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Factorial Series Representation of Stieltjes Series Converging Factors

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Abstract: The practical usefulness of Levin-type nonlinear sequence transformations as numerical tools for the summation of divergent series or for the convergence acceleration of slowly converging series is nowadays beyond dispute. The Weniger transformation, in particular, is able to accomplish spectacular results when used to overcome resummation problems, often outperforming better-known resummation techniques, like, for instance, Padé approximants. However, our theoretical understanding of Levin-type transformations is still far from being satisfactory and is particularly bad as far as the decoding of factorially divergent series is concerned. The Stieltjes series represent a class of power series of fundamental interest in mathematical physics. In the present paper, it is shown how the converging factor of any order of typical Stieltjes series can be expressed as an inverse factorial series, whose terms are analytically retrieved through a simple recursive algorithm. A few examples of applications are presented, in order to show the effectiveness and implementation ease of the algorithm itself. We believe that further investigations of the asymptotic forms of the remainder terms, encoded within the converging factors, could eventually lead toward a more general theory of the asymptotic behavior of the Weniger transformation when it is applied to Stieltjes series in high transformation orders. It is a rather ambitious project, which should be worthy of being pursued in the future.

Keywords: mathematical physics; divergent series; Stieltjes series; converging factors

MSC: 40A05; 65B10

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1. Prelude

The present paper is a fitting tribute to the memory of Ernst Joachim Weniger, mentor and friend, who passed away on 10 August 2022, sadly too soon. I had the good fortune to meet Ernst Joachim in 2009 at a conference organized at the Centre International de Rencontres Mathématiques in Marseille, France. Since then, I have had the pleasure of working with him for over a decade. In 2015, we published our first (and sole) joint paper, in which a convergence theory for the resummation of the Euler series via Levin-type nonlinear sequence transformations, was presented. In particular, we proved that his transformation, named the Weniger transformation, was not only able to resum the Euler series but also that, in accomplishing such a task, it turned out to be "exponentially faster" than Padé approximants. The Euler series is an example of the Stieltjes series, for which a well-established general convergence theory based on Padé approximants is well known to be available. After our 2015 joint paper, Ernst Joachim and I thought that trying to conceive a convergence theory also for Levin-type transformations, when applied to Stieltjes series, could have been an ambitious scientific project, but one worthy of being pursued.

What is contained in the present work should have been my second joint work with Ernst Joachim. In writing it, I tried to respect to the best of my ability all that Ernst Joachim had and would suggest to me, also thanks to the help of still unpublished material and several discussions that took place during his visits to my former Engineering Department at the University "Roma Tre."

2. Introduction

Power series are among the most important mathematical objects, not only in classical analysis, but also in the mathematical treatment of several scientific and engineering problems. Most power series have a *finite* radius of convergence. If the argument of a power series is close to the expansion point, convergence is usually (very) good, but once the argument approaches the boundary of the circle of convergence, convergence can become arbitrarily bad. Outside the circle of convergence, power series diverge [1].

Especially in theoretical physics, wildly divergent power series occur abundantly which cannot be replaced easily by alternative expressions. The Stirling series for the logarithm of the gamma function ([2] Equation (5.11.1)), as well as the Poincaré seminal work on asymptotic series [3], are classic examples of useful divergent series. In quantum physics, divergent series have become indispensable. Dyson [4] first argued that perturbation expansions in quantum electrodynamics must diverge factorially. Bender and Wu [5–7] showed that factorially divergent perturbation expansions occur also in nonrelativistic quantum mechanics. Later, it was found that factorially divergent perturbation expansions are actually the rule in quantum physics rather than the exception (see for example the articles by ([8] Table 1) and [9], or the articles reprinted in the book by [10]). In view of the conceptual and technical problems caused by divergent series both in applied mathematics and theoretical physics, it should not be surprising that there has been extensive research on summation techniques. Recent reviews including numerous references can be found in [11–14].

The most important summation techniques are the Borel summation [15], which replaces a divergent perturbation expansion by a Laplace-type integral, and Padé approximants [16], which transform the partial sums of a (formal) power series to rational functions. There is endless research on these summation techniques. Borel summation is discussed, for example, in Costin [17], Shawyer and Watson [18], Sternin and Shatalov [19], while the probably most complete treatment of Padé approximants can still be found in the monograph by Baker and Graves-Morris [20]. In recent years, a considerable deal of work has been invested on the so-called sequence transformations, which turned out to be very useful numerical tools for the summation of divergent series. The extended recent literature, in particular on *nonlinear* and *nonregular* sequence transformations, is available in [12,21–35].

Levin-type transformations [26,36,37], in particular, constitute an effective and powerful class of nonlinear sequence transformations, which differ substantially from other, better-known sequence transformations, such as, for example, the celebrated Wynn's epsilon algorithm [38]. So many successful applications of Levin-type transformations have been achieved and described in the literature (see, for instance, the references in ([11] Appendix B), [13,39], or in the articles in [12]), that they are also treated in NIST Handbook ([2] Chapter 3.9(v) Levin's and Weniger's Transformations). The practical usefulness of sequence transformations as computational tools for the acceleration of convergence as well as the summation of divergent series is now established beyond doubt. However, from a purely theoretical perspective, our understanding of the convergence properties of Levin-type sequence transformations is still far from being satisfactory.

In the case of Padé, the situation is much better. For example, if the input data are the partial sums of a Stieltjes series, it can be proved rigorously that certain subsequences of the Padé table converge to the value of the corresponding Stieltjes function. More precisely, if a factorially divergent power series is a Stieltjes series and if it also satisfies the so-called Carleman condition, such a series would be Padé summable. In [13], a *rigorous* convergence study of the summation of the Euler series, namely $\mathcal{E}(z) \sim \sum_{n=0}^{\infty} (-1)^n n! z^n$, which is the paradigm of the factorially divergent power series (for a compact description, see ([13] Section 2)), was carried out by comparing the performances of Padé approximants with those of a certain class of nonlinear sequence transformations.

Readers are encouraged to go through Section 4 of [13], where an extensive summary of the most relevant properties of Levin-type transformations can be found. Levin-type transformations *always* admit explicit analytical expressions, which might be used as a natural starting point for building up a convergence theory. However, there is a main conceptual drawback: the technical problems occurring in any convergence studies, substantially differ among the various Levin-type transformations. For example, in [13], the convergence analysis of the summation of the Euler series turns out to be, in the case of the Weniger transformation, much easier than in the case of other Levin-type transformations. The principal reason stems in the role played by the so-called *converging factor* [40] (Converging factors must not be mixed up with *convergence factors* for summable series, which are, for instance, discussed in an article and a book by [41,42]), whose theory has been extensively discussed in articles [43–48] and a book ([49] Sections 21–26) by Dingle.

The Euler series is a Stieltjes series, and its partial sums are Padé summable to the so-called Euler integral. In the present paper, it is proved that, when the series under investigation is of the Stieltjes type, an inverse factorial representation of its converging factor can always be analytically retrieved via an algorithm based on the following points:

(i) the converging factor of a typical Stieltjes series satisfies a first-order finite difference equation.
(ii) there is a tight connection between inverse factorial series and recurrence relationships, the former being the natural mathematical tools for representing the solution of linear finite difference equations, similarly as solutions of differential equations are naturally expressed in terms of power series. A detailed discussion of the most important features of factorial series plus additional references can be found, for instance, in [50] and a more condensed one in ([13] Appendix B).

This paper is structured as follows: in Section 3, after the main definitions and properties of the Stieltjes series and Stieltjes functions have been briefly resumed, the recurrence rule for the converging factors of the Stieljes series is derived and formally solved through inverse factorial series expansions. In Section 4, a somewhat didactical derivation of Weniger's transformation is proposed, in order to enlighten its connection with the factorial series representation of converging factors. Section 5 represents the core of this paper: the algorithm to generate the factorial series representation of a typical Stieltjes series converging factor is proposed and its derivation detailed. Section 6 is devoted to illustrate several examples of the application of the results established in Section 5. Finally, a few conclusive considerations and plans for future works are outlined in Section 7.

In order to improve this paper's readability, the most technical parts of the paper, together with those containing the main definitions and notations, have been confined within several appendices, with the first of them (A) being a shorter version of ([13] Appendix B).

3. Converging Factors of Stieltjes Series

Consider an increasing real-valued function $\mu(t)$ defined for $t \in [0, \infty]$, with infinitely many points of increase. The measure $\mathrm{d}\mu$ is then positive on $[0, \infty]$. Assume all positive quantities,

$$\mu_m = \int_0^\infty t^m \, \mathrm{d}\mu \,, \qquad m \geq 0 \,, \tag{1}$$

which will be called moments, to be finite. Then, the formal power series

$$\sum_{m=0}^{\infty} \frac{(-)^m}{z^{m+1}} \, \mu_m \,, \tag{2}$$

is called a *Stieltjes series*. Such a series turns out to be asymptotic, for $z \to \infty$, to the function $F(z)$ defined by

$$F(z) \;=\; \int_0^\infty \frac{\mathrm{d}\mu}{z+t} \,, \qquad |\arg(z)| < \pi \,, \tag{3}$$

which is called the *Stieltjes function*. In other words,

$$F(z) \sim \sum_{m=0}^{\infty} \frac{(-)^m}{z^{m+1}}\, \mu_m\,, \qquad z \to \infty\,. \tag{4}$$

Whether such a series converges or diverges depends on the behavior of the moment sequence $\{\mu_m\}_{m=0}^{\infty}$ as $m \to \infty$. Any Stieltjes function can always be expressed as the nth-order partial sum of the associated Stieltjes series (4), plus a truncation error term which has itself the form of a Stieltjes integral (see for example ([26] Theorem 13-1)), i.e.,

$$F(z) \,=\, \sum_{m=0}^{n} \frac{(-1)^m}{z^{m+1}}\, \mu_m \,+\, \left(-\frac{1}{z}\right)^{n+2} \int_0^\infty \frac{t^{n+1}\mathrm{d}\mu}{z+t}\,, \qquad |\arg(z)| < \pi\,. \tag{5}$$

Such a truncation error term can always be recast as the product of the first term neglected, namely $(-1/z)^{n+2}\,\mu_{n+1}$, and a *converging factor* [40,49], say $\varphi_{n+1}(z)$, which is defined as follows:

$$\varphi_m(z) \,=\, \frac{1}{\mu_m} \int_0^\infty t^m\, \frac{\mathrm{d}\mu}{t+z}\,, \qquad m \in \mathbb{N}_0\,, \qquad |\arg(z)| < \pi\,, \tag{6}$$

in such a way that $F(z)$ takes on the following form:

$$F(z) \,=\, \sum_{m=0}^{n} \frac{(-1)^m}{z^{m+1}}\, \mu_m \,+\, \frac{(-1)^{n+1}}{z^{n+2}}\, \mu_{n+1}\, \varphi_{n+1}(z)\,. \tag{7}$$

From Equation (7), it appears that the value of the Stieltjes function $F(z)$ could be retrieved, in principle, starting from the knowledge of a *finite* number of terms of the associated Stieltjes series, provided that the corresponding converging factor is to be estimated in some way. In particular, if a sufficient approximation to the converging factor could be obtained by avoiding the explicit evaluation of the Stieltjes integral into Equation (6), then, at least in principle, all problems related to the numerical evaluation of the Stieltjes function $F(z)$ via its asymptotic power series (4) would definitely be solved, even if this power series converges prohibitively slowly or not at all.

For example, the analysis in [13] was ultimately made possible thanks to the following explicit expansion obtained for the nth-order Euler series converging factor ([51] Equation (52)),

$$\varphi_n(z) \,=\, \sum_{k=0}^{\infty} \frac{L_k^{(-1)}(1/z)}{z}\, \frac{k!}{(n)_{k+1}}\,, \tag{8}$$

where $L_k^{(-1)}(\cdot)$ denotes a generalized Laguerre polynomial ([2] Table 18.3.1) and where the symbol $(n)_{k+1} = n(n+1)\ldots(n+k)$ denotes the so-called Pochhammer symbol. The expansion in Equation (8) is just an example of *inverse factorial series* in the discrete index $n \in \mathbb{N}_0$, which greatly facilitates the application of finite difference operators, on which Levin-type nonlinear transformations are ultimately based. For reader's convenience, the basic definitions and properties of factorial series are briefly recalled in Appendix A. A more extensive review can be found, as previously said, in ([13] Appendix B).

It is a remarkably simple task to prove that the converging factors $\varphi_m(z)$ satisfy a first-order linear recurrence rule. To this end, it is sufficient to start from the integral representation of φ_{m+1} given in Equation (6), i.e.,

$$\varphi_{m+1}(z) \,=\, \frac{1}{\mu_{m+1}} \int_0^\infty t^{m+1}\, \frac{\mathrm{d}\mu}{t+z}\,, \tag{9}$$

which can be recast as follows:

$$\varphi_{m+1}(z) \,=\, \frac{1}{\mu_{m+1}} \int_0^\infty t^m\, \frac{t}{t+z}\, \mathrm{d}\mu \,=\, \frac{1}{\mu_{m+1}} \int_0^\infty t^m \left(1 - \frac{z}{t+z}\right) \mathrm{d}\mu\,, \tag{10}$$

so that, after taking Equations (1) and (6) into account, we have

$$\varphi_{m+1} = \frac{\mu_m}{\mu_{m+1}} (1 - z\,\varphi_m)\,, \qquad m \geq 0\,, \tag{11}$$

Accordingly, the converging factor sequence $\{\varphi_m\}_{m=0}^{\infty}$ must depend only on z and on the moment ratio sequence $\left\{\frac{\mu_m}{\mu_{m+1}}\right\}_{m=0}^{\infty}$, a fact which can hardly be obtained in terms of the sole integral representation into Equation (9). In particular, the limiting value $\varphi_\infty = \lim_{m\to\infty} \varphi_m$ can be extracted directly from Equation (11), which yields

$$\varphi_\infty = \frac{\rho_0}{1 + z\,\rho_0}\,, \tag{12}$$

where

$$\rho_0 = \lim_{m\to\infty} \frac{\mu_m}{\mu_{m+1}}\,, \tag{13}$$

is directly related to the series convergence radius.

4. Inverse Factorial Series and Weniger Transformation

That Stieltjes series converging factors have to satisfy a recurrence relationship could represent the key to understanding why the Weniger transformation turns out to be particularly fit for summing series of the Stieltjes type.

Given the nth-order converging factor $\varphi_n(z)$, suppose there exists a sequence, say $\{c_k(z)\}_{k=0}^{\infty}$, whose terms are *independent of* n, such that the following representation holds:

$$\varphi_n(z) = \sum_{k=0}^{\infty} \frac{k!}{(n)_{k+1}}\, c_k(z)\,. \tag{14}$$

Then, on substituting from Equation (14) into Equation (7), we would have

$$F(z) = \sum_{m=0}^{n} a_m(z) + a_{m+1}(z) \sum_{k=0}^{\infty} \frac{k!}{(n+1)_{k+1}}\, c_k(z)\,, \tag{15}$$

where, for brevity, the quantity $a_m(z) = (-)^m\, \mu_m / z^{m+1}$ denotes the mth-order term of the Stieltjes series (4).

Suppose for a moment that the factorial series in Equation (14) consists only of a *finite* number of terms, say N. Then, it is easy to show that the correct value of $F(z)$ can be retrieved within a *finite* number of steps and, more importantly, *without explicitly knowing* the sequence $\{c_k(z)\}_{k=0}^{\infty}$, according to the following steps:

1. Equation (15) is first recast in the form

$$\sum_{k=0}^{N} \frac{k!}{(n+1)_{k+1}}\, c_k(z) = \frac{1}{a_{n+1}(z)}\, F(z) - \frac{s_n(z)}{a_{n+1}(z)}\,, \tag{16}$$

   where $s_n(z) = \sum_{m=0}^{n} a_m(z)$ denotes the nth-order partial sum of the series.

2. Multiply both sides of Equation (16) by $(n+1)_{N+1}$, which gives

$$(n+1)_{N+1} \sum_{k=0}^{N} \frac{k!}{(n+1)_{k+1}}\, c_k(z) = \frac{(n+1)_{N+1}}{a_{n+1}(z)}\, F(z) - \frac{(n+1)_{N+1}\, s_n(z)}{a_{n+1}(z)}\,. \tag{17}$$

3. The left side of Equation (17) is an Nth-order polynomial with respect to the variable n. This, in turn, implies that it can be annihilated by applying $N+1$ times the first forward difference operator with respect to n, defined as

$$\Delta f_n \,=\, f_{n+1} \,-\, f_n \,. \tag{18}$$

Accordingly, we have

$$\Delta^{N+1} \left\{ (n+1)_{N+1} \sum_{k=0}^{N} \frac{k!}{(n+1)_{k+1}}\, c_k(z) \right\} \,=\, 0\,, \tag{19}$$

where the iterated Δ^k operator turns out to be

$$\Delta^k f_n \,=\, (-1)^k \sum_{j=0}^{k} (-)^j \binom{k}{j} f_{n+j} \,. \tag{20}$$

4. Finally, substitution from Equation (19) into Equation (17) eventually gives

$$F(z) \,=\, \frac{\Delta^{N+1} \left\{ \dfrac{(n+1)_{N+1}\, s_n(z)}{a_{n+1}(z)} \right\}}{\Delta^{N+1} \left\{ \dfrac{(n+1)_{N+1}}{a_{n+1}(z)} \right\}}\,, \tag{21}$$

which proves the initial thesis.

It must be remarked again that Equation (21) represents an *exact* result only if the factorial expansion into Equation (14) reduces to a *finite* inverse factorial sum. If not, the quantity defined in the right side of Equation (21) can be interpreted as the Nth-order term of an infinite sequence, say $\{\delta_m^{(n)}\}_{m=1}^{\infty}$, which is built up from the partial sum sequence according to

$$\delta_m^{(n)} \,=\, \frac{\Delta^{m+1} \left\{ \dfrac{(n+1)_{m+1}\, s_n}{\Delta s_n} \right\}}{\Delta^{m+1} \left\{ \dfrac{(n+1)_{m+1}}{\Delta s_n} \right\}}\,, \qquad m \,\geq\, 1\,, n \,\geq\, 0\,, \tag{22}$$

where $\Delta s_n \,=\, a_{n+1}$, and where, for simplicity, any dependence on the variable z has to tacitly be intended. Equation (22) defines the so-called *Weniger transformation* of the partial sum sequence $\{s_m\}_{m=0}^{\infty}$ [26].

For the readers' convenience, the explicit expansion of the right side of Equation (22) is given,

$$\delta_m^{(n)} \,=\, \frac{\displaystyle\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (1+n+j)_{m+1} \frac{s_j}{\Delta s_{j+1}}}{\displaystyle\sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} (1+n+j)_{m+1} \frac{1}{\Delta s_{j+1}}}\,, \tag{23}$$

where Equation (20) has been taken into account.

What would be highly desirable is that, as $m \to \infty$, the δ-sequence defined in Equation (22) could converge, in some sense, to the correct value $F(z)$, i.e.,

$$\lim_{m\to\infty} \delta_m^{(n)} \,=\, F(z)\,, \tag{24}$$

regardless the value of the parameter n.

Presently, a general convergence theory does not exist that could validate, for a typical Stieltjes series, Equation (24). However, it seems reasonable that such a theory, assuming it does exist, should be based on the representation of the convergence factor given in (14). In the next section, an algorithm aimed at building up such a representation will be conceived.

5. On the Factorial Series Expansion of Stieltjes Series Converging Factor

Our idea consists in letting the solution of the difference Equation (11) in the form of an inverse factorial series, as follows:

$$\varphi_m(z) \,=\, \varphi_\infty \,+\, \sum_{k=0}^{\infty} \frac{k!}{(m+\beta)_{k+1}}\, c_k(z)\,, \tag{25}$$

where φ_∞ is given in Equation (12), β is a real nonnegative parameter, and $\{c_k(z)\}_{k=0}^{\infty}$ denotes an infinite sequence *independent of* m (Clearly, it is expected the sequence $\{c_k\}_{k=0}^{\infty}$ to depend also on β. However, for simplicity such a dependence has not been made explicit). For the moment, it will be assumed $\beta = 0$.

Suppose now that also the sequence $\{\mu_m/\mu_{m+1}\}_{m=0}^{\infty}$ admits itself an inverse factorial expansion of the form

$$\frac{\mu_m}{\mu_{m+1}} \,=\, \rho_0 \,+\, \sum_{k=0}^{\infty} \frac{k!}{(m)_{k+1}}\, b_k\,, \tag{26}$$

with $\{b_k\}_{k=0}^{\infty}$ also being a numerical sequence *independent of* m. As we shall see in a moment, Equation (26) can be taken for granted under rather general conditions. For example, consider again the Euler series, which is the asymptotic series (4) with $F(z) = \mathcal{E}(z)$, where

$$\mathcal{E}(z) \,=\, \int_0^\infty \frac{\exp(-t)}{t+z}\, \mathrm{d}t\,, \tag{27}$$

defines the so-called Euler integral. Euler series is a Stieltjes series, with $\mu(t) = 1 - \exp(-t)$ and $\mu_m \,=\, m!$, so that

$$\frac{\mu_m}{\mu_{m+1}} \,=\, \frac{1}{m+1} \,=\, \frac{1}{m} - \frac{1}{m} + \frac{1}{m+1} \,=\, \frac{1}{m} - \frac{1}{m(m+1)}\,. \tag{28}$$

Equation (28) is of the form (26) with $\rho_0 = 0$, $b_0 = 1$, $b_1 = -1$, and $b_k = 0$ for $k > 1$. It is worth noting that the factorial series expansion into Equation (26) can always be obtained, in principle, on assuming the moment ratio sequence to admit an inverse power series representation with respect to the index m,

$$\frac{\mu_m}{\mu_{m+1}} \,=\, \sum_{k=0}^{\infty} \frac{\rho_k}{m^k} \,=\, \rho_0 \,+\, \sum_{k=1}^{\infty} \frac{\rho_k}{m^k} \,=\, \rho_0 \,+\, \sum_{k=0}^{\infty} \frac{\rho_{k+1}}{m^{k+1}}\,, \qquad m \ge 1\,. \tag{29}$$

The last series expansion can always be transformed into an inverse factorial series, for example, by using the following conversion formula [50]:

$$\frac{\mu_m}{\mu_{m+1}} \,=\, \rho_0 \,+\, \sum_{k=0}^{\infty} \frac{\rho_{k+1}}{m^{k+1}} \,=\, \rho_0 \,+\, \sum_{k=0}^{\infty} \frac{k!}{(m)_{k+1}}\, b_k\,, \tag{30}$$

where

$$b_k \,=\, \frac{(-1)^k}{k!} \sum_{\mu=0}^{k} (-1)^\mu\, s(k,\mu)\, \rho_{\mu+1}\,, \tag{31}$$

and the symbol $s(k,\mu)$ denotes the Stirling number of the first order (Mathematica Notation for the Stirling number: $s \to \mathrm{S1}$). In order to solve Equation (11), the following formula will also play a key role:

$$\varphi_{m+1} \,=\, \varphi_\infty \,+\, \frac{c_0}{m} \,+\, \sum_{k=1}^{\infty} \frac{k!}{(m)_{k+1}}\, (c_k \,-\, c_{k-1})\,, \tag{32}$$

whose derivation is detailed in Appendix B. Another formula which will be revealed to be of pivotal importance is that concerning the *product* of two factorial series. In particular, it is not difficult to prove that

$$\left(\sum_{k=0}^{\infty} \frac{k!}{(m)_{k+1}}\, b_k\right)\left(\sum_{k=0}^{\infty} \frac{k!}{(m)_{k+1}}\, c_k\right) = \sum_{k=1}^{\infty} \frac{k!}{(m)_{k+1}}\, d_k\,, \tag{33}$$

with the sequence $\{d_k\}_{k=1}^{\infty}$ being defined as follows:

$$d_k = \frac{1}{k!}\sum_{\nu=0}^{k-1}\sum_{\mu=0}^{\nu} \nu!\,(\mu+1)_{k-1-\nu}\, b_{k-1-\nu}\, c_{\nu-\mu}\,, \tag{34}$$

or, equivalently,

$$d_k = \frac{1}{k!}\sum_{\nu=0}^{k-1}\sum_{\lambda=0}^{\nu} \nu!\,(\nu-\lambda+1)_{k-1-\nu}\, b_{k-1-\nu}\, c_{\lambda}\,. \tag{35}$$

Now, all we have to do is to substitute from Equations (30), (25), and (32) into Equation (11). Nontrivial mathematical steps, detailed in Appendix C, lead to

$$\begin{cases} c_{-1} = 0\,, \\ c_0 = \dfrac{b_0}{(1+z\rho_0)^2}\,, \\ c_k = \dfrac{c_{k-1}}{1+z\rho_0} + \dfrac{b_k}{(1+z\rho_0)^2} - \dfrac{z}{1+z\rho_0}\, d_k\,, \qquad k \geq 1\,, \end{cases} \tag{36}$$

which, together with Equations (26) and (34), represents the main result of the present paper.

We have proved that the nth-order truncation error of a Stieltjes series can be, under quite general hypotheses, expressed as the product of the first term of the series neglected by the nth-order partial sum, times an inverse factorial series whose single terms are defined through a triangular recurrence relation involving, apart from z, only the asymptotic expansion of the moment ratio sequence $\{\mu_m/\mu_{m+1}\}$ for $m \to \infty$.

It should be noted how, in the case of divergent series with $\rho_0 = 0$, Equation (36) takes on the following form:

$$\begin{cases} c_{-1} = 0\,, \\ c_0 = b_0\,, \\ c_k = c_{k-1} + b_k - z\, d_k\,, \qquad k > 1\,. \end{cases} \tag{37}$$

6. A Few Examples of Applications

6.1. Revisiting Euler Series

In Ref. [51], the factorial representation (25) of the converging factor of the Euler series was found for the particular case of $\beta = 0$, being

$$\varphi_m(z) = \sum_{k=0}^{\infty} \frac{k!}{(m)_{k+1}}\, z L_k^{(-1)}(z)\,, \tag{38}$$

where the symbol $L_k^{(\alpha)}(\cdot)$ denotes the associate Laguerre polynomial [52]. Equation (38) was obtained by using the general approach proposed in [53]. This led to a nontrivial derivation involving the use of Bernoulli as well as exponential polynomials [51]. In the present section, Equation (38) will now be re-derived by using the results previously obtained and generalized to arbitrary positive values of β. To this end, the first, preliminary step is to find the factorial expansion of the moment ratio in terms of $m+\beta$. This can be achieved, for instance, by recasting the moment ratio as follows:

$$\frac{\mu_m}{\mu_{m+1}} = \frac{1}{m+1} = \frac{1}{m+\beta-(\beta-1)}, \tag{39}$$

and then by using the fundamental expansion ([54] Equation (3) on p. 77): (In the books by (Milne-Thomson [55] p. 291) and by (Paris and Kaminski [56] p. 177), this expansion is called Waring's formula, but in Nielsen's book ([54] Footnote 2 on p. 77), this formula is attributed to Stirling [57], and in the book by (Nörlund [58] p. 199), it is attributed to Nicole (compare also (Tweedie [59] p. 30))).

$$\frac{1}{z-w} = \sum_{n=0}^{\infty} \frac{(w)_n}{(z)_{n+1}}, \qquad \mathrm{Re}(z-w) > 0, \tag{40}$$

which gives at once

$$\frac{\mu_m}{\mu_{m+1}} = \sum_{k=0}^{\infty} \frac{k!}{(m+\beta)_{k+1}} \frac{(\beta-1)_k}{k!}. \tag{41}$$

On comparing Equation (41) with Equation (26) and on taking into account that $\rho_0 = 0$, we have

$$b_k = \frac{(\beta-1)_k}{k!}, \qquad k \geq 0, \quad \beta \geq 0. \tag{42}$$

It is worth showing an alternative approach, based on the asymptotic expansion in Equation (29). In particular, the expanding coefficients $\{\rho_k\}_{k=0}^{\infty}$ can be found by writing Equation (39) as follows:

$$\frac{\mu_m}{\mu_{m+1}} = \frac{1}{m+\beta-(\beta-1)} = \frac{1}{m+\beta} \frac{1}{1-\dfrac{\beta-1}{m+\beta}}, \tag{43}$$

and by formally expanding the second factor as a geometric series, i.e.,

$$\frac{\mu_m}{\mu_{m+1}} = \sum_{k=0}^{\infty} \frac{(\beta-1)^k}{(m+\beta)^{k+1}}. \tag{44}$$

On comparing Equation (44) with Equation (29), we have $\rho_0 = 0$ and $\rho_k = (\beta-1)^{k-1}$, for $k \geq 1$, so that Equation (31) leads to

$$b_k = \frac{(-1)^k}{k!} \sum_{\mu=0}^{k} (-1)^{\mu} s(k,\mu) (\beta-1)^{\mu} = \frac{(\beta-1)^k}{k!}, \tag{45}$$

which proves again Equation (42).

Once the factorial expansion of the moment ratio has been obtained, it is sufficient to substitute from Equation (42) into Equation (36) to obtain, with a small amount of elementary algebra,

$$\begin{aligned} c_0 &= 1, \\ c_1 &= \beta - z = L_1^{(\beta-1)}(z), \\ c_2 &= \frac{1}{2}\left(\beta^2 + \beta + z^2 - 2(\beta+1)z\right) = L_2^{(\beta-1)}(z), \end{aligned} \tag{46}$$

and so on, for all $k > 2$. Accordingly, Equation (38) becomes

$$\varphi_m(z) = \sum_{k=0}^{\infty} \frac{k!}{(m+\beta)_{k+1}} \, z L_k^{(\beta-1)}(z) \,, \qquad \beta \geq 0 \,. \tag{47}$$

6.2. Error Function

The following Stieltjes series:

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{z^{m+1}} \, \Gamma\left(m + \frac{1}{2}\right), \tag{48}$$

where $\Gamma(\cdot)$ denotes the Gamma function [60], is asymptotic to the complementary error function $\mathrm{erfc}(\cdot)$ [60], being

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{z^{m+1}} \, \Gamma\left(m + \frac{1}{2}\right) \sim \frac{\pi \, \exp(z) \, \mathrm{erfc}(\sqrt{z})}{\sqrt{z}} \,, \qquad z \to \infty \,. \tag{49}$$

In order to find the factorial expansion of the corresponding converging factor, it is sufficient to recast the moment ratio as follows:

$$\frac{\mu_m}{\mu_{m+1}} = \frac{\Gamma\left(m + \frac{1}{2}\right)}{\Gamma\left(m + \frac{3}{2}\right)} = \frac{1}{m + \frac{1}{2}} = \frac{1}{\left(m - \frac{1}{2}\right) + 1} \,, \tag{50}$$

and to note that it is formally identical to Equation (39), once the transformation $m \to m - 1/2$ is applied. Accordingly, it is not difficult to prove that all steps leading to Equation (47) for the Euler series can be repeated for the asymptotic series (48), which yields

$$\varphi_m(z) = \sum_{k=0}^{\infty} \frac{k!}{\left(m + \beta - \frac{1}{2}\right)_{k+1}} \, z L_k^{(\beta-1)}(z) \,, \qquad m > 1 \,. \tag{51}$$

6.3. The Logarithm Function

Another important Stieltjes function is the natural logarithm. Consider the function $\mathcal{L}(z)$ defined as follows:

$$\mathcal{L}(z) = \int_0^1 \frac{\mathrm{d}t}{z + t} = \log\left(1 + \frac{1}{z}\right). \tag{52}$$

The Stieltjes series of the form (4) asymptotic to $\mathcal{L}(z)$, is characterized by the following moment sequence:

$$\mu_m = \frac{1}{m+1} \,, \qquad m \geq 0 \,, \tag{53}$$

so that

$$\frac{\mu_m}{\mu_{m+1}} = \frac{m+2}{m+1} = 1 + \frac{1}{m+1} \,. \tag{54}$$

Similarly, as for the error function, the factorial expansion of the converging factor of the logarithm can also be obtained in a surprisingly simple way. In fact, it is sufficient to compare Equation (54) with Equation (30), written with $m + \beta$ in place of m and, on taking Equation (41) into account, to have

$$\frac{\mu_m}{\mu_{m+1}} = 1 + \sum_{k=0}^{\infty} \frac{k!}{(m+\beta)^{k+1}} \, \frac{(\beta-1)^k}{k!} \,, \tag{55}$$

from which it follows that

$$\rho_0 = 1 \qquad b_k = \frac{(\beta-1)^k}{k!}\,. \tag{56}$$

Then, on substituting from Equation (56) into Equation (36) after simple algebra, we obtain

$$\begin{aligned} c_0 &= \frac{1}{(1+z)^2}\,,\\ c_1 &= \frac{\beta+(\beta-2)\,z}{(1+z)^3}\,,\\ c_2 &= \frac{\beta(1+\beta)\,+\,2(\beta-2)(\beta+1)\,z\,+\,(\beta-2)(\beta-1)\,z^2}{2\,(1+z)^4}\,, \end{aligned} \tag{57}$$

and so on, for all $k > 2$. Remarkably, it is also possible to give Equation (57) the following closed-form expression:

$$c_k(z) = \frac{(-1)^k}{(1+z)^2}\,P_k^{(2-\beta-k,\beta-1)}\left(\frac{z-1}{z+1}\right), \qquad k \geq 0\,, \tag{58}$$

where the symbol $P_n^{(\alpha,\beta)}(\cdot)$ denotes the Jacobi polynomial [60].

6.4. Lerch's Trascendental Function

The logarithm function is a particular case of a more general Stieltjes function, namely the Lerch trascendental [60], whose asymptotic Stieltjes series (4) is characterized by the following moment sequence:

$$\mu_m = \frac{1}{(m+\alpha)^s}\,, \qquad m \geq 0\,, \tag{59}$$

where, for simplicity, it will be assumed $\alpha > 0$ and $s > 1$. The moment ratio then turns out to be

$$\frac{\mu_m}{\mu_{m+1}} = \left(1+\frac{1}{m+\alpha}\right)^s = \left(1+\frac{1}{m+\beta+(\alpha-\beta)}\right)^s, \qquad m \geq 0\,, \tag{60}$$

whose inverse factorial expansion can be obtained by using Equations (29)–(31). To this end, consider first the case $\beta = 0$. Then, the asymptotic inverse power expansion with respect to m gives

$$\begin{aligned} \rho_0 &= 1\,,\\ \rho_1 &= s\,,\\ \rho_2 &= \frac{1}{2}\,s(s-1)\,-\,s\alpha\,,\\ \rho_3 &= \frac{1}{6}\,s(s-1)(s-2) - s(s-1)\alpha + s\alpha^2\,,\\ \rho_4 &= \frac{1}{24}\,s(s-1)(s-2)(s-3)\,-\,\frac{1}{2}\,s(s-1)(s-2)\alpha\,+\,\frac{3}{2}\,s(s-1)\alpha^2\,-\,s\alpha^3\,,\\ &\ldots \end{aligned} \tag{61}$$

Accordingly, on using Equation (31) again, the expanding coefficients b_k of the factorial expansion of $\frac{\mu_m}{\mu_{m+1}}$ turn out to be

$$\begin{aligned} b_0 &= s\,, \\ b_1 &= \frac{1}{2}\,s(s-1-2\alpha)\,, \\ b_2 &= \frac{1}{12}\,s(s^2-1-6s\alpha+6\alpha^2)\,, \\ b_3 &= \frac{1}{144}\,s\,(s(s(6+s)-1)-12s(s+3)\alpha\,+\,36(s+1)\alpha^2-24\alpha^3\,-\,6)\,, \\ &\dots \end{aligned} \tag{62}$$

and our recursive algorithm provides the following analytical expressions of the first terms of the sequence $\{c_k\}_{k=0}^{\infty}$:

$$\begin{aligned} c_0 &= \frac{s}{(1+z)^2}\,, \\ c_1 &= -\,\frac{(s((s+1)(z-1)+2(1+z)\alpha)}{2(1+z)^3}\,, \\ c_2 &= \frac{s((1+s)(5+s+s(z-4)z-z(8+z))+6(s(z-1)-2)(1+z)\alpha+6(1+z)^2\alpha^2)}{12(1+z)^4}\,, \\ &\dots \end{aligned} \tag{63}$$

Note that, in order to recover the more general factorial expansion of the converging factor (25) valid for a typical $\beta > 0$, it is sufficient, according to Equation (60), to replace α by $\alpha - \beta$ into Equation (63). Differently from Equation (57), we were not able to give Equation (63) a closed form expression.

7. Conclusions

Although the practical usefulness of divergent series has always been more important than concerns about mathematical rigor, especially in physics, a theoretical understanding of the mathematical machinery behind Levin-type sequence transformations would be highly desirable. An important achievement in this direction has been established in [13], where it was rigorously proved that a particular type of Levin-type transformation, the Weniger transformation [26], is not only able to resum the Euler series, but also that it turns out to be "exponentially faster" than Padé approximants in accomplishing such a task. The Euler series is the paradigm of factorial divergence. Several other series expansions occurring in mathematical physics are factorially or even hyperfactorially diverging. The unquestionable superiority of the Weniger transformation over Padé in resumming the Euler series proved in [13] was not limited to such a specific case, as it was shown in the past [61–65].

Levin-type transformations in general, and Weniger's transformation in particular, are able to dramatically accelerate the convergence of slowly convergent series as well as to resum wildly divergent series, some of them even not resummable via Padé. The retrieving capabilities of Weniger's transformation are strictly related to the possibility of representing the series converging factor of any finite order as a suitable inverse factorial series.

In the present paper, which is the natural continuation of Ref. [13], it has been proved that the nth-order converging factor of a Stieltjes series can be expressed through an inverse factorial series, whose terms are uniquely determined, starting from the knowledge of the series moment ratio, through a linear recurrence relation. Our constructive proof has the form of an algorithm, developed starting from the first-order difference equation which has to be satisfied by the convergent factors of *any* Stieltjes series. A few examples of applications of our algorithm have also been presented, in order to show its effectiveness and implementation ease.

The class of Stieltjes series is of fundamental interest in mathematical physics. Recently, it was rigorously shown that the factorially divergent perturbation expansion for the energy eigenvalue of the $\mathcal{PT}$-symmetric Hamiltonian $H(\lambda) = p^2 + \frac{1}{4}x^2 + \mathrm{i}\lambda x^3$ is a Stieltjes series [66,67], thus proving a conjecture formulated ten years before by Bender and Weniger [68]. A much more recent result is the proof that the character of the celebrated Bessel solution of Kepler's equation [69] is also Stieltjes [70]. Weniger's transformation turned out to be successful, in the past, for resumming extremely violently divergent perturbation expansions [71–74], where Padé approximants were not powerful enough to achieve anything substantial, as in the case of Rayleigh-Schrödinger perturbation series for the sextic anharmonic oscillator, or even not able to sum, as for the more challenging octic case [75]. There is no doubt that a better theoretical understanding of the summation of hyper-factorially divergent expansions via Weniger's transformation would also be highly desirable.

Ernst Joachim Weniger and I were strongly convinced that the results obtained in [13] could have been a preliminary step toward the development of a general convergence theory of the summation of Stieltjes series with Levin-type transformations in place of Padé approximants. What has been shown here is a small leap in this direction. The present work could also lead to a hypothetical general framework for the analysis of the convergence rate of Weniger's transformation. A possible path in this direction (I must thank one of the anonymous reviewer for such an interesting suggestion), i.e., to obtain an estimate for the error of a delta transformation, could be to look at the first term of the inverse factorial series not eliminated at the m-th step into the δ-sequence of Equation (22), and to try approximating the remainder term of the transformation in some compact asymptotic form. This is not an easy task at all but worthy of being pursued in the future.

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Appendix A. Basic Properties of Factorial Series

Let $\Omega(z)$ be a function that vanishes as $z \to +\infty$. A *factorial series* for $\Omega(z)$ is an expansion in terms of inverse Pochhammer symbols of the following kind:

$$\Omega(z) = \frac{b_0}{z} + \frac{b_1 1!}{z(z+1)} + \frac{b_2 2!}{z(z+1)(z+2)} + \cdots = \sum_{\nu=0}^{\infty} \frac{b_\nu \nu!}{(z)_{\nu+1}} . \tag{A1}$$

In general, $\Omega(z)$ will have simple poles at $z = -m$ with $m \in \mathbb{N}_0$.

Factorial series had been employed already in Stirling's classic book *Methodus Differentialis* [57,76], which appeared in 1730. In the nineteenth and the early twentieth century, the theory of factorial series was fully developed by a variety of authors. Fairly complete surveys of the older literature as well as thorough treatments of their properties can be found in books on finite differences by Meschkowski [77], Milne-Thomson [55], Nielsen [54], and Nörlund [58,78,79]. Factorial series are also discussed in the books by Knopp [80] and Nielsen [81] on infinite series.

Compared to inverse power series, factorial series possess vastly superior convergence properties. If we use (see for example (Olver et al. [2] Equation (5.11.12)))

$$\Gamma(z+a)/\Gamma(z+b) \;=\; z^{a-b}\left[1+\mathrm{O}(1/z)\right], \qquad z\to\infty, \tag{A2}$$

we obtain the asymptotic estimate $n!/(z)_{n+1} = \mathrm{O}(n^{-z})$ as $n\to\infty$. Thus, the factorial series (A1) converges with the possible exception of the points $z=-m$ with $m\in\mathbb{N}_0$ if and only if the associated Dirichlet series $\bar{\Omega}(z)=\sum_{n=1}^{\infty} b_n/n^z$ converges (see for example ([80] p. 262) or ([82] p. 167)). Accordingly, a factorial series converges for sufficiently large $\mathrm{Re}(z)$ even if the reduced series coefficients b_n in (A1) grow like a fixed power n^α with $\mathrm{Re}(\alpha)>0$ as $n\to\infty$. Thus, a factorial series can converge even if its effective coefficients $b_n n!$ diverge quite wildly as $n\to\infty$.

Factorial series can also be expressed in terms of the beta function $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ ([2] Equation (5.12.1)). Because of $B(z,n+1)=n!/(z)_{n+1}$, the factorial series in Equation (A1) can be interpreted as an expansion in terms of special beta functions (see for instance (Milne-Thomson [55] p. 288) or (Paris and Kaminski [56] [p. 175])):

$$\Omega(z) \;=\; \sum_{n=0}^{\infty} b_n\, B(z,n+1)\,. \tag{A3}$$

The beta function possesses the integral representation (see for example ([2] Equation (5.12.1)))

$$B(x,y) \;=\; \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\mathrm{d}t\,, \qquad \mathrm{Re}(x),\mathrm{Re}(y)>0\,, \tag{A4}$$

which implies

$$B(z,n+1) \;=\; \frac{n!}{(z)_{n+1}} \;=\; \int_0^1 t^{z-1}\,(1-t)^{n}\,\mathrm{d}t\,, \qquad \mathrm{Re}(z)>0 \quad n\in\mathbb{N}_0\,. \tag{A5}$$

If we insert this into Equation (A3) and interchange integration and summation, we obtain the following very important integral representation ([54] Section I on p. 244):

$$\Omega(z) \;=\; \int_0^1 t^{z-1}\,\varphi_\Omega(t)\,\mathrm{d}t\,, \qquad \mathrm{Re}(z)>0\,, \tag{A6a}$$

$$\varphi_\Omega(t) \;=\; \sum_{n=0}^{\infty} b_n\,(1-t)^n\,. \tag{A6b}$$

Often, the properties of $\Omega(z)$ can be studied more easily via this integral representation than via the defining factorial series (A1) (see for example ([54] Ch. XVII)).

Appendix B. Proof of Equation (32)

First of all, φ_{m+1} is recast as follows:

$$\varphi_{m+1} \;=\; \varphi_m \,+\, \Delta\,\varphi_m\,. \tag{A7}$$

Then, on applying the forward difference operator Δ to both sides of Equation (25), we have

$$\Delta\,\varphi_m \;=\; \Delta\sum_{k=0}^{\infty}\frac{k!}{(m)_{k+1}}\,c_k \;=\; \sum_{k=0}^{\infty} k!\,c_k\,\Delta\left(\frac{1}{(m)_{k+1}}\right), \tag{A8}$$

where (see ([26] Equation (13.2-12)))

$$\Delta\left(\frac{1}{(m)_{k+1}}\right) \;=\; -\frac{k+1}{(m)_{k+2}}\,, \tag{A9}$$

so that

$$\Delta\,\varphi_m \;=\; -\sum_{k=1}^{\infty}\frac{k!}{(m)_{k+1}}\,c_{k-1}\,. \tag{A10}$$

Then, on taking again Equation (A7) into account, from Equations (25) and (A10), Equation (32) follows after simple rearranging.

Appendix C. Proof of Equation (36)

We start by recasting Equation (11) as follows:

$$\varphi_{m+1} \;=\; \frac{\mu_m}{\mu_{m+1}} \,-\, z\,\frac{\mu_m}{\mu_{m+1}}\,\varphi_m\,, \tag{A11}$$

which, on taking Equations (25), (30), and (32) into account, can be recast as follows:

$$\begin{aligned}\varphi_\infty \,+\, \sum_{k=0}^{\infty}\frac{k!}{(m)_{k+1}}\,\Delta c_{k-1} \;=\; \rho_0 \,+\, \sum_{k=0}^{\infty}\frac{k!}{(m)_{k+1}}\,b_k \\ -\,z\left(\rho_0 \,+\, \sum_{k=0}^{\infty}\frac{k!}{(m)_{k+1}}\,b_k\right)\left(\varphi_\infty \,+\, \sum_{k=0}^{\infty}\frac{k!}{(m)_{k+1}}\,c_k\right),\end{aligned} \tag{A12}$$

where, for simplicity, it has been assumed formally $c_{-1} = 0$. Equation (A12) can be rearranged, on taking Equation (12) into account, as follows:

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{k!}{(m)_{k+1}}\,\Delta c_{k-1} \;=\; \frac{1}{1\,+\,z\rho_0}\,\sum_{k=0}^{\infty}\frac{k!}{(m)_{k+1}}\,b_k \\ -\,z\rho_0\,\sum_{k=0}^{\infty}\frac{k!}{(m)_{k+1}}\,c_k \,-\, z\left(\sum_{k=0}^{\infty}\frac{k!}{(m)_{k+1}}\,b_k\right)\left(\sum_{k=0}^{\infty}\frac{k!}{(m)_{k+1}}\,c_k\right),\end{aligned} \tag{A13}$$

Finally, on using Equation (33) into Equation (A13), after simple algebra, we obtain

$$\begin{cases} c_0 \;=\; \dfrac{b_0}{1+z\rho_0} \,-\, z\rho_0\,c_0\,, \\[2ex] \Delta c_{k-1} \;=\; \dfrac{b_k}{1+z\rho_0} \,-\, z\rho_0\,c_k \,-\, z\,d_k\,, \qquad k \ge 1\,, \end{cases} \tag{A14}$$

from which Equation (36) follows.

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