# $N$-tangle, Entangled Orthonormal Basis, and a Hierarchy of Spin Hamilton Operators

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Z. Naturforsch. **66a,** 615 – 619 (2011) / DOI: 10.5560/ZNA.2011-0025
Received April 30, 2010 / revised June 29, 2011

An $N$-tangle can be defined for the finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^{2^N}$, with $N = 3$ or $N$ even. We give an orthonormal basis which is fully entangled with respect to this measure. We provide a spin Hamilton operator which has this entangled basis as normalized eigenvectors if $N$ is even. From these normalized entangled states a Bell matrix is constructed and the cosine–sine decomposition is calculated. If $N$ is odd the normalized eigenvectors can be entangled or unentangled depending on the parameters.

*Key words:* Entanglement; Spin Hamilton Operators; Orthonormal Basis; Cosine–Sine Decomposition.

Two level and higher level quantum systems and their physical realization have been studied by many authors (see [1] and references therein). We consider a spin Hamilton operator acting in the finite-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^{2^N}$ and the normalized states

$$|\psi\rangle = \sum_{j_1,j_2,\ldots,j_N=0}^{1} c_{j_1,j_2,\ldots,j_N} |j_1\rangle \otimes |j_2\rangle \otimes \cdots \otimes |j_N\rangle$$

in this Hilbert space. Here $|0\rangle$, $|1\rangle$ denotes the standard basis. Let $\varepsilon_{jk}$ $(j,k = 0,1)$ be defined by $\varepsilon_{00} = \varepsilon_{11} = 0$, $\varepsilon_{01} = 1$, $\varepsilon_{10} = -1$.

Let $\sigma_x$, $\sigma_y$, $\sigma_z$ be the Pauli spin matrices. We consider entanglement for the eigenvectors of the hierarchy of spin Hamilton operators

$$\hat{H}_N = \hbar\omega(\overbrace{\sigma_z \otimes \sigma_z \otimes \cdots \otimes \sigma_z}^{N\text{-factors}}) + \Delta_1(\overbrace{\sigma_x \otimes \sigma_x \otimes \cdots \otimes \sigma_x}^{N\text{-factors}}) + \Delta_2(\overbrace{\sigma_y \otimes \sigma_y \otimes \cdots \otimes \sigma_y}^{N\text{-factors}})$$

with $N \geq 2$. Here $\otimes$ denotes the Kronecker product [2 – 4], $\omega > 0$, $\Delta_1, \Delta_2 \geq 0$, and $\sigma_x \otimes \cdots \otimes \sigma_x$, $\sigma_y \otimes \cdots \otimes \sigma_y$, $\sigma_z \otimes \cdots \otimes \sigma_z$ are elements of the Pauli group $\mathcal{P}_N$. The $N$-qubit Pauli group [5] is defined by

$$\mathcal{P}_N := \{I_2, \sigma_x, \sigma_y, \sigma_z\}^{\otimes N} \otimes \{\pm 1, \pm \mathrm{i}\},$$

where $I_2$ is the $2 \times 2$ identity matrix. The $N$-qubit Clifford group $\mathcal{C}_N$ is the normalizer of the Pauli group – a unitary matrix $U$ acting on $N$-qubits is contained in $\mathcal{C}_N$ if

$$UMU^{-1} \in \mathcal{P}_N \quad \text{for each} \quad M \in \mathcal{P}_N.$$

Thus the Hamilton operator $\hat{H}_N$ acts in the finite-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^{2^N}$. The eigenvalue problem for the case with $\Delta_2 = 0$ has been studied by Steeb and Hardy [6, 7].

Many authors developed methods for the detection and classification of entangled states in finite dimensional Hilbert spaces for mixed states and pure $N$-qubit states [8 – 23]. A pure $N$-partite state is separable if and only if all the reduced density matrices of the elementary subsystems describe pure states. In a bipartite case, separability can be determined by calculating the Schmidt decomposition of the state. The concept of the Schmidt decomposition cannot be straightforwardly generalized to the case of $N$ separate subsystems [12]. Besides these two well-known methods, a separability condition based on comparing the amplitudes and phases of the components of the state has been presented. There are some other approaches to detect the separability of pure states [10, 11]. Here we select the entanglement measure given by Wong and Christensen [18]. The tangle of Wong and Christensen [18] can only be used to detect entanglement, since there are entangled states on which it vanishes. We note that the $W$ state

$$|W\rangle = \frac{1}{\sqrt{3}}(|0\rangle\otimes|0\rangle\otimes|1\rangle+|0\rangle\otimes|1\rangle\otimes|0\rangle+|1\rangle\otimes|0\rangle\otimes|0\rangle)$$

has vanishing Wong–Christensen tangle and yet is not separable. Dür et al. [19] showed that three qubits can be entangled in two inequivalent ways. Acín et al. [20] described the classification of mixed three-qubit states. Verstraete et al. [21] showed that four qubits can be entangled in nine different ways. Osterloh and Siewert [22] constructed $N$-qubit entanglement from antilinear operators. Entanglement witness in spin models has been studied by Tóth [23]. A symbolic C++ program to calculate the tangle of Wong and Christensen [18] has been given by Steeb and Hardy [24].

Let $N$ be even or $N=3$. Wong and Christensen [18] introduced an $N$-tangle by

$$\tau_{1\ldots N} = 2\left|\sum_{\substack{\alpha_1,\ldots,\alpha_N=0\\ \cdots\\ \delta_1,\ldots,\delta_N=0}}^{1} c_{\alpha_1\ldots\alpha_N}c_{\beta_1\ldots\beta_N}c_{\gamma_1\ldots\gamma_N}c_{\delta_1\ldots\delta_N} \times \varepsilon_{\alpha_1\beta_1}\varepsilon_{\alpha_2\beta_2}\cdots\varepsilon_{\alpha_{N-1}\beta_{N-1}}\varepsilon_{\gamma_1\delta_1}\varepsilon_{\gamma_2\delta_2}\cdots \varepsilon_{\gamma_{N-1}\delta_{N-1}}\varepsilon_{\alpha_N\gamma_N}\varepsilon_{\beta_N\delta_N}\right|.$$

This includes the definition for the 3-tangle [8]. Let $N=4$. Consider the two states with $c_{0000}=1/\sqrt{2}$, $c_{1111}=1/\sqrt{2}$ (all other coefficients are 0), and $c_{0000}=1/\sqrt{2}$, $c_{1111}=-1/\sqrt{2}$ (all other coefficients are 0). Using this measure of entanglement, we find for both cases that the states are fully entangled, i.e. $\tau_{1234}=1$. Fourteen more states can be constructed with

$$c_{j_1j_2j_3j_4}=1/\sqrt{2}, \qquad c_{\bar{j}_1\bar{j}_2\bar{j}_3\bar{j}_4}=\pm 1/\sqrt{2},$$

where $\bar{j}$ denotes the NOT-operation, i.e. $\bar{0}=1$ and $\bar{1}=0$. These sixteen states form an orthonormal basis in the Hilbert space $\mathbb{C}^{16}$.

This result can be extended for $N\geq 4$ and $N$ even. The orthonormal basis would be given by

$$|\phi_{j_1\ldots j_N}\rangle = \frac{1}{\sqrt{2}}(|j_1\rangle\otimes|j_2\rangle\otimes\cdots\otimes|j_N\rangle\pm|\bar{j}_1\rangle\otimes|\bar{j}_2\rangle\otimes\cdots\otimes|\bar{j}_N\rangle).$$

These states are also fully entangled using the measure given above. These states are also related to a Hamilton operator described below.

Let us now find the spectrum of $\hat{H}_N$ and the unitary matrix $U_N(t)=\exp(-\mathrm{i}\hat{H}_N t/\hbar)$. Since $\mathrm{tr}\hat{H}_N=0$ for all $N$, we obtain

$$\sum_{j=1}^{2^N} E_j = 0,$$

where $E_j$ are the eigenvalues of $\hat{H}_N$. Consider the hermitian and unitary operators

$$\Sigma_{z,N} := \sigma_z\otimes\sigma_z\otimes\cdots\otimes\sigma_z,\quad \Sigma_{x,N} := \sigma_x\otimes\sigma_x\otimes\cdots\otimes\sigma_x,$$
$$\Sigma_{y,N} := \sigma_y\otimes\sigma_y\otimes\cdots\otimes\sigma_y\,.$$

We have to distinguish between the case $N$ even and the case $N$ odd. If $N$ is even then the commutators vanish, i.e.

$$[\Sigma_{x,N},\Sigma_{y,N}]=0,\quad [\Sigma_{y,N},\Sigma_{z,N}]=0,\quad [\Sigma_{z,N},\Sigma_{x,N}]=0\,.$$

If $N$ is odd then the anti-commutators vanish, i.e.

$$[\Sigma_{z,N},\Sigma_{x,N}]_+=0,\quad [\Sigma_{z,N},\Sigma_{x,N}]_+=0,\quad [\Sigma_{z,N},\Sigma_{x,N}]_+=0\,.$$

Note that $\Sigma_{x,N}$, $\Sigma_{y,N}$, and $\Sigma_{z,N}$ are elements of the Pauli group $\mathcal{P}_N$ described above.

Thus setting $\hat{H}_N=\hat{H}_{N0}+\hat{H}_{N1}+\hat{H}_{N2}$ with

$$\hat{H}_{N0}=\hbar\omega(\sigma_z\otimes\sigma_z\otimes\cdots\otimes\sigma_z),$$
$$\hat{H}_{N1}=\Delta_1(\sigma_x\otimes\sigma_x\otimes\cdots\otimes\sigma_x),$$
$$\hat{H}_{N2}=\Delta_2(\sigma_y\otimes\sigma_y\otimes\cdots\otimes\sigma_y),$$

we find that for $N$ even, owing to the result given above,

$$[\hat{H}_{N0},\hat{H}_{N1}]=0,\ [\hat{H}_{N1},\hat{H}_{N2}]=0,\ [\hat{H}_{N0},\hat{H}_{N2}]=0\,.$$

Then the unitary operator $U_N(t)=\exp(-\mathrm{i}\hat{H}_N t/\hbar)$ for $N$ even can easily be calculated since

$$U_N(t) = \exp(-\mathrm{i}\hat{H}_{N0}t/\hbar)\exp(-\mathrm{i}\hat{H}_{N1}t/\hbar) \cdot\exp(-\mathrm{i}\hat{H}_{N2}t/\hbar)\,.$$

If $N$ is odd, owing to the result given above, we have

$$[\hat{H}_{N0},\hat{H}_{N1}]_+=0,\quad [\hat{H}_{N1},\hat{H}_{N2}]_+=0,\quad [\hat{H}_{N0},\hat{H}_{N2}]_+=0\,.$$

Here too the time evolution $U_N(t)=\exp(-\mathrm{i}\hat{H}_N t/\hbar)$ can easily be calculated. We use the abbreviation

$$E := \sqrt{\hbar^2\omega^2+\Delta_1^2+\Delta_2^2}\,.$$

Consider now the general cases. If $N$ is odd the Hamilton operator has only two eigenvalues, namely $E$ and $-E$. Both are $2^{N-1}$ times degenerate. The unnormalized eigenvectors for $+E$ are given by

$$\begin{pmatrix} E+\hbar\omega \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \Delta_1-(-i)^N\Delta_2 \end{pmatrix}, \begin{pmatrix} 0 \\ E-\hbar\omega \\ 0 \\ \vdots \\ 0 \\ \Delta_1+(-i)^N\Delta_2 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ E+\hbar\omega \\ \Delta_1-(-i)^N\Delta_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The unnormalized eigenvectors for $-E$ are given by

$$\begin{pmatrix} E-\hbar\omega \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ -(\Delta_1-(-\mathrm{i})^N\Delta_2) \end{pmatrix}, \begin{pmatrix} 0 \\ E-\hbar\omega \\ 0 \\ \vdots \\ 0 \\ -(\Delta_1-(-\mathrm{i})^N\Delta_2) \\ 0 \end{pmatrix}, \dots \begin{pmatrix} 0 \\ \vdots \\ 0 \\ E-\hbar\omega \\ -(\Delta_1-(-\mathrm{i})^N\Delta_2) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The normalization factors are

$$\frac{1}{\sqrt{(E+\hbar\omega)^2+\Delta_1^2+\Delta_2^2}}, \qquad \frac{1}{\sqrt{(E-\hbar\omega)^2+\Delta_1^2+\Delta_2^2}},$$

respectively. For $N$ odd the time evolution is given by

$$\begin{aligned} U_N(t) &= \mathrm{e}^{-\mathrm{i}\omega t\Sigma_{z,N}-\mathrm{i}\Delta_1 t\Sigma_{x,N}/\hbar-\mathrm{i}\Delta_2 t\Sigma_{y,N}/\hbar} \\ &= I_{2^N}\cos(Et/\hbar) \\ &\quad -\mathrm{i}\frac{\hbar\omega\Sigma_{z,N}+\Delta_1\Sigma_{x,N}+\Delta_2\Sigma_{y,N}}{E}\cdot\sin(Et/\hbar). \end{aligned}$$

For $N$ even the four eigenvalues are given by

$$E_1=\hbar\omega+\Delta_1-\Delta_2,\ E_2=-\hbar\omega-\Delta_1+\Delta_2,$$

$$E_3=-\hbar\omega+\Delta_1+\Delta_2,\ E_4=\hbar\omega-\Delta_1-\Delta_2.$$

The eigenvalues are $2^{N-2}$ times degenerate. The corresponding $2^N$ normalized eigenvectors for the case $N$ even are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \pm1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \pm1 \\ 0 \end{pmatrix}, \dots, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \pm1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

They do not depend on $\Delta$ and $\hbar\omega$. The first vector is the Greenberger–Horne–Zeilinger (GHZ)-state. It is well-known that these $2^N$ eigenvectors form an orthonormal basis in the Hilbert space $\mathbb{C}^{2^N}$. As described above we apply the entanglement measure given by Wong and Christensen. It follows that these states are fully entangled. These states can also be generated from the GHZ-state by applying the unitary matrix

$$I_2\otimes\cdots\otimes I_2\otimes\sigma_x\otimes I_2\otimes\cdots\otimes I_2,$$

where $\sigma_x$ is at the $j$th position ($j=1,\dots,N$). Since these are local unitaries all states have the same entanglement as the GHZ-state. Since for $N$ even we have

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\omega t\Sigma_{z,N}/\hbar} &= I_{2^N}\cos(\omega t)-\mathrm{i}\Sigma_{z,N}\sin(\omega t), \\ \mathrm{e}^{-\mathrm{i}\Delta_1 t\Sigma_{x,N}\hbar} &= I_{2^N}\cos(\Delta_1 t/\hbar)-\mathrm{i}\Sigma_{x,N}\sin(\Delta_1 t/\hbar), \\ \mathrm{e}^{-\mathrm{i}\Delta_2 t\Sigma_{y,N}/\hbar} &= I_{2^N}\cos(\Delta_2 t/\hbar)-\mathrm{i}\Sigma_{y,N}\sin(\Delta_2 t/\hbar), \end{aligned}$$

it follows that for $N$ even the unitary operator $U_N(t)$ for the time evolution is given by

$$\begin{aligned} &\mathrm{e}^{-\mathrm{i}\hat{H}_N t/\hbar} \\ &= \mathrm{e}^{-\mathrm{i}\omega t\Sigma_{z,N}}\,\mathrm{e}^{-\mathrm{i}\Delta_1 t\Sigma_{x,N}/\hbar}\,\mathrm{e}^{-\mathrm{i}\Delta_2 t\Sigma_{y,N}/\hbar} \\ &= I_{2^N}\cos(\omega t)\cos(\Delta_1 t/\hbar)\cos(\Delta_2 t/\hbar) \\ &\quad -\mathrm{i}\Sigma_{z,N}\sin(\omega t)\cos(\Delta_1 t/\hbar)\cos(\Delta_2 t/\hbar) \\ &\quad -\mathrm{i}\Sigma_{x,N}\cos(\omega t)\sin(\Delta_1 t/\hbar)\cos(\Delta_2 t/\hbar) \end{aligned}$$

$$
\begin{aligned}
&- \mathrm{i}\Sigma_{y,N} \cos(\omega t)\cos(\Delta_1 t/\hbar)\sin(\Delta_2 t/\hbar) \\
&- \Sigma_{z,N}\Sigma_{x,N} \sin(\omega t)\sin(\Delta_1 t/\hbar)\cos(\Delta_2 t/\hbar) \\
&- \Sigma_{z,N}\Sigma_{y,N} \sin(\omega t)\cos(\Delta_1 t/\hbar)\sin(\Delta_2 t/\hbar) \\
&- \Sigma_{x,N}\Sigma_{y,N} \cos(\omega t)\sin(\Delta_1 t/\hbar)\sin(\Delta_2 t/\hbar) \\
&+ \mathrm{i}\Sigma_{z,N}\Sigma_{x,N}\Sigma_{y,N} \sin(\omega t)\sin(\Delta_1 t/\hbar)\sin(\Delta_2 t/\hbar).
\end{aligned}
$$

For this basis we can form the $2^N \times 2^N$ ($N$ even) unitary matrix

$$
B = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & 1 \\
0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & 0 \\
0 & 0 & 1 & \dots & 0 & 0 & \dots & 1 & 0 & 0 \\
\vdots & & & & \vdots & \vdots & & & & \\
0 & 0 & 0 & & 0 & 0 & & 0 & 0 & 0 \\
0 & 0 & 0 & & 1 & 1 & & 0 & 0 & 0 \\
0 & 0 & 0 & & 1 & -1 & & 0 & 0 & 0 \\
0 & 0 & 0 & & 0 & 0 & & 0 & 0 & 0 \\
\vdots & & & & \vdots & \vdots & & & & \\
0 & 0 & 1 & \dots & 0 & 0 & & -1 & 0 & 0 \\
0 & 1 & 0 & & 0 & 0 & & 0 & -1 & 0 \\
1 & 0 & 0 & & 0 & 0 & & 0 & 0 & -1
\end{pmatrix}.
$$

For implementations of $B$ as quantum gates the cosine–sine decomposition [2, 4] is useful. This matrix has the cosine–sine decomposition

$$
B = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \begin{pmatrix} C & S \\ -S & C \end{pmatrix} \begin{pmatrix} U_3 & 0 \\ 0 & U_4 \end{pmatrix},
$$

where the unitary matrices $U_1$, $U_2$, $U_3$, $U_4$ are given by

$$
U_2 = I_{2^{N-1}}, \qquad U_4 = I_{2^{N-1}},
$$

$$
U_1 = U_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
$$

and the invertible matrices $C$ and $S$ are the $2^{N-1} \times 2^{N-1}$ matrices

$$
C = S = \frac{1}{\sqrt{2}} I_{2^{N-1}}.
$$

Thus the unitary matrices $U_1$ and $U_3$ are Kronecker products of the NOT-gate.

If $N$ is odd then the eigenvectors are entangled if $\hbar\omega = 0$. If $\hbar\omega \to \infty$ the eigenvectors become unentangled, i.e. can be written as product states.

We have provided a spin Hamilton operator acting in the finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^{2^N}$ that provides a fully entangled basis if $N$ is even. If $N$ is odd we vary the parameters $\hbar\omega$, $\Delta_1$, $\Delta_2$ such that we can vary between entangled and unentangled states. Such Hamilton operators could also be investigated applying Riemannian geometry [25].


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