}}\hat{a}_{0\mathsf{j}} + \mu^{-\Delta}\pi\mathbb{K}^{-1}\sum_{\mathsf{i,j}=0}^{\mathsf{b}_{d-p-1}}\hat{b}_{0\mathsf{i}}\left[\mathbb{G}_{d-p-1}^{\partial R}\right]^{-1}_{\mathsf{ij}}\hat{b}_{0\mathsf{j}}$$

$$\hat{H}_{\text{osc}} = \pi\mathbb{K}^{-1}\sum_{\mathsf{n}\in\mathcal{N}_\perp^*}\left(\mu^{\Delta}\hat{a}_{\mathsf{n}}\hat{a}_{\mathsf{n}} + \mu^{-\Delta}\hat{b}_{\mathsf{n}}\hat{b}_{\mathsf{n}}\right) = 2\pi\mathbb{K}^{-1}\sum_{\mathsf{n}\in\mathcal{N}_\perp^*}\hat{J}_{\mathsf{n}}\hat{K}_{\mathsf{n}} + \mu E_0, \tag{3.4.38}$$

with E_0 a potentially divergent zero-point energy that will play no role in the entanglement entropy. This Hamiltonian is motivated by three very important points: firstly, it is positive-definite, and secondly, when expressed in terms of the ladder-operators it plays a natural algebraic role, extending (3.4.35) to

$$\left[\hat{J}_{\mathsf{n}}, \hat{K}_{\mathsf{m}}\right] = \frac{\mathbb{K}}{2\pi}\sqrt{\lambda_{\mathsf{n}}}\delta_{\mathsf{nm}} \tag{3.4.39}$$

$$\left[\hat{H}, \hat{K}_{\mathsf{n}}\right] = \sqrt{\lambda_{\mathsf{n}}}\hat{K}_{\mathsf{n}} \quad \text{and} \quad \left[\hat{H}, \hat{J}_{\mathsf{n}}\right] = -\sqrt{\lambda_{\mathsf{n}}}\hat{J}_{\mathsf{n}}. \tag{3.4.40}$$

It is evident in this writing, that the operator $\hat{J}_{\mathsf{n}}$, with $\mathsf{n} \in \mathcal{N}_\perp^*$, lowers the energy by $\sqrt{\lambda_{\mathsf{n}}}$ units and $\hat{K}_{\mathsf{n}}$ raises the energy by the same amount.

Thirdly, and perhaps most importantly, when expressed in terms of the original field variables on ∂R, $\hat{H}$ is *local*. Namely writing

$$i_{\partial X}^* A = a + \mathrm{d}t \wedge \tilde{a}, \tag{3.4.41}$$

$$i_{\partial X}^* B = b + \mathrm{d}t \wedge \tilde{b}, \tag{3.4.42}$$

with $a \in \Omega^0(\mathbb{R}) \otimes \Omega^p(\partial R)$, $\widetilde{a} \in \Omega^0(\mathbb{R}) \otimes \Omega^{p-1}(\partial R)$, $b \in \Omega^0(\mathbb{R}) \otimes \Omega^{d-p-1}(\partial R)$, and $\widetilde{b} \in \Omega^0(\mathbb{R}) \otimes \Omega^{d-p-2}(\partial R)$, $\hat{H}$ takes the form

$$\hat{H} = \frac{\mu^{\Delta}}{4\pi} \langle a, \mathbb{K}\, a \rangle_{\partial R} + \frac{\mu^{-\Delta}}{4\pi} \langle b, \mathbb{K}\, b \rangle_{\partial R} \,. \tag{3.4.43}$$

One can already recognise $\hat{H}$ as the Hamiltonian of the chiral mixed Maxwell theory appearing in section 3.3.[18]

The structure of the $\mathcal{H}_R$ factor of the extended Hilbert space is now clear. The eigenstates of the zero-modes $\hat{a}_{0\mathrm{i}}$ and $\hat{b}_{0\mathrm{j}}$ define primary, lowest-weight states, $|\omega\rangle$, that are annihilated by all $\hat{J}_{\mathrm{n}}$. We can act on each such state with $\hat{K}_{\mathrm{n}}$, repeatedly to construct a whole Verma module,

$$\mathcal{V}_\omega := \mathrm{span}\left\{|\omega, \{N_{\mathrm{n}}\}\rangle\right\} := \mathrm{span}\left\{ \prod_{\mathrm{n}\in\mathcal{N}_\perp^*} \prod_{\mathrm{l}=1}^{\kappa} \left(\hat{K}_{\mathrm{ln}}\right)^{N_{\mathrm{ln}}} |\omega\rangle \right\}. \tag{3.4.44}$$

The full Hilbert space is then a direct sum of all Verma modules

$$\mathcal{H}_R = \bigoplus_{\omega} \mathcal{V}_\omega. \tag{3.4.45}$$

The primary states are labeled by the eigenvalues of the current algebra zero-modes. These we can easily find by using Dirac quantisation restricting to ∂R:

$$\int_{\sigma_{\mathrm{i}}} a = 2\pi n^{\mathrm{i}}, \qquad \forall \sigma_{\mathrm{i}} \in \mathrm{H}_p(\partial R), \qquad n^{\mathrm{i}} \in \mathbb{Z}, \tag{3.4.46}$$

and

$$\int_{\eta_{\mathrm{i}}} b = 2\pi m^{\mathrm{i}}, \qquad \forall \eta_{\mathrm{i}} \in \mathrm{H}_p(\partial R), \qquad m^{\mathrm{i}} \in \mathbb{Z}. \tag{3.4.47}$$

Now, invoking the mode expansion, (3.4.23), this amounts to

$$\hat{a}_{0\mathrm{i}}\,|\omega\rangle = \mathbb{K} \sum_{\mathrm{j}=1}^{\mathrm{b}_p} n^{\mathrm{j}} \left[\mathbb{G}_p^{\partial R}\right]_{\mathrm{ij}} |\omega\rangle\,, \tag{3.4.48}$$

$$\hat{b}_{0\mathrm{i}}\,|\omega\rangle = \mathbb{K} \sum_{\mathrm{j}=1}^{\mathrm{b}_{d-p-1}} m^{\mathrm{j}} \left[\mathbb{G}_{d-p-1}^{\partial R}\right]_{\mathrm{ij}} |\omega\rangle\,. \tag{3.4.49}$$

[18]Indeed, this matching is made explicit by "adding in" the time component, $\widetilde{a}$, of $i_{\partial X}^* A$ to $\hat{H}$ by replacing b through the "chiral" boundary condition $b = \mu^{\Delta}(-1)^{(d-p)p} \star_{\partial R} \widetilde{a}$.

Therefore, the primary states, are labeled by two vectors of integers, $\boldsymbol{n} \in \mathbb{Z}^{\kappa\, \mathrm{b}_p(\partial R)}$ and $\boldsymbol{m} \in \mathbb{Z}^{\kappa\, \mathrm{b}_{d-p-1}(\partial R)}$, $|\omega\rangle \equiv |\boldsymbol{n}, \boldsymbol{m}\rangle$, and have energy

$$\Delta_{\boldsymbol{nm}} = \pi\mu\Big(\boldsymbol{n}\cdot\Big(\mathbb{K}\otimes\widetilde{\mathbb{G}}_p^{\partial R}\Big)\cdot\boldsymbol{n} + \boldsymbol{m}\cdot\Big(\mathbb{K}\otimes\widetilde{\mathbb{G}}_{d-p-1}^{\partial R}\Big)\cdot\boldsymbol{m}\Big), \tag{3.4.50}$$

where we have, again absorbed the powers of μ into the dimensionless Gram matrices $\widetilde{\mathbb{G}}_k^{\partial R}$. From here on we will also measure our energies in units of μ, i.e. $\sqrt{\lambda_{\mathrm{n}}} \to \mu\sqrt{\lambda_{\mathrm{n}}}$ with the new λ_{n}'s dimensionless.

We can now define extended characters of our algebra. Having decomposed the Hilbert space as

$$\mathcal{H}_R = \bigoplus_{\substack{\boldsymbol{n}\in\mathbb{Z}^{\kappa\, \mathrm{b}_p(\partial R)} \\ \boldsymbol{m}\in\mathbb{Z}^{\kappa\, \mathrm{b}_{d-p-1}(\partial R)}}} \mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}, \tag{3.4.51}$$

we can first compute the characters of each Verma module, $\mathrm{ch}_{\mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}}[q] := \mathrm{tr}_{\mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}}\, q^{\hat{H}/\mu}$. This is straightforward. Remembering that $\hat{K}_{\mathrm{n}}$ raises the energy by $\mu\sqrt{\lambda_{\mathrm{n}}}$ we obtain

$$\mathrm{ch}_{\mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}}[q] := \underbrace{q^{\kappa E_0}}_{\text{vacuum contribution}} \underbrace{\left(\prod_{\mathrm{n}\in\mathcal{N}_\perp^*}\sum_{N_{\mathrm{n}}=0}^{\infty} q^{N_{\mathrm{n}}\sqrt{\lambda_{\mathrm{n}}}}\right)^{\kappa}}_{\text{descendants}} \underbrace{q^{\Delta_{\boldsymbol{n},\boldsymbol{m}}}}_{\text{primaries}}, \tag{3.4.52}$$

Finally, summing over all the Verma modules gives us an *extended character*:

$$\begin{aligned}\mathbf{ch_0}[q] &:= \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{\mathrm{b}_p} \\ \boldsymbol{m}\in\mathbb{Z}^{\mathrm{b}_{d-p-1}}}} \mathrm{ch}_{\mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}}[q] \\ &= q^{\kappa E_0}\left(\prod_{\mathrm{n}\in\mathcal{N}_\perp^*}\sum_{N_{\mathrm{n}}=0}^{\infty} q^{N_{\mathrm{n}}\sqrt{\lambda_{\mathrm{n}}}}\right)^{\kappa} \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{\mathrm{b}_p} \\ \boldsymbol{m}\in\mathbb{Z}^{\mathrm{b}_{d-p-1}}}} q^{\Delta_{\boldsymbol{n},\boldsymbol{m}}}.\end{aligned} \tag{3.4.53}$$

Here, we once again encounter the generalised Dedekind eta function from section 3.3;

$$\eta_{\partial R}^{(p-1)}[q] := q^{-\frac{1}{2}E_0}\left(\prod_{\mathrm{n}\in\mathcal{N}_\perp^*}\sum_{N_{\mathrm{n}}=0}^{\infty} q^{N_{\mathrm{n}}\sqrt{\lambda_{\mathrm{n}}}}\right)^{-1/2}. \tag{3.4.54}$$

Additionally, recalling the form of the primary state energies, (3.4.50), the sum over the primaries of $q^{\Delta_{\boldsymbol{n},\boldsymbol{m}}}$ organises into the Siegel Theta functions appearing in section 3.3. Consequently, we find a curious and powerful correspondence: the partition function, (3.3.30), of the chiral mixed Maxwell theory, from section 3.3 takes the form of an extended character of our current algebra!

$$\mathbf{ch_0}[q] = \frac{\Theta\Big[q;\mathbb{K}\otimes\widetilde{\mathbb{G}}^{\partial R}_{p}\Big]}{\left(\eta^{(p-1)}_{\partial R}[q]\right)^{\kappa}}\,\frac{\Theta\Big[q;\mathbb{K}\otimes\widetilde{\mathbb{G}}^{\partial R}_{d-p-1}\Big]}{\left(\eta^{(d-p-2)}_{\partial R}[q]\right)^{\kappa}} = \mathcal{Z}_{\text{edge}}\big[\mathbb{S}^1_{\varepsilon}\times\partial R\big]. \tag{3.4.55}$$

3.4.5 The entanglement entropy

We have now finally set the stage to calculate the entanglement entropy. Consider partitioning the Cauchy slice, Σ, into two parts $\Sigma = R\cup_{\partial R} R^{\mathrm{c}}$, where $\cup_{\partial R}$ denotes gluing along their common boundary. We would like to compute the entanglement entropy of a state in $\mathcal{H}_\Sigma$ upon tracing out the degrees of freedom on R^{c}. However, we have seen that $\dim\mathcal{H}_\Sigma = |\det\mathbb{K}|^{\mathrm{b}_p(\Sigma)}$, which is obviously finite, but each of $\mathcal{H}_R$ and $\mathcal{H}_{R^{\mathrm{c}}}$ is a sum of Verma modules, (3.4.45), and thus infinite-dimensional. Thus we starkly see that $\mathcal{H}_\Sigma \neq \mathcal{H}_R\otimes\mathcal{H}_{R^{\mathrm{c}}}$. Instead, we will embed $\mathcal{H}_\Sigma$ inside the tensor product and impose the quantum gluing condition, (3.4.5), to ensure that gauge transformations acting on ∂R annihilate the physical states living in $\mathcal{H}_\Sigma\subset\mathcal{H}_R\otimes\mathcal{H}_{R^{\mathrm{c}}}$. Let us denote

$$\hat{Q}_R[\alpha] := (-1)^{d(p+1)}\frac{\mathbb{K}}{4\pi}\int_{\partial R}\alpha\wedge\hat{b} = \sum_{\mathrm{n}\in\mathcal{N}^*_\perp}\alpha^{\mathrm{n}}\hat{b}_{\mathrm{n}},\quad\text{and} \tag{3.4.56}$$

$$\hat{Q}_R[\beta] := \frac{\mathbb{K}}{4\pi}\int_{\partial R}\beta\wedge\hat{a} = \sum_{\mathrm{n}\in\mathcal{N}^*_\perp}\beta^{\mathrm{n}}\hat{a}_{\mathrm{n}}, \tag{3.4.57}$$

the operators with support on ∂R. Similarly for R^{c} there are charge generators

$$\hat{Q}_{R^{\mathrm{c}}}[\overline{\alpha}] := (-1)^{d(p+1)}\frac{\mathbb{K}}{4\pi}\int_{\partial R^{\mathrm{c}}}\overline{\alpha}\wedge\hat{b} = \sum_{\mathrm{n}\in\mathcal{N}^*_\perp}\overline{\alpha}^{\mathrm{n}}\hat{b}_{\mathrm{n}},\quad\text{and} \tag{3.4.58}$$

$$\hat{Q}_{R^{\mathrm{c}}}\Big[\overline{\beta}\Big] := \frac{\mathbb{K}}{4\pi}\int_{\partial R^{\mathrm{c}}}\overline{\beta}\wedge\hat{a} = \sum_{\mathrm{n}\in\mathcal{N}^*_\perp}\overline{\beta}^{\mathrm{n}}\hat{a}_{\mathrm{n}}, \tag{3.4.59}$$

with support on ∂R^{c}. Upon gluing R to R^{c} along their common boundary, we will identify α and β with their parity opposites, $\overline{\alpha}$ and $\overline{\beta}$, respectively. We can then define the generators of gauge transformations on the entire Σ as

$$\hat{\mathcal{Q}}[\alpha] := \hat{Q}_R[\alpha]\otimes\hat{1}_{R^{\mathrm{c}}} + \hat{1}_R\otimes\hat{Q}_{R^{\mathrm{c}}}[\overline{\alpha}], \tag{3.4.60}$$

$$\hat{\mathcal{Q}}[\beta] := \hat{Q}_R[\beta]\otimes\hat{1}_{R^{\mathrm{c}}} + \hat{1}_R\otimes\hat{Q}_{R^{\mathrm{c}}}\Big[\overline{\beta}\Big], \tag{3.4.61}$$

where $\hat{1}_R\left(\hat{1}_{R^{\mathrm{c}}}\right)$ is the identity operator on $\mathcal{H}_R$ ($\mathcal{H}_{R^{\mathrm{c}}}$). The condition that the physical states should be uncharged under the charge generators, identifies $\mathcal{H}_\Sigma$ as a specific subspace of $\mathcal{H}_R \otimes \mathcal{H}_{R^c}$, namely

$$\mathcal{H}_\Sigma = \ker \hat{\mathcal{Q}}[\alpha] \cap \ker \hat{\mathcal{Q}}[\beta] \subset \mathcal{H}_R \otimes \mathcal{H}_{R^c}. \tag{3.4.62}$$

At the level of the ladder operators, (3.4.36), the quantum gluing condition can be expressed as

$$\left(\hat{J}_{\mathsf{n}} \otimes \hat{1}_{R^c} + \hat{1}_R \otimes \hat{K}_{\mathsf{n}}\right)|\psi\rangle \stackrel{!}{=} 0, \tag{3.4.63}$$

$$\left(\hat{K}_{\mathsf{n}} \otimes \hat{1}_{R^c} + \hat{1}_R \otimes \hat{J}_{\mathsf{n}}\right)|\psi\rangle \stackrel{!}{=} 0. \tag{3.4.64}$$

which roughly states that raising operators on R are lowering operators on R^c and vice versa. Due to this, it is clear that $|\widetilde{\psi}\rangle$ will be *maximally entangled* in terms of the oscillator occupation numbers, $\{N_{\mathsf{n}}\}$. The quantum gluing condition, (3.4.5), applied at the level of zero-modes enforces a matching of the integer charges. The result of this is that physical states of this $\mathcal{H}_\Sigma$ embed into *generalised Ishibashi states* of $\mathcal{H}_R \otimes \mathcal{H}_{R^c}$:

$$|\widetilde{\psi}\rangle = |\mathbf{0}\rangle\rangle := \sum_{\boldsymbol{n},\boldsymbol{m}} \sum_{\{N_{\mathsf{n}}\}} |\boldsymbol{n}, \boldsymbol{m}, \{N_{\mathsf{n}}\}\rangle_R \otimes \overline{|-\boldsymbol{n}, -\boldsymbol{m}, \{N_{\mathsf{n}}\}\rangle_{R^c}}, \tag{3.4.65}$$

where the overline denotes an anti-linear conjugate of the state.[19] The reduced density matrix is given by tracing over the $\mathcal{H}_{R^c}$ tensor factor:

$$\widetilde{\rho}_R := \mathrm{tr}_{\mathcal{H}_{R^c}} \, |\mathbf{0}\rangle\rangle\langle\langle\mathbf{0}| \, . \tag{3.4.66}$$

However the above object is not well defined, per se. We need to first address the subtle issue of normalizability. Given that $|\mathbf{0}\rangle\rangle$ is maximally entangled over an infinite number of modes (labeled by occupation numbers, N_{n}), its norm is divergent. To obtain a normalizable vector in $\mathcal{H}_R \otimes \mathcal{H}_{R^c}$ we need to first regularise $|\mathbf{0}\rangle\rangle$ and we will do so with the Sugawara-type Hamiltonian we discussed earlier, (3.4.37):

$$|\varepsilon\rangle\rangle := \exp\left(-\frac{1}{4}\varepsilon\left(\hat{H}_R + \hat{H}_{R^c}\right)\right)|\mathbf{0}\rangle\rangle\,, \tag{3.4.67}$$

where we have explicitly denoted which factor of $\mathcal{H}_{\text{ext}}$ the Hamiltonians act.

From here it is easy to see that the regulated Ishibashi state, (3.4.67), takes the form of a canonical purification of a thermal density matrix with inverse temperature ε. This thermal density matrix is exactly the (unnormalised) reduced density matrix

$$\widetilde{\rho}_R(\varepsilon) := \mathrm{tr}_{\mathcal{H}_{R^c}} \, \mathrm{e}^{-\frac{\varepsilon}{4}\left(\hat{H}_R + \hat{H}_{R^c}\right)} \, |\varepsilon\rangle\rangle\langle\langle\varepsilon| \, \mathrm{e}^{-\frac{\varepsilon}{4}\left(\hat{H}_R + \hat{H}_{R^c}\right)}, \tag{3.4.68}$$

[19] We have written this for $|\mathbf{0}\rangle\rangle$, corresponding to the ground state without any Wilson surfaces piercing Σ. The computation with a generic $|\boldsymbol{r}\rangle\rangle$ can be easily obtained, by shifting the argument of the Siegel Theta functions by a constant fractional charge. This will not affect the end entanglement entropy.

and whose norm is given by the extended character, (3.4.53)

$$\operatorname{tr}_{\mathcal{H}_R} \widetilde{\rho}_R(\varepsilon) = \mathbf{ch_0}[q_\varepsilon], \qquad q_\varepsilon := \mathrm{e}^{-\varepsilon\mu}. \tag{3.4.69}$$

As $\varepsilon \to 0$ this is simply the regulated dimension of $\mathcal{H}_R$. Thus, the von Neumann entropy of the reduced density matrix is the regulated log dim $\mathcal{H}_R$. More suggestively, however, given the correspondence (3.4.55), this is *also* the high-temperature limit of the thermodynamic entropy of the edge mode theory, i.e.

$$\mathcal{S}_{\mathrm{EE}} = \lim_{\varepsilon\to 0} (1 - \varepsilon\partial_\varepsilon)\mathcal{Z}_{\mathrm{edge}}\big[\mathbb{S}^1_\varepsilon \times \partial R\big]. \tag{3.4.70}$$

At this point the technical computations follow that of subsection 3.3.2. We arrive again at the main result for the entanglement entropy:

$$\begin{aligned}\mathcal{S}_{\mathrm{EE}} = &\sum_{k=1}^{\lfloor \frac{d-1}{2} \rfloor} \mathrm{C}_k^{(p-1)} \left(\frac{\ell}{\varepsilon}\right)^{d-2k} + \frac{\kappa}{2}\left(\mathcal{I}^{(p-1)}_{\frac{d-2}{2}} + \mathcal{I}^{(d-p-2)}_{\frac{d-2}{2}}\right)\delta_{d,\mathrm{even}} \log\left(\frac{\ell}{\varepsilon}\right) \\ &- \frac{1}{2}\big(\mathrm{b}_p + \mathrm{b}_{d-p-1}\big) \log|\det \mathbb{K}|.\end{aligned} \tag{3.4.71}$$

where $\mathrm{C}_k^{(p-1)}$ are non-universal, dimensionless, numbers, $\mathcal{I}^{(k)}_{\frac{d-2}{2}}$ is the $\left(\frac{d-2}{2}\right)$-th heat kernel coefficient for the spectral zeta function, and we have exchanged $\ell = \mu^{-1}\mathrm{e}$ as a characteristic length scale (see section 3.3 for details).

3.5 Discussion

In this chapter, we considered the edge contributions to the entanglement entropy in higher-dimensional abelian topological phases described by p-form BF theories. These are phases whose ground states are condensates of p-form surface operators. We found that the entanglement entropy coming from localised edge modes at the entangling surface takes the form of a non-universal, divergent, area law decreasing in powers of two with a possible log divergence in even dimensions. The constant corrections to this area law are given in terms of topological features of the entangling surface, namely its $(p-1)$-st and $(d-p-2)$-st Betti numbers. Our result is upheld through two separate, but complementary, fronts: we have performed a replica path integral calculation where the entropy arises as a high-temperature thermal entropy of an edge mode partition function living on a regulated entangling surface (with the regulator playing the role of the inverse temperature). This is edge mode theory is a chiral combination of $(p-1)$- and $(d-p-2)$-form Maxwell theories, which we call "chiral mixed Maxwell theory." We followed this calculation with a more rigorous definition of the entanglement entropy through an extended Hilbert space and showed that this extended Hilbert space is organised by a novel infinite-dimensional current algebra, which has not appeared in the literature (to this degree of generality) before. We elucidated features of this current algebra and extracted the entanglement entropy through its representation characters. Along the way we have shown that these characters account for the spectrum of the Maxwell edge-theory and match its thermal partition function.

There are several features of our main result that require elaboration and that we would like to highlight at this point.

Comparing to the GTV result

Let us comment on the discrepancy of our result with the result found by Grover, Turner, and Vishwanath [188] in states of discrete gauge theories described by condensates of p-form membranes:

$$\mathcal{S}_{\text{GTV}} = \mathcal{S}_{\text{local}} - \sum_{n=0}^{p-1} (-1)^{p-1+n} \mathrm{b}_n(\partial R) \ \log|G|, \tag{3.5.1}$$

where $\mathcal{S}_{\text{local}}$ is built out of integrating local quantities (and so includes possible logs and Euler characteristics in even dimensions), and $|G|$ is the order of a discrete gauge group (this is the analogue of $|\det \mathbb{K}|$ in our computation). This result was arrived at by counting the constraints implied by the intersection of p-form membranes with the entangling surface.[20] In order to compare (3.1.2) and (3.5.1) and better highlight the discrepancy, it is useful to write our result in terms of the analogous alternating sum. We can do this either in the language of differential forms or in the language of chains. The calculations from sections 3.3 and 3.4 are natural in the language of differential forms, so let us start there. We note

$$\begin{aligned} -\frac{1}{2}\left(\mathrm{b}_{p-1} + \mathrm{b}_{d-p-2}\right) = &-\sum_{n=0}^{p-1} (-1)^{p-1+n} \mathrm{b}_n \\ &-\frac{1}{2}\sum_{n=0}^{p-2} (-1)^{p-2+n} \dim \Omega^n - \frac{1}{2}\sum_{n=0}^{d-p-3} (-1)^{d-p-3+n} \dim \Omega^n \\ &+\frac{1}{2}\dim \mathsf{E}^{p-1} + \frac{1}{2}\dim \mathsf{E}^{d-p-2}, \end{aligned} \tag{3.5.2}$$

where $\dim \Omega^k$ and $\dim \mathsf{E}^k$ are the dimensions of all k-forms and exact k-forms on ∂R, respectively; these are divergent quantities but can be regulated, say, on a lattice and regarding them as cochains. This equality follows from the short exact sequences.

$$0 \to \mathsf{C}^k \to \Omega^k(\Sigma) \xrightarrow{\mathrm{d}} \mathsf{E}^{k+1} \to 0, \qquad 0 \to \mathsf{E}^k \to \mathsf{C}^k \to \mathrm{H}^k \to 0, \tag{3.5.3}$$

where C^k is the space of closed k-forms, and utilising Poincaré duality on ∂R, $\mathrm{b}_k = \mathrm{b}_{d-2-k}$. The first line of (3.5.2) is the desired GTV result. It is plausible

[20] We are grateful the authors of [188] for correspondence and explaining their result to us.

that the second line involving $\dim \Omega^n$ can be absorbed into the definition of the path integral measures for the gauge fields and the ghosts (which also take an alternating form).[21] However, it is harder for us to argue away the third line involving $\dim \mathsf{E}^{p-1/d-p-2}$. This term would not be there if the object of interest were instead $-\frac{1}{2}(\dim \mathsf{C}^{p-1} + \dim \mathsf{C}^{d-p-2})$, i.e. a counting of closed forms as opposed to harmonic forms. However, this appears unnatural in our approach. For instance, recall that in the computation from section 3.3, the $|\det \mathbb{K}|$ arises hand-in-hand from an instanton sum and a counting of zero-modes of the Hodge Laplacian restricted to transversal $(p-1)/(d-p-2)$-forms. Both of these objects are counted by harmonic forms, not closed forms. This is mirrored in the computation of section 3.4 where the $|\det \mathbb{K}|$ arises from counting zero-modes of the current algebra. There we argued that exact forms give rise to exactly zero charges and so again a true count of the zero-modes naturally lands upon the $\mathrm{b}_{p-1} + \mathrm{b}_{d-p-2}$.

In terms of chains we can also express, through wholly similar manipulations,

$$\begin{aligned}
-\frac{1}{2}\left(\mathrm{b}_{p-1} + \mathrm{b}_{d-p-2}\right) = &-\sum_{n=0}^{p-1}(-1)^{p-1+n}\mathrm{b}_n \\
&+ \frac{1}{2}\sum_{n=0}^{p-1}(-1)^{p-1+n}\dim \mathsf{C}_n + \frac{1}{2}\sum_{n=0}^{d-p-2}(-1)^{d-p-2+n}\dim \mathsf{C}_n \\
&- \frac{1}{2}\dim \mathsf{Z}_{p-1} - \frac{1}{2}\dim \mathsf{Z}_{d-p-2},
\end{aligned} \tag{3.5.4}$$

where $\dim \mathsf{C}_n$ and $\dim \mathsf{Z}_n$ is the dimension of all n-chains on ∂R and the dimension of n-chains on ∂R without boundary, respectively. Again, the first line of (3.5.4) is commensurate with the GTV result. The second line is formed of locally integrated quantities on ∂R and it is feasible that they can be subtracted through a Kiteav–Preskill/Levin–Wen-like scheme. It is not clear whether the third line can be locally subtracted (we indeed believe not). The GTV result would follow if instead the object of interest were $\frac{1}{2}\left(\mathsf{B}_{p-1} + \mathsf{B}_{d-p-2}\right)$ where $\mathsf{B}_k = \mathsf{Z}_k - \mathrm{b}_k$ is the number of boundary-less k-chains that are the boundary of a $k+1$-chain. This counting, which is in fact the one undertaken in [188], is pictorially natural when viewing the ground state as a p-membrane condensate.

Further elucidating the origin of this gap is ongoing work. In this vein, having another independent mode of calculation, e.g. a higher-dimensional version of surgery, would help clarify things; see the discussion below.

On possible bulk contributions and essential topological entanglement

In this chapter we have focussed on the contribution from edge modes to the entanglement entropy. Since this theory is topological, it is natural to believe that this is

[21] See [250] for an inspirational manipulation of this sort in four-dimensional Maxwell theory.

the sole contribution to the entropy; however, let us revisit this assumption. In the replica computation of section 3.3, we did not undertake the computation of

$$\mathcal{Z}_{\text{bulk}}[\mathcal{X}_{n,\varepsilon}] = |\det \mathbb{K}|^{\text{h}_p(\mathcal{X}_{n,\varepsilon},\partial\mathcal{X}_{n,\varepsilon})}\text{T}_{\text{RS}}[\mathcal{X}_{n,\varepsilon},\partial\mathcal{X}_{n,\varepsilon}]^{(-1)^{p-1}}, \tag{3.5.5}$$

but schematically, since all of the oscillator contributions to T_{RS} come with Dirichlet boundary conditions, we expect them to scale with n and not contribute to the entropy. Instead, we need to worry about homological contributions of $\mathcal{Z}_{\text{bulk}}$, which can arise from cycles that can either pull back to or anchor on $\partial\mathcal{X}_{n,\varepsilon}$.

This counting is actually a bit cleaner in the extended Hilbert space discussion of section 3.4. There we focussed on variational charges, however there are also charge operators that cannot be written variationally. These are homological surface operators

$$\hat{\text{W}}^{\boldsymbol{w}_{\text{j}}}_{\eta^{\text{j}}} := \exp\left(\int_{\eta^{\text{j}}} w^{\text{J}}_{\text{j}}\, A_{\text{J}}\right), \qquad \hat{\text{V}}^{\boldsymbol{v}_{\text{i}}}_{\sigma^{\text{i}}} := \exp\left(\int_{\sigma^{\text{i}}} v^{\text{I}}_{\text{i}}\, B_{\text{I}}\right), \tag{3.5.6}$$

where η^{j} and σ^{i} are basis p- and $(d-p-1)$-homology cycles in R. These commute with the variational charges of section 3.4, however, do not commute with each other. Thus states in the $\mathcal{H}_R$ factor of extended Hilbert space should in fact also be labeled with the eigenvalues of one set of the operators. Without loss of generality, we can label the states by w_{j} for each p-cycle in R: $\left|\{w_{\text{j}}\}\right\rangle$. What happens to these quantum numbers when we solve for the true ground state, $|\widetilde{\psi}\rangle$, in $\mathcal{H}_R\otimes\mathcal{H}_{R^{\text{c}}}$? For w_{j}'s corresponding to cycles that live in the interior of R or R^{c}, i.e. those that do not pull back to or intersect ∂R, we simply match those charges with the charge of $|\psi\rangle$ for the corresponding cycles in Σ. However, for cycles of R that pull back to ∂R, they are unfixed: they must combine with a similar cycle from R^{c} to give the correct charge corresponding to a cycle in Σ. This suggests that $|\widetilde{\psi}\rangle$ is maximally correlated over cycles of Σ that pull back[22] to R. This strongly suggests that there is, in fact, a bulk entropy equal to $|\det\mathbb{K}|$ times the number of such p-cycles. This number was precisely calculated in [1] and is the magnetic p-form essential topological entanglement:

$$\begin{aligned}\mathcal{S}_{\text{bulk}} &\overset{?}{=} \mathcal{E}_{\text{mag}}\\ &= \left[\sum_{n=0}^{p-1}(-1)^{p-1-n}\text{b}_n(\partial R) + \sum_{n=0}^{p}\left(\text{b}_n(\Sigma) - \dim \text{H}_n(\Sigma,\partial R)\right)\right]\log|\det\mathbb{K}|.\end{aligned} \tag{3.5.7}$$

where $\text{H}_k(\Sigma,\partial R)$ are relative homology classes. Interestingly the GTV alternating sum makes an appearance (albeit with the opposite sign than in [188]), however we

[22] Cycles that intersect ∂R transversally must additionally "match up" with a cycle from R^{c} but they are not cancelled, instead their charge is fixed by the charge in $|\psi\rangle$. If there is a mismatch of either type of cycles between R and R^{c} they must be set to zero by hand, i.e. if an anchored cycle of R cannot be "completed" to a true cycle of Σ upon gluing on R^{c}, or in a cycle of R that pulls back to ∂R becomes trivial in R^{c}, then we simply set that charge to zero in $|\widetilde{\psi}\rangle$.

do not regard this as accounting for the above discrepancy. For instance, when Σ is topologically trivial (say $\mathbb{R}^{d-1}$ or $\mathbb{S}^{d-1}$) then the terms in (3.5.7) exactly cancel and $\mathcal{S}_{\text{bulk}}$ vanishes (this is by design since there are no non-trivial cycles in Σ to "count"). Thus, the above discrepancy remains in this simple example.

Surgery

Lastly, let us comment on the possibility of another, independent, manner of evaluating this entanglement entropy using surgery. We can proceed via replica path integral, very much in the spirit of section 3.3; however, instead of regulating X_n by excising a tubular neighborhood, we work to directly evaluate $\mathcal{Z}[X_n]$ on the branched cover over ∂R. For generic manifolds this seems quite difficult (although there may be some feasible benchmark examples, e.g. when Σ is a product of spheres and R is a product of a disc and spheres). However, one promising avenue is to develop a program to evaluate such manifolds systematically. Let us recall the procedure in three dimensions, which hinges upon the fact that the Hilbert space on a two-sphere is one-dimensional. Thus, the path integral on any three-geometry, M, with $\partial M = \mathbb{S}^2$, produces a state proportional to the one produced by the path integral on a three-ball, $\mathbb{B}^3$:

$$|M\rangle \propto \left|\mathbb{B}^3\right\rangle. \tag{3.5.8}$$

Thus, for any manifold that can be written as union of two manifolds across a common two-sphere, $X = M_1 \cup_{\mathbb{S}^2} M_2$, we find formally:

$$\mathcal{Z}[X] = \langle M_1|M_2\rangle_{\mathcal{H}_{\mathbb{S}^2}} = \frac{\left\langle M_1\middle|\mathbb{B}^3\right\rangle_{\mathcal{H}_{\mathbb{S}^2}}\left\langle\mathbb{B}^3\middle|M_2\right\rangle_{\mathcal{H}_{S^2}}}{\langle\mathbb{B}^3|\mathbb{B}^3\rangle_{\mathcal{H}_{\mathbb{S}^2}}} = \frac{\mathcal{Z}[\overline{M}_1]^*\mathcal{Z}[\overline{M}_2]}{\mathcal{Z}[\mathbb{S}^3]}, \tag{3.5.9}$$

where $\overline{M}_{1,2}$ are $M_{1,2}$ with their $\mathbb{S}^2$ boundaries "filled-in" with a $\mathbb{B}^3$. This effectively allows one to "cut open" path integrals defined on complicated manifolds along two-spheres and "cap them off" smoothly and evaluate them through simpler "ingredient" path integrals, e.g. $\mathcal{Z}[\mathbb{S}^3]$. In higher dimensions, topology is more involved, but we also potentially have more tools at our disposal: equation (3.2.22) indicates that there are potentially multiple choices of Σ with $\dim \mathcal{H}_\Sigma = 1$ which can provide an "ingredient" for surgery. Developing this further can provide an independent check on both of the above points: the discrepancy with the GTV result as well as the existence of bulk contributions.

4 State-operator correspondence for nonlocal operators

4.1 Introduction

Conformal Field Theory (CFT) is a fundamental framework in theoretical physics. It plays a crucial role in various distinct areas of theoretical physics. In statistical mechanics, it characterises the universality classes of different systems at criticality [251, 252]. Additionally, it describes the long-range behaviour of many quantum field theories (QFTs), and the short-range behaviour of ultraviolet-complete (UV-complete) QFTs via the renormalisation group [6–8]. Finally, it serves as a gateway to quantum gravity, most notably, through string theory and the AdS/CFT correspondence [253–255]. These points alone highlight the fundamental importance of a deep, non-perturbative understanding of CFT. In flat space a lot is known. In any given CFT, the complete set of scaling dimensions of local operators and operator product expansion (OPE) coefficients suffices to reconstruct arbitrary correlation functions, effectively solving the theory. Significant progress has been achieved using traditional methods like conformal perturbation theory, as well as non-perturbative approaches such as the conformal bootstrap [256–258].

However, this emphasis on local objects is not well suited for some physical situations of interest. For instance, condensed matter theory has seen renewed interest in nonlocal excitations, such as anyons [259] and fractons [118]. In quantum field theory, the question of quark confinement is connected to the physics of line operators [15, 260–263]. Finally, quantum gravity is inherently linked with nonlocal operators, based on the simple argument that diffeomorphism invariance forbids local operators.[1] The physics of nonlocal operators is most effectively probed by placing the corresponding physical system on a space with interesting topology.

[1] Topological local operators, corresponding to $(d-1)$-form symmetries, are an exception. However, arguments about the absence of (higher-form) global symmetries suggest their nonexistence [208].

Reconciling this with the theme of the first paragraph, necessitates an understanding of conformal field theory on topologically non-trivial spacetimes. This understanding is effectively complete for two-dimensional CFTs, where Moore and Seiberg [54] showed that in the rational case, modular covariance of torus one-point functions ensures that the CFT can be defined and solved on arbitrary Riemann surfaces. Central to this are (i) the geometry of Riemann surfaces [264], and (ii) the state-operator correspondence. The idea can be summarised as follows. Any Riemann surface can be viewed as a collection of discs and pair-of-pants geometries sewn together along circles. Alternatively, one can insert a resolution of the identity operator along a given circle, expressed in terms of states on that circle. Using the state-operator correspondence these states are mapped to local operators at the centre of a disc bounding that circle. Thus, the computation of any arbitrary correlation function is reduced to the two- and three-point functions on the sphere and one-point functions on the torus.

In higher dimensions, significantly less is known.[2] While the main ideas are the same, technical difficulties arise. More explicitly, locality suggests that QFT observables should be reconstructable from basic building blocks via cutting and sewing. However, this process is much more involved than its two-dimensional counterpart. Therefore, while local operators still completely determine the spectrum of states on a spatial $(d-1)$-sphere, this is insufficient for reconstructing the entire CFT. A natural question to ask is, then: how feasible is a state-operator correspondence that relates nonlocal operators to states on other spatial manifolds? Belin, de Boer, and Kruthoff [269] attempt to answer this question for the three-torus and argue that such a correspondence is not straightforward for a generic CFT.

What is the situation when additional symmetries are available? Recent years have seen an incredible surge of interest in the study of symmetries in quantum field theory. Starting with higher-form symmetries [15], and generalising onwards to higher-group, non-invertible symmetries and other generalisations,[3] these notions of symmetry provide powerful organising principles for quantum field theories. Thus, one may reformulate the question as follows: Is there a state-operator correspondence for nonlocal operators, in higher-dimensional CFTs with generalised global symmetries? In this chapter, we provide an affirmative answer to this question.

More precisely we consider unitary CFTs in $d = 2p + 2$ dimensions with continuous p-form symmetries (invertible or non-invertible). It turns out that this is a very powerful combination. The photonisation argument [37] relates them to circle or orbifold branches of theories of free p-forms. This further implies an infinite collection of codimension-one topological operators, labelled by chiral and anti-

[2] See, however, [265–272] for some of what is known.

[3] See, e.g. [67] for a more complete list of references

chiral p-forms. This is in complete analogy with two-dimensional CFTs, where holomorphic conserved currents can be dressed with arbitrary holomorphic functions, remaining conserved. This analogy goes further: the spectrum of states of such theories is organised by current algebras, generalising the Kac–Moody algebras of two-dimensional CFTs. We focus on the four-dimensional case, enjoying one-form symmetries, and realised by free Maxwell theory. We explicitly construct the representation theory of these algebras on generic closed spatial slices, Σ. A major result is the complete characterisation of the space of states on a generic spatial topology and geometry. This Hilbert space consists of Kac–Moody descendants built on top of primary states, charged under the one-form symmetries. As a non-trivial check, the extended character of those representations matches exactly the partition function of Maxwell theory on $\mathbb{S}^1_\beta \times \Sigma$, as obtained via path integral methods, and is given by strikingly simple formulas that are very reminiscent of two-dimensional rational CFTs:

$$\mathcal{Z}_{\text{Maxwell}}\left[\mathbb{S}^1_\beta \times \Sigma, \mathrm{t}\right] = \frac{\Theta_\Sigma(q, \mathrm{t})}{\eta_\Sigma(q)^2}, \qquad q = \mathrm{e}^{-\beta}. \tag{4.1.1}$$

In the above, $\Theta_\Sigma(q)$, defined in (4.4.47), generalises the Siegel–Narain Theta function and $\eta_\Sigma(q)$, defined in (4.4.45), generalises the Dedekind eta function, while t is the complexified coupling constant of Maxwell theory.[4] On $\Sigma = \mathbb{S}^2 \times \mathbb{S}^1$, we match this spectrum to states prepared by path integrals with insertions of line operators. This has two important consequences. Firstly, a classification of line operators in four-dimensional CFTs with global symmetries. There are *primary* operators, carrying charge under the one-form symmetry and preparing the highest-weight states of our current algebra. In the case of Maxwell theory these operators are given by Wilson–'t Hooft lines. Similarly to local operators, these primary line operators have a definite scaling weight, as defined in [273] and discussed in [274]. Then, there are *descendant* line operators, obtained by dressing the Wilson–'t Hooft lines with photon modes. Relatedly, the second consequence, and our main result is a state-operator correspondence stating the following:

In four-dimensional CFTs with a continuous one-form symmetry, states on $\mathbb{S}^2 \times \mathbb{S}^1$ are in one-to-one correspondence with line operators on $\mathbb{R}^3 \times \mathbb{S}^1$.

Interestingly, we find that the lack of Weyl transformation from $\mathbb{R}^3 \times \mathbb{S}^1$ to the Lorentzian cylinder, $\mathbb{R} \times \mathbb{S}^2 \times \mathbb{S}^1$, coupled with the two polarisations of the photon, implies that radial evolution is equivalent to a *squeezing transformation* of the photon states. This implies, in turn, that the vacuum state is not prepared by a path integral with no operator insertions, but rather, by a path integral with insertions of photon modes of all frequencies.

[4] See also [2] for a related story in generic dimensions.

An organising summary is as follows. In section 4.2 we explain the photonisation argument, and its implications. We show that invertible continuous p-form symmetries in $(2p+2)$-dimensional CFTs lead to a higher-dimensional generalisation of Kac–Moody algebras. We supplement this with a discussion on non-invertible continuous p-form symmetry, where we construct a non-invertible generalisation of these current algebras. In section 4.3 we focus on the case $(p=0)$ of two-dimensional CFTs, where we review the well-known state-operator correspondence for the free compact scalar, in terms of its organising Kac–Moody algebra. We then jump, in section 4.4, to the case of $p=1$ and free Maxwell theory, detailing the path integral on generic closed manfiolds. We then quantise the theory using our current algebras, matching it to the path integral expressions. This allows us to reach section 4.5, where we set up our state-operator correspondence on $\mathbb{S}^2 \times \mathbb{S}^1$, by performing a path integral on $\mathbb{B}^3 \times \mathbb{S}^1$ with line operator insertions. To that end, we first classify the line operators of the theory in terms of our current algebra. We then perform the radial evolution on $\mathbb{B}^3 \times \mathbb{S}^1$, and explain how it leads to squeezing the photon states, to finally land directly onto the nonlocal state-operator correspondence. We finish, in section 4.6 with a discussion on our results and interesting future directions. In appendices C.1–C.3 we collect details regarding the spectral analysis of the Hodge Laplacian, the current algebra on a generic manifold, and the radial evolution, respectively.

4.2 Photonisation and higher-dimensional current algebras

In this section we explore the photonisation argument of [37] and its various incarnations. We first show that a unitary conformal field theory in $2p+2$ dimensions with a continuous p-form symmetry (when $p=0$, we restrict to abelian 0-form symmetries) has a realisation as a theory of free p-forms. We go on to show that this gives rise to a current algebra, akin to the two-dimensional Kac–Moody algebras. We comment on non-abelian versions of the arguments for the $p=0$ case. Finally, we derive non-invertible current algebras, for theories enjoying continuous non-invertible symmetries and comment on their applications.

4.2.1 Photonisation

Our starting point is a unitary conformal field theory in $d=2p+2$ dimensions, with a continuous, invertible p-form symmetry. We focus on the case of a single $\mathrm{U}(1)^{[p]}$ symmetry.[5] Higher-form symmetries are abelian, so for $p \geqslant 1$, this is all we can have.[6] For $p=0$ we can also have non-abelian symmetries; we comment on those in subsection 4.2.3. The continuous symmetry induces a conserved $(p+1)$-form

[5]We denote a p-form symmetry group as $\mathrm{U}(1)^{[p]}$.

[6]We can also have a product of decoupled $\mathrm{U}(1)^{[p]}$s; the generalisation is trivial.

current, $J_{[p+1]} := J_{\mu_1\cdots\mu_{p+1}}(x)\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p+1}}$, satisfying

$$\mathrm{d} \star J_{[p+1]} = 0. \tag{4.2.1}$$

This current leads to a conserved charge, supported on a codimension-$(p+1)$ manifold

$$Q[\Sigma_{d-p-1}] := \int_{\Sigma_{d-p-1}} \star J_{[p+1]}, \tag{4.2.2}$$

or equivalently a codimension-$(p+1)$ topological operator

$$U_{\Sigma_{d-p-1}}(\alpha) := \exp\left(\mathrm{i}\alpha \int_{\Sigma_{d-p-1}} \star J_{[p+1]} \right). \tag{4.2.3}$$

Since this is a compact U(1) symmetry — as opposed to non-compact U(1), i.e. $\mathbb{R}$ —, the parameter α is circle-valued: $\alpha \sim \alpha + 2\pi$.

Let us now see the implications of conformality, reviewing the argument of [37]. We start by discussing the theory on Euclidean flat space, $\mathbb{R}^d$, with metric $g_{\mu\nu} = \delta_{\mu\nu}$. The $(p+1)$-form currents are primary operators of scaling dimension $p+1$, so their two-point function[7] is completely fixed by conformal symmetry, up to a constant, k [275]:

$$\left\langle J_{[p+1]}(x_1)\ J_{[p+1]}(x_2) \right\rangle = \frac{\mathrm{k}}{\|x_{12}\|^d}\, \mathrm{G}(x_{12}), \tag{4.2.4}$$

where $x_{12} := x_1 - x_2$ and $\mathrm{G}(x_{12})$ is a uniquely determined tensor structure:

$$\mathrm{G}^{\mu_1\cdots\mu_{p+1}\nu_1\cdots\nu_{p+1}}(x_{12}) = \delta^{\mu_1}_{[\rho_1}\cdots\delta^{\mu_{p+1}}_{\rho_{p+1}]}\delta^{\nu_1}_{[\sigma_1}\cdots\delta^{\nu_{p+1}}_{\sigma_{p+1}]}\prod_{\ell=1}^{p+1}\left(g^{\rho_\ell\sigma_\ell} - 2\frac{(x_{12})^{\rho_\ell}(x_{12})^{\sigma_\ell}}{\|x_{12}\|^2} \right), \tag{4.2.5}$$

where $g_{\rho\sigma}$ is the flat Euclidean metric and $g^{\rho\sigma}$ its inverse. From here on, it is easy to show that

$$\left\langle \mathrm{d}J_{[p+1]}(x_1)\ J_{[p+1]}(x_2) \right\rangle = 0, \tag{4.2.6}$$

which immediately implies

$$\left\langle \mathrm{d}J_{[p+1]}(x_1)\ \mathrm{d}J_{[p+1]}(x_2) \right\rangle = 0. \tag{4.2.7}$$

Using the standard state-operator correspondence, this represents the norm of a state on $\mathcal{H}_{\mathbb{S}^{d-1}}$, created by the local operator $\mathrm{d}J_{[p+1]} = 0$. Since this is a unitary CFT, this norm can be zero only if the operator $\mathrm{d}J_{[p+1]}$ is zero itself. Said differently,

[7]Here and in the following we write the correlation functions of products of currents in differential form languange to suppress indices. This is not a wedge product of forms, it should rather be seen as a form-valued form, or in other words a section of $\bigwedge^\bullet \mathrm{T}^*\mathbb{R}^d \otimes \bigwedge^\bullet \mathrm{T}^*\mathbb{R}^d$.

conformal invariance and unitarity implies that there is a dual conserved current, $\star J_{[p+1]}$, i.e.:

$$\mathrm{d}J_{[p+1]} = 0. \tag{4.2.8}$$

An equivalent, implicit argument, follows the reasoning of [276]. All the states in a conformal field theory are organised in representations of the Cartan subalgebra of the Euclidean conformal algebra, $\mathfrak{so}(d+1,1)$. This results in each state, $|\mathcal{O}\rangle \in \mathcal{H}_{\mathbb{S}^{d-1}}$, being labelled by its scaling dimension, $\Delta_{\mathcal{O}}$, and a set of highest weights, $\{h_1, h_2, \dots, h_{\lfloor d/2 \rfloor}\}_{\mathcal{O}}$, of the $\mathfrak{so}(d)$ irreducible representation. Let us focus on the primary states; the descendants can be obtained by acting with the ladder operators of the conformal algebra. Unitarity, i.e. $\langle \mathcal{O}|\mathcal{O}\rangle \geqslant 0$ for all primary states $\mathcal{O}$, imposes that $\Delta_{\mathcal{O}} \geqslant f\big(\{h_i\}_{\mathcal{O}}\big)$, for some function f, that depends on the dimension, d, and the $\mathfrak{so}(d)$ representation [277].

Now, consider the state $|J_{[p+1]}\rangle$, corresponding to a conserved current. Its scaling dimension is $\Delta_J = d - p - 1$. Moreover, the conservation equation $\mathrm{d} \star J_{[p+1]} = 0$, implies that this state belongs in a short conformal multiplet, since its first descendant is null. As such, the above unitarity bound, $\Delta_J \geqslant f(\{h_i\}_J)$, must be saturated on this state. In dimensions $d = 2p+2$, the conserved currents lie in reducible representations of $\mathfrak{so}(d)$: they can be decomposed into self-dual and anti-self-dual currents, $J^{\pm}_{[p+1]} := \frac{1}{2}\big(J_{[p+1]} \pm \mathrm{i}^{p-1} \star J_{[p+1]}\big)$, each of which is in an irreducible representation. To saturate the unitarity bound for $|J_{[p+1]}\rangle$, the unitarity bound for both $|J^{\pm}_{[p+1]}\rangle$ must be saturated. Therefore both $|J^{\pm}_{[p+1]}\rangle$ lie in short multiplets, which in turn implies that $\mathrm{d} \star J^{\pm}_{[p+1]} = 0$, or in other words, one has $\mathrm{d}J_{[p+1]} = 0$, on top of $\mathrm{d} \star J_{[p+1]} = 0$.

To summarise, what both of the above arguments show is that a unitary $(2p+2)$-dimensional CFT with a continuous p-form symmetry, must also have a dual p-form symmetry. In other words, it has:

$$\mathrm{d} \star J_{[p+1]} = 0 \quad \text{and} \quad \mathrm{d}J_{[p+1]} = 0. \tag{4.2.9}$$

This is nothing but the equations of motion of a p-form abelian gauge field, $a_{[p]}$, upon identifying $J_{[p+1]}$ with the field strength, $f_{[p+1]}$, of $a_{[p]}$. In other words, unitary $(2p+1)$-dimensional CFTs with a $\mathrm{U}(1)^{[p]}$ p-form symmetry can always be realised by p-form Maxwell theories. In the words of [37] we say that the CFT *photonises.*

4.2.2 Abelian current CFTs

We will now show that the symmetry is actually enhanced even more, into a current algebra — a higher-dimensional version of the two-dimensional Kac–Moody algebras. This algebra will turn out to be spectrum-generating (up to decoupled neutral dressing) and it will eventually be key to reaching the main result of this chapter in section 4.5. We will call CFTs with such a current algebra *current CFTs*. We will mainly analyse abelian current CFTs, but we will comment on non-abelian and non-invertible generalisations in subsections 4.2.3 and 4.2.4.

We go temporarily to a more general setup and relax conformality. Consider a $(d = 2p + 2)$-dimensional Euclidean quantum field theory, with two, dual, $\mathrm{U}(1)^{[p]}$ p-form symmetries, i.e. two conserved currents $J_{[p+1]}$, $\star J_{[p+1]}$:

$$\mathrm{d} \star J_{[p+1]} = 0 \quad \text{and} \quad \mathrm{d}J_{[p+1]} = 0. \tag{4.2.10}$$

The space of $(p+1)$-forms, $\Omega^{p+1}(X)$, admits a $\mathbb{Z}_2$ grading, $\Omega^{p+1}(X) = \Omega^{p+1}_{+}(X) \oplus \Omega^{p+1}_{-}(X)$, graded by the Hodge-star operator, or more precisely, by the operator $\mathrm{i}^{1-p}\star$. Namely, there exist projectors[8]

$$\mathbb{P}_{\pm} := \tfrac{1}{2}\big(\mathbb{1} \pm \mathrm{i}^{1-p} \star\big), \tag{4.2.11}$$

that allow us to write any $(p+1)$-form, $\omega_{[p+1]}$, as $\mathbb{P}_{+}\omega_{[p+1]} + \mathbb{P}_{-}\omega_{[p+1]}$, where $\mathbb{P}_{\pm}\omega_{[p+1]}$ satisfy

$$\mathrm{i}^{1-p} \star \mathbb{P}_{\pm}\omega_{[p+1]} = \pm\mathbb{P}_{\pm}\omega_{[p+1]}. \tag{4.2.12}$$

We will call equation (4.2.12), self-duality equation and will refer to $\mathbb{P}_{\pm}\omega_{[p+1]}$ as (anti-)self-dual forms, in all dimensions, even though, technically, they are (anti-)self-dual only when p is odd, or equivalently in dimensions $d = 4k$. In $d = 4k + 2$, corresponding to even p, they are just the eigenforms of the operator $\mathrm{i}\star$.

An important property of the projectors (4.2.11) is that the can pass through wedge products of $(p+1)$-forms:

$$\big(\mathbb{P}_{\pm}\omega_{[p+1]}\big) \wedge \eta_{[p+1]} = \omega_{[p+1]} \wedge \big(\mathbb{P}_{\pm}\eta_{[p+1]}\big). \tag{4.2.13}$$

To see that, it suffices to note that $\star\omega_{[p+1]} \wedge \eta_{[p+1]}$ is a $(2p+2)$-form and hence is proportional to the volume form. The simultaneous conservation, (4.2.10), of $J_{[p+1]}$ and $\star J_{[p+1]}$ is equivalent to the conservation of

$$J^{\pm}_{[p+1]} := \mathbb{P}_{\pm} J_{[p+1]}, \tag{4.2.14}$$

i.e.

$$\mathrm{d}J^{\pm}_{[p+1]} = 0. \tag{4.2.15}$$

Once we have these two conserved currents, we can construct an infinite number of conserved charges. In particular, note that the family of currents

$$\mathcal{J}^{\pm}_{\Lambda} := - \star \Big(J^{\pm}_{[p+1]} \wedge \Lambda^{\mp}_{[p]}\Big) \tag{4.2.16}$$

[8] Recall that we are in Euclidean signature; in Lorentzian signature there would be an extra i.

is conserved for any (anti)chiral p-form, i.e. for any $\Lambda^{\mp}_{[p]}$ such that

$$\mathbb{P}_{\pm}\mathrm{d}\Lambda^{\mp}_{[p]} = 0. \tag{4.2.17}$$

These can be thought of as p-form analogues of (anti)holomorphic functions.[9] Indeed,

$$\mathrm{d}\star\mathcal{J}^{\pm}_{\Lambda} = J^{\pm}_{[p+1]}\wedge\mathrm{d}\Lambda^{\mp}_{[p]} = J_{[p+1]}\wedge\mathbb{P}_{\pm}\mathrm{d}\Lambda^{\mp}_{[p]} = 0. \tag{4.2.18}$$

Integrating on a codimension-one closed manifold, $\Sigma_{d-1} = \Sigma_{2p+1}$, gives us two families of conserved charges:

$$Q^{\pm}_{\Lambda}[\Sigma_{d-1}] := \int_{\Sigma_{d-1}} J^{\pm}_{[p+1]}\wedge\Lambda^{\mp}_{[p]}, \tag{4.2.19}$$

or equivalently topological operators,

$$U^{\pm}_{\Sigma_{d-1}}(\Lambda) := \exp\left(\mathrm{i}\int_{\Sigma_{d-1}} J^{\pm}_{[p+1]}\wedge\Lambda^{\mp}_{[p]}\right). \tag{4.2.20}$$

Here we mention two important points. First, note that these are zero-form symmetries, regardless of p. As such, they can a priori, have non-trivial commutators. They stem, however from an abelian symmetry, therefore their commutation relations, if non-trivial, indicate a central extension. We will indeed find such a central extention in the next paragraphs. Second, note that there is a gauge redundancy in defining the conserved charges. Namely, we must identify $\Lambda^{\mp}_{[p]} \sim \Lambda^{\mp}_{[p]} + \lambda^{\mp}_{[p]}$, where $\mathrm{d}\lambda^{\mp}_{[p]} = 0$, since the shift by a closed p-form leaves the charges invariant.

Now we reinstate conformality. In subsection 4.2.1 we saw that a CFT with these two conserved currents obeys the dynamics of p-form Maxwell theory. We can therefore, locally, realise the currents as the curvature of a p-form gauge field $J_{[p+1]} = \mathrm{d}a_{[p]}$, whose equation of motion is a free wave equation $\mathrm{d}^{\dagger}\mathrm{d}a_{[p]} = 0$. We then recognise $a_{[p]}$ and $\star J_{[p+1]}$, restricted to a codimension-one slice, as conjugate phase-space variables, giving rise to a (pre-)symplectic form:

$$\Omega_{\Sigma_{d-1}} = \int_{\Sigma_{d-1}} \delta\mathbf{p}\wedge\delta\mathbf{q}\,, \tag{4.2.21}$$

identifying

$$\mathbf{q} = a_{[p]}\Big|_{\Sigma_{d-1}} \quad \text{and} \quad \mathbf{p} = 2\mathrm{k}\left(\star J^{\pm}_{[p+1]}\right)\Big|_{\Sigma_{d-1}}. \tag{4.2.22}$$

In the above, the factor 2k is conventional, and is related to the strength of the interaction, i.e. the electric charge of the p-form Maxwell theory realising the CFT. In fact, we should identify as the canonical position, $\mathbf{q}$, a conjugacy class, or a specific

[9] Equation (4.2.17) is a local definition, like the defining equations of (anti)holomorphic functions.

representative thereof, of the equivalence relation $a_{[p]} \sim a_{[p]} + \lambda_{[p]}$, for $\lambda_{[p]}$ flat. This is what takes care of the degeneracy of the symplectic form, corresponding to gauge transformations. We will proceed with (4.2.22), keeping this issue in mind.

We can then compute the commutation relations between the charges, simply by plugging the vector fields they generate on the phase space in the definition of the symplectic form. First we compute

$$\frac{\delta Q^{\pm}_{\Lambda}}{\delta \mathbf{q}} = -\frac{1}{2}\, \mathrm{d}\Lambda^{\mp}_{[p]} \quad \text{and} \quad \frac{\delta Q^{\pm}_{\Lambda}}{\delta \mathbf{p}} = \pm \mathrm{i}^{1-p} \mathrm{k}\, \Lambda^{\mp}_{[p]}. \tag{4.2.23}$$

With this, we immediately get the algebra:

$$\left[Q^{\sigma}_{\Lambda_1}, Q^{\sigma'}_{\Lambda_2}\right]_{\Sigma_{d-1}} = \frac{\mathrm{i}^{1-p}\mathrm{k}}{2}\left(\sigma + (-1)^p \sigma'\right) \int_{\Sigma_{d-1}} \Lambda_1^{-\sigma} \wedge \mathrm{d}\Lambda_2^{-\sigma'}\,, \tag{4.2.24}$$

where σ and σ' are signs, $\sigma, \sigma' \in \{+, -\}$. Here and in the following, we drop the subscript $[p]$ indicating the form-degree of Λ to declutter the notation. This algebra is a higher-dimensional generalistion of the familiar two-dimensional Kac–Moody algebra, and it will be one of the protagonists of the story that follows. To better illustrate (4.2.24) let us elaborate further on two specific values of p.

$p = 0$.

This case corresponds to the case of a two-dimensinal CFT, with a (zero-form) U(1) symmetry. There, $\Lambda_i^{\pm}$ are just scalar functions and the above commutators become,

$$\begin{aligned} \left[Q^{\pm}_{\Lambda_1}, Q^{\pm}_{\Lambda_2}\right]_{\Sigma_1} &= \pm \mathrm{i}\mathrm{k} \int_{\Sigma_1} \Lambda_1^{\mp}\, \mathrm{d}\Lambda_2^{\mp} \\ \left[Q^{+}_{\Lambda_1}, Q^{-}_{\Lambda_2}\right]_{\Sigma_1} &= 0. \end{aligned} \tag{4.2.25}$$

This is just the familiar $\widehat{\mathfrak{u}}(1) \times \widehat{\mathfrak{u}}(1)$ Kac–Moody algebra. Tracing back the logical tower that led us here, and running the arguments again, in language familiar from two-dimensional CFTs, we recover the following familiar statement. A conserved current, in a unitary two-dimensional CFT, can always be split into a holomorphic and an antiholomorphic piece, which are separately conserved. These can, in turn, be dressed with arbitrary holomorphic and antiholomorphic functions to give rise to a holomorphic and an antiholomorphic current algebra. Quantising on the theory on a spatial circle and expanding the currents in Fourier modes, gives rise to the usual abelian affine Kac–Moody algebra. Indeed, taking Σ_1 to be $\mathbb{S}^1$ and expanding in Fourier modes, $\mathrm{e}^{\pm \mathrm{i}\mathrm{n}\theta}$, gives the familiar form of the $\widehat{\mathfrak{u}}(1) \times \widehat{\mathfrak{u}}(1)$ Kac–Moody algebra:

$$\begin{aligned} \left[J^{\pm}_{\mathrm{n}}, J^{\pm}_{\mathrm{m}}\right] &= \mathrm{k}\, \mathrm{n}\, \delta_{\mathrm{n}+\mathrm{m},0}, \\ \left[J^{+}_{\mathrm{n}}, J^{-}_{\mathrm{m}}\right] &= 0, \qquad \mathrm{n}, \mathrm{m} \in \mathbb{Z}. \end{aligned} \tag{4.2.26}$$

Finally, recall that all these CFTs have a free-field realisation as a free compact scalar, $\phi \sim \phi + 2\pi R$. The level, k, of the algebra is related to the radius of the scalar.

$p = 1$.

This is the case of a four-dimensional CFT with two, dual, one-form symmetries. As we have argued before, this is realised by regular Maxwell theory. Here $\Lambda_i^{\pm}$ are one-forms, and the charges are defined on a three-dimensional manifold. The commutation relations become, in this case,

$$\begin{aligned} \left[Q^+_{\Lambda_1}, Q^+_{\Lambda_2}\right]_{\Sigma_3} &= 0 = \left[Q^-_{\Lambda_1}, Q^-_{\Lambda_2}\right]_{\Sigma_3} \\ \left[Q^+_{\Lambda_1}, Q^-_{\Lambda_2}\right]_{\Sigma_3} &= \mathrm{k}\int_{\Sigma_3} \Lambda_1^- \wedge \mathrm{d}\Lambda_2^+ \,. \end{aligned} \tag{4.2.27}$$

We recognise again the structure of a $\widehat{\mathfrak{u}}(1)\times\widehat{\mathfrak{u}}(1)$ current algebra, with the important difference that here the central extension mixes the two factors, instead of acting on each of them separately. Observing the pattern set by (4.2.24) it is clear that in $4n+2$ dimensions the chiral and antichiral components will go their own, centrally extended, way separately, whereas in $4n$ dimesions, they mix. This has to do with the existence of real self- and anti-self-dual forms, as noted earlier. Let us note that, just like the two-dimensional case, we can expand the currents in "modes," to obtain an algebra of the individual modes. We will do so in subsection 4.4.3. Moreover, the level, k, of the algebra is now related to the coupling of the Maxwell theory that realises the CFT, i.e. the electric charge. Finally, in this case, the algebra (4.2.27) was constructed in [37], where it was also arrived at through a twistor formalism, without appealing to the phase space structure of Maxwell theory.

Going back to the general case, let us mention yet another presentation of the current algebra, (4.2.27), that will be useful to obtain a non-invertible version thereof in subsection 4.2.4. This presentation is in terms of the topological operators, that act on the Hilbert space. A generic topological operator in a current CFT takes the form

$$U\left(\Lambda_1^-, \Lambda_2^+\right) := \exp\left(\mathrm{i}\int_{\Sigma_{d-1}} J^+_{[p+1]} \wedge \Lambda_1^- + \mathrm{i}\int_{\Sigma_{d-1}} J^-_{[p+1]} \wedge \Lambda_2^+\right), \tag{4.2.28}$$

where we supperss the (topological) dependence on Σ_{d-1}. The fusion of two such operators is

$$U\left(\Lambda_1^-, \Lambda_2^+\right) \otimes U\left(\Lambda_3^-, \Lambda_4^+\right) = f_p^{(\mathrm{k})}\left(\Lambda_1^-, \Lambda_2^+, \Lambda_3^-, \Lambda_4^+\right) U\left(\Lambda_1^- + \Lambda_3^-, \Lambda_2^+ + \Lambda_4^+\right), \tag{4.2.29}$$

where

$$f_p^{(\mathrm{k})}(\Lambda_1^-, \Lambda_2^+, \Lambda_3^-, \Lambda_4^+) := \exp\left(\frac{\mathrm{i}^{p+1}\,\mathrm{k}}{2}\int_{\Sigma_{d-1}}\left\{\delta_{p,\mathrm{even}}\left(\Lambda_1^- \wedge \mathrm{d}\Lambda_3^- + \Lambda_2^+ \wedge \mathrm{d}\Lambda_4^+\right) - \delta_{p,\mathrm{odd}}\left(\Lambda_1^- \wedge \mathrm{d}\Lambda_4^+ + \Lambda_2^+ \wedge \mathrm{d}\Lambda_3^-\right)\right\}\right). \tag{4.2.30}$$

This is a simple abelian fusion rule, with a central extension, proportional to k, specified by the function $f_p^{(\mathrm{k})}(\Lambda_1^-, \Lambda_2^+, \Lambda_3^-, \Lambda_4^+)$. As before, we will illustrate (4.2.29) in the cases $p = 0$ and $p = 1$.

At $\boldsymbol{p = 0}$, there is a central term when fusing topological operators with the same chirality:

$$U\left(\Lambda_1^-, 0\right) \otimes U\left(\Lambda_3^-, 0\right) = \exp\left(\frac{\mathrm{i}\,\mathrm{k}}{2}\int_{\Sigma_1} \Lambda_1^- \wedge \mathrm{d}\Lambda_3^-\right) U\left(\Lambda_1^- + \Lambda_3^-, 0\right), \tag{4.2.31}$$

and similarly for Λ_2^+ and Λ_4^+. Operators with opposite chirality do not see each other:

$$U\left(\Lambda_1^-, 0\right) \otimes U\left(0, \Lambda_4^+\right) = U\left(\Lambda_1^-, \Lambda_4^+\right), \tag{4.2.32}$$

and similarly for Λ_2^+ and Λ_3^-. This is, of course, a reflection of the fact we mentioned above, that in two dimensional CFTs the holomorphic and antiholomorphic sectors do not see each other in the central extension.

Coming to $\boldsymbol{p = 1}$, we see the central extension in the fusion of operators of opposite chirality:

$$U\left(\Lambda_1^-, 0\right) \otimes U\left(0, \Lambda_4^+\right) = \exp\left(\frac{\mathrm{k}}{2}\int_{\Sigma_3} \Lambda_1^- \wedge \mathrm{d}\Lambda_4^+\right) U\left(\Lambda_1^-, \Lambda_4^+\right), \tag{4.2.33}$$

while those with the same chirality have a simple abelian fusion:

$$U\left(\Lambda_1^-, 0\right) \otimes U\left(\Lambda_3^-, 0\right) = U\left(\Lambda_1^- + \Lambda_3^-, 0\right), \tag{4.2.34}$$

and similar equations involving the rest of the operators.

Let us also comment on the relation of (4.2.24), with other appearances of higher-dimensional generalisations of Kac–Moody algebras in the literature. As explained above, (4.2.24) is the natural generalisation of the four-dimensional Kac–Moody algebra of [37]. Moreover, it complements (and coincides with, in some cases) the current algebras that organise the edge-modes of topological field theories [2]. However, it is not the same as Mickelsson–Faddeev algebra [278–280], and by extension also not the same as the higher-dimensional loop algebras of [281]. Moreover it is also different from the higher Kac–Moody algebras of [282, 283].

4.2.3 Non-abelian current CFTs

We only comment very briefly on non-abelian current CFTs, as we have nothing to add besides what is already well-known. This case can only occur at $p = 0$, since $(p \geqslant 1)$-form symmetries are abelian. So, consider a two-dimensional CFT with a conserved one-form current $J_{[1]}$, valued in the Lie algebra, $\mathfrak{g}$, of a semi-simple Lie group G. The argument of subsection 4.2.1 goes through, and shows that $\star J_{[1]}$ must also be conserved. The consequence is that this time instead of photonisation, it corresponds to non-abelian bosonisation [284]. Namely, the conservation of the currents corresponds to the equations of motion of a Wess–Zumino–Witten (WZW) model. We can locally write the current $J_{[1]}$ as $g^{-1}\mathrm{d}g$, for g a G-valued scalar and the conservation of the current is

$$\mathrm{d} \star \left(g^{-1}\mathrm{d}g\right) = 0. \tag{4.2.35}$$

From here on, it is a standard exercise in two-dimensional CFT (see e.g. [285]) to derive the associated current algebra, which will naturally be the the affine Kac–Moody algebra $\widehat{\mathfrak{g}}_{\mathrm{k}} \otimes \widehat{\mathfrak{g}}_{\mathrm{k}}$.

4.2.4 Non-invertible current CFTs

A different option is to consider current CFTs whose underlying symmetry is non-invertible. The simplest way to obtain a continuous non-invertible symmetry is to begin with the setup of subsection 4.2.1 and impose the equivalence relation $J_{[p+1]} \sim -J_{[p+1]}$. From the point of view of the photonised theory, this corresponds to gauging the $\mathbb{Z}_2$ charge-conjugation symmetry $A_{[p]} \longmapsto -A_{[p]}$. This corresponds to the orbifold branch of p-form Maxwell theory, or equivalently, an $\mathrm{O}(2) = \mathrm{U}(1) \rtimes \mathbb{Z}_2$ p-form gauge theory. Variants of this theory (mostly for $p = 0$ and $p = 1$), and its (non-invertible) symmetries have been subject of intense study in recent years [63, 65, 76, 78–80, 82].

Let us first review the non-invertible symmetries of $\mathrm{O}(2)$ p-form gauge theory. It is clear that the operator

$$U(\alpha) = \exp\left(\mathrm{i}\alpha \int_{\Sigma_{d-p-1}} \star J_{[p+1]}\right), \tag{4.2.36}$$

albeit still topological, is no longer gauge-invariant, for generic values of α. Note that α is no longer circle-valued, but rather it is valued in the segment $\alpha \sim -\alpha \sim \alpha + 2\pi$. In what follows, it will be implictly assumed that we take a representative of the equivalence classes defined by the above equivalence relations in $[0, \pi]$. The operator $U(0) = \mathbf{1}_{d-p-1}$, is the identity $(d-p-1)$-dimensional operator and is, naturally, gauge-invariant. Moreover,

$$U(\pi) = \exp\left(\mathrm{i}\pi \int_{\Sigma_{d-p-1}} \star J_{[p+1]}\right) =: (-\mathbf{1})_{d-p-1} \tag{4.2.37}$$

is also gauge-invariant and gnerates a $\mathbb{Z}_2^{[p]}$ p-form symmetry. For the rest we can construct topological and gauge-invariant operators by taking a direct sum of the original topological operators. Explicitly, the operators

$$D(\alpha) := U(\alpha) \oplus U(\alpha)^{-1}, \tag{4.2.38}$$

for $\alpha \in (0, \pi)$, pass all the tests of being a symmetry of the theory.[10] What was sacrificed, is obviously invertibility. This is reflected on the fusion rules:

$$D(\alpha) \otimes D(\beta) = D(\alpha + \beta) \oplus D(\alpha - \beta), \tag{4.2.39}$$

for $\alpha \neq \beta \neq \pi - \beta \in (0, \pi)$. Moreover, these operators have quantum dimension two, namely, $\langle D(\alpha)\rangle_{\mathbb{S}^{p+1}} = 2$, in contrast to the invertible topological operators, which have quantum dimension one.

Furthermore, as was explained in [80], since charge-conjugation is a zero-form symmetry, after gauging there is a Pontryagin-dual (or quantum) $\mathbb{Z}_2^{[2p]}$ $2p$-form symmetry ($d - 0 - 2 = 2p$), generated by the $\mathbb{Z}_2$ charge-conjugation Wilson lines: $\mathbf{1}_1$ and $(-\mathbf{1})_1$. While their fusion rules are simply $\mathbb{Z}_2^{[2p]}$ fusion rules:

$$(-\mathbf{1})_1 \otimes (-\mathbf{1})_1 = \mathbf{1}_1, \tag{4.2.40}$$

they can also dress the $(d - p - 1)$-dimensional operators we discussed above, and appear in their fusion rules. The operator $(-\mathbf{1})_1$ is known as a "determinant line."

Therefore, the full set of codimension-$(p - 1)$ topological operators in O(2) p-form gauge theory is

$$\left\{\mathbf{1}_{d-p-1},\ (-\mathbf{1})_{d-p-1},\ \mathbf{1}_{d-p-1}^{(-\mathbf{1})_1},\ (-\mathbf{1})_{d-p-1}^{(-\mathbf{1})_1},\ D(\alpha),\ D(\alpha)^{(-\mathbf{1})_1}\ \middle|\ \alpha \in (0, \pi)\right\}, \tag{4.2.41}$$

where the superscript $(-\mathbf{1})_1$ indicates dressing with the determinant line. The rest of the fusion rules follow by reconciling (4.2.39) with the allowed dressings [76, 80].

[10] Provided that there are states on which they act non-trivially. In this case there are: e.g. gauge invariant sums of the original U(1) Wilson operators [80].

These are:

$$
\begin{aligned}
(-\mathbf{1})_{d-p-1} \otimes (-\mathbf{1})_{d-p-1} &= \mathbf{1}_{d-p-1}, \\
D(\alpha) \otimes (-\mathbf{1})_{d-p-1} &= D(\pi-\alpha), \\
D(\alpha) \otimes D(\alpha) &= \mathbf{1}_{d-p-1} \oplus \mathbf{1}_{d-p-1}^{(-\mathbf{1})_1} \oplus D(2\alpha), && \alpha \neq \frac{\pi}{2} \\
D(\alpha) \otimes D(\pi-\alpha) &= (-\mathbf{1})_{d-p-1} \oplus (-\mathbf{1})_{d-p-1}^{(-\mathbf{1})_1} \oplus D(2\alpha-\pi), && \alpha \neq \frac{\pi}{2} \\
D(\pi/2) \otimes D(\pi/2) &= \mathbf{1}_{d-p-1} \oplus \mathbf{1}_{d-p-1}^{(-\mathbf{1})_1} \oplus (-\mathbf{1})_{d-p-1} \oplus (-\mathbf{1})_{d-p-1}^{(-\mathbf{1})_1}.
\end{aligned}
\tag{4.2.42}
$$

Now, seeing this from the lens of conformal field theory, we can easily repeat the argument of subsection 4.2.2. If the theory before gauging was a CFT, we have that $\mathrm{d}J^{\pm}_{[p+1]} = 0$. After gauging, the operators

$$D^{\pm}(\alpha) := U^{\pm}(\alpha) \oplus U^{\pm}(\alpha)^{-1}, \quad \text{with} \tag{4.2.43}$$

$$U^{\pm}(\alpha) := \exp\left(\mathrm{i}\alpha \int_{\Sigma_{d-p-1}} J^{\pm}{}_{[p+1]}\right), \qquad \alpha \in (0,\pi) \tag{4.2.44}$$

are topological operators of the O(2) p-form gauge theory. Their fusion rules follow from (4.2.39) and (4.2.42). Moreover, there are non-invertible analogues of (4.2.20) and (4.2.28). These are

$$D\big(\Lambda_1^-, \Lambda_2^+\big) := U\big(\Lambda_1^-, \Lambda_2^+\big) \oplus U\big(\Lambda_1^-, \Lambda_2^+\big)^{-1}, \tag{4.2.45}$$

with $\Lambda^{\pm}_{1,2}$ (anti-)chiral p-forms, taking values in the open interval $(0,\pi)$. Their fusion rules define a non-invertible current algebra:

$$
\begin{aligned}
D\big(\Lambda_1^-, \Lambda_2^+\big) \otimes D\big(\Lambda_3^-, \Lambda_4^+\big) = f_p^{(\mathrm{k})}\, & D\big(\Lambda_1^- + \Lambda_3^-, \Lambda_2^+ + \Lambda_4^+\big) \\
& \oplus \left(f_p^{(\mathrm{k})}\right)^{-1} D\big(\Lambda_1^- - \Lambda_3^-, \Lambda_2^+ - \Lambda_4^+\big),
\end{aligned}
\tag{4.2.46}
$$

where $f_p^{(\mathrm{k})} = f_p^{(\mathrm{k})}\big(\Lambda_1^-, \Lambda_2^+, \Lambda_3^-, \Lambda_4^+\big)$, given by (4.2.30). Note that this fusion rule is valid on generic chiral forms, $\Lambda_i^{\pm}$. At special points, i.e. when the $\Lambda_i^{\pm}$ coincide at a point, or differ by π, (4.2.46) should be understood with the operators dressed appropriately, as in (4.2.42). Let us now illustrate and interpret this non-invertible current algebra in the cases $p = 0$ and $p = 1$.

$p = 0$.

At $p = 0$ we find ourselves on the orbifold branch of the two-dimensional compact boson CFT. The fusion ring (4.2.46) splits, again, into holomorphic and antiholomorphic and we have, for example:

$$D(\Lambda_1^-,0)\otimes D(\Lambda_3^-,0)=\exp\left(\frac{\mathrm{i}\,\mathrm{k}}{2}\int_{\Sigma_1}\Lambda_1^-\wedge \mathrm{d}\Lambda_3^-\right)D(\Lambda_1^-+\Lambda_3^-,0)\\ \oplus\exp\left(-\frac{\mathrm{i}\,\mathrm{k}}{2}\int_{\Sigma_1}\Lambda_1^-\wedge \mathrm{d}\Lambda_3^-\right)D(\Lambda_1^--\Lambda_3^-,0), \tag{4.2.47}$$

and similarly for Λ_2^+, Λ_4^+. In complete analogy with the circle branch, where the $\widehat{\mathfrak{u}}(1)$ Kac–Moody algebra contains all the information to construct the full spectrum of the theory, here too, (4.2.47) is, in principle, sufficient to reconstruct the full orbifold branch.[11] To do so, one has to study the representation theory of (4.2.47). The representation theory of non-invertible symmetries was recently studied in [287–291]. In order to study the representation theory of our non-invertible current algebra it is necessary to extend these results to continuous and centrally extended non-invertible symmetries. We will not attempt to do that here, but we will return to it in future work.

$p = 1$.

In this case we land on the orbifold branch of four-dimensional Maxwell theory, i.e. O(2) gauge theory. The central extension appears when one fuses chiral with anti-chiral operators:

$$D(\Lambda_1^-,0)\otimes D(0,\Lambda_4^+)=\exp\left(\frac{\mathrm{k}}{2}\int_{\Sigma_3}\Lambda_1^-\wedge \mathrm{d}\Lambda_4^+\right)D(\Lambda_1^-,\Lambda_4^+)\\ \oplus\exp\left(-\frac{\mathrm{k}}{2}\int_{\Sigma_3}\Lambda_1^-\wedge \mathrm{d}\Lambda_4^+\right)D(\Lambda_1^-,-\Lambda_4^+), \tag{4.2.48}$$

and similarly for Λ_2^+ with Λ_3^-. The chiral-chiral and anti-chiral-anti-chiral channels do not see the central extension:

$$D(\Lambda_1^-,0)\otimes D(\Lambda_3^-,0)=D(\Lambda_1^-+\Lambda_3^-,0)\oplus D(\Lambda_1^--\Lambda_3^-,0), \tag{4.2.49}$$

and similarly for Λ_2^+ and Λ_4^+. We will see later, in section 4.5, that the current algebra (4.2.24) is is spectrum-generating, i.e. we can solve the underlying theory by considering its representations (up to contributions from a decoupled neutral sector). In analogy with the comments on the $p = 0$ case, (4.2.48) and (4.2.49), contains in principle all the information to solve O(2) gauge theory. Again, we will come back to doing so in the future.

[11] Together with its special points, coming from (4.2.42). The special points will be important to obtain the twisted sectors (cf. also [286]).

4.3 Two-dimensional CFTs and the local state-operator correspondence

In this section we will review the usual state-operator correspondence for local operators. We will therefore consider a two-dimensional CFT with a zero-form U(1) symmetry. This corresponds to the case $p = 0$ in the notation of section 4.2 and the corresponding Kac–Moody algebra can be represented as a compact free scalar [292–295]. We will therefore be rederiving the standard state-operator correspondence for the compact free scalar, in order to build some muscle towards section 4.5. Along the way we will remind the reader of some standard facts in two-dimensional conformal field theories, in order to simplify and compare with the discussion of the four-dimensional case, in section 4.5.

4.3.1 Partition function and the spectrum

The view from the path integral

Consider a compact free boson, $\phi \sim \phi + 2\pi$, on a closed Riemann surface, X. Let us first review the path integral of the compact scalar. Its action reads

$$S[\phi] := \frac{1}{2g^2} \int_X f^\phi \wedge \star f^\phi , \tag{4.3.1}$$

where $f^\phi := f^\phi_{\text{harm}} + \mathrm{d}\phi$ is the curvature of the free scalar, with $f^\phi_{\text{harm}} \in \text{Harm}^1(X)$. In this notation, ϕ is a well-defined, single-valued function (a zero-form), subject to the compactness condition, and all the winding has been passed on to the harmonic piece of its curvature, f^ϕ_{harm}. The coupling constant, g, is related to the radius of the compact scalar as $R^2 = \frac{4\pi}{g^2}$, as can be easily seen by rescaling ϕ to $\Phi = R\phi$, so that $\Phi \sim \Phi + 2\pi R$.[12] The harmonic form, f^ϕ_{harm} can always be chosen uniquely to be orthogonal to $\mathrm{d}\phi$, so the action splits into a harmonic piece and an oscillator piece:

$$S[\phi] = \frac{1}{2g^2} \int_X f^\phi_{\text{harm}} \wedge \star f^\phi_{\text{harm}} + \frac{1}{2g^2} \int_X \mathrm{d}\phi \wedge \star \, \mathrm{d}\phi \, . \tag{4.3.2}$$

Let us evaluate the partition function of the compact scalar, i.e. the path integral

$$\mathcal{Z}[X] = \int \mathrm{D} f^\phi_{\text{harm}} \, \mathrm{D}\phi \, \mathrm{e}^{-S[\phi]} = \mathcal{Z}_{\text{harm}}[X] \mathcal{Z}_{\text{osc}}[X]. \tag{4.3.3}$$

We begin by analysing the harmonic piece. In our notation, integrality of the winding of the compact scalar is expressed as

$$\int_{\Sigma_1} f^\phi_{\text{harm}} \in 2\pi\mathbb{Z}, \tag{4.3.4}$$

[12]We use conventions in which the action for Φ is normalised as $S[\Phi] = \frac{1}{8\pi} \int \mathrm{d}^2 z \, \partial\Phi\bar{\partial}\Phi$.

on any one-cycle Σ_1 of X. We can choose a convenient basis, $\left\{\tau_{\mathrm{i}}^{(1)}\right\}_{\mathrm{i}=1}^{\mathrm{b}_1(X)}$ of harmonic one-forms of X such that, given a basis $\{C_{\mathrm{i}(1)}\}_{\mathrm{i}=1}^{\mathrm{b}_1(X)}$ of one-cycles, it obeys

$$\int_{C_{\mathrm{i}(1)}} \tau_{\mathrm{j}}^{(1)} = \delta_{\mathrm{ij}}. \tag{4.3.5}$$

Such a basis will be called hereafter the *topological basis*. In the above, $\mathrm{b}_1(X) := \dim \mathrm{H}^1(X)$ is the first Betti number of X. In this basis we can expand

$$f^{\phi}_{\mathrm{harm}} = 2\pi n^{\mathrm{i}} \tau_{\mathrm{i}}^{(1)}, \qquad n^{\mathrm{i}} \in \mathbb{Z}, \tag{4.3.6}$$

where a sum over the repeated index, i, from 1 to $\mathrm{b}_1(X)$ is implied. The harmonic piece of the action (4.3.2) can therefore be written

$$S_{\mathrm{harm}}[\phi] = \frac{1}{2}\left(\frac{2\pi}{\mathrm{g}}\right)^2 n^{\mathrm{i}} \left[\mathbb{G}^{(1)}\right]_{\mathrm{ij}} n^{\mathrm{j}}, \tag{4.3.7}$$

where

$$\left[\mathbb{G}^{(1)}\right]_{\mathrm{ij}} := \int_X \tau_{\mathrm{i}}^{(1)} \wedge \star \tau_{\mathrm{j}}^{(1)} \tag{4.3.8}$$

is the Gram matrix of the above topological basis. The harmonic piece of the path integral then reads

$$\mathcal{Z}_{\mathrm{harm}}[X] = \sum_{\boldsymbol{n}\in\mathbb{Z}^{\mathrm{b}_1(X)}} \exp\left(-\frac{1}{2}\left(\frac{2\pi}{\mathrm{g}}\right)^2 \boldsymbol{n}\cdot\mathbb{G}^{(1)}\cdot\boldsymbol{n}\right), \tag{4.3.9}$$

where we combined n^{i} into a $\mathrm{b}_1(X)$-dimensional vector, $\boldsymbol{n}$. The oscillator contribution is a straightforward Gaussian integral, yielding

$$\mathcal{Z}_{\mathrm{osc}}[X] = \frac{\mathrm{vol}_0}{\sqrt{\det{}'\triangle_0}} = \left(\frac{\det\left(\frac{2\pi}{\mathrm{g}^2}\mathbb{G}^{(0)}\right)}{\det{}'\triangle_0}\right)^{\frac{1}{2}}, \tag{4.3.10}$$

where $\mathbb{G}^{(0)}$ is defined similarly as $\mathbb{G}^{(1)}$, but with respect to the zero-form topological basis and $\triangle_0 = \mathrm{d}^\dagger\mathrm{d}$ is the Laplacian acting on zero-forms. In total, the full partition function reads

$$\mathcal{Z}[X] = \left(\frac{\det\left(\frac{2\pi}{\mathrm{g}^2}\mathbb{G}^{(0)}\right)}{\det{}'\triangle_0}\right)^{\frac{1}{2}} \sum_{\boldsymbol{n}\in\mathbb{Z}^{\mathrm{b}_1(X)}} \exp\left(-\frac{1}{2}\left(\frac{2\pi}{\mathrm{g}}\right)^2 \boldsymbol{n}\cdot\mathbb{G}^{(1)}\cdot\boldsymbol{n}\right). \tag{4.3.11}$$

In the next paragraph we will take a canonical approach and we will view the partition function as a thermal trace. In order to compare, let us write the answer for the torus partition function from the path integral. To that end, we now take X to be a torus $X = \mathbb{T}^2_\tau := \mathbb{C}/(\mathbb{Z} \oplus \tau\,\mathbb{Z})$, with τ in the upper-half plane, $\tau \in \mathbb{H}$. Using the standard homology basis of one-cycles of the torus in terms of the A- and B-cycles, we have that

$$\mathbb{G}^{(1)} = \frac{1}{\operatorname{Im}\tau}\begin{pmatrix} 1 & \operatorname{Re}\tau \\ \operatorname{Re}\tau & |\tau|^2 \end{pmatrix}. \tag{4.3.12}$$

The harmonic zero-forms are simply the constant functions, so $\mathbb{G}^{(0)} = \operatorname{Im}\tau$. Finally, on the torus it is straightforward to calculate, using zeta-function regularisation that

$$\det{}'\triangle_0 = (\operatorname{Im}\tau)^2|\eta(\tau)|^2, \tag{4.3.13}$$

where $\eta(\tau)$ is the Dedekind eta function. Altogether the torus partition function reads

$$\mathcal{Z}[\mathbb{T}^2_\tau] = \sqrt{\frac{2\pi}{\mathrm{g}^2\operatorname{Im}\tau}}\frac{1}{|\eta(\tau)|^2}\sum_{n,m\in\mathbb{Z}}\exp\left(-\frac{1}{2}\left(\frac{2\pi}{\mathrm{g}}\right)^2\frac{|n+\tau m|^2}{\operatorname{Im}\tau}\right). \tag{4.3.14}$$

Finally, we can use the Poisson summation formula on the n sum to obtain

$$\mathcal{Z}[\mathbb{T}^2_\tau] = \frac{\Theta(q,\mathrm{g})}{|\eta(q)|^2}, \tag{4.3.15}$$

where

$$\Theta(q,\mathrm{g}) := \sum_{n,m\in\mathbb{Z}} q^{\frac{\pi}{2}\left(\frac{m}{\mathrm{g}}+\frac{n\mathrm{g}}{2\pi}\right)^2}\,\overline{q}^{\frac{\pi}{2}\left(\frac{m}{\mathrm{g}}-\frac{n\mathrm{g}}{2\pi}\right)^2}, \tag{4.3.16}$$

is the Siegel–Narain theta function, with $q := \mathrm{e}^{2\pi\mathrm{i}\tau}$ being the nome, $\overline{q}$ its complex conjugate, and we wrote the eta function as a function of the nome, instead of the complex structure.

The view from the algebra

Let us now match (4.3.15), starting from the Kac–Moody algebra, (4.2.26) and considering its representations. For simplicity we will discuss the quantisation on a rectangular torus, with complex structure, $\tau = \mathrm{i}\beta$. We will reinstate the generic complex structure at the end of the section. The Hamiltonian of the free scalar, on the spatial $\mathbb{S}^1$, takes the Sugawara form

$$H = L_0^+ + E_0^+ + L_0^- + E_0^-, \tag{4.3.17}$$

with

$$L_0^\pm = \frac{1}{\mathrm{k}}\sum_{\mathrm{n}\geqslant 0} J^\pm_{-\mathrm{n}}\,J^\pm_{\mathrm{n}} \quad \text{and} \quad E_0^\pm = \frac{1}{2}\zeta(-1) = -\frac{1}{24}, \tag{4.3.18}$$

where $\zeta(s)$ is the Riemann zeta function, regularising the zero-point energy, and k, the Kac–Moody level, is $\frac{\mathrm{g}^2}{2}$. With this Hamiltonian, the non-zero-modes of the

Kac–Moody algebra, $J^{\pm}_{\mathsf{n}\neq 0}$ are just ladder operators, as can be seen by computing their commutation relations with the Hamiltonian. The negative modes are creation operators, while the positive modes annihilation operators:

$$\left[H, J^{\pm}_{-\mathsf{n}}\right] = \quad \mathsf{n}\, J^{\pm}_{-\mathsf{n}} \qquad \mathsf{n} > 0, \tag{4.3.19}$$
$$\left[H, J^{\pm}_{\mathsf{n}}\right] = -\mathsf{n}\, J^{\pm}_{\mathsf{n}} \qquad \mathsf{n} > 0. \tag{4.3.20}$$

The zero-modes, $J^{\pm}_0$, commute among themselves and with the Hamiltonian, and they label the various fixed momenutm and winding sectors. Their eigenstates,

$$J^{\pm}_0 \left| j^+, j^- \right\rangle = j^{\pm} \left| j^+, j^- \right\rangle, \tag{4.3.21}$$

are primary states, or in other words, the ground state on each sector:

$$J^{\pm}_{\mathsf{n}} \left| j^+, j^- \right\rangle = 0, \qquad \mathsf{n} > 0. \tag{4.3.22}$$

To obtain the charges, $j^{\pm}$, we simply have to invoke momentum and winding flux quantisation on the spatial $\mathbb{S}^1$, i.e.

$$\int_{\mathbb{S}^1} f^{\phi} \in 2\pi\mathbb{Z} \quad \text{and} \quad \int_{\mathbb{S}^1} \check{f}^{\phi} \in 2\pi\mathbb{Z}, \tag{4.3.23}$$

where $\check{f}^{\phi}$ is the widning, or magnetic dual of f^{ϕ}, defined as $\check{f}^{\phi} = \frac{\pi \mathrm{i}}{\mathrm{k}} \star f^{\phi}$.[13] With this, we immediately get, with a convenient overall normalisation of the current:

$$j^{\pm} = \frac{1}{\sqrt{8\pi}}\left(2\pi m \pm \mathrm{g}^2 n\right), \qquad n, m \in \mathbb{Z}. \tag{4.3.24}$$

The ground states are, then, labelled by two integers, n and m, and we will denote them as

$$\left| j^+, j^- \right\rangle =: |n, m\rangle\,. \tag{4.3.25}$$

Over each of the two families of ground states sits a Verma module, $\mathcal{V}^{\pm}_{n,m}$, generated by acting with the creation operators $J^{\pm}_{-\mathsf{n}}$. To each Verma module corresponds a character,

$$\mathrm{ch}_{\mathcal{V}^{\pm}_{n,m}}[q] := \mathrm{tr}_{\mathcal{V}^{\pm}_{n,m}}\, q^H\,. \tag{4.3.26}$$

Having the explicit form of the Hamiltonian, and the algebra in our disposal, it is trivial task to compute the characters. They read:

$$\mathrm{ch}_{\mathcal{V}^{\pm}_{n,m}}[q] = q^{h^{\pm}-\frac{1}{24}} \left(\prod_{\mathsf{n}=1}^{\infty} \sum_{N_{\mathsf{n}}=0}^{\infty} q^{\mathsf{n}\, N_{\mathsf{n}}} \right) = \frac{q^{h^{\pm}}}{\eta(q)}, \tag{4.3.27}$$

[13]The factor of i is so that the action, written in the dual frame is positive semidefinite in Euclidean signature.

where we recognise the form of the Dedekind eta function, upon performing the geometric sum. In the above, $h^{\pm} = \left(j^{\pm}\right)^2$, is the eigenvalue of $L_0^{\pm}$ acting on the ground states.

On each sector we have left- and right-moving states, i.e. a generic state on top of $|n, m\rangle$ lives in $\mathcal{V}^+_{n,m} \otimes \mathcal{V}^-_{n,m}$. The full Hilbert space of the theory is, then, a direct sum over all possible sectors:

$$\mathcal{H}_{\mathbb{S}^1} = \bigoplus_{n,m\in\mathbb{Z}} \mathcal{V}^+_{n,m} \otimes \mathcal{V}^-_{n,m}. \tag{4.3.28}$$

A generic state of this Hilbert space is

$$\left|n, m; \left\{N^+_{\mathrm{n}}, N^-_{\mathrm{m}}, \cdots\right\}\right\rangle := \left(J^+_{-\mathrm{n}}\right)^{N^+_{\mathrm{n}}} \left(J^-_{-\mathrm{n}}\right)^{N^-_{\mathrm{m}}} \cdots |n, m\rangle\,. \tag{4.3.29}$$

As such, we see that taking a trace, i.e. computing an extended character of all the Verma modules, lands us on the torus partition function, (4.3.15) (with $\tau = \mathrm{i}\beta$):

$$\mathbf{ch}[q] := \sum_{n,m\in\mathbb{Z}} \mathrm{ch}_{\mathcal{V}^+_{n,m}\otimes\mathcal{V}^-_{n,m}}[q] = \frac{1}{|\eta(q)|^2} \sum_{n,m\in\mathbb{Z}} q^{\Delta_{n,m}}, \tag{4.3.30}$$

with

$$\Delta_{n,m} = h^+ + h^- = \frac{\mathrm{g}^2 n^2}{4\pi} + \frac{m^2\pi}{\mathrm{g}^2}. \tag{4.3.31}$$

Reinstating the generic complex structure, $\tau \in \mathbb{H}$, we obtain the well-appreciated, yet remarkable result:

$$\mathcal{Z}\left[\mathbb{T}^2_\tau\right] = \mathbf{ch}[q] \quad \text{with} \quad q = \mathrm{e}^{2\pi\mathrm{i}\tau}. \tag{4.3.32}$$

4.3.2 The state-operator correspondence

As a preparatory exercise for the four-dimensional story of the later sections, we will review here the well-established state-operator correspondence for local operators. In particular we will show that the states in $\mathcal{H}_{\mathbb{S}^1}$, as in (4.3.28), can be prepared by a path integral, with insertions of local operators, on the disc, $\mathbb{D}^2$, whose boundary is the spatial $\mathbb{S}^1$. We will do so for the compact scalar, where both sides of the state operator correspondence can be explicitly identified and checked. Moreover, a crucial role is played by the Kac–Moody algebra of the compact scalar, organising the local operators into primaries and descendants. This will serve as an analogy for the later sections, where, as we will see, the ideas will be similar.

The general strategy towards a state-operator map for local operators (illustrated in figure 4.1), common to all CFTs is the following. We take a spherical slice (here $\mathbb{S}^1$), of the Euclidean cylinder (here $\mathbb{R} \times \mathbb{S}^1$) at $t = 0$. Each state on this slice is prepared by a choice of boundary conditions at $t = -\infty$. Mapping the Euclidean cylinder to the disc (rather, d-ball in higher dimensions) with a Weyl transformation, this choice gets mapped to a boundary condition at the centre of the disc.[14] This boundary condition can be satisfied by the insertion of a local operator. Then the path integral on the disc, with this operator insertion prepares the state that we are after. More precisely, the path integral on the disc, with boundary conditions $\Phi\big(\partial\mathbb{D}^2\big) = \Phi_\partial$ (where Φ denotes all dynamical fields) and an insertion $\mathcal{O}[\Phi(0)]$ at the centre of the disc produces a wavefunctional:

$$\Psi_{\mathcal{O}}[\Phi_\partial] = \int_{\mathcal{C}[\Phi_\partial]} \mathrm{D}\Phi\, \mathrm{e}^{-S[\Phi]}\, \mathcal{O}[\Phi(0)], \tag{4.3.33}$$

where $\mathcal{C}[\Phi_\partial]$ denotes an appropriate functional space over the disc, with boundary conditions Φ_∂. We will write the state produced by $\mathcal{O}$ as

$$|\mathcal{O}\rangle = \int_{\mathcal{C}[\,\cdot\,]} \mathrm{D}\Phi\, \mathrm{e}^{-S[\Phi]}\, \mathcal{O}[\Phi(0)], \tag{4.3.34}$$

to indicate that we need to provide a boundary condition to get out a number.

Here we note that it would be much easier to work in complex coordinates, and exploit the power of Cauchy's theorem, as is commonly done in two-dimensional conformal field theories. We will not do so, however, as a preparatory exercise for the four-dimensional case where we cannot afford that luxury. For a discussion in a very similar vein, written in complex variables see e.g. [296].

Before discussing the states and the operators, it will be useful to obtain an integral expression for the ladder operators, in terms of the currents, that we can insert at an intermediate point in the path integral. To that end, let $i_{\mathbb{S}^1_r} : \mathbb{S}^1_r \hookrightarrow \mathbb{D}^2$ be the map embedding a circle of radius r into the disc. The ladder operators can be expressed as:

$$J^\pm_{\mathrm{m}} = \frac{\mp \mathrm{i} r^{\mathrm{m}+1}}{\sqrt{2\pi}} \int_{\mathbb{S}^1_r} \mathrm{e}^{\pm \mathrm{i}\mathrm{m}\theta}\, i^*_{\mathbb{S}^1_r} J^\pm, \tag{4.3.35}$$

where $i^*_{\mathbb{S}^1_r} J^\pm$ is the pullback of $J^\pm \in \Omega^2\big(\mathbb{D}^2\big)$ along the map $i_{\mathbb{S}^1_r}$. This is arrived at by expanding $J^\pm$ in Fourier modes, solving the closedness condition, $\mathrm{d}J^\pm = 0$ and then inverting the Fourier transform. As a sanity check, we can verify that $J^\pm_{\mathrm{m}}$, given by (4.3.35), obey the Kac–Moody algebra, (4.2.26):

$$\big[J^\pm_{\mathrm{n}}, J^\pm_{\mathrm{m}}\big] = \mathrm{k}\,\mathrm{n}\,\delta_{\mathrm{n}+\mathrm{m},0}, \tag{4.3.36}$$

while the mixed commutators are zero.

[14] We will parametrise the disc by a radial coordinate, $r \in [0,1]$, and an angle $\theta \in [0, 2\pi)$, i.e. $\mathrm{d}s^2_{\mathbb{D}^2} = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$.

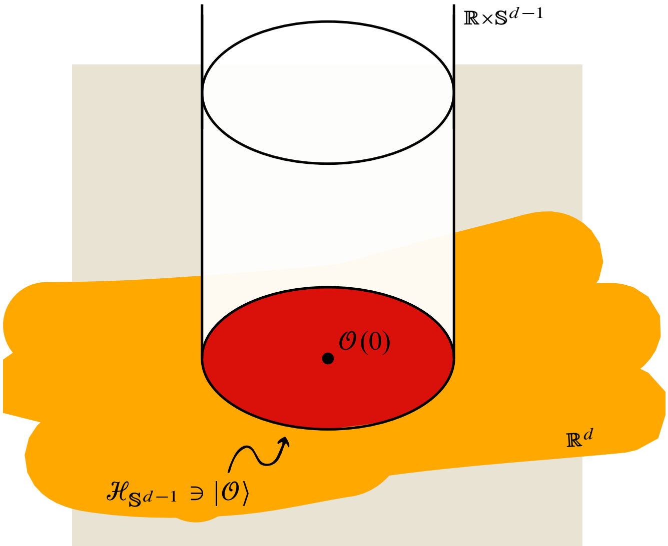

Figure 4.1: *The state-operator correspondence. Any state on $\mathbb{S}^{d-1}$ can be prepared by a path integral on $\mathbb{B}^d$ with a local operator inserted in the centre. The state then evolves in time on the Lorentzian cylinder $\mathbb{R}\times\mathbb{S}^{d-1}$.*

Let us start by discussing the ground states, $|n,m\rangle$. They are prepared by inserting a vertex operator, $\mathrm{V}_{n,m}$, of momentum n and winding m:

$$\mathrm{V}_{n,m}(x) := \exp\Big(\mathrm{i}\, n\, \phi(x) + \mathrm{i}\, m\, \check{\phi}(x)\Big), \tag{4.3.37}$$

where x is a point on the disc and $\check{\phi}$ is the winding dual of ϕ, i.e. the field whose curvature is $\check{f}^{\phi}$, as defined around (4.3.23). The states are then obtained as:

$$|n,m\rangle = \int_{\mathcal{C}[\cdot]} \mathrm{D}\phi\, \mathrm{e}^{-S[\phi]}\, \mathrm{V}_{n,m}(0). \tag{4.3.38}$$

In this case, $\mathcal{C}[\cdot]$ is the space of smooth functions on the disc. To verify that these are indeed the ground states, it suffices to show that they are indeed annihilated by the lowering operators, $J^{\pm}_{\mathrm{n}>0}$. To illustrate the argument, let us first focus on the state $|0,0\rangle$, which is the zero-momentum, zero-winding state, corresponding to the identity operator. There we have

$$J^{\pm}_{\mathrm{n}}\,|0,0\rangle \sim \lim_{r\to 0} \int_{\mathcal{C}[\cdot]} \mathrm{D}\phi\, \mathrm{e}^{-S[\phi]} \Big(\int_{\mathbb{S}^1_r} r^{\mathrm{n}+1} \mathrm{e}^{\pm \mathrm{i}\mathrm{n}\theta}\, i^*_{\mathbb{S}^1_r} J^{\pm} \Big), \tag{4.3.39}$$

where the twiddle indicates that we are ignoring purely numerical factors. Since we are integrating over smooth functions, and $J^{\pm} = \mathbb{P}_{\pm} f^{\phi}$, the integrand is smooth inside the disc, for $\mathsf{n} \geqslant 0$. Therefore, we have that

$$J_{\mathsf{n}}^{\pm}\,|0,0\rangle = 0, \qquad \forall\,\mathsf{n} > 0. \tag{4.3.40}$$

In this case, we also have that $J_0^{\pm}\,|0,0\rangle = 0$, as $|0,0\rangle$ is a state of zero momentum and winding. To show that the states $|n,m\rangle$ are annihilated by all positive modes the argument is essentially the same, save for the fact that before assessing the integrand one needs to look at the operator product between $i^*_{\mathbb{S}^1_r} J^{\pm}$ and $\mathrm{V}_{n,m}$. Since $J^{\pm}$ is a current and $\mathrm{V}_{n,m}$ is a primary operator, their OPE goes like $\frac{1}{r}$. Therefore, $|n,m\rangle$ are annihilated by all positive modes, but not by the zero-modes, as expected.

We have, so far, argued, that (4.3.38) produces all the highest-weight states, of our Verma module discussion in subsection 4.3.1. To complete the picture, what remains is a construction of the descendants. It suffices to consider the states $\left|n,m;\{1_{\mathsf{n}}^{\pm}\}\right\rangle$; a generic state is straightforward to arrive at, afterwards. The claim is, then, that these states are prepared by:

$$\left|n,m;\{1_{\mathsf{n}}^{\pm}\}\right\rangle = \int_{\mathcal{C}[\cdot]} \mathrm{D}\phi\, \mathrm{e}^{-S[\phi]}\;\; \partial_{\pm}^{\mathsf{n}}\phi\, \mathrm{V}_{n,m}(0), \qquad \mathsf{n} > 0, \tag{4.3.41}$$

where $\partial_{\pm}$ is the "(anti-)self-dual derivative"[15] and $\partial_{\pm}^{\mathsf{n}}\phi\mathrm{V}_{n,m}(0)$ denotes the operator obtained upon performing the OPE of $\partial_{\pm}^{\mathsf{n}}\phi$ with $\mathrm{V}_{n,m}$ and placing it at 0. To verify the claim (4.3.41), observe what happens if we act with a lowering operator. As before, we treat the descendants of the state dual to the identity operator, $|0,0\rangle$, first.

$$J_{\mathsf{m}}^{\pm}\left|0,0;\{1_{\mathsf{n}}^{\pm}\}\right\rangle \sim \lim_{r\to 0}\int_{\mathcal{C}[\cdot]} \mathrm{D}\phi\, \mathrm{e}^{-S[\phi]} \left(\int_{\mathbb{S}^1_r} \mathrm{d}\theta\, r^{\mathsf{m}+1}\mathrm{e}^{\pm\mathrm{i}(\mathsf{m}+1)\theta}\; \partial_{\pm}\phi(r,\theta)\, \partial_{\pm}^{\mathsf{n}}\phi(0)\right), \tag{4.3.42}$$

where we have also used the fact that, for all that matters here, $i^*_{\mathbb{S}^1_r} J^{\pm}$ is proportional to $\mathrm{e}^{\pm\mathrm{i}\theta}\partial_{\pm}\phi\,\mathrm{d}\theta$. We recognise the insertion as the $(\mathsf{n}-1)$-th (anti-)self-dual derivative of a current. So we can invoke the current-current OPE [285], to get

$$\partial_{\pm}\phi(r,\theta)\;\partial_{\pm}^{\mathsf{n}}\phi(0) \sim \frac{\mathsf{k}\,\mathsf{n}!}{r^{\mathsf{n}+1}}\,\mathrm{e}^{\mp\mathrm{i}(\mathsf{n}+1)\theta}. \tag{4.3.43}$$

This, in turn, implies:

$$J_{\mathsf{m}}^{\pm}\left|0,0;\{1_{\mathsf{n}}^{\pm}\}\right\rangle \sim \lim_{r\to 0}\int_{\mathcal{C}[\cdot]} \mathrm{D}\phi\, \mathrm{e}^{-S[\phi]} \left(\int_0^{2\pi} \mathrm{d}\theta\, r^{\mathsf{m}-\mathsf{n}}\mathrm{e}^{\pm\mathrm{i}(\mathsf{m}-\mathsf{n})\theta}\right) \sim \mathsf{k}\,\delta_{\mathsf{m},\mathsf{n}}\,|0,0\rangle\,. \tag{4.3.44}$$

[15] Intuitively ∂_+ is just the holomorphic derivative and ∂_- the antiholomorphic derivative. If we insist on avoiding complex analysis language we can also define them as follows. First define bases of the (anti-)self-dual one-forms, $\mathrm{d}x^{\pm} := \mathrm{e}^{\mathrm{i}\theta}(\mathrm{d}r \pm \mathrm{i}r\,\mathrm{d}\theta)$. Then dualise these to vectors fields, as $(\mathrm{d}x^{\pm})^{\sharp}$ and take an interior product with the exterior derivative, so $\partial_{\pm} := \left(\mathrm{d}x^{\pm}\right)^{\sharp} \lrcorner\, \mathrm{d}$.

The factor of n in the Kac–Moody algebra (4.2.26) is absorbed into the proportionality symbol, and is related to a rescaling of the operator insertion by $\frac{1}{(\mathsf{n}-1)!}$. However, we kept track of the Kac–Moody level, since it stems from the OPE of the currents.

Inserting back $\mathrm{V}_{n,m}$, it is now a matter of Wick contractions and restating the above arguments to show that

$$J^{\pm}_{\mathsf{m}} \left|n, m; \{1^{\pm}_{\mathsf{n}}\}\right\rangle = \mathsf{k}\,\mathsf{m}\,\delta_{\mathsf{m},\mathsf{n}} \left|n, m\right\rangle . \tag{4.3.45}$$

This concludes the proof that the operator $\mathrm{V}_{n,m}[\phi]\, \partial^{\mathsf{n}}_{\pm}\phi$ prepares the state $\left|n, m; \{1^{\pm}_{\mathsf{n}}\}\right\rangle$. We therefore have all the ingredients to write down the generic state. It is given by:

$$\left|n, m; \{N^{+}_{\mathsf{n}}, N^{-}_{\mathsf{m}}, \cdots\}\right\rangle = \int_{\mathcal{C}[\cdot]} \mathrm{D}\phi\, \mathrm{e}^{-S[\phi]} \left\{ \left(\partial^{\mathsf{n}}_{+}\phi\right)^{N^{+}_{\mathsf{n}}} \left(\partial^{\mathsf{m}}_{-}\phi\right)^{N^{-}_{\mathsf{m}}} \cdots \mathrm{V}_{n,m}(0) \right\}. \tag{4.3.46}$$

4.4 Four-dimensional Maxwell theory

Having dealt with the two-dimensional case in sufficient detail we now turn to the main focus of this chapter: 4d and the nonlocal state operator correspondence. We will illustrate this in four dimensions, although the results extend to generic even dimensions easily. We therefore, consider a unitary four-dimensional CFT, with a $\mathrm{U}(1)^{[1]}$ symmetry. As we have explained in section 4.2, this theory has necessarily a free photon description. We will, therefore focus on free Maxwell theory. We will mimick the structure of section 4.3: we will first discuss the path integral on arbitrary compact manifolds; then we will use the $p = 1$ form of the current algebra (4.2.24), study its representations and show that they explicitly reproduce the path integral expressions. This will then lead us to the nonlocal state-operator correspondence.

4.4.1 The Maxwell path integral

We consider Maxwell theory on a closed, compact four-dimensional manifold X.[16] Since we are interested in topologically non-trivial manifolds we must also allow for a theta angle. The action is:

$$S[a] := \frac{1}{2\mathrm{g}^2} \int_X f \wedge \star f - \frac{\mathrm{i}\theta}{16\pi^2} \int_X f \wedge f, \tag{4.4.1}$$

where the closed two-form f is the curvature of a U(1) gauge field, $a \in \Omega^1(X)$. We will often abbreviate the first term as $\| f \|^2$, where the norm is with respect to the usual

[16] We will take X to be torsion-free, throughout. All results can be generalised to manifolds with torsion, following, for example, the arguments in [220].

Hodge inner-product. Hodge decomposition instructs us to write $f = f_{\text{harm}} + \mathrm{d}a$, where f_{harm} is a harmonic representative of the second cohomology class of X, chosen uniquely to be orthogonal to $\mathrm{d}a$ in the Hodge inner product. Moreover, the theta term does not see the topologically trivial piece, so the action reads, with this decomposition

$$S[a] = \underbrace{\frac{1}{2\mathrm{g}^2}\|f_{\text{harm}}\|^2 - \frac{\mathrm{i}\theta}{16\pi^2}\int_X f_{\text{harm}} \wedge f_{\text{harm}}}_{\substack{\shortparallel \\ S_{\text{inst}}[f_{\text{harm}}]}} + \underbrace{\frac{1}{2\mathrm{g}^2}\|\mathrm{d}a\|^2}_{\substack{\shortparallel \\ S_{\text{osc}}[a]}}. \tag{4.4.2}$$

With a theta angle turned on, it is useful to introduce the complexified coupling constant:

$$\mathrm{t} := \frac{\theta}{4\pi} + \frac{2\pi\mathrm{i}}{\mathrm{g}^2}. \tag{4.4.3}$$

We refrain from using the more common τ for the complexified coupling constant to avoid confusion with the two dimensional case, where τ represents the complex structure of the base space. Sometimes we will use

$$\mathrm{t}^{\pm} := \operatorname{Re}\mathrm{t} \pm \mathrm{i}\operatorname{Im}\mathrm{t}, \tag{4.4.4}$$

to write some of our formulas more compactly.

Passing to the path integral, the partition function of Maxwell theory splits into an instanton contribution and an oscillator contribution

$$\mathcal{Z}[X;\mathrm{t}] = \int \frac{\mathrm{D}f_{\text{harm}}\,\mathrm{D}a}{\operatorname{vol}\mathcal{G}} \exp(-S_{\text{inst}}[f_{\text{harm}}] - S_{\text{osc}}[a]) = \mathcal{Z}_{\text{inst}}[X;\mathrm{t}]\mathcal{Z}_{\text{osc}}[X;\mathrm{t}], \tag{4.4.5}$$

where $\mathcal{G}$ is the group of gauge transformations; shifts of a by a flat connection. We will compute each piece separately and discuss the respective integration measures in detail.

Let us first focus on the oscillator part. By Hodge decomposition, and upon employing the Faddeev–Popov procedure, we can gauge-fix a to be coclosed. The Faddeev-Popov ghosts are, in this case, zero-forms with fermionic coefficients. The oscillator part of the action becomes then

$$S_{\text{osc}}[a] = \frac{1}{2\mathrm{g}^2}\langle a, \Box_1\, a\rangle\,, \tag{4.4.6}$$

where $\Box_p := \triangle_p\big|_{\ker \mathrm{d}^\dagger}$, is the transversal Laplacian, i.e. the Hodge Laplacian, $\triangle_p = \mathrm{d}^\dagger\mathrm{d} + \mathrm{d}\mathrm{d}^\dagger$, restricted to coclosed forms. Here we can expand the action in terms of eigen-one-forms of the transversal Laplacian,

$$a = \sum_{\lambda \in \operatorname{spec}_X(\Box_1)} c_\lambda a_\lambda. \tag{4.4.7}$$

In the above, $\lambda = 0$, and a_0 collects all the zero-mode spectrum of the Laplacian, $\Box_1 a_0 = 0$. We can shift all the g-dependence on the zero-mode path integral by normalising the measure as

$$\mathrm{D}a := \prod_{\lambda \in \mathrm{spec}_X(\Box_1)} \frac{\mathrm{d}c_\lambda}{\sqrt{2\pi}\,\mathrm{g}}. \tag{4.4.8}$$

Both the gauge field and the ghost path integrals are Gaussian and can be evaluated directly to give

$$\mathcal{Z}^{\mathrm{osc}}[X;\mathrm{t}] = \frac{\mathrm{vol}\big(\mathrm{H}^1(X;\mathbb{Z})\big)}{\mathrm{vol}(\mathrm{H}^0(X;\mathbb{Z}))} \left(\frac{\det{}'\Box_0}{\det{}'\Box_1}\right)^{1/2}. \tag{4.4.9}$$

Using the specrtral properties of the Hodge Laplacian, we can rewrite this in terms of the full Laplacian as

$$\mathcal{Z}^{\mathrm{osc}}[X;\mathrm{t}] = \frac{\mathrm{vol}\big(\mathrm{H}^1(X;\mathbb{Z})\big)}{\mathrm{vol}(\mathrm{H}^0(X;\mathbb{Z}))} \frac{\det{}'\triangle_0}{\sqrt{\det{}'\triangle_1}}. \tag{4.4.10}$$

In the above, $\mathrm{vol}\big(\mathrm{H}^1(X;\mathbb{Z})\big)$, and $\mathrm{vol}\big(\mathrm{H}^0(X;\mathbb{Z})\big)$ is the zero-mode volume of the gauge fields and the ghosts, respectively, computed with the measure (4.4.8). To see that, note that the zero-modes of $\triangle_1$ are precisely flat connections, or in other words elements of $\mathrm{H}^1(X)$. Since we identify two connections under large gauge transformations (these are packaged in the instanton sum), we must look at $\mathrm{H}^1(X;\mathbb{Z})$. We have assumed that the manifold, X, is torsion-free and thus $\mathrm{H}^1(X;\mathbb{Z}) \cong \mathbb{Z}^{\mathrm{b}_1(X)}$. We can therefore use the topological basis of the harmonic one-forms, $\left\{\tau_{\mathrm{i}}^{(1)}\right\}_{\mathrm{i}=1}^{\mathrm{b}_1(X)}$, defined similarly as in (4.3.5) to write

$$\left[\mathbb{G}^{(1)}\right]_{\mathrm{ij}} := \int_X \tau_{\mathrm{i}}^{(1)} \wedge \star \tau_{\mathrm{j}}^{(1)}. \tag{4.4.11}$$

With that we can integrate over $\mathrm{H}^1(X;\mathbb{Z})$ with the measure (4.4.8), to get

$$\mathrm{vol}\big(\mathrm{H}^1(X;\mathbb{Z})\big) = \det\!\left(\frac{2\pi}{\mathrm{g}^2}\mathbb{G}^{(1)}\right)^{1/2}. \tag{4.4.12}$$

For the ghosts we have similarly

$$\mathrm{vol}\big(\mathrm{H}^0(X;\mathbb{Z})\big) = \det\!\left(\frac{2\pi}{\mathrm{g}^2}\mathbb{G}^{(0)}\right)^{1/2}, \tag{4.4.13}$$

where $\mathbb{G}^{(0)}$ is defined analogously by the topological basis of $\mathrm{H}^0(X;\mathbb{Z})$. If $\mathrm{b}_1(X) = 0$, the Laplacian $\Box_1$ has no zero-modes and hence we compute the full determinant $\det \Box_1$.

It is convenient to disentangle base-space and target-space quantities to keep track of the modular properties and of electromagnetic duality separately. The only place where g appears is in the zero-mode volumes and therefore we have that

$$\mathcal{Z}_{\mathrm{osc}}[X;\mathrm{t}] = \left(\frac{2\pi}{\mathrm{g}^2}\right)^{\frac{1}{2}(\mathrm{b}_1(X)-\mathrm{b}_0(X))} \left(\frac{\det\mathbb{G}^{(1)}}{\det\mathbb{G}^{(0)}}\right)^{1/2} \frac{\det{}'\triangle_0}{\sqrt{\det{}'\triangle_1}}. \tag{4.4.14}$$

For the rest, we are only interested connected manifolds, therefore $\mathrm{b}_0(X) = 1$ and $\det\mathbb{G}^{(0)} = \mathrm{vol}(X)$. Furthermore, note that the prefactor is just $\mathrm{Im}\,\mathrm{t}$. So, all in all, the oscillator piece reads, finally

$$\mathcal{Z}_{\mathrm{osc}}[X;\mathrm{t}] = (\mathrm{Im}\,\mathrm{t})^{\frac{1}{2}(\mathrm{b}_1(X)-1)} \left(\frac{\det\mathbb{G}^{(1)}}{\mathrm{vol}(X)}\right)^{1/2} \frac{\det{}'\triangle_0}{\sqrt{\det{}'\triangle_1}}. \tag{4.4.15}$$

We now turn to the instanton contribution. Similarly as in section 4.3, f_{harm} is integrally quantised on two-cycles,

$$\int_{\Sigma_2} f_{\mathrm{harm}} \in 2\pi\mathbb{Z}, \tag{4.4.16}$$

so we can expand $f_{\mathrm{harm}} = 2\pi n^{\mathrm{i}}\tau^{(2)}_{\mathrm{i}}$ with $n^{\mathrm{i}} \in \mathbb{Z}$, and $\left\{\tau^{(2)}_{\mathrm{i}}\right\}_{\mathrm{i}=1}^{\mathrm{b}_2(X)}$ is the topological basis of hamronic two-forms. The first term of the instanton piece of the action becomes, then:

$$\frac{1}{2\mathrm{g}^2}\|f_{\mathrm{harm}}\|^2 = \frac{2\pi^2}{\mathrm{g}^2} n^{\mathrm{i}} \left[\mathbb{G}^{(2)}\right]_{\mathrm{ij}} n^{\mathrm{j}}, \qquad n^{\mathrm{i}} \in \mathbb{Z}, \tag{4.4.17}$$

where $\mathbb{G}^{(2)}$, is defined similarly as in (4.4.11). For the theta term we make use of the intersection bilinear on X:

$$\mathbb{Q}_{\mathrm{ij}} := \int_X \tau^{(2)}_{\mathrm{i}} \wedge \tau^{(2)}_{\mathrm{j}}, \tag{4.4.18}$$

to write

$$\frac{\mathrm{i}\theta}{16\pi^2}\int_X f_{\mathrm{harm}} \wedge f_{\mathrm{harm}} = \frac{\mathrm{i}\theta}{4} n^{\mathrm{i}}\,\mathbb{Q}_{\mathrm{ij}}\,n^{\mathrm{j}}. \tag{4.4.19}$$

Finally, the measure, $\mathrm{D}f_{\mathrm{harm}}$, becomes a sum over (vectors of) integers, $\boldsymbol{n} \in \mathbb{Z}^{\mathrm{b}_2(X)}$. Putting it all together, the instanton contribution reads

$$\mathcal{Z}_{\mathrm{inst}}[X;\mathrm{t}] = \sum_{\boldsymbol{n}\in\mathbb{Z}^{\mathrm{b}_2(X)}} \exp(\pi\,\mathrm{i}\,\boldsymbol{n}\cdot\mathbb{B}(t)\cdot\boldsymbol{n}), \tag{4.4.20}$$

where we have defined, for conciseness, the matrix

$$\mathbb{B}(\mathrm{t}) := \mathrm{Re}\,\mathrm{t}\;\mathbb{Q} + \mathrm{i}\,\mathrm{Im}\,\mathrm{t}\;\mathbb{G}^{(2)} = \frac{2\pi\,\mathrm{i}}{\mathrm{g}^2}\mathbb{G}^{(2)} + \frac{\theta}{4\pi}\mathbb{Q}. \tag{4.4.21}$$

Combining the oscillator and the instanton contributions we have in total the partition function of free Maxwell theory on a general, torsion-free, connected, closed four-manifold X:

$$\mathcal{Z}[X;\mathrm{t}] = (\operatorname{Im}\mathrm{t})^{\frac{1}{2}(\mathrm{b}_1(X)-1)} \left(\frac{\det \mathbb{G}^{(1)}}{\operatorname{vol}(X)}\right)^{1/2} \frac{\det{}'\triangle_0}{\sqrt{\det{}'\triangle_1}} \sum_{\boldsymbol{n}\in\mathbb{Z}^{\mathrm{b}_2(X)}} \exp(\pi\,\mathrm{i}\,\boldsymbol{n}\cdot\mathbb{B}(\mathrm{t})\cdot\boldsymbol{n}). \tag{4.4.22}$$

SL(2, ℤ) duality

Before moving on, we briefly comment on some of the duality properties of this theory. It is well-known, that Maxwell theory enjoys an SL(2, ℤ) duality group, generated by S-duality, $\mathrm{t}\longmapsto -1/\mathrm{t}$ and T-duality, $\mathrm{t}\longmapsto \mathrm{t}+1$. It is, however, also known that the SL(2, ℤ) duality group is afflicted with an anomaly [250, 297, 298]. On a generic manifold, one then either needs to sacrifice one of the two generators, or couple the system to a five-dimensional SPT phase, carrying the duality anomaly (as is done, e.g. in [298]). In this section we calculate an instantiation of the duality anomaly using the form (4.4.22), and discuss ways to guarantee that either S- or T-duality holds on the nose. We also show that, in fact, the manifolds required to obtain our state-operator map, are free from the duality anomaly.

Key to the duality properties, will be a version of the Poisson summation formula:

$$\sum_{\boldsymbol{n}\in\mathbb{Z}^N} \mathrm{e}^{-\boldsymbol{n}\cdot\mathbb{A}\cdot\boldsymbol{n}} = \frac{\pi^{N/2}}{\sqrt{\det\mathbb{A}}} \sum_{\boldsymbol{n}\in\mathbb{Z}^N} \mathrm{e}^{-\pi^2\boldsymbol{n}\cdot\mathbb{A}^{-1}\cdot\boldsymbol{n}}. \tag{4.4.23}$$

The form (4.4.22) is particularly handy to perform Poisson resummation. In particular, we can check that $-\mathbb{Q}^\top\mathbb{B}(-1/\mathrm{t})\mathbb{Q}^\top$ is the inverse of $\mathbb{B}(\mathrm{t})$, and the partition function can be expressed as:

$$\begin{aligned}\mathcal{Z}[X;\mathrm{t}] = &\left(\frac{(\operatorname{Im}\mathrm{t})^{(\mathrm{b}_1(X)-1)}}{\det(-\mathrm{i}\mathbb{B}(\mathrm{t}))}\right)^{1/2} \left(\frac{\det\mathbb{G}^{(1)}}{\operatorname{vol}(X)}\right)^{1/2} \frac{\det{}'\triangle_0}{\sqrt{\det{}'\triangle_1}}\times\\ &\times \sum_{\boldsymbol{n}\in\mathbb{Z}^{\mathrm{b}_2(X)}} \exp(\pi\,\mathrm{i}\,\boldsymbol{n}\cdot\mathbb{B}(-1/\mathrm{t})\cdot\boldsymbol{n}).\end{aligned} \tag{4.4.24}$$

Let us now make some simplifying assumptions for the rest of the calculation. Since $\mathrm{H}^2(X)$ is a middle cohomology group for a four-dimensional manifold its elements can be decomposed into self-dual and anti-self-dual pieces. The dimensions of the (anti)-self-dual parts of the cohomology group are $\mathrm{b}_2^\pm(X)$, such that $\mathrm{b}_2^+(X)+\mathrm{b}_2^-(X) = \mathrm{b}_2(X)$ and $\mathrm{b}_2^+(X)-\mathrm{b}_2^-(X)=\sigma$, with σ being the Hirzebruch signature of X. For all the manifolds that we are interested in (namely products of spheres of different dimensions), $\sigma=0$, therefore $\mathrm{b}_2^+(X)=\mathrm{b}_2^-(X)=\frac{1}{2}\mathrm{b}_2(X)$. We will assume this for the following, although it is straightforward to generalise the discussion to manifolds with $\sigma\neq 0$.

The determinant of $\mathbb{B}(\mathrm{t})$ can be calculated to be $\det(-\mathrm{i}\mathbb{B}(\mathrm{t})) = |\mathrm{t}|^{\mathrm{b}_2(X)}$. This follows immediately from the fact that the eigenvalues of $\mathbb{G}^{(1)}\mathbb{Q}^{-1}$ are ± 1, with multiplicity $\frac{1}{2}\mathrm{b}_2(X)$. Furthermore using $\mathrm{Im}(-1/\mathrm{t}) = |\mathrm{t}|^{-2}\,\mathrm{Im}(\mathrm{t})$, we get the S-dual form of the partition function:

$$\mathcal{Z}\left[X; -\frac{1}{\mathrm{t}}\right] = |\mathrm{t}|^{\frac{1}{2}\chi(X)}\mathcal{Z}[X;\mathrm{t}], \tag{4.4.25}$$

where $\chi(X)$ is the Euler characteristic of X. Moreover, we can immediately see that if $\mathbb{Q}$ is even

$$\mathcal{Z}[X;\mathrm{t}+1] = \mathcal{Z}[X;\mathrm{t}], \tag{4.4.26}$$

since its effect on (4.4.22) is to shift the exponent by $2\pi \times$ integer, whereas if $\mathbb{Q}$ is odd, $\mathcal{Z}[X;\mathrm{t}+2] = \mathcal{Z}[X;\mathrm{t}]$.[17] (4.4.25) and (4.4.26), correspond to the action of the S and T generators of the $\mathrm{SL}(2,\mathbb{Z})$ duality group. In its current form, the partition function of Maxwell theory favours T-duality, while the S-duality transformation, seemingly suffers from the aforementioned anomaly.

However, this is deceiving. Namely, we can add to the action a counterterm of the form

$$S_{\mathrm{ct}} = \frac{1}{32\pi^2}\int_X f(\mathrm{t})\,\epsilon^{abcd} R_{ab}\wedge R_{cd} = f(\mathrm{t})\,\chi(X), \tag{4.4.27}$$

where the last equation is only true if $f(\mathrm{t})$ does not depend on X. Choosing $f(\mathrm{t}) = -\frac{1}{4}\log \mathrm{Im}\,\mathrm{t}$, the partition function gets modified to

$$\widetilde{\mathcal{Z}}[X;\mathrm{t}] := (\mathrm{Im}\,\mathrm{t})^{\frac{1}{4}\chi(X)}\mathcal{Z}[X;\mathrm{t}]. \tag{4.4.28}$$

It is straightforward to check that indeed, both transformations are healthy, and

$$\widetilde{\mathcal{Z}}[X;\gamma\cdot\mathrm{t}] = \widetilde{\mathcal{Z}}[X;\mathrm{t}], \qquad {}^{\forall}\gamma \in \mathrm{SL}(2,\mathbb{Z}). \tag{4.4.29}$$

Written in terms of (4.4.22), the duality-corrected version of the partition function reads

$$\widetilde{\mathcal{Z}}[X;\mathrm{t}] = (\mathrm{Im}\,\mathrm{t})^{\frac{\mathrm{b}_2(X)}{4}}\left(\frac{\det \mathbb{G}_1}{\mathrm{vol}(X)}\right)^{1/2}\frac{\det{}'\Delta_0}{\sqrt{\det{}'\Delta_1}}\sum_{\boldsymbol{n}\in\mathbb{Z}^{\mathrm{b}_2(X)}}\exp\left(\frac{\pi\mathrm{i}}{2}\,\boldsymbol{n}\cdot\mathbb{B}(\mathrm{t})\cdot\boldsymbol{n}\right). \tag{4.4.30}$$

Finally, we mention for completeness, that for manifolds of non-vanishing Hirzebruch signature, absorbing the S-duality non-invariance into a counterterm is not for free. Instead it is now the T-transformation that acts non-trivially [297]:

$$\widetilde{\mathcal{Z}}[X;\mathrm{t}+1] = \mathrm{e}^{-\frac{1}{3}\pi\mathrm{i}\sigma}\widetilde{\mathcal{Z}}[X;\mathrm{t}]. \tag{4.4.31}$$

[17] This can be corrected for, by adding an extra, topological term, to the action, as in [297]

Thus in those cases the $\mathrm{SL}(2,\mathbb{Z})$ duality group indeed suffers from an anomaly. In what follows we will drop the tilde from the partition function, to simplify the notation. In most cases it will not even make a difference, since we will be mainly interested in manifolds with a circle factor where the tilded and the untilded partition functions coincide.

4.4.2 Partition functions on $\mathbb{S}^1 \times \Sigma$

Of special importance to our discussion are manifolds of the form $X = \mathbb{S}^1 \times \Sigma$, where Σ is some connected, closed, torsion-free, orientable three-manifold. These are the types of manifolds on which the partition function admits a trace interpretation and the radius β of the circle is the inverse temperature. We will therefore specialise the above discussion to those manifolds and obtain more explicit formulas for the partition functions.

We will make frequent use of various topological characteristics of such manifolds, so we outline those here. First of all, note that for these manifolds, the Künneth formula and Poincaré duality, imply that that all their Betti numbers are determined by just one of them. Namely:

$$\begin{aligned} \mathrm{b}_0\big(\mathbb{S}^1 \times \Sigma\big) &= \mathrm{b}_4\big(\mathbb{S}^1 \times \Sigma\big) = 1 \\ \mathrm{b}_1\big(\mathbb{S}^1 \times \Sigma\big) &= \mathrm{b}_3\big(\mathbb{S}^1 \times \Sigma\big) = 1 + \mathrm{b}_1(\Sigma) = 1 + \mathrm{b}_2(\Sigma) \\ \mathrm{b}_2\big(\mathbb{S}^1 \times \Sigma\big) &= 2\mathrm{b}_1(\Sigma) = 2\mathrm{b}_2(\Sigma). \end{aligned} \tag{4.4.32}$$

The topological basis of harmonic one-forms and two-forms on $\mathbb{S}^1 \times \Sigma$ will then be induced by the respective basis on Σ. For the one-forms, a basis is given by the unique normalised harmonic one-form on the circle, $\frac{\mathrm{d}\tau}{2\pi\beta}$ (in local coordinates), and the topological basis of harmonic one-forms on Σ. As such, the Gram matrix of this basis becomes:

$$\mathbb{G}^{(1)} = \mathrm{diag}\bigg(\frac{\mathrm{vol}(\Sigma)}{2\pi\beta}, 2\pi\beta\, \mathbb{G}^{(1)}_{\Sigma}\bigg), \tag{4.4.33}$$

where $\mathbb{G}^{(1)}_{\Sigma}$ is the Gram matrix associated with the topological basis on Σ. For the two-forms, we have similarly

$$\mathbb{G}^{(2)} = \mathrm{diag}\Big(\beta\,\mathbb{E}, (\beta\,\mathbb{E})^{-1}\Big), \tag{4.4.34}$$

where $\mathbb{E} = 2\pi\mathbb{G}^{(2)}_{\Sigma}$ and we have also used Poincaré duality to bring it in this, symmetric, form. As indicative from the choice of notation, the matrix $\mathbb{E}$ will bear the interpretation of an energy, namely that carried by Wilson and 't Hooft loops placed on various spatial one-cycles. Finally, the intersection matrix is given by

$$\mathbb{Q} = \begin{pmatrix} & \mathbb{1}_{\mathrm{b}_1(\Sigma)} \\ \mathbb{1}_{\mathrm{b}_1(\Sigma)} & \end{pmatrix}. \tag{4.4.35}$$

With this in mind, we now go to compute the partition function on $X = \mathbb{S}^1_\beta \times \Sigma$, starting from (4.4.31). First, the instanton contribution, (4.4.20), is

$$\mathcal{Z}_{\text{inst}}\big[\mathbb{S}^1 \times \Sigma; \mathrm{t}\big] = \sum_{\boldsymbol{n},\boldsymbol{m} \in \mathbb{Z}^{b^1(\Sigma)}} \exp\left(\pi \mathrm{i} \begin{pmatrix} \boldsymbol{n}^\top & \boldsymbol{m}^\top \end{pmatrix} \begin{pmatrix} \mathrm{i}\,\mathrm{Im}\,\mathrm{t}\,\beta\,\mathbb{E} & \mathrm{Re}\,\mathrm{t}\,\mathbb{1} \\ \mathrm{Re}\,\mathrm{t}\,\mathbb{1} & \mathrm{i}\,\mathrm{Im}\,\mathrm{t}\,(\beta\,\mathbb{E})^{-1} \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{m} \end{pmatrix}\right), \tag{4.4.36}$$

which becomes

$$\mathcal{Z}_{\text{inst}}\big[\mathbb{S}^1 \times \Sigma; \mathrm{t}\big] = (\mathrm{Im}\,\mathrm{t})^{-\frac{\mathrm{b}_1(\Sigma)}{2}} \beta^{\frac{\mathrm{b}_1(\Sigma)}{2}} (\det \mathbb{E})^{\frac{1}{2}} \sum_{\boldsymbol{n},\boldsymbol{m} \in \mathbb{Z}^{\mathrm{b}_1(\Sigma)}} q^{\Delta_{\boldsymbol{n},\boldsymbol{m}}}, \tag{4.4.37}$$

after a Poisson resummation. In the above we have defined, $q := \mathrm{e}^{-\beta}$, and

$$\Delta_{\boldsymbol{n},\boldsymbol{m}} := \frac{1}{2} \frac{(\boldsymbol{n} + \mathrm{t}\,\boldsymbol{m})^\dagger \mathbb{E} (\boldsymbol{n} + \mathrm{t}\,\boldsymbol{m})}{\mathrm{Im}\,\mathrm{t}}. \tag{4.4.38}$$

Adding to that the oscillator contribution, (4.4.10), and using that $\mathbb{G}^{(1)}_\Sigma = \left[\mathbb{G}^{(2)}_\Sigma\right]^{-1}$, by Poincaré duality, we get in total

$$\mathcal{Z}\big[\mathbb{S}^1 \times \Sigma; \mathrm{t}\big] = \beta^{\mathrm{b}_1(\Sigma)-1} \frac{\det{}'\triangle_0}{\sqrt{\det{}'\triangle_1}} \sum_{\boldsymbol{n},\boldsymbol{m} \in \mathbb{Z}^{b^1(\Sigma)}} q^{\Delta_{\boldsymbol{n},\boldsymbol{m}}}. \tag{4.4.39}$$

As far as the determinants are of concern, we have the following:

$$\mathrm{spec}\big(\triangle_0, \mathbb{S}^1 \times \Sigma\big) = \left\{ \frac{(2\pi)^2 k^2}{\beta^2} + \lambda_{\mathrm{n}_0},\ k \in \mathbb{Z}, \mathrm{n}_0 \in \mathcal{N}_0 \right\}, \tag{4.4.40}$$

where λ_{n_0} are the eigenvalues of the scalar laplacian on Σ and $\mathcal{N}_0$ is a countable index set. It is then straightforward to calculate, upon zeta-function regularising the determinants and employing Euler's product formula for the hyperbolic sine:

$$\det{}'_{\mathbb{S}^1 \times \Sigma} \triangle_0 = \beta^{2\mathrm{b}_0(\Sigma)} \prod_{\mathrm{n}_0 \in \mathcal{N}_0^*} \sinh^2\left(\frac{1}{2}\beta\sqrt{\lambda_{\mathrm{n}_0}}\right), \tag{4.4.41}$$

where $\mathcal{N}_0^* := \mathcal{N}_0 \setminus \big\{\mathrm{n}_0 \mid \lambda_{\mathrm{n}_0} = 0\big\}$. Similarly, the spectrum of the one-form Laplacian is

$$\mathrm{spec}\big(\triangle_1, \mathbb{S}^1 \times \Sigma\big) = \Bigg\{ \underbrace{\frac{(2\pi)^2 n^2}{\beta^2} + \lambda_{\mathrm{n}_1}}_{\overset{!!}{\lambda_{n,\mathrm{n}_1}}},\ n \in \mathbb{Z}, \mathrm{n}_1 \in \mathcal{N}_1 \Bigg\} \cup \Bigg\{ \underbrace{\frac{(2\pi)^2 k^2}{\beta^2} + \lambda_{\mathrm{n}_0}}_{\overset{!!}{\lambda_{k,\mathrm{n}_0}}},\ k \in \mathbb{Z}, \mathrm{n}_0 \in \mathcal{N}_0 \Bigg\}, \tag{4.4.42}$$

with λ_{n_1} being the eigenvalues of $\triangle_1$ on Σ and $\mathcal{N}_1$ a countable index set, counting the eignevalues of the one-form Hodge Laplacian on Σ. The determinant of $\triangle_1$ is then:

$$\det{}'_{\mathbb{S}^1\times\Sigma}\triangle_1 = \beta^{2(b_1(\Sigma)+b_0(\Sigma))}\left[\prod_{n_1\in\mathcal{N}_1^*}\sinh^2\left(\frac{1}{2}\beta\sqrt{\lambda_{n_1}}\right)\right]\times$$
$$\times\left[\prod_{n_0\in\mathcal{N}_0^*}\sinh^2\left(\frac{1}{2}\beta\sqrt{\lambda_{n_0}}\right)\right], \tag{4.4.43}$$

with $\mathcal{N}_1^*$ defined analogously as $\mathcal{N}_0^*$. Therefore (recall $b_0(\Sigma) = 1$)

$$\begin{aligned}\frac{\det{}'\triangle_0}{\sqrt{\det{}'\triangle_1}} &= \beta^{1-b_1(\Sigma)}\prod_{\substack{n_0\in\mathcal{N}_0^*\\ n_1\in\mathcal{N}_1^*}}\frac{\sinh(\frac{1}{2}\beta\sqrt{\lambda_{n_0}})}{\sinh(\frac{1}{2}\beta\sqrt{\lambda_{n_1}})}\\ &= \beta^{1-b_1(\Sigma)}\left[\prod_{n\in\mathcal{N}_\perp^*}q^{-\frac{1}{2}\sqrt{\lambda_n}}\left(1-q^{\sqrt{\lambda_n}}\right)\right]^{-1},\end{aligned} \tag{4.4.44}$$

where λ_n are the eigenvalues of the transversal Laplacian on one-forms on Σ, indexed by $n \in \mathcal{N}_\perp$, and $\mathcal{N}_\perp^*$ collects the non-zero modes. To reach the last equality we have used the spectral properties of the Hodge Laplacian. We will give this quantity a special name:

$$\eta_\Sigma(q) := \left[\prod_{n\in\mathcal{N}_\perp^*}q^{-\frac{1}{2}\sqrt{\lambda_n}}\left(1-q^{\sqrt{\lambda_n}}\right)\right]^{1/2}, \tag{4.4.45}$$

as it serves as a four-dimensional generalisation of the Dedekind eta function. Putting everything together, the partition function takes the very simple form:

$$\boxed{\mathcal{Z}[\mathbb{S}^1\times\Sigma;t] = \frac{\Theta_\Sigma(q,t)}{\eta_\Sigma(q)^2},} \tag{4.4.46}$$

where

$$\Theta_\Sigma(q,t) := \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^{b_1(\Sigma)}} q^{\Delta_{\boldsymbol{n},\boldsymbol{m}}}, \tag{4.4.47}$$

is a generalisation of the Siegel–Narain theta function (that depends on the topology of Σ). Observe that this form of the partition function exhibits a remarkable similarity with the two-dimensional formulas of section 4.3 (in particular with (4.3.30))! We will see, in the following section, that this partition function can, too, be interpreted as an extended character; an extended character of the higher-dimensional current algebras of section 4.2.

4.4.3 Current algebra and the spectrum

We now shift gears from the path integral and turn to the current algebras of section 4.2. In keeping with the previous section, we will focus on the $p = 1$ incarnation of the current algebra (4.2.24) and expand it in modes, to obtain the algebra of the individual modes. This will allow us to construct representations of the current algebra, and will lead, eventually, to identifying the exact Hilbert space of Maxwell theory on any compact spatial slice in terms of the representations of (4.2.24). Finally, we will compute the characters of the aforementioned representations and show that they reproduce exactly the path integral expressions for the partition functions of Maxwell theory.

The starting point is the current algebra (4.2.27):[18]

$$\left[Q^{\pm}_{\Lambda_1}, Q^{\mp}_{\Lambda_2}\right]_{\Sigma} = \pm \mathrm{k} \int_{\Sigma} \Lambda_1^{\mp} \wedge \mathrm{d}\Lambda_2^{\pm} \,, \tag{4.4.48}$$

where we remind the reader the definition of the codimension-one charges:

$$Q^{\pm}_{\Lambda}[\Sigma] := \int_{\Sigma} J^{\pm} \wedge \Lambda^{\mp}, \tag{4.4.49}$$

with $J^{\pm}$ being the self- and anti-self-dual conserved two-form currents, (4.2.14), and $\Lambda^{\mp}$ the chiral and anti-chiral one-forms, (4.2.17), while the (not necessarily integer) level k is related to the coupling constant of Maxwell theory as

$$\mathrm{k} = \frac{\mathrm{g}^2}{2} = \frac{\pi}{\mathrm{Im}\,\mathrm{t}}. \tag{4.4.50}$$

The strategy to obtain a mode algebra is to expand the (pullbacks on Σ of) the forms $J^{\pm}$ and $\Lambda^{\mp}$ in terms of a suitable basis. Such a basis is given by the eigenforms of the Hodge Laplacian, $\triangle_p$. However, since $J^{\pm}$ is closed, it will only have overlap with the longitudinal eigenform components of the two-form Hodge Laplacian. Relatedly, due to the gauge invariance in defining the charges, up to a shift of $\Lambda^{\mp}$ by a closed form, $\Lambda^{\mp}$ has overlap only with the transversal eigenforms of the one-form Hodge Laplacian. As such, we consider the (orthonormalised) basis of transversal one-forms on Σ, given as

$$\mathcal{B}^1_{\perp}(\Sigma) := \left\{\phi_{\mathrm{n}} \in \Omega^1(\Sigma) \;\middle|\; \Box_1 \phi_{\mathrm{n}} = \lambda_{\mathrm{n}}\phi_{\mathrm{n}}\right\}_{\mathrm{n}\in\mathcal{N}_{\perp}}, \tag{4.4.51}$$

with $\Box_1$ and $\mathcal{N}_{\perp}$ defined like in subsection 4.4.1. It is easy to see, that a basis for the longitudinal two-forms is given in terms of $\star_{\Sigma}\phi_{\mathrm{n}}$:

$$\mathcal{B}^2_{\parallel}(\Sigma) := \left\{\star_{\Sigma}\phi_{\mathrm{n}} \in \Omega^2(\Sigma) \;\middle|\; \Box_1 \phi_{\mathrm{n}} = \lambda_{\mathrm{n}}\phi_{\mathrm{n}}\right\}_{\mathrm{n}\in\mathcal{N}_{\perp}}. \tag{4.4.52}$$

[18] Having allowed for a topological theta term, the phase space variables that led to this algebra are slightly different. It is straightforward, however, to repeat the calculation with a theta term to show that the algebra is unchanged.

Therefore we expand:

$$i_\Sigma^* \Lambda^\mp = \sum_{n\in\mathcal{N}_\perp} \Lambda_n^\mp \phi_n, \tag{4.4.53}$$

$$i_\Sigma^* J^\pm = \sum_{n\in\mathcal{N}_\perp} J_n^\pm \star_\Sigma \phi_n. \tag{4.4.54}$$

In this basis, the current modes define individually conserved currents. In other words:

$$Q_n^\pm := Q^\pm_{\Lambda_n^\mp \phi_n} = J_n^\pm. \tag{4.4.55}$$

These current, or charge modes obey the following algebra, folowing from (4.4.48):

$$\left[Q_n^\pm, Q_m^\mp\right] = \pm k \left\langle \phi_n, \star\, d\phi_m \right\rangle_\Sigma . \tag{4.4.56}$$

There exists a basis which diagonalises the above mode algebra. Let us focus, momentarily, on the non-zero part of the spectrum. On a three-dimensional closed manifold, $\Box_1$ is the square of another self-adjoint operator, the Beltrami operator: $\star d$. The spectrum of $\star d$ consists of $\pm\sqrt{\lambda_n}$. The eigen-one-forms of $\star d$, and their Hodge stars provide bases for coclosed one-, and closed two-forms respectively, which simultanesously diagonalise $\Box_1$ and $\overline{\Box}_2 := \triangle_2\big|_{\ker d}$. These two bases and can be used to diagonalise (4.4.56). While, generically, the spectrum of $\Box_1$ is simple, meaning that there are no degeneracies of eigenvalues [299], for many of the physical applications we have in mind, and more importantly, in the case of this chapter, for the state-operator correspondence of section 4.5, we are interested in products of spheres. In such cases the spectrum of $\Box_1$ is actually twofold degenerate (cf. appendix C.1). Equivalently, both signs of $\pm\sqrt{\lambda_n}$ appear in the spectrum of $\star d$. We proceed in the main text with this assumption, in order to connect smoothly with section 4.5, and we treat the generic case in appendix C.2, where we show that the results of this section remain unchanged. With this disclaimer, we proceed with the basis:

$$\mathcal{V}_\perp^1(\Sigma) = \left\{\phi_{0i}, \phi_{n\sigma} \in \Omega^1(\Sigma) \;\middle|\; \star d\phi_{n\sigma} = \sigma\sqrt{\lambda_n}\phi_{n\sigma}\right\}_{n\in\mathcal{N}_\perp^*}^{\sigma=\pm}, \tag{4.4.57}$$

and its Hodge dual. In the above we have also included all the zero-modes of $\Box_1$, ϕ_{0i}, induced by those of $\triangle_1$. The number of zero-modes is $\dim\ker\triangle_1 = b_1(\Sigma) = b_2(\Sigma)$. Moreover, we have taken this basis to be orthonormalised. In this basis, $i_\Sigma^* J^\pm$ and

$i_\Sigma^* \Lambda^\mp$ read

$$i_\Sigma^* J^\pm = \sum_{\mathrm{i}=1}^{\mathrm{b}_2(\Sigma)} J_{0\mathrm{i}}^\pm \star_\Sigma \phi_{0\mathrm{i}} + \sum_{\substack{\mathrm{n}\in\mathcal{N}_\perp^* \\ \sigma=\pm}} J_{\mathrm{n}\sigma}^\pm \star_\Sigma \phi_{\mathrm{n}\sigma}, \tag{4.4.58}$$

$$i_\Sigma^* \Lambda^\mp = \sum_{\mathrm{i}=1}^{\mathrm{b}_1(\Sigma)} \Lambda_{0\mathrm{i}}^\mp \phi_{0\mathrm{i}} + \sum_{\substack{\mathrm{n}\in\mathcal{N}_\perp^* \\ \sigma=\pm}} \Lambda_{\mathrm{n}\sigma}^\mp \star_\Sigma \phi_{\mathrm{n}\sigma}, \tag{4.4.59}$$

where $\mathcal{N}_\perp^*$ denotes the non-zero part of the spectrum. As before:

$$Q_{0\mathrm{i}}^\pm := Q^\pm_{\Lambda_{0\mathrm{i}}^\mp \phi_{0\mathrm{i}}} = J_{0\mathrm{i}}^\pm, \quad \text{and} \tag{4.4.60}$$

$$Q_{\mathrm{n}\sigma} := Q^\pm_{\Lambda_{\mathrm{n}\sigma}^\mp \phi_{\mathrm{n}\sigma}} = J_{\mathrm{n}\sigma}^\pm. \tag{4.4.61}$$

The algebra of the modes $Q_{\mathrm{n}\sigma}^\pm$ is

$$\left[Q_{\mathrm{n}\sigma}^\pm, Q_{\mathrm{m}\tau}^\mp\right] = \pm \mathrm{k}\,\sigma \sqrt{\lambda_\mathrm{n}}\,\delta_{\mathrm{nm}}\delta_{\sigma\tau}. \tag{4.4.62}$$

This is a direct four-dimensional analogue of the two-dimensional $\widehat{\mathfrak{u}}(1) \times \widehat{\mathfrak{u}}(1)$ Kac–Moody algebra (4.2.26). Before moving on, let us note that we can redefine the modes as[19]

$$\mathcal{A}_{\mathrm{n}\pm} := Q_{\mathrm{n}\pm}^\pm, \quad \text{and} \quad \mathcal{A}_{\mathrm{n}\pm}^\dagger := Q_{\mathrm{n}\pm}^\mp. \tag{4.4.63}$$

This redefinition reduces the current algebra to a collection of harmonic oscillators:

$$\left[\mathcal{A}_{\mathrm{n}\sigma}, \mathcal{A}_{\mathrm{m}\tau}^\dagger\right] = \mathrm{k}\sqrt{\lambda_\mathrm{n}}\,\delta_{\mathrm{nm}}\delta_{\sigma\tau}, \tag{4.4.64}$$

and makes the quantisation of Maxwell theory on arbitrary spatial topology an exercise in quantum mechanics.

We now proceed with quantisation. The Hamiltonian on Σ is simply

$$H_\Sigma = \frac{1}{2\mathrm{g}^2}\left(\|\boldsymbol{E}\|_\Sigma^2 + \|\boldsymbol{B}\|_\Sigma^2\right), \tag{4.4.65}$$

where the electric and magnetic fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, are defined in terms of pullbacks on Σ, of $\star f$ and f, respectively, in the usual fashion. The underlying current algebra of Maxwell theory allows us to write the Hamiltonian in a Sugawara form. In particular, expressing $\boldsymbol{E}$ and $\boldsymbol{B}$ in terms of the modes $\mathcal{A}_{\mathrm{n}\sigma}^{(\dagger)}$, we have

$$H_\Sigma = \frac{1}{\mathrm{k}}\sum_{\mathrm{i}=1}^{\mathrm{b}_2(\Sigma)} Q_{0\mathrm{i}}^+ Q_{0\mathrm{i}}^- + \frac{1}{\mathrm{k}}\sum_{\substack{\mathrm{n}\in\mathcal{N}_\perp^* \\ \sigma=\pm}} \mathcal{A}_{\mathrm{n}\sigma}^\dagger \mathcal{A}_{\mathrm{n}\sigma} + E_0, \tag{4.4.66}$$

[19]Note that in Lorentzian signature, the projectors take the form $\mathbb{P}_\pm^{\mathrm{L}} = \frac{1}{2}(\mathbb{1} \pm \mathrm{i}\star)$. Demanding that the field strength, f, be real implies that $\overline{J^\pm} = J^\mp$, where the overline denotes complex conjugation. Therefore, $Q_{\mathrm{n}\sigma}^\mp$ is indeed the Hermitian conjugate $Q_{\mathrm{n}\sigma}^\pm$.

where $E_0 = \frac{1}{2}\sum_{\mathrm{n},\sigma}\sqrt{\lambda_\mathrm{n}}$, is the (potentially infinite) vacuum energy. The Hamiltonian is, then, that of a countably infinite collection of harmonic oscillators, with creation (resp. annihilation) operators $\mathcal{A}^\dagger_{\mathrm{n}\pm}$ ($\mathcal{A}_{\mathrm{n}\pm}$), raising (lowering) the energy by $\sqrt{\lambda_\mathrm{n}}$. From here on, the story follows closely its two dimensional analogue of section 4.3, with slight differences that can all be traced back to the different self-duality properties of middle forms on $4n$- versus $(4n+2)$-dimensional manifolds.

The zero-modes, $Q^\pm_{0\mathrm{i}}$, commute with the Hamiltonian and their eigenstates have a distinguished role. Let $|j\rangle$, be such a state, with

$$Q^\pm_{0\mathrm{i}}\,|j\rangle = j^\pm_\mathrm{i}\,|j\rangle\,, \qquad \mathrm{i}\in\{1,\cdots,\mathrm{b}_2(\Sigma)\}, \tag{4.4.67}$$

that is of highest-weight. Namely, all lowering operators should annihilate the state:

$$\mathcal{A}_{\mathrm{n}\pm}\,|j\rangle \overset{!}{=} 0, \qquad \forall\mathrm{n}\in\mathcal{N}^*. \tag{4.4.68}$$

These states are, then, the *primary states* of the current algebra (4.4.62). Similarly to the two-dimensional case, flux quantisation determines the eigenvalues $j^\pm_\mathrm{i}$. For any 2-cycle $C_{(2)\mathrm{i}}\in\mathrm{H}_2(\Sigma)$, the magnetic and electric fluxes are quantised:

$$\Phi_\mathrm{mag} := \int_{C_{(2)\mathrm{i}}} f \in 2\pi\,\mathbb{Z}, \tag{4.4.69}$$

$$\Phi_\mathrm{elec} := \int_{C_{(2)\mathrm{i}}} \check{f} \in 2\pi\,\mathbb{Z}. \tag{4.4.70}$$

In the above, $\check{f}$ is the magnetic dual field strength, which, in the presence of a theta term, is given by [57]:

$$\check{f} := \mathrm{Re}\,\mathrm{t}\; f + \mathrm{i}\,\mathrm{Im}\,\mathrm{t}\;\star f. \tag{4.4.71}$$

At the quantum level, we demand that the flux quantisation holds inside expectation values, i.e. $\langle j|\Phi_\mathrm{mag}|j\rangle\in 2\pi\mathbb{Z}$ and $\langle j|\Phi_\mathrm{elec}|j\rangle\in 2\pi\mathbb{Z}$. It is clear that only the zero-modes contribute to that expectation value. This fixes exactly the charges, $j^\pm_\mathrm{i}$. A few lines of linear algebra, to switch from the orthonormal basis with which $J^\pm_{0\mathrm{i}}$ were obtained to the topological basis, which is the natural basis for harmonic forms, show that

$$\boldsymbol{j}^\pm = \pm\pi\mathrm{i}\,\frac{\mathbb{J}\left(\boldsymbol{n}+\mathrm{t}^\mp\boldsymbol{m}\right)}{\mathrm{Im}\,\mathrm{t}}. \tag{4.4.72}$$

In the above we combined $j^\pm_\mathrm{i}$ into a $\mathrm{b}_2(\Sigma)$-dimensional vector $\boldsymbol{j}^\pm$. Moreover, $\boldsymbol{n}$ and $\boldsymbol{m}$ are $\mathrm{b}_2(\Sigma)$-dimensional vectors of integers, $\mathrm{t}^\pm := \mathrm{Re}\,\mathrm{t}\pm\mathrm{i}\,\mathrm{Im}\,\mathrm{t}$, and the matrix $\mathbb{J}$ satisfies $\mathbb{J}^2 = \mathbb{G}^{(2)}_\Sigma = \frac{1}{2\pi}\mathbb{E}$, in terms of the "energy" matrix, defined below (4.4.34).[20]

[20]Since $\mathbb{J}$ is a square root of the energy matrix, the reader might worry about the sign of the square root of each diagonal entry. This is immaterial, however, since the sign can be absorbed in a sign-flip of the components of $\boldsymbol{n}$ and $\boldsymbol{m}$, which simply results in a reshuffling/relabelling of the different sectors of the Hilbert space.

Since the eignevalues of the zero-modes are determined in terms of the vectors of integers, $\boldsymbol{n}$ and $\boldsymbol{m}$, we will henceforth denote the primary states, $|j\rangle$, as $|\boldsymbol{n},\boldsymbol{m}\rangle$. These states have an energy, above the vacuum energy, that can simply be determined by acting with the Hamiltonian to be

$$H_\Sigma\,|\boldsymbol{n},\boldsymbol{m}\rangle = \Delta_{\boldsymbol{n},\boldsymbol{m}}\,|\boldsymbol{n},\boldsymbol{m}\rangle\,, \tag{4.4.73}$$

with

$$\Delta_{\boldsymbol{n},\boldsymbol{m}} = \frac{1}{2}\frac{(\boldsymbol{n}+\mathrm{t}\,\boldsymbol{m})^\dagger\;\mathbb{E}\;(\boldsymbol{n}+\mathrm{t}\,\boldsymbol{m})}{\operatorname{Im}\mathrm{t}}. \tag{4.4.74}$$

Note that this is the exact same formula as (4.4.38). This is not a coincidence. We will clarify the relation between the instanton sectors and the current algebra primaries, in the next section. But first we should finish constructing the Hilbert space.

The remaining states are obtained by acting on $|\boldsymbol{n},\boldsymbol{m}\rangle$ with the raising operators, $\mathcal{A}^\dagger_{\mathrm{n}\pm}$. This results in a Verma module, $\mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}$, spanned by $|\boldsymbol{n},\boldsymbol{m}\rangle$ and its descendants:

$$\Big|\boldsymbol{n},\boldsymbol{m};\{N_{\mathrm{n}\sigma}\}^{\sigma=\pm}_{\mathrm{n}\in\mathcal{N}^*_\perp}\Big\rangle := \prod_{\substack{\mathrm{n}\in\mathcal{N}^*_\perp\\ \sigma=\pm}}\left(\mathcal{A}^\dagger_{\mathrm{n}\sigma}\right)^{N_{\mathrm{n}\sigma}}|\boldsymbol{n},\boldsymbol{m}\rangle\,, \tag{4.4.75}$$

where $N_{\mathrm{n}\sigma}$ are positive integers, indicating how many times each creation operator acts, to obtain the desired state. The energy gap between these states and the ground states, at the bottom of the Verma module, is

$$E_{\{N_{\mathrm{n}\sigma}\}} = \sum_{\substack{\mathrm{n}\in\mathcal{N}^*_\perp\\ \sigma=\pm}} N_{\mathrm{n}\sigma}\sqrt{\lambda_{\mathrm{n}}}. \tag{4.4.76}$$

Finally, as a matter of convention, we choose the states such that states at different level are orthogonal, and each state has unit norm, $\langle\boldsymbol{n},\boldsymbol{m};\{N_{\mathrm{n}\sigma}\}|\boldsymbol{n},\boldsymbol{m};\{M_{\mathrm{n}\sigma}\}\rangle = \delta_{\{N_{\mathrm{n}\sigma}\},\{M_{\mathrm{n}\sigma}\}}$. Moreover, states at different topological sectors are orthogonal by construction. Finally, the full Hilbert space of the theory, is simply a direct sum of the Verma modules:

$$\mathcal{H}_\Sigma = \bigoplus_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^{\mathrm{b}_2(\Sigma)}}\mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}. \tag{4.4.77}$$

To complete the study of the representation theory of our current algebra, what remains is to compute the characters of each of the Verma modules: $\mathrm{ch}_{\mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}}[q] := \mathrm{tr}_{\mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}}\,q^H$. We have already performed the difficult task of calculating the energies, so all we have to do now is sum them. This yields, at first:

$$\mathrm{ch}_{\mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}}[q] = q^{\Delta_{\boldsymbol{n},\boldsymbol{m}}}\left(\prod_{\mathrm{n}\in\mathcal{N}^*_\perp} q^{\frac{1}{2}\sqrt{\lambda_{\mathrm{n}}}}\right)\left(\prod_{\mathrm{n}\in\mathcal{N}^*_\perp}\sum_{N_{\mathrm{n}}=0}^{\infty} q^{N_{\mathrm{n}}\sqrt{\lambda_{\mathrm{n}}}}\right), \tag{4.4.78}$$

which, after resumming the geometric series, becomes

$$\mathrm{ch}_{\mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}}[q] = \frac{q^{\Delta_{\boldsymbol{n},\boldsymbol{m}}}}{\eta_\Sigma(q)^2}. \tag{4.4.79}$$

In the last equation we used the definition (4.4.45), of the four-dimensional eta function. The $\eta_\Sigma(q)$ term is resumming the contribution of the descendants, while the $q^{\Delta_{\boldsymbol{n},\boldsymbol{m}}}$ term is capturing the contribution of the primaries. Finally, summing over the Verma modules yields an extended character of our current algebra. All the descendants give the same contribution, while for the primaries we have to sum all their contributions. This gives:

$$\mathbf{ch}[q] := \sum_{\boldsymbol{n},\boldsymbol{m}\in\mathbb{Z}^{\mathrm{b}_2(\Sigma)}} \mathrm{ch}_{\mathcal{V}_{\boldsymbol{n},\boldsymbol{m}}}[q] = \frac{\Theta_\Sigma(q;\mathrm{t})}{\eta_\Sigma(q)^2}. \tag{4.4.80}$$

Comparing with (4.4.46), we see a remarkable identification of the $\mathbb{S}^1 \times \Sigma$ partition function, as computed through the path integral, with an extended character of an infinite-dimensional algebra:

$$\mathcal{Z}\Big[\mathbb{S}^1_\beta \times \Sigma; \mathrm{t}\Big] = \mathbf{ch}[q], \qquad q = \mathrm{e}^{-\beta}. \tag{4.4.81}$$

Let us comment on this result. Identifying the nome with $\mathrm{e}^{-\beta}$, gives a physical meaning to the character, $\mathbf{ch}[q]$. It is simply a thermal trace over the Hilbert space over Σ. From this point of view, it ought to reproduce the path integral expression. It is known, however, that there are subtleties in matching the two quantities. The compatibility of the manifestly covariant path integral formalism with the manifestly unitary canonical formalism has been a source of confusion [300–303], ultimately resolved in [304] by arguing the form of the canonical partition function. Our formula (4.4.81), provides a more explicit manifestation of Donnelly and Wall's unitarity proof [304]. Recovering unitarity is an important point, as it was a crucial ingredient for the photonisation argument of section 4.2, which ultimately led to the higher-dimensional current algebras.

It is interesting to keep track of which quantity is mapped to what in the identification (4.4.46) = (4.4.80). In (4.4.46), the Theta function stems from the sum over instantons. This counts, effectively, insertions of Wilson–'t Hooft operators in Maxwell theory [57]. On the other hand, on the canonical side, the Theta function comes from the primary states. Both of them are subsequently decorated with oscillators that give rise to the eta functions. This story is pointing at a special connection between the spectrum of operators and states in Maxwell theory, which we shall explicate in the following section.

4.4.4 The spectrum on $\mathbb{S}^2 \times \mathbb{S}^1$

Among all three-dimensional topologies, of particular interest to us is $\Sigma \cong \mathbb{S}^2 \times \mathbb{S}^1$, for reasons that will become evident in the next section. Here we briefly specialise the generic results of subsection 4.4.3 and collect the relevant formulas, for the case of $\mathbb{S}^2 \times \mathbb{S}^1$.

First, the geometry. Using a Weyl rescaling of the spacetime metric, we can set the radius of $\mathbb{S}^1$ to be unity and the only parameter is the radius of the sphere, or, really, the dimensionless ratio of sphere over the circle radius, which we will denote as r_0. We will refer to this spatial slice as Σ_{r_0}. Its metric is:

$$\mathrm{d}s^2_{\Sigma_{r_0}} = r_0^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\big) + \mathrm{d}\eta^2\,, \tag{4.4.82}$$

where $\theta \in [0,\pi)$ and $\varphi \in [0,2\pi)$ are coordinates on the sphere and $\eta \in [0,2\pi)$ is the coordinate on the circle.

As we review in appendix C.1, on $\mathbb{S}^2 \times \mathbb{S}^1$ there is a single zero-mode of the one- or the two-form Hodge Laplacian. The rest of the modes are labelled by the angular and magnetic quantum numbers, ℓ and m, on the sphere, and the momentum, k on the circle, as well as the Beltrami label, $\sigma = \pm$. The eigenvalue of the Laplacian of a mode labelled by $\mathsf{n} = (\ell, m, k)$ is

$$\lambda_{\mathsf{n}}(r_0) = \frac{\ell(\ell+1)}{r_0^2} + k^2. \tag{4.4.83}$$

The Hamiltonian on this slice reads:

$$H_{\Sigma_{r_0}} = \frac{1}{\mathrm{k}} Q_0^+ Q_0^- + \frac{1}{\mathrm{k}} \sum_{\substack{\mathsf{n}=(\ell,m,k)\\ \sigma=\pm}} \mathcal{A}^\dagger_{\mathsf{n}\sigma}\mathcal{A}_{\mathsf{n}\sigma} + E_0. \tag{4.4.84}$$

The primary states, are labelled by two integers: $|n,m\rangle$, with $n,m \in \mathbb{Z}$. The unique harmonic two-form on Σ_{r_0} (normalised as in (4.3.5)) is

$$\tau^{(2)} = \frac{1}{4\pi}\sin^2\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi\,, \tag{4.4.85}$$

and yields $\mathbb{G}^{(2)}_{\Sigma_{r_0}} = (2r_0^2)^{-1}$. This gives, via (4.4.72) and (4.4.74), the charge and the energy of the of the primary states, as:

$$j^\pm = \pm\pi\mathrm{i}\frac{1}{\sqrt{2}r_0}\frac{n + \mathrm{t}^\mp m}{\operatorname{Im}\mathrm{t}}, \quad \text{and} \tag{4.4.86}$$

$$\Delta_{n,m} = \frac{\pi}{2r_0^2}\frac{|n+\mathrm{t}\,m|^2}{\operatorname{Im}\mathrm{t}}, \tag{4.4.87}$$

respectively.[21] The rest of the states are just obtained by acting with creation operators, like in (4.4.75). For example, the first excited states are obtained by acting with $\mathcal{A}^\dagger_{\mathsf{n}\sigma}$, with $\mathsf{n} = (1, m, 0)$ or $\mathsf{n} = (0, 0, 1)$ depending on the value of r_0 and have energy

$$E_{1,m,0}(r_0) = \frac{\sqrt{2}}{r_0} \quad \text{and} \quad E_{0,0,1}(r_0) = 1, \tag{4.4.88}$$

above the ground-state energy, respectively.

4.5 The state-operator correspondence

We now have all the ingredients to arrive at the culmination of this chapter; the nonlocal state-operator correspondence. While the matching of the partition function and the current algebra extended character is valid on any connected, closed, torsion-free, orientable three-manifold, for the state-operator correspondence we need to choose a specific manifold. We will only treat the easiest case, deferring more complex cases for future investigation. We restrict ourselves with the topology $\Sigma \cong \mathbb{S}^2 \times \mathbb{S}^1$.

The picture we have in mind here is the following. States on $\mathcal{H}_\Sigma$ are prepared by a path integral on a manifold whose boundary is $\Sigma = \mathbb{S}^2 \times \mathbb{S}^1$, with various operator insertions, which we will clarify momentarily. Besides local operators, four-dimensional Maxwell theory is host to a plethora of line operators (which we classify in subsection 4.5.1). We place these operators on the $\mathbb{S}^1$. This leaves us with filling-in the $\mathbb{S}^2$ to obtain a three-ball, $\mathbb{B}^3$. As we will see, the path integral on $Y := \mathbb{B}^3 \times \mathbb{S}^1$, with line operators inserted on $\mathbb{S}^1 \times \{0\}$, where by $\{0\}$ we denote the origin on $\mathbb{B}^3$, produces all the states, (4.4.75), of $\mathcal{H}_\Sigma$ that we described above. See figure 4.2. In notation similar to section 4.3 we denote these states as

$$|\mathcal{L}\rangle := \int_{\mathcal{C}[\,\cdot\,]} \mathrm{D}a\, \mathrm{e}^{-S[a]}\, \mathcal{L}\big(\mathbb{S}^1 \times \{0\}\big), \tag{4.5.1}$$

where $\mathcal{L}$ is some line operator, and $\mathcal{C}[\,\cdot\,]$ is, again an appropriate functional space over Y (now encapsulating information about gauge equivalent configurations), pending additional input regarding boundary conditions on ∂Y. Once boundary conditions, say $a(\partial Y) = a_\partial$, are imposed, the path integral (4.5.1) produces the wavefunctional

$$\Psi_{\mathcal{L}}[a_\partial] = \langle a_\partial | \mathcal{L}\rangle\,. \tag{4.5.2}$$

Here we will show that these states reproduce the Hilbert space of subsection 4.4.3 entirely.

[21] This formulas may look funny, on dimensional grounds, but remember that r_0 is a dimensionless parameter. To reinstate dimensionful parameters, the energy is proportional to the Gram matrix, which, in this case, is $\frac{\mathrm{vol}(\text{circle})}{\mathrm{vol}(\text{sphere})}$, which has the correct dimensions to be an energy.

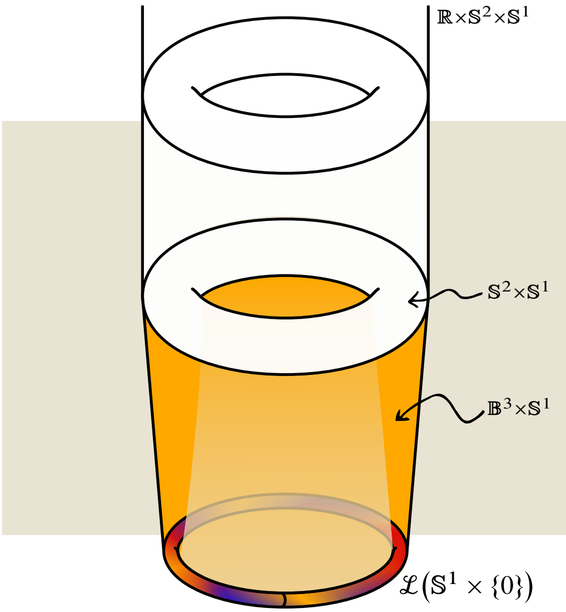

Figure 4.2: *The nonlocal state-operator correspondence. Any state on $\mathbb{S}^2 \times \mathbb{S}^1$ can be prepared by a path integral on $\mathbb{B}^3 \times \mathbb{S}^1$ with a line operator, $\mathcal{L}(\mathbb{S}^1 \times \{0\})$, at the centre of the ball, wrapping the $\mathbb{S}^1$. The state then evolves in time on the Lorentzian "cylinder" $\mathbb{R} \times \mathbb{S}^2 \times \mathbb{S}^1$.*

4.5.1 Line operators

First we outline the possible line operators in our disposal. Since in pure Maxwell theory there are no local charged operators, the only gauge-invariant line operators must be supported either on closed loops or infinitely extending lines. In what follows we will only consider closed loops (which we will denote as γ), as we are dealing with compact spaces. Infinitely extending lines can be obtained as a limit. The basic kind of line operator there is, is a Wilson loop:

$$\mathrm{W}_n(\gamma) := \exp\left(\mathrm{i}\, n \int_\gamma a\right), \qquad n \in \mathbb{Z}. \tag{4.5.3}$$

These operators are electrically charged under the one-form symmetry of Maxell theory, which served as the starting point of photonisation (cf. section 4.2). In particular, if

$$U_C^{\text{elec}} := \exp\left(\mathrm{i} \int_C \star f\right), \tag{4.5.4}$$

is the generator of the electric one-form symmetry, where C is a closed two dimensional surface, $\mathrm{W}_n(\gamma)$ carries charge n:

$$U_C^{\mathrm{elec}} \cdot \mathrm{W}_n(\gamma) = \exp\left(\frac{2\pi\,\mathrm{i}\,n}{\mathrm{Im}\,\mathrm{t}}\;\mathrm{Link}(C,\gamma)\right)\mathrm{W}_n(\gamma), \tag{4.5.5}$$

where $\mathrm{Link}(C,\gamma)$ denotes the linking number between the closed surface C and the loop γ.

Furthermore, there exist magnetically charged line operators. As we explained in section 4.2, and as is well-understood in Maxwell theory [15], there is also a magnetic one-form symmetry, with generators (in the absense of a theta term)

$$U_C^{\mathrm{mag}} := \exp\left(\mathrm{i}\int_C f\right). \tag{4.5.6}$$

The operators charged under this symmetry are 't Hooft loops. In the electric presentation of the theory these are described by excising a loop from spacetime, providing boundary conditions for the gauge-fields that fix the value of $\int f$ on a thin tube surrounding the loop. They can be written more compactly, however, in the magnetic presentation, where we exchange the gauge-field for a magnetic gauge-field, $\check{a}$, whose curvature is $\check{f} = \mathrm{Re}\,\mathrm{t}\, f + \mathrm{i}\,\mathrm{Im}\,\mathrm{t} \star f$. The 't Hooft loops become, in this presentation, Wilson loops for the magnetic photon:

$$\text{'tH}_m(\gamma) := \exp\left(\mathrm{i}\,m\int_\gamma \check{a}\right), \qquad m \in \mathbb{Z}. \tag{4.5.7}$$

and carry "electric" charge in the magnetic presentation:

$$\exp\left(\mathrm{i}\int_C \star\check{f}\right) \cdot \text{'tH}_m(\gamma) = \exp\left(\frac{2\pi\,\mathrm{i}\,m\,|\mathrm{t}|^2}{\mathrm{Im}\,\mathrm{t}}\mathrm{Link}(C,\gamma)\right)\text{'tH}_m(\gamma). \tag{4.5.8}$$

In the original, electric, presentation, they carry magnetic charge:

$$U_C^{\mathrm{mag}} \cdot \text{'tH}_m(\gamma) = \exp(2\pi\,\mathrm{i}\,m\;\mathrm{Link}(C,\gamma))\,\text{'tH}_m(\gamma). \tag{4.5.9}$$

In the presence of a theta term, they also carry electric charge, due to the Witten effect. This is given by:

$$U_C^{\mathrm{elec}} \cdot \text{'tH}_m(\gamma) = \exp\left(2\pi\,\mathrm{i}\,m\,\frac{\mathrm{Re}\,\mathrm{t}}{\mathrm{Im}\,\mathrm{t}}\;\mathrm{Link}(C,\gamma)\right)\text{'tH}_m(\gamma). \tag{4.5.10}$$

More generally, we can consider dyonic, or Wilson–'t Hooft, line operators labelled both by an electric and a magnetic charge:

$$\mathrm{W}_{n,m}(\gamma) := \exp\left(\mathrm{i}\,n\int_\gamma a + \mathrm{i}\,m\int_\gamma \check{a}\right), \qquad n,m \in \mathbb{Z}. \tag{4.5.11}$$

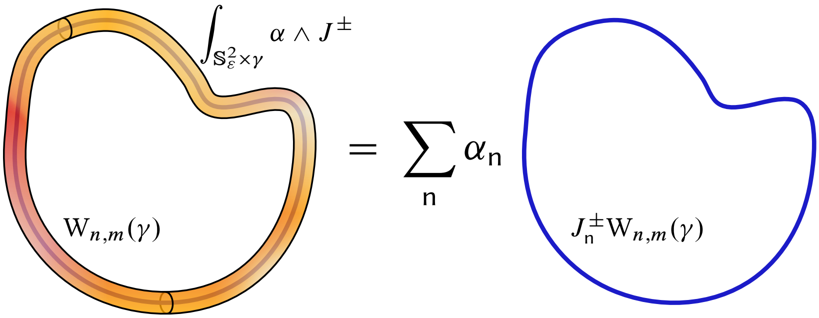

Figure 4.3: *An example of descendant line operators. On the left-hand-side of the equation, a small tube of the operator $J^\pm$ smeared over a function, α, surrounds the curve γ which supports a Wilson–'t Hooft line. On the right-hand-side is the result of shrinking the tube on the line, namely a series of new, descendant, line operators.*

Simultaneously, we will combine the one-form symmetries into their irreducible, self- and anti-self-dual, incarnations, as in section 4.2:

$$U_C^\pm := \exp\left(\mathrm{i}\int_C J^\pm\right). \tag{4.5.12}$$

The Wilson–'t Hooft lines carry charge under $U_C^\pm$ as:

$$U_C^\pm \cdot \mathrm{W}_{n,m} = \exp\left(\pm 2\pi \frac{n+\mathrm{t}^\pm m}{\operatorname{Im}\mathrm{t}}\operatorname{Link}(C,\gamma)\right)\mathrm{W}_{n,m}. \tag{4.5.13}$$

These are the basic, or primary, in a sense to be made precise below, line operators. But this is not the complete story. We can obtain composite, or descendant (in the same to-be-made-precise sense) line operators by smearing functions of gauge-invariant local operators on γ. The only gauge-invariant local operators in our disposal are polynomials of (derivatives of) $J^\pm$. An unambiguous way to smear the local operators over the basic line operators, is to excise an infinitesimal tube around γ, i.e. an $\mathbb{S}^2_\varepsilon\times\gamma$, on which we spread the desired local operator, and shrink it onto γ by taking the limit $\varepsilon\to 0$. This is completely analogous to how disorder operators, including the aforementioned basic 't Hooft loops (in the electric presentation), are defined in quantum field theory. Using the mode expansion of $J^\pm$ on a three-dimensional surface (4.4.58), and the OPE of the currents we can convert the complicated local operator into sums of powers of the modes, $J^\pm_{\mathsf{n}}$, of $J^\pm$. A sketch of this construction, for a specific operator, is shown in figure 4.3. In the next few sections we will make this picture precise.

4.5.2 Radial evolution

A crucial feature of the Hilbert space on $\mathbb{S}^2 \times \mathbb{S}^1$, is its Verma module structure under the Kac–Moody algebra. In order to reproduce this structure, via a path integral on $Y = \mathbb{B}^3 \times \mathbb{S}^1$, we need to find an expression of the current modes, $J^{\pm}_{\mathrm{n}}$, that can be inserted in the path integral. Key to doing so is solving the self-duality and conservation equations, for $J^{\pm}$ on Y. We take the metric on Y to be,

$$\mathrm{d}s^2_Y = \mathrm{d}r^2 + \underbrace{r^2\,\mathrm{d}s_{\mathbb{S}^2} + \mathrm{d}s^2_{\mathbb{S}^1}}_{\mathrm{d}s^2_{\Sigma_r}}, \tag{4.5.14}$$

i.e. we fix the $\mathbb{S}^1$ radius to be one, throughout, and label the spatial slice where the $\mathbb{S}^2$ is of radius $r \in [0, r_0]$ by Σ_r, where Σ_{r_0} is the slice on which the states, in subsection 4.4.4, are prepared. With these conventions, it is useful to decompose the currents as,

$$J^{\pm} = \mathrm{d}r \wedge J^{\pm}_r + J^{\pm}_{\Sigma_r}, \tag{4.5.15}$$

where $J^{\pm}_r \in \Omega^1(\Sigma_r)$ and $J^{\pm}_{\Sigma_r} \in \Omega^2(\Sigma_r)$. In this decomposition, the self-duality equations, $\star_Y J^{\pm} = \pm J^{\pm}$, are expressed as a relation between $J^{\pm}_r$ and $J^{\pm}_{\Sigma_r}$:

$$J^{\pm}_r = \pm \star_r J^{\pm}_{\Sigma_r}. \tag{4.5.16}$$

Here, and in the following, we denote by $\star_r$, the Hodge star on Σ_r, induced by the metric on the slices. Moreover, from the conservation equations, $\mathrm{d}J^{\pm} = 0$, we read a Gauss law and a dynamical equation controlling the radial evolution of the current. The Gauss law, written equivalently in terms of $J^{\pm}_{\Sigma_r}$ or $J^{\pm}_r$, using (4.5.16), reads:

$$\mathbf{d}J^{\pm}_{\Sigma_r} = 0 \qquad \Leftrightarrow \qquad \mathbf{d}^{\dagger} J^{\pm}_r = 0, \tag{4.5.17}$$

The bold differential, $\mathbf{d}$, denotes the differential on Σ_r and the codifferential is defined with respect to the Hodge-star on Σ_r. Similarly, the radial evolution equations both in terms of $J^{\pm}_{\Sigma_r}$ and $J^{\pm}_r$ read:

$$\partial_r J^{\pm}_{\Sigma_r} \mp \mathbf{d} \star_r J^{\pm}_{\Sigma_r} = 0 \qquad \Leftrightarrow \qquad \star_r \partial_r \star_r J^{\pm}_r \mp \star_r \mathbf{d} J^{\pm}_r = 0. \tag{4.5.18}$$

As explained in subsection 4.4.3 and appendix C.1, $J^{\pm}_{\Sigma_r}$, as a closed two-form, can be expanded in the (orthonormalised) basis, $\mathcal{V}^2_{\parallel}(\Sigma_r)$, given by (C.1.25), of Hodge-duals of eigen-one-forms of the Beltrami operator on Σ_r:

$$J^{\pm}_{\Sigma_r} = J^{\pm}_0(r) \star_r \phi_0(r) + \sum_{\substack{\mathrm{n}\in\mathcal{N}^*_{\perp} \\ \sigma=\pm}} J^{\pm}_{\mathrm{n}\sigma}(r) \star_r \phi_{\mathrm{n}\sigma}(r), \tag{4.5.19}$$

where we now explicitly denote the dependence of the coefficients, $J^{\pm}_{\mathsf{n}}(r)$, and the basis forms, $\phi_{\mathsf{n}\sigma}(r)$, on r. Plugging this expansion into (4.5.18), yields a system of first order ordinary differential equations for the coefficients:

$$\begin{aligned} \partial_r J^{\pm}_0(r) + \frac{1}{r} J^{\pm}_0(r) &= 0 \\ \partial_r J^{\pm}_{\mathsf{n}\sigma}(r) + \left[\mathbb{A}^{\pm}_{\mathsf{n}}(r)\right]_{\sigma\tau} J^{\pm}_{\mathsf{n}\tau}(r) &= 0, \end{aligned} \tag{4.5.20}$$

together with boundary conditions fixing $J^{\pm}_0(r_0) = Q^{\pm}_0$, and $J^{\pm}_{\mathsf{n}\sigma}(r_0) = Q^{\pm}_{\mathsf{n}\sigma}$. These are precisely the modes that were used to build creation and annihilation operators on Σ_{r_0}. Note that this differential equation factorises modes of different energy, i.e. different n, but mixes modes on the $\sigma\tau$-plane which we shall call the Beltrami plane.

The differential equation for the zero-mode, can be immediately integrated, yielding

$$J^{\pm}_0(r) = \frac{r_0}{r}\, Q^{\pm}_0 . \tag{4.5.21}$$

For the non-zero-modes, the matrix $\mathbb{A}^{\pm}_{\mathsf{n}}(r)$ is given as

$$\left[\mathbb{A}^{\pm}_{\mathsf{n}}(r)\right]_{\sigma\tau} := \langle \phi_{\mathsf{n}\sigma}, \star_r \partial_r \star_r \phi_{\mathsf{n}\tau}\rangle \mp \langle \phi_{\mathsf{n}\sigma}, \star_r \mathbf{d}\phi_{\mathsf{n}\tau}\rangle \,. \tag{4.5.22}$$

We provide details about the explicit construction of the basis $\mathcal{V}(\Sigma_r)$, as well as all the entries of the matrix $\mathbb{A}^{\pm}_{\mathsf{n}}$ in appendices C.1 and C.3. The solution of the above equation takes the form of a radially ordered integral:[22]

$$J^{\pm}_{\mathsf{n}\sigma}(r) = \left[\mathbb{U}^{\pm}_{\mathsf{n}}(r, r_0)\right]_{\sigma\tau} Q^{\pm}_{\mathsf{n}\tau}, \quad \text{with} \tag{4.5.23}$$

$$\mathbb{U}^{\pm}_{\mathsf{n}}(r, r_0) := \mathrm{Rexp}\left(\int_r^{r_0} \mathrm{d}r'\, \mathbb{A}^{\pm}_{\mathsf{n}}(r)\right). \tag{4.5.24}$$

Altogether $J^{\pm}_{\Sigma_r}$ reads, then

$$J^{\pm}_{\Sigma_r} = Q^{\pm}_0 \frac{r_0}{r} \star_r \phi_0(r) + \sum_{\mathsf{n},\sigma,\tau} Q^{\pm}_{\mathsf{n}\sigma} \left[\mathbb{U}^{\pm}_{\mathsf{n}}(r, r_0)\right]_{\sigma\tau} \star_r \phi_{\mathsf{n}\tau}(r). \tag{4.5.25}$$

For illustration purposes, let us concentrate temporarily on a subfamily of the basis $\mathcal{V}(\Sigma_r)$, returning subsequently to the general case, deferring the details to the appendix. This subfamily concerns the eigenforms with no momentum along the $\mathbb{S}^1$ of Σ_r, i.e. the one-forms

$$\phi_{\ell m\sigma}(r) := \frac{1}{\sqrt{2}}\left(\Psi_{\ell m}\, \mathrm{d}\eta + \sigma \left(\frac{\ell(\ell+1)}{r^2}\right)^{-1/2} \star_r \mathbf{d}(\Psi_{\ell m}\, \mathrm{d}\eta)\right). \tag{4.5.26}$$

[22] That is, simply, a time-ordered integral, with the radius playing the role of time.

In the above, $\Psi_{\ell m}$, are the real spherical harmonics, normalised on $\mathbb{S}^2_r$, with ℓ and m the usual angular momentum and magnetic quantum numbers, with $\ell \neq 0$, and η is the coordinate on $\mathbb{S}^1$. One can check that these are orthonormalised eigenforms of the Beltrami operator on Σ_r, with eigenvalue $\sigma\left(\frac{\ell(\ell+1)}{r^2}\right)^{1/2}$. For this family the matrix $\mathbb{A}^{\pm}_{\mathsf{n}}(r)$ becomes

$$\mathbb{A}^{\pm}_{\ell m}(r) = \frac{1}{2r}\begin{pmatrix} 1 \mp 2\sqrt{\ell(\ell+1)} & 1 \\ 1 & 1 \pm 2\sqrt{\ell(\ell+1)} \end{pmatrix}. \tag{4.5.27}$$

In this case $\mathbb{A}^{\pm}_{\ell m}(r)$ commutes with itself at different radii so the radially ordered exponential reduces to a regular exponential, yielding, finally as solution

$$\begin{aligned} \mathbb{U}^{\pm}_{\ell m}(r, r_0) = \frac{1}{2+4\ell}\Bigg[\left(\frac{r}{r_0}\right)^{-\ell-1}&\begin{pmatrix} 1+2\ell \mp 2\sqrt{\ell(\ell+1)} & 1 \\ 1 & 1+2\ell \pm 2\sqrt{\ell(\ell+1)} \end{pmatrix} \\ +\left(\frac{r}{r_0}\right)^{\ell}&\begin{pmatrix} 1+2\ell \pm 2\sqrt{\ell(\ell+1)} & -1 \\ -1 & 1+2\ell \mp 2\sqrt{\ell(\ell+1)} \end{pmatrix}\Bigg]. \end{aligned} \tag{4.5.28}$$

Let us return to the general case. With the solutions (4.5.21) and (4.5.23), we can equivalently express the operators acting on Σ_{r_0} in terms of the fields $J^{\pm}$, as

$$Q^{\pm}_0 = \int_{\Sigma_r} \frac{r}{r_0}\phi_0(r) \wedge J^{\pm}_{\Sigma_r} \tag{4.5.29}$$

$$Q^{\pm}_{\mathsf{n}\sigma} = \int_{\Sigma_r} \left[\mathbb{U}^{\pm}_{\mathsf{n}}(r, r_0)\right]^{-1}_{\sigma\tau} \phi_{\mathsf{n}\tau}(r) \wedge J^{\pm}_{\Sigma_r}. \tag{4.5.30}$$

Note how, as they should, the modes $Q^{\pm}_i$ (where i stands either for 0 or $\mathsf{n}\sigma$), are actually independent of r. In other words, connecting to the discussion of subsection 4.4.3, $\mathbb{U}^{\pm}_i(r, r_0)^{-1} \cdot \phi_i(r)$ provide a basis of (anti-)chiral one-forms on Y:

$$\mathbb{P}_{\pm}\mathrm{d}\left(\mathbb{U}^{\pm}_i(r, r_0)^{-1} \cdot \phi_i(r)\right) = 0, \tag{4.5.31}$$

subject to a gauge condition. In this case, the natural gauge condition is given by the Coulomb gauge, that they lie in the kernel of $\mathbf{d} \star_r i^*_{\Sigma_r}$.

Knowing the expression of the charges in terms of the field content, we can now obtain how such charges act on states, $|\mathcal{L}\rangle$, of the form (4.5.1). We simply perform the path integral inserting $Q^{\pm}_i$ on a small Σ_r, and perform the OPE with the path integral insertion sitting at the centre, sending subsequently $r \to 0$:

$$Q^{\pm}_i |\mathcal{L}\rangle = \lim_{r\to 0}\int_{\mathcal{C}[\cdot]} \mathrm{D}a\, \mathrm{e}^{-S[a]} \int_{\Sigma_r} \mathbb{U}^{\pm}_i(r, r_0)^{-1} \cdot \phi_i(r) \wedge J^{\pm}_{\Sigma_r} \times \mathcal{L}\left(\mathbb{S}^1 \times \{0\}\right), \tag{4.5.32}$$

where, in (4.5.32), only smooth configurations of $J^{\pm}_{\Sigma_r}$ contribute, for the same reasons as in section 4.3. This is a direct four-dimensional analogue of (4.3.39). The behaviour of this insertion as $r \to 0$, requires some clarification. Let us illustrate the situation for the empty state:

$$|\mathbf{1}\rangle := \int_{\mathcal{C}[\cdot]} \mathrm{D}a \, \mathrm{e}^{-S[a]}. \tag{4.5.33}$$

The radial evolution matrix, $\mathbb{U}^{\pm}_{\mathsf{n}}(r, r_0)$, gives rise to a smooth mode, behaving as

$$\sim \left(\frac{r}{r_0}\right)^{\ell}, \tag{4.5.34}$$

at the origin, $r \to 0$, and a divergent one, behaving as

$$\sim \left(\frac{r}{r_0}\right)^{-\ell-1}. \tag{4.5.35}$$

In order to single out the configurations of the currents that contribute to the path integral, we must project out the divergent modes. This is implemented by the projector to the kernel of $\mathbb{U}^{\pm}_{\mathsf{n}}(0, r_0)$, which we will denote as $\mathbf{\Pi}^{\pm}_{\mathsf{n}}$.[23] Therefore, the part of $J^{\pm}_{\Sigma_r}$ that contributes in the path integral has an expansion:

$$J^{\pm}_{\Sigma_r}\Big|_{\text{smooth}} = \underbrace{Q^{\pm}_0 \frac{r_0}{r} \star_r \phi_0(r)}_{\sim r \sin\theta \, \mathrm{d}\varphi \wedge \mathrm{d}\theta} + \sum_{\mathsf{n},\sigma,\rho,\tau} Q^{\pm}_{\mathsf{n}\sigma} \underbrace{\mathbf{\Pi}^{\pm}_{\mathsf{n}\sigma\tau}\left[\mathbb{U}^{\pm}_{\mathsf{n}}(r, r_0)\right]_{\tau\rho} \star_r \phi_{\mathsf{n}\rho}(r)}_{\text{basis of smooth two-forms}}, \tag{4.5.36}$$

From this it immediately follows that the empty state is uncharged:

$$Q^{\pm}_0 |\mathbf{1}\rangle = \lim_{r\to 0} \int_{\mathcal{C}[\cdot]} \mathrm{D}a \, \mathrm{e}^{-S[a]} \int_{\Sigma_r} \frac{r}{r_0} \phi_0(r) \wedge J^{\pm}_{\Sigma_r} = 0, \tag{4.5.37}$$

as the smooth part of $J^{\pm}_{\Sigma_r}$ goes to zero at $r \to 0$. The action of the non-zero modes is also straighforward:

$$\begin{aligned} Q^{\pm}_{\mathsf{n}\sigma} |\mathbf{1}\rangle &= \lim_{r\to 0} \int_{\mathcal{C}[\cdot]} \mathrm{D}a \, \mathrm{e}^{-S[a]} \int_{\Sigma_r} \left[\mathbb{U}^{\pm}_{\mathsf{n}}(r, r_0)\right]^{-1}_{\sigma\tau} \phi_{\mathsf{n}\tau}(r) \wedge J^{\pm}_{\Sigma_r} \\ &= \int_{\mathcal{C}[\cdot]} \mathrm{D}a \, \mathrm{e}^{-S[a]} \, \mathbf{\Pi}^{\pm}_{\mathsf{n}\sigma\tau} Q^{\pm}_{\mathsf{n}\tau}, \end{aligned} \tag{4.5.38}$$

where here, and henceforth, repeated indices $\{\sigma, \tau, \ldots\}$ on the Beltrami plane are contracted. From this it follows that, if $v^{\pm}_{\mathsf{n}}$ is any vector in the image of $\mathbb{U}^{\pm}_{\mathsf{n}}(0, r_0)$, i.e. orthogonal to $\mathbf{\Pi}^{\pm}_{\mathsf{n}}$, the combination $v^{\pm}_{\mathsf{n}\sigma} Q^{\pm}_{\mathsf{n}\sigma}$ annihilates the empty state:

$$v^{\pm}_{\mathsf{n}\sigma} Q^{\pm}_{\mathsf{n}\sigma} |\mathbf{1}\rangle = 0. \tag{4.5.39}$$

[23] Note that $\mathbb{U}^{\pm}_{\mathsf{n}}(0, r_0)$ from (4.5.28) (cf. also appendix C.3) has rank 1, reflecting precisely the fact that there is a smooth and a divergent mode.

This behaviour is in fact general. This linear combination of charges will take the role of annihilation operators, for path integral states, with its Hermitian conjugate becoming the new creation operators.

4.5.3 Squeeze, squeeze, squeeze

Let us make the above intuition precise. As before, we take $v_{\mathrm{n}}^{\pm}$ to be a vector orthogonal to $\mathbf{\Pi}_{\mathrm{n}}^{\pm}$,

$$\mathbf{\Pi}_{\mathrm{n}}^{\pm} \cdot v_{\mathrm{n}}^{\pm} = 0. \tag{4.5.40}$$

Of course, since $\mathbf{\Pi}_{\mathrm{n}}^{\pm}$ is a rank-1 projector on a two-dimensional plane, $v_{\mathrm{n}}^{\pm}$ is unique, up to rescaling and a global phase. We will take $v_{\mathrm{n}}^{\pm}$ to be such that

$$\left(v_{\mathrm{n}}^{\pm}\right)^{\dagger} \cdot \boldsymbol{\sigma}_z \cdot v_{\mathrm{n}}^{\pm} = \pm 1, \tag{4.5.41}$$

where $\boldsymbol{\sigma}_z$ is the Pauli z-matrix. With this choice of $v_{\mathrm{m}}^{\pm}$, it is straightforward to check that the operators

$$\begin{aligned} \mathcal{B}_{\mathrm{n}\pm} &:= v_{\mathrm{n}\sigma}^{\pm} Q_{\mathrm{n}\sigma}^{\pm} = v_{\mathrm{n}\pm}^{\pm} \mathcal{A}_{\mathrm{n}\pm} + v_{\mathrm{n}\mp}^{\pm} \mathcal{A}_{\mathrm{n}\mp}^{\dagger}, \\ \mathcal{B}_{\mathrm{n}\pm}^{\dagger} &:= \bar{v}_{\mathrm{n}\sigma}^{\pm} Q_{\mathrm{n}\sigma}^{\mp} = \bar{v}_{\mathrm{n}\pm}^{\pm} \mathcal{A}_{\mathrm{n}\pm}^{\dagger} + \bar{v}_{\mathrm{n}\mp}^{\pm} \mathcal{A}_{\mathrm{n}\mp}, \end{aligned} \tag{4.5.42}$$

where the overline denotes complex conjugation, satisfy the correct Kac–Moody algebra of ladder operators:

$$\left[\mathcal{B}_{\mathrm{n}\sigma}, \mathcal{B}_{\mathrm{m}\tau}^{\dagger}\right] = \mathrm{k}\sqrt{\lambda_{\mathrm{n}}(r_0)}\delta_{\mathrm{nm}}\delta_{\sigma\tau}, \tag{4.5.43}$$

with all other commutators vanishing.

Notice, however, that (4.5.42) is a Bogoliubov transformation. The new set of oscillators can be obtained by a unitary transformation of the old oscillators:

$$\begin{aligned} \mathfrak{S}_{\mathrm{n}}(v_{\mathrm{n}})\mathcal{A}_{\mathrm{n}\pm}\mathfrak{S}_{\mathrm{n}}^{\dagger}(v_{\mathrm{n}}) &= \mathcal{B}_{\mathrm{n}\pm}, \quad \text{and} \\ \mathfrak{S}_{\mathrm{n}}(\bar{v}_{\mathrm{n}})\mathcal{A}_{\mathrm{n}\pm}^{\dagger}\mathfrak{S}_{\mathrm{n}}^{\dagger}(\bar{v}_{\mathrm{n}}) &= \mathcal{B}_{\mathrm{n}\pm}^{\dagger}. \end{aligned} \tag{4.5.44}$$

The operator $\mathfrak{S}_{\mathrm{n}}(v_{\mathrm{n}})$ is known as a squeezing operator, and in particular as a as *non-separable two-mode* squeezing operator [305–308]. Its explicit form, in terms of the original set of oscillators, follows by an application of the Baker–Campbell–Hausdorff formula, and is given as:

$$\begin{aligned} \mathfrak{S}_{\mathrm{n}}(v_{\mathrm{n}}) = \exp&\Big(-\rho_{\mathrm{n}}\Big(\mathrm{e}^{\mathrm{i}\xi_{\mathrm{n}}}\mathcal{A}_{\mathrm{n}+}^{\dagger}\mathcal{A}_{\mathrm{n}-}^{\dagger} - \mathrm{e}^{-\mathrm{i}\xi_{\mathrm{n}}}\mathcal{A}_{\mathrm{n}+}\mathcal{A}_{\mathrm{n}-}\Big)\Big)\times \\ &\times \exp\Big(\mathrm{i}\psi_{\mathrm{n}+}\mathcal{A}_{\mathrm{n}+}^{\dagger}\mathcal{A}_{\mathrm{n}+} + \mathrm{i}\psi_{\mathrm{n}-}\mathcal{A}_{\mathrm{n}-}^{\dagger}\mathcal{A}_{\mathrm{n}-}\Big), \end{aligned} \tag{4.5.45}$$

where $\rho_{\mathsf{n}}, \xi_{\mathsf{n}}, \psi_{\mathsf{n}\pm} \in \mathbb{R}$ are related to $v_{\mathsf{n}}^{\pm}$ as

$$\begin{aligned} v_{\mathsf{n}\pm}^{\pm} &= \mathrm{e}^{\mathrm{i}\psi_{\mathsf{n}\pm}} \cosh(\rho_{\mathsf{n}}) \\ v_{\mathsf{n}\mp}^{\pm} &= \mathrm{e}^{\mathrm{i}(\psi_{\mathsf{n}\pm}+\xi_{\mathsf{n}})} \sinh(\rho_{\mathsf{n}}). \end{aligned} \tag{4.5.46}$$

Apart from the phases $\psi_{\mathsf{n}\pm}$ — which are nevertheless necessary to capture the correct vectors $v_{\mathsf{n}}^{\pm}$ — this corresponds to an $\mathfrak{su}(1,1)$ squeezing transformation.

A salient feature of squeezing transformations is that applying the squeezing operator to the a ground state of the original set of operators,

$$\mathcal{A}_{\mathsf{n}\pm} |\mathbf{0}\rangle_{\mathcal{A}_{\mathsf{n}}} = 0, \tag{4.5.47}$$

yields a ground state of the new set of operators. Namely, the state

$$|\mathbf{0}\rangle_{\mathcal{B}_{\mathsf{n}}} := \mathfrak{S}_{\mathsf{n}}(v_{\mathsf{n}}) |\mathbf{0}\rangle_{\mathcal{A}_{\mathsf{n}}}, \tag{4.5.48}$$

satisfies

$$\mathcal{B}_{\mathsf{n}\pm} |\mathbf{0}\rangle_{\mathcal{B}_{\mathsf{n}}} = 0. \tag{4.5.49}$$

These states are known as *squeezed vacua*. Excited states can be built in the standard way, by applying creation operators of the new set of modes on the squeezed vacuum. They are related to the excited states of the original set of modes by a squeezing transformation:

$$\mathcal{B}_{\mathsf{n}\pm}^{\dagger} |\mathbf{0}\rangle_{\mathcal{B}_{\mathsf{n}}} = \mathfrak{S}_{\mathsf{n}}(v_{\mathsf{n}}) \mathcal{A}_{\mathsf{n}\pm}^{\dagger} |\mathbf{0}\rangle_{\mathcal{A}_{\mathsf{n}}}. \tag{4.5.50}$$

In other words, they remain squeezed.

Finally, recall that our setup involves infinitely many modes. Demanding that the new ground states are annihillated by $\mathcal{B}_{\mathsf{n}\pm}$ for all $\mathsf{n} \in \mathcal{N}_{\perp}^{*}$, identifies it with an all-mode squeezed state that is only pairwise non-separable. In other words there exists a squeezing operator acting on all modes defined as:

$$\mathfrak{S} \underset{\text{for}}{\overset{\text{shorthand}}{:=\!=\!=}} \mathfrak{S}\Big(\{v_{\mathsf{n}}\}_{\mathsf{n}\in\mathcal{N}_{\perp}^{*}}\Big) := \prod_{\mathsf{n}\in\mathcal{N}_{\perp}^{*}} \mathfrak{S}_{\mathsf{n}}(v_{\mathsf{n}}), \tag{4.5.51}$$

Note here, that there is no ordering amgiguity in defining $\mathfrak{S}$, as modes at different levels commute. It is for the same reason that it also acts on each energy level separately, i.e.

$$\mathfrak{S}\mathcal{A}_{\mathsf{n}\pm}\mathfrak{S}^{\dagger} = \mathcal{B}_{\mathsf{n}\pm}, \qquad \forall \mathsf{n} \in \mathcal{N}_{\perp}^{*}, \tag{4.5.52}$$

without mixing ladder operators at different levels. As a consequence, there are all-mode squuezed vacua

$$|\mathbf{0}\rangle_{\mathcal{B}} := \mathfrak{S} |\mathbf{0}\rangle_{\mathcal{A}}, \tag{4.5.53}$$

annihillated by all $\mathcal{B}$ modes,

$$\mathcal{B}_{\mathsf{n}\pm} \left|\mathbf{0}\right\rangle_{\mathcal{B}} = 0, \qquad {}^{\forall}\mathsf{n} \in \mathcal{N}_{\perp}^{*}, \tag{4.5.54}$$

where $\left|\mathbf{0}\right\rangle_{\mathcal{A}}$ is the ground state annihilated by all $\mathcal{A}$ modes,

$$\mathcal{A}_{\mathsf{n}\pm} \left|\mathbf{0}\right\rangle_{\mathcal{A}} = 0, \qquad {}^{\forall}\mathsf{n} \in \mathcal{N}_{\perp}^{*}. \tag{4.5.55}$$

As an explicit illustration of this construction, let us focus again on the family without momentum on the $\mathbb{S}^1$, corresponding to $\mathsf{n} = (\ell, m, 0)$, and governed by the evolution matrix (4.5.28). The magnetic quantum number, m, does not enter any of the formulas, so we will suppress it, denoting the modes just by the angular momentum number, ℓ. The Bogoliubov coefficients are given by

$$v_\ell^{+} = \frac{-1}{2(\ell(\ell+1))^{1/4}} \begin{pmatrix} \sqrt{\ell} + \sqrt{\ell+1} \\ \sqrt{\ell} - \sqrt{\ell+1} \end{pmatrix}, \tag{4.5.56}$$

$$v_\ell^{-} = \frac{-\mathrm{i}}{2(\ell(\ell+1))^{1/4}} \begin{pmatrix} \sqrt{\ell} - \sqrt{\ell+1} \\ \sqrt{\ell} + \sqrt{\ell+1} \end{pmatrix}, \tag{4.5.57}$$

corresponding to the squeezing parameters

$$\rho_\ell = \frac{1}{4}\log\left(\frac{\ell+1}{\ell}\right), \quad \xi_\ell = \pi, \quad \psi_{\ell+} = \pi, \quad \text{and} \quad \psi_{\ell-} = \frac{3\pi}{2}. \tag{4.5.58}$$

This yields a two-mode squeezing operator as:

$$\begin{aligned} \mathfrak{S}_\ell(v_\ell) = \exp&\left(\frac{1}{4}\log\left(\frac{\ell+1}{\ell}\right)\left(\mathcal{A}_{\ell+}^{\dagger}\mathcal{A}_{\ell-}^{\dagger} - \mathcal{A}_{\ell-}^{\dagger}\mathcal{A}_{\ell-}\right)\right)\times \\ &\times \exp\left(\mathrm{i}\pi\left(\mathcal{A}_{\ell+}^{\dagger}\mathcal{A}_{\ell+} + \frac{3}{2}\mathcal{A}_{\ell-}^{\dagger}\mathcal{A}_{\ell-}\right)\right). \end{aligned} \tag{4.5.59}$$

4.5.4 The correspondence

After this interlude on squeezed states, we return to line operators, and the states they prepare, via the path integral. First we consider the states of the Wilson–'t Hooft operators, (4.5.11),

$$\left|\mathrm{W}_{n,m}\right\rangle := \int_{\mathcal{C}[\cdot]} \mathrm{D}a\, \mathrm{e}^{-S[a]}\, \mathrm{W}_{n,m}\big(\mathbb{S}^1 \times \{0\}\big). \tag{4.5.60}$$

Let us see the action of the charges on these states.

Starting with the zero-mode, we have:

$$Q_0^{\pm}\left|\mathrm{W}_{n,m}\right\rangle = \lim_{r\to 0}\int_{\mathcal{C}[\cdot]} \mathrm{D}a\, \mathrm{e}^{-S[a]} \int_{\Sigma_r} \frac{r}{r_0}\phi_0(r) \wedge J_{\Sigma_r}^{\pm} \times \mathrm{W}_{n,m}\big(\mathbb{S}^1 \times \{0\}\big). \tag{4.5.61}$$

We will need the OPE between $J^{\pm}_{\Sigma_r}$ and the Wilson–'t Hooft loop. In fact, here we will only need the Wick contraction, as all the regular terms are bound to vanish, by regularity of the rest of the zero-mode integrand. The desired OPE is:

$$J^{\pm}_{\Sigma_r} \times \mathrm{W}_{n,m}(\mathbb{S}^1 \times \{0\}) \sim \left(\mathrm{i}\, n \left\langle J^{\pm}_{\Sigma_r} \int_{\mathbb{S}^1} a \right\rangle + \mathrm{i}\, m \left\langle J^{\pm}_{\Sigma_r} \int_{\mathbb{S}^1} \check{a} \right\rangle \right) \mathrm{W}_{n,m}(\mathbb{S}^1 \times \{0\}), \quad (4.5.62)$$

where the twiddle indicates that it is considered up to regular terms. To continue on we will need the two-point function of the gauge-fields on $\mathbb{R}^3 \times \mathbb{S}^1$.[24] Parametrising $\mathbb{R}^3$ by coordinates x and the circle by an angle, η, the two-point function reads:

$$\left\langle a_\mu(x,\eta) a_\nu(0,0) \right\rangle = \frac{4}{\operatorname{Im} \mathrm{t}} \frac{g_{\mu\nu} \ \sinh \|x\|}{\|x\| (\cosh \|x\| - \cos \eta)} + \text{gauge-dependent terms.} \quad (4.5.63)$$

Integrating one of the gauge fields on the circle, to get the holonomy, and differentiating the other one to get the current yields immediately:

$$\left\langle \left(\int_{\Sigma_r} \frac{r}{r_0} \phi_0(r) \wedge J^{\pm}_{\Sigma_r} \right) \left(\int_{\mathbb{S}^1} a \right) \right\rangle = \pm \frac{\pi}{\sqrt{2}\, r_0} \frac{1}{\operatorname{Im} \mathrm{t}}. \quad (4.5.64)$$

From here, electric–magnetic duality implies

$$\left\langle \left(\int_{\Sigma_r} \frac{r}{r_0} \phi_0(r) \wedge J^{\pm}_{\Sigma_r} \right) \left(\int_{\mathbb{S}^1} \check{a} \right) \right\rangle = \pm \frac{\pi}{\sqrt{2}\, r_0} \frac{\mathrm{t}^{\mp}}{\operatorname{Im} \mathrm{t}}, \quad (4.5.65)$$

and therefore, finally:

$$Q^{\pm}_0 \left| \mathrm{W}_{n,m} \right\rangle = \pm \pi \mathrm{i} \, \frac{1}{\sqrt{2} r_0} \frac{n + \mathrm{t}^{\mp} m}{\operatorname{Im} \mathrm{t}} \left| \mathrm{W}_{n,m} \right\rangle . \quad (4.5.66)$$

Compare this to (4.4.86). The state $|\mathrm{W}_{n,m}\rangle$ has exactly the same zero-mode charges, as the Kac–Moody primaries $|n,m\rangle$. It is tempting, therefore, to identify the Wilson–'t Hooft state with $|n,m\rangle$.

This is almost the right answer. To iron out that almost we need the action of the non-zero modes. First, note that the OPE (4.5.62) cannot give any singular contribution upon integrated against any of the higher-modes. The reason is, simply, that $\left\langle J^{\pm}_{\Sigma_r} \int_{\mathbb{S}^1} a \right\rangle$ and $\left\langle J^{\pm}_{\Sigma_r} \int_{\mathbb{S}^1} a \right\rangle$ are proportional to the volume form on the sphere. But the one-forms $\phi_\mathsf{n}(r)$, that it will be integrated against, are periodic on the circle. Hence the net answer is zero. Therefore, as far as non-zero modes are concerned, the OPE produces only regular contributions. We are therefore in the same territory as for the empty state that we discussed previously. Hence again we have that

$$\mathcal{B}_{\mathsf{n}\pm} \left| \mathrm{W}_{n,m} \right\rangle = 0, \qquad \forall \mathsf{n} \in \mathcal{N}^*_\perp . \quad (4.5.67)$$

[24]Remember, we are taking this OPE in the limit $r \to \mathbf{0}$, so the three-ball, $\mathbb{B}^3_{r_0}$ is identical to the whole $\mathbb{R}^3$.

This is the precise sense, in which Wilson–'t Hooft lines are *primary operators*. They have definite one-form charge, and the states they generate are primary states of the Kac–Moody algebra (4.5.43), sitting at the bottom of the Verma module labelled by n and m.

Reconciling the above fact with our digression on squeezed states, the Wilson–'t Hooft lines correspond to *squeezed primary states*:

$$|\mathrm{W}_{n,m}\rangle = \mathfrak{S}\,|n,m\rangle\,. \tag{4.5.68}$$

Inverting this relation, we have, equivalently that the vacua $|n,m\rangle$ correspond to:

$$|n,m\rangle = \mathfrak{S}^\dagger\,|\mathrm{W}_{n,m}\rangle = \int_{\mathscr{C}[\cdot]} \mathrm{D}a\;\mathrm{e}^{-S[a]}\;\mathfrak{S}^\dagger \mathrm{W}_{n,m}\big(\mathbb{S}^1\times\{0\}\big), \tag{4.5.69}$$

where $\mathfrak{S}^\dagger \mathrm{W}_{n,m}$ denotes the line operator obtained by shrinking $\mathfrak{S}^\dagger$ — expressed in terms of the currents — onto the Wilson–'t Hooft loop, à la figure 4.3, or equivalently via the OPE:

$$\mathfrak{S}^\dagger \mathrm{W}_{n,m}\big(\mathbb{S}^1\times\{0\}\big) := \lim_{r\to 0}\mathfrak{S}^\dagger[J^\pm_{\Sigma_r}]\times \mathrm{W}_{n,m}\big(\mathbb{S}^1\times\{0\}\big). \tag{4.5.70}$$

Let us pause here to comment on the ground state, i.e. the state $|0,0\rangle$, of zero energy. By the above discussion, it corresponds to the operator

$$|0,0\rangle \quad \leftrightsquigarrow \quad \mathfrak{S}^\dagger\big(\mathbb{S}^1\times\{0\}\big). \tag{4.5.71}$$

This is clearly not the identity operator. In one sense, it is almost the identity operator, as this is the only primary line operator it sees. It is completely transparent to the one-form charges. However, it is also as far as one can get from the identity operator as it contains photon excitations of arbitrary frequency — indeed, of all frequencies. This was, to some extent, anticipated in [269], where it was shown that, in a generic CFT, the identity operator, cannot prepare the vacuum state on the torus (or generally, on any spatial slice other than the sphere). Our result is consistent with that statement, while still tractable in this example.

Continuing on, and moving up in the Verma module, all the other states can be reached by acting with $\mathscr{B}^\dagger_{\mathrm{n}\pm}$ on the squeezed primaries, i.e.

$$\begin{aligned}\Big|\mathrm{W}_{n,m};\{N_{\mathrm{n}\sigma}\}^{\sigma=\pm}_{\mathrm{n}\in\mathscr{N}^*_\perp}\Big\rangle &:= \prod_{\substack{\mathrm{n}\in\mathscr{N}^*_\perp\\ \sigma=\pm}}\Big(\mathscr{B}^\dagger_{\mathrm{n}\sigma}\Big)^{N_{\mathrm{n}\sigma}}|\mathrm{W}_{n,m}\rangle\\ &= \int_{\mathscr{C}[\cdot]}\mathrm{D}a\;\mathrm{e}^{-S[a]}\prod_{\substack{\mathrm{n}\in\mathscr{N}^*_\perp\\ \sigma=\pm}}\Big(\mathscr{B}^\dagger_{\mathrm{n}\sigma}\Big)^{N_{\mathrm{n}\sigma}}\mathrm{W}_{n,m}\big(\mathbb{S}^1\times\{0\}\big).\end{aligned} \tag{4.5.72}$$

The modes $\mathscr{B}_{\mathrm{n}\pm}$ and $\mathscr{B}^\dagger_{\mathrm{n}\pm}$ can be written, in terms of the local operators $J^\pm$ as

$$\mathscr{B}_{\mathrm{n}\pm}(\mathbb{S}^1\times\{0\}) = \lim_{r\to 0}\int_{\Sigma_r} v^{\pm}_{\mathrm{n}\sigma}\left[\mathbb{U}^{\pm}_{\mathrm{n}}(r,r_0)\right]^{-1}_{\sigma\tau}\phi_{\mathrm{n}\tau}(r)\wedge J^{\pm}_{\Sigma_r}, \tag{4.5.73}$$

$$\mathscr{B}^\dagger_{\mathrm{n}\pm}(\mathbb{S}^1\times\{0\}) = \lim_{r\to 0}\int_{\Sigma_r} \bar{v}^{\pm}_{\mathrm{n}\sigma}\left[\mathbb{U}^{\mp}_{\mathrm{n}}(r,r_0)\right]^{-1}_{\sigma\tau}\phi_{\mathrm{n}\tau}(r)\wedge J^{\pm}_{\Sigma_r}, \tag{4.5.74}$$

respectively. Acting them on Wilson–'t Hooft operators gives precisely the *descendant operators* that we discussed in subsection 4.5.1. They are related to the descendant states (4.4.75), by a squeezing transformation. For example, a single excited mode is

$$\mathscr{A}^\dagger_{\mathrm{n}\pm}\left|n,m\right\rangle = \mathfrak{S}^\dagger\mathscr{B}^\dagger_{\mathrm{n}\pm}\left|\mathrm{W}_{n,m}\right\rangle = \int_{\mathcal{C}[\cdot]}\mathrm{D}a\,\mathrm{e}^{-S[a]}\,\mathfrak{S}^\dagger\mathscr{B}^\dagger_{\mathrm{n}\pm}\mathrm{W}_{n,m}(\mathbb{S}^1\times\{0\}), \tag{4.5.75}$$

and similarly for the higher-excited states.

To recap, we have just constructed a one-to-one map between line operators on $\mathbb{R}^3\times\mathbb{S}^1$ and states on $\mathbb{S}^2\times\mathbb{S}^1$. The line operators that we have in our disposal are the Wilson–'t Hooft lines and modulated versions thereof, i.e. smeared with modes of the basic gauge-invariant operator, f, or equivalently $J^\pm$. Each different allowed smearing, that is, each different smooth configuration of $J^\pm$ gives a different state. The Wilson–'t Hooft lines are the primary operators of the Kac–Moody algebra. Modulated operators are their descendants. They prepare states on $\mathbb{S}^2\times\mathbb{S}^1$, that are orthonormal and span the entire Hilbert space. These states are related to the standard energy eigenstates on the Hilbert space by a squeezing transformation. This map is displayed in figure 4.4.

Energies and overlaps

In this last paragraph, we will, briefly, compare the squeezed states we have arrived at to the energy eigenstates. We begin by calculating the average energy of the squeezed primary states $\left|\mathrm{W}_{n,m}\right\rangle$. A quick computation shows that their average energy is

$$\left\langle\mathrm{W}_{n,m}\right|H_{\Sigma_{r_0}}\left|\mathrm{W}_{n,m}\right\rangle = \Delta_{n,m} + E_{0,\mathfrak{S}}(r_0), \tag{4.5.76}$$

where

$$E_{0,\mathfrak{S}}(r_0) = \sum_{\mathrm{n},\sigma}\sinh^2(\rho_{\mathrm{n}})\sqrt{\lambda_{\mathrm{n}}(r_0)}, \tag{4.5.77}$$

is essentially measuring a zero-point energy, which has to do with the fact that the Hamiltonian $H_{\Sigma_{r_0}}$ is not normal-ordered with respect to the new set of ladder operators. More importantly,

$$\Delta_{n,m} = \frac{\pi}{2r_0^2}\frac{|n+\mathrm{t}m|^2}{\operatorname{Im}\mathrm{t}} \tag{4.5.78}$$

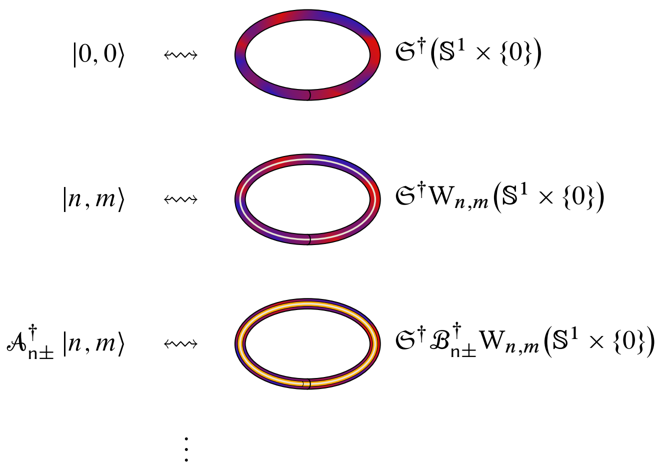

Figure 4.4: *The states and their corresponding operators. (Top) The squeezing operator, $\mathfrak{S}^\dagger(\mathbb{S}^1 \times \{0\})$, containing photons of all frequencies, represented by a multi-coloured line prepares the vacuum state. (Middle) The squeezing operator, surrounding a Wilson–'t Hooft line of charges n, m prepares the primary state $|n,m\rangle$. (Bottom) The squeezing operator, on top of the mode $\mathcal{B}^\dagger_{\mathrm{n}\pm}$ on top of a Wilson–'t Hooft line prepares the descendant $\mathcal{A}^\dagger_{\mathrm{n}\pm} |n,m\rangle$.*

is the same as the energy of the (unsqueezed) primaries (4.4.87), as well as the instanton weights in the path integral (4.4.38). This property reflects and refines the observation of Verlinde [274] and the argument of Kapustin [273], that the Wilson–'t Hooft lines have a quantum number $\Delta_{n,m}$ akin to a scaling weight.[25]

For the first excited states, an elementary computation reveals their average energy

$$\langle \mathrm{W}_{n,m}| \mathcal{B}_{\mathrm{n}\sigma} H_{\Sigma_{r_0}} \mathcal{B}^\dagger_{\mathrm{n}\sigma} |\mathrm{W}_{n,m}\rangle = \Delta_{n,m} + \Big(\cosh^2(\rho_{\mathrm{n}}) + \sinh^2(\rho_{\mathrm{n}})\Big)\sqrt{\lambda_{\mathrm{n}}(r_0)} + E_{0,\mathfrak{S}}(r_0). \tag{4.5.79}$$

In the no-squeezing limit, $\rho_{\mathrm{n}} \to 0$, this becomes precisely the energy of the first excited energy eigenstates.

Finally we can also compute overlaps between the squeezed and the unsqueezed primaries. This is very much facilitated by the disentangling formula of $\mathfrak{su}(1,1)$

[25] Do notice, however, that these are not the eigenvalues of the dilation operator of the theory under consideration.

squeezing operators [309, 310]:

$$\exp\left(\alpha_n \mathcal{X}_n^+ - \bar{\alpha}_n \mathcal{X}_n^-\right) = \exp\left(e^{i\arg(\alpha_n)} \tanh|\alpha_n| \mathcal{X}_n^+\right) \times \exp\left(-2\log\cosh|\alpha_n| \mathcal{X}_n^0\right) \exp\left(-e^{-i\arg(\alpha_n)} \tanh|\alpha_n| \mathcal{X}_n^-\right), \tag{4.5.80}$$

where

$$\mathcal{X}_n^+ := \frac{1}{\sqrt{2}} \frac{\mathcal{A}_{n+}^\dagger \mathcal{A}_{n-}^\dagger}{k\sqrt{\lambda_n}}, \tag{4.5.81}$$

$$\mathcal{X}_n^- := \frac{1}{\sqrt{2}} \frac{\mathcal{A}_{n+} \mathcal{A}_{n-}}{k\sqrt{\lambda_n}}, \tag{4.5.82}$$

$$\mathcal{X}_n^0 := \frac{1}{2}\left(1 + \frac{\mathcal{A}_{n+}^\dagger \mathcal{A}_{n+} + \mathcal{A}_{n-}^\dagger \mathcal{A}_{n-}}{k\sqrt{\lambda_n}}\right) \tag{4.5.83}$$

are the generators of $\mathfrak{su}(1,1)$:

$$[\mathcal{X}_n^+, \mathcal{X}_n^-] = -2\mathcal{X}_n^0, \qquad [\mathcal{X}_n^0, \mathcal{X}_n^\pm] = \pm\mathcal{X}_n^\pm. \tag{4.5.84}$$

In this case, the phases, $\psi_{n\pm}$, of the squeezing operator, (4.5.45), do not matter as the phase-shift operator they furnish is normal ordered, and thus, acts as the identity on $|n,m\rangle$. In total we find:

$$\langle n,m|W_{n,m}\rangle = \prod_{n\in\mathcal{N}_\perp^*} (\cosh(\rho_n))^{-2}. \tag{4.5.85}$$

4.6 Discussion

In this chapter, we studied CFTs with continuous generalised global symmetries. We showed that an invertible continuous $(p+1)$-form symmetry in a $(2p+2)$-dimensional unitary CFT automatically enhances to an infinite collection of codimension one, i.e. zero-form, conserved charges, labelled by (anti-)chiral p-forms. The algebra of these charges is spectrum-generating (up to decoupled neutral factors) and characterises completely the CFT. The dynamics of the CFT are those of free p-forms. Along a similar vein, we showed that a non-invertible continuous $(p+1)$-form symmetry, leads to a non-invertible current algebra, which we describe in terms of the fusion rules of the symmetry generators. As before, it characterises completely the dynamics, leading, in this case, to an O(2) p-form gauge theory. In the invertible case, and focussing on $p=1$, hence in four-dimensional CFTs, we constructed the representation theory of this algebra (for the invertible case), which we showed reproduces the path integral calculation, as a non-trivial check. This allowed us to describe explicitly the Hilbert space of Maxwell theory on arbitrary closed spatial manifolds. On non-trivial topologies, the full Hilbert space consists of the usual photon Hilbert space, as well as of Verma modules, built on top of states with one-form symmetry charge. Additionally, we showed that the full spectrum of states on $\mathbb{S}^2 \times \mathbb{S}^1$ can be obtained by a path integral on $\mathbb{B}^3 \times \mathbb{S}^1$, with various operator insertions. The radial evolution on the ball acts as a squeezing transformation between the path integral states and the energy eigenstates, requiring that line operators are subsequently dressed with a squeezing operator, containing all photon frequencies, to reach the energy eigenstates. Notably, the vacuum state of Maxwell theory on $\mathbb{S}^2 \times \mathbb{S}^1$ is not produced by the path integral with no insertions, but by the path integral with a squeezing operator inserted. Nevertheless, this construction leads to a one-to-one correspondence between line operators on $\mathbb{R}^3 \times \mathbb{S}^1$ and states on $\mathbb{S}^2 \times \mathbb{S}^1$ and a classification of line operators in terms of the current algebra. In short, Wilson–'t Hooft lines,

$$\mathrm{W}_{n,m}(\mathbb{S}^1) = \exp\left(\mathrm{i}n \int_{\mathbb{S}^1} a + \mathrm{i}m \int_{\mathbb{S}^1} \check{a}\right), \tag{4.6.1}$$

are charged under the one-form symmetries, $Q_0^{\pm}$, (4.5.29):

$$Q_0^{\pm} \times \mathrm{W}_{n,m}(\mathbb{S}^1) \sim \pm \frac{\pi \mathrm{i}}{\sqrt{2} r_0} \frac{n + \mathrm{t}^{\mp} m}{\mathrm{Im}\,\mathrm{t}} \mathrm{W}_{n,m}(\mathbb{S}^1), \tag{4.6.2}$$

and annihilated by the lowering operators $\mathcal{B}_{\mathrm{n}\pm}$, (4.5.73), of the Kac–Moody algebra:

$$\mathcal{B}_{\mathrm{n}\pm} \times \mathrm{W}_{n,m}(\mathbb{S}^1) \sim 0. \tag{4.6.3}$$

This defines them as primary operators. They have definite scaling weight, as defined previously, given by

$$\frac{1}{\mathrm{k}} Q_0^+ Q_0^- \times \mathrm{W}_{n,m}(\mathbb{S}^1) \sim \Delta_{n,m} \mathrm{W}_{n,m}(\mathbb{S}^1), \tag{4.6.4}$$

$$\Delta_{n,m} = \frac{\pi}{2 r_0^2} \frac{|n + \mathrm{t}\, m|^2}{\mathrm{Im}\,\mathrm{t}}. \tag{4.6.5}$$

The path integral on $\mathbb{B}^3 \times \mathbb{S}^1$ with an insertion of $\mathrm{W}_{n,m}(\mathbb{S}^1)$, dressed by a squeezing operator, (4.5.70), prepares the primary energy eigenstates, $|n, m\rangle$ (4.4.67). Descendant line operators are given by acting with raising operators, $\mathcal{B}^{\dagger}_{\mathrm{n}\pm}$, (4.5.74), on the primaries. Again, the path-inegral on $\mathbb{B}^3 \times \mathbb{S}^1$, with a descendant, dressed with a squeezing operator, produces the descendant states (4.4.75).

We comment on several questions we have left unanswered and possible generalisations.

Non-invertible symmetries and orbifold branches

In subsection 4.4.3 we described a non-invertible current algebra. This was described in terms of the fusion rules of non-invertible topological operators. The free theory that realises the algebra is, in that case, that of an O(2) p-form gauge theory. One very interesting question is to construct the representation theory of these non-invertible current algebras. At $p = 0$, or equivalently in $d = 2$, an important role is played by tube algebras [311, 312] and lasso actions [60] associated to the fusion of the topological operators. A careful study of this, should land exactly on the orbifold branch of a compact scalar, which has a much simpler algebraic description [286]. Similarly, exploiting and generalising representation theory of higher-dimensional non-invertible symmeties [287–290] should give an algebraic point of view to orbifold branches of gauge-theories, operning a window to new BKT-like phase transitions at points of enhanced symmetry.

Non-abelian current algebras

Despite the absence of non-abelian higher-form symmeries, one could still consider the physics of a four-dimensional CFT with a non-abelian version of our higher-dimensional current algebra, schematically of the form:

$$[\mathcal{Q}^{\mathfrak{a}}_{\mathsf{n}}, \mathcal{Q}^{\mathfrak{b}}_{\mathsf{m}}] = \mathsf{f}^{\mathfrak{a}\mathfrak{b}}{}_{\mathfrak{c}}\mathsf{T}^{\mathsf{l}}{}_{\mathsf{nm}}\mathcal{Q}^{\mathfrak{c}}_{\mathsf{l}} \; + \; \mathsf{k}\sqrt{\lambda_{\mathsf{n}}}\delta^{\mathfrak{a}\mathfrak{b}}\delta_{\mathsf{nm}}, \tag{4.6.6}$$

where the $\mathfrak{u}(1) \times \mathfrak{u}(1)$ indices, $\pm$, got replaced by some Lie algebra, $\mathfrak{g}$, indices, $\{\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \cdots\}$. There are a few indicators that such an algebra might be hiding behind four-dimensional superconformal field theories. For instance, the form of the Vafa–Witten partition functions [313], for $\mathcal{N} = 4$ super-Yang–Mills (SYM) theory and their relation to two-dimensional RCFTs is reminiscent of our exact formulas for the Maxwell partition fuction and their two-dimensional analogues. A more concrete indicator is Kapustin's definition of scaling weight for $\frac{1}{2}$BPS operators in $\mathcal{N} = 4$ SYM [273], which is arrived at in a similar way as the one for Wilson–'t Hooft operators, which as we saw, is intimately linked to the current algebra.

Modularity and factorisation

The form, (4.4.46) of the partition function begs for an investigation of the modularity properties of the partition function, as well as a possible "holomorphic factorisation" of the partition function. Both of these directions deserve separate attention. On the modularity side, if higher-dimensional CFTs behave like two-dimensional, then swapping the thermal circle for a spatial one-cycles should leave the partition function unaffected. Harnessing that statement leads to a generalised Cardy formula [265] and a universal form of the Casimir energy [270]. Our exact form of the partition function of Maxwell theory, can serve as a testing ground for modularity in higher-dimensional CFTs. Relatedly, but with a different goal in mind, one could imagine making q complex and study whether the Maxwell partition function factorises in a holomorphic and anti-holomorphic piece:

$$\mathcal{Z}_{\text{Maxwell}}(q,\bar{q}) \stackrel{?}{=} \mathcal{Z}(q)\,\mathcal{Z}(\bar{q}). \tag{4.6.7}$$

This would open a new window towards the physics of chiral one-forms (or more generally chiral p-forms), whose importance in six-dimensional superconformal field theories and in M-theory makes them subject of constant study [2, 232–235, 240, 314–316].

Other topologies

The choice of $\mathbb{S}^2 \times \mathbb{S}^1$ for our state-operator correspondence is a natural one for that the unique one-cycle allows line operators wrapping it and the unique two-cycle allows a single flux. It is not clear that this is the only choice, though. For example, states on a $\mathbb{T}^3$ topology are prepared via a four-dimensional path integral with some two-cycle filled in. It is likely that to get the complete set of states we have to consider a mixture of states prepared by filling in every two-cycle. Therefore, the state-operator map, would, in that case, include a sum over topologies, with a given boundary, in the path integral. This is not the case here, because of the requirement that the $\mathbb{S}^1$ remains non-trivial.

Surgery and overlaps

Given our state-operator correspondence, we now have a resolution of the identity on $\mathbb{S}^2 \times \mathbb{S}^1$, in terms of line operators:

$$\mathbb{1} = \sum_{n,m\in\mathbb{Z}} \sum_{\{N_{\mathrm{n}\sigma}\}} |\mathrm{W}_{n,m};\{N_{\mathrm{n}\sigma}\}\rangle\langle \mathrm{W}_{n,m};\{N_{\mathrm{n}\sigma}\}|\,. \tag{4.6.8}$$

We can use this to simplify correlation functions on manifolds that can be cut along an $\mathbb{S}^2 \times \mathbb{S}^1$. Relatedly, states can be glued to produce partition functions or correlation functions on manifolds surgered along $\mathbb{S}^2 \times \mathbb{S}^1$. This fact opens up a new perspective on partition functions. Take, for example, partition functions in d Euclidean dimensions with the topology of the sphere. These are known to be related to counting problems through F-theorem arguments [317], at least in $d = 3$. On the other hand, by the logic above, this partition function should be related to the overlap between states prepared by computing path integrals on some half spaces which do not necessarily produce the vacuum of the theory. An example would amount to considering the Hopf fibration of the sphere and consider states defined on the $\mathbb{T}^2$ gluing surface. Now we know those states, while having a simple path integral construction, are described by complicated superpositions of high-energy eigenstates. Therefore, these partition functions could have a more natural counting interpretation shedding light on the exact nature of the F-theorem. At this point this is just a plausibility argument, as we have only focused on theories with higher-form symmetries in even dimensions. It would be interesting to study this in more depth.

5 Conclusions

In this thesis we have considered aspects of generalised symmetries and their applications in topological and conformal quantum field theories. In this last chapter we briefly summarise the salient features and main results of this thesis and discuss interesting future directions and open questions.

Symmetries have served as an extermely useful guiding principle in the development and study of quantum field theory. In recent years, a transformative generalisation of the notion of symmetries has taken place. This includes, among others, higher-form, higher-group, non-invertible and subsystem symmetries. All of these serve as new organising principles for quantum field theories, the phases of matter, and (at a more philosophical level) nature as a whole. Throughout this thesis, generalised symmetries — in particular higher-form symmetries for the most part — have been crucial in obtaining universal results about the entanglement properties of topological quantum field theories, and about the underlying structure of conformal field theories.

From a condensed-matter-theoretic point of view, topological field theories arise as low-energy effective descriptions of topologically ordered systems. A physical mechanism for topological order in three (spacetime) dimensions, is given by the condensation of networks of line operators, known as string-net condensation. In higher-dimensions, analogous models — condensing networks of p-dimensional surface operators — give rise to topologically ordered ground states. This is intimately connected to generalised symmetries, as taking the generators of a discrete p-form symmetry as the condensing network, provides a description of deconfined, discrete (higher-)gauge theories. The long distance behaviour of these models is described by a specific topological quantum field theory, known as p-form BF theory. Moreover, topological orders correspond to and are classified by different patterns of entanglement. This is most cleanly showcased by the celebrated topological entanglement entropy [165, 166]. Understanding patterns of entanglement in p-form BF theory, gives direct low-energy access to a systematic understanding of topological order in higher-dimensions. This was the main motivating question for [1, 2] and correspondingly chapters 2 and 3.

In contrast to discrete theories, deconfined phases of (higher-)gauge theories with continuous gauge groups are gapless. Behind this fact lie, again, generalised symmetries. In particular, such phases can be understood as spontaneous symmetry breaking phases of higher-form global symmetries. A prominent example of that is given by electromagnetism (free Maxwell theory), and provides an explanation of the gaplessness of the photon in our world. In specific dimensions, these gapless phases become conformal. The combination of conformal invariance with generalised symmetries turns out to be extremely strong and leads to universal statements about the structure of conformal field theory. These ideas underlie much of the motivation and provide the groundwork for [3] and correspondingly chapter 4.

5.1 Summary

Having described the main conceptual underpinnings of this thesis, we will go into more detail, outlining the main results.

Algebraic and essential entanglement

Entanglement in gauge theories is tricky business. The main reason is non factorisability of the Hilbert space. Let us elaborate. Suppose one is interested in how the degrees of freedom in a region, R, of space are entangled with the rest: $R^{c} = \text{space} \setminus R$. A quantitative measure of that is the von Neumann entropy of a state — commonly taken to be the ground state, or some other pure state — reduced in the region, R, of interest, i.e. having traced out all the degrees of freedom outside. This step already consists in bipartitioning the space of states. In gauge theories such a factorisation is not available, the reason being that elementary excitations are associated with closed loops, rather than points. A natural way to define entanglement, bypassing the problem is via operator algebras [183]. There, the lack of factorisation presents itself as a non-trivial centre of the algebra.

Topologically ordered systems, and their low-energy descriptions, also exhibit lack of factorisation of their space of states, due to the presence of nonlocal operators. In chapter 2 we argued that a modification of the above reasoning and the techniques that come along is very suitable in the study of entanglement in topological field theories, and we showed that it leads to a concrete measure of entanglement in p-form BF theory. More precisely, we considered the *algebra of topological operators* that act on a spatial slice of the theory. We showed that associated with these algebras there are two natural and consistent (non-trivial) choices of subalgebras restricted to a subregion, R. These can be roughly described as follows:

1. The set of operators that can be deformed to act entirely in R, and
2. the set of operators that cannot be deformed to act entirely outside of R.

We call these topological algebras *magnetic* and *electric*, respectively, owing to their connections to analogous algebras in lattice gauge theories. They are related to one another by a form of subregion electric-magnetic duality.

We then put these algebras to use and assigned an entanglement entropy to the ground states of the theory under consideration. The topological nature of the theory allowed us to evaluate this entropy in arbitrary dimensions, arbitrary spatial slices, and subregions of arbitrary topology. We termed this entropy *essential topological entanglement* (ETE). The main features of ETE for p-form BF theory are the following:

1. *It is positive.* This is in contrast to the traditional topological entanglement entropy [165, 166], which is negative and can only be thought of as an entropy difference.
2. *It is finite.* ETE is bounded from above by the dimension of the Hilbert space. This is not straightforward in theories lacking Hilbert space factorisation, but crucial for TQFTs, where the Hilbert space is finite-dimensional.
3. *It is topological.* ETE depends solely on the topology of the spatial slice and the topology entangling surface, i.e. the interface between the region of interest and the rest of the spatial slice.

Moreover it possesses additional qualities which make it a useful tool to study topological order: it probes the total quantum dimension of the system, it cannot be distilled into Bell pairs, and it is defined intrinsically in the infrared description.

Edge-modes, current algebras, and entanglement spectrum

A more traditional approach to understanding the entanglement structure of topologically ordered phases relies on studying their spectrum of edge-modes. The logic goes as follows. TQFTs, when placed on manifolds with boundary are host to gapless edge-modes. The prototypical example is Chern–Simons theory supporting a compact chiral scalar on its boundary in the abelian case, or Wess–Zumino–Witten scalars in the non-abelian case. The density matrix of the edge-mode theory, reduced in a region of interest is maximally mixed among an infinite number of modes, giving rise to an entropy controlled by a divergent, geometry-dependent, area law. Importantly, however, the subleading constant correction is universal and captures presicely the topological entanglement entropy of Kitaev and Preskill [165], and Levin and Wen [166]. Thus, the edge-mode spectrum captures, entirely, the entanglement pattern defining the topological order. This goes by the name of *edge-entanglement spectrum* *correspondence.* What truly lies behind, is that the large gauge transformations of the Chern–Simons theory on a manifold with boundary give rise to a chiral current algebra that generates the spectrum of the edge theory.

In chapter 3 we presented a story along very similar lines, that is valid in arbitrary dimensions, and arbitrary topology — restricting, however, to abelian topological orders. The setup is similar to chapter 2, namely p-surface net condensation topological orders and their low-energy p-form BF theory description, but the result has a different flavour. Namely, we described how BF theory supports chiral gapless edge-modes given by higher-form abelian gauge theories, in what we called *chiral mixed Maxwell theory.* Moreover, we showed that the spectrum of this theory is generated by a higher-dimensional generalisation of the chiral Kac–Moody algebras. We further used this theory to extract an entanglement entropy. Firstly by studying the partition function of chiral mixed Maxwell theory on replicated manifolds, and secondly by an extended Hilbert space approach [176, 182], using the higher-dimenional chiral current algebra. This resulted in divergent area and sub-area contributions, controlled by geometric coefficients, and subleading corrections. We argued that a higher-dimensional Kitaev–Perskill / Levin–Wen subtraction scheme reveals a universal, topological subleading correction generalising the topological entanglement entropy (in a different way than ETE presented above) to higher-dimensions. Finally, the relation of the edge-mode theory to the entaglement entropy, via the current algebra, presents a higher-dimensional edge-entanglement spectrum correspondence.

The structure of conformal field theories

In chapter 4 we stayed within the theme of higher-dimensional current algebras, this time in conformal field theories. The key observation, originally due to [37], is that unitary conformal field theories in even spacetime dimensions with continuous higher-form symmetries have a realisation in terms of free higher-form gauge fields. We exploited this observation, including also continuous non-invertible symmetries, to show that the spectrum of these theories is organised in terms of a current algebra, very similar to the current algebras of chapter 3. The difference with chapter 3, is that there the algebra comes from bulk gauge transformations pushed to the boundary, while here the theory and its current algebra are intrinsically defined. This is akin to the difference between two chiral bosons and one, non-chiral, compact boson. Nonetheless, the techniques are similar. We constructed the representation theory of this current algebra allowing us to compute the spectrum of higher-form abelian gauge theory on arbitrary topology.[1]

[1]Actually in chapter 4 we did so for one-form gauge fields, namely just free photons. However, the spectrum for generic higher-form gauge field follows for free, by adapting the techniques of this chapter, with chapter 3 and appendix B.1.

Focussing in four-dimensional conformal field theories, we showed that the current algebras organise not only the spectrum of states, but also that of nonlocal operators. This led to a classification of line operators in free Maxwell theory, into *primary* and *descendant* operators, with respect to the organising current algebra. More importantly, however, it led to our main result: a direct and explicit *one-to-one correspondence between states on* $\mathbb{S}^2 \times \mathbb{S}^1$ *and line operators*. While such a correspondence was achieved in the presence of symmetries that organise the CFT, it adds an essential piece to our understanding of conformal field theory beyond flat spacetimes.

5.2 Outlook

The research presented in this thesis provided just a miniscule piece of the puzzle of understanding nature. Most important open questions remained unsolved, and most of the combined understanding of humankind remained unaffected. Even so, several interesting future research directions arose. Several of them are briefly explored in the discussion sections of the relevant chapters. In this final section, we further expand on some of these ideas introduce a few new ones that were not previously addressed.

Entanglement imprint of symmetries

Given the overall power of symmetries in quantum field theory, a natural question to ask is whether they leave an imprint on the entanglement spectrum of a theory. In other words, do symmetries constrain the nonlocal correlations of the states of a theory? The telltale sign of such a constraint would be to identify a measure of entanglement — an entanglement entropy — that is robust, meaning that it does not depend on fine geometric of the setup, and universal, meaning that quantum field theories sharing the same symmetries would have to necessarily share this entanglement. The interplay of symmetries and entanglement is studied in the literature under the moniker of *symmetry resolved entanglement* (SRE) [318, 319].[2] However, despite it being a potentially useful measure of entanglement, SRE does not exhibit the desired features of robustness and universality. For example, in two dimensional conformal field theories, SRE depends logarithmically on the ratio between the length of the subregion and a non-universal cutoff scale. Moreover, it actually only depends very weakly on the symmetry spectrum itself, a property known as equipartition [322].[3]

A different strategy is needed. What we would like to advocate for, here, is that the

[2] See also [320] for a review, and [321] for an approach incorportating non-invertible symmetries.

[3] To be more precise, the premise in SRE is that one calculates an entanglement entropy associated to a sector of fixed charge of the theory. It turns out that this entropy depends on the charge only at sub-sub-sub-subleading order.

topological operator algebras of chapter 2 seem to be a promising direction. We can argue for that in the following way. Consider a non-topological theory that enjoys some symmetry. In the modern description (cf. section 1.1) this is encoded in a set of topological operators that implement the symmetry. More precisely, it is an *algebra* of topological operators. This algebra is shared by all theories enjoying the same symmetry. Since we are interested in entanglement, we have to restrict this algebra to a subregion, which leads directly to the subregion topological operator algebras of chapter 2. The same techniques can be applied to defined a measure of entanglement, analogous to essential topological entanglement, but associated with the symmetries of the theory. Of course, difficulties may arise in the process. For example, it is not entirely clear how to separate the topological subregion algebras, from the non-topological ones, in a mathematically concrete way. This has to do with the fact that topological subregion algebras form naturally a type I von Neumann algebra, while the algebra of all operators, restricted to a subregion, is typically type III. The difficulty then is that of identifying and separating a type I subalgebra inside the full algebra. Nonetheless, assuming that these hurdles can be overcome, the result will necessarily be a robust and universal measure of entanglement reflecting the symmetry structure of the theory.

One way to bypass the issues mentioned above, while still in keeping with the same conceptual theme is to "grow an extra dimension." The algebras of topological operators are neatly packed in the symmetry topological field theory (symTFT) (cf. subsection 1.1.4) of the theory. As a reminder for the reader, the symTFT is a one-higher-dimensional topological field theory, associated with any quantum field theory enjoying some symmetry. Its primary advantage is that it decouples the action of the symmetries from the dynamical content of the theory under consideration. As such one can exploit its power to obtain universal results. It would be very interesting to apply the symTFT philosophy in the study of entanglement. The proposal is that the essential topological entanglement, for a TQFT associated with a fusion (higher-)category $\mathcal{C}$ (in the sense of [151]) contains robust information about the entanglement spectrum of a quantum field theory with symmetry $\mathcal{C}$. This proposal is exactly commensurate with the ideas of the above paragraph.

The general statement is speculative. In order to turn it into a concrete physical proposal, one needs to understand subregion topological operator algebras for objects, morphisms, and higher-morphisms of a higher-category. However, besides the arguments presented above there are indications about its correctness and usefulness, at least in the case of invertible symmetries. Consider, for instance, four-dimensional $\mathcal{N} = 4$ SU(N) super-Yang–Mills theory. It enjoys a one-form $\mathbb{Z}_N^{[1]}$ symmetry. The symTFT associated with it is a five-dimensional 2-form BF theory with level N.

Consequently the essential topological entanglement for this theory is

$$\mathcal{E} = \# \log N, \tag{5.2.1}$$

where # is a number depending on the topology of the subregion of interest and the rest of the Cauchy slice. Looking at this result from a holographic point of view, it has the flavour of logarithmic corrections to black hole entropy, which are known to be very robust. This approach offers a potential symmetry origin of this fact.

Chiral physics

Chiral higher-gauge fields are higher-form gauge fields with self-dual field strength.[4] They appear prominently in various areas in theoretical physics. A single chiral field carries anomalies. The simplest example is a single chiral scalar field in two dimensions. It has an 't Hooft anomalous U(1) symmetry, while also exhibits gravitational anomalies. The modern approach to treating anomalies consists in placing the system at the boundary of an anomaly inflow theory, or equivalently at the boundary of an SPT phase. In this particular case, it has been known and well-appreciated for decades, and is well-appreciated that placing an integer quantum Hall state[5] on a region with a boundary induces gapless chiral edge modes on its boundary. In the higher-form case the most notable appearances of chiral gauge fields is in string and M-theory. In type IIB string theory they appear through the Ramond–Ramond four-form, which is constrained to have self-dual flux. In M-theory, a two-form gauge field with self-dual field-strength propagates on the worldvolume of an M5-brane. Relatedly, such a field appears in six-dimensional $\mathcal{N} = (2, 0)$ superconformal field theories. A full quantum mechanical understanding of these theories is imperative to attest the viability of string or M-theory as a fully-fledged theory of quantum gravity. While there are various results that tie together pieces of this puzzle, a complete understanding has not yet been achieved.

The first problem one encounters when attempting to deal with chiral p-forms is the lack of a Lagrangian description. There are two main approaches attempting at bypassing this problem and treating chiral p-forms in a unified way. Let us briefly review them, outlining their strengths and pitfalls, before proposing a new approach. The first approach consists in a form of *holomorphic factorisation* [324–326]. Namely, one considers the partition function of a non-chiral p-form and tries to write it as

$$\mathcal{Z}_{\text{non-chiral}} \stackrel{?}{=} |\Theta|^2, \tag{5.2.2}$$

[4]Chiral p-form gauge fields are allowed in $d = 2p + 2$ dimensions.

[5]The integer quantum Hall effect is an example of an invertible topological order. According to the original definition of topological order [133] it is not an SPT phase. According to Kitaev's later definition [323], however, it is.

for some function, Θ, of the coupling constant. Then Θ bears immediately the interpretation of a chiral-form partition function. The main problem with that is that actually the non-chiral partition function takes the form of a sum of squares (cf. chapter 4) rather than a single square. This is a manifestation of the anomaly carried by the chiral form, namely its partition function should more properly be thought of as a partition vector, indicating a relative theory. The second approach is aimed at treating exactly this issue. Motivated by the quantum Hall case and the more general understanding of relative theories living at the edge of a gapped bulk, one writes a Chern–Simons-type topological theory in one dimension higher and evaluates its partition function on a space with boundary, supporting the chiral edge modes [237, 239, 327, 328]. While this is, to date, the most promising attempt, it does not always work, and even when it does it depends on intricate choices in the higher-dimensional theory, which seem unnatural from the lower-dimensional point of view. For example, in $d = 2 \bmod 4$ dimensions, there is a bulk description given by Chern–Simons theory, with a specific choice of quadratic refinement of the differential cohomology pairing. However, such a description is not straightforward in $d = 0 \bmod 4$ dimensions, and indeed it is expected that the analogous description of the Ramond–Ramond field in type IIB string theory should be through differential K-theory [329].

For the chiral scalar there is a third approach, that is not shared with its higher-form counterparts. That is, the chiral sclar enjoys a spectrum-generating chiral algebra. This allows for a Hamiltonian quantisation, with which one obtains the full spectrum of the theory on the circle and the partition function on the torus. In chapters 3 and 4 we showed that higher-form gauge fields also enjoy such current algebras. In those cases the algebra concerned either a non-chiral field or a doubled mixture of chiral fields. Nevertheless, this approach leads to a straightforward proposal. That is, simply, to consider and quantise half of the chiral algebras presented there. This is, in some sense, the link between the holomorphic factorisation approach and the bulk approach to chiral physics. Furthermore, it gives a natural conjecture for the form of the thermal partition function of a chiral p-form,[6] namely one half of (3.3.30), or, schematically:

$$\mathcal{Z}_{\text{chiral } p\text{-form}}[\mathbb{S}^1 \times \Sigma] \overset{?}{=} \frac{\Theta_\Sigma(q;k)}{\eta_\Sigma^{(p)}(q)},$$

with the participating functions as defined in the body of the thesis. Moreover, this approach may be appropriate in understanding non-abelian self-dual fields, such as the ones appearing in $\mathcal{N} = (2,0)$ superconformal field theories, by generalising the current algebras to non-abelian versions thereof (see also the outlook in chapter 4).

[6] Of course, this depends explicitly on a choice of quantisation of the fluxes of the fields. Such a choice is natural with respect to a higher-dimensional theory, but ad hoc on the chiral theory, commensurate with its relative nature.

The space of CFTs

Another possibility relating to the aforementioned current algebras, and inspired by well-known two-dimensional physics concerns the space of CFTs. More precisely, the space of CFTs in the universality class of Maxwell theory. This is an analogue to the space of two-dimensional CFTs with central charge $c = 1$. In two dimensions, this space consists of (apart from three isolated points) two continuous branches, corresponding to CFTs realised as a free compact scalar at some radius, and CFTs realised by a $\mathbb{Z}_2$ orbifold of a free compact scalar. Moreover, the two branches meet at a point of enhanced symmetry, which is intimately linked to the Berezinskii–Kosterlitz–Thouless (BKT) [330–332] phase transition. From a modern point of view these two branches correspond precisely to two-dimensional CFTs enjoying a continuous U(1) (zero-form) symmetry and a continuous abelian non-invertible symmetry, while the BKT point corresponds to the unique CFT enjoying both.[7]

In chapter 4 we developed tools which are equipped to attack similar questions in higher dimensions. Wearing the current algebra glasses, Maxwell theory is the four-dimensional version of the circle branch of a compact scalar CFT, while the O(2) gauge theory takes the roles of the orbifold branch. So one first question one may ask, is whether the two branches meet, similarly to what happens in two dimensions. Here however, there is an additional sensitivity related to the existence of multiple spatial slices one can quantise the theory on. For example, there may be a point where they meet when quantised say on a three-sphere, but not on a three-torus. Or vice versa. Such a scenario would open a window for truly topological phase transition, viewing Maxwell theory at this intersection point as the infrared fixed point of a higher-dimensional BKT-like phase transition. Relatedly, one may ask, whether these two branches exhaust the space of CFTs of this universality class, like the $c = 1$ theories do in two dimensions. On the one hand, the expectation is yes, given the general rarity of non-supersymmetric CFTs in higher dimensions. On the other hand, an approach like the one that revealed the three isolated points in two-dimensional $c = 1$ [293] may lead to CFTs outside the two branches. The reason is simple: there are more allowed topological manipulations. For instance, besides gauging a discrete symmetry, one can also higher-gauge [333], revealing new non-invertible defects in the resulting theory that do not match those of either the circle or the orbifold branch.

A related issue that this general approach reveals is related to the very notion of a marginal operator. Let us recall the stroy from two dimensions. At a generic point along the circle branch of the compact scalar the operator $\bar{\partial}\phi\, \partial\phi$ is exactly marginal and moves us along the circle branch. At the T-duality self-dual point, where the

[7] The story is subtle but beautiful. See [63] for details.

symmetry is enhanced to SU(2),[8] two more marginal operators appear: $\mathrm{e}^{\pm \mathrm{i}\sqrt{2}\phi}$. A linear combination of them, $\cos\left(\sqrt{2}\phi\right)$ is even under the charge-conjugation $\mathbb{Z}_2$ symmetry and upon survives upon its gauging. This is the operator that moves us along the orbifold branch. Coming back to Maxwell theory, a very similar story unfolds, only now the role of $\cos\left(\sqrt{2}\phi\right)$ is played by a charge-conjugation invariant line operator:

$$\mathcal{V}(\gamma) = \exp\left(\mathrm{i}\int_{\gamma} a\right) \oplus \exp\left(-\mathrm{i}\int_{\gamma} a\right), \tag{5.2.3}$$

where a is the photon of Maxwell theory. This operator should have the task of moving along the orbifold branch, or O(2) gauge theory. What is the precise way in which such operators are marginal, and most importantly, how does one couple them consistently to a conformal field theory? There are two natural guesses: one is to consider a non-genuine local operator, constrained to live on the curve γ [63] and couple that to the theory. In that case marginality is inherited from the usual marginal property of local operators. Another one, is that we have to take a more radical approach and write a theory of line operators, along the lines of [42]. Although these last few paragraphs are mostly posing questions and do not propose or attempt solutions, such questions indicate at a significant gap in our understanding of CFTs in higher-dimensions.

[8]We are treating only the holomorphic part here. The antiholomorphic part behaves similarly.

A Appendices for chapter 2

A.1 Precision BF

In this appendix we take a closer look at BF theory in cases where the usual definitions are insufficient, such as when the theory is defined on a manifold with torsion. The appropriate language to define BF theory precisely is that of differential cohomology. We use this language to write down the BF action on a generic manifold and give a precise definition of the path-integral measure and its properties. For the following we will take X to be a d-dimensional manifold, possibly with boundary and we will denote by $\iota_\partial : \partial X \hookrightarrow X$ the embedding of the boundary.

When studying p-form gauge theories on a non-trivial manifold the p-form gauge field is insufficient to capture all the topological properties of the theory. Instead, the relevant degrees of freedom can be captured by a $(p+1)$-cocycle in differential cohomology, $\check{\mathrm{H}}^{p+1}(X)$[1], i.e. a triplet $\check{A} = (A, N_A, F_A)$, where A is a regular p-cochain (the gauge field), N_A is a $(p+1)$-cocycle (giving rise to the flux, upon integration) and F_A is a closed $(p+1)$-form (the field strength). The constituents of the triplet are related by a constraint: $F_A = \mathrm{d}A + N_A$. To define actions, we also need a product in differential cohomology, $\vee : \check{\mathrm{H}}^p(X) \times \check{\mathrm{H}}^q(X) \to \check{\mathrm{H}}^{p+q}(X)$, that will replace the usual wedge product of differential forms. For a gentle introduction on the usage of differential cohomology in higher-gauge theories see [328], while for a mathematically rigorous approach see [334]. The bottomline, is that using the product $\vee$, one can write the BF action, on a generic d-dimensional manifold X, as a pairing in differential cohomology:

$$S_{\mathrm{BF}}\left[\check{A}, \check{B}\right] := \int_X \check{B} \vee \check{A}, \tag{A.1.1}$$

[1] More precisely, differential cohomology is categorical in spirit, so this is a workable model of it, known as Deligne or Deligne–Beilinson cohomology.

The above action action can we written in more user-friendly way as

$$S_{\mathrm{BF}}\left[\check{A},\check{B}\right]=\int_X B_{\mathrm{flat}}\wedge \mathrm{d}A+\int_{\mathrm{PD}[N_A]}B, \tag{A.1.2}$$

where B_{flat} is, as the name suggests the flat part of B, with $F_{B_{\mathrm{flat}}}=0$ and $\mathrm{PD}[N_A]\in \mathrm{H}_{d-p-1}(X)$ is the Poincaré dual of N_A. Here let us note that while differential cohomology is the correct framework to define BF theory, in practice only the torsion part of X can have a non-trivial effect. This is most readily seen in (A.1.2). Since $\mathrm{H}_{d-p-1}(X)\cong \mathrm{Tor}_{d-p-1}(X)\oplus \mathbb{Z}^{\mathrm{b}_{d-p-1}(X)}$, Dirac quantisation implies that the non-torsion part of the second term contributes as $2\pi\times$integer, which is trivial upon exponentiation. Therefore the action is effectively $S=\int_X B\wedge \mathrm{d}A+$torsion, reducing to the usual BF action, (2.2.1), whenever X has no torsion.

The final step to close this intellectual detour is to define the BF path-integral. The formal path-integral measure is

$$\mathrm{d}\mu\left(\check{A},\check{B}\right):=\mathrm{D}\check{A}\,\mathrm{D}\check{B}\,\mathrm{e}^{\mathrm{i}S_{\mathrm{BF}}\left[\check{A},\check{B}\right]}, \tag{A.1.3}$$

where we are summing over differential cohomology elements. The functional measure $\mathrm{D}\check{A}$ instructs us to integrate over all closed $(p+1)$-forms $F_A\in\Omega^{p+1}_{\mathrm{cl}}(X)$, integrate over all p-cochains $A\in \mathrm{C}^p(X)$, and finally sum over all $(p+1)$-cohomology classes, $N_A\in \mathrm{H}^{p+1}(X)$, respecting the constraint $F_A=\mathrm{d}A+N_A$. Moreover, to avoid overcounting we need to identify configurations that differ by a flat $(p-1)$-form field that vanishes on the boundary. The latter we will do simply by dividing out by the volume of these gauge transformations. The integral over F_A is trivial, due to the constraint, so we are left with

$$\int_{\check{\mathrm{H}}^{p+1}(X)}\mathrm{D}\check{A}\,\mathcal{O}[\check{A}]=\sum_{N_A\in \mathrm{H}^{p+1}(X;\mathbb{Z})}\int_{\mathrm{C}^p(X)}\frac{\mathrm{D}A}{\mathrm{vol}\big(\mathcal{G}_p(X)\big)}\mathcal{O}[(A,N_A,\delta A+N_A)], \tag{A.1.4}$$

where $\mathcal{O}[\check{A}]$ is an arbitrary test-functional.

We will only be dealing with a continuous structure group, therefore, we can safely regard A as a p-form. In this case, $\mathcal{G}_p$ is defined recursively as (taking $\mathcal{G}_0=\varnothing$)

$$\mathcal{G}_p(X):=\left\{\alpha\in\Omega^{p-1}(X)\;\middle|\;\iota^*_\partial\alpha=0\right\}\Big/\mathcal{G}_{p-1}(X). \tag{A.1.5}$$

Note that the gauge transformations act in principle also on N_A. Their action has been, however, absorbed into constraining N_A to be in the cohomology rather than in $\mathrm{Z}^{p+1}(X;\mathbb{Z})$. Obviously, the story is the same for $\mathrm{D}\check{B}$, with the appropriate changes in form-degrees.

One important property of the measure (A.1.3), which we use in the main text, is that if we shift $\check{A}$ and $\check{B}$ by some fixed $\check{\alpha}$ and $\check{\beta}$ respectively, the measure transforms as

$$\mathrm{d}\mu\,(\check{A}+\check{\alpha},\check{B}+\check{\beta}) = \mathrm{d}\mu\,(\check{A},\check{B})\;\mathrm{e}^{\mathrm{i}S_{\mathrm{BF}}\left[\check{\alpha},\check{B}\right]}\;\mathrm{e}^{\mathrm{i}S_{\mathrm{BF}}\left[\check{A},\check{\beta}\right]}\;\mathrm{e}^{\mathrm{i}S_{\mathrm{BF}}\left[\check{\alpha},\check{\beta}\right]}. \tag{A.1.6}$$

A.2 Proofs for section 2.3.

Proof of subregion electric-magnetic duality

Subregion electric-magnetic duality as described in subsection 2.3.1 is the statement that the surface operators generating $\mathfrak{A}_{\mathrm{mag}}[R]$ are in 1-1 correspondence with surface operators generating $\mathfrak{A}_{\mathrm{elec}}[R]$. Let us prove it. To do so, it suffices to show the following:

Claim: The pairing

$$\begin{aligned}\mathbb{L}:\mathrm{H}_p(\Sigma)\times\mathrm{H}_{d-p-1}(\Sigma) &\longrightarrow \mathbb{R}\\ (\eta,\sigma) &\longmapsto \int_{\eta\cap\sigma} 1\end{aligned}$$

is non-degenerate restricted to $\operatorname{im} i_p^R \times \operatorname{coker} i_{d-p-1}^{R^{\mathrm{c}}}$, where, as explained in the main text $i_k^R : \mathrm{H}_k(R)\to\mathrm{H}_k(\Sigma)$ is the pushforward in homology of the map that embeds R into Σ and similarly for R^{c}.

Proof. Let us assume the opposite of the claim. That is, either,

(i) There exists an $\hat{\eta}\in\operatorname{im} i_p^R$, such that $\mathbb{L}(\hat{\eta},\sigma)=0$, for all $\sigma\in\operatorname{coker} i_{d-p-1}^R$ or

(ii) There exists a $\hat{\sigma}\in\operatorname{coker} i_p^R$, such that $\mathbb{L}(\eta,\hat{\sigma})=0$, for all $\eta\in\operatorname{im} i_{d-p-1}^R$.

Note that since $\eta\in\operatorname{im} i_p^R$, it can always be homotopically deformed to lie entirely in R. We can therefore restrict our attention to cycles restricted in R. Then $\eta|_R\in\mathrm{H}_p(R)$, while $\sigma|_R\in\mathrm{H}_{d-p-1}(R,\partial R)$.

Let us assume case (i). By Poincaré–Lefschetz duality, $\mathrm{H}_{d-p-1}(R,\partial R)\cong\mathrm{H}^p(R\setminus\partial R)$ $\cong\mathrm{H}^p(\overline{R})$, we have

$$0=\int_{\hat{\eta}|_R\cap\sigma|_R} 1 = \int_{\hat{\eta}|_R}\mathrm{PD}[\sigma|_R],\qquad {}^{\forall}\mathrm{PD}[\sigma|_R]\in\mathrm{H}^p\big(\overline{R}\big), \tag{A.2.1}$$

where PD[·] denotes the Poincaré(–Lefschetz) dual of a cycle. In words, there exists a p-cycle of R that is orthogonal to all p-cohomology classes of $\overline{R}$, which is impossible. The proof of (ii) is wholly similar, with the difference being that the conclusion is that there exists a p-cohomology class in $\overline{R}$ orthogonal to all p-cycles of R, which is again impossible. ■

Proof of claim (2.3.9)

Claim:

$$\left(\mathfrak{A}_{\mathrm{mag}}[R]\right)^{\mathrm{c}} = \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_{\mathrm{i}}}_{\eta^{\mathrm{i}}}, \hat{\mathrm{V}}^{\boldsymbol{v}_{\mathrm{j}}}_{\sigma^{\mathrm{j}}} \;\middle|\; \eta^{\mathrm{i}}, \sigma^{\mathrm{j}} \,\tilde{\in}\, R^{\mathrm{c}}\right\} \equiv \mathfrak{A}_{\mathrm{mag}}[R^{\mathrm{c}}]. \tag{A.2.2}$$

Proof. Consider first the commutant of the operators $\hat{\mathrm{V}}^{\boldsymbol{v}}_{\sigma} \in \mathfrak{A}_{\mathrm{mag}}[R]$. We want to show that $\hat{\mathrm{W}}^{\boldsymbol{w}}_{\eta}$ commutes with all $\hat{\mathrm{V}}^{\boldsymbol{v}}_{\sigma} \in \mathfrak{A}_{\mathrm{mag}}[R]$ iff $\eta \in \operatorname{im} i_p^{R^{\mathrm{c}}}$. Firstly note that by the definition of the algebra, (2.2.34), $\hat{\mathrm{W}}^{\boldsymbol{w}}_{\eta}$ commutes with $\hat{\mathrm{V}}^{\boldsymbol{v}}_{\sigma}$ iff $\mathbb{L}(\eta, \sigma) = 0$. The statement we need to prove reduces, then, to the following:

$$\mathbb{L}(\eta, \sigma) = 0, \;\; {}^{\forall}\sigma \in \operatorname{im} i^{R}_{d-p-1} \quad \text{iff} \quad \eta \in \operatorname{im} i_p^{R^{\mathrm{c}}}. \tag{A.2.3}$$

(⇒) If $\eta \in \operatorname{im} i_p^{R^{\mathrm{c}}}$, it is clear that $\mathbb{L}(\eta, \sigma) = 0$, ${}^{\forall}\sigma \in \operatorname{im} i^{R}_{d-p-1}$, since we can homotopically move σ and η to lie within the interior of R and R^{c} respectively.

(⇐) We will prove the only if direction ad absurdum. For that, suppose that $\eta \notin \operatorname{im} i_p^{R^{\mathrm{c}}}$. Then, since $\mathrm{H}_p(\Sigma) = \operatorname{im} i_p^{R^{\mathrm{c}}} \oplus \operatorname{coker} i_p^{R^{\mathrm{c}}}$, $\eta \in \operatorname{coker} i_p^{R^{\mathrm{c}}}$. Restricted to R then, $\eta|_R \in \mathrm{H}_p(R, \partial R)$ and $\sigma|_R \in \mathrm{H}_{d-p-1}(R)$. By Poincaré–Lefschetz duality, $\mathrm{H}_p(R, \partial R) \cong \mathrm{H}^{d-p-1}(R \setminus \partial R) = \mathrm{H}^{d-p-1}(\overline{R})$ and we have then,

$$0 \overset{!}{=} \mathbb{L}(\eta, \sigma) = \mathbb{L}(\eta|_R\,, \sigma|_R) = \int_{\sigma|_R} \mathrm{PD}[\eta|_R], \;\; {}^{\forall}\sigma \in \operatorname{im} i^{p}_{R}. \tag{A.2.4}$$

where, $\mathrm{PD}[\eta|_R] \in \mathrm{H}^{d-p-1}(\overline{R})$ is the Poincaré–Lefschetz dual of $\eta|_R$. That is to say, $\mathrm{PD}[\eta|_R]$ must be orthogonal to all $(d-p-1)$-cycles of R, which is impossible. Therefore the assumption that $\eta \notin \operatorname{im} i_p^{R^{\mathrm{c}}}$ was absurd.

The proof for the commutant of operators $\hat{\mathrm{W}}^{\boldsymbol{w}}_{\eta} \in \mathfrak{A}_{\mathrm{mag}}[R]$ is wholly similar, concluding thus the proof of the claim. ■

Proof of claim (2.3.15)

Claim:

$$(\mathfrak{A}_{\mathrm{elec}}[R])^{\mathrm{c}} = \mathfrak{U}\left\{\hat{\mathrm{W}}^{\boldsymbol{w}_{\mathrm{i}}}_{\eta^{\mathrm{i}}}, \hat{\mathrm{V}}^{\boldsymbol{v}_{\mathrm{j}}}_{\sigma^{\mathrm{j}}} \;\middle|\; \eta^{\mathrm{i}}, \sigma^{\mathrm{j}} \,\tilde{\not\in}\, R\right\} = \mathfrak{A}_{\mathrm{elec}}[R^{\mathrm{c}}]. \tag{A.2.5}$$

Proof. Consider first the commutant of the operators $\hat{\mathrm{V}}^{\boldsymbol{v}}_{\sigma} \in \mathfrak{A}_{\mathrm{elec}}[R]$. We want to show that $\hat{\mathrm{W}}^{\boldsymbol{w}}_{\eta}$ commutes with all $\hat{\mathrm{V}}^{\boldsymbol{v}}_{\sigma} \in \mathfrak{A}_{\mathrm{elec}}[R]$ iff $\eta \in \operatorname{coker} i_p^{R}$. As before, by the definition of the algebra, (2.2.34), the statement we need to prove reduces to:

$$\mathbb{L}(\eta, \sigma) = 0, \;\; {}^{\forall}\sigma \in \operatorname{coker} i^{R^{\mathrm{c}}}_{d-p-1} \quad \text{iff} \quad \eta \in \operatorname{coker} i_p^{R}. \tag{A.2.6}$$

(⇐) We will first prove the only if direction. This we will prove again by contradiction. For that, suppose that $\eta \notin \operatorname{coker} i_p^{R}$. Then, $\eta \in \operatorname{im} i_p^{R}$. Restricting to R we have $\eta|_R \in \mathrm{H}_p(R)$ and $\sigma|_R \in \mathrm{H}_{d-p-1}(R, \partial R)$. By Poincaré–Lefschetz duality, $\mathrm{H}_{d-p-1}(R, \partial R) \cong \mathrm{H}^p(R \setminus \partial R) = \mathrm{H}^p\left(\overline{R}\right)$ hence,

$$0 \overset{!}{=} \mathbb{L}(\eta, \sigma) = \mathbb{L}(\eta|_R\,, \sigma|_R) = \int_{\eta|_R} \mathrm{PD}[\sigma|_R], \ \forall\sigma \in \operatorname{coker} i^p_R. \tag{A.2.7}$$

where, $\mathrm{PD}[\sigma|_R] \in \mathrm{H}^p\left(\overline{R}\right)$ is the Poincaré–Lefschetz dual of $\sigma|_R$. That is to say, every p-cocycle in the interior of R must be orthogonal to $\eta|_R$, which is impossible. Therefore the assumption that $\eta \notin \operatorname{coker} i^{R^c}_p$ was absurd.

($\Rightarrow$) The if direction follows immediately from subregion electric-magnetic duality, proven above, and the fact that the rank of $\mathbb{L}$ is $\mathrm{b}_p(\Sigma)$ (since it is inverse to $\mathbb{G}_p$ as a matrix). The restrictions of $\mathbb{L}$ to the subspaces $\operatorname{im} i^R_p \times \operatorname{coker} i^{R^c}_{d-p-1}$ and $\operatorname{coker} i^R_p \times \operatorname{im} i^{R^c}_{d-p-1}$ saturate the rank of $\mathbb{L}$, hence the rank of $\mathbb{L}$ restricted to $\operatorname{coker} i^R_p \times \operatorname{coker} i^{R^c}_{d-p-1}$ is zero. In other words for any $\eta \in \operatorname{coker} i^R_p$, $\mathbb{L}(\eta, \sigma) = 0$, $\forall\sigma \in \operatorname{coker} i^{R^c}_{d-p-1}$, concluding the proof.

The proof for the commutant of operators $\hat{\mathrm{W}}^{\boldsymbol{w}}_\eta \in \mathfrak{A}_{\mathrm{elec}}[R]$ is wholly similar, concluding thus the proof of the claim. ■

Counting the magnetic and electric centers

Let us first focus on the magnetic center. Consider, as in the main text, the pushout square that embeds ∂R into Σ through R and R^c:

$$\begin{array}{ccc} \partial R & \xrightarrow{\ j^R\ } & R \\ {\scriptstyle j^{R^c}}\big\downarrow & \searrow^{\ell^{\partial R}} & \big\downarrow{\scriptstyle i^R} \\ R^c & \xrightarrow[\ i^{R^c}\ ]{} & \Sigma. \end{array}$$

Each of these maps induces a push-forward on the homology,

$$j^{R^{(c)}}_k : \mathrm{H}_k\left(\partial R^{(c)}\right) \to \mathrm{H}_k\left(R^{(c)}\right), \tag{A.2.8}$$

$$i^{R^{(c)}}_k : \mathrm{H}_k\left(R^{(c)}\right) \to \mathrm{H}_k(\Sigma). \tag{A.2.9}$$

The operators in the magnetic algebra of R, $\mathfrak{A}_{\mathrm{mag}}[R]$, are generated by surface operators whose cycles lie in the image of i^R_p and i^R_{d-p-1} and similarly for $\mathfrak{A}_{\mathrm{mag}}[R^c]$, replacing R by R^c. The cycles generating the center then lie in $\operatorname{im} i^R_p \cap \operatorname{im} i^{R^c}_p$ and $\operatorname{im} i^R_{d-p-1} \cap \operatorname{im} i^{R^c}_{d-p-1}$. Since R and R^c share only ∂R, it is evident that the cycles generating the center lie in $\operatorname{im}\left(i^R \circ j^R\right)_k \cong \operatorname{im}\left(i^{R^c} \circ j^{R^c}\right)_k =: \operatorname{im} \ell^{\partial R}_k$, as illustrated by the above pushout square. Therefore, the dimension of the magnetic center is

$$\left|\mathfrak{Z}_{\mathrm{mag}}[R]\right| = |\det \mathbb{K}|^{\left(\mathrm{h}^p_{\mathrm{mag}} + \mathrm{h}^{d-p-1}_{\mathrm{mag}}\right)}, \qquad \mathrm{h}^k_{\mathrm{mag}} = \dim \operatorname{im} \ell^{\partial R}_k. \tag{A.2.10}$$

Since $\partial R \subset \Sigma$, we can calculate $\dim \operatorname{im} \ell_k^{\partial R}$ by the long exact sequence of relative homology [335]:

$$\cdots \to \mathrm{H}_k(\partial R) \xrightarrow{\ell_k} \mathrm{H}_k(\Sigma) \xrightarrow{r_k} \mathrm{H}_k(\Sigma, \partial R) \xrightarrow{\delta_{k-1}} \mathrm{H}_{k-1}(\partial R) \to \cdots . \tag{A.2.11}$$

Using exactness of the sequence, we find that

$$\begin{aligned} \dim \operatorname{im} \ell_k &= (-1)^{k-1} \sum_{n=0}^{k-1} (-1)^n \dim \mathrm{H}_n(\partial R) \\ &\quad + (-1)^k \sum_{n=0}^{k} (-1)^n \big(\dim \mathrm{H}_n(\Sigma) - \dim \mathrm{H}_n(\Sigma, \partial R)\big), \end{aligned} \tag{A.2.12}$$

which leads to (2.3.12).

It is useful to pause at this point and illustrate how the bulk dependent terms of (A.2.12) ensure the correct counting of operators when Σ is topologically trivial. For example, consider the case when $\Sigma = \mathbb{S}^2$, R is collection of q disks such that $\partial R = \bigsqcup^q \mathbb{S}^1$, and let $k = 1$. In this case, since there are no non-trivial 1-cycles on $\mathbb{S}^2$, there are no 1-cycle operators in $\mathfrak{A}[\Sigma]$ to count. We should find $\dim \operatorname{im} \ell_1 = 0$. Applying (2.3.12) we find

$$\begin{aligned} \dim \operatorname{im} \ell_1 &= \dim \mathrm{H}_0(\partial R) - \dim \mathrm{H}_0(\mathbb{S}^2) \\ &\quad + \dim \mathrm{H}_0(\mathbb{S}^2, \partial R) + \dim \mathrm{H}_1(\mathbb{S}^2) - \dim \mathrm{H}_1(\mathbb{S}^2, \partial R) \\ &= q - 1 + \dim \mathrm{H}_0(\mathbb{S}^2, \partial R) - \dim \mathrm{H}_1(\mathbb{S}^2, R) \end{aligned} \tag{A.2.13}$$

To calculate the dimensions of the relative homologies, we note that $\mathrm{H}_n(\Sigma, \partial R) \cong \widetilde{\mathrm{H}}_n(\Sigma/\partial R)$ where $\widetilde{\mathrm{H}}_n(\cdot)$ denotes reduced homology and $\Sigma/\partial R$ is the quotient space. By definition of the reduced homology $\dim \widetilde{\mathrm{H}}_0(\mathbb{S}^2/\partial R) = 0$ and we can easily calculate $\dim \widetilde{\mathrm{H}}_1(\mathbb{S}^2/\partial R) = q - 1$, as illustrated in figure A.1. Thus indeed

$$\dim \operatorname{im} \ell_1 = 0 \tag{A.2.14}$$

in this case.

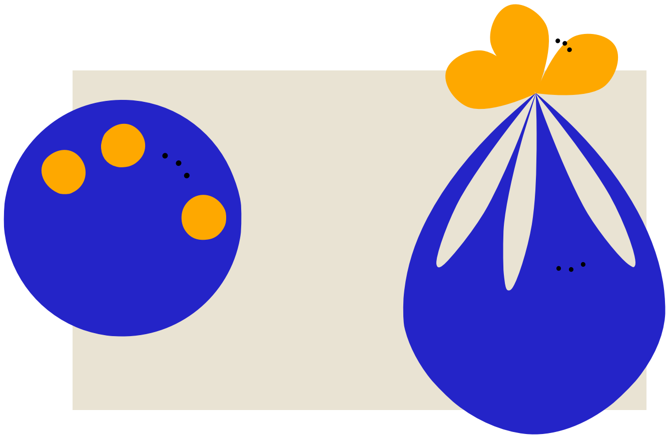

Figure A.1: *(Left) The setup of the above example:* $\Sigma = \mathbb{S}^2$ *and* $R = \bigsqcup^q \mathbb{B}^2$. *(Right)* $\Sigma/\partial R$.

Coming back to the general case, we can now sum the $\mathrm{h}^p_{\mathrm{mag}}$ and $\mathrm{h}^{d-p-1}_{\mathrm{mag}}$, to find

$$\begin{aligned}
\mathrm{h}_{\mathrm{mag}} &:= \dim \operatorname{im} \ell^{\partial R}_p + \dim \operatorname{im} \ell^{\partial R}_{d-p-1} \\
&= 2\sum_{n=0}^{p-1} (-1)^{p-1-n} \dim \mathrm{H}_n(\partial R) + (-1)^{p-1}\chi(\partial R) + \dim \mathrm{H}_p(\Sigma) + (-1)^p \chi(\Sigma) \\
&\quad + \sum_{n=0}^{p} (-1)^{p-1-n} \dim \mathrm{H}_n(\Sigma, \partial R) + \sum_{n=0}^{d-p-1} (-1)^{d-p-n} \dim \mathrm{H}_n(\Sigma, \partial R) \\
&= 2\sum_{n=0}^{p-1} (-1)^{p-1-n} \dim \mathrm{H}_n(\partial R) + (-1)^{p-1}\chi(\partial R) + \dim \mathrm{H}_p(\Sigma) + (-1)^p \chi(\Sigma) \\
&\quad + (-1)^{p-1}\chi(\Sigma, \partial R) - \dim \mathrm{H}_p(\Sigma, \partial R) \\
&\quad + (-1)^{d-p-1} \left(\dim \mathrm{H}_{d-1}(\Sigma, \partial R) - \dim \mathrm{H}_0(\Sigma, \partial R)\right) \\
&= 2\sum_{n=0}^{p-1} (-1)^{p-1-n} \mathrm{b}_n(\partial R) + \big(\mathrm{b}_p(\Sigma) - \dim \mathrm{H}_p(\Sigma, \partial R)\big) \\
&\quad + (-1)^{d-p-1} \left(\dim \mathrm{H}_{d-1}(\Sigma, \partial R) - \dim \mathrm{H}_0(\Sigma, \partial R)\right).
\end{aligned} \tag{A.2.15}$$

which leads to (2.3.13). In the second line above we have used $\mathrm{H}_n(\cdot) \cong \mathrm{H}_{D-n}(\cdot)$ for absolute homologies on D-dimensional compact spaces. In the third line we've used the similar relation for the relative homology $\mathrm{H}_n(\Sigma, \partial R)$ which holds for all degrees except the top and bottom degrees, $\mathrm{H}_{d-1}(\Sigma, \partial R)$ and $\mathrm{H}_0(\Sigma, \partial R)$, respectively[2].

[2]To see that, we note again that $\mathrm{H}_n(\Sigma, \partial R) \cong \widetilde{\mathrm{H}}_n(\Sigma/\partial R)$. For the reduced homology, it holds that $\widetilde{\mathrm{H}}_n(\cdot) \cong \mathrm{H}_n(\cdot)$, whenever $n \neq 0$. On $\Sigma/\partial R$, one can define a non-degenerate pairing $\mathbb{L} : \mathrm{H}_n(\Sigma/\partial R) \times$

Additionally,

$$\chi(\Sigma, \partial R) := \sum_{n=0}^{d-1}(-1)^n \dim \mathrm{H}_n(\Sigma, \partial R)$$

is the relative Euler characteristic of the pair $(\Sigma, \partial R)$, and $\chi(\partial R) - \chi(\Sigma) + \chi(\Sigma, \partial R) = 0$, as it is simply the rank-nullity relation of (A.2.11). Lastly, we've used $\dim \mathrm{H}_n(\cdot) = \dim \mathrm{H}^n(\cdot) \equiv \mathrm{b}_n(\cdot)$ for absolute homologies on compact spaces.

For counting the dimension of $\mathfrak{Z}_{\mathrm{elec}}[R]$, let us show

$$\mathrm{h}^p_{\mathrm{elec}} = \mathrm{h}^{d-p-1}_{\mathrm{mag}}. \tag{A.2.16}$$

That is, the number of p-cycles, η, such that $\eta \not\tilde{\in} R, \not\tilde{\in} R^c$ is equal to the number of $(d-p-1)$-cycles, σ with $\sigma \tilde{\in} R, \tilde{\in} R^c$. To show this we need is suffices to show the following.

Claim: The pairing

$$\mathbb{L}(\eta, \sigma) := \int_{\eta \cap \sigma} 1 \tag{A.2.17}$$

is non-degenerate restricted to $(\operatorname{coker} i^R_p \cap \operatorname{coker} i^{R^c}_p) \times (\operatorname{im} i^R_{d-p-1} \cap \operatorname{im} i^{R^c}_{d-p-1})$.

Proof. Suppose not. That is either

(i) there exists a $\hat{\eta} \in \operatorname{coker} i^R_p \cap \operatorname{coker} i^{R^c}_p$ such that $\mathbb{L}(\hat{\eta}, \sigma) = 0$, for all $\sigma \in \operatorname{im} i^R_{d-p-1} \cap \operatorname{im} i^{R^c}_{d-p-1}$ or

(ii) there exists a $\hat{\sigma} \in \operatorname{im} i^R_{d-p-1} \cap \operatorname{im} i^{R^c}_{d-p-1}$ such that $\mathbb{L}(\eta, \hat{\sigma}) = 0$, for all $\eta \in \operatorname{coker} i^R_p \cap \operatorname{coker} i^{R^c}_p$.

If we suppose (i), then there is no homotopic obstruction to deforming $\hat{\eta}$ to either entirely R or R^c which contradicts it lying in the cokernels of i^R and i^{R^c}. So let us suppose (ii) and pick a $\hat{\sigma}$ satisfying (ii). Because $\mathbb{L}$ is a non-degenerate pairing on $\mathrm{H}_p(\Sigma) \times \mathrm{H}_{d-p-1}(\Sigma)$ there exists a $\check{\eta} \in \mathrm{H}_p(\Sigma)$ such that $\mathbb{L}(\check{\eta}, \hat{\sigma}) \neq 0$. By assumption, $\check{\eta}$ must lie in either $\operatorname{im} i^R_p$ or $\operatorname{im} i^{R^c}_p$ and so is completely deformable within Σ to either R or R^c. But then $\mathbb{L}(\check{\eta}, \hat{\sigma})$ must actually vanish because $\hat{\sigma}$, which lies in $\operatorname{im} i^R_{d-p-1} \cap \operatorname{im} i^{R^c}_{d-p-1}$, can be deformed to the respective complementary region so that it has no intersection with $\check{\eta}$. This contradiction completes the proof that $\mathbb{L}$ is non-degenerate on $(\operatorname{coker} i^R_p \cap \operatorname{coker} i^{R^c}_p) \times (\operatorname{im} i^R_{d-p-1} \cap \operatorname{im} i^{R^c}_{d-p-1})$. ■

Similar arguments show that $\mathrm{h}^{d-p-1}_{\mathrm{elec}} = \mathrm{h}^p_{\mathrm{mag}}$. As a consequence

$$|\mathfrak{Z}_{\mathrm{elec}}| = |\mathfrak{Z}_{\mathrm{mag}}|. \tag{A.2.18}$$

$\mathrm{H}_{D-n}(\Sigma/\partial R) \to \mathbb{R}$, as $\mathbb{L}(\alpha, \beta) := \int_{\alpha\cap\beta} 1$. This renders $\mathrm{H}_n(\Sigma/\partial R) \cong \mathrm{H}_{D-n}(\Sigma/\partial R)$ whenever $n \notin \{0, D\}$ and shows the desired statement for relative homology.

A.3 Decomposition of the reduced density matrix

Given a state

$$\rho = \sum_{\mathfrak{v},\mathfrak{v}'} \rho_{\mathfrak{v},\mathfrak{v}'} \left|\mathfrak{v}\middle\rangle\!\middle\langle\mathfrak{v}'\right|, \qquad \rho^*_{\mathfrak{v},\mathfrak{v}'} = \rho_{\mathfrak{v}',\mathfrak{v}}, \qquad \sum_{\mathfrak{v}} \rho_{\mathfrak{v},\mathfrak{v}} = 1, \tag{A.3.1}$$

we wish to write down the reduced density matrix, ρ_R, corresponding to the subregion algebra, $\mathfrak{A}_{\text{mag}}[R]$. The general ansatz for this reduced density matrix is given by

$$\rho_R = \sum_{\{\boldsymbol{w}^R_{\mathrm{i}}\}} \sum_{\{\boldsymbol{v}^R_{\mathrm{j}}\}} C_{\{\boldsymbol{w}^R_{\mathrm{i}}\},\{\boldsymbol{v}^R_{\mathrm{j}}\}} \prod_{\eta^{\mathrm{i}}\tilde{\in} R} \hat{\mathrm{W}}^{\boldsymbol{w}^R_{\mathrm{i}}}_{\eta^{\mathrm{i}}} \prod_{\sigma^{\mathrm{j}}\tilde{\in} R} \hat{\mathrm{V}}^{\boldsymbol{v}^R_{\mathrm{j}}}_{\sigma^{\mathrm{j}}}, \tag{A.3.2}$$

for some coefficients C. We notate the charges $\boldsymbol{w}^R$ and $\boldsymbol{v}^R$ to indicate that they are for surface operators deformable into R. Given that $\hat{\mathrm{W}}^{\boldsymbol{w}_{\mathrm{i}}=\boldsymbol{0}}_{\eta^{\mathrm{i}}}$ and $\hat{\mathrm{V}}^{\boldsymbol{v}_{\mathrm{j}}=\boldsymbol{0}}_{\sigma^{\mathrm{j}}}$ both act as the identity, it will be notationally useful to extend $\{\boldsymbol{w}^R_{\mathrm{i}}\}$ and $\left\{\boldsymbol{v}^R_{\mathrm{j}}\right\}$ to full charge vectors $\mathfrak{w}^R \in (\Lambda_A)^{\mathrm{h}}$ and $\mathfrak{v}^R \in (\Lambda_B)^{\mathrm{h}}$, respectively, with zero entries for all cycles not deformable into R:

$$\boldsymbol{w}^R_{\mathrm{i}} = \boldsymbol{v}^R_{\mathrm{j}} = 0 \qquad \forall \eta^{\mathrm{i}}, \sigma^{\mathrm{j}} \tilde{\notin} R. \tag{A.3.3}$$

We then write

$$\rho_R = \sum_{\mathfrak{w}^R} \sum_{\mathfrak{v}^R} C_{\mathfrak{w}^R,\mathfrak{v}^R}\, \hat{\mathrm{W}}^{\mathfrak{w}^R}\, \hat{\mathrm{V}}^{\mathfrak{v}^R}. \tag{A.3.4}$$

Hermiticity and unit-trace (with respect to $\mathscr{H}_\Sigma$) imply

$$C^*_{\mathfrak{w}^R,\mathfrak{v}^R} = C_{-\mathfrak{w}^R,-\mathfrak{v}^R}\, \mathrm{e}^{\mathrm{i}\Gamma(\mathfrak{v}^R,\mathfrak{w}^R)}, \tag{A.3.5}$$

and

$$C_{\boldsymbol{0},\boldsymbol{0}} = (\dim\mathscr{H}_\Sigma)^{-1} = |\det\mathbb{K}|^{-\mathrm{b}_p(\Sigma)} \equiv \mathscr{N}_\Sigma^{-1}, \tag{A.3.6}$$

respectively. We can solve for the coefficients $C_{\mathfrak{w}^R,\mathfrak{v}^R}$ in terms of the coefficients of the state, $\rho_{\mathfrak{v},\mathfrak{v}'}$, by enforcing

$$\mathrm{Tr}\left(\rho_R\, \mathcal{O}_R\right) = \mathrm{Tr}\left(\rho\, \mathcal{O}_R\right) = \sum_{\mathfrak{v},\mathfrak{v}'} \rho_{\mathfrak{v},\mathfrak{v}'} \left\langle \mathfrak{v}' \middle| \mathcal{O}_R \middle| \mathfrak{v} \right\rangle, \tag{A.3.7}$$

for all $\mathcal{O}_R \in \mathfrak{A}_{\text{mag}}[R]$. By considering a generic element $\mathcal{O}_R = \hat{\mathrm{W}}^{\hat{\mathfrak{w}}^R}\hat{\mathrm{V}}^{\hat{\mathfrak{v}}^R}$ (for fixed $\hat{\mathfrak{w}}^R$ and $\hat{\mathfrak{v}}^R$) we can easily work out

$$C_{\mathfrak{w}^R,\mathfrak{v}^R} = \mathscr{N}_\Sigma^{-1} \sum_{\bar{\mathfrak{v}}} \rho_{\bar{\mathfrak{v}},\bar{\mathfrak{v}}-\mathfrak{v}^R} \mathrm{e}^{-\mathrm{i}\Gamma(\bar{\mathfrak{v}},\mathfrak{w}^R)}. \tag{A.3.8}$$

Thus we can write a generic reduced density matrix as

$$\rho_R = \mathscr{N}_\Sigma^{-1} \sum_{\mathfrak{w}^R} \sum_{\mathfrak{v}^R} \sum_{\bar{\mathfrak{v}}} \rho_{\bar{\mathfrak{v}},\bar{\mathfrak{v}}-\mathfrak{v}^R} \mathrm{e}^{-\mathrm{i}\Gamma(\bar{\mathfrak{v}},\mathfrak{w}^R)}\, \hat{\mathrm{W}}^{\mathfrak{w}^R}\, \hat{\mathrm{V}}^{\mathfrak{v}^R}. \tag{A.3.9}$$

In particular for a pure state (as in subsection 2.3.4),

$$\rho = \psi_{\mathfrak{v}} \psi^*_{\mathfrak{v}'} \left|\mathfrak{v}\middle\rangle\!\middle\langle\mathfrak{v}'\right|, \tag{A.3.10}$$

the reduced density matrix is written as

$$\rho_R = \mathcal{N}_\Sigma^{-1} \sum_{\mathfrak{w}^R} \sum_{\mathfrak{v}^R} \sum_{\bar{\mathfrak{v}}} \psi_{\bar{\mathfrak{v}}} \psi^*_{\bar{\mathfrak{v}}-\mathfrak{v}^R} \mathrm{e}^{-\mathrm{i}\Gamma\left(\bar{\mathfrak{v}},\mathfrak{w}^R\right)} \,\hat{\mathrm{W}}^{\mathfrak{w}^R} \,\hat{\mathrm{V}}^{\mathfrak{v}^R}. \tag{A.3.11}$$

B Appendices for chapter 3

B.1 Partition functions of higher-form gauge theories

In this appendix, we provide details on the computations of the edge-mode partition function of Section 3.3.

B.1.1 The partition function of $(p-1)$-form Maxwell theory

First we will quantise, by performing the Euclidean path integral, $(p-1)$-form Maxwell theory on $M_{d-1} = \mathbb{S}^1_\beta \times Y_{d-2}$, where β is the radius of $\mathbb{S}^1$. Its action is simply

$$S[a] = \frac{1}{2}k\mu^\Delta \|f\|^2 := \frac{1}{2}k\mu^\Delta \int_M f \wedge \star f, \tag{B.1.1}$$

where k is a dimensionless constant, μ is an energy scale and $\Delta = d-1-2p$. $f \in \Omega^p_{\text{cl}}(M)$ is a closed p-form on M and is identified with the curvature of a $(p-1)$-form gauge field $a \in \Omega^{p-1}(M)$. As such, it can be Hodge-decomposed as $f = f_\text{h} + \mathrm{d}a$, where $f_\text{h} \in \text{Harm}^p(M)$ is a harmonic form, which we can (uniquely) choose to be orthogonal to $\mathrm{d}a$. f_h labels the instantons of the theory. Moreover, Dirac quantisation condition, implies that f_h takes values in cohomology with coefficients in $2\pi\mathbb{Z}$: $f_\text{h} \in \text{H}^p(M; 2\pi\mathbb{Z})$. The partition function of $(p-1)$-form Maxwell theory on M takes the form

$$\mathcal{Z}^{(p-1)}_{\text{Maxwell}}[M] = \mathcal{Z}_{\text{inst}}[M]\mathcal{Z}_{\text{osc}}[M], \tag{B.1.2}$$

where

$$\mathcal{Z}_{\text{inst}}[M] := \sum_{f_\text{h} \in \text{H}^p(M;2\pi\mathbb{Z})} \exp\left(-\frac{1}{2}k\mu^\Delta \|f_\text{h}\|^2\right) \tag{B.1.3}$$

$$\mathcal{Z}_{\text{osc}}[M] := \int \frac{\mathrm{D}a}{\text{vol}(\mathcal{G}_{p-1})} \exp\left(-\frac{1}{2}k\mu^\Delta \|\mathrm{d}a\|^2\right). \tag{B.1.4}$$

In the above, $\mathcal{G}_{p-1}$ is the group of reducible gauge transformations, characteristic of higher-gauge theories, generated by shifts by closed $(p-1)$ forms, modulo their own gauge transformations.

In order to evaluate the above partition function, it will be helpful to introduce the topological basis of harmonic forms, $\left\{\tau_{\mathrm{i}}^{(k)}\right\}_{\mathrm{i}=0}^{\mathrm{b}_k(M)}$ defined as in (3.2.14). On $M = \mathbb{S}^1_\beta \times Y$ this is

$$\left\{\tau_{\mathrm{i}}^{(k)}\right\}_{\mathrm{i}=0}^{\mathrm{b}_k(M)} = \left\{\left\{\overline{\tau}_{\mathrm{i}}^{(k)}\right\}_{\mathrm{i}=0}^{\mathrm{b}_k(Y)}, \left\{\sigma \wedge \overline{\tau}_{\mathrm{i}}^{(k-1)}\right\}_{\mathrm{i}=0}^{\mathrm{b}_k(Y)}\right\}, \tag{B.1.5}$$

where $\sigma := (2\pi\beta)^{-1}\,\mathrm{vol}_\beta$ is the unique normalised harmonic 1-form on the $\mathbb{S}^1_\beta$. The Gram matrix for this basis:

$$\left[\mathbb{G}_k^M\right]_{\mathrm{ij}} := \int_M \tau_{\mathrm{i}}^{(k)} \wedge \star\tau_{\mathrm{j}}^{(k)}, \tag{B.1.6}$$

becomes, with the above decomposition

$$\mathbb{G}_k^M = \begin{pmatrix} 2\pi\beta\mathbb{G}_k^Y & \\ & (2\pi\beta)^{-1}\mathbb{G}_{k-1}^Y \end{pmatrix}, \tag{B.1.7}$$

with $\mathbb{G}_k^Y$ being the analogous Gram matrix for Y, defined over the basis $\left\{\overline{\tau}_{\mathrm{i}}^{(k)}\right\}_{\mathrm{i}=0}^{\mathrm{b}_k(M)}$.

The instanton, f_{h}, can be therefore written as

$$f = 2\pi n^{\mathrm{i}}\overline{\tau}_{\mathrm{i}}^{(p)} + 2\pi m^{\mathrm{j}}\sigma \wedge \overline{\tau}_{\mathrm{j}}^{(p)}, \tag{B.1.8}$$

with $n^{\mathrm{i}}, m^{\mathrm{j}} \in \mathbb{Z}$. Combining n^{i} and m^{j} into vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ of length $\mathrm{b}_p(Y)$ and $\mathrm{b}_{p-1}(Y)$ respectively, the instanton contribution, (B.1.3), reads:

$$\mathcal{Z}_{\mathrm{inst}}[M] = \sum_{\substack{\boldsymbol{n}\in\mathbb{Z}^{\mathrm{b}_p(Y)}\\ \boldsymbol{m}\in\mathbb{Z}^{\mathrm{b}_{p-1}(Y)}}} \exp\left(-\pi\mu^\Delta k\left(2\pi\beta\boldsymbol{n}\cdot\mathbb{G}_p^Y\cdot\boldsymbol{n} + \frac{1}{2\pi\beta}\boldsymbol{m}\cdot\mathbb{G}_{p-1}^Y\cdot\boldsymbol{m}\right)\right). \tag{B.1.9}$$

We can Poisson resum the sum over $\boldsymbol{m}$ to obtain

$$\mathcal{Z}_{\mathrm{inst}}[M] = (2\pi\beta)^{\frac{1}{2}\mathrm{b}_{p-1}(Y)}\det\left(k\mu^\Delta\mathbb{G}_{p-1}^Y\right)^{-\frac{1}{2}}\Theta\left[q; k\,\widetilde{\mathbb{G}}_p^Y\right]\Theta\left[q; k^{-1}\,\widetilde{\mathbb{G}}_{d-p-1}^Y\right], \tag{B.1.10}$$

where $q := \mathrm{e}^{-\beta\mu}$, the Siegel-type Theta functions are defined as in (3.3.29), the tilded versions of the Gram matrices are their dimensionless incarnations, having absorbed the relevant powers of μ, and we have also used the duality $\mathbb{G}_k^Y = \left[\mathbb{G}_{d-2-k}^Y\right]^{-1}$.

Coming to the oscillator contribution, this is given by [220, 236, 336]

$$\mathcal{Z}_{\mathrm{osc}}[M] = \prod_{k=0}^{p-1}\det\left(\mu^\Delta k\,\mathbb{G}_k^M\right)^{\frac{1}{2}(-1)^{p-k-1}}\det{}'(\triangle_k)^{\frac{k}{2}(p-k)(-1)^{p-k}}, \tag{B.1.11}$$

where $\triangle_k := \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}$ is the Hodge Laplacian on k-forms on M. and $\det'$ excludes zero-modes. These are packaged in the first factor, the alternating product over $\mathbb{G}_k^M$. In our case, $M = \mathbb{S}^1_\beta \times Y$, we can vastly simplify this expression. Applying (B.1.7) in the zero-mode product we get

$$\prod_{k=0}^{p-1} \det\left(\mu^\Delta k\, \mathbb{G}_k^M\right)^{\frac{1}{2}(-1)^{p-k-1}} = (2\pi\beta)^{-\frac{1}{2}\mathrm{b}_{p-1}(Y)+(-1)^{p-1}\sum_{n=0}^{p-2}(-1)^n \mathrm{b}_n(Y)} \times \times \det\left(k\mu^\Delta \mathbb{G}^Y_{p-1}\right)^{\frac{1}{2}}. \tag{B.1.12}$$

Moving to the contribution of the Laplacian, note that its spectrum on $M = \mathbb{S}^1_\beta \times Y$ is

$$\operatorname{spec}(\triangle_k) = \left\{\left(\frac{2\pi \mathsf{n}}{\beta}\right)^2 + \lambda_{\mathsf{n}_k},\ \mathsf{n} \in \mathbb{Z},\ \mathsf{n}_k \in \mathcal{N}_k\right\} \cup \left\{\left(\frac{2\pi \mathsf{n}}{\beta}\right)^2 + \lambda_{\mathsf{n}_{k-1}},\ \mathsf{n} \in \mathbb{Z},\ \mathsf{n}_{k-1} \in \mathcal{N}_{k-1}\right\}, \tag{B.1.13}$$

where $\mathcal{N}_k$ is the index-set of the eigenvalues of the k-form Laplacian on Y. From this it follows that the determinant of the Laplacian takes the form

$$\begin{aligned}\det{}'\triangle_k = &\prod_{\substack{\mathsf{n}\in\mathbb{Z} \\ \left(\frac{2\pi\mathsf{n}}{\beta}\right)^2+\lambda_{\mathsf{n}_k}\neq 0}} \prod_{\mathsf{n}_k\in\mathcal{N}_k} \left(\left(\frac{2\pi \mathsf{n}}{\beta}\right)^2 + \lambda_{\mathsf{n}_k}\right) \times \prod_{\substack{\mathsf{n}\in\mathbb{Z} \\ \left(\frac{2\pi\mathsf{n}}{\beta}\right)^2+\lambda_{\mathsf{n}_{k-1}}\neq 0}} \prod_{\mathsf{n}_{k-1}\in\mathcal{N}_{k-1}} \left(\left(\frac{2\pi \mathsf{n}}{\beta}\right)^2 + \lambda_{\mathsf{n}_{k-1}}\right) = \\ &= \beta^{2(\mathrm{b}_k(Y)+\mathrm{b}_{k-1}(Y))}\left[\prod_{\mathsf{n}_k\in\mathcal{N}_k^*} \sinh^2\left(\frac{1}{2}\beta\sqrt{\lambda_{\mathsf{n}_k}}\right)\right] \times \qquad \text{(B.1.14)}\\ &\times \prod_{\mathsf{n}_{k-1}\in\mathcal{N}_{k-1}^*} \sinh^2\left(\frac{1}{2}\beta\sqrt{\lambda_{\mathsf{n}_{k-1}}}\right), \qquad \text{(B.1.15)}\end{aligned}$$

where we zeta-regularised the infinite products[1], using the spectral zeta-function, $\zeta_Y^{(k)}(s)$, for the k-form Laplacian on Y, and $\mathcal{N}_k^*$ excludes the zero-modes. Lastly, note that $\mathcal{N}_k^* = \mathcal{N}_k^\perp \oplus \mathcal{N}_k^\parallel = \mathcal{N}_k^\perp \oplus \mathcal{N}_{k-1}^\perp$, meaning that the non-zero spectrum of the full Laplacian splits into a direct sum of the non-zero spectrum of the transversal Laplacian plus that of the longitudinal Laplacian. The latter is equal to the spectrum

[1]Since we are interested in computing a path integral, we throw purely numerical coefficients, in particular various floating powers of 2, in this and the following expressions.

of the transversal Laplacian acting on $(k-1)$-forms. With that, if we denote

$$\begin{aligned} S_k &:= \prod_{\mathsf{n}_k \in \mathcal{N}_k^*} \sinh^2\left(\frac{1}{2}\beta\sqrt{\lambda_{\mathsf{n}_k}}\right) \\ &= \prod_{\mathsf{n}_k \in \mathcal{N}_k^\perp} \sinh^2\left(\frac{1}{2}\beta\sqrt{\lambda_{\mathsf{n}_k}}\right) \times \prod_{\mathsf{n}_k \in \mathcal{N}_{k-1}^\perp} \sinh^2\left(\frac{1}{2}\beta\sqrt{\lambda_{\mathsf{n}_{k-1}}}\right) \\ &=: S_k^\perp S_{k-1}^\perp, \end{aligned}$$

we have that

$$\begin{aligned} \prod_{k=0}^{p-1} \left(\det{}'\triangle_k\right)^{\frac{1}{2}(p-k)(-1)^{p-k}} &= \beta^{(-1)^p \sum_{k=0}^{p-1}(-1)^k \mathrm{b}_k(Y)} \prod_{k=0}^{p-1} \left(S_k S_{k-1}\right)^{\frac{\kappa}{2}(p-k)(-1)^{p-k}} \\ &= \beta^{(-1)^p \sum_{k=0}^{p-1}(-1)^k \mathrm{b}_k(Y)} \prod_{k=0}^{p-1} S_k^{\frac{\kappa}{2}(-1)^{p-k}} \\ &= \beta^{(-1)^p \sum_{k=0}^{p-1}(-1)^k \mathrm{b}_k(Y)} \left(S_{p-1}^\perp\right)^{-\frac{\kappa}{2}} \\ &= \beta^{(-1)^p \sum_{k=0}^{p-1}(-1)^k \mathrm{b}_k(Y)} \left(\eta_Y^{(p-1)}[q]\right)^{-2}, \end{aligned} \tag{B.1.16}$$

where we used (3.3.24), to express the result in terms of the $(p-1)$-form η function associated with Y. Putting everything together, the oscillator contribution to the partition function is

$$\mathcal{Z}_{\mathrm{osc}}[M] = (2\pi\beta)^{\frac{1}{2}\mathrm{b}_{p-1}(Y)} \det\left(k\mu^\Delta \mathbb{G}_{p-1}^Y\right)^{\frac{1}{2}} \left(\eta_Y^{(p-1)}[q]\right)^{-2}, \tag{B.1.17}$$

Combining with the instanton contribution, all the factors of β and k exactly cancel and we get:

$$\mathcal{Z}_{\mathrm{Maxwell}}^{(p-1)}\left[\mathbb{S}_\beta^1 \times Y\right] = \frac{\Theta\left[q; k\,\widetilde{\mathbb{G}}_p^Y\right]}{\eta_Y^{(p-1)}[q]} \frac{\Theta\left[q; k^{-1}\,\widetilde{\mathbb{G}}_{p-1}^Y\right]}{\eta_Y^{(d-p-2)}[q]}, \tag{B.1.18}$$

where we also used the fact that the eta function associated with $(p-1)$-forms is equal to that for $(d-p-2)$-forms, in order to write the result in a symmetric fashion.

B.1.2 The partition function of chiral mixed Maxwell theory

Now that we have all the details of the vanilla, higher-form gauge theory under our belt, we will quantise the chiral mixed Maxwell theory on $M = \mathbb{S}_\beta^1 \times Y$. This is a theory with action

$$S[\mathcal{A}] = \frac{k\mu^\Delta}{2}\|\mathcal{A}\|^2 = \frac{k\mu^{-\Delta}}{2}\|\mathcal{B}\|^2, \tag{B.1.19}$$

where $\mathcal{A}$ and $\mathcal{B}$ are closed p- and $(d-p-1)$-forms, respectively, satisfying the generalised chiral condition $\mathcal{B} + \mu^{\Delta}(-1)^{(d-p-1)(p+1)} \star \mathcal{A} \stackrel{!}{=} 0$. In the main text we argued that this leads to fractionally quantised fluxes along cycles involving the thermal circle. Let us elaborate on that. Dirac quantisation imposes that magnetic and electric fluxes are quantised as:

$$\Phi^{\mathcal{A}}_{\text{mag}}[\eta] := \frac{1}{2\pi}\int_{\eta} \mathcal{A} \in \mathbb{Z}, \qquad \Phi^{\mathcal{A}}_{\text{elec}}[\gamma] := \frac{k}{2\pi}\int_{\gamma} \star\mathcal{A} \in \mathbb{Z}, \tag{B.1.20}$$

$$\Phi^{\mathcal{B}}_{\text{mag}}[\gamma] := \frac{1}{2\pi}\int_{\gamma} \mathcal{B} \in \mathbb{Z}, \qquad \Phi^{\mathcal{B}}_{\text{elec}}[\eta] := \frac{k}{2\pi}\int_{\eta} \star\mathcal{B} \in \mathbb{Z}, \tag{B.1.21}$$

for all cycles $\eta \in \mathrm{H}_p(Y)$ and $\gamma \in \mathrm{H}_{d-p-1}(Y)$. Note that the electric fluxes are defined including a factor of the coupling constant, here k [274]. The generalised chiral condition then requires that magnetic fluxes of $\mathcal{A}$ along cycles of the form $\mathbb{S}^1_\beta \times \widetilde{\eta}$, for $\widetilde{\eta} \in \mathrm{H}_{p-1}(Y)$ are quantised in units of $\frac{1}{k}$. Consequently, decomposing $\mathcal{A}$, as in the Maxwell story, as $\mathcal{A} = \mathcal{A}_{\mathrm{h}} + \mathrm{d}a$, where $\mathcal{A}_{\mathrm{h}}$ is harmonic and then further decomposing $\mathcal{A}_{\mathrm{h}}$ in terms of the basis (B.1.5), we get

$$\mathcal{A}_{\mathrm{h}} = 2\pi n^{\mathrm{i}} \overline{\tau}^{(p)}_{\mathrm{i}} + \frac{2\pi}{k} m^{\mathrm{j}} \sigma \wedge \overline{\tau}^{(p)}_{\mathrm{j}}. \tag{B.1.22}$$

From here on, the story is completely analogous to the vanilla Maxwell case. The instanton contribution is

$$\mathcal{Z}_{\text{inst}}[M] = (2\pi\beta)^{\frac{1}{2}\mathrm{b}_{p-1}(Y)} \det\!\left(k\mu^{\Delta}\mathbb{G}^{Y}_{p-1}\right)^{-\frac{1}{2}} \Theta\!\left[q; k\,\widetilde{\mathbb{G}}^{Y}_{p}\right] \Theta\!\left[q; k\,\widetilde{\mathbb{G}}^{Y}_{d-p-1}\right]. \tag{B.1.23}$$

The oscillators are completely unaffected by this story. It is cleaner to write their contribution in terms of the magnetic photon, $\widetilde{a}$, such that $\mathrm{d}\widetilde{a} = k\mu^{-\Delta} \star \mathrm{d}a$. It is, of course, not necessary to do that; this way we will just avoid the necessity of adding a counterterm. Doing so results in an oscillator contribution as

$$\mathcal{Z}_{\text{osc}}[M] = (2\pi\beta)^{-\frac{1}{2}\mathrm{b}_{p-1}(Y)} \det\!\left(k\mu^{\Delta}\mathbb{G}^{Y}_{p-1}\right)^{\frac{1}{2}} \left(\upeta^{(p-1)}_{Y}[q]\right)^{-2}. \tag{B.1.24}$$

In total, the partition function reads:

$$\mathcal{Z}^{(p-1,d-p-2)}_{\text{chiral}}\left[\mathbb{S}^1_\beta \times Y\right] = \frac{\Theta\!\left[q; k\,\widetilde{\mathbb{G}}^{Y}_{p}\right]}{\upeta^{(p-1)}_{Y}[q]} \, \frac{\Theta\!\left[q; k\,\widetilde{\mathbb{G}}^{Y}_{d-p-1}\right]}{\upeta^{(d-p-2)}_{Y}[q]}. \tag{B.1.25}$$

B.2 Laplacians, zeta functions, and heat kernels

In this appendix, we discuss some spectral properties of the Hodge Laplacian.

Let (Y, g) be a closed, compact, $(d-2)$-dimensional Riemannian manifold and let $\triangle_\ell := \mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger$ be the Hodge Laplacian acting on $\Omega^\ell(Y)$. Denoting by spec′ the non-zero-mode spectrum, it holds that

$$\begin{aligned}\mathrm{spec}'\left(\triangle_\ell, X\right) &= \mathrm{spec}'\left(\triangle_\ell\Big|_{\ker \mathrm{d}^\dagger}, Y\right) \oplus \mathrm{spec}'\left(\triangle_\ell\Big|_{\ker \mathrm{d}}, Y\right) = \\ &= \mathrm{spec}'\left(\triangle_\ell\Big|_{\ker \mathrm{d}^\dagger}, Y\right) \oplus \mathrm{spec}'\left(\triangle_{\ell-1}\Big|_{\ker \mathrm{d}^\dagger}, Y\right).\end{aligned} \tag{B.2.1}$$

Therefore, we will focus on the spectral properties of the operator $\triangle_\ell\Big|_{\ker \mathrm{d}^\dagger}$ which we will call $\Box_\ell$. Consider the eigenvalue equation for $\Box_\ell$:

$$\Box_\ell \varphi_{\mathsf{n}} = \lambda_{\mathsf{n}} \varphi_{\mathsf{n}}, \qquad \mathsf{n} \in \mathcal{N}_\ell, \tag{B.2.2}$$

with $\mathcal{N}_\ell$ being a countable set. We can define the spectral zeta function of $\Box_\ell$:

$$\zeta_W^{(\ell)}(s) := \sum_{\mathsf{n} \in \mathcal{N}_\ell^*} \lambda_{\mathsf{n}}^{-s}, \qquad s \in \mathbb{C}, \text{ with } \mathrm{Re}\, s > \frac{d-2}{2}, \tag{B.2.3}$$

where $\mathcal{N}_\ell^* := \{\mathsf{n} \in \mathcal{N}_\ell \mid \lambda_{\mathsf{n}} \neq 0\}$. For $\ell = 0$ this reduces to the well-known Minakshisundaram–Pleijel zeta function [337].

It will be useful to introduce the heat kernel:

$$\mathcal{K}_Y^{(\ell)}(t) := \mathrm{tr}\left(\mathrm{e}^{-t\Box_p}\right) = \sum_{\mathsf{n} \in \mathcal{N}_\ell} \mathrm{e}^{-t\lambda_{\mathsf{n}}}. \tag{B.2.4}$$

The heat kernel admits a small-t expansion, as

$$\mathcal{K}_Y^{(\ell)}(t) = \frac{1}{(4\pi t)^{\frac{d-2}{2}}} \sum_{k=0}^{\infty} \mathcal{I}_k^{(\ell)} t^k, \tag{B.2.5}$$

where $\mathcal{I}_k^{(\ell)}$ are given by integrals of geometric data of W. We can now invoke a Mellin transform in order to write

$$\lambda_{\mathsf{n}}^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-t\lambda_{\mathsf{n}}} t^{s-1}, \tag{B.2.6}$$

and hence

$$\zeta_W^{(\ell)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1} \left(\mathcal{K}_Y^{(\ell)}(t) - \dim\ker \Box_\ell\right) =: \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, t^{s-1}\, \widetilde{\mathcal{K}}_Y^{(\ell)}(t). \tag{B.2.7}$$

where $\dim\ker\Box_\ell$ counts the number of of zero-modes.

We will now use the following trick. We will split the above integral as follows.

$$\zeta_W^{(\ell)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, \Theta(t)\, t^{s-1}\, \widetilde{\mathcal{K}}_Y^{(\ell)}(t) + \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}t\, (1-\Theta(t))\, t^{s-1}\, \widetilde{\mathcal{K}}_Y^{(\ell)}(t), \tag{B.2.8}$$

where $\Theta(t)$ is the Heaviside Theta function.[2] The merit of this is that the second integral in (B.2.8) is now manifestly analytic in s, while the first integral is explicitly computable. Plugging in the asymptotic expansion, (B.2.5), of $\mathcal{K}_Y^{(\ell)}$, into the first integral, one gets

$$\zeta_Y^{(\ell)}(s) = \sum_{k=0}^{\infty} \frac{\mathcal{I}_k^{(\ell)}}{(4\pi)^{\frac{d-2}{2}}\Gamma(s)} \frac{1}{s - \frac{d-2}{2} + k} + \frac{\dim\ker\Box_\ell}{s\Gamma(s)} - \frac{H(s)}{\Gamma(s)}, \tag{B.2.9}$$

where $H(s)$ is analytic in s.

From this we get the pole structure of the spectral zeta function. It has simple poles at $s = \frac{d-2}{2} - k$, with $k \in \mathbb{Z}_{\geqslant 0}$, with residue

$$\operatorname{Res}\zeta_Y^{(\ell)}\left(\frac{d-2}{2} - k\right) = \frac{\mathcal{I}_k^{(\ell)}}{(4\pi)^{\frac{d-2}{2}}\Gamma\left(\frac{d-2}{2} - k\right)}. \tag{B.2.10}$$

When d is even, the spectral zeta-function has poles only at $s > 0$, i.e. $k \in \left\{0, 1, \cdots \frac{d-2}{2} - 1\right\}$, because of the Gamma function in the denominator.

Moreover, notice that at $s = 0$, we have

$$\zeta_Y^{(\ell)}(0) = \begin{cases} \mathcal{I}_{\frac{d-2}{2}}^{(\ell)} - \dim\ker\Box_\ell, & \text{when } d \text{ is even,} \\ \qquad - \dim\ker\Box_\ell, & \text{when } d \text{ is odd.} \end{cases} \tag{B.2.11}$$

This follows immediately from $\Gamma(s) \sim \frac{1}{s}$ as $s \to 0$, since the analytic piece $H(s)/\Gamma(s)$ vanishes as $s \to 0$; the topological piece, $-\dim\ker\Box_\ell$, is universal, and the first term only contributes when ℓ is even. Finally we have that $\dim\ker\Box_\ell = \dim\ker\Delta_\ell = \mathrm{b}_\ell(Y)$, so

$$\zeta_Y^{(\ell)}(0) = \begin{cases} \mathcal{I}_{\frac{d-2}{2}}^{(\ell)} - \mathrm{b}_\ell(Y), & \text{when } d \text{ is even,} \\ \qquad - \mathrm{b}_\ell(Y), & \text{when } d \text{ is odd.} \end{cases} \tag{B.2.12}$$

[2]For continuity purposes, one could use a smoothened version of the Heaviside Theta function.

C Appendices for chapter 4

C.1 The transversal Laplacian

In this appendix we will give some outline some spectral properties of the transversal Laplacian on three dimensional manifolds, which we denote generically by Σ, and, subsequently, explicitly construct its spectrum on $\Sigma_r = \mathbb{S}^2_r \times \mathbb{S}^1$.

C.1.1 Spectral properties

First, by transversal Laplacian, on p-forms, on a d-dimensional, closed manifold M, we mean:

$$\square_p^M := \triangle\Big|_{\Omega^p(M)\cap\ker \mathrm{d}^\dagger}, \tag{C.1.1}$$

where $\triangle$ denotes the Hodge Laplacian, $\triangle = \left(\mathrm{d}+\mathrm{d}^\dagger\right)^2$. On non-zero-modes, the transversal Laplacian is equivalent to $\mathrm{d}^\dagger\mathrm{d}$ on p-forms. However these operators have, potentially, different number of zero-modes. We care about $\square_p^M$ which has

$$\dim\ker\square_p^M = \dim\ker\triangle_p = \mathrm{b}_p(M), \tag{C.1.2}$$

where $\mathrm{b}_p(M)$ is the p-th Betti number of M. The zero-modes are given, simply, by the harmonic p-forms on M. Finally, the transversal Laplacian is a self-adjoint operator, and hence, its eigenforms provide a basis for coclosed (also known as transversal) p-forms. By Poincaré duality, the spectrum of the Hodge Laplacian for $0 \leqslant p \leqslant d$ is determined by the spectrum of the transversal Laplacian for $0 \leqslant p \leqslant \left\lfloor \frac{d-1}{2} \right\rfloor$.

Let us now focus on three dimensional manifolds, Σ, one-forms (i.e. $p=1$), and non-zero-modes.[1] There is another seld-adjoint operator acting on one-forms, the

[1] The story can be adapted, more generally, to p-forms on $(2p+1)$-dimensional manifolds, mutatis mutandis [338].

Beltrami operator:

$$\star\mathrm{d} : \Omega^1(\Sigma) \to \Omega^1(\Sigma). \tag{C.1.3}$$

The transversal Laplacian is simply the square of the Beltrami operator, $\Box_1^\Sigma = (\star\mathrm{d})^2$. As such, they commute, so they can be diagonalised simultaneously, and, moreover, the non-zero spectrum of the transversal Lalpacian consists of the squares of the eigenvalues of the Beltrami operator. Since $\star\mathrm{d}$ is self-adjoint its eigen-one-forms provide an (orthonormalisable) basis of coclosed one-forms.

On a generic closed, oriented three-dimensional manifold, Σ, the Beltrami operator has simple spectrum [299]. The same holds for the Hodge Laplacian [299] with an appropriate clarification on the word generic.[2] For most of our applications, and importantly, for the state-operator correspondence, we are interested in very non-generic manifolds, with high-degree of symmetry, and therefore degeneracy, such as products of spheres. There, the spectrum of the Hodge Laplacian is actually (at least) twofold degenerate. On $\mathbb{S}^3$ it is a classic result, see e.g. [340–342] for an account of it. On $\mathbb{S}^2 \times \mathbb{S}^1$, we construct explicitly the spectrum below, in appendix C.1.2. Finally, on $\mathbb{T}^3$, it is straightforward to construct the spectrum and see that it is spanned by

$$\exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{\theta})\, \omega(\boldsymbol{k}).$$

In the above, $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_3)$ are the three angles of the torus, $\boldsymbol{k} = (k_1, k_2, k_3)$, with $k_i = \frac{n_i}{L_i}$, where $n_i \in \mathbb{Z}$ and L_i are the radii of each circle, and $\omega(\boldsymbol{k}) = \omega(\boldsymbol{k})_i\, \mathrm{d}x^i$ is an eigenform — or equivalently $\omega(\boldsymbol{k})_i$ is an eigenvector — of the matrix $\mathrm{i}\, \varepsilon_{ij\ell} k^\ell$. It is then easy to verify that there are two such eigenvectors, both corresponding to the eigenvalues $\|\boldsymbol{k}\|^2$ for the transversal Laplacian, verifying its twofold degeneracy.

C.1.2 Eigenforms and eigenvalues on $\mathbb{S}^2 \times \mathbb{S}^1$

WNow we move on to constructing the spectrum on $\Sigma_r = \mathbb{S}^2_r \times \mathbb{S}^1$. We use the following metric and coordinates:

$$\mathrm{d}s^2_{\Sigma_r} = r^2\big(\mathrm{d}\theta^2 + \sin\theta\, \mathrm{d}\varphi^2\big) + \mathrm{d}\eta^2\,, \tag{C.1.4}$$

where $\varphi \in [0, 2\pi)$ and $\theta \in [0, \pi)$ are the angles on the sphere and $\eta \in [0, 2\pi)$ is the angle on the circle. We will denote the eigen-one-forms of $\Box_1^{\Sigma_r}$ as Φ, subscripted by their various quanutm numbers.

Zero-mode

The easiest to construct is the zero-mode. This is given by the unique harmonic form on Σ_r, $\mathrm{d}\eta$. Therefore, the normalised zero-mode is given by

$$\Phi_0 := \frac{\mathrm{d}\eta}{\sqrt{\mathrm{vol}(\Sigma_r)}} = \frac{\mathrm{d}\eta}{2\sqrt{2}\pi\, r}. \tag{C.1.5}$$

[2] More precisely, it is simple on the complement of a codimension-1 set of metrics, but not on the complement of a codimension-2 set of metrics [339].

Non-zero modes

Let us introduce some convenient notation. First, we write

$$T_k(\eta) := \begin{cases} \frac{1}{\sqrt{\pi}} \cos(k\eta), & k > 0, \\ \frac{1}{\sqrt{2\pi}}, & k = 0, \\ \frac{1}{\sqrt{\pi}} \sin(|k|\eta), & k < 0, \end{cases} \tag{C.1.6}$$

with $\eta \in [0, 2\pi)$. These are the real, orthonormal eigenfunctions of the Laplacian on $\mathbb{S}^1$, with eigenvalue k^2:

$$\Box_0^{\mathbb{S}^1} T_k(\eta) = -\nabla^2 T_k(\eta) = k^2 \; T_k(\eta). \tag{C.1.7}$$

Similarly, we denote by $\Upsilon_{\ell m}(\theta, \varphi)$ the real spherical harmonics on $\mathbb{S}^2_r$:

$$\Upsilon_{\ell m}(\theta, \varphi) := \frac{\mathcal{N}_{\ell m}}{r} \mathrm{P}_\ell^{|m|}(\cos\theta) \begin{cases} \cos(m\varphi), & m > 0, \\ 1, & m = 0, \\ \sin(|m|\varphi), & m < 0, \end{cases} \tag{C.1.8}$$

where,

$$\mathcal{N}_{\ell m} = (-1)^m \sqrt{2} \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - |m|)!}{(\ell + |m|)!}}\,, \tag{C.1.9}$$

and $\mathrm{P}_\ell^m(x)$ are the associated Legendre polynomials. In the above, $\ell \in \mathbb{Z}_{\geqslant 0}$ and m are integers, with $-\ell \leqslant m \leqslant \ell$. These are the real, orthonormalised eigenfunctions of the Laplacian on $\mathbb{S}^2_r$:

$$\Box_0^{\mathbb{S}^2_r} \Upsilon_{\ell m}(\theta, \varphi) = -\nabla^2 \Upsilon_{\ell m}(\theta, \varphi) = \underbrace{\frac{\ell(\ell + 1)}{r^2}}_{=:\,\rho_{\ell m}(r)} \Upsilon_{\ell m}(\theta, \varphi). \tag{C.1.10}$$

The degeneracy of the eigenvalue $\rho_{\ell m}(r)$ is

$$D_{\ell m} = 2\ell + 1. \tag{C.1.11}$$

Using these two building blocks, we can build all the non-zero eigen-one-forms.

Momentumless

A first family is given by eigenforms with no momentum along the $\mathbb{S}^1$, i.e. $k = 0$. These take the form

$$\Phi^{(1)}_{\ell m} := \Upsilon_{\ell m}(\theta, \varphi) \frac{\mathrm{d}\eta}{\sqrt{2\pi}}, \tag{C.1.12}$$

with $\ell > 0$ and $-\ell \leqslant m \leqslant \ell$, and have eigenvalue $\rho_{\ell m}(r)$:

$$\Box_1^{\Sigma_r} \Phi^{(i)}_{\ell m} = \frac{\ell(\ell+1)}{r^2} \Phi^{(i)}_{\ell m}. \tag{C.1.13}$$

One can explicitly check that

$$\begin{aligned} \Phi^{(2)}_{\ell m} &:= \frac{1}{\sqrt{\rho_{\ell m}(r)}} \star_r \mathbf{d}\Phi^{(1)}_{\ell m} \\ &= \frac{1}{\sqrt{2\pi}} \frac{r}{\sqrt{\ell(\ell+1)}} \left(\frac{1}{\sin\theta}\, \partial_\varphi \Upsilon_{\ell m}(\theta,\varphi)\, \mathrm{d}\theta - \sin\theta\, \partial_\theta \Upsilon_{\ell m}(\theta,\varphi)\, \mathrm{d}\varphi \right), \end{aligned} \tag{C.1.14}$$

is also an eigen-one-form of $\Box_1^{\Sigma_r}$, with the same eigenvalue, $\rho_{\ell m}(r)$. All of these modes are orthonormalised:

$$\left\langle \Phi^{(i)}_{\ell m}, \Phi^{(i')}_{\ell' m'} \right\rangle = \delta_{\ell\ell'}\delta_{mm'}\delta_{ii'}, \qquad i, i' \in \{1,2\}. \tag{C.1.15}$$

Momentumful

The rest of the modes, are modes with momentum along the $\mathbb{S}^1$, i.e. $k \neq 0$. They are given by

$$\begin{aligned} \Phi^{(1)}_{\ell m k} &:= \frac{1}{\sqrt{\rho_{\ell m}(r) + k^2}} \left(\sqrt{\frac{\rho_{\ell m}(r)}{k^2}} \Upsilon_{\ell m}(\theta,\varphi) \mathrm{d}T_k(\eta) - \sqrt{\frac{k^2}{\rho_{\ell m}(r)}} \mathrm{d}\Upsilon_{\ell m}(\theta,\varphi)\, T_k(\eta) \right) \\ &= \left(\frac{\ell(\ell+1)}{r^2} + k^2 \right)^{-\frac{1}{2}} \left[\frac{\sqrt{\ell(\ell+1)}}{r} \frac{\partial_\eta T_k(\eta)}{k} \Upsilon_{\ell m}(\theta,\varphi)\, \mathrm{d}\eta \right. \\ &\qquad \left. - k T_k(\eta) \frac{r}{\sqrt{\ell(\ell+1)}} \left(\partial_\varphi \Upsilon_{\ell m}(\theta,\varphi)\ \mathrm{d}\varphi + \partial_\theta \Upsilon_{\ell m}(\theta,\varphi)\, \mathrm{d}\theta \right) \right], \end{aligned} \tag{C.1.16}$$

and

$$\begin{aligned} \Phi^{(2)}_{\ell m k} &:= \frac{1}{\sqrt{\rho_{\ell m}(r) + k^2}} \star_r \mathbf{d}\Phi^{(2)}_{\ell m k} \\ &= \frac{1}{\sqrt{\rho_{\ell m}(r)\, k^2}} \star_r [\mathrm{d}\Upsilon_{\ell m}(\theta,\varphi) \wedge \mathrm{d}T_k(\eta)] \\ &= \frac{r}{\sqrt{\ell(\ell+1)}} \frac{\partial_\eta T_k(\eta)}{k} \left(\frac{1}{\sin\theta}\, \partial_\varphi \Upsilon_{\ell m}(\theta,\varphi)\ \mathrm{d}\theta - \sin\theta\, \partial_\theta \Upsilon_{\ell m}(\theta,\varphi)\, \mathrm{d}\varphi \right). \end{aligned} \tag{C.1.17}$$

Their eigenvalue is $\lambda_{\ell m k}(r) := \rho_{\ell m}(r) + k^2$:

$$\Box_1^{\Sigma_r} \Phi^{(i)}_{\ell m k} = \left(\frac{\ell(\ell+1)}{r^2} + k^2 \right) \Phi^{(i)}_{\ell m k} =: .\lambda_{\ell m k}(r) \Phi^{(i)}_{\ell m k} \tag{C.1.18}$$

and are again orthonormalised:

$$\left\langle \Phi^{(i)}_{\ell mk}, \Phi^{(i')}_{\ell' m' k} \right\rangle = \delta_{\ell\ell'}\delta_{mm'}\delta_{kk'}\delta_{ii'}. \tag{C.1.19}$$

We write collectively, both for the case with momentum and without momentum, the modes as $\Phi^{(i)}_{\ell mk}$, by defining $\Phi^{(i)}_{\ell m0} := \Phi^{(i)}_{\ell m}$. Therefore, a complete orthonormal basis of the space of coclosed one-forms on Σ_r is given by

$$\mathcal{B}^1_\perp(\Sigma_r) := \left\{ \Phi_0, \left\{ \Phi^{(i)}_{\ell mk} \right\}^{i=1,2}_{\ell\in\mathbb{Z}_{>0},\, m\in[-\ell,\ell],\, k\in\mathbb{Z}} \right\}. \tag{C.1.20}$$

The degeneracy of the eigenvalue $\lambda_{\ell mk}(r)$ on one-forms is

$$D^{(1)}_{\ell mk} = 2(2\ell+1). \tag{C.1.21}$$

Finally, the Hodge duals of the above forms provide the spectrum of the longitudinal Laplacian on two-forms, $\triangle\big|_{\Omega^2(\Sigma_r)\cap\ker\mathbf{d}}$. Relatedly, an orthonormal basis for those is:

$$\mathcal{B}^2_\parallel(\Sigma_r) := \left\{ \star_r\, \Phi_0, \left\{ \star_r\, \Phi^{(i)}_{\ell mk} \right\}^{i=1,2}_{\ell\in\mathbb{Z}_{>0},\, m\in[-\ell,\ell],\, k\in\mathbb{Z}} \right\}. \tag{C.1.22}$$

C.1.3 A convenient basis

As we discussed above, the transversal Laplacian commutes with the operator $\star_r\mathbf{d}$. Therefore they can be diagonalised simultaneously. The one-forms that diagonalise the operator $\star_r\mathbf{d}$ are related to the above basis as follows:

$$\phi_{\ell mk\sigma} := \frac{1}{\sqrt{2}}\left(\Phi^{(1)}_{\ell mk} + \sigma\, \Phi^{(2)}_{\ell mk} \right), \qquad \sigma = \pm. \tag{C.1.23}$$

They are again orthonormal and statisfy

$$\star_r \mathbf{d}\phi_{\ell mk\sigma} = \sigma \sqrt{\lambda_{\ell mk}(r)}\, \phi_{\ell mk\sigma}. \tag{C.1.24}$$

Therefore, we can also write the following two bases for coclosed one-forms and closed two-forms respectively:

$$\begin{aligned} \mathcal{V}^1_\perp(\Sigma_r) &:= \left\{ \phi_0(r), \left\{ \phi_{\mathrm{n}\sigma}(r) \right\}^{\sigma=\pm}_{\mathrm{n}=(\ell,m,k)} \right\}, \\ \mathcal{V}^2_\parallel(\Sigma_r) &:= \left\{ \star_r\, \phi_0(r), \left\{ \star_r\, \phi_{\mathrm{n}\sigma}(r) \right\}^{\sigma=\pm}_{\mathrm{n}=(\ell,m,k)} \right\}, \end{aligned} \tag{C.1.25}$$

where we also renamed the zero-mode to $\phi_0(r) \equiv \Phi_0(r)$. When we need to expand both one- and two- forms in these bases, we will refer to them collectively as $\mathcal{V}(\Sigma_r)$. This basis has the advantage that it makes the Kac–Moody structure clear.

C.2 Current algebra in the generic case

In this appendix we explain how the Kac–Moody algebra of subsection 4.4.3 gets modified in the generic — in the sense of [299, 339] — case of a closed three-dimensional manifold, where the spectrum of the transversal Laplacian is simple, i.e. the eigenvalues are non-degenerate (cf. appendix C.1). We denote the three-dimensional Riemannian manifold we are treating, by a slight abuse of notation, as Σ, where it should be understood that we are actually considering the pair (Σ, g) where g is a Riemannian metric on Σ used to define the Hodge-star. We are not interested in varying the metric to obtain genericity results, like in [299, 339], so we suppress it.

The Beltrami operator, $\star\mathrm{d}$, will still be used to diagonalise the current algebra (4.2.27)/(4.4.48). However, in the generic case, if $\sqrt{\lambda_{\mathsf{n}}}$ appears in the spectrum of $\star\mathrm{d}$, $-\sqrt{\lambda_{\mathsf{n}}}$ doesn't, and vice versa. So we will label the orthonormalised eigen-one-forms of $\star\mathrm{d}$ as

$$\{\phi_{\mathsf{n}_\pm}\}_{\mathsf{n}_\pm\in\mathcal{N}_\pm}, \tag{C.2.1}$$

(where $\mathcal{N}_\pm$ is the index-set containing the labels $\mathsf{n}_\pm$) satisfying

$$\star\,\mathrm{d}\phi_{\mathsf{n}_\pm} = \pm\sqrt{\lambda_{\mathsf{n}_\pm}}\phi_{\mathsf{n}_\pm}, \tag{C.2.2}$$

where $\lambda_{\mathsf{n}_\pm} > 0$ is the corresponding (non-zero) eigenvalue of the transversal Laplacian on $\phi_{\mathsf{n}_\pm}$. The labelling above, is such that eigenforms with label n_- contain the negative spectrum of the Beltrami operator, and those with n_+, the positive spectrum. The genericity result mentioned above, implies that typically, $\lambda_{\mathsf{n}_+} \neq \lambda_{\mathsf{n}_-}$ for all $\mathsf{n}_+, \mathsf{n}_-$. This is unlike the case we have treated in the main text — valid in highly symmetric, and fine-tuned, manifolds, such as products of spheres. Moreover, on a three-dimensional manifold, the spectrum of $\star\mathrm{d}$ accumulates at $+\infty$ and at $-\infty$ [343–345], which guarantees that both signs appear in (C.2.2). As before, we exclude the kernel of $\star\mathrm{d}$, as this is treated, unambiguously, via the harmonic forms, $\{\phi_{0\mathsf{i}}\}_{\mathsf{i}=1}^{\mathsf{b}_1(\Sigma)}$, of Σ. The one-forms $\phi_{\mathsf{n}_\pm}$ are such that

$$\langle\phi_{\mathsf{n}_\pm}, \phi_{\mathsf{m}_\pm}\rangle_\Sigma = \delta_{\mathsf{n}_\pm\mathsf{m}_\pm} \quad \text{and} \quad \langle\phi_{\mathsf{n}_+}, \phi_{\mathsf{n}_-}\rangle_\Sigma = 0. \tag{C.2.3}$$

As explained in appendix C.1, $\{\phi_{0\mathsf{i}}, \phi_{\mathsf{n}_\pm}\}$ provide a complete basis of coclosed one-forms on Σ. We use this basis to expand $\Lambda^\pm$ in (4.2.27)/(4.4.48), and the dual, two-form basis, given by the Hodge stars of the above, to expand $J^\pm$:

$$i_\Sigma^*\Lambda^\mp = \sum_{\mathsf{i}=1}^{\mathsf{b}_1(\Sigma)} \Lambda_{0\mathsf{i}}^\mp\phi_{0\mathsf{i}} + \sum_{\mathsf{s}=\pm}\sum_{\mathsf{n}_\mathsf{s}\in\mathcal{N}_\mathsf{s}} \Lambda_{\mathsf{n}_\mathsf{s}}^\mp \star_\Sigma \phi_{\mathsf{n}_\mathsf{s}}, \tag{C.2.4}$$

$$i_\Sigma^* J^\pm = \sum_{\mathsf{i}=1}^{\mathsf{b}_2(\Sigma)} Q_{0\mathsf{i}}^\pm \star_\Sigma \phi_{0\mathsf{i}} + \sum_{\mathsf{s}=\pm}\sum_{\mathsf{n}_\mathsf{s}\in\mathcal{N}_\mathsf{s}} Q_{\mathsf{n}_\mathsf{s}}^\pm \star_\Sigma \phi_{\mathsf{n}_\mathsf{s}}. \tag{C.2.5}$$

The difference with (4.4.58) is subtle, but important. This expansion leads to the mode algebra:

$$\left[Q^{\pm}_{n_s}, Q^{\mp}_{m_t}\right] = \pm k s\sqrt{\lambda_{n_s}}\delta_{n_s m_t}. \tag{C.2.6}$$

Defining ladder operators

$$\mathcal{A}_{n_s} := \begin{cases} Q^{+}_{n_s} & s=+, \\ Q^{-}_{n_s} & s=-, \end{cases} \quad \text{and} \quad \mathcal{A}^{\dagger}_{n_s} := \begin{cases} Q^{-}_{n_s} & s=+, \\ Q^{+}_{n_s} & s=-, \end{cases}, \tag{C.2.7}$$

leads to the algebra

$$\left[\mathcal{A}_{n_s}, \mathcal{A}^{\dagger}_{m_t}\right] = k\sqrt{\lambda_{n_s}}\delta_{n_s m_t}. \tag{C.2.8}$$

Note that we still have two decoupled algebras, one for each sign of s, i.e. one for each side of the spectrum of $\star d$. In terms of these modes, the Hamiltonian takes the form

$$H_\Sigma = \frac{1}{k}\sum_{i=1}^{b_2(\Sigma)} Q^{+}_{0i}Q^{-}_{0i} + \frac{1}{k}\sum_{s=\pm}\sum_{n_s\in\mathcal{N}_s}\mathcal{A}^{\dagger}_{n_s}\mathcal{A}_{n_s} + E_0, \tag{C.2.9}$$

with

$$E_0 = \frac{1}{2}\sum_{s}\sum_{n_s}\sqrt{\lambda_{n_s}}. \tag{C.2.10}$$

As can be seen from (C.2.8), the operators $\mathcal{A}^{\dagger}_{n_s}$ raise the energy by $\sqrt{\lambda_{n_s}}$, and $\mathcal{A}_{n_s}$ lower it by the same amount.

From here on, it is an easy exercise to repeat the steps explained in the main text and arrive at the conclusion that the Hilbert space is spanned by

$$\left|\boldsymbol{n}, \boldsymbol{m}; \{N_{n_s}\}^{s=\pm}_{n_s\in\mathcal{N}_s}\right\rangle := \prod_{s=\pm}\prod_{n\in\mathcal{N}_s}\left(\mathcal{A}^{\dagger}_{n_s}\right)^{N_{n_s}}|\boldsymbol{n}, \boldsymbol{m}\rangle\,, \tag{C.2.11}$$

with $|\boldsymbol{n}, \boldsymbol{m}\rangle$ the primary states, given by (4.4.67) and (4.4.68), with the appropriate tweak that they are annihilated by all $\mathcal{A}_{n_s}$. The rest of the discussion, including (4.4.80) and its implications follow immediately.

C.3 Details on the radial evolution

Here we collect some details about the radial evolution on $\mathbb{B}^3\times\mathbb{S}^1$. For the reader's convenience we repeat the radial evolution equation, (4.5.20) here:

$$\partial_r J^{\pm}_{n\sigma}(r) + \left[\mathbb{A}^{\pm}_{n}(r)\right]_{\sigma\tau} J^{\pm}_{n\tau}(r) = 0. \tag{C.3.1}$$

The matrix $\mathbb{A}^{\pm}_{n}(r)$ is given, in the basis $\mathcal{V}(\Sigma_r)$ (C.1.25), by

$$\begin{aligned}\mathbb{A}^{\pm}_{n}(r) &= \langle\phi_{n\sigma}, \star_r\partial_r\star_r\phi_{n\tau}\rangle \mp \langle\phi_{n\sigma}, \star_r \mathbf{d}\phi_{n\tau}\rangle \\ &= \frac{1}{2r}\frac{\frac{\ell(\ell+1)}{r^2}}{\frac{\ell(\ell+1)}{r^2}+k^2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \mp \sqrt{\frac{\ell(\ell+1)}{r^2}+k^2}\begin{pmatrix}1 & \\ & -1\end{pmatrix},\end{aligned} \tag{C.3.2}$$

where $\mathsf{n} = (\ell, m, k)$. Writing

$$J^{\pm}_{\mathsf{n}\sigma}(r) = \left[\mathbb{U}^{\pm}_{\mathsf{n}}(r, r_0)\right]_{\sigma\tau} J^{\pm}_{\mathsf{n}\sigma}(r_0), \tag{C.3.3}$$

the initial value problem consisting of (C.3.1) and its boundary conditions at $r = r_0$ gets mapped to the matrix ordinary differential equation:

$$\dot{\mathbb{U}}^{\pm}_{\mathsf{n}}(r, r_0) + \mathbb{A}^{\pm}_{\mathsf{n}}(r)\mathbb{U}^{\pm}_{\mathsf{n}}(r, r0) = 0, \tag{C.3.4}$$

where the dot indicates derivative with respect to r, together with boundary conditions

$$\mathbb{U}^{\pm}_{\mathsf{n}}(r_0, r_0) = \mathbb{1}, \tag{C.3.5}$$

This has the unique solution

$$\mathbb{U}^{\pm}_{\mathsf{n}}(r, r_0) = \mathrm{Rexp}\left(\int_r^{r_0} \mathrm{d}r' \, \mathbb{A}^{\pm}_{\mathsf{n}}\big(r'\big)\right), \tag{C.3.6}$$

where Rexp denotes the radially ordered exponential:

$$\begin{aligned}\mathrm{Rexp}\left(\int_r^{r_0} \mathrm{d}r' \, \mathbb{O}\big(r'\big)\right) := \sum_{N=0}^{\infty} \frac{1}{N!} \int_r^{r_0} \mathrm{d}r_1 \int_r^{r_0} \mathrm{d}r_2 \cdots \\ \cdots \int_r^{r_0} \mathrm{d}r_N \, \mathrm{R}\big(\mathbb{O}(r_1)\mathbb{O}(r_2) \cdots \mathbb{O}(r_N)\big),\end{aligned} \tag{C.3.7}$$

with

$$\mathrm{R}(\mathbb{O}_1(r_1)\mathbb{O}_2(r_2)) := \begin{cases} \mathbb{O}_1(r_1)\mathbb{O}_2(r_2) & \text{if } r_1 < r_2, \\ \mathbb{O}_2(r_2)\mathbb{O}_1(r_1) & \text{if } r_2 < r_1. \end{cases} \tag{C.3.8}$$

When $k = 0$, it holds that $\left[\mathbb{A}^{\pm}_{\mathsf{n}}(r_1), \mathbb{A}^{\pm}_{\mathsf{n}}(r_2)\right] = 0$ for all radii, and hence the ordered exponential reduces to a regular one. The solution is given in this case by (4.5.28):

$$\begin{aligned}\mathbb{U}^{\pm}_{\ell m}(r, r_0) = \frac{1}{2 + 4\ell}\Bigg[\left(\frac{r}{r_0}\right)^{-\ell-1} \begin{pmatrix} 1 + 2\ell \mp 2\sqrt{\ell(\ell+1)} & 1 \\ 1 & 1 + 2\ell \pm 2\sqrt{\ell(\ell+1)} \end{pmatrix} \\ + \left(\frac{r}{r_0}\right)^{\ell} \begin{pmatrix} 1 + 2\ell \pm 2\sqrt{\ell(\ell+1)} & -1 \\ -1 & 1 + 2\ell \mp 2\sqrt{\ell(\ell+1)} \end{pmatrix}\Bigg].\end{aligned} \tag{C.3.9}$$

Here and onwards we suppress the magnetic quantum number m in the labelling of the radial evolution data, as they don't depend on it. The eigenvalues of $\mathbb{U}^{\pm}_{\ell}(r, r_0)$ are

$$\delta^{\pm}_{(1)\ell} = \left(\frac{r}{r_0}\right)^{-\ell-1} \quad \text{and} \quad \delta^{\pm}_{(2)\ell} = \left(\frac{r}{r_0}\right)^{\ell}, \tag{C.3.10}$$

with eigenvectors

$$u^{\pm}_{(1)\ell} = \begin{pmatrix} 1+2\ell \mp 2\sqrt{\ell(\ell+1)} \\ 1 \end{pmatrix} \quad \text{and} \quad u^{\pm}_{(2)\ell} = \begin{pmatrix} -1-2\ell \mp 2\sqrt{\ell(\ell+1)} \\ 1 \end{pmatrix}, \tag{C.3.11}$$

respectively. It is evident that at $r \to 0$ one of the eigenvalues vanishes and, thus, the evolution matrix becomes rank one. Its kernel is spanned by $v^{\pm}_{(2)\ell}$. The projector $\mathbf{\Pi}^{\pm}_{\ell}$, appearing from (4.5.36) onwards, is, in this case:

$$\mathbf{\Pi}^{\pm}_{\ell} = \frac{u^{\pm}_{(2)\ell} \otimes u^{\pm}_{(2)\ell}}{\left\| u^{\pm}_{(2)\ell} \right\|^2} = \frac{1}{2(1+2\ell)} \begin{pmatrix} 1+2\ell \pm 2\sqrt{\ell(\ell+1)} & -1 \\ -1 & 1+2\ell \mp 2\sqrt{\ell(\ell+1)} \end{pmatrix}. \tag{C.3.12}$$

For $k \neq 0$, $\mathbb{A}^{\pm}_{\mathrm{n}}(r)$ does not commute with itself at different radii and thus we have to use the radially ordered exponential. While this we cannot solve exactly for arbitrary radius, the crucial feature for our state-operator correspondence is its behaviour as $r \to 0$. In that limit, the differential equation (C.3.4) reduces to the differential equation for $k = 0$, and reveals that $\mathbb{U}^{\pm}_{\ell m k}(r, r_0)$, behaves identically to its $k = 0$ eigenpart. Namely it has a singular and a regular part, scaling as

$$\sim \left(\frac{r}{r_0}\right)^{-\ell-1} \quad \text{and} \quad \sim \left(\frac{r}{r_0}\right)^{\ell}, \tag{C.3.13}$$

respectively. Correspondingly, $\mathbb{U}^{\pm}_{\ell m k}(0, r_0)$ becomes rank-1.

Short summary in english

Symmetry is a fundamental organising principle of nature and, by extension, of theoretical physics. Physical phenomena are characterised by their symmetries and how they represent them. In the study of quantum field theory (QFT) the symmetries of a physical system are associated with conserved quantities, such as energy, momentum, or the number of particles in a given space. Recently, it has been understood that the notion of symmetry can be generalised in several ways. Central to this understanding is the fact that symmetry is intimately linked to topology.

One generalisation of particular significance is higher-form symmetry. This generalisation concerns conserved quantities of extended objects, such as the number of lines piercing a surface. Higher-form symmetries are present in many theories of interest, such as electromagnetism and the standard model of particle physics. Another important generalisation is that of non-invertible symmetries. This type of symmetry is somewhat esoteric in that the symmetry action cannot be undone. Nevertheless, such symmetries find applications in a variety of models, most notably in the critical Ising model which describes the long distance behaviour of many physical systems.

Such *generalised symmetries* offer new organising and guiding principles for theoretical physics. This thesis focusses on utilising the power of generalised symmetries to obtain universal results in quantum field theory and the phases of matter. The main findings of this thesis fall in two categories. Firstly, obtaining universal measures of entanglement in topologically ordered systems and topological quantum field theory (TQFT), in generic dimensions and topology. Secondly, elucidating the underlying structure of conformal field theories (CFTs), with a particular emphasis on nonlocal operators.

Topological order

From a condensed-matter-theoretic point of view, TQFTs arise as low-energy effective descriptions of topologically ordered systems. A physical mechanism for topological order in (2+1) dimensions is given by the condensation of networks of line operators, known as string-net condensation. In higher dimensions, analogous models — condensing networks of p-dimensional surface operators — give rise to topologically ordered ground states. This is intimately connected to generalised symmetries, since taking the generators of a discrete p-form symmetry as the condensing network, provides a description of deconfined, discrete gauge theories. The long distance behaviour of these models is described by a specific topological quantum field theory, known as p-form BF theory. Moreover, topological orders correspond to and are classified by different patterns of entanglement. This is most cleanly showcased by the celebrated topological entanglement entropy. Understanding patterns of entanglement in p-form BF theory, gives direct, low-energy access to a systematic understanding of topological order in higher-dimensions. This is the main motivating question for chapters 2 and 3.

More precisely, chapter 2 presents an algebraic study of the entanglement structure of p-form BF theory in arbitrary dimensions. This is done directly in the low-energy topological quantum field theory by considering the algebras of topological surface operators restricted to subregions. Two relevant notions of subregion operator algebras are defined, which are related by a form of electric-magnetic duality. It is subsequently shown that with each subregion algebra, there is an associated entaglement entropy, termed *essential topological entanglement* (ETE). This is a refinement of the topological entanglement entropy. ETE is intrinsic to the theory, inherently finite, positive, and sensitive to more intricate topological features of the state and the entangling region.

Then, in chapter 3 an alternative perspective is explored. Remaining within the setup of p-form abelian BF theory, the entanglement entropy arising from edge modes, i.e. excitations localised on the boundary of the region of interest, is considered. This is done on arbitrary spatial topology and across arbitrary entangling surfaces. The entropy contains a series of terms that scale as powers of the area of the entangling surface (area and subarea laws), plus universal corrections proportional to the topology of the entangling surface. The calculation comes in two flavours: firstly, through an induced edge-mode theory, appearing on the regulated entangling surface in a replica path integral, and secondly through a more rigorous definition of the entanglement entropy via an extended Hilbert space. Along the way several key results are presented, that are of their own merit. The edge-mode theory is given by a novel combination of $(p-1)$-form and $(d-p-2)$-form electrodynamics linked by a chirality condition, in what is termed *chiral mixed Maxwell theory*. The thermal partition function of this theory is explicitly evaluated. Additionally, it is shown that the extended Hilbert space is organised into representations of an infinite-dimensional, centrally extended current algebra which naturally generalises Kac–Moody algebras to arbitrary dimension and topology. Lastly, the two approaches are connected, showing that the thermal partition function of the chiral mixed Maxwell theory is precisely an extended representation character of the current algebra, establishing an exact correspondence of the edge-mode theory and the entanglement spectrum.

Conformal field theory

In contrast to discrete theories, deconfined phases of (higher-)gauge theories with continuous gauge groups are gapless. Behind this fact lie, again, generalised symmetries. In particular, such phases can be understood as spontaneous symmetry breaking phases of higher-form global symmetries. The most prominent example of that is given by electromagnetism, and provides an explanation of the masslessness of the photon in our world. In specific dimensions, these gapless phases become conformal. The combination of conformal invariance with generalised symmetries turns out to be extremely strong and leads to universal statements about the structure of conformal field theory. These ideas underlie much of the motivation for chapter 4.

The main result of chapter 4 is a *one-to-one correspondence between line operators and states* in four-dimensional CFTs with continuous 1-form symmetries. Such CFTs enjoy an infinite dimensional current algebra, closely related to the algebras of chapter 3. The representation theory of this current algebra is constructed, and the space of states on an arbitrary closed spatial slice is described in detail. Then, the spectrum on $\mathbb{S}^2 \times \mathbb{S}^1$ is rederived via a path integral on $\mathbb{B}^3 \times \mathbb{S}^1$ with insertions of line operators. This leads to a direct and explicit correspondence between the line operators of the theory and the states on $\mathbb{S}^2 \times \mathbb{S}^1$. An interesting conclusion is that the ground state does not correspond to the identity operator, but to a particular operator, known in quantum optics as a squeezing operator. Additionally, some of the above results are generalised in two directions. Firstly, universal current algebras and their representation theory are constructed in $(2p+2)$-dimensional CFTs, with continuous p-form symmetry, and secondly extensions pertaining to non-invertible symmetries are provided.

Samenvatting

Symmetrie is een fundamenteel organiserend principe van de natuur en daarmee van de theoretische fysica. Fysische verschijnselen worden gekarakteriseerd door hun symmetrieën en hoe zij deze representeren. In de studie van de kwantumveldentheorie (QFT) worden de symmetrieën van een fysisch systeem in verband gebracht met behouden grootheden, zoals energie, impuls of het aantal deeltjes in een bepaald gebied. Recentelijk is ontdekt dat het begrip symmetrie op verschillende manieren kan worden gegeneraliseerd. Centraal in deze ontwikkeling staat het feit dat symmetrie nauw verbonden is met topologie.

Een generalisatie van bijzonder belang is de hogere-vormsymmetrie. Deze generalisatie betreft behouden grootheden van hoger-dimensionale objecten, zoals het aantal lijnen dat een oppervlak doorboort. Hogere-vormsymmetrieën zijn aanwezig in vele belangrijke theorieën, zoals elektromagnetisme en het standaardmodel van de deeltjesfysica. Een andere belangrijke generalisatie is die van niet-inverteerbare symmetrieën. Dit type symmetrie is enigszins esoterisch omdat de symmetrieactie niet ongedaan kan worden gemaakt. Desalniettemin vinden dergelijke symmetrieën toepassingen in verschillende modellen, in het bijzonder in het kritische Ising-model, dat het gedrag op lange afstand van veel fysische systemen beschrijft.

Dergelijke *gegeneraliseerde symmetrieën* bieden nieuwe organiserende en leidende principes voor de theoretische fysica. Dit proefschrift richt zich op het benutten van de kracht van gegeneraliseerde symmetrieën om universele resultaten te verkrijgen in de kwantumveldentheorie en de fasen van materie. De belangrijkste bevindingen van dit proefschrift vallen in twee categorieën. Ten eerste, het verkrijgen van universele maatstaven van verstrengeling in topologisch geordende systemen en topologische kwantumveldentheorie (TQFT), in generieke dimensies en topologie. Ten tweede, het verduidelijken van de onderliggende structuur van conforme veldentheorieën (CFT's), met een bijzondere nadruk op niet-lokale operatoren.

Topologische ordening

Vanuit het perspectief van de theorie van gecondenseerde materie, ontstaan TQFT's als effectieve beschrijvingen van topologisch geordende systemen voor lage energieën. Een fysisch mechanisme voor topologische ordening in $(2+1)$ dimensies is gegeven door de condensatie van netwerken van lijnoperatoren, bekend als snaarnet-condensatie. In hogere dimensies geven analoge modellen — condenseerbare netwerken van p-dimensionale oppervlak-operatoren — aanleiding tot topologisch geordende grondtoestanden. Dit is nauw verbonden met gegeneraliseerde symmetrieën, aangezien het nemen van de generatoren van een discrete p-vorm-symmetrie als het condenseerbare netwerk een beschrijving geeft van 'deconfined,' discrete ijktheorieën. Het langeafstandsgedrag van deze modellen wordt beschreven door een specifieke topologische kwantumveldentheorie, bekend als p-vorm BF-theorie. Bovendien komen topologische ordeningen overeen met en worden zij geclassificeerd door verschillende patronen van verstrengeling. Dit wordt het duidelijkst geïllustreerd door de beroemde topologische verstrengelingsentropie. Het begrijpen van verstrengelingspatronen in p-vorm BF-theorie geeft directe toegang bij lage energieën tot een systematisch begrip van topologische ordening in hogere dimensies. Dit is de belangrijkste motivatie voor hoofdstukken 2 and 3.

Hoofdstuk 2 presenteert een algebraïsche studie van de verstrengelingsstructuur van p-vorm BF-theorie in generieke dimensies. Dit wordt direct gedaan in de lage-energie beschrijving in termen van topologische kwantumveldentheorie door de algebra's van topologische oppervlakoperatoren te beschouwen, beperkt tot subgebieden. Twee relevante begrippen van subgebiedoperator-algebra's worden gedefinieerd, die gerelateerd zijn door een vorm van elektrisch-magnetische dualiteit. Vervolgens wordt aangetoond dat bij elke subgebiedalgebra een bijbehorende verstrengelingsentropie hoort, genaamd *essentiële topologische verstrengeling* (ETE). Dit is een verfijning van de topologische verstrengelingsentropie. ETE is intrinsiek aan de theorie, inherent eindig, positief en gevoelig voor meer complexe topologische kenmerken van de toestand en het verstrengelde gebied.

Vervolgens wordt in hoofdstuk 3 een alternatief perspectief verkend. In de context van p-vorm abelse BF-theorie wordt de verstrengelingsentropie beschouwd van randmodi, dat wil zeggen, excitaties die gelokaliseerd zijn op de rand van het betreffende gebied. Dit wordt gedaan voor generieke ruimtelijke topologie en verstrengelende gebieden. De entropie bevat een reeks termen die schalen als machten van het oppervlak van het verstrengelende gebied (oppervlakte- en suboppervlaktewetten), plus universele correcties evenredig met de topologie van het verstrengelende gebied. De berekening komt in twee smaken: ten eerste via een geïnduceerde randmodustheorie, die verschijnt op het gereguleerde verstrengelende oppervlak in een replica padintegraal, en ten tweede via een meer rigoureuze definitie van de verstrengelingsentropie via een uitgebreide Hilbertruimte. Gaandeweg worden verschillende sleutelresultaten gepresenteerd, die op zichzelf waardevol zijn. De randmodustheorie wordt gekarakteriseerd door een nieuwe combinatie van $(p-1)$-vorm en $(d-p-2)$-vorm elektrodynamica, verbonden door een chirale voorwaarde, in wat we *chirale gemengde Maxwell-theorie* noemen. De thermische partitiefunctie van deze theorie wordt expliciet geëvalueerd. Bovendien wordt bewezen dat de uitgebreide Hilbertruimte georganiseerd wordt door representaties van een oneindig-dimensionale, centraal uitgebreide stroomalgebra die Kac–Moody-algebra's op natuurlijke wijze generaliseert naar willekeurige dimensie en topologie. Tenslotte worden de twee benaderingen verbonden, waarbij wordt aangetoond dat de thermische partitiefunctie van de chirale gemengde Maxwell-theorie precies een uitgebreid representatiekarakter is van de stroomalgebra, waarmee een exacte overeenkomst wordt vastgesteld tussen de randmodustheorie en het verstrengelingsspectrum.

Conforme veldentheorie

In tegenstelling tot discrete theorieën zijn 'deconfined' fasen van (hogere-)ijktheorieën met continue ijkgroepen kloofloos. Dit feit is wederom onderbouwd door gegeneraliseerde symmetrieën. In het bijzonder kunnen dergelijke fasen worden begrepen als spontaan symmetriebrekende fasen van hogere-vorm globale s ymmetrieën. Het meest prominente voorbeeld hiervan is het elektromagnetisme, dat een verklaring geeft voor de massaloosheid van het foton in onze wereld. In specifieke dimensies worden deze kloofloze fasen conform. De combinatie van conforme invariantie met gegeneraliseerde symmetrieën blijkt extreem krachtig te zijn en leidt tot universele uitspraken over de structuur van conforme veldentheorie. Deze ideeën vormen grotendeels de motivatie voor hoofdstuk 4.

Het belangrijkste resultaat van hoofdstuk 4 is een *een-op-een correspondentie tussen lijnoperatoren en toestanden* in vierdimensionale CFT's met continue 1-vormsymmetrieën. Dergelijke CFT's genieten een oneindig dimensionale stroomalgebra, nauw verwant aan de algebra's beschreven in hoofdstuk 3. De representatietheorie van deze stroomalgebra wordt geconstrueerd, en de ruimte van toestanden op een willekeurige gesloten ruimtelijke snede wordt in detail beschreven. Vervolgens wordt het spectrum op $\mathbb{S}^2\times\mathbb{S}^1$ herafgeleid via een padintegraal over $\mathbb{B}^3\times\mathbb{S}^1$ in de aanwezigheid van lijnoperatoren. Dit leidt tot een directe en expliciete correspondentie tussen de lijnoperatoren van de theorie en de toestanden op $\mathbb{S}^2\times\mathbb{S}^1$. Een interessante conclusie is dat de grondtoestand niet overeenkomt met de identiteitoperator, maar met een specifieke operator, bekend in de kwantumoptica als een knijpoperator. Daarnaast worden enkele van de bovenstaande resultaten in twee richtingen gegeneraliseerd. Ten eerste worden universele stroomalgebra's en hun representatietheorie geconstrueerd in $(2p+2)$-dimensionale CFT's met continue p-vormsymmetrie, en ten tweede worden uitbreidingen met betrekking tot niet-inverteerbare symmetrieën gegeven.

Acknowledgements

Pursuing a PhD can often feel isolating, but it is far from a solo journey. Many people have contributed to this achievement, and I am deeply grateful to each of them.

Firstly, I want to express my heartfelt gratitude to Diego for his mentorship throughout my PhD. Diego, I have learnt a lot from you and you have, to a great degree, shaped my thinking in physics. I've learnt from you how to be a theoretical physicist. Equally importantly, I've learnt what it means to do research: when to persevere and when to let go of ideas; when to trust my physics intuition; what a concrete computation really means. I hope I have absorbed some of your ways around research and around physics and have matured as a physicist. Thank you for the opportunities you've given me, and the time and effort you dedicated to me. Thank you for being my advisor.

Central throughout my PhD have been two more people: Dio and Jackson. I have found mentors in both of you, and our interactions have significantly contributed to my growth as a physicist.

Dio, I always like to tell the story of our first meeting. I had just started my master's in Amsterdam, and you were preparing to move to London. You told me, "Nice to meet you; stay away from the crazy things I'm doing." Fast forward about two years, and my first research project ended up being with you. I didn't stay away, and I'm very glad I didn't. Your approach to physics is truly inspiring, and your enthusiasm is contagious. You focus on profound problems, ask deep and grounding questions, and provide insightful answers that connect various areas of physics. You get excited over small pieces of new information, often in the form of a seemingly trivial computation, pondering their impact on our understanding of the field. Every time I've visited you in London, I have returned brimming with enthusiasm for physics.

Jackson, working with you has made research a joy. Our long discussions — whether online, at the blackboard, while cycling, scribbling on a napkin in a restaurant, or over beers in a pub — have been incredibly fun and enriching. I've loved the thrill of thinking we found the end of the rope, as well as the letdown when one of us spots a mistake. From our spidey senses tingling about minus signs and chiralities to the hive-mind realisations that we'd been barking up the wrong tree, your enthusiasm and dedication have been infectious, making all problems enjoyable to tackle. Your ability to balance between physical relevance and mathematical precision has taught me a lot. I look forward to many more projects with you after we finish our ongoing ones. More than all that, I've found a friend in a collaborator, and this is hard to overstate.

In addition, I would like to thank Jan for agreeing to be my co-supervisor. While this role has been mostly formal in my case, what has been invaluable is that you were always welcoming to questions — both about physics and navigating academia and research — and have been extremely helpful on several occasions during my PhD. Needless to say, that on the physics side, although we did not collaborate directly, I have learnt a great deal from our interactions.

I would also like to extend my gratitude to my committee members, Erik, Kareljan, Nabil, Dio, and Lorenz, for taking on this role and for challenging my understanding of physics.

The vibrant institute for theoretical physics in Amsterdam has made this journey both fun and stimulating. The members of the string theory group have been instrumental in enriching my experience here. I'm especially grateful to Jay, Alejandra, Lorenz, Ben, Andrea, Erik, Marcel, and guest-member David for their invaluable advice, physics discussions, and for making the group both engaging and lively. I've also greatly benefited from my interactions with the condensed matter theory group, particularly Vladimir, Kareljan, and Jasper. My heartfelt thanks also go to the PhDs and postdocs, whose presence made my time at the institute thoroughly enjoyable. So, thanks Greg, Antonio, Beatrix, Carlos, Ricardo, Dora, Evita, Rebekka, Bahman, Jeremy, Ramesh, Edward, Andrei, Art, Facundo, Boris, Giorgos, Gian-Piero, Karthik, Juan, Jildou, Sarah, Alessandro, Aude, Jingxin, Lizzy, Danny, and Shira, Damián, Tarek, Chiara, Claire, Alex, Ioana, Mert, Eleni, Dominik, Soner, Akash, Ignacio, Luis, Andrew, Ana-Maria, Masataka, Marija, Kamran, Prahar, and special thanks to Alberto and Pietro. Crucial have also been the support staff of the institute: Anne-Marieke, Klaartje, Astrid, Jiřina, and Rose, who have always been extremely helpful.

A special mention goes to those with whom I shared this journey: Davide, Alexander, Suzanne, and Diego. It was truly wonderful to share the office with you, become good friends, and support each other through all the significant moments of our PhDs. From welcomed distractions to sharing the stress of momentous times, your companionship has been invaluable. I will miss you all, but I am sure our bond will stay.

Throughout my PhD, I've had the fortune to travel extensively for physics and meet many remarkable people who have significantly enriched my experiences. I want to extend a special thank you to Alan and Valdo. A special mention also goes to the "Viet Cong": Gabriele, Ioanna, Martí, Tal, Veronica, Victor, Yorgo, Felipe, Michelle, and Keivan.

In fact, I want to particularly acknowledge Keivan, for his unwavering friendship. Even though we've ended up on different sides of the planet since our master's, we've managed to stay in touch with remarkable frequency, sharing the good and the bad, and finding support in each other. Similarly, I'm grateful for my friendship with Kyros, with whom I've grown very close to and shared a wide range of experiences and thoughts. I am also grateful to Vasilis, for our friendship, which somehow grew stronger after we both left Athens, and which has been key to navigating this journey, through lengthy conversations (that almost always started with "I only have five minutes."). A special mention goes to Daphné, who has been a significant part of my life, first as a partner and then as a very close friend, always providing steadfast support. Jemma, who entered my life more recently, has quickly become very close to me and holds a special place in my heart. Lastly, I want to express my deepest thanks to Ezra for all the cherished moments we've shared, including key moments of my PhD, and for her love and support.

Finally, I want to thank my family. I am especially grateful to my grandpa, who, when I was a child, sparked my curiosity about the nature of the world and introduced me to popular physics books — setting me on the path to study physics. Above all, my deepest gratitude goes to my parents. Your countless sacrifices made it possible for me to pursue my studies at every stage. Thank you for providing me with everything I needed to achieve this goal, for your love, belief in my abilities, and constant encouragement. I am truly grateful for your support and for always being there for me.

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