Angular momentum at null infinity in five dimensions

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Abstract

Using the Bondi coordinates, we discuss the angular momentum at null infinity in five dimensions and address the Poincare covariance of the Bondi angular momentum. We also show the angular momentum loss/gain law due to gravitational waves. In four dimensions, the angular momentum at null infinity has the supertranslational ambiguity and then it is known that we cannot construct well-defined angular momentum there. On the other hand, we would stress that we can define angular momentum at null infinity without any ambiguity in higher dimensions. This is because of the non-existence of supertranslations in higher dimensions.

1 Introduction

Inspired by the recent progress of the string theory, the importance of the gravity theory in higher dimensional space-times is steadily growing. However, there are still many remaining issues to be investigated in higher dimensions. One issue among them is the asymptotic structure. For asymptotically flat spacetimes, the asymptotic structure is defined at spatial and null infinites. The asymptotic structure at spatial infinity (spi) is well-defined by conformal embedding in four and higher dimensions [1]. The asymptotic structure at null infinity in four dimensions is well studied by many authors [2, 3]. On the other hand, there are only a few work about the asymptotic structure at null infinity in higher dimensions. Indeed, asymptotic flatness has been defined by using conformal completion method in only even dimensions [4–6] and by using the Bondi coordinates in five dimensions [7].

The asymptotic symmetry at null infinity in four dimensions is semi-direct product of the Lorentz group and the supertranslational group, which is an infinite dimensional translational group. The presence of supertranslations implies the infinite number of the direction of translation, while the Poincare group has only four directions in four dimensions. Because of this infinite directions of translation, we cannot construct well-defined angular momentum in four dimensions. There are many attempts to define of angular momentum at null infinity in four dimensions, whereas all those definitions are suffered from supertranslational ambiguity. On the other hands, in five dimensions, since asymptotic symmetry is the Poincare group [7], we can expect that angular momentum at null infinity can be defined without any ambiguities. The purpose of this paper is to discuss angular momentum at null infinity in five dimensions. We will see that angular momentum can be defined well and show the Poincare covariance of the Bondi angular momentum.

2 Bondi coordinate and Einstein equations

In the Bondi coordinates $x^a = (u, r, \theta, \phi, \psi)$ the metric can be written as

$$ds^2 = -\frac{Ve^B}{r^2}du^2 - 2e^B dudr + r^2 h_{AB}(dx^A + U^A du)(dx^B + U^B du), \tag{1}$$

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where

$$h_{AB} = \begin{pmatrix} e^{C_1} & \sin\theta\sinh D_1 & \cos\theta\sinh D_2 \\ \sin\theta\sinh D_1 & e^{C_2}\sin^2\theta & \sin\theta\cos\theta\sinh D_3 \\ \cos\theta\sinh D_2 & \sin\theta\cos\theta\sinh D_3 & e^{C_3}\cos^2\theta \end{pmatrix}, \tag{2}$$

where $x^A = (\theta, \phi, \psi)$ and we adopted the gauge condition satisfying $\det h_{AB} = \sin^2\theta\cos^2\theta$. $u = \text{const.}$ are null hypersurfaces and the periods of the coordinates θ, ϕ and ψ are $\pi/2$, 2π and 2π, respectively. From the gauge condition, e^{C_3} can be written as

$$e^{C_3} = \frac{1 + e^{C_2}\sinh^2 D_2 + e^{C_1}\sinh^2 D_3 - 2\sinh D_1 \sinh D_2 \sinh D_3}{e^{C_1+C_2} - \sinh^2 D_1}. \tag{3}$$

Then h_{AB} have five functional freedom. In the following we will identify C_1, C_2, D_1, D_2, D_3 as those freedom. In this coordinate system, null infinity is represented by $r = \infty$ and the metric at null infinity is

$$ds^2 = -du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2). \tag{4}$$

To investigate the asymptotic structure at null infinity, we have to solve the Einstein equations near null infinity. Here note that five dimensional space-times have five degree of freedom of gravitational fields. If we identify h_{AB} as the freedom of gravitational field, C_1, C_2, D_1, D_2, D_3 can be expanded as

$$C_1(u,r,x^A) = \frac{C_{11}(u,x^A)}{r\sqrt{r}} + \frac{C_{12}(u,x^A)}{r^2} + \frac{C_{13}(u,x^A)}{r^2\sqrt{r}} + \frac{C_{14}(u,x^A)}{r^3} + O(r^{-7/2}) \tag{5}$$

$$C_2(u,r,x^A) = \frac{C_{21}(u,x^A)}{r\sqrt{r}} + \frac{C_{22}(u,x^A)}{r^2} + \frac{C_{23}(u,x^A)}{r^2\sqrt{r}} + \frac{C_{24}(u,x^A)}{r^3} + O(r^{-7/2}) \tag{6}$$

$$D_1(u,r,x^A) = \frac{D_{11}(u,x^A)}{r\sqrt{r}} + \frac{D_{12}(u,x^A)}{r^2} + \frac{D_{13}(u,x^A)}{r^2\sqrt{r}} + \frac{D_{14}(u,x^A)}{r^3} + O(r^{-7/2}) \tag{7}$$

$$D_2(u,r,x^A) = \frac{D_{21}(u,x^A)}{r\sqrt{r}} + \frac{D_{22}(u,x^A)}{r^2} + \frac{D_{23}(u,x^A)}{r^2\sqrt{r}} + \frac{D_{24}(u,x^A)}{r^3} + O(r^{-7/2}) \tag{8}$$

$$D_3(u,r,x^A) = \frac{D_{31}(u,x^A)}{r\sqrt{r}} + \frac{D_{32}(u,x^A)}{r^2} + \frac{D_{33}(u,x^A)}{r^2\sqrt{r}} + \frac{D_{34}(u,x^A)}{r^3} + O(r^{-7/2}). \tag{9}$$

From Einstein equations $R_{rA} = 0$, we can see that U^A can be expanded as

$$U^A = \frac{U_1^A(u,x^A)}{r^2\sqrt{r}} + \frac{U_2^A(u,x^A)}{r^3} + \frac{U_3^A(u,x^A)}{r^3\sqrt{r}} + \frac{U_4^A(u,x^A)}{r^4} + O(r^{-9/2}), \tag{10}$$

and coefficients U_n^A can be written by $C_{1n}, C_{2n}, D_{1n}, D_{2n}, D_{3n}$, for example,

$$U_1^\theta = \frac{2}{5}\left[\frac{1}{\sin\theta\cos^2\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos^2\theta C_{11}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}D_{11} + \frac{1}{\cos\theta}\frac{\partial}{\partial\psi}D_{21} - \frac{1}{\sin\theta\cos\theta}C_{21}\right]. \tag{11}$$

Next, let us define the Bondi angular momentum from uA components of the metric. Near null infinity, $g_{u\phi}$ and $g_{u\psi}$ are expanded as

$$g_{u\phi} = \frac{1}{\sqrt{r}}\sin^2\theta U_1^\phi + \frac{1}{r}\sin^2\theta U_2^\phi + \frac{1}{r\sqrt{r}}\sin^2\theta U_3^\phi + \frac{1}{r^2}j^\phi + O(r^{-5/2}) \tag{12}$$

$$g_{u\psi} = \frac{1}{\sqrt{r}}\cos^2\theta U_1^\psi + \frac{1}{r}\cos^2\theta U_2^\psi + \frac{1}{r\sqrt{r}}\cos^2\theta U_3^\psi + \frac{1}{r^2}j^\psi + O(r^{-5/2}), \tag{13}$$

where

$$j^\phi = \sin\theta D_{11}U_1^\theta + \sin^2\theta C_{21}U_1^\phi + \sin\theta\cos\theta D_{31}U_1^\psi + \sin^2\theta U_4^\phi \tag{14}$$

$$j^\psi = \cos\theta D_{21}U_1^\theta + \sin\theta\cos\theta D_{31}U_1^\phi - \cos^2\theta(C_{11}+C_{21})U_1^\psi + \cos^2\theta U_4^\psi. \tag{15}$$

We define the Bondi angular momenta, $J^{\phi}_{\rm Bondi}$ and $J^{\psi}_{\rm Bondi}$, will be naturally defined by

$$J^{\phi}_{\rm Bondi}(u) \;=\; -\frac{1}{4\pi}\int_{S^3} j^{\phi} d\Omega\,,\; J^{\psi}_{\rm Bondi}(u) \;=\; -\frac{1}{4\pi}\int_{S^3} j^{\psi} d\Omega. \tag{16}$$

From $R_{uA}=0$, we can derive the evolution equations of angular momentum $J^{\phi}_{\rm Bondi}$ and $J^{\psi}_{\rm Bondi}$. The evolution equation of $J^{\phi}_{\rm Bondi}$ by gravitational waves can be expressed as

$$\frac{d}{du}J^{\phi}_{\rm Bondi}(u) \;=\; -\frac{1}{4\pi}\int_{S^3}\left[\left(\frac{\partial j^{\phi}}{\partial u}\right)_{\rm radiation}+\left(\frac{\partial j^{\phi}}{\partial u}\right)_{\rm total\ derivative}\right]d\Omega, \tag{17}$$

where $(\partial j^{\phi}/\partial u)_{\rm radiation}$ is the radiation part given by

$$\begin{aligned}
\left(\frac{\partial j^{\phi}}{\partial u}\right)_{\rm radiation} \;=\;& -\frac{1}{4}\frac{\partial C_{11}}{\partial\phi}\frac{\partial C_{11}}{\partial u}-\frac{1}{8}\frac{\partial C_{21}}{\partial\phi}\frac{\partial C_{11}}{\partial u}-\frac{1}{8}\frac{\partial C_{11}}{\partial\phi}\frac{\partial C_{21}}{\partial u}-\frac{1}{4}\frac{\partial C_{21}}{\partial\phi}\frac{\partial C_{21}}{\partial u}\\
&-\frac{1}{4}\frac{\partial D_{21}}{\partial\phi}\frac{\partial D_{21}}{\partial u}+\frac{1}{10}\tan\theta\frac{\partial D_{31}}{\partial u}\left(\frac{\partial C_{11}}{\partial\psi}+\frac{\partial C_{21}}{\partial\psi}\right)+\frac{3}{20}\tan\theta\frac{\partial D_{31}}{\partial\psi}\frac{\partial C_{11}}{\partial u}\\
&+\frac{2}{5}\tan\theta\frac{\partial D_{31}}{\partial\psi}\frac{\partial C_{21}}{\partial u}-\frac{1}{4}\frac{\partial D_{11}}{\partial\phi}\frac{\partial D_{11}}{\partial u}+\frac{3}{20}\tan\theta\frac{\partial D_{11}}{\partial u}\frac{\partial D_{21}}{\partial\psi}-\frac{3}{20}\tan\theta\frac{\partial D_{11}}{\partial\psi}\frac{\partial D_{21}}{\partial u}\\
&-\frac{1}{4}\tan\theta\frac{\partial C_{11}}{\partial\psi}\frac{\partial D_{31}}{\partial u}-\frac{1}{10}\tan\theta\frac{\partial C_{21}}{\partial u}\frac{\partial D_{31}}{\partial\psi}-\frac{2}{5}\tan\theta\frac{\partial D_{31}}{\partial u}\frac{\partial C_{21}}{\partial\psi}-\frac{1}{4}\frac{\partial D_{31}}{\partial u}\frac{\partial D_{31}}{\partial\phi}\\
&+\frac{3}{20\cos^2\theta}\frac{\partial D_{31}}{\partial u}\frac{\partial}{\partial\theta}(\sin\theta\cos^2\theta D_{21})-\frac{3}{20}\cos\theta\frac{\partial}{\partial\theta}(\tan\theta D_{31})\frac{\partial D_{21}}{\partial u}\\
&+\frac{3}{20\cos^2\theta}\frac{\partial D_{11}}{\partial u}\frac{\partial}{\partial\theta}(\sin\theta\cos^2\theta C_{11})-\frac{3}{20}\cos\theta\frac{\partial C_{11}}{\partial u}\frac{\partial}{\partial\theta}(\tan\theta D_{11})\\
&+\frac{3}{20\sin\theta\cos\theta}\frac{\partial C_{21}}{\partial u}\frac{\partial}{\partial\theta}(\sin^2\theta\cos\theta D_{11})-\frac{3}{20}\sin\theta\frac{\partial C_{21}}{\partial\theta}\frac{\partial D_{11}}{\partial u}\\
&-\frac{3}{20\cos\theta}\frac{\partial}{\partial u}(C_{21}D_{11})
\end{aligned} \tag{18}$$

and since $(\partial j^{\phi}/\partial u)_{\rm total\ derivative}$ is the total derivative part which has no contribution to the evolution for the angular momentum by gravitational waves, we do not write its explicit form here. The evolution equation for $J^{\psi}_{\rm Bondi}$ has same form. For the details, see [8].

3 Asymptotic symmetry and Poincare covariance of angular momentum

Asymptotic symmetry is defined as the transformation group which preserve the boundary conditions of at null infinity. By infinitesimal transformations ξ^a, the metric is transformed as $g_{ab}\to g_{ab}+\delta g_{ab}$, where

$$\delta g_{ab} \;=\; \nabla_a\xi_b+\nabla_b\xi_a. \tag{19}$$

In Ref. [7], it was shown that the asymptotic symmetry at null infinity in five dimensions is Poincare group and the generator of the translations is represented as $\xi^u=-f(x^A)$ where the function $f(x^A)$ is

$$f(x^A) \;=\; a_u+a_x\sin\theta\cos\phi+a_y\sin\theta\sin\phi+a_z\cos\theta\cos\psi+a_w\cos\theta\sin\psi. \tag{20}$$

The global charges associated with asymptotic symmetry (the Poincare group), which is angular momentum M_{ab} should be transformed under translations as

$$M_{ab}\to M_{ab}+2P_{[a}\omega_{b]}, \tag{21}$$

where $f=\hat{x}^a\omega_a$ and $\hat{x}^a=(1,\hat{x}^i)$ and $\hat{x}^i=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta\cos\psi,\cos\theta\sin\psi)$. P^a is a Bondi momentum [8]. However, we are considering dynamical space-times which has no exact timelike Killing vector. This means that the quantities M_{ab} would change due to gravitational waves under translations ($u \to u - f(x^A)$). Then, the expected transformations of M_{ab} under translation are

$$M_{ab}(u) \quad \to \quad M_{ab}(u-f) + 2P_{[a}(u)\omega_{b]} \;=\; M_{ab}(u) + 2P_{[a}(u)\omega_{b]} - \left(f\frac{d}{du}M_{ab}(u)\right)_{\text{radiation}}. \qquad (22)$$

In fact, we can show this transformation as follows. By the translations, j^ϕ is transformed as

$$j^\phi \quad \to \quad j^\phi - \alpha(x^A)\frac{\partial j^\phi}{\partial u} + (\delta j^\phi)_{\text{non radiation}} + (\text{total derivative terms}), \qquad (23)$$

where we do not write the explicit form of above terms. For the details, see [8]. Using the solutions for U^A, we can show that

$$J^\phi_{\text{Bondi}} \quad \to \quad J^\phi_{\text{Bondi}} - \frac{1}{4\pi}\int_{S^3}\left[-\alpha(x^A)\left(\frac{\partial j^\phi}{\partial u}\right)_{\text{radiation}}\right]d\Omega + \frac{3}{16\pi}\int_{S^3} m(u,x^A)\frac{\partial\alpha}{\partial\phi}d\Omega. \qquad (24)$$

This transformation is equivalent to $M_{\hat{x}\hat{y}} \to M_{\hat{x}\hat{y}} + 2P_{[\hat{x}}\omega_{\hat{y}]} - (f dM_{\hat{x}\hat{y}}/du)_{\text{radiation}}$. This stands for the Poincare covariance of the Bondi angular momentum. In the same way, we can show the Poincare covariance of angular momentum J^ψ_{Bondi}.

4 Summary

In this paper, we defined the Bondi angular momentum at null infinity in five dimensions and showed the Poincare covariance of the Bondi angular momentum. In addition, we successfully confirmed the Bondi angular momentum loss/gain due to gravitational wave. Asymptotic symmetry at null infinity is an infinite dimensional translational group (supertranslations) in four dimensions, not a four dimensional group. Then this implies that the angular momentum at null infinity has always ambiguities. Contrasted with this, it is shown that asymptotic symmetry at null infinity is the Poincare group in five dimensions. Then we can define the Bondi angular momentum at null infinity in a Poincare covariant way without any ambiguities.

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