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Topological string amplitudes for the local $\frac{1}{2}$K3 surface

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We study topological string amplitudes for the local $\frac{1}{2}$K3 surface. We develop a method of computing higher genus amplitudes along the lines of the direct integration formalism, making full use of the Seiberg–Witten curve expressed in terms of modular forms and E_8-invariant Jacobi forms. The Seiberg–Witten curve was constructed previously for the low-energy effective theory of the non-critical E-string theory in $\mathbb{R}^4 \times T^2$. We clarify how the amplitudes are written as polynomials in a finite number of generators expressed in terms of the Seiberg–Witten curve. We determine the coefficients of the polynomials by solving the holomorphic anomaly equation and the gap condition, and construct the amplitudes explicitly up to genus three. The results encompass topological string amplitudes for all local del Pezzo surfaces.

Subject Index B27, B33

1. Introduction

Topological string theory on the local $\frac{1}{2}$K3 surface provides us with a unified description of the low-energy effective theory of four-dimensional $\mathcal{N} = 2$ SU(2) gauge theories [1,2] and their extensions to five and six dimensions. The local $\frac{1}{2}$K3 surface is a non-compact Calabi–Yau threefold in which the $\frac{1}{2}$K3 surface appears as a divisor. By blowing down exceptional curves, one can reduce $\frac{1}{2}$K3 to any del Pezzo surface $\mathcal{B}_n$ ($n \leq 8$), including $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Topological string theory on the local $\frac{1}{2}$K3 describes the low-energy effective theory of the six-dimensional $(1, 0)$ supersymmetric non-critical E-string theory in $\mathbb{R}^4 \times T^2$ [3–8]. Similarly, topological string theory on the local $\mathcal{B}_n$ corresponds to the non-critical E_n string theory in $\mathbb{R}^4 \times T^2$ with one of the cycles of the T^2 shrinking to zero size [9,10]. This theory shares the same moduli space with the five-dimensional $\mathcal{N} = 1$ SU(2) gauge theory on $\mathbb{R}^4 \times S^1$ with $n - 1$ fundamental matters [11–13]. For the toric case ($n \leq 5$), the topological string amplitudes have been well studied. In particular, the all-genus topological string partition function in this case is given by the Nekrasov partition function for the above five-dimensional gauge theory [14–17].

For toric Calabi–Yau threefolds, the construction of topological string amplitudes has been well understood. One can use the topological vertex formalism [18] to construct the all-genus partition function as a sum over partitions on the A-model side. The "remodeling the B-model" conjecture [19], which is based on the topological recursion for matrix models [20], enables us to generate the amplitudes on the B-model side recursively with respect to the genus [21,22]. Indeed, for toric local del Pezzo surfaces, topological string amplitudes have been studied both in the former approach [23] and in the latter approach [24–26]. For non-toric Calabi–Yau threefolds, however, such a universal prescription is lacking at present. The purpose of this paper is to formulate a method of constructing the topological string amplitudes for the most general local $\frac{1}{2}$K3 surface.

A generalization of the topological vertex formalism was proposed [27] and applied to the construction of the topological string partition functions for non-toric local del Pezzo surfaces [28]. (See also [29] for another construction for the local $\mathcal{B}_6$.) Remarkably, this formalism enables us to construct the all-genus partition function as a sum over partitions. The partition function in this form is, however, not suitable for obtaining the topological string amplitude at each genus in a closed form. Also these constructions do not seem to apply directly to the case of the general local $\frac{1}{2}$K3 surface. On the other hand, one can construct the topological string amplitude at each genus by solving the holomorphic anomaly equation [30]. Higher-genus amplitudes have been constructed explicitly for some special cases with one or two moduli parameters [31–33]. Moreover, a simple, specific form of the holomorphic anomaly equation was proposed for the topological string amplitudes for the local $\frac{1}{2}$K3 surface [8,32]. By solving this equation one can construct higher-genus amplitudes for the most general case with manifest affine E_8 symmetry [8,34]. In this construction, however, the amplitudes are obtained not in a closed form, but rather in the form of an instanton expansion with respect to one of the Kähler moduli parameters.

Recently, it has been discovered and proved that topological string amplitudes for any Calabi–Yau threefold are polynomials in a finite number of generators [35,36]. By making use of this remarkable fact and taking account of the symmetry, in particular modular properties of the amplitudes [37], one can directly solve the holomorphic anomaly equation and efficiently determine the amplitude at each genus in a closed form [38]. This method, which we will call the direct integration method, is applicable, in principle, to topological strings on any Calabi–Yau threefold. It has also been applied to the gravitational corrections to Seiberg–Witten theories [38–41].

There are many examples of non-compact Calabi–Yau threefolds for which the mirror geometries are essentially described by Seiberg–Witten curves. In this case, the symmetry of the topological string amplitudes can naturally be understood in terms of the Seiberg–Witten curve. The Seiberg–Witten curve turns out to be useful to construct the topological string amplitude not only at genus zero, but also at higher genus. All these arguments apply to the local $\frac{1}{2}$K3 surface: The mirror geometry in this case is described by the Seiberg–Witten curve for the E-string theory [7,42]. In particular, the most general form expressed in terms of modular forms and E_8-invariant Jacobi forms was constructed [42]. Making full use of this Seiberg–Witten curve, we are able to formulate a method of constructing the topological string amplitudes at higher genus in a closed form for the most general local $\frac{1}{2}$K3.

Let us briefly summarize our construction in the following. We first clarify the polynomial structure of the higher-genus amplitudes and identify the generators of the polynomials. The generators are expressed in terms of one of the periods and the complex structure modulus of the torus associated with the Seiberg–Witten curve. We elucidate the modular anomaly of the generators, which can be interpreted as the holomorphic anomaly. This enables us to evaluate the holomorphic anomaly of the ansätze for the higher-genus amplitudes. Each time we solve the holomorphic anomaly equation, there appears a holomorphic ambiguity that cannot be fixed by the equation. We fix them by imposing a gap condition. The gap condition for the topological strings on the local $\frac{1}{2}$K3 surface is known [8]. This comes from the geometric property of the local $\frac{1}{2}$K3. Using this method, we construct the amplitudes explicitly up to genus three.

While the basic idea of our construction is the same as that of the direct integration method, ours is rather different from the standard one in appearance. We start from the holomorphic anomaly equation of Hosono–Saito–Takahashi [32] specific to the present model, rather than that of Bershadsky–Cecotti–Ooguri–Vafa (BCOV) [30]. We use our original generators when constructing ansätze for the amplitudes. In terms of these generators the amplitudes can be concisely expressed. Despite these differences, both methods should be essentially equivalent. We show that the amplitudes and the holomorphic anomaly equation can be written in a form akin to what have been obtained for other models by the standard direct integration method [37–40].

As we mentioned in the beginning, the topological string theory on the local $\frac{1}{2}$K3 surface encompasses that on all local del Pezzo surfaces. Remarkably, when the topological string amplitudes for the local $\frac{1}{2}$K3 are expressed in terms of the Seiberg–Witten curve, their forms are universal to all local del Pezzo surfaces. To obtain the amplitudes for any local del Pezzo surface, we have only to reduce the Seiberg–Witten curve correspondingly [10,43]. By way of illustration, we present explicit forms of amplitudes for three basic examples, the massless local $\mathcal{B}_8$, the local $\mathbb{P}^2$ and the local $\mathbb{P}^1 \times \mathbb{P}^1$.

This paper is organized as follows. In Sect. 2, we review some basic properties of the topological string amplitudes for the local $\frac{1}{2}$K3 surface. In Sect. 3, we describe the method of constructing topological string amplitudes for the local $\frac{1}{2}$K3 in a closed form. First we review how the topological string amplitude at genus zero is constructed from the Seiberg–Witten curve. We then study the modular anomaly of fundamental quantities and interpret them as the holomorphic anomaly. With these data, we solve the holomorphic anomaly equation at low genus. We make a conjecture on the general structure of the amplitudes, which greatly simplifies the problem of solving the holomorphic anomaly equation. We present two other expressions for the amplitudes and the holomorphic anomaly equation. In particular, the last expression is similar to what is found in the standard direct integration method. In Sect. 4, we study how to reduce our general results to the topological string amplitudes for all local del Pezzo surfaces. We present explicit forms of amplitudes for three basic examples: the massless local $\mathcal{B}_8$, the local $\mathbb{P}^2$, and the local $\mathbb{P}^1 \times \mathbb{P}^1$. Section 5 is devoted to the conclusion and discussion. In Appendix A, we present explicitly the generators of E_8-invariant Jacobi forms and the Seiberg–Witten curve for the present model. Appendix B is a collection of derivative formulas. In Appendix C, we present the explicit form of the amplitude at genus three. In Appendix D, we summarize our conventions of special functions.

2. Properties of topological string amplitudes for local $\frac{1}{2}$K3

In this section we review some basic properties of the topological string amplitudes for the local $\frac{1}{2}$K3 surface. The reader is referred to Refs. [8,32,33] for further details.

The $\frac{1}{2}$K3 surface, also known as the rational elliptic surface or the almost del Pezzo surface $\mathcal{B}_9$, is obtained by blowing up nine base points of a pencil of cubic curves in $\mathbb{P}^2$. The $\frac{1}{2}$K3 surface admits an elliptic fibration over $\mathbb{P}^1$. A generic $\frac{1}{2}$K3 surface has 12 singular fibers, while a generic elliptic K3 surface has 24 singular fibers.

The second homology group $H_2(\frac{1}{2}\mathrm{K3}, \mathbb{Z})$ is generated by the class of a line in $\mathbb{P}^2$ and the nine classes of the exceptional curves. With an inner product given by the intersection number, $H_2(\frac{1}{2}\mathrm{K3}, \mathbb{Z})$ acquires the structure of the ten-dimensional odd unimodular Lorentzian lattice $\Gamma^{9,1}$ (also denoted by $\mathrm{I}_{9,1}$). The automorphism group of $\Gamma^{9,1}$ contains the Weyl group of the affine E_8 root system. This property is crucial to our construction of the topological string amplitudes for the local $\frac{1}{2}$K3. It is also useful to note that the lattice decomposes as $\Gamma^{9,1} = \Gamma^{1,1} \oplus \Gamma_8$, where $\Gamma^{1,1}$ is the two-dimensional odd unimodular Lorentzian lattice and Γ_8 is the E_8 root lattice. $\Gamma^{1,1}$ is generated by [B], [E] with

$[\mathrm{B}] \cdot [\mathrm{E}] = 1$, $[\mathrm{B}] \cdot [\mathrm{B}] = -1$, $[\mathrm{E}] \cdot [\mathrm{E}] = 0$, where [B] and [E] can be viewed as the classes of the base and the fiber of the elliptic fibration. The automorphism group of Γ_8 is given by the Weyl group of the E_8 root system, which will be denoted by $W(E_8)$.

By a local $\frac{1}{2}$K3 surface we mean the total space of the canonical bundle of a generic $\frac{1}{2}$K3 surface. It is a non-compact Calabi–Yau threefold. We consider the A-model topological string theory on it. In this paper we let F_g denote *the instanton part* of the topological string amplitude at genus g. What we mean by *the instanton part* will be explained soon. We consider the amplitudes in real polarization, namely, F_g are holomorphic functions. As we will see below, the holomorphic anomaly of the amplitudes can be read from the modular anomaly.

Let F denote the all-genus topological string partition function defined as

$$F = \sum_{g=0}^{\infty} F_g x^{2g-2}. \tag{2.1}$$

F can be viewed as the generating function of the Gopakumar–Vafa invariants [44]. By taking account of the $W(E_8)$ symmetry, F can be expressed as

$$F(\varphi, \tau, \boldsymbol{\mu}; x) = \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{\boldsymbol{\lambda} \in P_+} \sum_{\boldsymbol{w} \in \mathcal{O}_{\boldsymbol{\lambda}}} N^r_{n,k,\boldsymbol{\lambda}} \sum_{m=1}^{\infty} \frac{1}{m} \left(2 \sin \frac{mx}{2}\right)^{2r-2} e^{2\pi i m(n\varphi + k\tau + \boldsymbol{w}\cdot\boldsymbol{\mu})}. \tag{2.2}$$

Here, P_+ denotes the set of all dominant weights of E_8, and the sum with respect to weights $\boldsymbol{w}$ is taken over the Weyl orbit of $\boldsymbol{\lambda}$; φ and τ denote the Kähler moduli corresponding to the base and the fiber of the elliptic fibration, respectively, while $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_8)$ denote the orthogonal coordinates for the complexified root space of E_8. The Gopakumar–Vafa invariants $N^r_{n,k,\boldsymbol{\lambda}}$ are integers. They count the BPS multiplicities of the five-dimensional $\mathcal{N} = 1$ supersymmetric theory obtained by compactifying the M-theory on the local $\frac{1}{2}$K3 surface. This five-dimensional theory is identified with the effective theory of the six-dimensional E-string theory on $\mathbb{R}^5 \times S^1$.

We defined F_g as *the instanton part*, which means that F_g is expanded as

$$F_g(\varphi, \tau, \boldsymbol{\mu}) = \sum_{n=1}^{\infty} Z_{g,n}(\tau, \boldsymbol{\mu})\, e^{2\pi i n\varphi} \tag{2.3}$$

and does not contain any polynomial (including constant) term in φ. From the point of view of the E-string theory, $Z_n := e^{-\pi i n\tau} Z_{0,n}$ is the BPS partition function of the E-strings wound n times [5,8]. Z_n is also interpreted as the partition function of $\mathcal{N} = 4$ U(n) topological Yang–Mills theory on $\frac{1}{2}$K3 [8]. Throughout this paper, we refer to this F_g as the topological string amplitude at genus g.

At present, the most general way of computing higher-genus amplitudes applicable to any Calabi–Yau threefold is to solve the BCOV holomorphic anomaly equation [30]. In this paper we define the topological string amplitudes as holomorphic functions, but one could adopt the standard definition in terms of twisted $\mathcal{N} = 2$ superconformal field theories, in which the amplitudes also possess anti-holomorphic dependence on moduli parameters. It is well known that this anti-holomorphic dependence, or the holomorphic anomaly, is governed by the BCOV holomorphic anomaly equation

$$\bar{\partial}_{\bar{\imath}} F_g = \frac{1}{2} \bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}} e^{2K} G^{j\bar{\jmath}} G^{k\bar{k}} \left(D_j D_k F_{g-1} + \sum_{h=1}^{g-1} D_j F_h D_k F_{g-h} \right). \tag{2.4}$$

Here, F_g denotes the amplitudes at genus g, K is the Kähler potential, $G_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K$ is the Kähler metric, D_i denotes a certain covariant derivative, and $\bar{C}_{\bar{i}\bar{j}\bar{k}} = \overline{C_{ijk}}$ with $C_{ijk} = D_i D_j D_k F_0$. One can recursively solve this differential equation to construct higher-genus amplitudes F_g up to holomorphic ambiguities. It is worth noting that F_g are polynomials in a finite number of generators [35,36], which greatly helps the construction.

In practice, however, it is rather hard to solve a topological string model with ten Kähler moduli parameters, in particular when the target space is not a local toric Calabi–Yau threefold. Nevertheless, in the case of the local $\frac{1}{2}$K3 one can make full use of the symmetry to construct the amplitudes much more efficiently than in generic cases. As is explained below, F_g at low g are fully characterized by the symmetry, the holomorphic anomaly equation, and the gap condition.

Let us start with the symmetry. Due to the automorphism of the homology lattice of $\frac{1}{2}$K3, the partition function exhibits the affine E_8 symmetry. Moreover, it possesses good modular properties in τ. It is known that $Z_{g,n}$ has the following structure [33]:

$$Z_{g,n}(\tau, \boldsymbol{\mu}) = \frac{T_{g,n}(\tau, \boldsymbol{\mu})}{\left[\prod_{k=1}^{\infty}(1-q^k)\right]^{12n}}, \tag{2.5}$$

where

$$q = e^{2\pi i \tau}. \tag{2.6}$$

$T_{g,n}$ is a $W(E_8)$-invariant quasi-Jacobi form of weight $2g-2+6n$ and index n. The reader is referred to Appendix A for the basic properties of the $W(E_8)$-invariant Jacobi form. By $W(E_8)$-invariant *quasi*-Jacobi forms we mean those which are generated by the generators of the ordinary $W(E_8)$-invariant Jacobi forms and the Eisenstein series $E_2(\tau)$.

$E_2(\tau)$ is not strictly a modular form, as it transforms as

$$E_2\left(-\frac{1}{\tau}\right) = \tau^2\left(E_2(\tau) + \frac{6}{\pi i \tau}\right). \tag{2.7}$$

However, the non-holomorphic function

$$\hat{E}_2(\tau, \bar{\tau}) := E_2(\tau) + \frac{6}{\pi i(\tau - \bar{\tau})} \tag{2.8}$$

transforms as a modular form of weight 2,

$$\hat{E}_2\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}\right) = \tau^2 \hat{E}_2(\tau, \bar{\tau}). \tag{2.9}$$

By replacing all $E_2(\tau)$ by $\hat{E}_2(\tau, \bar{\tau})$, the amplitude F_g transforms as a modular function of weight $2g-2$ at the cost of losing holomorphicity. This non-holomorphicity is regarded as the holomorphic anomaly of the amplitude. In other words, the modular/holomorphic anomaly of the amplitude always appears through E_2. For later convenience, we introduce a normalized notation $\xi := \frac{1}{24}E_2$ and let

$$\partial_\xi = 24\partial_{E_2} \tag{2.10}$$

measure the holomorphic anomaly. We also introduce a normalized variable $\phi = 2\pi i \varphi + \phi_0$, so that

$$\partial_\phi = \frac{1}{2\pi i}\partial_\varphi. \tag{2.11}$$

The precise relation between ϕ and φ will be given in Sect. 3. Throughout this paper we hold τ and $\boldsymbol{\mu}$ constant when we take partial derivatives with respect to ξ and ϕ. In terms of these normalized variables, the holomorphic anomaly equation for the partition function F is written as [32]

$$\partial_\xi e^F = x^2 \partial_\phi(\partial_\phi + 1)e^F. \tag{2.12}$$

By expanding the equation in x, it becomes a set of recursive equations:

$$\partial_\xi F_g = \partial_\phi^2 F_{g-1} + \partial_\phi F_{g-1} + \sum_{h=0}^{g} \partial_\phi F_h \partial_\phi F_{g-h}. \tag{2.13}$$

The equation for $g = 0$ should be understood with $F_{-1} = 0$. In terms of $Z_{g,n}$, the holomorphic anomaly equations read

$$\partial_\xi Z_{g,n} = n(n+1)Z_{g-1,n} + \sum_{h=0}^{g}\sum_{k=1}^{n-1} k(n-k)Z_{h,k}Z_{g-h,n-k}. \tag{2.14}$$

Again, the equation for $g = 0$ should be understood with $Z_{-1,n} = 0$.

The above form of holomorphic anomaly equation was first proposed for $g = 0$ [8] and later extended for general g [32]. The validity of the equation has been further confirmed in Refs. [33,34]. It is expected that the above equation is equivalent to the BCOV holomorphic anomaly equation for the local $\frac{1}{2}$K3 [34,45].

As the holomorphic anomaly equation is a differential equation, one needs to fix the integration constant, i.e. the holomorphic ambiguity, at each genus. For the present model, it is known that the following gap condition can be used for this purpose:

$$F = \sum_{n=1}^{\infty} e^{2\pi i n\varphi}\left(\frac{1}{n(2\sin\frac{nx}{2})^2} + \mathcal{O}(q^n)\right). \tag{2.15}$$

This condition is equivalent to the following constraint on the Gopakumar–Vafa invariants:

$$N^g_{n,k,\boldsymbol{\lambda}} = 0 \quad \text{for} \quad k < n \quad \text{except} \quad N^0_{1,0,\mathbf{0}} = 1. \tag{2.16}$$

This follows from the geometric structure of the local $\frac{1}{2}$K3 [8]. In terms of $Z_{g,n}$ the gap condition reads

$$Z_{g,n} = \beta_g n^{2g-3} + \mathcal{O}(q^n), \tag{2.17}$$

where β_g are rational numbers defined by the following expansion:

$$\begin{aligned}\sum_{g=0}^{\infty}\beta_g x^{2g} &= \frac{x^2}{4\sin^2\frac{x}{2}} \\ &= 1 + \frac{1}{12}x^2 + \frac{1}{240}x^4 + \frac{1}{6048}x^6 + \mathcal{O}(x^8).\end{aligned} \tag{2.18}$$

It has been checked [8,33,34] for low g and n that $Z_{g,n}$ can be determined uniquely by the symmetry (2.5), the holomorphic anomaly equations (2.14), and the gap conditions (2.17).[1] Based on this fact, we will develop a method of constructing F_g in a closed form in the next section.

[1] For general g, however, these conditions are not likely to be sufficient for determining the amplitude completely. See the discussion at the end of Sect. 3.3.

3. Closed expressions for amplitudes

3.1. Genus zero amplitude and instanton expansion

It is known that the genus zero amplitude F_0 for the local $\frac{1}{2}$K3 surface is obtained as the prepotential associated with the Seiberg–Witten curve of the form

$$y^2 = 4x^3 - fx - g, \tag{3.1}$$

with

$$f = \sum_{j=0}^{4} a_j u^{4-j}, \qquad g = \sum_{j=0}^{6} b_j u^{6-j}. \tag{3.2}$$

Actually, a Seiberg–Witten curve of this form itself describes an elliptic fibration of the $\frac{1}{2}$K3 surface. It can be viewed as a sort of local mirror symmetry between one $\frac{1}{2}$K3 and another $\frac{1}{2}$K3 [8,33]. We present the explicit form of the Seiberg–Witten curve in Appendix A. It was determined in Ref. [42] so that the instanton expansion of the prepotential correctly reproduces $Z_{0,n}$ at low n calculated by the method of [8], which we summarized in the last section.

Let us recall how the prepotential is obtained from the Seiberg–Witten curve of the above general form. Given the Seiberg–Witten curve (3.1), the expectation value of the scalar component of the $\mathcal{N} = 2$ vector multiplet is expressed as

$$\phi = -\frac{1}{2\pi} \int du \oint_{\alpha} \frac{dx}{y}, \tag{3.3}$$

where α is one of the fundamental cycles of the curve. The complexified gauge coupling constant $\tilde{\tau}$ is given by the complex structure modulus of the Seiberg–Witten curve. On the other hand, $\tilde{\tau}$ is given by the second derivative of the prepotential. In terms of the instanton part F_0 of the prepotential, $\tilde{\tau}$ is expressed as

$$\tilde{\tau} = \tau + \frac{i}{2\pi} \partial_{\phi}^2 F_0, \tag{3.4}$$

where τ is the bare gauge coupling constant. By solving these relations, one obtains the prepotential from the Seiberg–Witten curve.

The practical calculation can be organized as follows [10,42,46]. Since the present Seiberg–Witten curve is elliptic, one can make full use of the explicit map between an elliptic curve and a torus. Let $(2\pi\omega, 2\pi\omega\tilde{\tau})$ denote the fundamental periods of the torus. The map from the torus to the elliptic curve in the Weierstrass form (3.1) is given in terms of the Weierstrass $\wp$-function by

$$x = \wp(z; 2\pi\omega, 2\pi\omega\tilde{\tau}), \qquad y = \partial_z \wp(z; 2\pi\omega, 2\pi\omega\tilde{\tau}). \tag{3.5}$$

The coefficients of the elliptic curve and the periods of the torus are related as

$$f = \frac{1}{12}\frac{\tilde{E}_4}{\omega^4}, \qquad g = \frac{1}{216}\frac{\tilde{E}_6}{\omega^6}, \tag{3.6}$$

where we use the notation

$$\tilde{E}_{2n} := E_{2n}(\tilde{\tau}). \tag{3.7}$$

One can express $\tilde{\tau}$ and ω in terms of the Seiberg–Witten curve by inverting the modular functions. First, we eliminate ω from the two equations (3.6) by taking the ratio f^3/g^2. Equivalently, we can look at the j-invariant. We expand it in u^{-1} as

$$\frac{1}{\tilde{j}} = \frac{\tilde{E}_4^3 - \tilde{E}_6^2}{1728\tilde{E}_4^3} = \frac{f^3 - 27g^2}{1728 f^3}$$
$$= \frac{1}{j} - \frac{E_6 b_1}{4E_4^3}\frac{1}{u} + \mathcal{O}\left(\frac{1}{u^2}\right). \tag{3.8}$$

Here we have used $a_0 = \frac{1}{12}E_4,\ b_0 = \frac{1}{216}E_6,\ a_1 = 0.$[2] On the other hand, the j-invariant has the following expansion:

$$\tilde{j} = \frac{1}{\tilde{q}} + 744 + 196884\tilde{q} + \mathcal{O}\left(\tilde{q}^2\right), \qquad \tilde{q} = e^{2\pi i\tilde{\tau}}. \tag{3.9}$$

Inverting this expansion and using (3.8), we obtain the expansion of $\tilde{\tau}$ in u^{-1}. By introducing the notation

$$t := 2\pi i(\tilde{\tau} - \tau), \tag{3.10}$$

the expansion is expressed as

$$t = -\frac{E_4 b_1}{4\Delta}\frac{1}{u} + \left(\frac{E_6 a_2}{48\Delta} - \frac{E_4 b_2}{4\Delta} - \frac{E_4(E_4E_2 + 5E_6){b_1}^2}{192\Delta^2}\right)\frac{1}{u^2} + \mathcal{O}\left(\frac{1}{u^3}\right). \tag{3.11}$$

Substituting this into (3.6), one obtains the expansion of ω in u^{-1}. We choose the sign of ω in such a way that ω is expanded as

$$\omega = \frac{1}{u} - \frac{(E_4E_2 - E_6)b_1}{48\Delta}\frac{1}{u^2} + \mathcal{O}\left(\frac{1}{u^3}\right). \tag{3.12}$$

Integrating this by u, one obtains ϕ. We define ϕ with the normalization

$$\phi := -\int \omega du \tag{3.13}$$

so that e^ϕ has the expansion

$$e^\phi = \frac{1}{u} - \frac{(E_4E_2 - E_6)b_1}{48\Delta}\frac{1}{u^2} + \mathcal{O}\left(\frac{1}{u^3}\right). \tag{3.14}$$

Inverting this relation, we have

$$\frac{1}{u} = e^\phi + \frac{(E_4E_2 - E_6)b_1}{48\Delta}e^{2\phi} + \mathcal{O}\left(e^{3\phi}\right). \tag{3.15}$$

Substituting this into (3.11), we obtain

$$t = -\frac{E_4 b_1}{4\Delta}e^\phi + \left(\frac{E_6 a_2}{48\Delta} - \frac{E_4 b_2}{4\Delta} - \frac{E_4(E_4E_2 + 2E_6)b_1^2}{96\Delta^2}\right)e^{2\phi} + \mathcal{O}\left(e^{3\phi}\right). \tag{3.16}$$

[2] This is the convention of the Seiberg–Witten curve adopted in Ref. [42]. By suitable rescaling and shift of variables one can always recast a generic Seiberg–Witten curve of the form (3.1), (3.2) as this form without loss of generality.

Similarly, from (3.12) and (3.15) we obtain

$$\ln \omega = \phi + \left(\frac{(E_6 E_2 - E_4^2) a_2}{1152\Delta} - \frac{(E_4 E_2 - E_6) b_2}{96\Delta} + \frac{(-2E_4^2 E_2^2 - 8E_6 E_4 E_2 + 5E_4^3 + 5E_6^2) b_1^2}{9216\Delta^2} \right) e^{2\phi} + \mathcal{O}\left(e^{3\phi}\right), \tag{3.17}$$

which will be used later. As explained in the beginning, the instanton part of the prepotential is given by

$$F_0 = -\partial_\phi^{-2} t. \tag{3.18}$$

Here ∂_ϕ^{-1} denotes integration with respect to ϕ of a power series in e^ϕ.

This prepotential is identified with the genus zero amplitude F_0 for the local $\frac{1}{2}$K3 surface. The bare gauge coupling τ is interpreted as the Kähler modulus τ of the original $\frac{1}{2}$K3. The scalar expectation value ϕ is identified with the Kähler modulus φ as [46]

$$e^\phi = -q \left[\prod_{k=1}^\infty (1 - q^k) \right]^{12} e^{2\pi i \varphi}. \tag{3.19}$$

Taking this into account, F_0 is expanded as

$$F_0 = -\frac{E_4 b_1}{4 \left[\prod_{k=1}^\infty (1 - q^k) \right]^{12}} e^{2\pi i \varphi} + \frac{-2E_6 \Delta a_2 + 24 E_4 \Delta b_2 + (E_4^2 E_2 + 2E_6 E_4) b_1^2}{384 \left[\prod_{k=1}^\infty (1 - q^k) \right]^{24}} e^{4\pi i \varphi} + \mathcal{O}\left(e^{6\pi i \varphi}\right). \tag{3.20}$$

By substituting the coefficients a_n, b_n of the Seiberg–Witten curve presented in Appendix A, one obtains the genus zero amplitude for the local $\frac{1}{2}$K3 as a series expansion in $e^{2\pi i \varphi}$ up to any desired order.

3.2. *Modular anomaly*

The Seiberg–Witten curve transforms as a $W(E_8)$-invariant Jacobi form (see Appendix A). On the other hand, the genus zero amplitude F_0 contains E_2 and therefore exhibits the modular anomaly. The E_2's appear when one expands the j-invariant $j(\tilde{\tau})$ around $\tilde{\tau} = \tau$. Thus, the modulus $\tilde{\tau}$ and the period ω of the Seiberg–Witten curve do exhibit the modular anomaly when expanded in u^{-1}. In Ref. [46], the modular anomaly of $\tilde{\tau}$ and ω was studied in the course of proving the holomorphic anomaly equation for the genus zero amplitude. Extending the analysis, here we study the modular anomaly of various quantities derived from the Seiberg–Witten curve. We will use this to solve the holomorphic anomaly equation for higher-genus amplitudes.

As mentioned above, the Seiberg–Witten curve transforms as a Jacobi form. This means that the modulus $\tilde{\tau}(u, \tau, \boldsymbol{\mu})$ of the curve transforms in precisely the same way as τ does under the action of $\mathrm{SL}(2, \mathbb{Z})$. It then follows that $t = 2\pi i(\tilde{\tau} - \tau)$ is invariant under $\tau \to \tau + 1,\ \tilde{\tau} \to \tilde{\tau} + 1$, while it transforms as

$$\frac{1}{t} \to \tau^2 \left(\frac{1}{t} + \frac{1}{2\pi i \tau} \right) \qquad \text{for} \qquad \tau \to -\frac{1}{\tau},\ \tilde{\tau} \to -\frac{1}{\tilde{\tau}}. \tag{3.21}$$

This anomalous behavior is expected since E_2's appear in the coefficients of the expansion (3.11). Moreover, one finds that the transformation of t^{-1} is very similar to that of E_2 as in Eq. (2.7). This suggests that t^{-1} depends on E_2 as

$$t^{-1} = \frac{1}{12}E_2 + (\text{modular function of weight 2}). \tag{3.22}$$

One can explicitly check this using the series expansion (3.11). Let us express it as

$$\left(\partial_\xi t^{-1}\right)_u = 2, \tag{3.23}$$

or

$$\left(\partial_\xi t\right)_u = -2t^2. \tag{3.24}$$

Here, $\left(\partial_\xi t\right)_u$ denotes the partial derivative of t with respect to ξ, holding u constant.

Next let us consider modular properties of the combination

$$\omega t = \left(\frac{\tilde{E}_4}{12f}\right)^{1/4} t. \tag{3.25}$$

We have used (3.6). One can see that this transforms as a modular form of weight 4 in τ, since the constituents transform as

$$\tilde{E}_4 \to \tilde{\tau}^4 \tilde{E}_4, \qquad f \to \tau^{-20} f, \qquad t \to \tilde{\tau}^{-1}\tau^{-1} t \tag{3.26}$$

under the S-transformation $\tau \to -1/\tau,\ \tilde{\tau} \to -1/\tilde{\tau}$. This means that the combination ωt is free of the modular anomaly, namely

$$\left(\partial_\xi (\omega t)\right)_u = 0. \tag{3.27}$$

Using (3.24), one obtains

$$\left(\partial_\xi \omega\right)_u = 2\omega t. \tag{3.28}$$

Furthermore, combining (3.28) with (3.13) one obtains

$$\left(\partial_\xi \phi\right)_u = 2\partial_\phi^{-1} t, \tag{3.29}$$

$$\left(\partial_\xi^2 \phi\right)_u = 0. \tag{3.30}$$

Based on these formulas and (3.6), one can evaluate the modular anomaly of various quantities. We present a list of formulas in Appendix B.

So far in this subsection, we have regarded u and ξ as independent variables and taken the derivative ∂_ξ holding u constant. Let us say we are in the (u, ξ) frame. On the other hand, the holomorphic anomaly equations (2.13) are given in the (ϕ, ξ) frame. It is useful to see how expressions in these frames are transformed into each other. The derivative of a function A with respect to ξ is transformed between these frames by the simple chain rule

$$\left(\partial_\xi A\right)_\phi = -(\partial_\xi \phi)_u \left(\partial_\phi A\right)_\xi + \left(\partial_\xi A\right)_u, \tag{3.31}$$

$$= -2(\partial_\phi^{-1} t) \left(\partial_\phi A\right) + \left(\partial_\xi A\right)_u. \tag{3.32}$$

We have used (3.29) in the second equality. We sometimes omit the subscript ξ, as we always hold ξ constant when we take derivatives ∂_u and ∂_ϕ. Applying this formula to $\partial_\phi F_0 = -\partial_\phi^{-1} t$ and using (3.29), (3.30), we see that

$$\left(\partial_\xi \left(\partial_\phi F_0\right)\right)_\phi = 2 \left(\partial_\phi F_0\right) \left(\partial_\phi^2 F_0\right). \tag{3.33}$$

By integrating both sides by ϕ, we obtain the holomorphic anomaly equation (2.13) at $g = 0$,

$$\partial_\xi F_0 = \left(\partial_\phi F_0\right)^2. \tag{3.34}$$

3.3. *Higher-genus amplitudes*

The expression (2.13) of the holomorphic anomaly equation is not convenient for practical purposes, since derivatives of F_g appear on both sides of the equation. Using $\partial_\phi F_0 = -\partial_\phi^{-1} t$ and the chain rule (3.32), one can rewrite the equation into the recursive form

$$\left(\partial_\xi F_g\right)_u = \partial_\phi^2 F_{g-1} + \partial_\phi F_{g-1} + \sum_{h=1}^{g-1} \partial_\phi F_h \partial_\phi F_{g-h} \tag{3.35}$$

for $g \geq 1$. In the following, we solve this equation and construct F_g for low g.

Let us first consider the case of $g = 1$. In this case, the equation simply reads

$$\begin{aligned} \left(\partial_\xi F_1\right)_u &= \partial_\phi^2 F_0 + \partial_\phi F_0 \\ &= -t - \partial_\phi^{-1} t. \end{aligned} \tag{3.36}$$

With the help of the derivative formulas (B.5)–(B.9), one immediately finds a solution of the form

$$F_1 = c_1 \ln \omega - \left(\frac{c_1}{12} + \frac{1}{24}\right) \ln \tilde{\Delta} - \frac{1}{2}\phi + f_1(\tau). \tag{3.37}$$

The constant c_1 and the function $f_1(\tau)$ can be determined by the condition that F_1 takes the form (2.3), namely it does not contain any polynomial term in ϕ. From (3.16) and (3.17) we see that

$$\ln \tilde{\Delta} = \ln \Delta + \mathcal{O}(e^{2\pi i \varphi}), \qquad \ln \omega = \phi + \mathcal{O}(e^{4\pi i \varphi}). \tag{3.38}$$

Using these we can determine the unknowns as $c_1 = 1/2, f_1 = (\ln \Delta)/12$ and obtain

$$F_1 = \frac{1}{2} \ln \omega - \frac{1}{12} \ln \tilde{\Delta} + \frac{1}{12} \ln \Delta - \frac{1}{2}\phi. \tag{3.39}$$

While this is not a rigorous derivation, we have checked that the above form is the correct answer. Combined with (3.15)–(3.17) and (3.19), the above expression correctly reproduces $Z_{1,n}$, which we explicitly calculated up to $n = 5$ using the method explained in the last section. It also reproduces the result for $\boldsymbol{\mu} = \mathbf{0}$ presented in Ref. [34]. Note that a similar expression has been presented for four-dimensional Seiberg–Witten theories [17,40].

To compute amplitudes for $g \geq 2$ by solving (3.35), we point out an interesting fact that the term $-\phi/2$ in F_1 precisely cancels the linear term $\partial_\phi F_{g-1}$ on the right-hand side of (3.35). Therefore, if we introduce the notation

$$\mathcal{F}_1 = F_1 + \frac{1}{2}\phi, \qquad \mathcal{F}_2 = F_2 + \frac{1}{96}E_2, \qquad \mathcal{F}_g = F_g \quad \text{for} \quad g \geq 3, \tag{3.40}$$

the holomorphic anomaly equation (3.35) turns into the very simple form

$$\left(\partial_\xi \mathcal{F}_g\right)_u = \partial_\phi^2 \mathcal{F}_{g-1} + \sum_{h=1}^{g-1} \partial_\phi \mathcal{F}_h \partial_\phi \mathcal{F}_{g-h} \tag{3.41}$$

for $g \geq 2$. Note that this form has already been presented in Ref. [40] in the case of four-dimensional SU(2) Seiberg–Witten theories. It is natural that the holomorphic anomaly equation takes the same form in the present case, since the definition (3.13) of ϕ through the Seiberg–Witten curve is common in both cases.

Based on this simple form, let us construct the amplitude at $g = 2$. Equation (3.41) in this case reads

$$\begin{aligned}(\partial_\xi \mathcal{F}_2)_u &= \partial_\phi^2 \mathcal{F}_1 + (\partial_\phi \mathcal{F}_1)^2 \\ &= \frac{1}{2}\partial_\phi^2 \ln\omega + \frac{1}{4}(\partial_\phi \ln\omega)^2 - \frac{1}{12}\tilde{E}_2\, \partial_\phi t\, \partial_\phi \ln\omega - \frac{1}{12}\tilde{E}_2\, \partial_\phi^2 t + \frac{1}{144}\tilde{E}_4(\partial_\phi t)^2.\end{aligned} \tag{3.42}$$

Note that the last expression is a polynomial in (quasi-)modular forms $\tilde{E}_{2k}$ and derivatives $\partial_\phi^m \ln\omega$, $\partial_\phi^n t$. The polynomial is constrained so that each term contains two ∂_ϕ's and is of weight 0. Note that after every E_2 is replaced by $\hat{E}_2$, a modular function in $\tilde{\tau}$ transforms as that in τ with the same weight. Thus, the weights of the generators of the polynomial read

$$[\tilde{E}_{2k}] = 2k, \qquad [\partial_\phi^m \ln\omega] = 0, \qquad [\partial_\phi^n t] = -2. \tag{3.43}$$

We see from (3.42) that $\mathcal{F}_2$ is of weight 2, since ξ is of weight 2. Let us make an ansatz that $\mathcal{F}_2$ has the same polynomial structure as (3.42), namely a polynomial in $\tilde{E}_{2k}$, $\partial_\phi^m \ln\omega$, $\partial_\phi^n t$ with two ∂_ϕ's. Explicitly, the ansatz reads

$$\begin{aligned}\mathcal{F}_2 = {}& c_1\tilde{E}_2\, \partial_\phi^2 \ln\omega + c_2\tilde{E}_2(\partial_\phi \ln\omega)^2 + (c_3\tilde{E}_2^2 + c_4\tilde{E}_4)\partial_\phi t\, \partial_\phi \ln\omega \\ &+ (c_5\tilde{E}_2^2 + c_6\tilde{E}_4)\partial_\phi^2 t + (c_7\tilde{E}_2^3 + c_8\tilde{E}_4\tilde{E}_2 + c_9\tilde{E}_6)(\partial_\phi t)^2.\end{aligned} \tag{3.44}$$

Substituting this ansatz into (3.42), one can partly determine the coefficients c_j. The derivatives of the generators with respect to ξ are summarized in Appendix B. One has to be careful when taking derivatives of $\partial_\phi^n \ln\omega$ and $\partial_\phi^n t$ with respect to ξ. We differentiate them in the (u, ξ) frame, where ∂_ξ and ∂_ϕ do not commute. The explicit forms of these derivatives for general n are given in Eq. (B.10), (B.11), which can be shown by using the chain rule (3.32).

The holomorphic anomaly equation (3.42) reduces the number of undetermined parameters to three. These remaining parameters can be fixed by the condition that F_2 takes the form (2.3) and by the gap conditions (2.17) at $n = 1, 2$. In the end, one obtains

$$\begin{aligned}\mathcal{F}_2 = {}& \frac{1}{48}\tilde{E}_2\, \partial_\phi^2 \ln\omega + \frac{1}{96}\tilde{E}_2(\partial_\phi \ln\omega)^2 - \frac{1}{576}(\tilde{E}_2^2 - \tilde{E}_4)\partial_\phi t\, \partial_\phi \ln\omega \\ &- \frac{1}{1920}(5\tilde{E}_2^2 + 3\tilde{E}_4)\partial_\phi^2 t - \frac{1}{207360}(35\tilde{E}_2^3 + 51\tilde{E}_4\tilde{E}_2 - 86\tilde{E}_6)(\partial_\phi t)^2.\end{aligned} \tag{3.45}$$

In the same way, we are able to determine the amplitude at genus three. The most general ansatz for $\mathcal{F}_3$ is written with 68 unknown parameters. The holomorphic anomaly equation gives 45 relations and leaves 23 undetermined parameters. These are fixed completely by the condition that F_3 takes the form (2.3) and by the gap conditions (2.17) up to $n = 4$. The explicit form of F_3 is presented in Appendix C. We checked for low n that $Z_{g,n}$ calculated from the above-obtained $\mathcal{F}_2, \mathcal{F}_3$ are in agreement with the results obtained by the method described in Sect. 2. In the next section we will also reproduce the higher-genus amplitudes for local del Pezzo surfaces from these results, which serves as another consistency check.

Based on the above explicit construction of $\mathcal{F}_g$ at low genera, we propose the following conjecture:

$\mathcal{F}_g$ $(g \geq 2)$ is a polynomial in $\tilde{E}_{2k},\ \partial_\phi^m \ln\omega,\ \partial_\phi^n t\ \ (k = 1,2,3,\ m,n \in \mathbb{Z}_{>0})$, in which each term contains $2g-2$ ∂_ϕ's and is of weight $2g-2$.

(3.46)

Note that the form of the polynomial is no longer unique for $g \geq 4$, since not all of $\partial_\phi^m \ln\omega,\ \partial_\phi^n t$ are independent. Actually, they are finitely generated. This can be seen as follows. Recall that f, g are polynomials of degree 4, 6 in u, respectively. Since $\partial_u = -\omega\partial_\phi$, this means that

$$\left(\omega\partial_\phi\right)^k \frac{\tilde{E}_4}{\omega^4} = 0, \qquad \text{for} \quad k > 4, \tag{3.47}$$

$$\left(\omega\partial_\phi\right)^k \frac{\tilde{E}_6}{\omega^6} = 0, \qquad \text{for} \quad k > 6. \tag{3.48}$$

These relations give rise to non-trivial relations among the derivatives $\partial_\phi^m \ln\omega,\ \partial_\phi^n t$. Using these relations, one can express all $\partial_\phi^m \ln\omega$ and $\partial_\phi^n t$ with $m, n \in \mathbb{Z}_{>0}$ in terms of those with $m = 1, \ldots, 6$ and $n = 1, \ldots, 4$ and $\tilde{E}_2, \tilde{E}_4, \tilde{E}_6$. Therefore, by assuming the above conjecture, $\mathcal{F}_g$ can also be expressed in terms of these generators. In this expression $\mathcal{F}_g$ $(g \geq 4)$ is still a polynomial in $\partial_\phi^m \ln\omega$ and $\partial_\phi^n t$, but becomes a rational function in $\tilde{E}_{2k}$. In Sect. 3.5 we will introduce another expression in which $\mathcal{F}_g$ is indeed written as a polynomial of a finite number of generators.

The polynomial structure of $\mathcal{F}_g$ is expected, since topological string amplitudes for any Calabi–Yau threefold are polynomials in a finite number of generators [36]. The significance of the conjecture (3.46) is that the generators are explicitly given in terms of the Seiberg–Witten curve. This allows us to study topological string amplitudes for not only the local $\frac{1}{2}$K3 but all local del Pezzo surfaces in a unified way, as we will see in the next section. The conjecture (3.46) provides us with a systematic construction of the ansatz for general $\mathcal{F}_g$. In particular, the holomorphic anomaly equations and the gap conditions will be sufficient for completely fixing the form of $\mathcal{F}_g$ at low genus, as we have explicitly seen for $g = 2, 3$.

On the other hand, it is not likely that these conditions suffice to determine $\mathcal{F}_g$ at general g. In fact, if we apply the method to the special case with $\boldsymbol{\mu} = \mathbf{0}$, where $T_{g,n}(\tau, \mathbf{0})$ are now ordinary quasi-modular forms, an undetermined coefficient appears already at $g = 4$. To determine the amplitude completely at general g, one needs additional conditions, such as gap conditions imposed at the conifold loci of the moduli space. This is indeed the case for some simple local del Pezzo surfaces [47,48].

3.4. Expression in (u, ξ) frame

Equation (3.41) looks somewhat irregular, as the left-hand side is written in the (u, ξ) frame while the right-hand side is in the (ϕ, ξ) frame. For practical purposes, this is actually a convenient form since the derivatives of $\mathcal{F}_g$ are assembled in a single term in the (u,ξ) frame while the gap condition can be explicitly expressed in the (ϕ,ξ) frame. On the other hand, it is also useful to express the equation entirely in the (u,ξ) frame. By using $\partial_\phi = -\frac{1}{\omega}\partial_u$, (3.41) can be rewritten as

$$\left(\partial_\xi \mathcal{F}_g\right)_u = \frac{1}{\omega^2}\left(\partial_u^2\mathcal{F}_{g-1} - \partial_u \ln\omega\,\partial_u\mathcal{F}_{g-1} + \sum_{h=1}^{g-1}\partial_u\mathcal{F}_h\partial_u\mathcal{F}_{g-h}\right) \tag{3.49}$$

for $g \geq 2$. In this frame, it is easier to take derivatives with respect to ξ, while the gap condition cannot be expressed in a simple manner. $\mathcal{F}_2$ is expressed as

$$\begin{aligned}\mathcal{F}_2 = \frac{1}{\omega^2}\bigg(&\frac{1}{48}\tilde{E}_2\,\partial_u^2\ln\omega - \frac{1}{96}\tilde{E}_2(\partial_u\ln\omega)^2 + \frac{1}{5760}(5\tilde{E}_2^2 + 19\tilde{E}_4)\partial_u t\,\partial_u\ln\omega\\ &- \frac{1}{1920}(5\tilde{E}_2^2 + 3\tilde{E}_4)\partial_u^2 t - \frac{1}{207360}(35\tilde{E}_2^3 + 51\tilde{E}_4\tilde{E}_2 - 86\tilde{E}_6)(\partial_u t)^2\bigg),\end{aligned} \tag{3.50}$$

which is almost as simple as the previous expression (3.45).

3.5. *Expression in direct integration style*

There is another interesting expression of the topological string amplitudes and the holomorphic anomaly equation. One can express the amplitudes directly in terms of the coefficients of the Seiberg–Witten curve. To see this, let us start with studying the transformation rules for the generators.

From (3.6) and

$$D := f^3 - 27g^2 = \frac{\tilde{\Delta}}{\omega^{12}}, \tag{3.51}$$

we see that

$$\begin{aligned}\partial_u \ln f &= \partial_u \ln \tilde{E}_4 - 4\partial_u\ln\omega\\ &= \frac{1}{3}\left(\tilde{E}_2 - \frac{\tilde{E}_6}{\tilde{E}_4}\right)\partial_u t - 4\partial_u\ln\omega,\end{aligned} \tag{3.52}$$

$$\begin{aligned}\partial_u \ln D &= \partial_u \ln\tilde{\Delta} - 12\partial_u\ln\omega\\ &= \tilde{E}_2\,\partial_u t - 12\partial_u\ln\omega.\end{aligned} \tag{3.53}$$

Solving these relations, one obtains

$$\partial_u t = \frac{1}{\omega^2}\frac{-6fg' + 9f'g}{2D}, \tag{3.54}$$

$$\partial_u\ln\omega = \frac{(-2fg' + 3f'g)X + (-2f^2f' + 36gg')}{8D}, \tag{3.55}$$

where

$$X := \frac{\tilde{E}_2}{\omega^2}. \tag{3.56}$$

The derivative of X in u is computed as

$$\partial_u X = \frac{(2fg' - 3f'g)X^2 + (4f^2f' - 72gg')X + (24f^2g' - 36ff'g)}{8D}. \tag{3.57}$$

With the help of these relations and $\partial_\phi = -\frac{1}{\omega}\partial_u$, it is straightforward to express higher derivatives of t and $\ln\omega$ in terms of X and $f^{(m)}(u),\ g^{(n)}(u)$. Note that $\tilde{E}_4, \tilde{E}_6$ can also be rewritten in terms of $f,\ g,\ \omega$ by using (3.6). After all this, $\mathcal{F}_2$ is expressed as

$$\begin{aligned}\mathcal{F}_2 = \frac{1}{92160D^2}\Big(&(100f^2g'^2 - 300ff'gg' + 225f'^2g^2)X^3\\ &+ (240f^4g'' - 780f^3f'g' - 360f^3f''g + 990f^2f'^2g\\ &- 6480fg^2g'' + 11880fgg'^2 - 14580f'g^2g' + 9720f''g^3)X^2\\ &+ (-480f^5f'' + 420f^4f'^2 + 8640f^3gg'' + 10080f^3g'^2 - 54000f^2f'gg'\\ &+ 12960f^2f''g^2 + 29160ff'^2g^2 - 233280g^3g'' + 213840g^2g'^2)X\\ &+ (5184f^5g'' - 12816f^4f'g' - 7776f^4f''g + 15336f^3f'^2g - 139968f^2g^2g''\\ &+258336f^2gg'^2 - 428976ff'g^2g' + 209952ff''g^3 + 167184f'^2g^3)\Big).\end{aligned} \tag{3.58}$$

Note that ω does not appear explicitly in this expression. The same type of expression for F_3 is immediately obtained by rewriting the result in Appendix C. We do not present its lengthy expression here, but the calculation is straightforward.

The holomorphic anomaly equations can be written in a form more suited to the above expression. Observe that when the $\mathcal{F}_g$ are expressed as in Eq. (3.58), holomorphic anomalies appear only through X. Hence, one can simply replace ∂_ξ by

$$\partial_\xi = (\partial_\xi X)_u \partial_X = \frac{24}{\omega^2}\partial_X \tag{3.59}$$

in the holomorphic anomaly equation (3.49). For $g = 2$, the equation is now written as

$$24\partial_X\mathcal{F}_2 = \partial_u^2\mathcal{F}_1 - \partial_u \ln\omega\, \partial_u\mathcal{F}_1 + (\partial_u\mathcal{F}_1)^2\,. \tag{3.60}$$

By using

$$\begin{aligned}\partial_u\mathcal{F}_1 &= -\frac{1}{2}\partial_u\ln\omega - \frac{1}{12}\partial_u\ln D\\ &= \frac{(2fg' - 3f'g)X + (-2f^2f' + 36gg')}{16D}\end{aligned} \tag{3.61}$$

and (3.55), (3.57), one can evaluate the right-hand side of (3.60) as a quadratic polynomial in X. Then, integrating directly both sides of (3.60) in X, one obtains (3.58) up to the “constant” part in X. For $g \geq 3$, let us introduce the notation

$$\check{\mathcal{F}}_1 = \mathcal{F}_1 - \frac{1}{2}\ln\omega, \qquad \check{\mathcal{F}}_g = \mathcal{F}_g \quad \text{for} \quad g \geq 2. \tag{3.62}$$

The holomorphic anomaly equations can then be written again in a very simple form:

$$24\partial_X\check{\mathcal{F}}_g = \partial_u^2\check{\mathcal{F}}_{g-1} + \sum_{h=1}^{g-1}\partial_u\check{\mathcal{F}}_h\partial_u\check{\mathcal{F}}_{g-h}, \qquad g \geq 3. \tag{3.63}$$

Regarding the explicit form of $\mathcal{F}_g$ at $g = 2, 3$ and the above equation, we present our conjecture on the structure of the amplitudes in another form: $\mathcal{F}_g$ $(g \geq 2)$ can be expressed as

$$\mathcal{F}_g = \frac{1}{D^{2g-2}}\sum_{k=0}^{3g-3}P_{g,k}[\partial_u^{2g-2}, f, g]X^k. \tag{3.64}$$

Here, $P_{g,k}[\partial_u^{2g-2}, f, g]$ denotes a polynomial in $\partial_u^m f$, $\partial_u^n g$ $(m, n \in \mathbb{Z}_{\geq 0})$ in which each term contains $2g-2$ ∂_u's. The form of the polynomial is constrained so that $\mathcal{F}_g$ transforms as a $W(E_8)$-invariant quasi-Jacobi form of index 0. Note that the constituents of the amplitudes are of the following indices (see Appendix A):

$$[X] = 2, \quad [f] = 4, \quad [g] = 6, \quad [D] = 12, \quad [\partial_u] = -1. \tag{3.65}$$

Recall that f, g are polynomials of degree 4, 6 in u, respectively. Therefore, in this expression it is manifest that $\mathcal{F}_g$ is a polynomial in a finite number of generators, namely,

$$\frac{1}{D}, \quad X, \quad \partial_u^m f, \quad \partial_u^n g, \qquad m = 0, \ldots, 4, \quad n = 0, \ldots, 6. \tag{3.66}$$

We find that the above structure of the amplitudes and the holomorphic anomaly equation is akin to what has been obtained for other models by the direct integration method [37–40]. While we have taken a different path from the standard approach, both constructions should be essentially equivalent.

4. Topological string amplitudes for local del Pezzo surfaces

The topological string amplitudes for the local $\frac{1}{2}$K3 surface encompass those for all local del Pezzo surfaces. In this section we see how the former reduce to the latter. In fact, when the topological string amplitudes for the local $\frac{1}{2}$K3 are expressed in terms of the Seiberg–Witten curve, their forms are universal to all local del Pezzo surfaces. We obtain the amplitudes for any local del Pezzo surface by merely replacing the Seiberg–Witten curve with the corresponding one. The mirror pair of the local del Pezzo surface $\mathcal{B}_n$ is given by the Seiberg–Witten curve for the five-dimensional E_n strings [10]. It is also easy to reduce the most general Seiberg–Witten curve to that for any del Pezzo surface [10,43,49]. We first discuss the general cases and then present explicit forms of amplitudes for three basic examples: the massless local $\mathcal{B}_8$, the local $\mathbb{P}^2$, and the local $\mathbb{P}^1 \times \mathbb{P}^1$.

4.1. *General cases*

The Seiberg–Witten curve for the local $\mathcal{B}_8$ is obtained from that for the local $\frac{1}{2}$K3 by simply taking the limit $q \to 0$. Curves for the other local $\mathcal{B}_n$ $(n \leq 7)$ are immediately obtained by a suitable rescaling [10,43]. The construction of the topological string amplitudes from the Seiberg–Witten curve is essentially the same as in the case of the local $\frac{1}{2}$K3. In particular, the mirror map between u and ϕ for $\mathcal{B}_n$ $(n \leq 8)$ is simply given by the $q \to 0$ limit of (3.15).

Below we present the minor modifications needed for the local $\mathcal{B}_n$ $(n \leq 8)$. The instanton parts of the topological string amplitudes at $g = 0, 1$ are slightly modified as follows:

$$F_0 = -\partial_\phi^{-2} t + \frac{9-n}{6}\phi^3, \tag{4.1}$$

$$F_1 = \frac{1}{2}\ln\omega - \frac{1}{12}\ln\tilde{\Delta} - \frac{1}{2}\phi + \frac{9-n}{12}\phi, \tag{4.2}$$

with

$$t := 2\pi i\tilde{\tau} \tag{4.3}$$

instead of (3.10). Expressions for higher-genus amplitudes $\mathcal{F}_g$ ($g \geq 2$) hold as they stand, where the $\mathcal{F}_g$ are now related to F_g as

$$\mathcal{F}_1 = F_1 + \frac{1}{2}\phi, \qquad \mathcal{F}_2 = F_2 + \frac{1}{96}, \qquad \mathcal{F}_g = F_g \quad \text{for} \quad g \geq 3. \tag{4.4}$$

We also need to modify the relation (3.19) between ϕ and φ, since it is no longer valid in the limit $q = 0$. Instead of (3.19), we identify them by

$$e^{\phi} = -e^{2\pi i \varphi}. \tag{4.5}$$

4.2. *Massless local $\mathcal{B}_8$*

As an illustration we first consider the case of local $\mathcal{B}_8$ with $\boldsymbol{\mu} = \mathbf{0}$. In this case the corresponding Seiberg–Witten curve is extremely simple. The coefficients are given by

$$f = \frac{1}{12}u^4, \qquad g = \frac{1}{216}u^6 - 4u^5. \tag{4.6}$$

The amplitudes at $g = 0, 1$ are given by (4.1), (4.2) with $n = 8$. By substituting the above f, g into (3.58) one obtains

$$\begin{aligned}\mathcal{F}_2 = \frac{1}{207360u^4(u-432)^2}\Big(&25X^3 + 15u(-25u + 6048)X^2 \\ &+ 75u^2(29u^2 - 22464u + 5225472)X + u^5(335u - 273888)\Big).\end{aligned} \tag{4.7}$$

Similarly, from the expression of F_3 in Appendix C, one obtains

$$\begin{aligned}\mathcal{F}_3 = \frac{1}{5016453120u^8(u-432)^4}\Big(&525X^6 - 8400u^2X^5 \\ &+ 315u^2(175u^2 + 5184u + 5225472)X^4 \\ &+ 560u^3(-325u^3 + 18360u^2 - 89859456u + 11851370496)X^3 \\ &+ 63u^4(4625u^4 - 5008896u^3 + 8491143168u^2 \\ &- 2300402073600u + 260052929740800)X^2 \\ &+ 672u^7(-325u^3 + 284796u^2 - 623837376u + 7054387200)X \\ &+ u^8(61775u^4 - 96755904u^3 + 219325750272u^2 \\ &+ 15910182715392u + 9788763779629056)\Big).\end{aligned} \tag{4.8}$$

From these expressions one can compute Gopakumar–Vafa invariants. The instanton expansions in this case read

$$\frac{1}{u} = e^{\phi} - 60e^{2\phi} - 1530e^{3\phi} - 274160e^{4\phi} - 50519055e^{5\phi} + \mathcal{O}\left(e^{6\phi}\right), \tag{4.9}$$

$$\omega = e^{\phi} + 5130e^{3\phi} + 1347520e^{4\phi} + 372046365e^{5\phi} + \mathcal{O}\left(e^{6\phi}\right), \tag{4.10}$$

$$t = \phi + 252e^{\phi} + 36882e^{2\phi} + 7637736e^{3\phi} + 1828258569e^{4\phi} + \mathcal{O}\left(e^{5\phi}\right). \tag{4.11}$$

Table 1. Gopakumar–Vafa invariants for the massless local $\mathcal{B}_8$.

N_n^r n	1	2	3	4	5	$\cdots$
r						
0	252	−9252	848628	−114265008	18958064400	
1	−2	760	−246790	76413833	−23436186176	
2	0	-4	30464	−26631112	16150498760	
3	0	0	-1548	5889840	−7785768630	
$\vdots$						$\ddots$

The Gopakumar–Vafa invariants are computed by recasting F as

$$\sum_{g=0}^{\infty} F_g x^{2g-2} = \sum_{r=0}^{\infty}\sum_{n=1}^{\infty} N_n^r \sum_{m=1}^{\infty} \frac{1}{m}\left(2\sin\frac{mx}{2}\right)^{2r-2} e^{2\pi i m n\varphi}. \tag{4.12}$$

We present the Gopakumar–Vafa invariants N_n^r at low degrees in Table 1. This reproduces the known result, for example found in Refs. [5,33].[3] Moreover, it is easy to compute N_n^r up to an arbitrarily large degree of n, as we now have the exact form of the amplitudes $\mathcal{F}_g$.

We have performed the expansion around the large volume point $u = \infty$ to compute the Gopakumar–Vafa invariants, but we could expand the amplitudes at arbitrary u. It would be interesting to study the behavior of the amplitudes around the other points such as the orbifold point, as in Ref. [25].

4.3. Local $\mathbb{P}^2$

The coefficients of the Seiberg–Witten curve are given by

$$f = \frac{1}{12}u^4 - 2u, \qquad g = \frac{1}{216}u^6 - \frac{1}{6}u^3 + 1. \tag{4.13}$$

The amplitudes at $g = 0, 1$ are given by (4.1), (4.2) with $n = 0$. By substituting the above f, g into (3.58), one obtains

$$\mathcal{F}_2 = \frac{75X^3 - 165u^2X^2 + 125u^4X + 9(5u^6 - 464u^3 + 6192)}{7680(u^3-27)^2}. \tag{4.14}$$

Similarly, from the expression of F_3 in Appendix C, one obtains

$$\begin{aligned}\mathcal{F}_3 = \frac{1}{20643840(u^3-27)^4}\Big(&14175X^6 - 75600u^2X^5 + 315u(533u^3 + 3024)X^4\\ &- 560(355u^6 + 6750u^3 + 8748)X^3\\ &+ 21u^2(6305u^6 + 257472u^3 + 1181952)X^2\\ &- 672u(70u^9 + 5007u^6 + 49086u^3 + 34992)X\\ &+ 6965u^{12} + 774992u^9 + 13201920u^6 + 27993600u^3 + 20155392\Big).\end{aligned} \tag{4.15}$$

[3] The Gopakumar–Vafa invariants N_n^r at $r = 1$ and the instanton numbers $\tilde{N}_n^g$ for $g = 1$ curves found in [5,33] are related by $N_n^1 = \sum_{k|n} \tilde{N}^1_{(n/k)}$ [33].

Table 2. Gopakumar–Vafa invariants for the local $\mathbb{P}^2$.

N_n^r	n	1	2	3	4	5	$\cdots$
r							
0		3	−6	27	−192	1695	
1		0	0	−10	231	−4452	
2		0	0	0	−102	5430	
3		0	0	0	15	−3672	
$\vdots$							$\ddots$

The instanton expansions in this case are given by

$$\frac{1}{u} = e^{\phi} - 2e^{4\phi} - e^{7\phi} - 20e^{10\phi} - 177e^{13\phi} + \mathcal{O}\left(e^{16\phi}\right), \tag{4.16}$$

$$\omega = e^{\phi} + 4e^{4\phi} + 41e^{7\phi} + 520e^{10\phi} + 7275e^{13\phi} + \mathcal{O}\left(e^{16\phi}\right), \tag{4.17}$$

$$t = 9\phi + 27e^{3\phi} + \frac{405}{2}e^{6\phi} + 2196e^{9\phi} + \frac{110997}{4}e^{12\phi} + \mathcal{O}\left(e^{15\phi}\right). \tag{4.18}$$

The all-genus topological string partition function can be expressed as

$$\sum_{g=0}^{\infty} F_g x^{2g-2} = \sum_{r=0}^{\infty}\sum_{n=1}^{\infty} N_n^r \sum_{m=1}^{\infty} \frac{1}{m}\left(2\sin\frac{mx}{2}\right)^{2r-2} Q^{mn}, \tag{4.19}$$

where

$$Q = e^{6\pi i\varphi} = -e^{3\phi}. \tag{4.20}$$

Table 2 shows the Gopakumar–Vafa invariants N_n^r at low r and n. These are in agreement with the known result (see [31,50], for example) of the Gopakumar–Vafa invariants for local $\mathbb{P}^2$.

4.4. Local $\mathbb{P}^1 \times \mathbb{P}^1$

The coefficients of the Seiberg–Witten curve are given by

$$\begin{aligned} f &= \frac{1}{12}u^4 - \frac{2}{3}\chi u^2 + \frac{4}{3}\chi^2 - 4, \\ g &= \frac{1}{216}u^6 - \frac{1}{18}\chi u^4 + \left(\frac{2}{9}\chi^2 - \frac{1}{3}\right)u^2 + \left(-\frac{8}{27}\chi^3 + \frac{4}{3}\chi\right), \end{aligned} \tag{4.21}$$

where

$$\chi = e^{2\pi i\mu} + e^{-2\pi i\mu}. \tag{4.22}$$

The amplitudes at $g = 0, 1$ are given by (4.1), (4.2) with $n = 1$. Substituting the above f, g into (3.58), one obtains

$$\begin{aligned} \mathcal{F}_2 = \frac{1}{12960(u^2-4\chi+8)^2(u^2-4\chi-8)^2}\Big(&100u^2X^3 \\ &+ 120\left(-2u^4 + 5\chi u^2 + 12\chi^2 - 48\right)X^2 \end{aligned}$$

$$
\begin{aligned}
&+15\left(13u^6-80\chi u^4+(16\chi^2+768)u^2+384\chi^3-1536\chi\right)X\\
&+8\left(10u^8-201u^6\chi+(1452\chi^2-2808)u^4\right.\\
&\left.+(-4528\chi^3+20304\chi)u^2+5184\chi^4-36288\chi^2+62208\right)\Big). \qquad (4.23)
\end{aligned}
$$

We do not present the explicit form of $\mathcal{F}_3$ since it is slightly lengthy, but the calculation is straightforward. The instanton expansions in this case read

$$\frac{1}{u}=e^{\phi}-\chi e^{3\phi}+(\chi^2-3)e^{5\phi}+(-\chi^3+\chi)e^{7\phi}+\mathcal{O}\left(e^{9\phi}\right), \qquad (4.24)$$

$$\omega=e^{\phi}+\chi e^{3\phi}+(\chi^2+9)e^{5\phi}+(\chi^3+43\chi)e^{7\phi}+\mathcal{O}\left(e^{9\phi}\right), \qquad (4.25)$$

$$t=8\phi+8\chi e^{2\phi}+(4\chi^2+56)e^{4\phi}+\left(\frac{8}{3}\chi^3+208\chi\right)e^{6\phi}+\mathcal{O}\left(e^{8\phi}\right). \qquad (4.26)$$

The all-genus topological string partition function can be expressed as

$$\sum_{g=0}^{\infty}F_g x^{2g-2}=\sum_{r=0}^{\infty}\sum_{n_1,n_2=0}^{\infty}N^r_{n_1,n_2}\sum_{m=1}^{\infty}\frac{1}{m}\left(2\sin\frac{mx}{2}\right)^{2r-2}Q_1^{mn_1}Q_2^{mn_2}, \qquad (4.27)$$

where

$$Q_1=e^{2\pi i(2\varphi+\mu)}, \qquad Q_2=e^{2\pi i(2\varphi-\mu)}. \qquad (4.28)$$

We checked that $N^r_{n_1,n_2}$ are in agreement with the known data of the Gopakumar–Vafa invariants for the local $\mathbb{P}^1\times\mathbb{P}^1$ (see [50], for example).

5. Conclusion and discussion

In this paper we have developed a general method of computing topological string amplitudes for the local $\frac{1}{2}$K3 surface. We have demonstrated that the amplitudes can be concisely expressed in terms of the Seiberg–Witten curve, which manifestly exhibits good modular properties and the affine E_8 Weyl group invariance. We have clarified the general structure of the amplitudes. The amplitudes at $g=0,1$ are given in Eqs. (3.18), (3.39), while higher-genus amplitudes $\mathcal{F}_g$ ($g\geq 2$) are written as a polynomial in generators expressed in terms of the Seiberg–Witten curve. Given the structure, one can determine the coefficients of the polynomials by solving the holomorphic anomaly equation and the gap condition. We have explicitly computed the form of the amplitudes for $g=2,3$. We have also found that the holomorphic anomaly equation takes a very simple form if we adopt notations in which the amplitudes at low genus are slightly modified.

The topological strings on the local $\frac{1}{2}$K3 surface encompass those on all local del Pezzo surfaces. We have elucidated how to reduce the amplitudes to those for the local del Pezzo surfaces. By way of illustration, we have explicitly constructed the amplitudes for three simple cases. These amplitudes correctly reproduce the known Gopakumar–Vafa invariants.

There are several directions for further investigation. We have proposed that the conjectures (3.46), (3.64) on the structure of the amplitudes hold for general g. It is important to prove them and clarify how they are related to the general scheme of the polynomial structure [36]. Another point to be clarified is the precise conditions needed to determine the amplitudes at arbitrarily high genus. For general g, the gap condition used in this paper is not likely to be sufficient for fixing the amplitude.

On the other hand, it is known that regularity at the orbifold point and the large radius point and the leading behavior at the conifold points suffice to determine the holomorphic ambiguities at least for local del Pezzo surfaces with one or two moduli parameters [47]. We expect that the same sort of argument will apply to the case of the most general local $\frac{1}{2}$K3 surface.

The direct integration method has been applied to the four-dimensional SU(2) Seiberg–Witten theories with matters [39–41]. We know from Nekrasov partition functions that by taking a certain limit topological string amplitudes on the toric del Pezzo surfaces reproduce the prepotential and the gravitational corrections of the four-dimensional theories. It is interesting to see how our general formulas reproduce those results. The cases of non-toric local del Pezzo surfaces are of particular interest. In terms of the Seiberg–Witten curves, we know how the four-dimensional SU(2) theories with an E_n global symmetry [51–53] are reproduced from the five-dimensional ones [10,42]. It would be interesting to construct the gravitational corrections to these four-dimensional theories with an E_n flavor symmetry.

The topological recursion [20], or more specifically the "remodeling the B-model" conjecture [19], is a powerful method of computing topological string amplitudes. This method is free of the holomorphic ambiguity and also computes the open string amplitudes. It would be very interesting if our expressions for the amplitudes $\mathcal{F}_g$ can be derived by a method similar to the topological recursion.

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Appendix A. Seiberg–Witten curve for E-string theory

The low-energy effective theory of the E-string theory in $\mathbb{R}^4 \times T^2$ is described as SU(2) Seiberg–Witten theory with nine parameters, τ and $\boldsymbol{\mu} = (\mu_1, \dots, \mu_8)$. τ is regarded as the bare gauge coupling and $\boldsymbol{\mu}$ are the masses of fundamental matters. The theory possesses an E_8 flavor symmetry, and the Weyl group $W(E_8)$ acts on $\boldsymbol{\mu}$ as an automorphism. On the other hand, from the point of view of the six-dimensional theory, τ is the modulus of the T^2 in the 5,6-directions and the $\boldsymbol{\mu}$ are interpreted as Wilson lines along these directions. The theory therefore admits modular properties in τ and double periodicity in $\boldsymbol{\mu}$. These symmetries become manifest if we express the dependence on these parameters through $W(E_8)$-invariant Jacobi forms.

A.1. $W(E_8)$-invariant Jacobi forms

Let $\varphi_{k,m}(\tau, \boldsymbol{\mu})$ denote $W(E_8)$-invariant Jacobi forms of weight k and index m. They are holomorphic in τ ($\mathrm{Im}\,\tau > 0$), $\boldsymbol{\mu} \in \mathbb{C}^8$, and satisfy the following properties [54,55]:

(i) Weyl invariance:

$$\varphi_{k,m}(\tau, w(\boldsymbol{\mu})) = \varphi_{k,m}(\tau, \boldsymbol{\mu}), \qquad w \in W(E_8). \tag{A.1}$$

(ii) Quasi-periodicity:

$$\varphi_{k,m}(\tau, \boldsymbol{\mu} + \boldsymbol{v} + \tau \boldsymbol{w}) = e^{-m\pi i(\tau \boldsymbol{w}^2 + 2\boldsymbol{\mu}\cdot \boldsymbol{w})}\varphi_{k,m}(\tau, \boldsymbol{\mu}), \qquad \boldsymbol{v}, \boldsymbol{w} \in \Gamma_8. \tag{A.2}$$

(iii) Modular properties:

$$\varphi_{k,m}\left(\frac{a\tau + b}{c\tau + d}, \frac{\boldsymbol{\mu}}{c\tau + d}\right) = (c\tau + d)^k \exp\left(m\pi i \frac{c}{c\tau + d}\,\boldsymbol{\mu}^2\right)\varphi_{k,m}(\tau, \boldsymbol{\mu}). \tag{A.3}$$

(iv) $\varphi_{k,m}(\tau, \boldsymbol{\mu})$ admit a Fourier expansion as

$$\varphi_{k,m}(\tau, \boldsymbol{\mu}) = \sum_{l=0}^{\infty} \sum_{\substack{\boldsymbol{v} \in \Gamma_8 \\ \boldsymbol{v}^2 \le 2ml}} c(l, \boldsymbol{v}) e^{2\pi i(l\tau + \boldsymbol{v}\cdot\boldsymbol{\mu})}. \tag{A.4}$$

Here, Γ_8 is the E_8 root lattice and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. Note that in this convention the index m coincides with the level of the affine E_8 Lie algebra.

Among others, the most fundamental $W(E_8)$-invariant Jacobi form is the theta function associated with the lattice Γ_8,

$$\Theta(\tau, \boldsymbol{\mu}) = \sum_{\boldsymbol{w}\in\Gamma^8} \exp\left(\pi i \tau \boldsymbol{w}^2 + 2\pi i \boldsymbol{\mu}\cdot\boldsymbol{w}\right) = \frac{1}{2}\sum_{k=1}^{4}\prod_{j=1}^{8} \vartheta_k(\mu_j, \tau). \tag{A.5}$$

One can see from the properties of the Jacobi theta functions that $\Theta(\tau, \boldsymbol{\mu})$ is of weight 4 and index 1. Jacobi forms of higher indices can be constructed from $\Theta(\tau, \boldsymbol{\mu})$ as follows.

To construct more general $W(E_8)$-invariant Jacobi forms, we introduce the functions

$$\begin{aligned} e_1(\tau) &= \tfrac{1}{12}\left(\vartheta_3(\tau)^4 + \vartheta_4(\tau)^4\right), \\ e_2(\tau) &= \tfrac{1}{12}\left(\vartheta_2(\tau)^4 - \vartheta_4(\tau)^4\right), \\ e_3(\tau) &= \tfrac{1}{12}\left(-\vartheta_2(\tau)^4 - \vartheta_3(\tau)^4\right), \end{aligned} \tag{A.6}$$

and

$$h(\tau) = \vartheta_3(2\tau)\vartheta_3(6\tau) + \vartheta_2(2\tau)\vartheta_2(6\tau). \tag{A.7}$$

Let us then define the following nine $W(E_8)$-invariant Jacobi forms:

$$\begin{aligned} &A_1(\tau, \boldsymbol{\mu}) = \Theta(\tau, \boldsymbol{\mu}), \quad A_2(\tau, \boldsymbol{\mu}) = \tfrac{8}{9}\mathscr{H}\{\Theta(2\tau, 2\boldsymbol{\mu})\}, \quad A_3(\tau, \boldsymbol{\mu}) = \tfrac{27}{28}\mathscr{H}\{\Theta(3\tau, 3\boldsymbol{\mu})\}, \\ &A_4(\tau, \boldsymbol{\mu}) = \Theta(\tau, 2\boldsymbol{\mu}), \ A_5(\tau, \boldsymbol{\mu}) = \tfrac{125}{126}\mathscr{H}\{\Theta(5\tau, 5\boldsymbol{\mu})\}, \\ &B_2(\tau, \boldsymbol{\mu}) = \tfrac{32}{5}\mathscr{H}\{e_1(\tau)\Theta(2\tau, 2\boldsymbol{\mu})\}, \qquad B_3(\tau, \boldsymbol{\mu}) = \tfrac{81}{80}\mathscr{H}\left\{h(\tau)^2\Theta(3\tau, 3\boldsymbol{\mu})\right\}, \\ &B_4(\tau, \boldsymbol{\mu}) = \tfrac{16}{15}\mathscr{H}\left\{\vartheta_4(2\tau)^4\Theta(4\tau, 4\boldsymbol{\mu})\right\}, \quad B_6(\tau, \boldsymbol{\mu}) = \tfrac{9}{10}\mathscr{H}\left\{h(\tau)^2\Theta(6\tau, 6\boldsymbol{\mu})\right\}. \end{aligned} \tag{A.8}$$

Here, $\mathscr{H}\{\cdot\}$ denotes the sum of all possible distinct $\mathrm{SL}(2, \mathbb{Z})$ transforms of the argument. Explicitly, they read

$$\begin{aligned} &A_1(\tau, \boldsymbol{\mu}) = \Theta(\tau, \boldsymbol{\mu}), \qquad A_4(\tau, \boldsymbol{\mu}) = \Theta(\tau, 2\boldsymbol{\mu}), \\ &A_n(\tau, \boldsymbol{\mu}) = \tfrac{n^3}{n^3+1}\left(\Theta(n\tau, n\boldsymbol{\mu}) + \tfrac{1}{n^4}\textstyle\sum_{k=0}^{n-1}\Theta(\tfrac{\tau+k}{n}, \boldsymbol{\mu})\right), \qquad n = 2, 3, 5, \end{aligned}$$

$$B_2(\tau,\boldsymbol{\mu}) = \tfrac{32}{5}\Big(e_1(\tau)\Theta(2\tau,2\boldsymbol{\mu}) + \tfrac{1}{2^4}e_3(\tau)\Theta(\tfrac{\tau}{2},\boldsymbol{\mu}) + \tfrac{1}{2^4}e_2(\tau)\Theta(\tfrac{\tau+1}{2},\boldsymbol{\mu})\Big),$$
$$B_3(\tau,\boldsymbol{\mu}) = \tfrac{81}{80}\Big(h(\tau)^2\Theta(3\tau,3\boldsymbol{\mu}) - \tfrac{1}{3^5}\textstyle\sum_{k=0}^{2}h(\tfrac{\tau+k}{3})^2\Theta(\tfrac{\tau+k}{3},\boldsymbol{\mu})\Big),$$
$$B_4(\tau,\boldsymbol{\mu}) = \tfrac{16}{15}\Big(\vartheta_4(2\tau)^4\Theta(4\tau,4\boldsymbol{\mu}) - \tfrac{1}{2^4}\vartheta_4(2\tau)^4\Theta(\tau+\tfrac{1}{2},2\boldsymbol{\mu}) - \tfrac{1}{2^2\cdot 4^4}\textstyle\sum_{k=0}^{3}\vartheta_2(\tfrac{\tau+k}{2})^4\Theta(\tfrac{\tau+k}{4},\boldsymbol{\mu})\Big),$$
$$B_6(\tau,\boldsymbol{\mu}) = \tfrac{9}{10}\Big(h(\tau)^2\Theta(6\tau,6\boldsymbol{\mu}) + \tfrac{1}{2^4}\textstyle\sum_{k=0}^{1}h(\tau+k)^2\Theta(\tfrac{3\tau+3k}{2},3\boldsymbol{\mu}) - \tfrac{1}{3\cdot 3^4}\textstyle\sum_{k=0}^{2}h(\tfrac{\tau+k}{3})^2\Theta(\tfrac{2\tau+2k}{3},2\boldsymbol{\mu}) - \tfrac{1}{3\cdot 6^4}\textstyle\sum_{k=0}^{5}h(\tfrac{\tau+k}{3})^2\Theta(\tfrac{\tau+k}{6},\boldsymbol{\mu})\Big). \quad (A.9)$$

A_n, B_n are of index n and weight $4, 6$, respectively. If we set $\boldsymbol{\mu} = \mathbf{0}$, these Jacobi forms reduce to ordinary modular forms. We have determined the normalization of A_n, B_n so that they reduce to the Eisenstein series

$$A_n(\tau,\mathbf{0}) = E_4(\tau), \qquad B_n(\tau,\mathbf{0}) = E_6(\tau). \quad (A.10)$$

A_n, B_n generate all the $W(E_8)$-invariant Jacobi forms appearing in the coefficients of the Seiberg–Witten curve.[4]

A.2. Seiberg–Witten curve

The Seiberg–Witten curve for the E-string theory was constructed in Ref. [42]. Here we present the same curve expressed in terms of the $W(E_8)$-invariant Jacobi forms introduced above:

$$y^2 = 4x^3 - fx - g, \quad (A.11)$$

$$f = \sum_{j=0}^{4} a_j u^{4-j}, \qquad g = \sum_{j=0}^{6} b_j u^{6-j}, \quad (A.12)$$

$$a_0 = \frac{1}{12}A_0, \qquad a_1 = 0, \qquad a_2 = \frac{6}{E_4\Delta}\Big(-A_0A_2 + A_1^2\Big),$$
$$a_3 = \frac{1}{9E_4^2\Delta^2}\Big(-7A_0^2B_0A_3 - 20A_0^3B_3 - 9A_0B_0A_1A_2 + 30A_0^2A_1B_2 + 6B_0A_1^3\Big),$$
$$a_4 = \frac{1}{864E_4^3\Delta^3}\Big((A_0^6 - A_0^3B_0^2)A_4 + (56A_0^5 - 56A_0^2B_0^2)A_1A_3 - 27A_0^5A_2^2 - 90A_0^3B_0A_2B_2 - 75A_0^4B_2^2 + (180A_0^4 - 36A_0B_0^2)A_1^2A_2 + 240A_0^2B_0A_1^2B_2 + (-210A_0^3 + 18B_0^2)A_1^4\Big),$$
$$b_0 = \frac{1}{216}B_0, \qquad b_1 = -\frac{4}{E_4}A_1, \qquad b_2 = \frac{5}{6E_4^2\Delta}\Big(A_0^2B_2 - B_0A_1^2\Big),$$

[4] There are alternative choices for the generators A_n, B_n. For instance, one can take $\frac{256}{45}\mathscr{H}\{e_1(\tau)\Theta(4\tau,4\boldsymbol{\mu})\}$ instead of B_4 and/or $\frac{54}{55}\mathscr{H}\big\{h(2\tau)^2\Theta(6\tau,6\boldsymbol{\mu})\big\}$ instead of B_6.

$$b_3 = \frac{1}{108E_4^3\Delta^2}\Big(-7A_0^5A_3 - 20A_0^3B_0B_3$$
$$- 9A_0^4A_1A_2 + 30A_0^2B_0A_1B_2 + (16A_0^3 - 10B_0^2)A_1^3\Big),$$
$$b_4 = \frac{1}{1728E_4^4\Delta^3}\Big((-5A_0^7 + 5A_0^4B_0^2)B_4 + (80A_0^6 - 80A_0^3B_0^2)A_1B_3$$
$$+ 9A_0^5B_0A_2^2 + 30A_0^6A_2B_2 + 25A_0^4B_0B_2^2 - 48B_0A_0^4A_1^2A_2$$
$$+ (-140A_0^5 + 60A_0^2B_0^2)A_1^2B_2 + (74A_0^3B_0 - 10B_0^3)A_1^4\Big),$$
$$b_5 = \frac{1}{72E_4^5\Delta^3}\Big((-21A_0^7 + 21A_0^4B_0^2)A_5 - 294A_0^6A_2A_3 - 770A_0^4B_0B_2A_3$$
$$- 840A_0^4B_0A_2B_3 - 2200A_0^5B_2B_3 + 168A_0^5A_1^2A_3 + 480B_0A_0^3A_1^2B_3$$
$$- 621A_0^5A_1A_2^2 + 3525A_0^4A_1B_2^2 + 1224A_0^4A_1^3A_2 - 240A_0^2B_0A_1^3B_2$$
$$+ (-456A_0^3 + 24B_0^2)A_1^5\Big),$$
$$b_6 = \frac{1}{13436928E_4^6\Delta^5}\Big((-20A_0^{12} + 40A_0^9B_0^2 - 20A_0^6B_0^4)B_6$$
$$+ (-189A_0^{10}B_0 + 378A_0^7B_0^3 - 189A_0^4B_0^5)A_1A_5$$
$$+ (-9A_0^{10}B_0 + 9A_0^7B_0^3)A_2A_4 + (-15A_0^{11} + 15A_0^8B_0^2)B_2A_4$$
$$+ (-180A_0^{11} + 180A_0^8B_0^2)A_2B_4 + (-300A_0^9B_0 + 300A_0^6B_0^3)B_2B_4$$
$$+ (22A_0^9B_0 - 22A_0^6B_0^3)A_1^2A_4 + (150A_0^{10} + 120A_0^7B_0^2 - 270A_0^4B_0^4)A_1^2B_4$$
$$+ (196A_0^{10}B_0 - 196A_0^7B_0^3)A_3^2 + (1120A_0^{11} - 1120A_0^8B_0^2)A_3B_3$$
$$+ (1600A_0^9B_0 - 1600A_0^6B_0^3)B_3^2 + (-2982A_0^9B_0 + 2982A_0^6B_0^3)A_1A_2A_3$$
$$+ (-2520A_0^{10} - 4410A_0^7B_0^2 + 6930A_0^4B_0^4)A_1B_2A_3$$
$$+ (3360A_0^{10} - 10920A_0^7B_0^2 + 7560A_0^4B_0^4)A_1A_2B_3$$
$$+ (-19800A_0^8B_0 + 19800A_0^5B_0^3)A_1B_2B_3 + (2016A_0^8B_0 - 2016A_0^5B_0^3)A_1^3A_3$$
$$+ (-5920A_0^9 + 7360A_0^6B_0^2 - 1440A_0^3B_0^4)A_1^3B_3 + (405A_0^9B_0 + 162A_0^6B_0^3)A_2^3$$
$$+ (1215A_0^{10} + 1620A_0^7B_0^2)A_2^2B_2 + 4725A_0^8B_0A_2B_2^2$$
$$+ (1125A_0^9 + 1500A_0^6B_0^2)B_2^3 + (-9477A_0^8B_0 + 5103A_0^5B_0^3)A_1^2A_2^2$$
$$+ (-9180A_0^9 - 5400A_0^6B_0^2)A_1^2A_2B_2 + (20925A_0^7B_0 - 33075A_0^4B_0^3)A_1^2B_2^2$$
$$+ (20304A_0^7B_0 - 9072A_0^4B_0^3)A_1^4A_2$$
$$+ (12780A_0^8 + 5400A_0^5B_0^2 + 540A_0^2B_0^4)A_1^4B_2$$
$$+ (-11076A_0^6B_0 + 1512A_0^3B_0^3 - 36B_0^5)A_1^6\Big). \quad \text{(A.13)}$$

Note that a_n, b_n satisfy most of the properties of the $W(E_8)$-invariant Jacobi forms except the condition $\boldsymbol{v}^2 \leq 2ml$ in the Fourier expansion. a_n, b_n are of index n and weight $4 - 6n, 6 - 6n$, respectively. It is useful to let the variables u, x, y transform formally as Jacobi forms of weights

$-6, -10, -15$ and index $1, 2, 3$, respectively. The whole curve then transforms as a Jacobi form of weight -30 and index 6. f, g are of weight $-20, -30$ and index $4, 6$, respectively.

Appendix B. Derivative formulas

$$q\frac{d}{dq}\ln\Delta = E_2, \tag{B.1}$$

$$q\frac{d}{dq}E_2 = \frac{1}{12}(E_2^2 - E_4), \tag{B.2}$$

$$q\frac{d}{dq}E_4 = \frac{1}{3}(E_4E_2 - E_6), \tag{B.3}$$

$$q\frac{d}{dq}E_6 = \frac{1}{2}(E_6E_2 - E_4^2). \tag{B.4}$$

$$\left(\partial_\xi t\right)_u = -2t^2, \tag{B.5}$$

$$\left(\partial_\xi \omega\right)_u = 2\omega t, \tag{B.6}$$

$$\left(\partial_\xi \phi\right)_u = 2\partial_\phi^{-1} t, \tag{B.7}$$

$$\left(\partial_\xi \ln\tilde{\Delta}\right)_u = 24t, \tag{B.8}$$

$$\partial_\xi \tilde{E}_{2k} = 4kt\tilde{E}_{2k} + 24\delta_{1,k}\,. \tag{B.9}$$

$$\left(\partial_\xi\left(\partial_\phi^n \ln\omega\right)\right)_u = -2\sum_{k=0}^{n-1}\binom{n}{k+1}\partial_\phi^k t\,\partial_\phi^{n-k}\ln\omega + 2\partial_\phi^n t, \tag{B.10}$$

$$\left(\partial_\xi\left(\partial_\phi^n t\right)\right)_u = -2\sum_{k=0}^{n-1}\left[\binom{n}{k+1} + 2\binom{n-1}{k}\right]\partial_\phi^k t\,\partial_\phi^{n-k} t \qquad (n \geq 1). \tag{B.11}$$

Appendix C. Genus three amplitude

$$\begin{aligned}
F_3 = \ & (\partial_\phi^4 \ln\omega)\left(\tfrac{1}{2304}\tilde{E}_2^2 + \tfrac{1}{2592}\tilde{E}_4\right) \\
&+ (\partial_\phi^3 \ln\omega)(\partial_\phi \ln\omega)\left(\tfrac{1}{1152}\tilde{E}_2^2 - \tfrac{1}{6912}\tilde{E}_4\right) \\
&+ (\partial_\phi^2 \ln\omega)^2\left(\tfrac{1}{2304}\tilde{E}_2^2 + \tfrac{11}{20736}\tilde{E}_4\right) \\
&+ (\partial_\phi^2 \ln\omega)(\partial_\phi \ln\omega)^2\left(\tfrac{1}{2304}\tilde{E}_2^2 + \tfrac{1}{20736}\tilde{E}_4\right) \\
&+ (\partial_\phi^3 \ln\omega)(\partial_\phi t)\left(\tfrac{1}{41472}\tilde{E}_2^3 + \tfrac{11}{82944}\tilde{E}_4\tilde{E}_2 - \tfrac{13}{82944}\tilde{E}_6\right) \\
&+ (\partial_\phi^2 \ln\omega)(\partial_\phi \ln\omega)(\partial_\phi t)\left(\tfrac{1}{13824}\tilde{E}_2^3 - \tfrac{25}{248832}\tilde{E}_4\tilde{E}_2 + \tfrac{7}{248832}\tilde{E}_6\right) \\
&+ (\partial_\phi \ln\omega)^3(\partial_\phi t)\left(\tfrac{1}{41472}\tilde{E}_2^3 - \tfrac{1}{31104}\tilde{E}_4\tilde{E}_2 + \tfrac{1}{124416}\tilde{E}_6\right) \\
&+ (\partial_\phi^2 \ln\omega)(\partial_\phi^2 t)\left(\tfrac{7}{62208}\tilde{E}_4\tilde{E}_2 - \tfrac{7}{62208}\tilde{E}_6\right) \\
&+ (\partial_\phi^2 \ln\omega)(\partial_\phi t)^2\left(-\tfrac{1}{331776}\tilde{E}_2^4 + \tfrac{79}{1492992}\tilde{E}_4\tilde{E}_2^2 - \tfrac{35}{373248}\tilde{E}_6\tilde{E}_2 + \tfrac{131}{2985984}\tilde{E}_4^2\right)
\end{aligned}$$

$$
\begin{aligned}
&+ (\partial_\phi \ln\omega)^2(\partial_\phi^2 t)\left(\tfrac{5}{248832}\tilde{E}_4\tilde{E}_2 - \tfrac{5}{248832}\tilde{E}_6\right)\\
&+ (\partial_\phi \ln\omega)^2(\partial_\phi t)^2\left(-\tfrac{1}{331776}\tilde{E}_2^4 + \tfrac{43}{2985984}\tilde{E}_4\tilde{E}_2^2 - \tfrac{25}{1492992}\tilde{E}_6\tilde{E}_2 + \tfrac{1}{186624}\tilde{E}_4^2\right)\\
&+ (\partial_\phi \ln\omega)(\partial_\phi^3 t)\left(-\tfrac{1}{13824}\tilde{E}_2^3 + \tfrac{1}{19440}\tilde{E}_4\tilde{E}_2 + \tfrac{13}{622080}\tilde{E}_6\right)\\
&+ (\partial_\phi \ln\omega)(\partial_\phi^2 t)(\partial_\phi t)\left(-\tfrac{5}{165888}\tilde{E}_2^4 + \tfrac{35}{497664}\tilde{E}_4\tilde{E}_2^2 - \tfrac{5}{248832}\tilde{E}_6\tilde{E}_2 - \tfrac{5}{248832}\tilde{E}_4^2\right)\\
&+ (\partial_\phi \ln\omega)(\partial_\phi t)^3\left(-\tfrac{1}{497664}\tilde{E}_2^5 + \tfrac{29}{2985984}\tilde{E}_4\tilde{E}_2^3\right.\\
&\qquad\qquad \left. - \tfrac{1}{110592}\tilde{E}_6\tilde{E}_2^2 - \tfrac{1}{995328}\tilde{E}_4^2\tilde{E}_2 + \tfrac{7}{2985984}\tilde{E}_6\tilde{E}_4\right)\\
&+ (\partial_\phi^4 t)\left(-\tfrac{1}{20736}\tilde{E}_2^3 - \tfrac{121}{1244160}\tilde{E}_4\tilde{E}_2 - \tfrac{173}{8709120}\tilde{E}_6\right)\\
&+ (\partial_\phi^3 t)(\partial_\phi t)\left(-\tfrac{11}{497664}\tilde{E}_2^4 - \tfrac{287}{3732480}\tilde{E}_4\tilde{E}_2^2 + \tfrac{421}{6531840}\tilde{E}_6\tilde{E}_2 + \tfrac{361}{10450944}\tilde{E}_4^2\right)\\
&+ (\partial_\phi^2 t)^2\left(-\tfrac{1}{55296}\tilde{E}_2^4 - \tfrac{19}{331776}\tilde{E}_4\tilde{E}_2^2 + \tfrac{19}{387072}\tilde{E}_6\tilde{E}_2 + \tfrac{61}{2322432}\tilde{E}_4^2\right)\\
&+ (\partial_\phi^2 t)(\partial_\phi t)^2\left(-\tfrac{13}{1990656}\tilde{E}_2^5 - \tfrac{1}{27648}\tilde{E}_4\tilde{E}_2^3\right.\\
&\qquad\qquad \left. + \tfrac{25}{331776}\tilde{E}_6\tilde{E}_2^2 - \tfrac{19}{1990656}\tilde{E}_4^2\tilde{E}_2 - \tfrac{23}{995328}\tilde{E}_6\tilde{E}_4\right)\\
&+ (\partial_\phi t)^4\left(-\tfrac{7}{23887872}\tilde{E}_2^6 - \tfrac{181}{71663616}\tilde{E}_4\tilde{E}_2^4 + \tfrac{19}{2239488}\tilde{E}_6\tilde{E}_2^3\right.\\
&\qquad\qquad \left. - \tfrac{47}{7962624}\tilde{E}_4^2\tilde{E}_2^2 - \tfrac{1}{559872}\tilde{E}_6\tilde{E}_4\tilde{E}_2 + \tfrac{73}{71663616}\tilde{E}_4^3 + \tfrac{1}{995328}\tilde{E}_6^2\right).
\end{aligned} \tag{C.1}
$$

Appendix D. Conventions

We define the Eisenstein series, the modular discriminant, and the j-invariant by their Fourier expansion:

$$E_{2n}(\tau) = 1 + \frac{(2\pi i)^{2n}}{(2n-1)!\,\zeta(2n)}\sum_{k=1}^{\infty}\frac{k^{2n-1}q^k}{1-q^k}, \qquad q = e^{2\pi i\tau}, \tag{D.1}$$

$$\Delta(\tau) = q\left[\prod\nolimits_{k=1}^{\infty}(1-q^k)\right]^{24} = \frac{1}{1728}\Big(E_4(\tau)^3 - E_6(\tau)^2\Big), \tag{D.2}$$

$$j(\tau) = \frac{E_4(\tau)^3}{\Delta(\tau)}. \tag{D.3}$$

We often omit the argument of these functions, as far as it is τ. When the argument is $\tilde{\tau}$, we use the following abbreviations:

$$\tilde{E}_{2n} := E_{2n}(\tilde{\tau}), \qquad \tilde{\Delta} := \Delta(\tilde{\tau}), \qquad \tilde{j} := j(\tilde{\tau}). \tag{D.4}$$

The Weierstrass $\wp$-function is defined as

$$\wp(z|2\pi\omega, 2\pi\omega\tau) = \frac{1}{z^2} + \sum_{m,n\in\mathbb{Z}^2_{\neq(0,0)}}\left[\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{{\Omega_{m,n}}^2}\right], \quad \Omega_{m,n} = 2\pi\omega(m+n\tau). \tag{D.5}$$

This function satisfies the differential equation

$$(\partial_z \wp)^2 = 4\wp^3 - \frac{E_4(\tau)}{12\omega^4}\wp - \frac{E_6(\tau)}{216\omega^6}. \tag{D.6}$$

The Jacobi theta functions are defined as

$$\vartheta_1(z,\tau) = i\sum_{n\in\mathbb{Z}}(-1)^n y^{n-1/2} q^{(n-1/2)^2/2}, \tag{D.7}$$

$$\vartheta_2(z,\tau) = \sum_{n\in\mathbb{Z}} y^{n-1/2} q^{(n-1/2)^2/2}, \tag{D.8}$$

$$\vartheta_3(z,\tau) = \sum_{n\in\mathbb{Z}} y^n q^{n^2/2}, \tag{D.9}$$

$$\vartheta_4(z,\tau) = \sum_{n\in\mathbb{Z}}(-1)^n y^n q^{n^2/2}, \tag{D.10}$$

where $y = e^{2\pi i z}$, $q = e^{2\pi i\tau}$. We also use the following abbreviated notation:

$$\vartheta_k(\tau) := \vartheta_k(0,\tau). \tag{D.11}$$

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