

On generally covariant mathematical formulation of Feynman integral in Lorentz signature

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Abstract

It is widely accepted that the Feynman integral is one of the most promising methodologies for defining a generally covariant formulation of nonperturbative interacting quantum field theories (QFTs) without a fixed prearranged causal background. Recent literature suggests that if the spacetime metric is not fixed, e.g. because it is to be quantized along with the other fields, one may not be able to avoid considering the Feynman integral in the original Lorentz signature, without Wick rotation. Several mathematical phenomena are known, however, which are at some point showstoppers to a mathematically sound definition of Feynman integral in Lorentz signature. The Feynman integral formulation, however, is known to have a differential reformulation, called to be the master Dyson–Schwinger (MDS) equation for the field correlators. In this paper it is shown that a particular presentation of the MDS equation can be cast into a mathematically rigorously defined form: the involved function spaces and operators can be strictly defined and their properties can be established. Therefore, MDS equation can serve as a substitute for the Feynman integral, in a mathematically sound formulation of constructive QFT, in arbitrary signature, without a fixed background causal structure. It is also shown that even in such a generally covariant setting, there is a canonical way to define the Wilsonian regularization of the MDS equation. The main result of the paper is a necessary and sufficient condition for the regularized MDS solution space to be nonempty, for conformally invariant Lagrangians. This theorem also provides an iterative approximation algorithm for obtaining regularized MDS solutions, and is guaranteed to be convergent whenever

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the solution space is nonempty. The algorithm could eventually serve as a method for putting Lorentz signature QFTs onto lattice, in the original metric signature.

Keywords: Feynman integral, master Dyson–Schwinger equation, generally covariant, constructive field theory

 Supplementary material for this article is available [online](#)

1. Introduction

By now, a lot is known about the mathematically sound formulation of interacting quantum field theory (QFT), using perturbation theory [1]. However, still until now, there is no widely accepted concise mathematical formulation known for nonperturbative interacting QFT. Strictly speaking, as of now, it is only conjectured that eventually one could well-define an interacting QFT model in a nonperturbative manner, in a constructive way, e.g. as specified by a Lagrangian. A well known promising attempt for the nonperturbative approach is the algebraic quantum field theory (AQFT) [2, 3]. AQFT is known to capture several important qualitative aspects of the QFT formalism in physics, such as the spin-statistics theorem, but there are no known concrete AQFT constructions in the complexity of e.g. a $3 + 1$ dimensional full quantum electrodynamics. Concrete AQFT models, as of now, are only known for free particles in arbitrary dimensions, or for simple systems, such as discrete Ising models in $1 + 1$ dimensional and discrete spacetimes, or for particular simple systems in spacetime dimensions typically lower than $3 + 1$. There are also recent advances of perturbative AQFT on causal sets, in which framework concrete interacting models are constructed by now [4], assuming a finite system of causal sets. Due to the difficulties of nonperturbative formulation, the perturbative rigorous formulation of constructive QFT (pQFT) was seriously considered by a number of authors [5–9]. In particular [10], proves the perturbative renormalizability of Yang–Mills interactions over globally hyperbolic spacetimes. Moreover, a generally covariant framework was already developed [11]. However, it is generally thought that the only promising framework, which could be capable of formalizing *nonperturbative* interacting generally covariant QFT models in the continuum limit, is likely to be the Feynman integral formulation [12].

A lot is known about Feynman integrals [13, 14], but in Lorentz signature, without taking a Wick rotation, it seems to be still not a completely understood mathematical construction, although the modern literature seems to tighten the noose on the measure theoretically well defined Feynman integral [15, 16]. Other authors [17] argue, that Feynman integral should not be, strictly speaking, understood in the measure theoretical sense, i.e. in the sense of infinitesimal summation, but in a more generalized sense. That kind of picture is indeed supported by the fact that e.g. for a fermionic system, the Feynman integral is defined as a Berezin integral, which indeed has little link with integration in terms of infinitesimal summation. To complicate the picture, recent literature suggests [10, 18–20] that in order to apply Feynman integral formalism to a generally covariant setting, in which case the a background spacetime metric is not fixed, the applicability of the usual Wick rotation from Lorentzian to Euclidean signature can be problematic.

The above issues with the Lorentz signature Feynman integral formulation can be circumvented using the well known differential reformulation of Feynman integral formalism, called to be the master Dyson–Schwinger (MDS) equations for the field correlators (see e.g. [21]

for a didactic review). From the usually presented form of the MDS equation in the QFT literature, it is not immediately evident that the function spaces and operators involved in the MDS equation are well defined, and are not merely symbolical summaries of heuristic QFT protocols. In this paper, however, it is argued that with the right choice of variables, these objects can be made mathematically well defined, and as such, the MDS equation can be used to substitute the Feynman integral for a mathematically sound definition of constructive nonperturbative generally covariant QFT. It will be also shown, that in these variables the Wilsonian regularized version of the MDS equation can also be canonically defined in a generally covariant setting, which is not yet described in the literature. The main result of the paper is a theorem about a necessary and sufficient condition for the regularized MDS equation to have nonempty solution space, for theories with classically conformally invariant Lagrangians. The pertinent theorem is constructive in the sense that it provides a (probably slowly converging) iterative algorithm for approximating MDS solutions, which is guaranteed to be convergent whenever the solution space is nonempty. This method can eventually be also employed for doing lattice QFT-like calculations in arbitrary signatures, in particular, in the original Lorentz signature.

The structure of the paper is as follows. In section 2 the heuristic form of the MDS equation is recalled, as derived from the heuristic Feynman integral formulation in usual QFT. The rest of the paper intends to keep mathematical rigor. In section 3 the function spaces and operators needed to define the (unregularized) MDS operator are presented. In section 4, these are generalized in the distributional sense, and the Wilsonian regularized version of the MDS equation is invoked and justified. Section 5 is dedicated for the main theorem of the paper about a necessary and sufficient condition for the existence of solutions of the regularized MDS equation, for conformally invariant Lagrangians. Appendix A was added in order to pin down the precise continuity properties of a typical Euler–Lagrange (EL) functional in a standard classical field theory, which is key in the construction. Appendix B was added for completeness, in order to define the Wilsonian renormalizability in a generally covariant way, given the notion of Wilsonian regularization. The paper heavily relies on the theory of non-normable topological vector spaces (TVS), and therefore a supplementary material (<https://stacks.iop.org/CQG/39/185004/mmedia>) [22] is provided for a recollection of important and sometimes counterintuitive theorems on these, for readers not specialized in the theory of TVS.

2. Feynman integral and the heuristic form of the MDS equation

We briefly recall the justification of the MDS equation in the Feynman integral formulation of QFT. Let F denote the space of all (that is, off-shell) smooth classical field configurations. As expanded in appendix A, in most models it is safe to assume that F is a topological affine space, such that its subordinate vector space \mathbb{F} , the space of smooth field variations, carries a nuclear Fréchet topology. The affineness of F is necessary in order to naturally accommodate gauge fields. \mathbb{F}^* will denote the topological dual of \mathbb{F} , understood with the standard strong dual topology. In the Feynman integral formulation of QFT, it is postulated that the evaluation method for Feynman type (i.e., causally ordered) quantum vacuum expectation value of observables in a (not necessarily unique) vacuum state ρ is the following. Given a fixed reference field $\psi_0 \in F$ and test functionals $J_1, \dots, J_n \in \mathbb{F}^*$, the causally ordered quantum vacuum expectation value of the polynomial observable $(J_1 | \cdot - \psi_0) \dots (J_n | \cdot - \psi_0) : F \rightarrow \mathbb{R}$ is

declared to be

$$\int_{\psi \in F} (J_1|\psi - \psi_0) \dots (J_n|\psi - \psi_0) e^{i\frac{1}{\hbar}S(\psi)} d\rho(\psi) \Big/ \int_{\psi \in F} e^{i\frac{1}{\hbar}S(\psi)} d\rho(\psi), \quad (1)$$

where the symbol $d\rho(\cdot)$ denotes the hypothetical Feynman measure corresponding to a vacuum state ρ , $(\cdot|\cdot)$ denotes the duality pairing form between \mathbb{F}^* and \mathbb{F} , whereas $S : F \rightarrow \mathbb{R}$ is the action functional of the underlying classical field theory. In the heuristic calculations, $d\rho(\cdot)$ is handled as if it were a Lebesgue measure on F , and as if $e^{i\frac{1}{\hbar}S(\cdot)} d\rho(\cdot)$ were a finite measure, having finite moments and analytic Fourier transform. A sign change $\hbar \mapsto -\hbar$ would correspond to a reversal in the causal ordering, if there were any *a priori* causal structure over the spacetime manifold (which in fact, is not needed to be assumed at this point). The hypothetical *partition function* condenses all these information about the state ρ , and would be a mapping

$$Z_{\hbar, \psi_0} : \mathbb{F}^* \rightarrow \mathbb{C}, \quad J \mapsto Z_{\hbar, \psi_0}(J) := \int_{\psi \in F} e^{i(J|\psi - \psi_0)} e^{i\frac{1}{\hbar}S(\psi)} d\rho(\psi), \quad (2)$$

i.e. the formal Fourier transform of the hypothetical measure $e^{i\frac{1}{\hbar}S(\cdot)} d\rho(\cdot)$. The collection of *n-field correlators*

$$G_{\hbar, \psi_0}^{(n)} := \left((-i)^n \frac{1}{Z_{\hbar, \psi_0}(J)} D^{(n)} Z_{\hbar, \psi_0}(J) \right) \Big|_{J=0} \quad (3)$$

is an other means to rephrase these information about the state ρ , and also can be used to evaluate the quantum expectation values equation (1) by simple duality pairing, like $(J_1 \otimes \dots \otimes J_n | G_{\hbar, \psi_0}^{(n)})$. Here $D^{(n)} Z_{\hbar, \psi_0}$ is assumed to behave like the *n*th Fréchet derivative of the partition function $J \mapsto Z_{\hbar, \psi_0}(J)$, implicitly assuming that Z_{\hbar, ψ_0} is *n*-times continuously Fréchet differentiable (and for fermion fields, this differentiation is assumed to be a graded differentiation). Since the partition function would be a map $Z_{\hbar, \psi_0} : \mathbb{F}^* \rightarrow \mathbb{C}$, the collection of field correlators $G_{\hbar, \psi_0} := \left(G_{\hbar, \psi_0}^{(0)}, G_{\hbar, \psi_0}^{(1)}, \dots, G_{\hbar, \psi_0}^{(n)}, \dots \right)$ would sit in $\mathcal{T}(\mathbb{F}) := \bigoplus_{n \in \mathbb{N}_0} \mathbb{F}^{\otimes n}$, i.e. in the tensor algebra of \mathbb{F} , or more precisely in a graded-symmetrized subspace of $\mathcal{T}(\mathbb{F})$.

Let $E(\psi) := D_F S(\psi)$ denote the EL functional, i.e. the derivative of the action functional S , evaluated at the classical field configuration $\psi \in F$. It would be a map $E : F \times \mathbb{F} \rightarrow \mathbb{R}$, $(\psi, \delta\psi) \mapsto (E(\psi)|\delta\psi) := (D_F S(\psi)|\delta\psi)$, being linear in its second variable, since it is a derivative. In the usual QFT protocol it is assumed that the EL functional E is multipolynomial, and thus so is the real valued map $\psi \mapsto (E(\psi)|\delta\psi)$ for any fixed field variation $\delta\psi \in \mathbb{F}$. Let $\mathbf{E}((-i)D_{\mathbb{F}^*} + \psi_0)$ be the multipolynomial differential operator defined by the polynomial coefficients of the Euler–Lagrange functional E . Applying the usual rules of formal Fourier transform, a function $Z : \mathbb{F}^* \rightarrow \mathbb{C}$ is of the form equation (2), up to a complex multiplier, if and only if it satisfies the MDS equation

$$(\mathbf{E}((-i)D_{\mathbb{F}^*} + \psi_0)Z)|_J = -\hbar JZ(J) \quad (\forall J \in \mathbb{F}^*), \quad (4)$$

see e.g. [21] for a didactic derivation. The operational meaning of this usual presentation of the MDS equation might not seem immediately evident. However, expressing Z_{\hbar, ψ_0} via its formal Taylor series, encoded by the collection of field correlators $G_{\hbar, \psi_0} \in \mathcal{T}(\mathbb{F})$, the MDS

equation (4) is seen to be equivalent to

we search for $G \in \mathcal{T}(\mathbb{F})$ such that:

$$G^{(0)} = 1 \quad \text{and} \quad \iota_{(\mathbf{E}_{\psi_0}|\delta\psi)}G = i\hbar L_{\delta\psi}G \quad (\forall \delta\psi \in \mathbb{F}). \quad (5)$$

The symbols of this equation mean the following. $L_{\delta\psi}$ denotes the left-multiplication operator in the tensor algebra $\mathcal{T}(\mathbb{F})$ by the one-vector $\delta\psi \in \mathbb{F}$. The symbol ι_p denotes the left-insertion operator by some element p from the topological dual space of $\mathcal{T}(\mathbb{F})$. The map $E_{\psi_0} : \mathbb{F} \rightarrow \mathbb{F}^*$ is defined via $E_{\psi_0} := E \circ (\mathbf{I}_{\mathbb{F}} + \psi_0)$ from the original EL functional $E : F \rightarrow \mathbb{F}^*$, i.e. it is the EL functional with respect to a fixed reference field $\psi_0 \in F$, re-expressed on the space of field variations \mathbb{F} . Since it was assumed to be multipolynomial, it can eventually be regarded as a linear map $E_{\psi_0} : \mathcal{T}(\mathbb{F}) \rightarrow \mathbb{F}^*$. As such, it may be identified with an element $\mathbf{E}_{\psi_0} \in (\mathcal{T}(\mathbb{F}))^* \otimes \mathbb{F}^*$, and correspondingly $(E_{\psi_0}|\delta\psi)$ with $(\mathbf{E}_{\psi_0}|\delta\psi) \in (\mathcal{T}(\mathbb{F}))^*$ ($\forall \delta\psi \in \mathbb{F}$), which then has the corresponding left-insertion operator $\iota_{(\mathbf{E}_{\psi_0}|\delta\psi)}$ acting over $\mathcal{T}(\mathbb{F})$. The spaces and operators involved in equation (5) would be perfectly meaningful if the space of fields F were finite dimensional, and could be used as a substitute for Feynman integral formulation equation (1), regardless of e.g. a metric signature or other auxiliary information on the details of the underlying classical theory described by the EL functional E . In section 3 it shall be shown that the pertinent objects can be made well-defined even when F is indeed the infinite dimensional space of smooth off-shell field configurations in a realistic field theory. The equation (5) presentation of the MDS equation does not seem to be described in the literature.

In QFT, it is also necessary to consider the Wilsonian regularized version of the Feynman integral. Wilsonian regularization means performing the Feynman integral equation (1) on a subspace of off-shell fields with their high frequency modes suppressed. In a generally covariant setting the meaning of this might not seem immediately evident, but Wilsonian regularized Feynman type expectation value of the observable $(J_1 | \cdot - \psi_0) \dots (J_n | \cdot - \psi_0) : F \rightarrow \mathbb{R}$ can be postulated as

$$\int_{\delta\psi \in \mathcal{R}[\mathbb{F}]} (J_1|\delta\psi) \dots (J_n|\delta\psi) d\mathcal{R}_* \mu_{\psi_0}(\delta\psi) \Big/ \int_{\delta\psi \in \mathcal{R}[\mathbb{F}]} 1 d\mathcal{R}_* \mu_{\psi_0}(\delta\psi) \quad (6)$$

with $\psi_0 \in F$ and $J_1, \dots, J_n \in \mathbb{F}^*$ as previously, where $\mathcal{R} : \mathbb{F} \rightarrow \mathbb{F}$ is some continuous linear operator, $\mathcal{R}[\mathbb{F}] \subset \mathbb{F}$ denotes the image of \mathbb{F} by \mathcal{R} , the symbol μ_{ψ_0} stands for the pushforward of the hypothetical finite measure $e^{i\frac{1}{\hbar}S(\cdot)} d\rho(\cdot)$ on F via the map $F \rightarrow \mathbb{F}, \psi \mapsto (\psi - \psi_0)$, and $\mathcal{R}_* \mu_{\psi_0}$ stands for the pushforward of the measure μ_{ψ_0} on \mathbb{F} to $\mathcal{R}[\mathbb{F}]$ by \mathcal{R} . The map \mathcal{R} can be called a *regulator*, and typically it is a convolution operator by some test function in case of theories over an affine spacetime (can be generalized for arbitrary spacetimes as well), and equation (6) means nothing but the natural pushforward Feynman integration on the subspace $\mathcal{R}[\mathbb{F}] \subset \mathbb{F}$, given that the original Feynman integration equation (1) on F was meaningful. The map \mathcal{R} implements the high frequency damping. Using the fundamental formula of integral substitution, one infers that the Wilsonian regularized MDS equation on the field correlators reads

$$G^{(0)} = 1 \quad \text{and} \quad \iota_{(\mathbf{E}_{\psi_0}|\delta\psi)}G = i\hbar L_{\mathcal{R}\delta\psi}G \quad (\forall \delta\psi \in \mathbb{F}) \quad (7)$$

in the analogy of equation (5), where again $L_{\mathcal{R}\delta\psi}$ is the left-multiplication in $\mathcal{T}(\mathbb{F})$ by the one-vector $\mathcal{R}\delta\psi \in \mathbb{F}$. As shall be expanded in section 4, the pertinent objects can be made

well-defined similarly to that of the unregularized MDS equation. The Wilsonian regularized MDS equation (7) does not seem to be described in the literature.

From this point on, we drop the heuristic arguments, and all the statements and formulas are intended to be mathematically rigorous. The aim of this paper is to show that the objects involved in equations (5) and (7) are mathematically well defined, and to establish the fundamental properties of the solution spaces of the pertinent equations.

3. Mathematically rigorous definition of the unregularized MDS operator

As detailed in appendix A, in a generic classical field theory, it is safe to assume that the space of off-shell fields F is the affine space of smooth sections of a real finite dimensional affine bundle over a real finite dimensional smooth base manifold. The space of field variations \mathbb{F} are comprised of differences of elements in F , and as such it is the vector space of smooth sections of the real finite dimensional vector bundle subordinate to our affine bundle, understood with the standard \mathcal{E} smooth function topology, which is known to be nuclear Fréchet. Within \mathbb{F} , there is the space of test field variations \mathbb{F}_T , comprised of compactly supported smooth sections, with the standard \mathcal{D} test function topology. For the sake of genericity, in this section we avoid using the knowledge that F , \mathbb{F} and \mathbb{F}_T are these concrete spaces, they will be considered abstract spaces instead. The symbol $*$ shall denote strong topological dual. See [22] and the appendix of [11] for a condensed summary on the theory of TVS.

Definition 1. Let F be a real affine space, with a subordinate real topological vector space \mathbb{F} . Let the topology on \mathbb{F} be nuclear Fréchet (in short, NF space, see also [22]-remark 2, the \mathcal{E} smooth function space is the archetype of an NF space). We call F the *space of classical field configurations* and the subordinate vector space \mathbb{F} the *space of classical field variations*. Let $\mathbb{F}_T \subset \mathbb{F}$ be some subspace of \mathbb{F} , endowed with a topology not weaker than \mathbb{F} . Let \mathbb{F}_T be either nuclear Fréchet or the strict inductive limit of a countable system of nuclear Fréchet spaces with closed adjacent images (in short, LNF space, see also [22]-remark 2, the \mathcal{D} test function space is the archetype of an LNF space). Then, we call \mathbb{F}_T the *space of test field variations*.

As detailed in appendix A, in a generic concrete classical field theory, the EL functional is the derivative of the action functional, with its linear variable restricted to the space of test field variations, so that the EL functional becomes an everywhere defined map. It is also shown to be a jointly sequentially continuous map in its two variables. This justifies the following abstract definition.

Definition 2. Let $E : F \times \mathbb{F}_T \rightarrow \mathbb{R}$, $(\psi, \delta\psi_T) \mapsto E(\psi, \delta\psi_T)$ be a jointly sequentially continuous map which is linear in its second variable. Then, E will be called a *classical EL functional*. (By means of appendix A theorem 45(b) and (c), then E is also separately continuous in its two variables, and when viewed as a map $E : F \rightarrow \mathbb{F}_T^*$, $\psi \mapsto E(\psi)$, it is continuous.) Given a $\delta\psi_T \in \mathbb{F}_T$, when the second argument of E is evaluated, it will be denoted by $(E|\delta\psi_T) : F \rightarrow \mathbb{R}$, which is then a continuous map. When that map is evaluated at some $\psi \in F$, we denote it by $(E(\psi)|\delta\psi_T) \in \mathbb{R}$. We call the equation

$$\text{we search for } \psi \in F \text{ such that : } \quad \forall \delta\psi_T \in \mathbb{F}_T : \quad (E(\psi)|\delta\psi_T) = 0 \quad (8)$$

the *classical EL equation*. When E is viewed as a map $E : F \rightarrow \mathbb{F}_T^*$, given any fixed field $\psi_0 \in F$, we use the notation $E_{\psi_0} := E \circ (\mathbb{I}_F + \psi_0)$, which will then be a continuous map $E_{\psi_0} : \mathbb{F} \rightarrow \mathbb{F}_T^*$, and ψ_0 will be called a *reference field*. (By construction, for all $\psi \in F$ the identity $E(\psi) = E_{\psi_0}(\psi - \psi_0)$ holds.)

In order to define the MDS operator, we will need to invoke the notion of a topologized tensor algebra made out of \mathbb{F} . For that, recall the below facts.

Remark 3. In this remark block let \mathbb{U} denote a nuclear Fréchet (NF) or strong dual of a nuclear Fréchet (DNF) space. (See also [22]-remark 2.)

- (a) For all $n \in \mathbb{N}_0$, the completed topological tensor product $\overset{n}{\otimes} \mathbb{U}$ is meaningful (e.g. understood with the projective tensor product topology), and is NF or DNF, respectively. Moreover, in the analogy of finite dimensional vector spaces, the pertinent tensor product can be implemented via the multiplicative realization. That is, it is topologically isomorphic to the space of the jointly continuous n -fold multilinear forms on the strong dual space of \mathbb{U} . (See also [22]-remark 2.)
- (b) With the same assumptions, one has that for all $n \in \mathbb{N}_0$, the identity $\left(\overset{n}{\otimes} \mathbb{U}\right)^* \equiv \overset{n}{\otimes} \mathbb{U}^*$ holds. (See also [22]-remark 2.)
- (c) Given a countable system of NF or a countable system of DNF spaces, their Cartesian product can be equipped with a vector space structure and with the product (also called Tychonoff or initial or projective) topology. This is the weakest topology such that the canonical projections of the Cartesian product are continuous. With this, it will become an NF or DNF space, respectively. (See also [22]-remark 4, [22]-remark 2.) Therefore, the Tychonoff tensor algebra $\mathcal{T}(\mathbb{U}) := \bigoplus_{n=0}^{\infty} \overset{n}{\otimes} \mathbb{U}$ is meaningful and is NF or DNF, respectively. (The symbol $\bigoplus_{n=0}^{\infty} := \bigtimes_{n=0}^{\infty}$ as set operation, but we use rather \bigoplus for vector spaces.)
- (d) Given a countable system of NF or a countable system of DNF spaces, in their Cartesian product vector space there is the subspace of the elements with all zero except for finite entries, which subspace is called the algebraic direct sum space. This can be equipped with the locally convex direct sum (also called final or injective) topology. This is the strongest topology such that the canonical injections of the Cartesian product are continuous. With this, it will become an NF or DNF space, respectively (see also [22]-remark 4, [22]-remark 2). Therefore, the algebraic tensor algebra $\mathcal{T}_a(\mathbb{U}) := \bigoplus_{n=0}^{\infty} \overset{n}{\otimes} \mathbb{U}$ with the locally convex direct sum topology is meaningful and is NF or DNF, respectively.
- (e) One has that $(\mathcal{T}(\mathbb{U}))^* \equiv \mathcal{T}_a(\mathbb{U}^*)$ and $(\mathcal{T}_a(\mathbb{U}))^* \equiv \mathcal{T}(\mathbb{U}^*)$. (See also [22]-remark 4.)
- (f) The Tychonoff tensor algebra has a jointly continuous bilinear map $\otimes : \mathcal{T}(\mathbb{U}) \times \mathcal{T}(\mathbb{U}) \rightarrow \mathcal{T}(\mathbb{U})$, the tensor algebra multiplication, with a unit element $\mathbb{1} := (1, 0, 0 \dots) \in \mathcal{T}(\mathbb{U})$ (consequences of [22]-remark 3). The subspaces of k -tensors provide a grading of $\mathcal{T}(\mathbb{U})$. Quite trivially, the left multiplication operator for all $u \in \mathcal{T}(\mathbb{U})$ is a continuous linear map $L_u : \mathcal{T}(\mathbb{U}) \rightarrow \mathcal{T}(\mathbb{U})$.
- (g) Similarly, the algebraic tensor algebra has a jointly continuous bilinear map $\mathcal{T}_a(\mathbb{U}) \times \mathcal{T}_a(\mathbb{U}) \rightarrow \mathcal{T}_a(\mathbb{U})$, the tensor algebra multiplication, with a corresponding unit element (consequences of [22]-remark 3). The subspaces of k -tensors provide a grading of $\mathcal{T}_a(\mathbb{U})$. Quite trivially, the left multiplication operator is a continuous linear map $\mathcal{T}_a(\mathbb{U}) \rightarrow \mathcal{T}_a(\mathbb{U})$.
- (h) Since $\mathcal{T}(\mathbb{U})$ and $\mathcal{T}_a(\mathbb{U}^*)$ are strong duals to each-other, and both of these are graded unital associative algebras with jointly continuous multiplications, by transposing the algebra multiplication and unit from the duals, one infers that both $\mathcal{T}(\mathbb{U})$ and $\mathcal{T}_a(\mathbb{U}^*)$ are bialgebras, with corresponding coproduct and counit. The counit of $\mathcal{T}(\mathbb{U})$ is $b : \mathcal{T}(\mathbb{U}) \rightarrow \mathbb{R}$, $G := (G^{(0)}, G^{(1)}, \dots) \mapsto bG := G^{(0)}$, i.e. extraction of the scalar component, the symbol ‘ b ’ standing for ‘base’ or ‘bottom form’.

- (i) Due to the bialgebra nature of $\mathcal{T}(\mathbb{U})$, i.e. due to the existence of a continuous coproduct on $\mathcal{T}(\mathbb{U})$, for all $p \in \mathcal{T}_a(\mathbb{U}^*)$ the corresponding left insertion operator $\iota_p : \mathcal{T}(\mathbb{U}) \rightarrow \mathcal{T}(\mathbb{U})$ is meaningful, and is a continuous linear operator. More concretely, the left insertion operator $\iota_{p^{(n)}}$ with a $p^{(n)} \in \overset{n}{\otimes}\mathbb{U}^*$ ($n \in \mathbb{N}_0$) exists, because for all $m \in \mathbb{N}_0$ ($m \geq n$) the tensor product $\overset{m}{\otimes}\mathbb{U}$ can be identified with the space of $\overset{n}{\otimes}\mathbb{U}^* \times \overset{m-n}{\otimes}\mathbb{U}^* \rightarrow \mathbb{R}$ jointly continuous bilinear forms, as stated in (i). Similarly, the left insertion operators make sense in $\mathcal{T}_a(\mathbb{U})$, and is a continuous linear operator. (For the sake of distinction in terminology, we call merely the operators $\iota_{p^{(1)}}$ with $p^{(1)} \in \mathbb{U}^* \equiv \overset{1}{\otimes}\mathbb{U}^* \subset \mathcal{T}_a(\mathbb{U}^*)$ as insertion operators, whereas for generic $p^{(n)} \in \overset{n}{\otimes}\mathbb{U}^* \subset \mathcal{T}_a(\mathbb{U}^*)$ ($n \in \mathbb{N}_0$) or more generally for $p \in \mathcal{T}_a(\mathbb{U}^*)$, we call the corresponding $\iota_{p^{(n)}}$ or ι_p as multipolynomial insertion operator.) For all $p \in \mathcal{T}_a(\mathbb{U}^*)$, one has the identity $p = b\iota_p$. For the left insertion operator, we use the normalization convention such that for all $G^{(n)} \in \overset{n}{\otimes}\mathbb{U}$ and $u \in \mathbb{U}$ and $p \in \mathbb{U}^*$ one has $\iota_p L_u G^{(n)} = (n+1)(p|u)G^{(n)}$.
- (j) A historical note: over an affine (Minkowski) spacetime, one can define the space of rapidly decreasing (Schwartz) functions \mathcal{S} , which is an NF space. The tensor algebra $\mathcal{T}_a(\mathcal{S})$ is referred to as Borchers–Uhlmann algebra (original papers: [23, 24], and including a short review: [25]). The Wightman functionals in QFT are understood to be in the space $(\mathcal{T}_a(\mathcal{S}))^* \equiv \mathcal{T}(\mathcal{S}^*)$.
- (k) By construction, the \mathcal{T}_a topology is strongest tensor algebra topology, whereas \mathcal{T} is the weakest. It is possible to define a natural topological tensor algebra which is in between the \mathcal{T} and \mathcal{T}_a , in terms of topology strength. It will be motivated and introduced later, in section 5, and will be key to the presented construction, if one wishes to quantize analytic EL functionals, and not only polynomial ones.

Definition 4. Assume that the space of field variations as real nuclear Fréchet spaces has a direct sum splitting $\mathbb{F} = \mathbb{F}_r \oplus \mathbb{F}_c$, called *the real-complex splitting*, where both \mathbb{F}_r and \mathbb{F}_c are closed (and therefore nuclear Fréchet), and \mathbb{F}_c has a complex structure (i.e. it can be regarded as a complex nuclear Fréchet space). Denote by $\mathbb{F}_{r\mathbb{C}} := \mathbb{F}_r \otimes \mathbb{C}$ the complexification of \mathbb{F}_r . Then, we use the notation $\mathbb{F}_{(\mathbb{C})} := \mathbb{F}_{r\mathbb{C}} \oplus \mathbb{F}_c$, and call it the *space of field variations with complex structure*. (We assume that also $\mathbb{F}_T \subset \mathbb{F}$ respects this splitting.)

The above definition is necessary, because in field theory, certain fields (like variations of Dirac fields) sit in an inherently complex vector space, whereas other fields (like variations of gauge fields) sit in an inherently real vector space, and QFT assumes that the sectors not being inherently complex are complexified. In the most simple case, one has merely $\mathbb{F}_{(\mathbb{C})} = \mathbb{F}$ if \mathbb{F} was complex, or $\mathbb{F}_{(\mathbb{C})} = \mathbb{F} \otimes \mathbb{C}$ if \mathbb{F} was real.

Definition 5. Let the vector space of field variations admit a real-complex splitting $\mathbb{F} = \mathbb{F}_r \oplus \mathbb{F}_c$, as in definition 4. Furthermore, assume a direct sum structure $\mathbb{F} = \bigoplus_{i=1}^f \mathbb{F}_i$, such that for each $i = 1, \dots, f$ the subspace \mathbb{F}_i is either entirely within \mathbb{F}_r or in \mathbb{F}_c and are closed (thus, also nuclear Fréchet), and let there be integers $s_i \in \{0, 1\}$ associated to each subspace \mathbb{F}_i ($i = 1, \dots, f$). Then, the subspaces $\mathbb{F}_1, \dots, \mathbb{F}_f$ are called the *flavor sectors*, and their associated integers s_1, \dots, s_f are called *bosonic or fermionic labels*. (We assume that also $\mathbb{F}_T \subset \mathbb{F}$ respects this splitting.)

In the most simple case, there is only one single flavor sector, globally endowed with a bosonic or fermionic label. For invoking the MDS equation, we will need the graded-symmetrized subspace of $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$, according to the bosonic and fermionic labels. In order to establish that algebra, the following remark is useful.

Remark 6. Whenever \mathbb{F} is split as $\mathbb{F} = \bigoplus_{i=1}^f \mathbb{F}_i$ into flavor sectors with bosonic/fermionic labels s_i ($i = 1, \dots, f$), as in definition 5, then for all $n \in \mathbb{N}_0$ one may introduce a continuous linear representation U_π of a permutation group element $\pi \in \Pi_n$ on the space $\bigotimes^n \mathbb{F}_{(\mathbb{C})}$ as follows (see also [26] chapter 4). Take an element $x_1 \otimes \dots \otimes x_n \in \bigotimes^n \mathbb{F}_{(\mathbb{C})}$, where each factor x_i ($i = 1, \dots, n$) resides in some $\mathbb{F}_{(\mathbb{C})j}$ ($j = 1, \dots, f$). Then, set

$$U_\pi(x_1 \otimes \dots \otimes x_n) := (-1)^{s_1\sigma_1(\pi) + \dots + s_f\sigma_f(\pi)} x_{\pi(1)} \otimes \dots \otimes x_{\pi(n)},$$

where $\sigma_i(\pi) \in \{0, 1\}$ ($i = 1, \dots, f$) is the parity of the permutation π within each index block. The map U_π can then be linearly extended in $\bigotimes^n \mathbb{F}_{(\mathbb{C})}$. Due to the NF property of the involved spaces, the topology defining seminorms on $\bigotimes^n \mathbb{F}_{(\mathbb{C})}$ may be taken to be such that U_π are continuous ([26] chapter 4), therefore can uniquely be extended as acting as a continuous linear map $U_\pi : \bigotimes^n \mathbb{F}_{(\mathbb{C})} \rightarrow \bigotimes^n \mathbb{F}_{(\mathbb{C})}$, thus defining the signed permutation operator on the entire space $\bigotimes^n \mathbb{F}_{(\mathbb{C})}$. Therefore, on each space $\bigotimes^n \mathbb{F}_{(\mathbb{C})}$ the continuous linear projection operator

$$P_n := \frac{1}{n!} \sum_{\pi \in \Pi_n} U_\pi$$

can be defined. The family of operators P_n ($n \in \mathbb{N}_0$) on the spaces $\bigotimes^n \mathbb{F}_{(\mathbb{C})}$ can be joined as a single grading preserving continuous linear projection operator $P : \mathcal{T}(\mathbb{F}_{(\mathbb{C})}) \rightarrow \mathcal{T}(\mathbb{F}_{(\mathbb{C})})$. This signed symmetrizer projection operator P has the following properties against the tensor algebra multiplication:

$$P(xy) = P(P(x)y) = P(xP(y)) = P(P(x)P(y)) \quad (\forall x, y \in \mathcal{T}(\mathbb{F}_{(\mathbb{C})})).$$

Therefore, the closed subspace $\text{Ker}(P)$ is a two-sided ideal in $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$. (The presented approach was inspired by [26] chapter 4.)

Using the fact that the closed subspace of an NF space is also NF and that the factor space of an NF space with a closed subspace is also NF (see also [22]-remark 2), the following definition is meaningful.

Definition 7. Let the space of field variations \mathbb{F} admit flavor sectors \mathbb{F}_i and bosonic/fermionic labels s_i ($i = 1, \dots, f$), as in definition 5, and corresponding signed symmetrization projector P as in remark 6. Then the factor algebra $A(\mathbb{F}_{(\mathbb{C})}) := \mathcal{T}(\mathbb{F}_{(\mathbb{C})})/\text{Ker}(P)$ is called the **field algebra**. Clearly, it is a unital associative algebra, and a nuclear Fréchet (NF) topological vector space, with jointly continuous algebra multiplication $\bullet : A(\mathbb{F}_{(\mathbb{C})}) \times A(\mathbb{F}_{(\mathbb{C})}) \rightarrow A(\mathbb{F}_{(\mathbb{C})})$. The topological transpose P^* of P allows the analogous construction in the strong dual of $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$, which makes it also a unital associative algebra with jointly continuous algebra multiplication, and therefore $A(\mathbb{F}_{(\mathbb{C})})$ retains the bialgebra structure from $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$.

Since the complementing projection operator $I - P$ to P is also continuous, as TVS one may naturally identify $A(\mathbb{F}_{(\mathbb{C})})$ with the closed subspace $\text{Ker}(I - P) = \text{Ran}(P) \subset \mathcal{T}(\mathbb{F}_{(\mathbb{C})})$. Using this linear topological identification, the algebraic product \bullet may be pushed forward from $A(\mathbb{F}_{(\mathbb{C})})$ to the subspace $\text{Ran}(P) \subset \mathcal{T}(\mathbb{F}_{(\mathbb{C})})$. That is, as usual, the algebra $A(\mathbb{F}_{(\mathbb{C})})$ may be

regarded as a closed subspace of $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$. On that space the product \bullet can be traced back to the tensor algebra product \otimes , with the identity: for all $x \in \otimes^m \mathbb{F}_{(\mathbb{C})}$ and $y \in \otimes^n \mathbb{F}_{(\mathbb{C})}$, one has $x \bullet y = \frac{(m+n)!}{m!n!} P(x \otimes y)$. The unit element, the counit map, as well as the insertion operator by a one-form $p^{(1)} \in \mathbb{F}_{(\mathbb{C})}^*$ coincides to the one defined on $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$. The strong dual of $A(\mathbb{F}_{(\mathbb{C})})$ may be identified with the corresponding subspace of $\mathcal{T}_a(\mathbb{F}_{(\mathbb{C})}^*)$. Whenever not confusing, we will suppress the multiplication symbol \bullet .

The above definition was necessary, because in QFT the Feynman type field correlators are graded-symmetrized, i.e. they sit rather in $A(\mathbb{F}_{(\mathbb{C})})$ than in $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$. (In the most simple case one has that $A(\mathbb{F}_{(\mathbb{C})})$ is $\vee(\mathbb{F}_{(\mathbb{C})})$ or $\wedge(\mathbb{F}_{(\mathbb{C})})$.) As expanded above, the left multiplication operator (given some $\delta\psi \in \mathbb{F}$) is the same as $L_{\delta\psi}$ in $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$, with a subsequent graded-symmetrization and combinatorial normalization. It shall be denoted by the same symbol $L_{\delta\psi}$, when not confusing. According to the chosen normalization conventions, in the algebra $A(\mathbb{F}_{(\mathbb{C})}) \subset \mathcal{T}(\mathbb{F}_{(\mathbb{C})})$, the counit map b and the left-insertion operator ι_p by a one-form $p \in \mathbb{F}_{(\mathbb{C})}^*$ literally coincide with the corresponding operators in $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$. Due to the graded-symmetrization, one has that for all $G \in A(\mathbb{F}_{(\mathbb{C})})$, and for all $\delta\psi \in \mathbb{F}_{(\mathbb{C})}$, $\delta J \in \mathbb{F}_{(\mathbb{C})}^*$ from the same fermionic sector $(\iota_{\delta J} L_{\delta\psi} + L_{\delta\psi} \iota_{\delta J})G = (\delta J | \delta\psi)G$ holds, whereas $(\iota_{\delta J} L_{\delta\psi} - L_{\delta\psi} \iota_{\delta J})G = (\delta J | \delta\psi)G$ holds otherwise.

Until section 5, for the sake of simplicity we assume that the EL functional $E : F \rightarrow \mathbb{F}_T^*$ is multipolynomial, which is defined as follows.

Definition 8. Let $E : F \times \mathbb{F}_T \rightarrow \mathbb{R}$ be an EL functional as in definition 2. We say that the EL functional E is **multipolynomial**, whenever there exists a reference field $\psi_0 \in F$, such that there exists an element $\mathbf{E}_{\psi_0} \in (A(\mathbb{F}_{(\mathbb{C})}))^* \otimes \mathbb{F}_T^* \subset \mathcal{T}_a(\mathbb{F}_{(\mathbb{C})}^*) \otimes \mathbb{F}_T^*$, for which

$$\forall \psi \in F, \delta\psi_T \in \mathbb{F}_T : (\mathbf{E}_{\psi_0} | \psi - \psi_0) | \delta\psi_T = \left(\mathbf{E}_{\psi_0} \left| \left(1, \overset{1}{\otimes}(\psi - \psi_0), \overset{2}{\otimes}(\psi - \psi_0), \dots \right) \otimes \delta\psi_T \right. \right)$$

holds. (Note that then for all $\psi_0 \in F$ there exists the corresponding element \mathbf{E}_{ψ_0} .) When an element $\delta\psi_T \in \mathbb{F}_T$ is contracted with \mathbf{E}_{ψ_0} in its last tensorial entry, we will use the notation $(\mathbf{E}_{\psi_0} | \delta\psi_T)$ to denote the corresponding element of $(A(\mathbb{F}_{(\mathbb{C})}))^* \subset \mathcal{T}_a(\mathbb{F}_{(\mathbb{C})}^*)$.

Given $(\mathbf{E}_{\psi_0} | \delta\psi_T)$ as above, it has a corresponding multipolynomial insertion operator over the tensor algebra $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$, as stated in remark 3(i). We shall denote that by the symbol $\iota_{(\mathbf{E}_{\psi_0} | \delta\psi_T)}$.

Definition 9. Let \hbar be a fixed real number. Let $F, \mathbb{F}, \mathbb{F}_T$ as in definition 1. Let $E : F \times \mathbb{F}_T \rightarrow \mathbb{R}$ as in definition 2, and assume that it is multipolynomial as in definition 8. Let $A(\mathbb{F}_{(\mathbb{C})})$ be the field algebra as in definition 7. Then, for some fixed reference field $\psi_0 \in F$ and fixed test field $\delta\psi_T \in \mathbb{F}_T$ the operator

$$\mathbf{M}_{\hbar, \psi_0, \delta\psi_T} : A(\mathbb{F}_{(\mathbb{C})}) \rightarrow A(\mathbb{F}_{(\mathbb{C})}), \quad G \mapsto \mathbf{M}_{\hbar, \psi_0, \delta\psi_T} G := \left(\iota_{(\mathbf{E}_{\psi_0} | \delta\psi_T)} - i\hbar L_{\delta\psi_T} \right) G \tag{9}$$

is called the **unregularized MDS operator**. We call the below equation the **unregularized MDS equation**:

$$\begin{aligned} & \text{we search for } (\psi_0, G_{\psi_0}) \in F \times A(\mathbb{F}_{(\mathbb{C})}), \quad \text{such that:} \\ & bG_{\psi_0} = 1, \quad \text{and} \quad \forall \delta\psi_T \in \mathbb{F}_T : \mathbf{M}_{\hbar, \psi_0, \delta\psi_T} G_{\psi_0} = 0. \end{aligned} \tag{10}$$

The MDS formulation of QFT can be thought of as a construction, where the objects of interest are elements of $F \times A(\mathbb{F}_{(\mathbb{C})})$, and the selection equation for the physically realized

such elements is the MDS equation. In section 4 it shall be shown that some finetuning (regularization) to this idea is needed, as is well known in the QFT literature.

Definition 10. Any continuous map $O : F \rightarrow \mathbb{R}$ is called an *observable*, similarly as in a classical field theory. Given a fixed $\psi_0 \in F$, we use the notation $O_{\psi_0} := O \circ (\mathbb{I}_F + \psi_0)$, which is then a continuous map $O_{\psi_0} : F \rightarrow \mathbb{R}$, and one has $O(\psi) = O_{\psi_0}(\psi - \psi_0)$ for all $\psi \in F$ and observable O . An observable $O : F \rightarrow \mathbb{R}$ is called *multipolynomial observable*, whenever for some reference fields $\psi_0 \in F$, there exists an element $\mathbf{O}_{\psi_0} \in \mathcal{T}_a(\mathbb{F}^*)$, such that for all $\psi \in F$, one has $O_{\psi_0}(\psi - \psi_0) = \left(\mathbf{O}_{\psi_0} \left| \left(1, \overset{1}{\otimes} (\psi - \psi_0), \overset{2}{\otimes} (\psi - \psi_0), \dots \right) \right. \right)$. (If it holds, it then holds for any $\psi_0 \in F$.)

Definition 11. Given a solution $(\psi_0, G_{\psi_0}) \in F \times A(\mathbb{F}_{(\mathbb{C})})$ of the MDS equation, the *(Feynman type) quantum expectation value* of the multipolynomial observable $O : F \rightarrow \mathbb{R}$ at the solution (ψ_0, G_{ψ_0}) is $\mu_{(\psi_0, G_{\psi_0})}(O) := (\mathbf{O}_{\psi_0} | G_{\psi_0})$.

We note that the above construction can be extended also to non-polynomial but analytic EL functionals and observables as well. For that, however, a stronger topology is needed on the tensor algebra of \mathbb{F} , which we will address later in section 5.

Example 12. For a scalar φ^4 model over a fixed Minkowski spacetime \mathcal{M} , the MDS operator reads as follows. Let \mathbf{v} be the affine constant maximal form over \mathcal{M} (corresponding to the Lebesgue measure). Denote by \square the Minkowski wave operator. Set $F := \mathbb{F} := C^\infty(\mathcal{M}, \mathbb{R})$ and $\mathbb{F}_T := C_c^\infty(\mathcal{M}, \mathbb{R})$. Then, the EL functional is $E : F \times \mathbb{F}_T \rightarrow \mathbb{R}, (\psi, \delta\psi_T) \mapsto \int_{\mathcal{M}} \delta\psi_T \square \psi \mathbf{v} + \int_{\mathcal{M}} \delta\psi_T \psi^3 \mathbf{v}$. For any fixed test field $\delta\psi_T \in \mathbb{F}_T$ the corresponding MDS operator can be expressed as

$$\begin{aligned} & (\mathbf{M}_{\hbar, \psi_0, \delta\psi_T} G)^{(n)}(x_1, \dots, x_n) \\ &= \int_{y \in \mathcal{M}} \delta\psi_T(y) \square_y G^{(n+1)}(y, x_1, \dots, x_n) \mathbf{v}(y) + \int_{y \in \mathcal{M}} \delta\psi_T(y) G^{(n+3)}(y, y, y, x_1, \dots, x_n) \mathbf{v}(y) \\ & - i\hbar n \frac{1}{n!} \sum_{\pi \in \Pi_n} \delta\psi_T(x_{\pi(1)}) G^{(n-1)}(x_{\pi(2)}, \dots, x_{\pi(n)}) \end{aligned} \tag{11}$$

at the reference field $\psi_0 = 0$ (for all $G \in A(\mathbb{F}_{(\mathbb{C})}) = \mathbb{V}(\mathbb{F} \otimes \mathbb{C})$ and $n \in \mathbb{N}_0$ and $x_1, \dots, x_n \in \mathcal{M}$, where Π_n denotes the set of permutations of the symbols $1, \dots, n$).

Example 13. For a pure Yang–Mills model (possibly non-abelian, i.e. self-interacting) over a fixed spacetime (\mathcal{M}, g_{ab}) , the MDS operator reads as follows (Penrose abstract indices $abc\dots$ and $abc\dots$ are used for tangent tensors and their duals, respectively). Let \mathbf{v} be the canonical volume form associated to the spacetime metric g_{ab} . Let F denote the affine space of covariant derivation operators over some vector bundle $V(\mathcal{M})$ with some given structure group \mathcal{G} (internal or gauge group). Then, any two covariant derivations $\nabla, \nabla' \in F$ has a difference tensor (Yang–Mills potential) $A := \nabla' - \nabla$ residing in the space of smooth sections of $T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M})$, denoted by \mathbb{F} . Let \mathbb{F}_T denote the space compactly supported sections from \mathbb{F} . Then, the EL functional is $E : F \times \mathbb{F}_T \rightarrow \mathbb{R}, (\nabla, A_T) \mapsto \int_{\mathcal{M}} A_{Td} \cdot \left(-\tilde{\nabla}_e \mathbf{v} g^{ec} g^{db} P(\nabla)_{cb} \right)$, the symbol $P(\nabla)_{ab}$ denoting the curvature tensor of ∇ and \cdot denoting the pointwise trace form on the sections of $V(\mathcal{M}) \otimes V^*(\mathcal{M})$, whereas $\tilde{\nabla}$ denoting an extension of ∇ to the mixed tensor algebra of $V(\mathcal{M})$ and $T(\mathcal{M})$ with an arbitrary torsion-free covariant derivation on $T(\mathcal{M})$. (The pertinent differential operator expression involving $\tilde{\nabla}$ is known to be uniquely defined, see also remark 42 in appendix A.) Specially, fix a covariant derivation $\nabla \in F$ as a reference field, then

the EL functional with respect to this reference field ∇ reads as $E_\nabla : \mathbb{F} \times \mathbb{F}_T \rightarrow \mathbb{R}, (A, A_T) \mapsto \int_{\mathcal{M}} A_T d \cdot \left(-\tilde{\nabla}_e v g^{ec} g^{db} (\tilde{\nabla}_c A_b - \tilde{\nabla}_b A_c + [A_c, A_b]) \right)$. Given a test field variation $A_T \in \mathbb{F}_T$, the corresponding MDS operator is

$$\begin{aligned} & (\mathbf{M}_{\hbar, \nabla, A_T} G)^{(n)}(x_1, \dots, x_n)_{a_1 \dots a_n}^{\alpha_1 \dots \alpha_n} \\ &= \int_{y \in \mathcal{M}} \left(-A_T(y)_d \gamma K(y)_{\gamma\beta} \tilde{\nabla}_e^y v(y) g(y)^{ec} g(y)^{db} \tilde{\nabla}_c^y G^{(n+1)}(y, x_1, \dots, x_n)_{ba_1 \dots a_n}^{\beta \alpha_1 \dots \alpha_n} \right. \\ & \quad + A_T(y)_d \gamma K(y)_{\gamma\beta} \tilde{\nabla}_e^y v(y) g(y)^{ec} g(y)^{db} \tilde{\nabla}_b^y G^{(n+1)}(y, x_1, \dots, x_n)_{ca_1 \dots a_n}^{\beta \alpha_1 \dots \alpha_n} \\ & \quad \left. - A_T(y)_d \gamma K(y)_{\gamma\beta} \tilde{\nabla}_e^y v(y) g(y)^{ec} g(y)^{db} G^{(n+2)}(y, y, x_1, \dots, x_n)_{[cb]a_1 \dots a_n}^{\delta \varepsilon \alpha_1 \dots \alpha_n} C(y)_{\delta \varepsilon}^\beta \right) \\ & \quad - i \hbar n \frac{1}{n!} \sum_{\pi \in \Pi_n} A_T(x_{\pi(1)})_{a_{\pi(1)}}^{\alpha_{\pi(1)}} G^{(n-1)}(x_{\pi(2)}, \dots, x_{\pi(n)})_{a_{\pi(2)} \dots a_{\pi(n)}}^{\alpha_{\pi(2)} \dots \alpha_{\pi(n)}} \end{aligned} \quad (12)$$

for all $G \in A(\mathbb{F}_{(\mathbb{C})}) = \mathbb{V}(\mathbb{F} \otimes \mathbb{C})$ and $n \in \mathbb{N}_0$ and $x_1, \dots, x_n \in \mathcal{M}$, where Π_n denotes the set of permutations of the symbols $1, \dots, n$, and the Penrose abstract indices $\alpha^\beta \gamma \delta \varepsilon \dots$ were used for the Lie algebra of the structure group \mathcal{G} , with $K_{\alpha\beta}$ denoting the index notation of the trace form, and $C_{\alpha\beta}^\gamma$ denoting the index notation of the commutator.

4. The weak (distributional) and the Wilsonian regularized MDS operator

Definition 14. Let $F, \mathbb{F}, \mathbb{F}_T, A(\mathbb{F}_{(\mathbb{C})}), E$ be as in definition 9. We call the EL functional $E : F \times \mathbb{F}_T \rightarrow \mathbb{R}$ to be **free** or **non-interacting**, whenever the corresponding continuous map $E : F \rightarrow \mathbb{F}_T^*$ is affine. We call the Euler–Lagrangian functional **interacting** otherwise. (Note that by construction, for a free EL functional, given any reference field $\psi_0 \in F$, the map $E_{\psi_0}(\cdot) - E_{\psi_0}(0) : \mathbb{F} \rightarrow \mathbb{F}_T^*$ is linear. If in addition, ψ_0 were an EL solution, then $E_{\psi_0}(\cdot) : \mathbb{F} \rightarrow \mathbb{F}_T^*$ is linear.)

Remark 15. It is seen that if $(\psi_0, G_{\psi_0}) \in F \times A(\mathbb{F}_{(\mathbb{C})})$ were a solution to the unregularized MDS equation (10), and the reference field $\psi_0 \in F$ is an EL solution, and E is non-interacting, then $\iota_{(E_{\psi_0} | \delta \psi_T)} G_{\psi_0}^{(2)} = i \hbar \delta \psi_T$ holds for all test fields $\delta \psi_T \in \mathbb{F}_T$.

Corollary 16. Let the EL functional E be the one of the free wave or Klein–Gordon equation over Minkowski spacetime. In that case, the solution space of the unregularized MDS equation (10) is empty, whenever $\hbar \neq 0$.

The above is rather evident by means of remark 15: the correlator $G_{\psi_0}^{(2)}$ would need to be proportional to a fundamental solution (Green’s functional), which does not sit in the space of smooth correlators $\mathbb{F} \otimes \mathbb{F}$, but is at best a distribution. It is thus tempting to extend the definition of the MDS equation in the weak (distributional) sense, so that free theories can have MDS solutions. In order to define the distributional sense fields, one needs to use the information that the EL functional $E : F \times \mathbb{F}_T \rightarrow \mathbb{R}$ is actually that of a concrete classical field theory. Namely, that F is the space of smooth sections of an affine bundle, \mathbb{F} is the space of smooth sections of its subordinate vector bundle, and \mathbb{F}_T is the space of compactly supported smooth sections of that vector bundle.

Remark 17. In order to define the weak MDS operator, we will need to substitute $A(\mathbb{F}_{(\mathbb{C})})$ with its distributional version, which is expanded below.

- (a) Assume that F is the space of smooth sections of an affine bundle over the base manifold \mathcal{M} , with subordinate vector bundle $U(\mathcal{M})$, whose smooth sections span the space \mathbb{F} . Take the densitised dual of that vector bundle, $U^\times(\mathcal{M}) := U^*(\mathcal{M}) \otimes \wedge^{\dim(\mathcal{M})} T^*(\mathcal{M})$, and denote the space of its smooth sections by \mathbb{F}^\times correspondingly. Take the n -fold external tensor product bundle $U^\times(\mathcal{M}) \boxtimes \cdots \boxtimes U^\times(\mathcal{M})$ of that, which will then be a vector bundle over the n -fold Cartesian product $\mathcal{M} \times \cdots \times \mathcal{M}$ as base manifold. The space of smooth sections of this vector bundle shall be denoted by \mathbb{F}_n^\times , which has its natural \mathcal{E} topology which is nuclear Fréchet (NF), and is topologically isomorphic to $\overset{n}{\otimes} \mathbb{F}^\times$ by means of Schwartz kernel theorem. It has the subspace of compactly supported sections, denoted by $\mathbb{F}_{T_n}^\times$ and is a dense subspace within \mathbb{F}_n^\times in the \mathcal{E} topology. The space $\mathbb{F}_{T_n}^\times$ with its natural \mathcal{D} topology becomes a countable strict inductive limit of nuclear Fréchet spaces with closed adjacent images (LNF space) whenever the base manifold \mathcal{M} is noncompact, and is nuclear Fréchet (NF) if \mathcal{M} is compact (see [22]-remark 2). The strong dual of the space $\mathbb{F}_{T_n}^\times$ is denoted by $(\mathbb{F}_{T_n}^\times)^*$ with its natural \mathcal{D}^* topology. It is a DLNF space when \mathcal{M} is noncompact, and DNF when \mathcal{M} is compact. One has that $\overset{n}{\otimes} \mathbb{F} \subset (\mathbb{F}_{T_n}^\times)^*$, i.e. the latter space can be regarded as the space of distributional n -field correlators.
- (b) In the above construction we avoided using completed topological tensor product $\overset{n}{\otimes} \mathbb{F}_T^\times$, as that space is topologically not isomorphic to $\mathbb{F}_{T_n}^\times$ whenever we are in the realm of LNF spaces, i.e. when \mathcal{M} is noncompact (although they are isomorphic as linear spaces, the latter has a stronger topology). This slight complication is mentioned in more details in [22]-remark 3(d). The pertinent issue is absent, whenever \mathcal{M} is compact ($\mathbb{F}_{T_n}^\times \equiv \overset{n}{\otimes} \mathbb{F}_T^\times$ topologically, in that case, i.e. one does not need to distinguish them on compact manifolds).
- (c) One can form the algebraic tensor algebra $\mathcal{T}_a(\mathbb{F}_T^\times)$, defined as the algebraic direct sum $\bigoplus_{n=0}^\infty \mathbb{F}_{T_n}^\times$ equipped with the locally convex direct sum topology. Its topology will be LNF whenever \mathcal{M} is noncompact, and NF if \mathcal{M} is compact. $\mathcal{T}_a(\mathbb{F}_T^\times)$ forms a unital associative algebra, with (at least) separately continuous multiplication.
- (d) The tensor algebra of distributional field variations $\mathcal{T}((\mathbb{F}_T^\times)^*)$ is defined to be the space $(\mathcal{T}_a(\mathbb{F}_T^\times))^*$. It is topologically isomorphic to $\bigoplus_{n=0}^\infty (\mathbb{F}_{T_n}^\times)^*$, by means of [22]-remark 4. It is a DLNF space when \mathcal{M} is noncompact, and DNF space if \mathcal{M} is compact. It is also a unital associative algebra, with an (at least) separately continuous algebra multiplication.
- (e) The distributional graded-symmetrized field algebra $A((\mathbb{F}_{T(\mathbb{C})}^\times)^*) \subset \mathcal{T}((\mathbb{F}_{T(\mathbb{C})}^\times)^*)$ can be defined in the analogy of definition 7. Clearly, the smooth field algebra $A(\mathbb{F}_{(\mathbb{C})})$, is dense in the distributional sense field algebra $A((\mathbb{F}_{T(\mathbb{C})}^\times)^*)$.

Remark 18. The MDS operator of a non-interacting EL functional can be naturally extended in the distributional sense, as follows.

- (a) A continuous linear operator $A : \mathbb{F} \rightarrow \mathbb{F}$ is said to possess a **formal transpose**, if there exists a continuous linear operator $A^t : \mathbb{F}_T^\times \rightarrow \mathbb{F}_T^\times$, such that for all $\delta\psi \in \mathbb{F}$ and $p_T \in \mathbb{F}_T^\times$ one has that $\int_{\mathcal{M}} (A\delta\psi) p_T = \int_{\mathcal{M}} \delta\psi (A^t p_T)$, with \mathcal{M} being the underlying manifold. The topological transpose $(A^t)^* : (\mathbb{F}_T^\times)^* \rightarrow (\mathbb{F}_T^\times)^*$ of the formal transpose operator is called the **distributional extension** of A .
- (b) The notion of formal transpose can be generalized to operators $A : \mathcal{T}(\mathbb{F}) \rightarrow \mathcal{T}(\mathbb{F})$, being of the type $A^t : \mathcal{T}_a(\mathbb{F}_T^\times) \rightarrow \mathcal{T}_a(\mathbb{F}_T^\times)$, and $(A^t)^* : (\mathcal{T}_a(\mathbb{F}_T^\times))^* \rightarrow (\mathcal{T}_a(\mathbb{F}_T^\times))^*$ being the distributional extension of A .

- (c) One may note that for all $\delta\psi_T \in \mathbb{F}_T$ and $G \in A(\mathbb{F}_{(C)})$ and $p \in \mathcal{T}_a(\mathbb{F}_T^\times)$ one has that $(p|L_{\delta\psi_T}G) = (\iota_{\delta\psi_T}p|G)$. Moreover, the linear map $\iota_{\delta\psi_T} : \mathcal{T}_a(\mathbb{F}_T^\times) \rightarrow \mathcal{T}_a(\mathbb{F}_T^\times)$ is continuous. Therefore, $\iota_{\delta\psi_T}$ is the formal transpose of $L_{\delta\psi_T}$. Consequently, the operator $L_{\delta\psi_T}$ admits a distributional extension $A((\mathbb{F}_{T(C)}^\times)^*) \rightarrow A((\mathbb{F}_{T(C)}^\times)^*)$, being the topological transpose of $\iota_{\delta\psi_T}$.
- (d) Whenever E is the EL functional of a non-interacting classical field theory, and $\psi_0 \in F$ is fixed, then for each $\delta\psi_T \in \mathbb{F}_T$ there exists a unique element $\pi_T \in \mathbb{F}_T^\times$, such that $(E_{\psi_0}(\delta\psi)|\delta\psi_T) - (E(\psi_0)|\delta\psi_T) = \int_{\mathcal{M}} \pi_T \delta\psi$ for all $\delta\psi \in \mathbb{F}$ (see also appendix A). Therefore, one has that $\iota_{(E_{\psi_0}|\delta\psi_T) - (E(\psi_0)|\delta\psi_T)} = \iota_{\int_{\mathcal{M}} \pi_T(\cdot)}$. Because of that, for any $G \in A(\mathbb{F}_{(C)})$ and $p \in \mathcal{T}_a(\mathbb{F}_T^\times)$ one has the identity $(p|\iota_{(E_{\psi_0}|\delta\psi_T) - (E(\psi_0)|\delta\psi_T)}G) = (L_{\pi_T}p|G)$, i.e. the formal transpose of $\iota_{(E_{\psi_0}|\delta\psi_T)}$ exists, being the continuous linear map $L_{\pi_T} + (E(\psi_0)|\delta\psi_T)I : \mathcal{T}_a(\mathbb{F}_T^\times) \rightarrow \mathcal{T}_a(\mathbb{F}_T^\times)$. Consequently, the operator $\iota_{(E_{\psi_0}|\delta\psi_T)}$ admits a distributional extension $A((\mathbb{F}_{T(C)}^\times)^*) \rightarrow A((\mathbb{F}_{T(C)}^\times)^*)$.
- (e) The above construction clearly fails for interacting classical field theories, since then the formal transpose of $\iota_{(E_{\psi_0}|\delta\psi_T)}$ as a continuous linear map $\mathcal{T}_a(\mathbb{F}_T^\times) \rightarrow \mathcal{T}_a(\mathbb{F}_T^\times)$ cannot be defined. See e.g. the interaction term in equation (11) as an example.
- (f) Let $E : F \times \mathbb{F}_T \rightarrow \mathbb{R}$ be the EL functional of a classical field theory, and $J \in \mathbb{F}_T^*$, then we call an element $K_J \in F$ a **solution with a source J** whenever $\forall \delta\psi_T \in \mathbb{F}_T : (E(K_J)|\delta\psi_T) = (J|\delta\psi_T)$ holds. Specially, one may consider only $J \in \mathbb{F}_T^\times \subset \mathbb{F}^\times \subset \mathbb{F}_T^*$. If $\mathbb{K} : \mathbb{F}_T^\times \rightarrow F$ is a continuous map, such that for all $J \in \mathbb{F}_T^\times$ the field $\mathbb{K}(J) \in F$ is a solution with a source J , then \mathbb{K} is called a **fundamental solution**. (It may or may not exist, and if exists, it is typically not unique.)

Definition 19. Let $F, \mathbb{F}, \mathbb{F}_T, A(\mathbb{F}_{(C)}), E, \hbar$ be as in definition 9, and let $E : F \times \mathbb{F}_T \rightarrow \mathbb{R}$ the EL functional of a non-interacting classical field theory as in the definition 14. Fix a reference field $\psi_0 \in F$. Then, by means of remark 18, for all $\delta\psi_T \in \mathbb{F}_T$, the MDS operator can be extended as a continuous linear operator $\mathbf{M}_{\hbar, \psi_0, \delta\psi_T} : A((\mathbb{F}_{T(C)}^\times)^*) \rightarrow A((\mathbb{F}_{T(C)}^\times)^*)$, called to be the **weak or distributional MDS operator**. We call the equation

$$\begin{aligned} &\text{we search for } G_{\psi_0} \in A((\mathbb{F}_{T(C)}^\times)^*), \quad \text{such that:} \\ &bG_{\psi_0} = 1, \quad \text{and} \quad \forall \delta\psi_T \in \mathbb{F}_T : \quad \mathbf{M}_{\hbar, \psi_0, \delta\psi_T} G_{\psi_0} = 0, \end{aligned} \tag{13}$$

the **weak or distributional MDS equation**.

Remark 20. With the above notations, assume that the EL equation admits a fundamental solution $\mathbb{K} : \mathbb{F}_T^\times \rightarrow F, J \mapsto \mathbb{K}(J)$ as in remark 18(f). Then, $E_{\psi_0} : \mathbb{F} \rightarrow \mathbb{F}_T^*$ also has a corresponding fundamental solution $\mathbb{K}_{\psi_0} : \mathbb{F}_T^\times \rightarrow \mathbb{F}, J \mapsto \mathbb{K}_{\psi_0}(J) := \mathbb{K}(J) - \psi_0$. Let ψ_0 be an EL solution, in which case $E_{\psi_0} : \mathbb{F} \rightarrow \mathbb{F}_T^*$ becomes linear, and assume that \mathbb{K}_{ψ_0} can be chosen to be linear. Such a linear fundamental solution $\mathbb{K}_{\psi_0} : \mathbb{F}_T^\times \rightarrow \mathbb{F}$ can be naturally considered as an element $\mathbb{K}_{\psi_0}^{(2)} \in \mathcal{L}(\mathbb{F}_T^\times, \mathbb{F}) \subset (\mathbb{F}_{T_2}^\times)^*$. Assume moreover, that $\mathbb{K}_{\psi_0}^{(2)}$ can be chosen to be invariant to the permutation symmetry of the field algebra. (E.g. for a wave or Klein–Gordon equation over Minkowski spacetime, the Feynman propagator would be such.)

- (a) Given these conditions, one may define the element $K_{\psi_0} := (0, 0, i\hbar\mathbb{K}_{\psi_0}^{(2)}, 0, 0, \dots) \in A((\mathbb{F}_{T(C)}^\times)^*)$, called to be the connected correlator, and one can take the ansatz $G_{\psi_0} := \exp(K_{\psi_0}) \in A((\mathbb{F}_{T(C)}^\times)^*)$. Then, $(\psi_0, G_{\psi_0}) \in F \times A((\mathbb{F}_{T(C)}^\times)^*)$ solves the weak (distributional sense) MDS equation (13).

- (b) For the bosonic case, the above statement is seen trivially, by the fact that for all $\delta\psi_T \in \mathbb{F}_T$ the insertion operator $\iota_{(\mathbf{E}_{\psi_0}|\delta\psi_T)}$ is an algebra derivation, and the field algebra is commutative, so one can use the formula for the derivative of exponential. If \mathbb{F} has fermionic flavor sectors as well, then one can still trace the problem back to derivations acting on exponential: whenever $\delta\psi_T \in \mathbb{F}_T$ resides in a single flavor sector, then for all $\delta\psi'_T$ from the same flavor sector, the linear map $L_{\delta\psi'_T} \iota_{(\mathbf{E}_{\psi_0}|\delta\psi_T)}$ is also an algebra derivation.
- (c) Rather evidently, the above do not necessarily exhaust all the possible solutions. Typically, a fundamental solution K_{ψ_0} satisfying the above is not unique. Moreover, one may add any term δK_{ψ_0} to K_{ψ_0} satisfying $b\delta K_{\psi_0} = 0$ and $\forall \delta\psi_T \in \mathbb{F}_T : \iota_{(\mathbf{E}_{\psi_0}|\delta\psi_T)}\delta K_{\psi_0} = 0$, in which case $\exp(K_{\psi_0} + \delta K_{\psi_0})$ will still solve the weak MDS equation. In usual QFTs, these ambiguities are removed by further invariance requirements on G_{ψ_0} , which are not dealt with in the present paper.
- (d) The existence of the assumed type of fundamental solution is guaranteed for any EL functional over an affine base manifold, whenever E_{ψ_0} corresponds to a linear PDE with a multipolynomial differential operator, having constant coefficients. This is ensured by the celebrated Malgrange–Ehrenpreis theorem ([27] chapitre I.1 théorème 1 and [28] chapter 6 theorem 10).

Corollary 21. *Let the EL functional E be the one of a non-interacting classical field theory, which has a fundamental solution as in remark 20. In that case, the solution space of the weak (distributional) MDS equation (13) is **not empty**.*

According to remark 20, given a reference field $\psi_0 \in F$, the solutions of the weak MDS equation (13) are not unique for free EL functionals. In the usual QFT constructions, this ambiguity is removed by additional requirements, such as Poincaré invariance of the solutions. In the presented construction, however such auxiliary conditions are not imposed, since in a generally covariant setting, it is not evident that the vacuum state should be required to be unique or not.

Remark 22. For interacting models, the followings can be stated.

- (a) On one hand, there is a negative result: for a generic non-interacting EL functional, the unregularized MDS equation (10) has no solutions.
- (b) On the other hand, there is a positive result: for a generic non-interacting EL functional of a classical field theory having appropriate fundamental solution, the weak MDS equation (13) does have solutions, just as is the common wisdom in heuristic QFT.
- (c) For interacting models, the weak MDS operator cannot be an everywhere defined continuous operator acting on the space of distributional correlators. For instance, in a φ^4 model over Minkowski spacetime, the interacting part of the MDS operator does not have a formal transpose, as seen in equation (11). This phenomenon occurs because the diagonal evaluation map of smooth functions $((x, y) \mapsto G(x, y)) \mapsto (z \mapsto G(z, z))$ $x, y, z \in \mathcal{M}$ cannot be extended to the distributions, in general.
- (d) In order to remedy the above problem, one is tempted to view the everywhere defined continuous bilinear operator $\mathbf{M}_{\hbar, \psi_0} : A(\mathbb{F}_{(\mathbb{C})}) \times \mathbb{F}_T \rightarrow A(\mathbb{F}_{(\mathbb{C})})$ as a densely defined bilinear operator $\hat{\mathbf{M}}_{\hbar, \psi_0} : A((\mathbb{F}_{T(\mathbb{C})}^\times)^*) \times \mathbb{F}_T \rightarrow A((\mathbb{F}_{T(\mathbb{C})}^\times)^*)$, via the natural dense linear embedding $A(\mathbb{F}_{(\mathbb{C})}) \subset A((\mathbb{F}_{T(\mathbb{C})}^\times)^*)$ of the function sense correlators to the distributional sense correlators. Then, one is tempted to take its maximally extended operator, understood by its sequential closure. That is, a distributional correlator would be in the domain of the

extended $\hat{\mathbf{M}}_{\hbar, \psi_0}$, whenever it admits a function sense approximating sequence converging to it in the distributional sense, such that the evaluated $\mathbf{M}_{\hbar, \psi_0}$ on the approximator sequence is convergent in the distributional sense. The operator would be closable, whenever any two such approximator sequence of the same domain element yielded the same result. This strategy is made impossible by the fact that for all interacting EL functionals one can show that the MDS operator is not sequentially closable. (This occurs because the above diagonal evaluation map is so-called maximally non-closable, see [22]-remark 6 for more details.)

- (e) The celebrated Hörmander’s criterion [29] on the wave front set gives a sufficient condition for diagonal evaluation of multivariate distributions, but that condition is not applicable for the present problem. (E.g. already the wave front set of a solution to the distributional MDS equation generated from the Minkowski wave or Klein–Gordon equation is known to fail Hörmander’s sufficiency criterion, see [30] chapter 4 and [31].)
- (f) One can prove that the solution space of the unregularized MDS equation (understood over the smooth correlators) is always empty, regardless of the structure of the underlying base manifold \mathcal{M} and the interactions in the EL functional (we plan to detail the proof in a different paper).

In summary, the problem is that for interacting models, only the function sense MDS operator is well defined, but its solution space is always empty. In the non-interacting case, the MDS operator can be extended in the distributional sense, and its solution space has the right properties. However, the distributional extension of the MDS operator cannot be achieved for interacting models. In order to overcome this difficulty, one needs the regularized MDS operator, introduced below.

Definition 23. Let $F, \mathbb{F}, \mathbb{F}_T, A(\mathbb{F}_{(\mathbb{C})}), E, \hbar$ be as in definition 9. Fix a continuous linear operator $\mathcal{R} : \mathbb{F} \rightarrow \mathbb{F}$. Given these, we call the operator

$$\begin{aligned} \mathbf{M}_{\hbar, \psi_0, \mathcal{R}, \delta\psi_T} : A(\mathbb{F}_{(\mathbb{C})}) &\rightarrow A(\mathbb{F}_{(\mathbb{C})}), \\ G \mapsto \mathbf{M}_{\hbar, \psi_0, \mathcal{R}, \delta\psi_T} G &:= \left(\iota_{(\mathbf{E}_{\psi_0} | \delta\psi_T)} - i\hbar L_{\mathcal{R}\delta\psi_T} \right) G \end{aligned} \tag{14}$$

the \mathcal{R} -regularized MDS operator. Moreover, we call

$$\begin{aligned} \text{we search for } (\psi_0, G_{\psi_0}) \in F \times A(\mathbb{F}_{(\mathbb{C})}), \quad &\text{such that:} \\ bG_{\psi_0} = 1, \quad \text{and } \forall \delta\psi_T \in \mathbb{F}_T : \mathbf{M}_{\hbar, \psi_0, \mathcal{R}, \delta\psi_T} G_{\psi_0} &= 0 \end{aligned} \tag{15}$$

the \mathcal{R} -regularized MDS equation.

The above definition is motivated by the Wilsonian regularization, heuristically stated in equation (7). If the base manifold \mathcal{M} were an affine space, in order to achieve a Wilsonian regularization (UV frequency damping), the regularizer operator \mathcal{R} should be chosen as the convolution operator by a test function on \mathcal{M} . It is not difficult to see that in such a setting, for the non-interacting case, the Wilsonian regularized MDS equation (15) does have solutions in the space of smooth field correlators. Thus, definition 23 is expected to make sense also for interacting theories, since there is no problem with the diagonal evaluation map on the space of multivariate smooth functions. In order to adapt this construction to generic, non-affine manifolds \mathcal{M} , we invoke a notion of generalized convolution on smooth manifolds, see also [32–34].

Remark 24. The following terminologies are standard in the theory of pseudodifferential operators [32–34], and generalizes the notion of convolution to manifolds.

- (a) A continuous linear map $C : (\mathbb{F}^\times)^* \rightarrow \mathbb{F}$ is called a **smoothing operator**, their space is denoted by $\Psi^{-\infty}$ in the literature. By Schwartz kernel theorem, such an operator can be identified with an element $\kappa \in \mathbb{F} \otimes \mathbb{F}^\times$, i.e. κ is a smooth section of the vector bundle $U(\mathcal{M}) \boxtimes U^\times(\mathcal{M})$ over the base $\mathcal{M} \times \mathcal{M}$, with \mathbb{F} being the space of smooth sections of $U(\mathcal{M})$. This is emphasized by writing C_κ instead, where $(C_\kappa \delta\psi_T)(x) = \int_{y \in \mathcal{M}} \kappa(x, y) \delta\psi_T(y)$ for all $\delta\psi_T \in \mathbb{F}_T \subset (\mathbb{F}^\times)^*$ and for all $x \in \mathcal{M}$.
- (b) A smoothing operator C_κ is called **properly supported**, whenever the canonical projections from $\text{supp}(\kappa) \subset \mathcal{M} \times \mathcal{M}$ onto each factor \mathcal{M} is proper, i.e. the inverse images of compact sets are compact. In other words, for all compact subsets $\mathcal{K} \subset \mathcal{M}$ the closure of the sets $\{(x, y) \in \mathcal{M} \times \mathcal{M} | x \in \mathcal{K}, \kappa(x, y) \neq 0\}$ and $\{(x, y) \in \mathcal{M} \times \mathcal{M} | y \in \mathcal{K}, \kappa(x, y) \neq 0\}$ are compact. In that case, the map C_κ can act as continuous linear maps $\mathbb{F}_T \rightarrow \mathbb{F}_T$, $\mathbb{F} \rightarrow \mathbb{F}$, $(\mathbb{F}^\times)^* \rightarrow (\mathbb{F}^\times)^*$, $(\mathbb{F}_T^\times)^* \rightarrow (\mathbb{F}_T^\times)^*$.

Definition 25. Let C_κ be a properly supported smoothing operator as in remark 24. If it preserves the flavor sectors, then κ is called a **mollifying kernel**.

Remark 26. A special example can shed some light on the role of κ in definition 25. Let \mathcal{M} be a finite dimensional real affine space (‘Minkowski spacetime’), the subordinate finite dimensional real vector space denoted by T (‘tangent space’). Let the vector bundle of fields be trivial, and trivialized compatibly with the affine structure. In that case, the fields, i.e. the elements of \mathbb{F} are simply smooth functions from \mathcal{M} to a finite dimensional real vector space, in which the classical fields take their values. Let us denote the identity operator of that finite dimensional real vector space by I . Due to the affineness of \mathcal{M} , up to a positive multiplier there exists a unique positive volume form field v which is parallel against the affine parallel transport (this corresponds to the Lebesgue measure). Take a compactly supported C^∞ real valued scalar field $\rho : T \rightarrow \mathbb{R}$. Then, the field $(x, y) \mapsto \kappa(x, y) := \rho(x - y)v(y)I$ is called a convolution kernel, and defines a mollifying kernel. For any element $\delta\psi \in \mathbb{F}$ one has then that $C_\kappa \delta\psi = \rho \star \delta\psi$, i.e. C_κ is the convolution operator by ρ . Similarly, for any element $p \in \mathbb{F}^\times$ one has that $C_\kappa^t p = \rho^t \star p$, where ρ^t is the reflected ρ (for all $v \in T$, $\rho^t(v) := \rho(-v)$). Due to the compact support of ρ , the κ is indeed properly supported. Moreover, by construction, it is flavor sector preserving.

With the notion of mollifying kernel, one can define the Wilsonian regularization (UV frequency cutoff) also over generic manifolds. Namely, in definition 23, one sets $\mathcal{R} = C_\kappa$ for some mollifying kernel κ . In that case, we use the abbreviation $\mathbf{M}_{\hbar, \psi_0, \kappa, \delta\psi_T}$ for $\mathbf{M}_{\hbar, \psi_0, \mathcal{R}, \delta\psi_T} = \mathbf{M}_{\hbar, \psi_0, C_\kappa, \delta\psi_T}$. It is seen that the regularized MDS equation is the analogy of the unregularized MDS equation (10), but with a smoothing appearing in it.

Remark 27. The following observation is useful for constructing concrete solutions of the regularized MDS equation. Let $\mathcal{R} : \mathbb{F} \rightarrow \mathbb{F}$ be a continuous linear operator (typically, a smoothing operator C_κ in our example). Then, it can be uniquely extended as a continuous grading preserving algebra derivation $\mathcal{R} : \mathcal{T}(\mathbb{F}) \rightarrow \mathcal{T}(\mathbb{F})$ of the unital associative topological graded algebra $\mathcal{T}(\mathbb{F})$ via requiring the annihilation of unity ($\mathcal{R}\mathbb{1} = 0$), the preservation of the space of n -tensors ($n \in \mathbb{N}_0$), the Leibniz rule over tensor product, and coincidence with \mathcal{R} on the one-vectors. If \mathcal{R} is also preserving flavor sectors of \mathbb{F} , then it can be restricted to $A(\mathbb{F}_{(\mathbb{C})}) \subset \mathcal{T}(\mathbb{F}_{(\mathbb{C})})$ as an algebra derivation. Similarly, the topological transpose operator $\mathcal{R}^* : \mathbb{F}^* \rightarrow \mathbb{F}^*$ extends as a continuous linear operator $\mathcal{R}^* : \mathcal{T}_a(\mathbb{F}^*) \rightarrow \mathcal{T}_a(\mathbb{F}^*)$. Assume moreover, that the pertinent operator \mathcal{R} on \mathbb{F} has a formal transpose $\mathcal{R}^t : \mathbb{F}_T^\times \rightarrow \mathbb{F}_T^\times$. Then, for the same reason it extends uniquely to $\mathcal{T}_a(\mathbb{F}_T^\times)$ in the above manner, and as $(\mathcal{R}^t)^*$ to $\mathcal{T}((\mathbb{F}_T^\times)^*)$, and thus also to $A((\mathbb{F}_T^\times(\mathbb{C}))^*) \subset \mathcal{T}((\mathbb{F}_T^\times(\mathbb{C}))^*)$, if \mathcal{R} was flavor sector preserving. The operator $(\mathcal{R}^t)^*$

will not be distinguished in notation from \mathcal{R} , since the former is the distributional extension of the latter. Similarly, \mathcal{R}^t will in general be denoted by \mathcal{R}^* , since the latter is the distributional extension of the former.

Remark 28. Use the assumptions of remark 20, and let κ be a mollifying kernel. In that case, the κ -regularized fundamental solution $\frac{1}{2}C_\kappa K_{\psi_0}^{(2)}$ resides in $\overset{2}{\otimes} \mathbb{F}$, and it is compatible with the permutation symmetry of the field algebra $A(\mathbb{F}_{(\mathbb{C})})$. Let the base manifold be affine, and let κ be specifically a convolution kernel by a symmetric test function. Then, $\frac{1}{2}C_\kappa K_{\psi_0}^{(2)}$ satisfies $\forall \delta \psi_T \in \mathbb{F}_T : \iota_{(\mathbb{E}_{\psi_0} | \delta \psi_T)} \frac{1}{2}C_\kappa K_{\psi_0}^{(2)} = L_{C_\kappa \delta \psi_T} \mathbb{1}$. Define the element $K_{\psi_0, \kappa} := (0, 0, i\hbar \frac{1}{2}C_\kappa K_{\psi_0}^{(2)}, 0, 0, \dots) \in A(\mathbb{F}_{(\mathbb{C})})$, called to be the smoothed connected correlator. Define the smoothed correlator with the ansatz $G_{\psi_0, \kappa} := \exp(K_{\psi_0, \kappa}) \in A(\mathbb{F}_{(\mathbb{C})})$. Then, $(\psi_0, G_{\psi_0, \kappa}) \in F \times A(\mathbb{F}_{(\mathbb{C})})$ solves the κ -regularized MDS equation (15). In order to see this, one merely needs to repeat the proof of remark 20. (The object $\frac{1}{2}C_\kappa K_{\psi_0}^{(2)}$ e.g. for a wave or Klein–Gordon model over Minkowski spacetime, would correspond to the smoothed Feynman propagator.)

It is seen that in the above definition the trick is that although the fundamental solution $K_{\psi_0}^{(2)} \in \mathcal{L}(\mathbb{F}_T^\times, \mathbb{F}) \subset (\mathbb{F}_T^\times)^*$ is merely defined in the distributional sense, but its κ -regularized version $\frac{1}{2}C_\kappa K_{\psi_0}^{(2)}$ sits in the space of smooth field correlators $\mathbb{F} \otimes \mathbb{F}$. Therefore, free theories of such kind will have smooth solutions of the κ -regularized MDS equation, and one does not need to go to the realm of distributional sense MDS equation, which is not applicable to interacting models. The replacement of the unregularized MDS operator with the κ -regularized MDS operator is called *regularization*. A further sanity check on the presented Wilsonian regularization scheme of the MDS equation is the fact that such an equation would always have formal perturbative solutions if the field algebra were $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$, and the base manifold \mathcal{M} were affine. This can be seen to be an immediate consequence of the Malgrange–Ehrenpreis surjectivity theorem ([27] chapitre I.1 théorème 1 and [28] chapter 6 theorem 10), but will be expanded in a different paper.

Having a rigorous and generally covariant formulation of Wilsonian regularization at hand, it is natural to ask the question whether it is possible to formulate Wilsonian renormalization using that. The answer is affirmative, and is addressed in appendix B.

5. An existence condition for regularized MDS solutions

In this section, we present an existence condition for the solutions of the Wilsonian regularized MDS equation.

Remark 29. The followings spell out some facts about the topology of \mathbb{F} and \mathbb{F}_T .

- (a) Assume that the base manifold \mathcal{M} underlying a concrete classical field theory is compact (with or without boundary, and if with boundary, we assume the cone condition). This is a realistic assumption for conformally invariant models, as for those, the theory can be reformulated on the compact manifold with boundary, underlying the conformally compactified spacetime.
- (b) With the assumption as above, rather obviously, the space \mathbb{F}_T shall become also an NF space, similarly to \mathbb{F} . That is, \mathbb{F}_T becomes metrizable with all of its benefits: its topology will be sequential, and separately continuous multilinear maps from it will become jointly continuous. (Recall that if \mathcal{M} is compact then, either $\mathbb{F}_T = \mathbb{F}$, or \mathbb{F}_T may be chosen to be the closed subspace of \mathbb{F} consisting of fields vanishing at $\partial\mathcal{M}$ together with all of their derivatives. Its natural topology will become an \mathcal{E} function topology instead of \mathcal{D} type.)

- (c) It is also an elementary fact that over a compact base manifold \mathcal{M} , the space \mathbb{F} and \mathbb{F}_T will not only be nuclear Fréchet, but also will admit continuous norms instead of merely continuous seminorms. By means of [22]-remark 7, then they become nuclear Fréchet spaces with a countable increasing system of topology defining Hilbertian norms. Since these norms are simply Sobolev norms, they are Gel'fand compatible (see [22]-remark 7 and [22]-remark 8), therefore by means of [22]-remark 7 they become NF spaces with the countably Hilbert (CH) property. Recall that if \mathcal{H} is a CH type NF space, then there exists a countable family

$$H_0 \supset H_1 \supset \dots \supset H_n \supset \dots$$

of TVS, such that $\mathcal{H} = \bigcap_{n \in \mathbb{N}_0} H_n$ and is dense in all of the spaces H_n ($n \in \mathbb{N}_0$), moreover their topologies are gradually strictly strengthening

$$\tau_0|_{\mathcal{H}} \subset \tau_1|_{\mathcal{H}} \subset \dots \subset \tau_n|_{\mathcal{H}} \subset \dots,$$

and all of their topologies are complete and generated by a Hilbertian scalar product (that is, for each $n \in \mathbb{N}_0$, the space H_n can be taken to be a Hilbert space), and for all $n \in \mathbb{N}_0$ there exists an integer $m \geq 1$ such that the inclusion maps $i_{n+m,n} : H_{n+m} \rightarrow H_n$ are nuclear. (Specially, the spaces H_0, H_1, \dots may be chosen such that all adjacent inclusion maps are nuclear.) The respective topology generating Hilbertian norms thus form an increasing system

$$\|\cdot\|_0 \leq C_{0,1} \|\cdot\|_1 \leq \dots \leq C_{n-1,n} \|\cdot\|_n \leq \dots \quad (\text{for some } C_{0,1}, \dots, C_{n-1,n}, \dots \in \mathbb{R}^+).$$

The corresponding Hilbertian scalar products shall be denoted by

$$\langle \cdot, \cdot \rangle_0, \langle \cdot, \cdot \rangle_1, \dots, \langle \cdot, \cdot \rangle_n, \dots$$

The proposed existence theorem will hinge on the fact that on the field algebra $A(\mathbb{F}_{(\mathbb{C})})$ it is possible to naturally define a reasonable topology, somewhat stronger than the Tychonoff topology, such that it preserves the NF property coming from \mathbb{F} , and if present, the eventual CH property of \mathbb{F} as well.

Remark 30. We recall some findings on topologies of the tensor algebra of \mathbb{F} [35].

- (a) Dubin and Hennings in their work ([35] chapter 3.1) introduces the notion of tensor algebra topology of the following kind. Let \mathbb{F} be a nuclear Fréchet space. Then, they define the vector space $\mathcal{T}(\mathbb{F}, \lambda, \mathfrak{p})$, where λ is some topological subspace of the space of $\mathbb{N}_0 \rightarrow \mathbb{C}$ sequences (it is a so-called *Köthe echelon space*), and \mathfrak{p} is a family of Hilbertian seminorms on \mathbb{F} defining its NF topology (recall that multiple seminorm families \mathfrak{p} can define the same topology on \mathbb{F}). As a vector space, it is defined as follows:

$$\mathcal{T}(\mathbb{F}, \lambda, \mathfrak{p}) := \left\{ G \in \bigoplus_{n=0}^{\infty} \overset{n}{\otimes} \mathbb{F} \mid \forall \|\cdot\| \in \mathfrak{p} : (\|G^{(n)}\|^{\otimes n})_{n \in \mathbb{N}_0} \in \lambda \right\},$$

where for all topology defining Hilbertian seminorms $\|\cdot\| \in \mathfrak{p}$ on \mathbb{F} , the symbol $\|\cdot\|^{\otimes n}$ denotes the n -fold cross norm over $\overset{n}{\otimes} \mathbb{F}$ originating from $\|\cdot\|$, which is then also a Hilbertian seminorm ([35] chapter 3.1). The locally convex vector topology on $\mathcal{T}(\mathbb{F}, \lambda, \mathfrak{p})$ is defined by the system of seminorms $(G \mapsto \sum_{n=0}^{\infty} |u_n| \|G^{(n)}\|^{\otimes n})_{u \in \lambda^\times, \|\cdot\| \in \mathfrak{p}}$, where λ^\times denotes the so-called Köthe dual (which is, under, mild conditions, the strong topological dual) of the sequence space λ .

(a) Notable Köthe echelon spaces include ([35] chapter 2.4):

$$\phi := \{(u_n)_{n \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} \mid \exists m \in \mathbb{N}_0 : \forall n > m : u_n = 0\} \quad (\text{terminating sequences}),$$

$$h := \{(u_n)_{n \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} \mid \forall m \in \mathbb{N}_0 : \exists C_m \in \mathbb{R}^+ : \forall n \in \mathbb{N}_0 : 2^{nm} |u_n| \leq C_m\},$$

$$h' := \{(u_n)_{n \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} \mid \exists c \in \mathbb{N} : \exists m_c \in \mathbb{N}_0 : \forall n \in \mathbb{N}_0 : |u_n| \leq c 2^{m_c n}\},$$

$$\omega := \{(u_n)_{n \in \mathbb{N}_0} \in \mathbb{C}^{\mathbb{N}_0} \mid \text{any } u\} \quad (\text{all sequences}).$$

These are understood with their so-called normal topologies ([35] chapter 2.1). The space ω is the space of all sequences with the natural Tychonoff topology, the space ϕ is the space of finitely terminating sequences with the natural locally convex direct sum topology, whereas the space h is known to be topologically isomorphic to the space $H(\mathbb{C})$ of entire complex functions ([35], p 978). All of them are Hausdorff locally convex TVS, and the pairs (ϕ, ω) , (h, h') , (h', h) , (ω, ϕ) are strong dual to each-other, and Köthe duals to each-other. The spaces ϕ, h, ω are metrizable, and thus Fréchet. The spaces ϕ, h, ω have the so-called ‘h’ property, because of which they are nuclear ([35] chapter 2.5). Therefore, ϕ, h, ω are NF, and h' is DNF. Specially, the space h also has the countably Hilbert property.

- (c) It is shown in [35] chapter 3.3 that specially for the sequence spaces $\lambda = \phi$ or $\lambda = h$ or $\lambda = \omega$, the tensor algebra $\mathcal{T}(\mathbb{F}, \lambda, \mathfrak{p})$ is independent of the choice of the representant \mathfrak{p} of the topology defining Hilbertian seminorms on \mathbb{F} , thus one may write merely $\mathcal{T}(\mathbb{F}, \lambda)$ instead. Moreover, they inherit the NF property of \mathbb{F} , making $\mathcal{T}(\mathbb{F}, \lambda)$ a unital associative algebra with jointly continuous multiplication, with NF topology.
- (d) It is rather easy to see that $\mathcal{T}(\mathbb{F}, \phi)$ is simply the algebraic tensor algebra $\mathcal{T}_a(\mathbb{F})$ with its natural locally convex direct sum topology, $\mathcal{T}(\mathbb{F}, \omega)$ is simply the Tychonoff tensor algebra $\mathcal{T}(\mathbb{F})$ with its natural Tychonoff topology.
- (e) For λ of the above types, it is shown in [35] chapter 3.3 that the topology defining seminorms on $\mathcal{T}(\mathbb{F}, \lambda)$ may be chosen to be Hilbertian seminorms

$$G \mapsto \sqrt{\sum_{n=0}^{\infty} |u_n|^2 (\|G^{(n)}\|^{\otimes n})^2} \quad (u \in \lambda^\times, \|\cdot\| \in \mathfrak{p}), \tag{16}$$

where \mathfrak{p} is a representant of a topology defining family of Hilbertian seminorms on \mathbb{F} . From this, it is explicitly seen that whenever \mathbb{F} is an NF space admitting a continuous Hilbertian norm, then the NF space $\mathcal{T}(\mathbb{F}, h)$ also admits a continuous Hilbertian norm.

- (f) The explicit form of a representant of a topology defining countable family of increasing Hilbertian seminorms on $\mathcal{T}(\mathbb{F}, h)$, encoding its NF topology, can be given by:

$$G \mapsto \sqrt{\sum_{n=0}^{\infty} 2^{mn} (\|G^{(n)}\|_m^{\otimes n})^2} \quad (m \in \mathbb{N}_0), \tag{17}$$

where $\|\cdot\|_0, \|\cdot\|_1, \dots$ is a representant of a topology defining countable system of increasing Hilbertian seminorms on \mathbb{F} , defining its NF topology. (The formula is the consequence of [35] proposition 3.7, but is also explicitly used in [36].) From the above formula it is seen that whenever \mathbb{F} is of CH type, then $\mathcal{T}(\mathbb{F}, h)$ is also of CH type (see also [22]-remark 7). We will use the abbreviation $\mathcal{T}_h(\mathbb{F}) := \mathcal{T}(\mathbb{F}, h)$, and will call it the analytic tensor algebra of \mathbb{F} , since the topology defining sequence space h is isomorphic to the space of entire functions $H(\mathbb{C})$.

- (g) From equation (17) it is trivially read off that the counit map $b : \mathcal{T}_h(\mathbb{F}) \rightarrow \mathbb{R}, G \mapsto bG := G^{(0)}$ is continuous. Therefore, the corresponding projection operator $\mathbb{1}b$ onto the scalar sector and its complement $I - \mathbb{1}b$ is also continuous. Moreover, the pertinent complementing projection operators $\mathbb{1}b$ and $I - \mathbb{1}b$ are orthogonal projections with respect to the representants of Hilbertian sesquilinear forms from equation (17), and $bG = \langle \mathbb{1}, G \rangle_m$ holds for all $G \in \mathcal{T}_h(\mathbb{F})$ and all $m \in \mathbb{N}_0$.
- (h) From the equation (17) form of the Hilbertian seminorms on $\mathcal{T}_h(\mathbb{F})$ it is seen that this representant family has the property that whenever \mathbb{F} is CH type NF space, then if its representant Hilbertian norm family is chosen to be such that the adjacent norms are nuclear against each-other, the adjacent Hilbertian norms defined by equation (17) are also nuclear against each-other. Similarly, whenever the adjacent norms on \mathbb{F} are Hilbert–Schmidt against each-other, then the adjacent norms equation (17) are also Hilbert–Schmidt against each-other.

In order to state our existence condition for the solutions of the regularized MDS equation, we will need to reconsider the space of field correlators to be based on $\mathcal{T}_h(\mathbb{F})$ with the analytic topology, and not on $\mathcal{T}(\mathbb{F})$ with the Tychonoff direct sum topology. The reason is that for the construction to work, we need the eventual CH property of \mathbb{F} to be inherited by its tensor algebra. Therefore, from this point on, the field algebra $A(\mathbb{F}_{(\mathbb{C})})$ will be defined to be the appropriately symmetrized subspace of $\mathcal{T}_h(\mathbb{F}_{(\mathbb{C})})$ instead of $\mathcal{T}(\mathbb{F}_{(\mathbb{C})})$ (see again remark 6 and definition 7 for the technical construction of the symmetrized algebra).

Remark 31. It is worth to verify that the tensor algebra topology on the new field algebra $A(\mathbb{F}_{(\mathbb{C})})$, inherited from $\mathcal{T}_h(\mathbb{F}_{(\mathbb{C})})$, is not overly strict. For instance, one would like a typical solution of the regularized MDS equation for a non-interacting theory to be not excluded from our new, smaller field algebra $A(\mathbb{F}_{(\mathbb{C})})$. By recalling remark 28, one sees from equation (17) that the pertinent existent solution of the regularized MDS equation for a non-interacting theory indeed resides in the new, stricter field algebra as well.

In order to state an existence condition, we shall assume, like in remark 29, that the base manifold \mathcal{M} under the concrete theory is compact (with or without boundary, and if with boundary, we assume the cone condition, so that Sobolev and Maurin theorems hold, see [22]-remark 8). As stated before, this is a realistic assumption in a conformally invariant theory, in which case the theory can be re-defined over a compact manifold with boundary (the conformal compactification of the would-be-spacetime).

Remark 32. Assume that the base manifold \mathcal{M} of the model is compact and its boundary, if not empty, has the cone condition. Then, the followings hold.

- (a) With such assumption, \mathbb{F} and \mathbb{F}_T become countably Hilbert type NF spaces, which is then inherited by $\mathcal{T}_h(\mathbb{F}_{(\mathbb{C})})$, and thus by the field algebra $A(\mathbb{F}_{(\mathbb{C})})$. From now on, let us use the abbreviation $\mathcal{H} := A(\mathbb{F}_{(\mathbb{C})})$.
- (b) In its original definition, the regularized MDS operator was a separately continuous bilinear map $\mathbf{M}_{\psi_0} : \mathcal{H} \times \mathbb{F}_T \rightarrow \mathcal{H}, (G, \delta\psi_T) \mapsto \mathbf{M}_{\psi_0, \delta\psi_T} G$ (see also the original definition equation (14), we suppress \hbar and the fixed mollifying kernel in the notation in this chapter). Due to the compactness assumption on \mathcal{M} , the space \mathbb{F}_T becomes also metrizable, therefore the map \mathbf{M}_{ψ_0} becomes jointly continuous ([22]-remark 3). Therefore, the regularized MDS operator, may be also viewed as a continuous linear map $\mathbf{M}_{\psi_0} : \mathcal{H} \otimes \mathbb{F}_T \rightarrow \mathcal{H}$.
- (c) Due to our compactness assumption on the base manifold \mathcal{M} , both \mathbb{F}_T and \mathcal{H} became countably Hilbert NF spaces, which technically means that in both spaces as well as on their tensor product, the properties remark 29(c) hold. Denote an associated chain of

Hilbert spaces subordinate to \mathcal{H} by H_0, H_1, \dots , their Hilbertian norms by $\|\cdot\|_0, \|\cdot\|_1, \dots$ and their Hilbertian scalar products by $\langle \cdot, \cdot \rangle_0, \langle \cdot, \cdot \rangle_1, \dots$. Similarly, for \mathbb{F}_T denote by F_0, F_1, \dots an associated chain of Hilbert spaces, their corresponding Hilbertian norms by $\|\cdot\|_0^F, \|\cdot\|_1^F, \dots$, and their Hilbertian scalar products by $\langle \cdot, \cdot \rangle_0^F, \langle \cdot, \cdot \rangle_1^F, \dots$. The associated chain of Hilbert spaces subordinate to $\mathcal{H} \otimes \mathbb{F}_T$ can be taken to be the Hilbert–Schmidt tensor product of the spaces $H_m \otimes_{\text{HS}} F_n$ ($m, n \in \mathbb{N}_0$), with their canonical Hilbertian cross-norms and crossed Hilbertian scalar products. (Eventually, a subfamily of this, with strictly growing norms may also be considered instead.)

- (d) Because of the nuclearity of the spaces \mathcal{H} and \mathbb{F}_T , each Hilbertian norm in the above chains will have a stronger norm in the chain for which the embedding map becomes Hilbert–Schmidt, and eventually becomes nuclear, for large enough norms in the chain. (This can also be seen less abstractly on our concrete spaces as a consequence of the Maurin embedding theorem [22] remark 8(b).)
- (e) The continuity of the linear map $\mathbf{M}_{\psi_0} : \mathcal{H} \otimes \mathbb{F}_T \rightarrow \mathcal{H}$ in terms of these Hilbert space chains means that

$$\begin{aligned} \forall k \in \mathbb{N}_0 : \exists m_k, n_k \in \mathbb{N}_0 : \exists C_{k,m_k,n_k} \in \mathbb{R}^+ : \\ \forall (G, \delta\psi_T) \in G \times \mathbb{F}_T : \quad \|\mathbf{M}_{\psi_0}(G \otimes \delta\psi_T)\|_k \leq C_{k,m_k,n_k} \|G\|_{m_k} \|\delta\psi_T\|_{n_k} \end{aligned} \tag{18}$$

holds. Since the norms were ordered, the above identity implies that once it holds, it holds for all $m \geq m_k$ and $n \geq n_k$ as well with some constants $C_{k,m,n} \in \mathbb{R}^+$. That is, the map \mathbf{M}_{ψ_0} is a continuous linear map $(\mathcal{H} \otimes \mathbb{F}_T) \cap (H_m \otimes_{\text{HS}} F_n) \rightarrow \mathcal{H} \cap H_k$ for large enough indices $m, n \in \mathbb{N}_0$, given the index $k \in \mathbb{N}_0$. The continuous extension of the map \mathbf{M}_{ψ_0} will be denoted by the same symbol for brevity, and it is then a continuous linear map $\mathbf{M}_{\psi_0} : H_m \otimes_{\text{HS}} F_n \rightarrow H_k$, for such indices.

- (f) By means of (d), between distant enough indices, the inclusion maps $H_m \supset \dots \supset H_{m'}$ ($m < m'$) and $F_n \supset \dots \supset F_{n'}$ ($n < n'$) become Hilbert–Schmidt, and eventually become nuclear. Therefore, given $k \in \mathbb{N}_0$, for large enough indices $m, n \in \mathbb{N}_0$ the map $\mathbf{M}_{\psi_0} : H_m \otimes_{\text{HS}} F_n \rightarrow H_k$ becomes Hilbert–Schmidt, and eventually becomes nuclear. (In concrete spaces, Maurin embedding theorem gives the concrete index bounds, see [22]-remark 8(b).)
- (g) As a particular case of the above statement, for all large enough indices $m, n \in \mathbb{N}_0$ one has that the linear map $\mathbf{M}_{\psi_0} : H_m \otimes_{\text{HS}} F_n \rightarrow H_0$ is Hilbert–Schmidt. The adjoint of this map $\mathbf{M}_{\psi_0}^\dagger : H_0 \rightarrow H_m \otimes_{\text{HS}} F_n$ is then also Hilbert–Schmidt. Therefore, the operator $\mathbf{M}_{\psi_0}^\dagger \mathbf{M}_{\psi_0} : H_m \otimes_{\text{HS}} F_n \rightarrow H_m \otimes_{\text{HS}} F_n$, becomes a positive nuclear (trace class) operator.
- (h) Fix a complete orthonormal basis $(e_i)_{i \in I}$ in F_n (since F_n is separable, one may set $I \equiv \mathbb{N}$). Then, for all $G \in H_m$ the estimate

$$B(G, G) := \sum_{i \in I} \left\langle (G \otimes_{\text{HS}} e_i), \mathbf{M}_{\psi_0}^\dagger \mathbf{M}_{\psi_0} (G \otimes_{\text{HS}} e_i) \right\rangle_{H_m \otimes_{\text{HS}} F_n} < \infty \tag{19}$$

is valid. That is because of the Hilbert–Schmidt property of the map $\mathbf{M}_{\psi_0} : H_m \otimes_{\text{HS}} F_n \rightarrow H_0$. Namely, for some (and therefore: for any) complete orthonormal basis $(g_j)_{j \in J}$ in H_m , one has that $\sum_{j \in J} \sum_{i \in I} \|\mathbf{M}_{\psi_0}(g_j \otimes_{\text{HS}} e_i)\|_{H_m \otimes_{\text{HS}} F_n}^2 < \infty$ holds (one may set $J \equiv \mathbb{N}$ as well, due to the separability of H_m). Taking specially an orthonormal basis $(g_j)_{j \in J}$ in H_m , such that one of its elements is $G/\|G\|_{H_m}$, one infers that indeed the estimate equation (19) holds.

- (i) Given $G \in H_m$, the corresponding expression equation (19) is independent of the chosen complete orthonormal basis $(e_i)_{i \in I}$ in F_n . That is because for a Hilbert–Schmidt operator

A and an unitary operator U in a Hilbert space, one has that the Hilbert–Schmidt norm of A and U^*AU is the same.

- (j) Due to the Hilbert–Schmidt property of $\mathbf{M}_{\psi_0} : H_m \otimes_{\text{HS}} F_n \rightarrow H_0$, the quadratic form $H_m \rightarrow \mathbb{C}, G \mapsto B(G, G)$ is continuous, and therefore by the polarization formula it gives rise to a corresponding continuous sesquilinear form $H_m \times H_m \rightarrow \mathbb{C}, (G_1, G_2) \mapsto B(G_1, G_2)$. Therefore, by Riesz representation theorem, there is a corresponding unique continuous linear map $\hat{\mathbf{M}}_{\psi_0}^2 : H_m \rightarrow H_m$, such that for all $G_1, G_2 \in H_m$, the identity $B(G_1, G_2) = \langle G_1, \hat{\mathbf{M}}_{\psi_0}^2 G_2 \rangle_{H_m}$ holds.
- (k) Due to the positive semidefiniteness of B , the map $\hat{\mathbf{M}}_{\psi_0}^2$ is a positive operator. Moreover, due to the Hilbert–Schmidt property of $\mathbf{M}_{\psi_0} : H_m \otimes_{\text{HS}} F_n \rightarrow H_0$, the map $\hat{\mathbf{M}}_{\psi_0}^2$ is nuclear (trace class). The nuclear (trace) norm of $\hat{\mathbf{M}}_{\psi_0}^2 : H_m \rightarrow H_m$, by construction, equals to the Hilbert–Schmidt norm of $\mathbf{M}_{\psi_0} : H_m \otimes_{\text{HS}} F_n \rightarrow H_0$. One can see that the operator $\hat{\mathbf{M}}_{\psi_0}^2$ is simply the absolute value squared version of the MDS operator, with its \mathbb{F}_T variable traced out.
- (l) It is obvious from the construction that $\bigcap_{\delta\psi_T \in \mathbb{F}_T} \text{Ker}(\mathbf{M}_{\psi_0, \delta\psi_T}) = \text{Ker}(\hat{\mathbf{M}}_{\psi_0}^2)$.

If the reference field $\psi_0 \in F$ was chosen to be such that it satisfies the EL equation (8), then one has that $\mathbf{M}_{\psi_0, \delta\psi_T} \mathbb{1} = -iL_{\delta\psi_T} \mathbb{1} \ (\forall \delta\psi_T \in \mathbb{F}_T)$. Because of that, in this situation, $b\hat{\mathbf{M}}_{\psi_0}^2 \mathbb{1} = \langle \mathbb{1}, \hat{\mathbf{M}}_{\psi_0}^2 \mathbb{1} \rangle_m > 0$, and therefore $\mathbb{1} \notin \text{Ker}(\hat{\mathbf{M}}_{\psi_0}^2)$. Thus, generally, the trivial correlator $\mathbb{1}$ cannot be a solution of the regularized MDS equation. One could still aim to find a projection of $\mathbb{1}$ which (up to normalization) satisfies the regularized MDS equation. Let us denote the orthogonal projection onto $\text{Ker}(\hat{\mathbf{M}}_{\psi_0}^2)$ in H_m by P . Then, $\text{Ker}(\hat{\mathbf{M}}_{\psi_0}^2) = \text{Ran}(P)$. One can state the following theorem on $P\mathbb{1}$.

Theorem 33. *Let P denote the orthoprojection in H_m onto $\text{Ker}(\hat{\mathbf{M}}_{\psi_0}^2)$. Then, the following statements are equivalent.*

- (a) *The solution space of the regularized MDS equation in H_m is not empty.*
- (b) *One has that $P\mathbb{1} \neq 0$.*
- (c) *One has that $bP\mathbb{1} \neq 0$.*

Proof. By construction, the MDS equation has solutions in H_m if and only if $b \text{Ker}(\hat{\mathbf{M}}_{\psi_0}^2) \neq \{0\}$, i.e. if and only if there exists some $G \in H_m$, such that $bPG \neq 0$. (That is because $\text{Ker}(\hat{\mathbf{M}}_{\psi_0}^2) = \text{Ran}(P)$ and because remark 32(1).) However, the identity $bPG = \langle \mathbb{1}, PG \rangle_m = \langle P\mathbb{1}, G \rangle_m$ holds, because of remark 30(g), and because P was orthoprojection in H_m . Therefore, (a) \Leftrightarrow (b).

Moreover, one has that $\langle P\mathbb{1}, P\mathbb{1} \rangle_m = \langle \mathbb{1}, P\mathbb{1} \rangle_m = bP\mathbb{1}$, since P was an orthoprojection in H_m , and because of remark 30(g). Therefore, (b) \Leftrightarrow (c). □

It is seen that the orthoprojection P in H_m onto $\text{Ker}(\hat{\mathbf{M}}_{\psi_0}^2)$ plays an important role in the problematics of existence of MDS solutions. One can approximate P as below.

Theorem 34. *For all $T > 0$ parameter, which is not smaller than the operator norm of $\hat{\mathbf{M}}_{\psi_0}^2$, and with the notation $\mathcal{P} := I - T^{-1}\hat{\mathbf{M}}_{\psi_0}^2$, the operator sequence $k \mapsto \mathcal{P}^k$ converges strongly (pointwise) to P in H_m .*

Proof. The operator $\mathcal{P} = I - T^{-1}\hat{\mathbf{M}}_{\psi_0}^2$ is a positive continuous operator with spectrum in $[0, 1]$. Therefore, $k \mapsto \mathcal{P}^k$ is a monotonically decreasing sequence of such operators, bounded

from below by the zero operator. Therefore the sequence $k \mapsto \mathcal{P}^k$ converges strongly (pointwise). Since it converges strongly, it converges also weakly (i.e. matrix element-wise), and its weak limit equals to the strong limit. We evaluate its strong limit via evaluating its weak limit, below.

Take any $f, g \in H_m$, then there exists a unique complex valued bounded variation Radon measure $\mu_{\mathcal{P},f,g}$ over \mathbb{C} with $\text{supp}(\mu_{\mathcal{P},f,g}) \subset \text{Sp}(\mathcal{P}) \subset [0, 1]$, and $\langle f, \mathcal{P}g \rangle_m = \int_{\lambda \in [0,1]} \lambda \, d\mu_{\mathcal{P},f,g}(\lambda)$ holds. Moreover, for any non-negative integer k , one has that $\langle f, \mathcal{P}^k g \rangle_m = \int_{\lambda \in [0,1]} \lambda^k \, d\mu_{\mathcal{P},f,g}(\lambda)$ holds. One has that $\int_{\lambda \in [0,1]} \lambda^k \, d\mu_{\mathcal{P},f,g}(\lambda) = \int_{\lambda \in [0,1[} \lambda^k \, d\mu_{\mathcal{P},f,g}(\lambda) + \int_{\lambda \in \{1\}} \lambda^k \, d\mu_{\mathcal{P},f,g}(\lambda)$, where the second term equals to $\langle f, Pg \rangle_m$ by construction. The function $\lambda \mapsto \lambda^k$ converges to zero pointwise on $[0, 1[$, and is bounded by the constant 1 function which is $\mu_{\mathcal{P},f,g}$ absolute integrable on $[0, 1[$. Therefore, by Lebesgue's theorem of dominated convergence, the integral $\int_{\lambda \in [0,1[} \lambda^k \, d\mu_{\mathcal{P},f,g}(\lambda)$ tends to zero as a function of k . Therefore, $\langle f, \mathcal{P}^k g \rangle_m$ converges to $\langle f, Pg \rangle_m$ in k , i.e. \mathcal{P}^k converges weakly to P . □

By combining theorems 33 and 34, one can draw the following conclusion.

Corollary 35. *For all $T > 0$ parameter, which is not smaller than the operator norm of $\hat{\mathbf{M}}_{\psi_0}^2$, one has that the iteration*

$$G_0 := \mathbb{1}, \quad G_{k+1} := G_k - T^{-1} \hat{\mathbf{M}}_{\psi_0}^2 G_k \tag{20}$$

converges in H_m . Therefore, there exists the finite real number $\nu := \lim_{k \rightarrow \infty} bG_k \in \mathbb{R}$.

The MDS equation has solutions in H_m if and only if $\nu \neq 0$.

Moreover, if $\nu \neq 0$, then $\frac{1}{\nu} \lim_{k \rightarrow \infty} G_k$ is a solution of the MDS equation in H_m .

Proof. Clearly, by construction we have that $G_k = \mathcal{P}^k \mathbb{1}$ for all $k \in \mathbb{N}_0$, and we have just shown in theorem 34, that $k \mapsto \mathcal{P}^k$ converges strongly to P . Therefore, $\lim_{k \rightarrow \infty} G_k = P \mathbb{1}$, and $\nu = bP \mathbb{1}$. Applying then theorem 33, we get the stated result. □

Remark 36. The following identities are useful for technical evaluation.

- (a) The minimal factor $T > 0$, which can be used in the above existence test is the operator norm of $\hat{\mathbf{M}}_{\psi_0}^2$. The operator norms are generally hard to estimate. However, it can be estimated from above by the trace norm of $\hat{\mathbf{M}}_{\psi_0}^2$, or equivalently, by the Hilbert–Schmidt norm of \mathbf{M}_{ψ_0} , which are technically easier to evaluate.
- (b) It is a useful fact that the indicator sequence $k \mapsto \nu_k := bG_k \in \mathbb{R}$ consists of non-negative numbers, and is monotonically decreasing. That is because $\nu_k = bG_k = \langle \mathbb{1}, G_k \rangle_m = \langle \mathbb{1}, \mathcal{P}^k \mathbb{1} \rangle_m$. The operator sequence $k \mapsto \mathcal{P}^k$ consists of a sequence of positive operators, which are monotonically decreasing. Therefore $k \mapsto \nu_k$ inherits this property. Thus, it is enough to test whether the indicator sequence $k \mapsto \nu_k$ is bounded away from zero. Moreover, the scalar component bG_k of the approximants G_k start from 1, they do stay real, and they do not flip sign from positive to negative, and they monotonically decrease.
- (c) By means of corollary 35, for concrete models, evaluating whether the indicator ν is bounded away from zero, is expected to involve elaborate Sobolev estimates. The corollary, however, pinpoints a well defined point where one has to invoke these estimates, and therefore this can be considered as a useful existence test condition.

- (d) Since corollary 35 is a necessary and sufficient condition, and not merely a sufficient condition, one may also use it in the reverse direction. Namely, if for a concrete model the regularized MDS equation had any solutions, then the iteration scheme of corollary 35 is guaranteed to be good enough to be convergent, and to produce one particular MDS solution. This is a useful piece of information, even without actually performing the above Sobolev estimates.

6. Concluding remarks

In the QFT literature, the MDS equation on the field correlators is known to be a differential reformulation of the Feynman integral formalism. In this paper it is shown that the MDS equation can be cast into a particular presentation, in which the involved function spaces and operators are perfectly well defined, regardless of a fixed background spacetime metric, or causal structure, or signature. Moreover, the Wilsonian regularized version of the construction is also shown to be well defined in such a generally covariant setting. A necessary and sufficient condition is proved for the solution space of the regularized MDS equation to be nonempty, for conformally invariant Lagrangians. The pertinent theorem is constructive in the sense that it provides an iterative algorithm to obtain an MDS solution. The algorithm is guaranteed to converge whenever the solution space is nonempty, and could be eventually used for a lattice QFT-like nonperturbative numerical solution scheme, capable of working in the original metric signature.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Continuity properties of the Euler–Lagrange functional

Our presentation of the MDS operator heavily relies on the precise definition of the EL functional of a classical field theory. In order to pin down the topological properties of the involved spaces and precise continuity property of their operators, we need to briefly recall the standard

variational formulation of a classical field theory. For the sake of simplified treatment, we will use neither the jet formalism, nor the theory of general connections over fiber bundles [37, 38]. We will rather concentrate on the TVS theory side [22], i.e. we keep the differential geometric treatment to a reasonable appropriate minimum. Let \mathcal{M} denote throughout the paper a finite dimensional real smooth orientable and oriented manifold, and let $m := \dim(\mathcal{M})$. It may be compact or noncompact, and may be with or without boundary (if with boundary, we assume the cone condition for it, so that locally the Sobolev and Maurin embedding theorems hold, see also [22]-section 9). The manifold \mathcal{M} is meant to model the spacetime manifold, or eventually, the compact manifold with boundary underlying the conformal compactification (Penrose diagram) of a spacetime. Let $V(\mathcal{M})$ be some real vector bundle over \mathcal{M} with finite dimensional fibers. Denote, as usual, by $\Gamma(\cdot)$ the space of smooth sections. In particular, the space $\Gamma(V(\mathcal{M}))$ denotes the real vector space of smooth sections of $V(\mathcal{M})$ (these are meant to model the matter fields). The covariant derivation operators over $\Gamma(V(\mathcal{M}))$ (which are meant to model the mediator fields) form an affine space with subordinate vector space $\Gamma(T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M}))$, as it is common knowledge. More particularly, covariant derivation operators can be considered as sections of an affine bundle over \mathcal{M} , which we will denote by $DV(\mathcal{M})$, and then a covariant derivation is an element of $\Gamma(DV(\mathcal{M}))$. The bundle $DV(\mathcal{M})$ is an affine bundle with subordinate vector bundle $T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M})$, the sections of which are the difference tensors (or gauge potentials, in field theory). As usual, the symbols \otimes shall denote pointwise tensor product of vector bundles over the same base, whereas \oplus will denote pointwise direct sum of vector bundles over the same base. The fibered product (pointwise Cartesian product, or Whitney sum) of two generic fiber bundle over the same base shall be denoted by $V(\mathcal{M}) \times_w U(\mathcal{M})$, and their elements by $(v, u)_w$ in order to indicate that these are pairs of fields, over the same base points. The subscript w is used in order to distinguish the above from sections (v, u) of the Cartesian product bundle $V(\mathcal{M}) \times U(\mathcal{M}')$, which would be a bundle over the product manifold $\mathcal{M} \times \mathcal{M}'$ (eventually $\mathcal{M}' = \mathcal{M}$).

Definition 37. Let \mathcal{M} , $V(\mathcal{M})$ and $DV(\mathcal{M})$ as above.

A **Lagrange form** is a base point preserving, smooth fiber bundle homomorphism

$$L : V(\mathcal{M}) \oplus T^*(\mathcal{M}) \otimes V(\mathcal{M}) \oplus T^*(\mathcal{M}) \wedge T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M}) \longrightarrow \wedge^m T^*(\mathcal{M}).$$

By construction, a Lagrange form takes some sections

$$\begin{aligned} v &\in \Gamma(V(\mathcal{M})), & Dv &\in \Gamma(T^*(\mathcal{M}) \otimes V(\mathcal{M})), \\ P &\in \Gamma(T^*(\mathcal{M}) \wedge T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M})) \end{aligned}$$

into a maximal form field $L(v, Dv, P) \in \Gamma(\wedge^m T^*(\mathcal{M}))$.

An element $(v, \nabla)_w \in \Gamma(V(\mathcal{M}) \times_w DV(\mathcal{M}))$ is called a **field configuration**. The field configurations form an affine space over the real vector space $\Gamma(V(\mathcal{M}) \oplus T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M}))$. An element $(\delta v, \delta C)_w$ from that space is called a **field variation**.

The map

$$\Gamma(V(\mathcal{M}) \times_w DV(\mathcal{M})) \longrightarrow \Gamma(\wedge^m T^*(\mathcal{M})), \quad (v, \nabla)_w \longmapsto L(v, \nabla v, P(\nabla))$$

is called the **Lagrangian expression**, where ∇v is the covariant derivative of the section v , and $P(\nabla)$ is the curvature tensor of ∇ . (Note that the expression $(v, \nabla v, P(\nabla))_w$ encodes the same information as the first jet of a field configuration $(v, \nabla)_w$, but we do not intend to use the jet formalism in the present paper.)

Given a Lagrange form L , its *action functional* is the real Radon measure valued map

$$S^L : \Gamma(V(\mathcal{M}) \times_w DV(\mathcal{M})) \rightarrow \text{Rad}(\mathcal{M}, \mathbb{R}), \quad (v, \nabla)_w \mapsto S^L(v, \nabla)_w$$

where on compact subsets $\mathcal{K} \subset \mathcal{M}$ the definition is $S^L_{\mathcal{K}}(v, \nabla)_w := \int_{\mathcal{K}} L(v, \nabla v, P(\nabla))$, i.e. the action functional is the Radon measure defined by local integrals of the Lagrangian expression, as usual.

We use the shorthand notation $F := \Gamma(V(\mathcal{M}) \times_w DV(\mathcal{M}))$ for the space of field configurations, moreover $\mathbb{F} := \Gamma(V(\mathcal{M}) \oplus T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M}))$ for the space of field variations. The space F is an affine space over the real vector space \mathbb{F} . The real vector space \mathbb{F} may be naturally endowed with the standard \mathcal{E} smooth function topology. (The \mathcal{E} topology is defined by the family of arbitrary order Sobolev norms of over compact patches of \mathcal{M} .) With this topology \mathbb{F} and thus F become Hausdorff locally convex topological vector and affine spaces, respectively. It is also common knowledge ([22]-remark 2), that \mathbb{F} with the \mathcal{E} topology becomes a *nuclear Fréchet space*, which fact will be an important detail in the QFT construction.

The real vector space of real valued Radon measures $\text{Rad}(\mathcal{M}, \mathbb{R})$ can be also naturally endowed with a topology, defined by compact setwise total variations as family of seminorms, or equivalently, by the convergence of measure sequences over compact sets. With this, $\text{Rad}(\mathcal{M}, \mathbb{R})$ becomes a Hausdorff locally convex topological vector space.

Remark 38. It is not true in general that the continuity of a map between topological spaces is equivalent to its sequential continuity. It is common knowledge, however, that metrizable topological spaces are *sequential* ([22]-remark 5), i.e. their topology is completely characterized by the convergence of sequences. Since \mathbb{F} is Fréchet space, by construction its topology is metrizable in a translationally invariant way, and therefore also is the topology of F . In particular, a map $S : F \rightarrow Y$ to any topological space Y is continuous if and only if S is sequentially continuous, i.e. it maps convergent sequences in F to convergent sequences in Y .

Remark 39. The following can be observed.

- (a) The action functional was defined to be a Radon measure valued map. That was motivated by the fact that no asymptotics was prescribed on the field configurations F , nor it was assumed that \mathcal{M} is compact. Because of that, one cannot guarantee that the smooth maximal form field $L(v, \nabla v, P(\nabla))$ is integrable throughout the full \mathcal{M} for sufficiently many field configurations $(v, \nabla)_w \in F$. It is, however, always locally integrable, hence the action functional as a Radon measure valued map is meaningful, and everywhere defined.
- (b) Due to Lebesgue's theorem of dominated convergence, the action functional is sequentially continuous, and therefore by means of remark 38, it is continuous.

The action functional is everywhere differentiable in the Fréchet–Hadamard sense (see also [22]-section 1), as it is common knowledge in Lagrangian field theory. In order to show its explicit form, we recall some differential geometric identities. We will use Penrose abstract indices for the tangent tensors throughout the section.

Remark 40. If ∇ is a covariant derivation over $T(\mathcal{M})$, then there is a unique covariant derivation $\tilde{\nabla}$ over $T(\mathcal{M})$ associated to it, having vanishing torsion tensor and having the

same affine parametrized geodesics as ∇ . The covariant derivation $\tilde{\nabla}$ is called the *torsion-free part* of ∇ . In explicit formulae: whenever v^b is a smooth section of $T(\mathcal{M})$, then one has $\tilde{\nabla}_a v^b = \nabla_a v^b + \frac{1}{2}T(\nabla)_{ac}^b v^c$, where $T(\nabla)_{ac}^b$ denotes the torsion tensor of ∇ .

Theorem 41. *The action functional S^L is everywhere differentiable, and its derivative at some fixed $(v, \nabla)_W \in F$ is a continuous linear map $DS^L(v, \nabla)_W : \mathbb{F} \rightarrow \text{Rad}(\mathcal{M}, \mathbb{R})$, given by the formula*

$$\begin{aligned}
 (\delta v, \delta C)_W &\mapsto (DS^L_{\mathcal{K}}(v, \nabla)_W | (\delta v, \delta C)_W) \\
 &= \int_{\mathcal{K}} \left(D_1 L(v, \nabla v, P(\nabla)) \delta v + D_2^a L(v, \nabla v, P(\nabla)) (\nabla_a \delta v + \delta C_a v) \right. \\
 &\quad \left. + 2D_3^{[ab]} L(v, \nabla v, P(\nabla)) \tilde{\nabla}_{[a} \delta C_{b]} \right), \tag{A.1}
 \end{aligned}$$

when evaluated on some compact subset $\mathcal{K} \subset \mathcal{M}$. Here, $D_1 L, D_2 L, D_3 L$ denote the spacetime pointwise partial derivative of L against its first, second and third field variable, respectively. It also follows that the derivative map $DS^L : F \times \mathbb{F} \rightarrow \text{Rad}(\mathcal{M}, \mathbb{R})$ is jointly continuous in its two variables.

Proof. This is a simple consequence of the below elementary facts.

- The Lagrange form evaluation as a map $(v, Dv, P)_W \mapsto L(v, Dv, P)$ acting on the space of sections is continuously differentiable in the \mathcal{E} topology, and the map $(v, \nabla)_W \mapsto (v, \nabla v, P(\nabla))_W$ is also continuously differentiable in the \mathcal{E} topology. Therefore, their composition, being the Lagrangian expression $(v, \nabla)_W \mapsto L(v, \nabla v, P(\nabla))$, is also differentiable in the \mathcal{E} topology, and its derivative is given by the integrand of equation (A.1).
- The local integral evaluation of a smooth maximal form over a compact subset $\mathcal{K} \subset \mathcal{M}$ is sequentially continuous map in the \mathcal{E} topology due to Lebesgue theorem of dominated convergence, and therefore by means of remark 38 it is continuous in the $\mathcal{E} \rightarrow \text{Rad}(\mathcal{M}, \mathbb{R})$ topologies. Due to its linearity then it is differentiable, and its derivative is itself.
- Chain rule for the differentiation of composite functions made out of the above two maps implies the first part of the theorem.
- Lebesgue’s theorem of dominated convergence implies joint sequential continuity of DS^L . Therefore, by means of remark 38, the derivative functional is jointly continuous as a $DS^L : F \times \mathbb{F} \rightarrow \text{Rad}(\mathcal{M}, \mathbb{R})$ map, since F, \mathbb{F} and thus their product $F \times \mathbb{F}$ is metrizable. This proves the second statement of the theorem. □

Remark 42. Let us also recall the following differential geometric identities.

- (a) Let $J^a_{[c_1 \dots c_m]}$ be a smooth section of $T(\mathcal{M}) \otimes \wedge^m T^*(\mathcal{M})$, i.e. a maximal form valued tangent vector field (the symbol $[\]$ denotes index antisymmetrization). Then, given any covariant derivation ∇ on $T(\mathcal{M})$, one has that the expression $\tilde{\nabla}_a J^a_{[c_1 \dots c_m]}$ is independent of the choice of the covariant derivation ∇ , where $\tilde{\nabla}$ denotes the torsion-free part of ∇ . That is, the divergence of a maximal form valued vector field is naturally defined without further assumptions. Similarly, for a smooth section $K^{[ab]}_{[c_1 \dots c_m]}$ of $(T(\mathcal{M}) \wedge T(\mathcal{M})) \otimes \wedge^m T^*(\mathcal{M})$ one has that $\tilde{\nabla}_a K^{[ab]}_{[c_1 \dots c_m]}$ is independent of the choice of the covariant derivation ∇ , and thus the divergence of such field is naturally defined without further assumptions.

(b) Let $J^a_{[c_1 \dots c_m]}$ be a smooth section of $T(\mathcal{M}) \otimes \wedge^m T^*(\mathcal{M})$, i.e. a maximal form valued tangent vector field. Then, given any covariant derivation ∇ on $T(\mathcal{M})$, one has that $\tilde{\nabla}_a J^a_{[c_1 \dots c_m]} = \text{md}_{[c_1} (J^a_{ac_2 \dots c_m]})$, where d denotes exterior differentiation (see [39]).

Theorem 43. *The derivative $DS^L(v, \nabla)_W$ of the action functional S^L at a fixed $(v, \nabla)_W \in F$ can be re-expressed as*

$$\begin{aligned} (\delta v, \delta C)_W &\mapsto (DS^L_{\mathcal{K}}(v, \nabla)_W | (\delta v, \delta C)_W) \\ &= \int_{\mathcal{K}} \left(D_1 L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \delta v - \left(\tilde{\nabla}_a D_2^a L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \right) \delta v \right) \\ &\quad + \left(D_2^a L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \delta C_a v - 2 \left(\tilde{\nabla}_a D_3^{[ab]} L(v, \nabla v, P(\nabla))_{[c_1 \dots c_m]} \right) \delta C_b \right) \\ &\quad + m \int_{\partial \mathcal{K}} \left(D_2^a L(v, \nabla v, P(\nabla))_{[ac_1 \dots c_{m-1}]} \delta v + 2 D_3^{[ab]} L(v, \nabla v, P(\nabla))_{[ac_1 \dots c_{m-1}]} \delta C_b \right), \quad (\text{A.2}) \end{aligned}$$

when evaluated over some compact subset $\mathcal{K} \subset \mathcal{M}$ with cone property boundary $\partial \mathcal{K}$.

Proof. This can be proved as usual in Lagrangian field theory. Namely, we start out from the expression in equation (A.1), use Leibniz rule, apply the differential geometric identities of remark 42, and then apply Stokes theorem for the boundary term. \square

Let us introduce $\mathbb{F}_T \subset \mathbb{F}$ to be either the vector space of compactly supported sections from \mathbb{F} , or if $\partial \mathcal{M} \neq \emptyset$ optionally they may be even required to vanish on $\partial \mathcal{M}$ together with all of their derivatives. Elements of \mathbb{F}_T will be called the **test field variations**. The space \mathbb{F}_T can be endowed with the standard \mathcal{D} test function topology, being stronger than the \mathcal{E} topology, defined by the restricted \mathcal{E} topology for sections with their supports within each fixed compact set of \mathcal{M} . It is common knowledge ([22]-remark 2), that \mathbb{F}_T with its natural \mathcal{D} test function topology is a *strict inductive limit of a countable system of nuclear Fréchet spaces with closed adjacent images (LNF space)* whenever \mathcal{M} is noncompact, and it is *nuclear Fréchet (NF space)* if \mathcal{M} is compact. These are important details in the QFT construction. It is seen that due to Lebesgue’s theorem of dominated convergence the integrand within the expression $(DS^L_{\mathcal{M}}(v, \nabla)_W | (\delta v_T, \delta C_T)_W)$, see again equations (A.1) and (A.2), is absolutely integrable for all fields $(v, \nabla)_W \in F$ and all test field variations $(\delta v_T, \delta C_T)_W \in \mathbb{F}_T$. In other words: the measure $\mathcal{K} \mapsto (DS^L_{\mathcal{K}}(v, \nabla)_W | (\delta v_T, \delta C_T)_W)$ has bounded total variation, and thus $(DS^L_{\mathcal{M}}(v, \nabla)_W | (\delta v_T, \delta C_T)_W) \in \mathbb{R}$ is finite. Consequently, the following definition is meaningful.

Definition 44. Let $\mathcal{M}, V(\mathcal{M}), L, S^L$ as before. The map

$$E^L : F \times \mathbb{F}_T \rightarrow \mathbb{R}, (\psi, \delta \psi_T) \mapsto (E^L(\psi) | \delta \psi_T) := (DS^L_{\mathcal{M}}(\psi) | \delta \psi_T) \quad (\text{A.3})$$

is called the **EL functional**. (Here, we used a shorthand notation $\psi := (v, \nabla)_W \in F$ for a field, and $\delta \psi_T := (\delta v_T, \delta C_T)_W \in \mathbb{F}_T$ for a test field variation.)

Note that it was possible to define the EL functional as real valued at the price of restricting its second argument to compactly supported field variations. This setting also explains why one can automatically discard the EL boundary terms in classical variational problems over non-compact manifolds without boundary. It is clear that for all $\psi \in F$ the map $(E^L(\psi) | \cdot) : \mathbb{F}_T \rightarrow \mathbb{R}$ is well defined. Moreover, it is linear, and continuous in the \mathcal{D} topology due to Lebesgue’s

theorem of dominated convergence. Therefore the EL functional may be viewed either as map $E^L : F \times \mathbb{F}_T \rightarrow \mathbb{R}$, or alternatively as a distribution valued map $E^L : F \rightarrow \mathbb{F}_T^*$, where $*$ denotes the strong dual. About their continuity properties, one can state the following.

Theorem 45. *The EL functional $E^L : F \times \mathbb{F}_T \rightarrow \mathbb{R}$, with F and \mathbb{F}_T carrying the standard \mathcal{E} and \mathcal{D} topologies, respectively, has the following continuity properties.*

- (a) *It is jointly sequentially continuous.*
- (b) *It is separately continuous in each variable.*
- (c) *It is continuous as a $E^L : F \rightarrow \mathbb{F}_T^*$ map.*

Proof. Property (a) is obviously seen via applying Lebesgue theorem of dominated convergence in the joint variables.

To see (b), take first a fixed $\delta\psi_T \in \mathbb{F}_T$. Then, the map $E^L(\cdot, \delta\psi_T) : F \rightarrow \mathbb{R}$ is sequentially continuous by means of (a), and due to the metrizability of F , by means of remark 38, then it is continuous. Take than a fixed $\psi \in F$. The linear map $E^L(\psi, \cdot) : \mathbb{F}_T \rightarrow \mathbb{R}$ is sequentially continuous by means of (a). Due to the facts in [22]-remark 5, the space \mathbb{F}_T carries the bornological property, by means of which the sequentially continuous linear map $E^L(\psi, \cdot) : \mathbb{F}_T \rightarrow \mathbb{R}$ is continuous.

To see (c), observe that due to (a) the map $E^L : F \rightarrow \mathbb{F}_T^*$ is sequentially continuous, whenever \mathbb{F}_T^* is endowed with the weak (pointwise) topology. Due to the facts in [22]-remark 5, the space \mathbb{F}_T^* carries the Montel property, therefore weakly convergent sequences are also strongly convergent in \mathbb{F}_T^* . Thus, the pertinent map is also sequentially continuous when the target space \mathbb{F}_T^* is endowed with its standard strong dual topology (\mathcal{D}^* topology). Due to the metrizability of F , by means of remark 38, then it is $\mathcal{E} \rightarrow \mathcal{D}^*$ continuous. \square

Definition 46. A tuple $(\mathcal{M}, V(\mathcal{M}), F, \mathbb{F}, \mathbb{F}_T, E, \mathcal{C})$ is called a **classical field theory**, where \mathcal{M} and $V(\mathcal{M})$ is as in definition 37, F is the space of smooth sections of the affine bundle $V(\mathcal{M}) \times_w DV(\mathcal{M})$, the space \mathbb{F} consists of the smooth sections of the subordinate vector bundle $V(\mathcal{M}) \oplus T^*(\mathcal{M}) \otimes V(\mathcal{M}) \otimes V^*(\mathcal{M})$, the space \mathbb{F}_T consists of compactly supported sections from \mathbb{F} (if $\partial\mathcal{M} \neq \emptyset$, optionally, elements of \mathbb{F}_T may be required to vanish on $\partial\mathcal{M}$ together with all of their derivatives—variation with boundary included/excluded). Furthermore, the object E is a map $F \times \mathbb{F}_T \rightarrow \mathbb{R}$, such that there exists a Lagrange form L as in definition 37, such that $E = E^L$. Finally, $\mathcal{C} := \{\psi \in F \mid \forall \delta\psi_T \in \mathbb{F}_T : (E(\psi) \mid \delta\psi_T) = 0\}$. The set \mathcal{C} is called the **solution space** of the classical field theory.

Definition 47. Let $(\mathcal{M}', V'(\mathcal{M}'), F', \mathbb{F}', \mathbb{F}'_T, E', \mathcal{C}')$ and $(\mathcal{M}, V(\mathcal{M}), F, \mathbb{F}, \mathbb{F}_T, E, \mathcal{C})$ be two classical field theories. These are called **isomorphic**, if and only if there exists a vector bundle isomorphism $V'(\mathcal{M}') \rightarrow V(\mathcal{M})$ with an underlying diffeomorphism $\mathcal{M}' \rightarrow \mathcal{M}$ of the base manifold, such that L subordinate to E is pulled back to L' subordinate to E' . (Isomorphic classical field theories are postulated to describe the same physics.) Quite naturally, isomorphisms of a classical field theory with itself are called **automorphisms**, or **symmetries**.

A classical field theory $(\mathcal{M}, V(\mathcal{M}), F, \mathbb{F}, \mathbb{F}_T, E, \mathcal{C})$ is called **generally covariant**, if and only if all the vector bundle automorphisms $V(\mathcal{M}) \rightarrow V(\mathcal{M})$ are automorphisms of the classical field theory.

A classical field theory $(\mathcal{M}, V(\mathcal{M}), F, \mathbb{F}, \mathbb{F}_T, E, \mathcal{C})$ is called **diffeomorphism invariant**, if and only if for all the diffeomorphisms $\mathcal{M} \rightarrow \mathcal{M}$ of the base manifold there exists a vector bundle automorphism $V(\mathcal{M}) \rightarrow V(\mathcal{M})$, such that it is an automorphism of the classical field theory.

Those automorphisms of a classical field theory $(\mathcal{M}, V(\mathcal{M}), F, \mathbb{F}, \mathbb{F}_T, E, \mathcal{C})$, for which the underlying $\mathcal{M} \rightarrow \mathcal{M}$ diffeomorphism is the identity of \mathcal{M} , are called **internal symmetries** or gauge transformations.

Definition 48. The **observables** of a classical field theory $(\mathcal{M}, V(\mathcal{M}), F, \mathbb{F}, \mathbb{F}_T, E, \mathcal{C})$ are the continuous maps $O : F \rightarrow \mathbb{R}$.

Remark 49. The presented formulation of a classical Lagrangian field theory formalizes the Palatini type variational principle, when applied to a setting eventually containing general relativity. That is: the spacetime metric field or its ingredients, if present in the theory, is treated just like any other of the fields. In particular, it is not assumed *a priori* that on $T(\mathcal{M})$ a Levi-Civita covariant derivation is present associated to some spacetime metric. If a metric and a covariant derivation on $T(\mathcal{M})$ is present, they are varied independently in the presented formulation. We also remark, that in this formulation, the Lagrange form of general relativity can be chosen to be polynomial in the field variables: one variable can be chosen to be the inverse spacetime metric densitised with the metric volume form, i.e. a field $\mathbf{g}_{[cdef]}^{ab}$ (this is in one-to-one correspondence with the ordinary spacetime metric field g_{ab}), and the other variable can simply be the $T(\mathcal{M})$ covariant derivation ∇_a . The Einstein–Hilbert Lagrangian expression is then $(\mathbf{g}_{[cdef]}^{ab}, \nabla_h) \mapsto \mathbf{g}_{[cdef]}^{ab} R(\nabla)_{abh}{}^h$ which is a third degree polynomial of its field variables, where $R(\nabla)_{abc}{}^d$ denotes the Riemann tensor of ∇ .

Appendix B. The Wilsonian renormalization

Remark 50. It is rather straightforward to see that the space of mollifying kernels form a real vector space, naturally carrying a Hausdorff sequential convergence (CVS) structure which is Cauchy complete. A sequence $(\kappa_n)_{n \in \mathbb{N}}$ of mollifying kernels is said to converge to zero iff for all compact sets $\mathcal{K} \subset \mathcal{M}$ there exists some compact set $\mathcal{K}' \subset \mathcal{M}$, such that for all $n \in \mathbb{N}$ the closure of the sets $\{(x, y) \in \mathcal{M} \times \mathcal{M} | x \in \mathcal{K}, \kappa_n(x, y) \neq 0\}$ and $\{(x, y) \in \mathcal{M} \times \mathcal{M} | y \in \mathcal{K}, \kappa_n(x, y) \neq 0\}$ are contained in $\mathcal{K} \times \mathcal{K}'$ and $\mathcal{K}' \times \mathcal{K}$, respectively, moreover the sections $(\kappa_n)_{n \in \mathbb{N}}$ along with all their polynomial derivatives converge uniformly to zero over the compact sets $\mathcal{K} \times \mathcal{K}' \subset \mathcal{M} \times \mathcal{M}$ and $\mathcal{K}' \times \mathcal{K} \subset \mathcal{M} \times \mathcal{M}$. We do not address in the present note whether this convergence structure originates from a TVS structure or not, since we do not need it. It is also rather easy to see, that whenever the base manifold \mathcal{M} is affine, the convolution kernels form a sequentially closed vector subspace within the space of all mollifying kernels.

Definition 51. On the set of mollifying kernels, one may introduce a natural, vector bundle automorphism invariant pre-ordering relation. Namely, for mollifying kernels κ'' and κ we say that $\kappa'' \lesssim \kappa$ (in words: **κ'' is less ultraviolet than κ**) iff either $C_{\kappa''} = C_{\kappa}$ or there exists some mollifying kernel κ' such that $C_{\kappa''} = C_{\kappa'} C_{\kappa}$ holds. It is evidently seen from the construction, that indeed this defines a pre-order, i.e. a relation which is transitive and reflexive. It is also seen that such relation may be also formulated on the set of convolution kernels, whenever convolution is meaningful, i.e. whenever the base manifold \mathcal{M} is affine (and in that case, the pertinent relation is invariant to affine transformations of \mathcal{M}).

Theorem 52. For a real valued smooth compactly supported test function φ_T over \mathcal{M} , denote by M_{φ_T} the multiplication operator by φ_T . The pre-order relation \lesssim , introduced in definition 51, when restricted to the set of mollifying kernels κ which admit some φ_T such that $C_{\kappa} M_{\varphi_T}$ is not finite rank, becomes a partial order, i.e. it is antisymmetric.

Proof. Let κ and κ' be any two mollifying kernels. We need to show that $\kappa' \succsim \kappa$ and $\kappa \succsim \kappa'$ implies $\kappa' = \kappa$ under the conditions of the theorem.

Writing out the condition $\kappa' \succsim \kappa$ and $\kappa \succsim \kappa'$ explicitly, there exist continuous linear operators $A, B : \mathbb{F} \rightarrow \mathbb{F}$, such that $C_{\kappa'} = AC_{\kappa}$ and $C_{\kappa} = BC_{\kappa'}$, where $A = I$ or $A = C_{\alpha}$ with some mollifying kernel α , and $B = I$ or $B = C_{\beta}$ with some mollifying kernel β . Putting these together, they imply $C_{\kappa'} = ABC_{\kappa'}$ and $C_{\kappa} = BAC_{\kappa}$. Taking any real valued compactly supported smooth test function φ_T over \mathcal{M} , these imply $C_{\kappa'}M_{\varphi_T} = ABC_{\kappa'}M_{\varphi_T}$ and $C_{\kappa}M_{\varphi_T} = BAC_{\kappa}M_{\varphi_T}$. Since κ and κ' was properly supported, then there exists some large enough compact region $\mathcal{K} \subset \mathcal{M}$ containing $\text{supp}(\varphi_T)$, such that the supports of the images of $C_{\kappa}M_{\varphi_T}$ and $C_{\kappa'}M_{\varphi_T}$ are also contained within \mathcal{K} . Let η_T be a real valued smooth compactly supported test function, which takes the value 1 within this set \mathcal{K} . Then, one has

$$C_{\kappa'}M_{\varphi_T} = ABM_{\eta_T}C_{\kappa'}M_{\varphi_T} \quad \text{and} \quad C_{\kappa}M_{\varphi_T} = BAM_{\eta_T}C_{\kappa}M_{\varphi_T}. \tag{B.1}$$

One can choose an even larger compact region $\mathcal{K}' \subset \mathcal{M}$, which contains $\text{supp}(\eta_T)$ and also contains the supports of the images of ABM_{η_T} and BAM_{η_T} . Under such conditions, the kernel function of $C_{\kappa'}M_{\varphi_T}$ and of $C_{\kappa}M_{\varphi_T}$ are square integrable, and therefore are Hilbert–Schmidt on the space of L^2 sections over \mathcal{K}' , so they are compact operators. If any of A or B is not the unit operator, then it is a mollifying operator by our assumptions, and then both ABM_{η_T} and BAM_{η_T} are also Hilbert–Schmidt on the above L^2 function space over \mathcal{K}' , for the same above reason, so they are also compact. Equation (B.1) implies that $\text{Ran}(C_{\kappa'}M_{\varphi_T})$ is contained in the eigenspace of ABM_{η_T} with eigenvalue one, and $\text{Ran}(C_{\kappa}M_{\varphi_T})$ is contained in the eigenspace of BAM_{η_T} with eigenvalue one. But since nonzero eigenvalue eigenspaces of compact operators are finite dimensional, $\text{Ran}(C_{\kappa'}M_{\varphi_T})$ and $\text{Ran}(C_{\kappa}M_{\varphi_T})$ must be finite dimensional if any of A or B are not the unity operator. But it was assumed that κ admitted some φ_T such that $\text{Ran}(C_{\kappa}M_{\varphi_T})$ is not finite dimensional (and for κ' the same was assumed with some φ'_T). Therefore, both A and B must be the unity operator, i.e. $\kappa' = \kappa$. \square

Remark 53. The pre-ordering \succsim becomes a partial order, under mild conditions.

- (a) If for some test function φ_T the mollifying kernel κ is such that $C_{\kappa}M_{\varphi_T}$ is injective on an infinite dimensional linear subspace of the L^2 sections, then $C_{\kappa}M_{\varphi_T}$ is not finite rank. That is because in the pertinent case, the L^2 adjoint of the continuous operator $C_{\kappa}M_{\varphi_T}$ is evidently non-finite rank, due to which the operator itself cannot be finite rank.
- (b) If the mollifying kernel κ is such that the operator C_{κ} is injective over the space of test field variations \mathbb{F}_T , then it satisfies the above condition, and thus $C_{\kappa}M_{\varphi_T}$ is not finite rank, for any test function φ_T .
- (c) If the base manifold \mathcal{M} is affine, then the convolutions are meaningful, and the convolution kernels by test functions are such that their C_{κ} operators are injective over the space of test field variations \mathbb{F}_T . That claim can be verified in Fourier space, using a consequence of the Paley–Wiener–Schwartz theorem ([29] theorem 7.3.1), namely the fact that the Fourier transform of a compactly supported distribution (and hence, of a function) is an analytic function. (Alternatively, it also follows from [40] theorem 4.4.) Therefore, if κ were a convolution kernel, by means of the above observation, $C_{\kappa}M_{\varphi_T}$ is not finite rank.

The above leads us to the following conclusion.

Corollary 54. *On the set of mollifying kernels which are injective on the space of test field variations, the pre-ordering \succsim is antisymmetric, i.e. it is a partial order.*

In particular, when the base manifold is affine, over the set of nonvanishing convolution kernels the pre-ordering \succsim is a partial order.

In such cases, we may use the symbol \preceq instead of \lesssim for clarity.

Remark 55. In section 2 it was argued that the Wilsonian regularization justifies our regularized MDS equation (15). Applying the heuristic integral substitution (measure pushforward) formula for composite maps in the Wilsonian Feynman integral equation (6), it would follow that if $(\psi_0, G_{\psi_0, \kappa}) \in F \times A(\mathbb{F}_{(\mathbb{C})})$ were a solution of the κ -regularized MDS equation, and $\kappa'' \lesssim \kappa$, then there should exist a solution $(\psi_0, G_{\psi_0, \kappa''}) \in F \times A(\mathbb{F}_{(\mathbb{C})})$ of the κ'' -regularized MDS equation, such that the identity

$$H_{C_{\kappa'}} G_{\psi_0, \kappa} = G_{\psi_0, \kappa''} \tag{B.2}$$

holds, where κ' is the corresponding mollifying kernel satisfying $C_{\kappa''} = C_{\kappa'} C_{\kappa}$ (because of $\kappa'' \lesssim \kappa$), and $H_{C_{\kappa'}}$ is defined as $\overset{n}{\otimes} C_{\kappa'}$ on the n -vectors of $A(\mathbb{F}_{(\mathbb{C})})$, and is thus a unital algebra homomorphism of $A(\mathbb{F}_{(\mathbb{C})})$ generated by the continuous linear operator $C_{\kappa'} : \mathbb{F} \rightarrow \mathbb{F}$. This equation is called the exact renormalization equation (ERGE) in the QFT literature, and $H_{C_{\kappa'}}$ is called a blocking transformation. It is seen that if Feynman integrals existed as a proper finite measure, the ERGE equation would be just the consequence of the fundamental formula for integral substitution, for the pushforward measures. In our rigorous formalism, defined on the field correlators, one needs to impose that by hand, as stated below.

Definition 56. Let the index set \mathcal{I} be the set of mollifying kernels, and denote by $A(\mathbb{F}_{(\mathbb{C})})^{\mathcal{I}}$ the set of all maps $\mathcal{I} \rightarrow A(\mathbb{F}_{(\mathbb{C})})$. Then, the *solution space of the Wilsonian renormalized MDS equation* is

$$\begin{aligned} Q_r := & \{ (\psi_0, G_{\psi_0, \cdot}) \in F \times A(\mathbb{F}_{(\mathbb{C})})^{\mathcal{I}} \mid \forall \kappa, \kappa'' \in \mathcal{I} : \kappa'' \lesssim \kappa \text{ (with } \kappa') \Rightarrow H_{C_{\kappa'}} G_{\psi_0, \kappa} = G_{\psi_0, \kappa''} \\ & \text{and } \forall \kappa \in \mathcal{I} : \forall \delta \psi_T \in \mathbb{F}_T : bG_{\psi_0, \kappa} = 1, \mathbf{M}_{\hbar, \psi_0, \kappa, \delta \psi_T} G_{\psi_0, \kappa} = 0 \}, \end{aligned} \tag{B.3}$$

i.e. they are the solution families of the regularized MDS equation, satisfying the ERGE relation. We say that a model is *Wilsonian renormalizable*, if Q_r is not empty.

One may recognize that the solution families satisfying the ERGE relation are so-called projective families, and therefore, the solution space of the Wilsonian renormalized MDS equation is the corresponding projective limit ([41] chapter 4.21). The theory is Wilsonian renormalizable whenever the corresponding projective limit exists as a nonempty set.

Remark 57. It is not uncommon in QFT that running coupling factors need to be introduced. In that case, it is assumed that the EL function can be specified as a finite sum $E = g_1 E_1 + \dots + g_n E_n$, with each $E_i : F \times \mathbb{F}_T \rightarrow \mathbb{R}$ being jointly sequentially continuous, called the EL terms, and g_i being nonzero real numbers, called to be the coupling factors ($i = 1, \dots, n$). Recall that the space of mollifying kernels \mathcal{I} was a Hausdorff complete sequential convergence vector space, due to which one can define (sequentially) continuous functions from \mathcal{I} to other convergence vector spaces. Given some (sequentially) continuous functionals $\gamma_i : \mathcal{I} \rightarrow \mathbb{R}$ ($i = 1, \dots, n$), one may define the running regularized MDS operator as

$$\begin{aligned} \mathbf{M}_{\hbar, \psi_0, \kappa, (\gamma_1, \dots, \gamma_n), \delta \psi_T} : & A(\mathbb{F}_{(\mathbb{C})}) \rightarrow A(\mathbb{F}_{(\mathbb{C})}), \\ G \mapsto \mathbf{M}_{\hbar, \psi_0, \kappa, (\gamma_1, \dots, \gamma_n), \delta \psi_T} G := & \left(\iota_{\gamma_1(\kappa)(\mathbf{E}_{1, \psi_0} | \delta \psi_T) + \dots + \gamma_n(\kappa)(\mathbf{E}_{n, \psi_0} | \delta \psi_T)} - i \hbar L_{C_{\kappa} \delta \psi_T} \right) G, \end{aligned} \tag{B.4}$$

for fixed $\hbar \in \mathbb{R}$, reference field $\psi_0 \in F$, test field variation $\delta\psi_T \in \mathbb{F}_T$, mollifying kernel $\kappa \in \mathcal{I}$ and running couplings γ_i ($i = 1, \dots, n$). The solution space of the Wilsonian renormalized MDS equation with running couplings is then

$$\begin{aligned} & \{(\psi_0, (\gamma_1, \dots, \gamma_n), G_{\psi_0, \cdot}) \in F \times C(\mathcal{I}, \mathbb{R})^n \times A(\mathbb{F}_{(\mathbb{C})})^{\mathcal{I}} \mid \\ & \quad \forall \kappa, \kappa'' \in \mathcal{I} : \kappa'' \lesssim \kappa (\text{with } \kappa') \Rightarrow H_{C, \kappa'} G_{\psi_0, \kappa} = G_{\psi_0, \kappa''} \\ & \quad \text{and } \forall \kappa \in \mathcal{I} : \forall \delta\psi_T \in \mathbb{F}_T : bG_{\psi_0, \kappa} = 1, \mathbf{M}_{\hbar, \psi_0, \kappa, (\gamma_1, \dots, \gamma_n), \delta\psi_T} G_{\psi_0, \kappa} = 0\}. \end{aligned} \tag{B.5}$$

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