

Symmetries of the squeezed Kerr oscillator

Francesco Iachello

Center for Theoretical Physics, Sloane Physics Laboratory, Yale University,
New Haven, CT 06520-8120, USA

francesco.iachello@yale.edu

Abstract. The algebraic structure of the squeezed Kerr oscillator $sp(2, \mathbb{R})-su(1,1)$ is described. It is shown that the Kerr oscillator has a novel and hitherto unknown symmetry $su(2)$ which occurs for integer values of the ratio $\eta = \Delta/K$ of detuning Δ to the Kerr coefficient K . The effect of quadratic, cubic and quartic squeezing of the Kerr oscillator is discussed. The results are of importance for the generation and stabilization of states useful for quantum computing.

1. Introduction

Recently, considerable interest has arisen in the development of quantum computers based on Kerr parametric oscillators [1-3]. It has also been possible to realize these parametric oscillators by sinusoidally driven superconducting circuits [4-6] and to measure their spectra as a function of the parameters appearing in the associated static effective Hamiltonian

$$\hat{H} = -\Delta \hat{a}^\dagger \hat{a} + K \hat{a}^{\dagger 2} \hat{a}^2 - \varepsilon_2 (\hat{a}^{\dagger 2} + \hat{a}^2) \quad (1)$$

where Δ is the detuning, K the Kerr coefficient and ε_2 the squeezing amplitude. Very recently, it has been found that the observed spectra have remarkable symmetry properties [7]. In this presentation, after introducing the algebraic structure of the squeezed Kerr oscillator, its newly found symmetries and their stability to perturbations are discussed.

2. Algebraic structure of the squeezed Kerr oscillator

The time dependent Hamiltonian of the sinusoidally driven circuit, can be written [4], in the rotating wave approximation RWA, as a time-independent expansion in terms of one-dimensional creation and annihilation operators, \hat{a}^\dagger, \hat{a} , with $[\hat{a}, \hat{a}^\dagger] = 1$, $\hat{H} = \sum_{n \geq 1} \hat{H}^{(n)}$. At orders 1 and 2, the boson expansion of the effective Hamiltonian is

$$\hat{H} = -\Delta (\hat{a}^\dagger \hat{a}) + K (\hat{a}^\dagger \hat{a}) (\hat{a}^\dagger \hat{a} - 1) - \varepsilon_2 (\hat{a}^\dagger \hat{a}^\dagger + \hat{a} \hat{a}). \quad (2)$$

The three operators

$$\hat{F}_+ = \hat{a}^\dagger \hat{a}^\dagger, \hat{F}_- = \hat{a} \hat{a}, \hat{F}_z = \hat{a}^\dagger \hat{a} \quad (3)$$

form a closed algebra g with commutation relations

$$[\hat{F}_z, \hat{F}_\pm] = \pm 2 \hat{F}_\pm, [\hat{F}_+, \hat{F}_-] = -4 \hat{F}_z - 2. \quad (4)$$

This algebra is the symplectic algebra $sp(2, \mathbb{R})$

$$g \equiv sp(2, \mathbb{R}) \sim su(1,1) \quad (5)$$



which is isomorphic to $su(1,1)$, the non-compact version of $su(2)$. In terms of elements of this algebra

$$\hat{H} = -\Delta\hat{F}_z + K\hat{F}_z(\hat{F}_z - 1) - \varepsilon_2(\hat{F}_+ + \hat{F}_-) = -\Delta\hat{F}_z + K\hat{F}_z(\hat{F}_z - 1) - 2\varepsilon_2\hat{F}_x \quad (6)$$

To order 3 and 4, the additional contributions to the effective Hamiltonian can be written as

$$\begin{aligned} \hat{H}^{(3)} &= -\Delta^{(3)}(\hat{a}^\dagger\hat{a}) - K^{(3)}(\hat{a}^\dagger\hat{a})(\hat{a}^\dagger\hat{a} - 1) + \varepsilon_2^{(3)}(\hat{a}^{\dagger 2} + \hat{a}^2) + \varepsilon_2'(\hat{a}^{\dagger 2}(\hat{a}^\dagger\hat{a}) + (\hat{a}^\dagger\hat{a})\hat{a}^2) \\ \hat{H}^{(4)} &= -\Delta^{(4)}(\hat{a}^\dagger\hat{a}) - K^{(4)}(\hat{a}^\dagger\hat{a})(\hat{a}^\dagger\hat{a} - 1) - \lambda^{(4)}((\hat{a}^\dagger\hat{a})^3 - 3(\hat{a}^\dagger\hat{a})^2 - 2(\hat{a}^\dagger\hat{a})) + \varepsilon_4^{(4)}(\hat{a}^{\dagger 4} + \hat{a}^4) \end{aligned} \quad (7)$$

Since again the contributions $\hat{H}^{(3)}, \hat{H}^{(4)}$ can be written in terms of elements of $sp(2, R)$, this algebra is the spectrum generating algebra (SGA) of the Kerr oscillator at order four, i.e. $\hat{H} = \sum_{4 \geq n \geq 1} \hat{H}^{(n)}$ is a polynomial in the elements of $sp(2, R)$.

3. Alternative algebraic structure

An alternative algebraic structure is obtained by introducing an auxiliary boson s and constructing the algebra of $u(2)$ [8]

$$\hat{F}'_- = \hat{s}^\dagger\hat{a}, \hat{F}'_+ = \hat{a}^\dagger\hat{s}, \hat{F}'_z = \frac{1}{2}(\hat{s}^\dagger\hat{s} - \hat{a}^\dagger\hat{a}), \hat{N} = \hat{s}^\dagger\hat{s} + \hat{a}^\dagger\hat{a}. \quad (8)$$

The three operators $\hat{F}'_-, \hat{F}'_+, \hat{F}'_z$ form the algebra of $su(2)$. Introducing operators $\hat{F}'_- = \hat{s}^\dagger\hat{a}, \hat{F}'_+ = \hat{a}^\dagger\hat{s}, \hat{n} = \hat{a}^\dagger\hat{a}, \hat{n}_s = \hat{s}^\dagger\hat{s}$, replacing \hat{s}, \hat{s}^\dagger by \sqrt{N} and considering

$$\frac{\hat{F}'_-}{\sqrt{N}} = \hat{a}, \frac{\hat{F}'_+}{\sqrt{N}} = \hat{a}^\dagger, \hat{n} = \hat{a}^\dagger\hat{a}, \frac{\hat{n}_s}{N} = 1 \quad (9)$$

one obtains the Heisenberg algebra $h(2)$. This algebra is also called the contracted algebra of $u(2)$, $u(2) \rightarrow_c h(2)$, with commutation relations

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= \hat{1}, [\hat{a}^\dagger, \hat{1}] = [\hat{a}, \hat{1}] = 0 \\ [\hat{a}, \hat{a}^\dagger\hat{a}] &= \hat{a}, [\hat{a}^\dagger, \hat{a}^\dagger\hat{a}] = -\hat{a}^\dagger \end{aligned} \quad (10)$$

The algebra $h(2)$ is an alternative SGA of the squeezed Kerr oscillator. Since $h(2)$ is a contracted algebra of $u(2)$, calculations of eigenvalues and eigenvectors of the Hamiltonian can be done by using $u(2)$ in the limit $N \rightarrow \infty$. In the $u(2)$ basis

$$\hat{H} = -\Delta\hat{n} + K\hat{n}(\hat{n} - 1) - \varepsilon_2(\hat{a}^\dagger\hat{a}^\dagger\hat{s}\hat{s} + \hat{s}^\dagger\hat{s}^\dagger\hat{a}\hat{a}) \quad (11)$$

with contracted form

$$\hat{H} = -\Delta\hat{n} + K\hat{n}(\hat{n} - 1) - \varepsilon_2'(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a}) \quad (12)$$

where $\varepsilon_2' = \varepsilon_2 N$, identical to equation (1).

4. Symmetry and classification of states

It is convenient here to introduce the dimensionless Hamiltonian

$$\hat{H} / K = -\eta\hat{n} + \hat{n}(\hat{n} - 1) - \xi\hat{P}_2 \quad (13)$$

where $\eta = \frac{\Delta}{K}, \xi = \frac{\varepsilon_2}{K}$ are called control parameters, $\hat{n} = \hat{a}^\dagger\hat{a}$ is the number operator and

$\hat{P}_2 = (\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a})$ is the pairing operator of order 2.

4.1 Kerr oscillator

The spectrum of the Kerr oscillator with Hamiltonian

$$\hat{H} / K = -(\eta + 1)\hat{n} + \hat{n}^2 = -\eta' \hat{n} + \hat{n}^2 \tag{14}$$

is shown in figure 1.

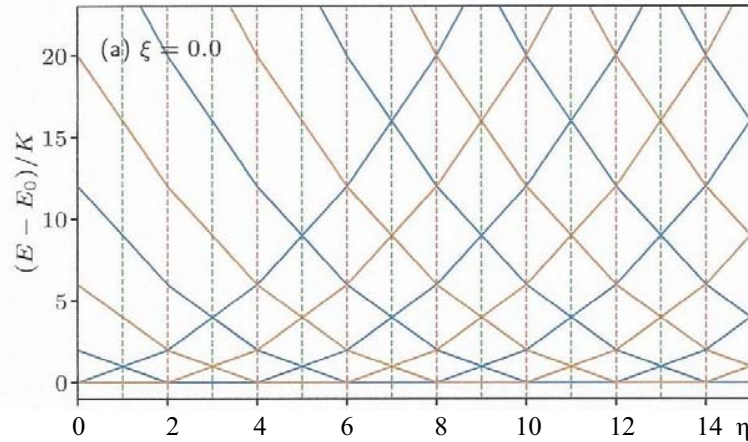


Figure 1. Energy levels of the Kerr oscillator counted from the energy of the lowest state as a function of $\eta = \Delta / K$. The parity of the levels is shown in blue (+) and orange(-). Adapted from [7].

The degeneracies in figure 1 are due to a remarkable (and hitherto unknown) quasi-spin symmetry $su(2)$ at $\eta' = \eta + 1$ integer. The degeneracy points can be classified by quasi-spin quantum numbers

$|j, m\rangle$ with $j = \frac{\eta'}{2} = \frac{\eta + 1}{2}$ and $m = \pm j, \pm(j - 1), \dots, \pm \frac{1}{2}$ for $j = \text{half-integer}$, $\eta = \text{even}$ and $m = \pm j, \pm(j - 1), \dots, 0$ for $j = \text{integer}$, $\eta = \text{odd}$. The eigenvalues E of H/K counted from the lowest state are $E = m^2 - \frac{1}{4}$, $m = \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \dots, \pm j$ for $j = \text{half-integer}$, $\eta = \text{even}$ and $E = m^2$, $m = 0, \pm 1, \pm 2, \dots, \pm j$ for $j = \text{integer}$, $\eta = \text{odd}$. The degeneracies of figure 1 can then be classified in terms of quasi-spin j and component m as in figure 2.

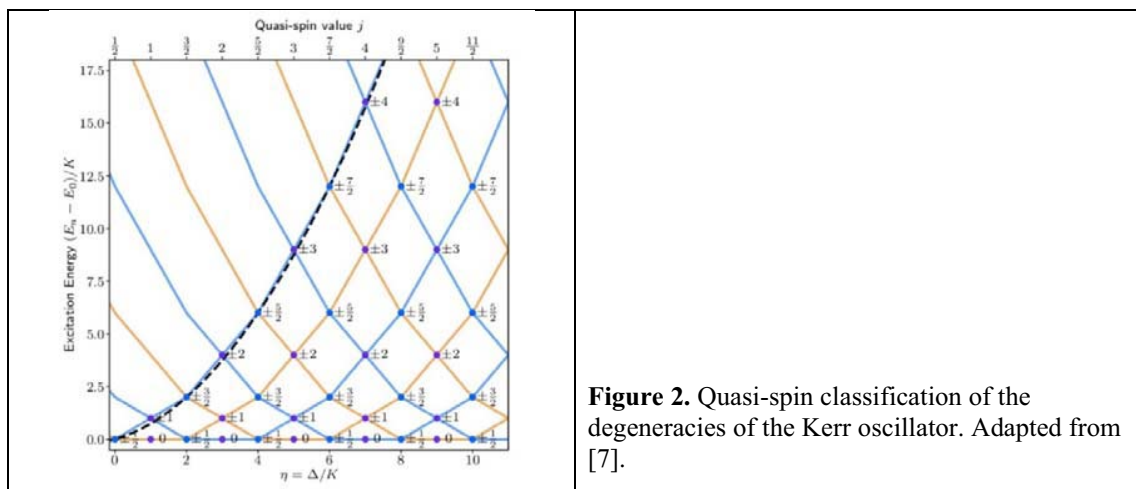


Figure 2. Quasi-spin classification of the degeneracies of the Kerr oscillator. Adapted from [7].

This symmetry is a *dynamic symmetry* $su(2) \supset so(2)$. A dynamic symmetry [9] is a situation in which the eigenvalues of H can be written in explicit analytic form in terms of labels of an irreducible representation of a chain of algebras $g \supset g' \supset g'' \supset \dots$ and often display degeneracies associated with subalgebras g', g'', \dots of g . Dynamic symmetries have played a major role in molecular [10], atomic, nuclear [11] and particle physics.

In order to elucidate this quasi-spin symmetry, it is convenient to construct the representations $|j, m\rangle$ with two boson operators \hat{b}_1, \hat{b}_2 , with eigenvalues of the number operators satisfying $n_1 + n_2 = N$. The values of j and m are [8]

$$j = \left(\frac{n_2 + n_1}{2} \right), m = \left(\frac{n_2 - n_1}{2} \right). \quad (15)$$

The degeneracies occur because there are two values of n_1 and n_2 satisfying $n_1 + n_2 = N$, except for $N=\text{even}$, $n_1 = n_2 = N/2$ when the two values merge into a single value. The dynamic symmetry arises from the simple identity

$$m^2 = \left(\frac{n_2 - n_1}{2} \right)^2 = \frac{N^2}{4} + (n_1^2 - Nn_1) \quad (16)$$

which, for $\eta = N = \text{integer}$, gives excitation energies $E = n_1^2 - \eta n_1$. An explicit analytic form of the quasi-spin states is

$$B : \frac{1}{\sqrt{(j-m)!(j+m)!}} (\hat{b}_2^\dagger)^{j+m} (\hat{b}_1^\dagger)^{j-m} |0\rangle, \quad (17)$$

also called the Schwinger representation of $su(2)$. The degenerate states are related by the transformation $n_2 \leftrightarrow n_1, m \leftrightarrow -m$. The parity in the two boson construction is $P \equiv \pi = (-)^{n_1}$. Therefore, for $N=\text{odd}$, $\eta=\text{even}$, the two degenerate states have opposite parity, while for $N=\text{even}$, $\eta=\text{odd}$, the two degenerate states have the same parity.

4.2 Squeezed quadratic oscillator

Consider next the Hamiltonian

$$\hat{H} / K = \hat{n}(\hat{n} - 1) - \xi \hat{P}_2. \quad (18)$$

The spectrum generating algebra of the squeezed Kerr oscillator is given in equation (6). The algebra $sp(2, \mathbb{R})$ has two subalgebras

$$\begin{aligned} sp(2, \mathbb{R}) &\supset u(1) \\ sp(2, \mathbb{R}) &\supset so(1,1) \end{aligned} \quad (19)$$

with generators $u(1) \doteq \hat{a}^\dagger \hat{a}$, $so(1,1) \doteq \frac{1}{2}(\hat{a}^\dagger \hat{a}^\dagger + \hat{a} \hat{a})$. Since $sp(2, \mathbb{R})$ is non-compact, it is convenient

to consider the alternative algebraic structure $u(2) \rightarrow_c h(2)$ with subalgebras

$$\begin{aligned} u(2) &\supset su(2) \supset u(1) \\ u(2) &\supset su(2) \supset so(2) \end{aligned} \quad (20)$$

States can be characterized by the quantum numbers [12] $u(1) : |[N], n\rangle$ with $n = 0, 1, \dots, N$ and $so(2) : |[N], \sigma\rangle$ with $\sigma = \pm N, \pm(N-2), \dots, \pm 1, \text{or}, 0$ for $N=\text{odd}$ or even respectively. The notation $|[N], \sigma\rangle$ can be converted to the usual $|j', m'\rangle$ by $|j' = N/2, m' = \sigma/2\rangle$ with

$m' = \pm \frac{N}{2}, \pm \left(\frac{N}{2} - 1\right), \dots, \pm \frac{1}{2}, \text{ or } 0$. The value of j' is half-integer for $N(\text{mod } 2) = \text{odd}$ and integer for $N(\text{mod } 2) = \text{even}$. The notation $|j', m'\rangle$ is introduced here to distinguish it from that $|j, m\rangle$ introduced in Section 4.1. The spectrum of the Hamiltonian (18) is shown in figure 3. This spectrum displays an Excited State Quantum Phase Transition (ESQPT) [13] as shown in [14]. In phase I states are singly degenerate and characterized by parity (+) blue and (-) orange. In phase II states are doubly degenerate and characterized by a vibrational quantum number v . The separatrix between the two phases is given by $E_s = \xi^2$.

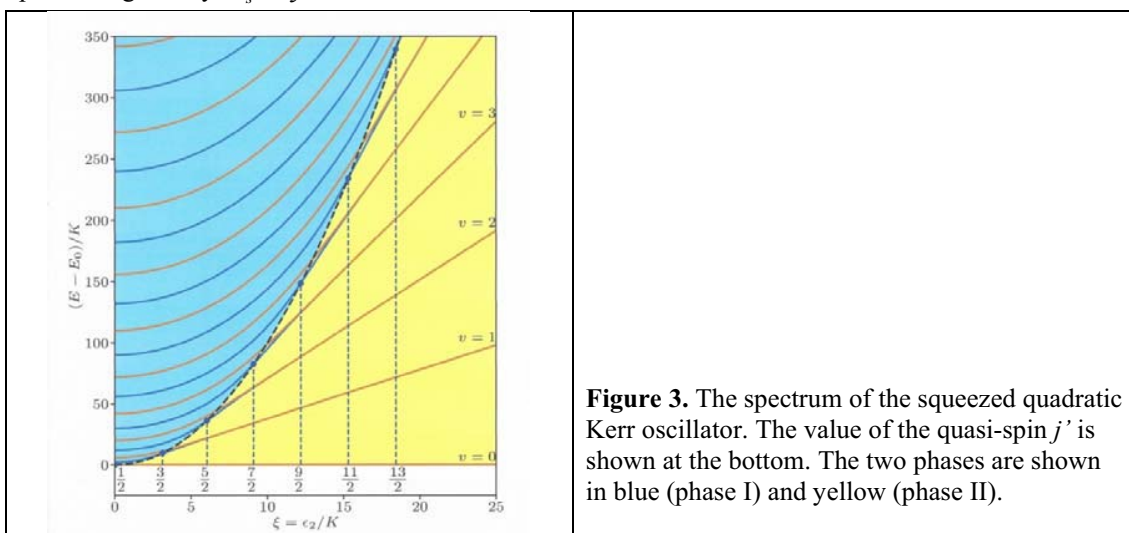


Figure 3. The spectrum of the squeezed quadratic Kerr oscillator. The value of the quasi-spin j' is shown at the bottom. The two phases are shown in blue (phase I) and yellow (phase II).

4.3. Squeezed Kerr oscillator

The Hamiltonian is given here by equation (13). Its spectrum is shown in figure 4. It can be separated into three phases with approximate separatrices $E_s = \frac{\eta}{2} + \frac{\eta^2}{4} + \eta\xi$ and $E'_s = \eta\xi$.

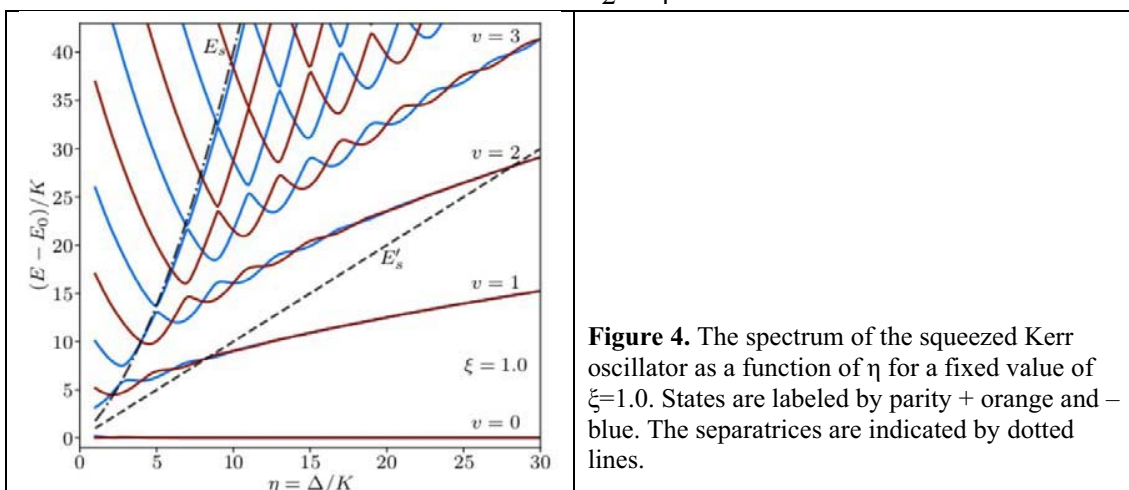


Figure 4. The spectrum of the squeezed Kerr oscillator as a function of η for a fixed value of $\xi=1.0$. States are labeled by parity + orange and - blue. The separatrices are indicated by dotted lines.

5. Stability of solutions

For applications to quantum computing, it is of importance to consider the stability of solutions to squeezing. Consider the Hamiltonian

$$\hat{H} / K = -\eta \hat{n} + \hat{n}(\hat{n} - 1) - \xi_i \hat{P}_i \quad (21)$$

where the squeezing terms are $\hat{P}_2 = (\hat{a}^{\dagger 2} + \hat{a}^2)$, $\hat{P}_3 = (\hat{a}^{\dagger 3} + \hat{a}^3)$ and $\hat{P}_4 = (\hat{a}^{\dagger 4} + \hat{a}^4)$. All these terms can be engineered and experimentally studied.

5.1 Quadratic squeezing

The effect of quadratic squeezing is shown in figure 5 for various values of ξ . A remarkable result here is that the degeneracies which occur at even values of η (j =half-integer) are not affected by the P_2 term, i.e. they occur at the same values of η . Also, in the limit of large η and ξ , states have a two-fold degeneracy, with states characterized by representations of the cyclic group C_2 .

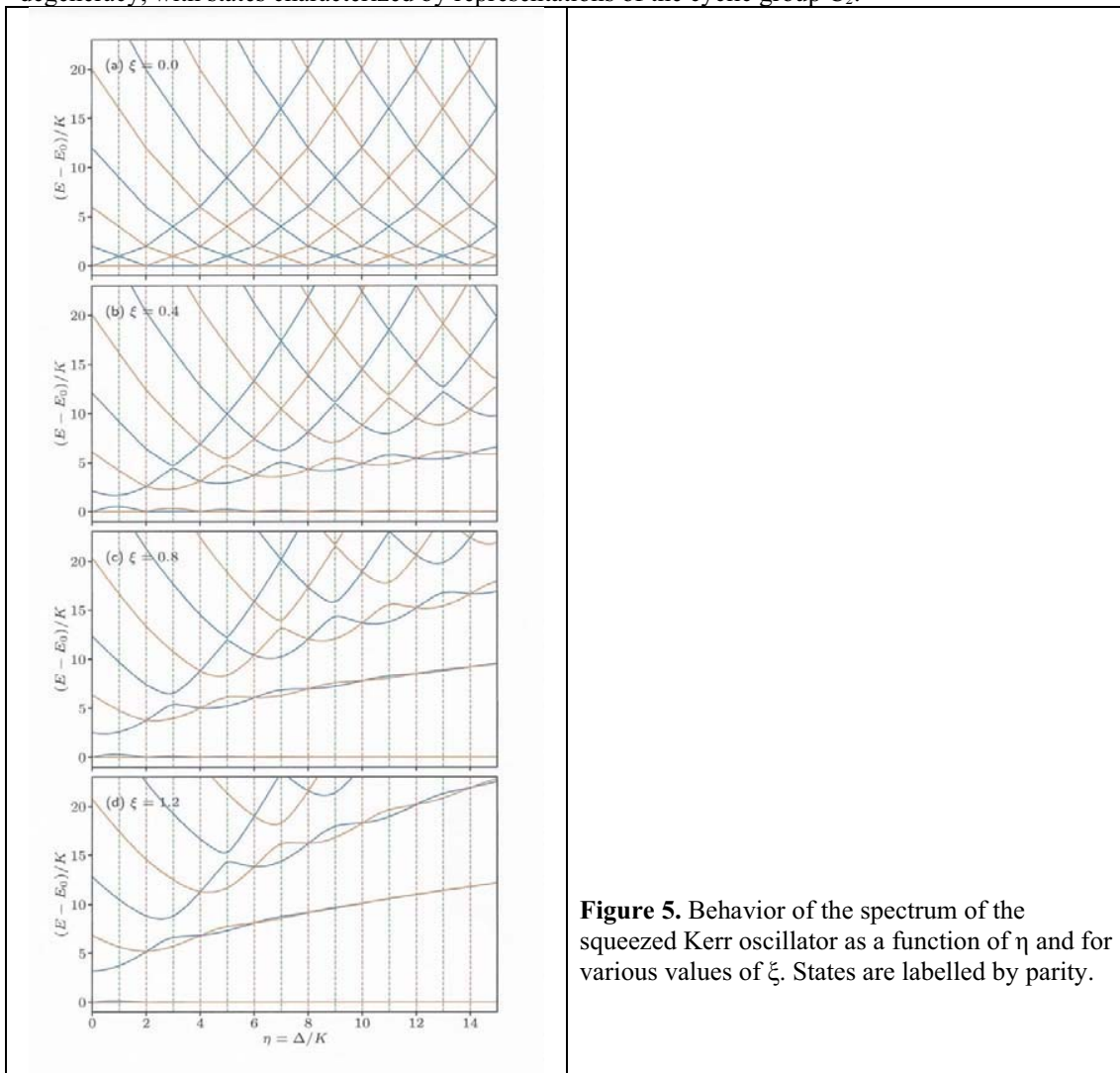


Figure 5. Behavior of the spectrum of the squeezed Kerr oscillator as a function of η and for various values of ξ . States are labelled by parity.

5.2 Cubic squeezing

The degeneracies here do not occur at the same η =even-integer values. The spectrum has a more complex structure as shown in figure 6. States in the asymptotic limit are triple degenerate and characterized by representations of the cyclic group C_3 . Also parity here is not a good quantum number. The braiding seen in this figure is also of particular interest.

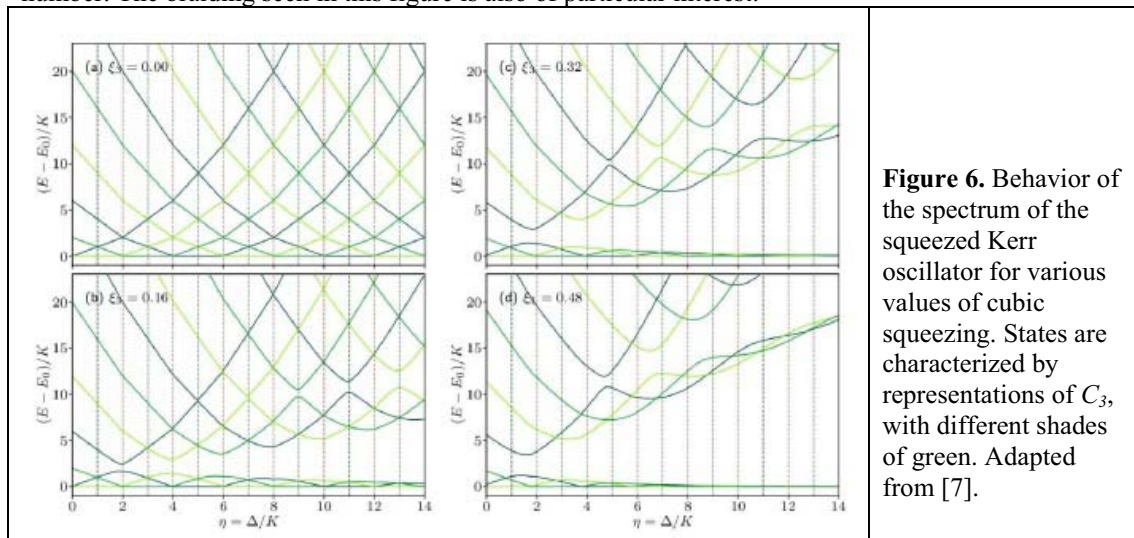


Figure 6. Behavior of the spectrum of the squeezed Kerr oscillator for various values of cubic squeezing. States are characterized by representations of C_3 , with different shades of green. Adapted from [7].

5.3 Quartic squeezing

Also here the degeneracies occur at values of the parameter η not equal to an even integer. States in the asymptotic limit are four-fold degenerate and form representations of C_4 .

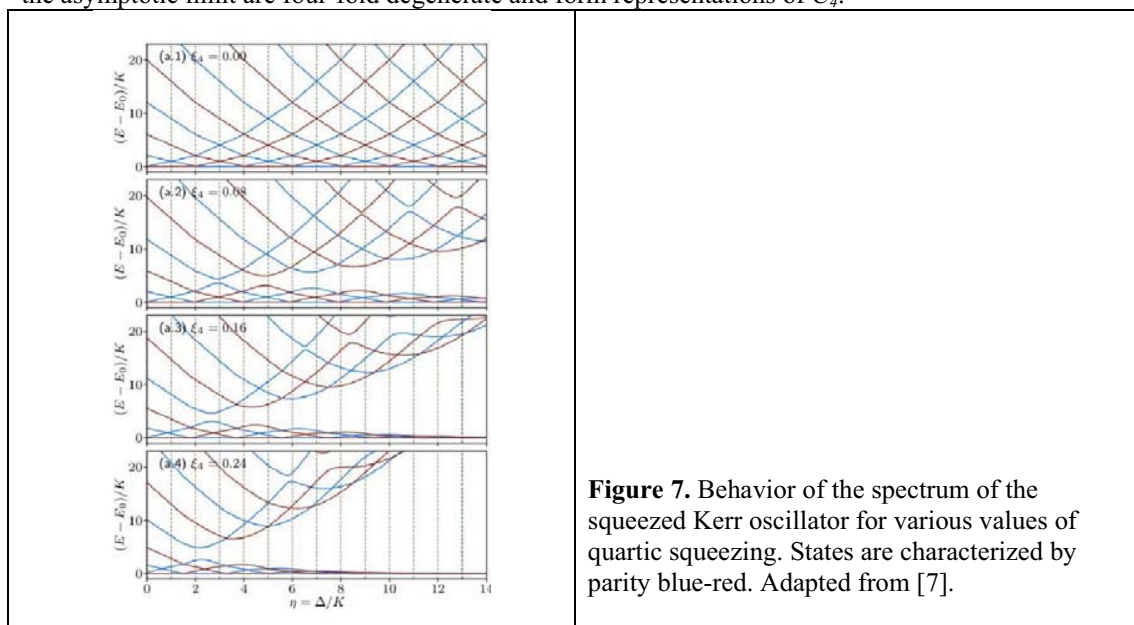


Figure 7. Behavior of the spectrum of the squeezed Kerr oscillator for various values of quartic squeezing. States are characterized by parity blue-red. Adapted from [7].

6. Summary and conclusions

In this contribution, the newly found quasi-spin symmetry $su(2)$ of the Kerr oscillator has been presented and the stability of this symmetry to perturbations induced by squeezing has been discussed.

The results presented here have major implications for the use of the Kerr oscillator in quantum computing.

The study of the symmetries of the Hamiltonian operator of the Kerr oscillator reported here is, at the present time, being expanded to the study of the symmetries of the Liouville operator which appears in the calculation of the half-lives of the states. Further planned studies include (i) two coupled oscillators with parametric symmetry $su(2)_1 \oplus su(2)_2$ and (ii) a large number of Kerr oscillators on a lattice, $\sum_i \oplus su(2)_i$. These studies are an application to the Kerr oscillator of previous methods used in connection with the Interacting Boson Model [10], $su(6)_1 \oplus su(6)_2$, the Vibron Model [11], $su(4)_1 \oplus su(4)_2$, and of the algebraic cluster model [15]. Finally, the symmetries discussed here may be present in other parametric models used in simulations of quantum computation, such as the Lipkin model [16] and the Rabi and Dicke models [17].

7. Acknowledgements

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