OKINAWA INSTITUTE OF SCIENCE AND TECHNOLOGY
GRADUATE UNIVERSITY

PhD Thesis

The Topology, Geometry and Physics of Non-Hausdorff Manifolds

by

David O'Connell

Supervisor: **Yasha Neiman**

March 2025

Declaration of Original and Sole Authorship

I, David O'Connell, declare that this thesis entitled *The Topology, Geometry and Physics of Non-Hausdorff Manifolds* and the data presented in it are original and my own work. I confirm that:

- No part of this work has previously been submitted for a degree at this or any other university.
- References to the work of others have been clearly acknowledged. Quotations from the work of others have been clearly indicated, and attributed to them.
- In cases where others have contributed to part of this work, such contribution has been clearly acknowledged and distinguished from my own work.
- None of this work has been previously published and pre-printed elsewhere, with the exception of the following:
  - **D. O'Connell**. Non-Hausdorff manifolds via adjunction spaces. Topology and its Applications, 326:108388, 2023
  - **D. O'Connell**. Vector bundles over non-Hausdorff manifolds. Topology and its Applications, page 108982, 2024
  - Y. Neiman and **D. O'Connell**. Topology change from pointlike sources. Physical Review D, 110(6):064026, 2024.

    Contribution: This paper was written in collaboration with Prof. Yasha Neiman. Regarding my contribution to the paper, I was the primary author, in the sense that I (i) wrote the paper and (ii) derived the majority of the results present. Prof. Neiman, being my PhD supervisor, initiated and supervised the project. In accordance with the guidelines mandated by the OIST graduate school, Prof. Neiman has submitted a co-authorisation form allowing the inclusion of our paper within this thesis.
  - **D. O'Connell**. A non-Hausdorff de Rham cohomology, an unpublished manuscript which can be found on the ArXiv as arXiv:2310.17151, 2023.

Date: March 2025
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Abstract

In this thesis we investigate various mathematical and physical problems surrounding the theory of non-Hausdorff manifolds. We start by introducing a topological theory of non-Hausdorff manifolds from first principles. We then pass our discussion into a smooth category by defining various structures of geometric interest on non-Hausdorff manifolds, all whilst circumventing the technical issues surrounding partitions of unity. We complete our mathematical contribution by describing de Rham cohomology for non-Hausdorff manifolds, ultimately proving a generalized version of de Rham's Theorem. For the physical contribution, we focus our attention on certain two-dimensional non-Hausdorff manifolds that may be interpreted as a type of topology-changing spacetime. We define a gravitational action for 2d non-Hausdorff spacetimes, and determine the angular conventions required to suppress their contribution within a Lorentzian-signature path integral for gravity that sums over topologies.

Acknowledgments

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List of Publications

This thesis, being a *Thesis by publication*, is the concatenation of four publications, whose original titles and abstracts are listed below.

1. **D. O'Connell**. *Non-Hausdorff manifolds via Adjunction spaces*, Topology and its Applications, 326:108388, 2023

   Link: https://www.sciencedirect.com/science/article/abs/pii/S0166864122003935

   Contribution: I am the sole author.

2. **D. O'Connell**. *Vector bundles over non-Hausdorff manifolds*, Topology and its Applications, page 108982, 2024

   Link: https://www.sciencedirect.com/science/article/abs/pii/S0166864124001676

   Contribution: I am the sole author.

3. **D. O'Connell**. *A non-Hausdorff de Rham cohomology*, arXiv:2310.17151, 2023.

   Link: https://arxiv.org/abs/2310.17151

   Contribution: I am the sole author.

4. Y. Neiman and **D. O'Connell**. *Topology change from pointlike sources*, Physical Review D, 110(6):064026, 2024.

   Link: https://journals.aps.org/prd/abstract/10.1103/PhysRevD.110.064026

   Contribution: This paper was written in collaboration with Prof. Yasha Neiman. Regarding my contribution to the paper, I was the primary author, in the sense that I (i) wrote the paper and (ii) derived the majority of the results present. Prof. Neiman, being my PhD supervisor, initiated and supervised the project. In accordance with the guidelines mandated by the OIST graduate school, Prof. Neiman has submitted a co-authorisation form allowing the inclusion of our paper within this thesis.

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Chapter 1

Introduction

In the theory of general relativity, space may change its geometry over time according to the whims of gravity. Naturally, many authors have wondered whether the fundamental topology of space may also fluctuate over time. This could mean that space splits in multiple, causally-disconnected pieces, or perhaps some more-nuanced process in which space gains holes or other higher-dimensional topological features. Over the years there have been several considerations of these ideas, ranging between classical and quantum theories of gravity [4, 12, 16, 24, 28, 29, 31, 32, 36, 48, 49, 67, 70, 97–99, 107].

Classically speaking, the influence that the underlying topology of spacetime has on the resulting geometric behaviour is relatively well-understood [52, 86, 88]. Using these ideas, various authors have derived certain *no-go theorems* on the nature of topology change. These results typically describe some kind of causal misbehaviour as a kinematical obstruction arising from the topology changing spacetime [42, 52, 96], though may also manifest as more geometric abnormalities involving energy conditions [11, 103] or spin structures [43, 44]. The standard consensus from these results is that, due to their undesirable physical properties, topology change in general relativity is forbidden – see [18, 33] for a more thorough review.

Despite being inappropriate from the classic view of general relativity, some authors nonetheless pursue topology change within a quantum theory of gravity. This theme dates back to Wheeler [106], who suggested that at the Planck scale, the curvature of spacetime may fluctuate so violently that it separates itself into topologically disconnected pieces. Motivated by this idea, some authors make the interesting claim that topology change may be an integral feature of the universe, despite being classically forbidden [56].

Among the studies of topology change within quantum gravity, there are some explicit attempts to include some topology-changing processes into a path integral. The most natural narrative for these considerations is the *sum-over-histories* approach to quantum gravity, in which the computation of transition amplitudes involves a sum over all topologies that permit a "physically reasonable" spacetime structure [28, 51, 99]. An exact list of such desiderata is not clear, and considerations may vary depending on the commitments of each approach to quantum gravity. In any case, one would expect that topology-changing spacetimes ought to be exponentially suppressed relative to classical configurations in the path integral, with more elaborate models receiving an even stronger suppression.

In the case of a naive sum-over-histories approach to gravity, topology changing transition amplitudes have been studied to various degrees. Typically, these occur in discrete settings such as in simplicial theories of quantum gravity [3, 5, 58], or in the theory of *causal sets* [28, 30]. In these approaches to quantum gravity, one typically leverages the extra assumption of discreteness of spacetime at a fundamental level (in varying senses) to obtain more tractable descriptions of their path integrals. Aside from these, explicit mentions of the exponential suppression of topology-changing spacetimes in other simple contexts can be found in [19].

Somewhat adjacently, there also exist several theories of *Euclidean* quantum gravity in which distinct topologies are frequently summed over in the path integral. Perhaps the most well-known is in the genus expansion of closed string theory [26, 90]. By virtue of being two dimensional, the worldsheets of closed strings have total scalar curvatures proportional to their Euler characteristic – a connection induced by the famous Gauss-Bonnet theorem for Riemannian surfaces. In the bosonic path integral for closed strings, a sum over distinct topologies is reduced to a sum of Euler characteristics, scaled explicitly by an overall weight induced by the dilaton field. Through further arguments, the dilaton field is then justified to be a fixed parameter of the theory, and thus suppression rules for the genus expansion are completely determined by the theory.

In reality, we may expect there to be a Lorentzian analogue to these Euclidean path integrals in which topology-changing *spacetimes* are summed over. In some sense, a prototype of this idea was explored by Sorkin and Louko in [67], who studied the potential inclusion of the *Trousers space* into a naive path integral summing over topologies in Lorentzian signature. This space can be seen as a basic building block of topology change in $2d$ [1, 31], as it represents a Lorentzian analogue of a closed string splitting into two. Sorkin and Louko evaluated the basic gravitational action of the Trousers space, and then determined the correct rules that would entail its suppression relative to the Lorentzian cylinder. Interestingly, they concluded that no inherent suppression mechanism exists, and instead needs to be fix by hand via a sign convention.

Despite topology change having been extensively studied in various degrees, there is an interesting gap in the literature that has existed for over half a century. In a brief article discussing time asymmetry, Penrose sketched an idea for a process of topology change which is markedly distinct from the standard mathematical models used by other authors. In his picture, found as Figure 12.3(a) in [89], the topology of space changes at the scale of an individual *point* in spacetime, rather than along spacelike hypersurfaces. In the simplest case points are doubled, and this topology change is then taken to grow along along nullrays. If taken within a compact spacetime with spacelike boundaries, i.e. the natural setting for transition amplitudes, then this pointlike topology change will grow to cover the entirety of space in finite time and result in an overall change of spatial topology.

Penrose's spacetimes are interesting in that they seem to localize topology change all the way down to a single event in spacetime. This process results in a causal oddity: the doubled points will have the same causal past, but causally-disconnected futures. These multi-lightcone structures may agree with certain Everettian interpretations of wavefunction collapse in quantum mechanics [34, 95], and the growth along nullrays is reminiscent of inflationary models of the early universe. Despite the intrigue, there is a rather important subtlety that Penrose correctly identifies: mathematically speaking, in order for such spaces to be manifolds, then they must necessarily violate the Hausdorff property. Heuristically speaking, the doubled points will be "superimposed on top of each other" in the resulting manifold, and in fact so will their future nullcones. This atypical mathematical feature explains the lack of attention surrounding Penrosian topology change: non-Hausdorffness makes these spaces fall outside the scope of ordinary differential geometry, and at his time of writing it was unclear whether these atypical spaces were tractable in the first place.

In this thesis we will take seriously the topology-changing spacetimes of Penrose by studying their inclusion within a path integral for quantum gravity in Lorentzian signature. We will work towards the same setting as that of Sorkin and Louko [67], that is, we will focus on the specific case of non-Hausdorff transitions from one circle to two. However, before studying their physical properties, we must first take on the important task of verifying their *mathematical* plausibility. Simply put, we cannot determine the gravitational action of non-Hausdorff spacetimes if they do not admit meaningful notions of calculus and geometry in the first place. As such, a large portion of this thesis is first devoted to the foundational development of a non-Hausdorff differential geometry, through which me may then formulate our physical questions.

Throughout the remainder of this introduction we will elaborate on the observations mentioned above. We will first summarise the mathematical formulation of (Hausdorff) topology changing spacetimes commonly found in the literature, together with the causal abnormalities and the aforementioned analysis of Sorkin and Louko. We will then explain the idea of Penrose in detail, and flesh out some observations regarding non-Hausdorffness in mathematics and physics. Finally, we will provide a broad outline of this thesis, as well as some important remarks regarding the formatting choices present in this document.

1.1 Topology Change via Lorentz Cobordisms

Broadly speaking, we may think of a spacetime[1] (M,g) as exhibiting topology change whenever it contains maximal spacelike hypersurfaces that are not homeomorphic. Since we would like spacelike hypersurfaces to change *through time*, it is appropriate to describe this process via a global time function $f : M \to \mathbb{R}$. This function f will assign to each event in M a numerical value in such a way that it preserves the causal ordering of events in M. By construction, each of the preimages $f^{-1}(t)$ will be a maximal spacelike hypersurface of M. Our intuition for topology change may then be refined by expecting there to be two times t and t' such that $f^{-1}(t)$ and $f^{-1}(t')$ are not homeomorphic in M.

[1]Throughout this thesis we will reserve the term "spacetime" for locally-Euclidean, second-countable, Hausdorff spaces equipped with a smooth atlas, a Lorentzian metric and a time-orientation, as per usual. We will also assume that spacetimes consist of a single connected component, though its boundary may not generally be connected.

With the computation of transition amplitudes in the back of our minds, we will consider the specific case in which M is a compact manifold with spacelike boundary. This implies that global time functions on (M,g) will take image on some closed interval $[t_i,t_f]$ instead of the whole real line. Formally, we then say that a spacetime (M,g) with spacelike boundary exhibits topology change whenever (M,g) admits a global time function $t : M \to [t_i,t_f]$ for which the preimages $f^{-1}(t_i)$ and $f^{-1}(t_f)$ are not homeomorphic. This allows us to arrange the distinct spatial topologies to be on the boundary of spacetime, with (M,g) smoothly interpolating between the two. We will now set about describing these topology-changing spacetimes in detail. To begin with we will start with their properties as smooth manifolds.

Within mathematical literature, a compact n-dimensional oriented smooth manifold M is called a *cobordism* between two codimension-1 closed manifolds N_1 and N_2 whenever the boundary of M is diffeomorphic to $N_1 \sqcup N_2$. When this is the case, N_1 and N_2 are referred to as *cobordant*. Historically, this notion of cobordism was initially proposed as an attempt to classify all closed manifolds of fixed dimension – being cobordant is weaker than diffeomorphic or homeomorphic equivalence, yet relates a pair of manifolds if one can be smoothly interpolated into the other inside the "space of manifolds" [6, 73, 102].

The cobordism relation (which is an equivalence relation) allows the demarcation of manifolds into a ring-structure called the *cobordism ring*, and the properties of this ring detail interesting dimension-dependent questions about manifolds [102]. Moreover, this notion of cobordism is related to the construction of one manifold from another by handle attachments, a process known as *surgery*. Indeed, it can be shown that a pair of closed manifolds are cobordant if one is constructible from the other by a finite series of handle attachments or removals. In general, a pair of $(n-1)$-dimensional closed manifolds need not be cobordant. However, in low dimensions there are positive existence results: it can be shown that any pair of $(n-1)$-dimensional closed manifolds are cobordant when $n = 2,3$ or 4 [73].

This notion of a smooth cobordism can be elegantly described via *Morse theory* [74, 81]. In this approach, one foliates a manifold by maximal codimension-1 hypersurfaces, and studies their evolution into each other. The manifold M is described via a *Morse function*, a smooth map $f : M \to [0,1]$. Tracing through the level sets of f (i.e. the preimages $f^{-1}(t)$) will then describe a foliation of M. Importantly, not every preimage $f^{-1}(t)$ will be an embedded submanifold of M. In some cases, the preimage $f^{-1}(t)$ may be an *immersed* submanifold, meaning that there may be singular behavior on this slice of M. These special slices correspond to critical points of the Morse function f, and the exact nature of these critical points encodes which type of handle to attach to change the topology of the slices of M. Thus from this Morse-theoretic data, one may rebuild M from a sequence of handle attachments applied to N_1.

1.1.1 Lorentz Cobordisms as Morse Geometries

We may formulate topology-changing spacetimes within the language of cobordisms: an n-dimensional compact topology-changing spacetime is a smooth cobordism between inequivalent codimension-1 hypersurfaces N_1 and N_2, for which the interpolating bulk manifold M

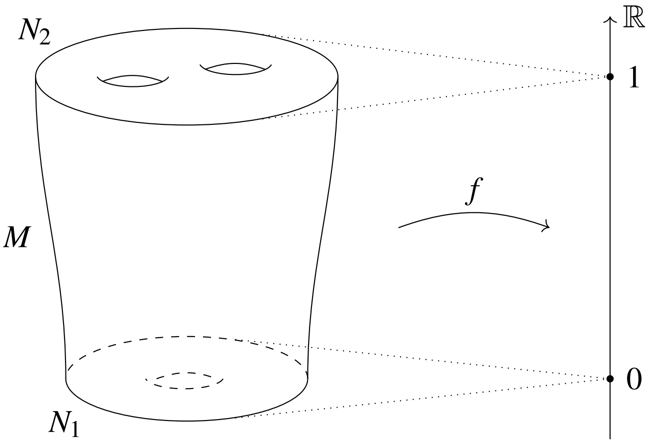


Figure 1.1: A smooth cobordism M between a pair of surfaces N_1 and N_2. This procedure can be described via a Morse function $f : M \to [0,1]$, where $N_1 \cong f^{-1}(0)$ and $N_2 \cong f^{-1}(1)$. The critical points of f indicate changes in the topology of M.

admits a spacetime structure. Such spacetimes are known in the literature as *Lorentz cobordisms* [110, 111]. Given the Morse-theoretic description of cobordisms and their guaranteed existence in low dimensions, it seems natural to try and induce a spacetime structure on a smooth cobordism M in such a manner that the Morse function $f : M \to [0,1]$ becomes a global time function. This approach dates back to Yodzis [110, 111], though some basic observation of the idea already exists in [42]. We will now demonstrate some shortcomings inherent to this approach.

We first recall that although any smooth manifold always admits a metrics of Euclidean signature, it is not the case that every manifold may be turned into a spacetime. A well-known theorem states that a manifold M admits a globally-defined Lorentzian metric g with time orientation whenever it admits a globally non-vanishing vector field [86]. In particular, one may create a Lorentzian metric g on M from a Euclidean metric h by using a vector field v to modify its signature via:

$$g(u,w) := h(u,w) - 2\frac{h(v,u)h(v,w)}{h(v,v)}. \tag{1.1}$$

In the above, the global definition of h requires the global non-vanishing of the vector field v, since otherwise the metric has a singularity in the latter term. Moreover if one opts to remove the normalization altogether, then vanishing of v would allow the signature of the metric to defer back to Euclidean signature at some points in M.

If we apply the above prescription to the gradient vector field $v = \nabla f$ arising from a Morse function f, we see that any topology-changing spacetime will inevitably admit singularities in its induced Lorentzian metrics at the critical points of its Morse function. Thus, in the situation where a topology-changing spacetime can be smoothly described via Morse theory, the resulting Lorentzian structure will be singular at certain points in the manifold.

Even in the case where we do not explicitly consider a Morse-theoretic approach, results such as the Hairy Ball Theorem demonstrate that not every smooth manifold admits a globally non-vanishing vector field [50]. In fact, arguments from cohomology tell us that such a vector field only exists provided the underlying topological space has zero Euler characteristic. Putting this together, we see that a compact, smooth manifold admits a spacetime structure whenever its Euler characteristic vanishes. As we will see in a moment, it may well be the case that simple topology-changing spacetimes have non-zero Euler characteristic, and therefore do not admit any non-singular Lorentzian metrics, Morse-theoretic or otherwise.

Before discussing our prototypical example of topology change, we will first outline a sort-of converse argument to the above. In [42], Geroch assumes the existence of a desired Lorentz cobordism, and then shows that it cannot be simultaneously topology-changing and causally well-behaved. Suppose that M is a Lorentz cobordism between codimension-1 manifolds N_1 and N_2. If we demand that this cobordism be non-trivial, i.e. that N_1 be not homeomorphic to N_2, then Geroch shows that a globally-defined spacetime structure causes (M,g) to have closed timelike curves. The argument itself is fairly simple: the spacetime structure on M defines a globally non-vanishing vector field v, and we may use the integral curves associated to v to construct a family of maximal timelike curves. If we assume that there are no closed timelike curves in M, then these integral curves will have to start on N_1 and end on N_2. Flowing N_1 along these curves will then yield a diffeomorphism with N_2, and thus there is no topology change. Simply put, Geroch's result shows that topology change is simultaneously incompatible with a non-singular metric and the absence of closed timelike curves.

1.1.2 The Trousers Space

Perhaps the simplest example of a topology-changing spacetime is the *Trousers space*, which is a smooth cobordism from one circle to two. Topologically speaking, an individual circle is being pulled apart from itself, until it discontinuously rips into two copies, as pictured in Figure 1.2. The Trousers space is the two-dimensional manifold that traces out this splitting process. As a smooth manifold, the Trousers space is well-defined and can easily constructed as a quotient space in a manner analogous to the construction of the cylinder from a square [4, 36].

Firstly, we observe that the Euler characteristic of the Trousers space is equal to -1. According to the previous discussion, this means that the Trousers space can never admit a non-singular Lorentzian metric. Additionally, any modification of this topology to include any further complications to its branching (i.e. adding extra genera or extra legs to the Trousers) will only ever decrease its Euler characteristic, and will never make it vanish. This means that all two-dimensional trouser-like transitions will be subject to some kind of abnormality in their causal structures. That being said, it is still possible to describe a metric on the Trousers space whose causality is well-behaved *almost* everywhere.

In Figure 1.2, we see some level sets of a Morse function for the Trousers. At some stage in the evolution there is a level set that is homeomorphic to the wedge sum $S^1 \vee S^1$; a sort-of "figure-8" configuration. At this stage, the spatial slice will be an immersed submanifold,

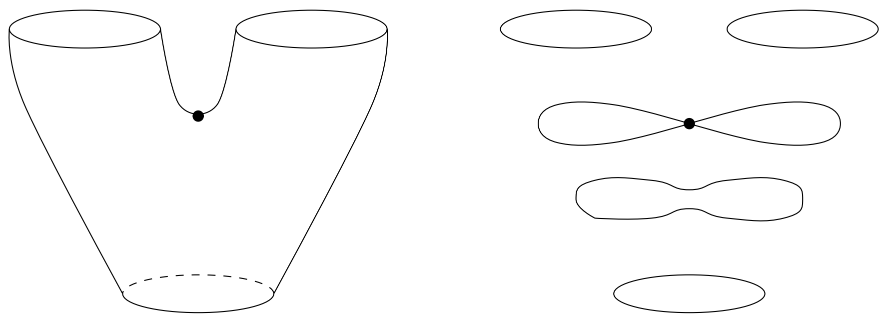


Figure 1.2: The Trousers space, together with a sample of its spacelike slices. Due to Morse theory, there is an inevitable singularity in any Lorentzian metric induced from a Morse function describing the Trousers space.

and will contain a critical point in the Morse function. This particular critical point is known as the *crotch singularity*, pictured as the marked point in Figure 1.2. It should be noted that by virtue of the Trousers space exhibiting topology change, *any* Morse function will induce a singularity of this kind. Nonetheless, the Trousers space will still admit a global time function, and thus exhibits at least some well-behaved causality.

Although any Morse-inspired metric on the Trousers space will inevitably possess a crotch singularity, there is still a sense in which one may attribute at least some causal structure to the whole space. In [41], the authors introduce the notion of *causal closure*, which permits the extension of well-defined causalities to their boundaries (in some sense of the term). This technique can be successfully applied to the Trousers space, and yields an unusual lightcone structure at the crotch singularity, depicted in Figure 1.3. Here there are four lightcones at the singularity – two past and two future. Physically speaking, we may interpret this as a local breakdown of the Einstein Equivalence Principle.

Despite being slightly unusual from a causal perspective, there is still a sense in which one may define and evaluate an Einstein-Hilbert action for the Trousers space. In two dimensions, the gravitational action reduces to the total scalar curvature of the manifold, together with boundary terms. In [67], Sorkin and Louko demonstrate that the Trousers is flat almost everywhere, but there is a δ-like singularity in the curvature of the Trousers space of strength $\pm 2\pi i$. In distinction to the dilaton field of closed bosonic string theory, the sign ambiguity in the total scalar curvature is a genuine choice, with sign dictated by a choice of branch cut of their analytic continuation scheme. Sorkin and Louko advocate for the sign choice $-2\pi i$, since this results in the suppression of the Trousers in a resulting gravitational path integral. Note that the consideration of the Trousers space may be justified on causal grounds, by virtue of its conformal structure inducing a *causal poset* relation [28].

The argument of Sorkin and Louko was later corroborated by the discrete Regge arguments of [100], where the curvature contribution at the crotch singularity is computed via an angular deficit. In this version of the argument, the Trousers space is triangulated and its total scalar is determined by a Lorentzian version of the Gauss-Bonnet theorem. In Lorentzian signature, angles are complex-valued, and in particular, a vector will collect a contribution of $\pm\frac{\pi i}{2}$ every time that it crosses a null ray and passes into an adjacent quadrant

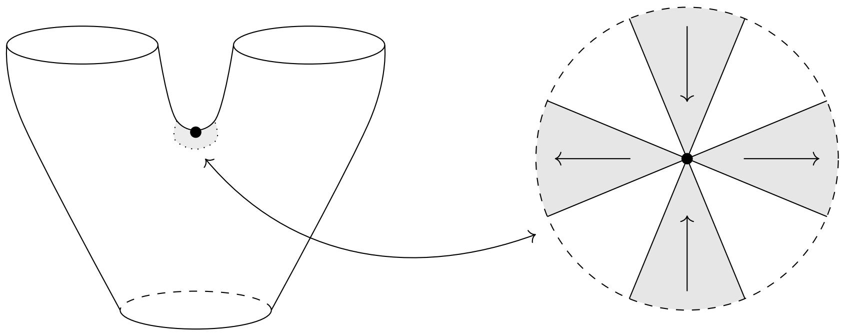


Figure 1.3: The Trousers space together with it's crotch singularity. Here, the causal completion induces four distinct lightcones (right); two future-directed and two past-directed.

of Minkowski space. By arranging a triangulation to take a vertex at the crotch singularity, one may determine the curvature contribution at the crotch singularity to be $\pm 2\pi i$ due to the extra lightcones present. Again, it is argued by hand that $-2\pi i$ be the correct choice of sign convention, as this entails the desired suppression behaviour.

Analyses such as [67] and [100] suggest that topology-change may be causally well-behaved enough to be included within a path integral. However, it should be noted that the causal abnormalities of the Trousers space are still prominent enough to pose problems for other aspects of physics. In particular the Trousers space is not globally-hyperbolic, and this poses significant problems for any resulting quantum field theories placed upon it. In [4, 16, 70] it has been argued that a scalar quantum field placed on the Trousers space will be misbehaved, ultimately due to the crotch singularity. Thus the Trousers space appears to be "physically unreasonable" in at least some important sense.

1.2 Penrosian Topology Change

We now present a model of topology change which is markedly different to the Lorentz cobordisms discussed above. This style of topology-changing spacetime was first introduced by Penrose in [89]. In a sense, the picture of Penrose's topology change occurs at the scale of individual points, viewed as zero-dimensional submanifolds. Since cardinality is the lone topological invariant of a zero-dimensional manifold, the only way to change the topology of an individual point is to increase its number. In the simplest case, we can imagine that an individual point in spacetime is doubled, with this topology change growing along null rays, as pictured in Figure 1.4.

In the article [89], Penrose proposes this idea as a remark regarding time-asymmetry. He likens these models to an Everettian interpretation of quantum mechanics, in which the branching out accompanies a type of wavefunction collapse. However, this is all speculative and throughout this thesis we will remain agnostic with regards to such ideas. Instead, we will now concern ourselves with the *mathematical* nature of a spacetime like this. As Penrose correctly identifies, the doubled-point structure of topology change renders these spaces outside the scope of ordinary differential geometry: these spaces are not Hausdorff.

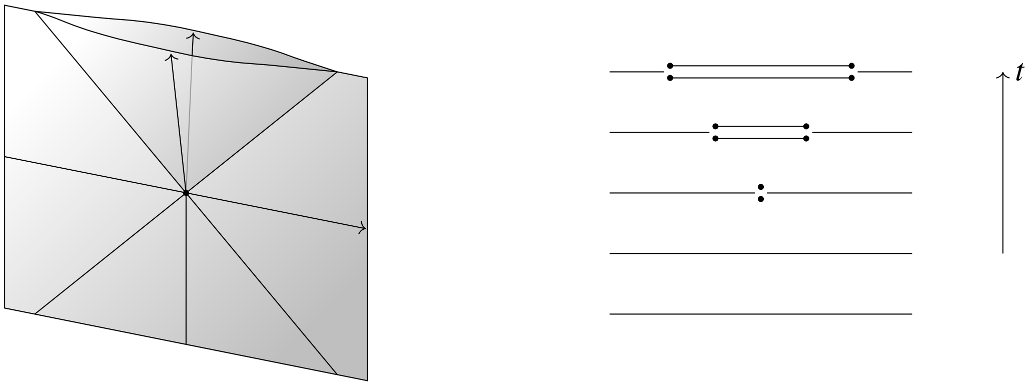


Figure 1.4: A simplification of Penrose's non-Hausdorff topology changing spacetime (left), together with a sample of its level sets (taken from [80, Fig. 2]). Although impossible to depict in ordinary Euclidean space, here there are two copies of the origin superimposed on top of each other, together with two copies of the future null cone.

1.2.1 Non-Hausdorff Manifolds

In contrast to the Hausdorff property, a topological space is called non-Hausdorff whenever there exists a pair of points whose open neighbourhoods always intersect.[2] Typical examples arise when trying to topologise a discrete collection of points, such as the Zariski space of Algebraic Geometry, the digital line of pointset topology, and the non-Hausdorff spaces of Domain theory. Aside from this, the point-set topological properties of non-Hausdorff spaces are seemingly well-studied, typically going under the name of *non-metrizable manifolds* [39, 71].

Within differential geometry, non-Hausdorffness is rarely considered. Perhaps the best-known occurrence of non-Hausdorff manifolds comes from the study of foliations [39]. In this setting, one foliates a Hausdorff manifold by immersed submanifolds (called leaves), and then introduces an equivalence relation that identifies points if they lie within the same leaf of the foliation. As each leaf collapses to a single point in the quotient space, a new structure emerges, called the *leaf space* of the foliation. This leaf space may be non-Hausdorff, as Figure 1.5 (adapted from [69]) demonstrates for the *line with two origins*.

Although every group action induces an equivalence relation by which one may quotient, the converse is not true in general. In fact, generally speaking, every equivalence relation is dual not to a group, but to a group*oid*. To each groupoid, one may associate a C^*-algebra, and thus we are lead to the study of *non-commutative geometry* à la Connes [22]. Applying this observation to the non-Hausdorff leaf spaces of foliations, we may assign a groupoid to this structure, known as the *holonomy groupoid*, and then use non-commutative algebras to study the resulting objects. In this context, non-Hausdorff manifolds have been studied to some extent.

Aside from the advanced studies of foliation theory and non-commutative geometry, there is a gap in the literature regarding basic properties of non-Hausdorff manifolds from

[2] Throughout this thesis we will reserve the "manifold" for the ordinary usage, and will use the term "non-Hausdorff manifold" to mean a locally-Euclidean, second-countable, non-Hausdorff space.

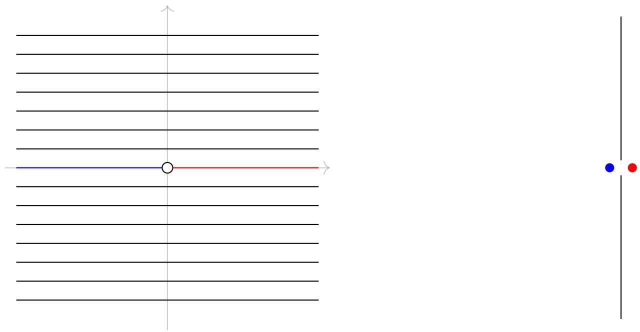


Figure 1.5: A foliation of the space $\mathbb{R}^2\backslash\{0\}$ (left), and the leaf space of this foliation (right). This leaf space is a non-Hausdorff manifold, known as the line with two origins.

their foundation up. Typically, standard texts on differential geometry may not even mention non-Hausdorff manifolds at all, or perhaps they will provide a few remarks as an afterthought. Older texts such as Lang [62] or Hirsch [54] prefer to start their treatment of manifolds locally and in as much generality as possible, so may not invoke the Hausdorff property until several chapters in. Nonetheless, manifolds are usually defined to be Hausdorff, for the following reason.

The passage from local to global lies at the heart of differential geometry. Essentially every geometric object one may consider on a manifold may be described by performing a construction locally in each chart, and then consistently combining these local constructions into something that makes sense across the whole manifold. There are several techniques for gluing together locally-defined objects into a global one, and the most relevant for our purposes involves the use of partitions of unity. Roughly put, partitions of unity provide a way to smoothly "patch together" local data across overlapping subsets of a manifold by assigning a function to each subset whose total sums to one over the entire space. Typically, these functions are taken to have compact support in the subsets they are tied to, and thus in most cases a partition of unity can be seen as a system of consistently-defined bump functions that sum to one over the whole manifold.

Often times, we define a linear quantity chartwise, and then take linear combinations of chartwise definitions, scaled by smooth functions. The precise properties of a partition of unity ensures that only finitely-many terms in this linear combination contribute, and the sum is finite and thus is well-defined. Using this idea, partitions of unity are applied in a variety of constructions within differential geometry, including in the extensions of functions, the locality of derivative operators, the definition of integration, the construction of a Riemannian metric on any manifold, and so on. Given their utility, partitions of unity are typically considered to be a crucial tool in differential geometry.

In a non-Hausdorff space, it is not always the case that a compact subset is topologically closed. This unusual fact causes issues for a partitions of unity, as the support of a function (which is defined to be a closed subset) may not fully lie in the intended open set anymore. In fact this issue is much more pervasive – later in this thesis we will show that *any* cover of a non-Hausdorff manifold by Hausdorff open sets will not admit a partition of unity sub-

Figure 1.6: A construction of the line with two origins as an adjunction of two copies of the real line.

ordinate to it. It is for this technical reason that Hausdorffness is usually taken as a defining property of manifolds.

1.2.2 Non-Hausdorff Spacetimes

Within the physics literature, non-Hausdorff manifolds are almost entirely unstudied. Perhaps the best-known examples come from the maximal extensions of the Misner space [92] and the Taub-NUT space [46, 52]. Note that here, the non-Hausdorffness occurs at the boundaries of these spacetimes, and not in the bulk, as in Figure 1.4. Other discussions of non-Hausdorff spacetimes exist in the philosophical literature [9, 53, 77]. However, up until recently, no complete geometrical understanding of non-Hausdorff manifolds was studied.

Throughout this thesis we will realize Penrose's idea by properly introducing the relevant geometric structures of interest for physics, all whilst circumventing the non-existence of arbitrary partitions of unity. In particular, we will describe the relevant topological, differentiable and Lorentzian structures on a non-Hausdorff manifold, with a later interest in applying these ideas in a path integral analysis similar to that of the Trousers space. Aside from filling in a gap in the literature, we are motivated by the possibility that Penrosian topology-changing spacetimes may admit non-singular Lorentzian metrics, at the cost of their separability. Indeed, by virtue of being non-Hausdorff, it appears as though these spacetimes fall outside of the no-go theorems for Lorentz cobordisms, and may therefore be a superior model of topology change in Lorentzian signature.

Underpinning almost every result in this thesis is the following observation, which probably dates back to [45], and exists in some form in [47]. In [45], Haefliger and Reeb study non-Hausdorff 1-manifolds as leaf spaces of codimension-1 foliations, and introduce an interesting representation of non-Hausdorff manifolds as adjunction spaces. Figure 1.6 depicts a simple example of their idea – essentially, one can construct the line with two origins by gluing together two copies of the real line everywhere except at their origins. Generally speaking, Haefliger and Reeb construct other non-Hausdorff 1-manifolds in this way, that is, by gluing together copies of the real line along open sets.

In principle, the gluing construction of Haefliger and Reeb offers a clue as how to circumvent the partition of unity problem detailed above. Roughly put: if we were to show that every non-Hausdorff manifold (perhaps subject to some additional constraints) were able to be represented by gluing together ordinary Hausdorff manifolds together, then we may be able to construct any geometric structure of interest piecewise on each Hausdorff subset. It seems natural to then try and "consistently glue" the Hausdorff pieces together to effectively transfer any geometric structure from local Hausdorff data into the global non-Hausdorff manifold. A large portion of this thesis is devoted to making precise this notion of gluing.

Using the idea of Haefliger and Reeb, we see that Penrose's spacetime in Figure 1.4 ought to be constructed by gluing together two copies of ordinary $2d$ Minkowski spacetime along the complements of the causal future of the origin. In flat spacetime causal cones are topologically closed, so their complements are open and thus the gluing should yield a non-Hausdorff space whose Hausdorff-violating points lie on the future null cones of the doubled origin. In principle, this type of gluing should be possible at the level of Lorentzian structures, and its curvature properties should be expressible in terms of the Hausdorff pieces of Minkowski space.

1.3 Outline of this Thesis

Throughout this thesis we will present a theory of non-Hausdorff topology-changing spacetimes. We will start from its mathematical foundations as a topological space, and then slowly work our way through all of the relevant structures until we can meaningfully refer to "non-Hausdorff spacetimes" as a modification of the familiar notion of a spacetime. After discussing topological matters, we will then refine our notion to include smooth manifolds, vector bundles, and then the sections of these bundles. With an eye towards reproducing the analysis of Sorkin and Louko for the Trousers space, we will prove a Gauss-Bonnet theorem and then use this to evaluate non-Hausdorff transitions between one circle and two. The chapters are organised as follows.

In Chapter 2 we will introduce the basic topological properties of non-Hausdorff manifolds. In particular, we flesh out the details of the idea depicted in Figure 1.6 by introducing a formalism for gluing together arbitrarily-many topological spaces together. As with the observation of Haefliger and Reeb, we will see that if one glues ordinary manifolds along open subsets then under certain circumstances one may construct a non-Hausdorff manifold. Importantly, we will see that *every* non-Hausdorff manifold can be expressed as a gluing of Hausdorff ones in this manner. Throughout this chapter we will also take asides into other topological elements of gluing spaces, in particular we also partially characterise maximal Hausdorff submanifolds that a non-Hausdorff topological manifold may admit.

In the third chapter, we extend our gluing formalism to an appropriate smooth category. We start by showing that if, in addition to the requirements of Chapter 2, one assumes all manifolds in the gluing construction are smooth, and all maps preserve these smooth structures in some precise sense, then the resulting quotient space assumes the structure of a *smooth* non-Hausdorff manifold. We use then extend this result to the setting of vector bundles as well: we show that all vector bundles over a non-Hausdorff manifold may be realised as a particular gluing of Hausdorff vector bundles. Importantly, we will also describe the space of smooth sections over a non-Hausdorff bundle in terms of sections over particular Hausdorff subbundles. Finally, we finish Chapter 3 by describing the Čech cohomology for non-Hausdorff manifolds, and use this to classify non-Hausdorff line bundles in certain cases.

In Chapter 4 we study differential forms and their integral properties in the non-Hausdorff case. Such an inquiry requires particular care: both integration and de Rham cohomology typically assume arbitrarily-existent partitions of unity in their standard treatment, so in the non-Hausdorff case several modifications of standard differential geometry need to be made. We will make these modifications, and then consequently prove an equivalence between de Rham and singular cohomologies in the non-Hausdorff case. We finish Chapter 4 with a proof a non-Hausdorff version of the Gauss-Bonnet theorem. The novelty here is that the integral expression for total curvature requires an additional counterterm which computes the geodesic curvatures of the Hausdorff-violating submanifolds sitting inside the non-Hausdorff surface – a clear difference from the standard Hausdorff case.

Finally, in Chapter 5 we will investigate the inclusion of non-Hausdorff manifolds within a path integral for gravity in Lorentzian signature. We consider the simplest possible setting, namely two dimensions. The reason for this ought to be clear – gravity in two dimensions is purely topological, so if there are any oddities due to the non-Hausdorffness of our manifolds, then these ought to already manifest in the $2d$ theory. Throughout this chapter we will prove a Lorentzian version of the Gauss-Bonnet theorem of Chapter 4, and then use this to motivate a gravitational action for non-Hausdorff manifolds. The result is an action similar to the ordinary Einstein-Hilbert action, however there is an extra Gibbons-Hawking boundary term for the Hausdorff-violating submanifolds sitting inside our space. In direct analogy to the Sorkin-Louko analysis for the Trousers space, we then derive the correct sign convention for Lorentzian angles that will entail appropriate suppression rules for non-Hausdorff transitions within the path integral.

We will then conclude the thesis with a technical summary of our results and some comments regarding future work. It should be remarked that as a general theme, we will assume the most generality in the beginning, and will slowly refine our notion of non-Hausdorff manifold to match our more-involved considerations. In Chapter 2, for instance, we will consider the case of non-Hausdorff manifolds formed from countably-many Hausdorff ones. Already by Chapter 3 we will restrict our notion to merely finite colimits. Then, in Chapter 4 we will add additional cohomological assumptions to the Hausdorff-violating submanifolds, and finally in Chapter 5 we will consider the specific example of the so-called *non-Hausdorff Trousers space*. Each assumption that results in a loss of generality has been made explicit in the technical summary found in the end matter of this document.

1.4 Remarks on Formatting

This document is a "Thesis by publication", defined according to OIST's Graduate School guidelines. The present work is a concatenation of the four papers [83–85] and [80], all of which I completed throughout my enrollment as a PhD student. According to the aforementioned guidelines, I have elected to leave the content of these four papers essentially unchanged, aside from the following modifications.

1. At the start of each Chapter I have included a one-page heuristic introduction that briefly introduces the context of the paper in regards to the overall logical flow of this thesis.

2. At the end of each Chapter I have included a small summary, entitled "Summary of Chapter" which summarizes the key points of the paper in regards to the overall context of the thesis.

3. The bibliography of each paper has been removed from the end of each Chapter and has been replaced with a total bibliography found at the end of this thesis.

4. The indexing of Theorems, Definitions, Lemmas, equations and so on have been reformatted to match the overall indexing of the chapters.

5. Any self-referential language has been modified to merge with the language of this thesis – for instance, terms like "In this paper..." have been replaced with "In this chapter...".

6. The abstracts have been removed from each paper.

7. The acknowledgments of each individual paper have been removed and merged with the overall acknowledgments at the start of this document.

8. Any appendices to the papers have been removed from the bulk text of the chapter and placed at the end of the overall document.

Aside from these formatting changes, there are no modifications to the intellectual content of the papers. In particular, I have opted to include the original introductions and conclusions within each chapter. This means that each Chapter, being an individual paper, is essentially self-contained. Due to the logical flow of my thesis topic, this means that each paper will inevitably contain repetitions of ideas, definitions, results, and so on. Therefore, any reader who chooses to read this document from start to finish is encouraged to keep this in mind, and to skip over repeated subject matter where necessary.

Chapter 2

Topological Properties of non-Hausdorff Manifolds

This chapter is published under the name “Non-Hausdorff Manifolds via Adjunction Spaces", Topology and its Applications, 326:108388, 2023 [84], and I am the sole author.

In this chapter we introduce the point-set topological properties of non-Hausdorff manifolds. We start by providing a general formalism of adjunction spaces which allows for arbitrarily-many topological spaces to be glued together. Within this formalism we will confirm various useful features of non-Hausdorff manifolds that will be used later on in this thesis. In particular, we show that

- Countably-many Hausdorff manifolds may be glued along pairwise homeomorphic open submanifolds to yield a second-countable, locally-Euclidean space.
- If these open submanifolds have pairwise homeomorphic boundaries (a condition that is made precise in Section 2.4.1), then the resulting topological space is always a non-Hausdorff manifold.
- In a similar spirit to Hajicek's study of H-submanifolds [47], we show that *every* non-Hausdorff manifold may be realised by our adjunction formalism.

Aside from these results, we also provide various examples of non-Hausdorff manifolds. In particular, we spend some time evaluating a 2-sphere in which the equatorial S^1 is twisted around itself so that its antipodes are Hausdorff-violating. We finish the chapter with a contribution towards classifying the maximal Hausdorff submanifolds that a given non-Hausdorff manifold may permit.

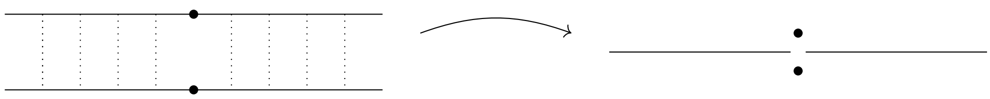


Figure 2.1: The line with two origins (right), constructed by gluing together two copies of the real line.

2.1 Introduction

In this chapter we will study non-Hausdorff manifolds. These are locally-Euclidean second-countable spaces that contain points that are "doubled" or superimposed on top of each other. As the name would suggest, such points cannot be separated by open sets, and thus violate the Hausdorff property. The prototypical example of a non-Hausdorff manifold is the so-called *line with two origins*, pictured below.

The line with two origins can be constructed by gluing together two copies of the real line together everywhere except at the origin. Other examples such as those figures found in [8] and [45] suggest a general theory for constructing non-Hausdorff manifolds by gluing together Hausdorff ones along open subspaces. A recent result of Placek and Luc confirms that any non-Hausdorff manifold can be built according to such a procedure [68].

In this chapter we will introduce and refine an approach similar to that of [68] and [82] to further study non-Hausdorff manifolds. We will start by introducing a calculus for adjoining countably-many Hausdorff manifolds together. We will show that any adjunction of countably-many Hausdorff manifolds M_i along open subsets A_{ij} with pairwise homeomorphic boundary components will yield a non-Hausdorff manifold $\mathbf{M}$ in which the Hausdorff-violation occurs precisely at the $\mathbf{M}$-relative boundaries of the subsets A_{ij}. Moreover, in such a situation the manifolds M_i will sit inside $\mathbf{M}$ as maximal Hausdorff (open) submanifolds.

Interestingly, a non-Hausdorff manifold will often admit infinitely-many maximal Hausdorff submanifolds [77]. This observation motivates the following question: in a non-Hausdorff manifold built by gluing together Hausdorff manifolds M_i, are there any topological properties that distinguish the M_i from the other maximal Hausdorff submanifolds? There is a well-known criterion due to Hajicek [47] that guarantees a given subset is a maximal Hausdorff submanifold. We will spend some time refining this idea, with the eventual conclusion being that if the non-Hausdorff manifold is "simple" (in a precise sense to be defined later) then the spaces M_i are the unique subspaces satisfying a stricter form of Hajicek's criterion.

This chapter is organised as follows. In the first section we will introduce a generalised theory of adjunction spaces as colimits of appropriate diagrams. We will identify various conditions which allow certain topological features to be preserved in the adjunction process. With an eye towards the rest of the chapter, special attention is paid to those adjunction spaces formed by gluing together topological spaces along open subspaces.

In Section 2.3 we will apply this formalism to the setting of manifolds. We will show that locally-Euclidean second-countable spaces can be formed by gluing together Hausdorff manifolds along homeomorphic open submanifolds. We will then spend some time studying the situation in which the gluing regions have homeomorphic boundary components. We will also argue that non-Hausdorff manifolds built in this way may be paracompact, however they will not admit partitions of unity subordinate to every open cover. We finish Section 2.3 with a discussion of some known results found regarding Hausdorff submanifolds.

In Section 2.4 we will introduce some examples of non-Hausdorff manifolds built from adjunction spaces. Most of these revolve around Euclidean space, with the exception of a non-Hausdorff sphere [25, 55], which we will construct from four copies of punctured spheres. Of particular interest is the branched Euclidean plane, which we will see has infinitely-many maximal Hausdorff submanifolds. Motivated by our examples, in Section 2.5 we will generalise the result of [77] and show that particularly simple non-Hausdorff manifolds admit only finitely-many maximal Hausdorff submanifolds satisfying a natural condition on their boundaries.

Throughout this chapter we will assume that all manifolds, Hausdorff or otherwise, are locally-Euclidean, second-countable and connected. We will denote Hausdorff manifolds using standard Latin letters, and we will use boldface characters to emphasise that the manifold in question is potentially non-Hausdorff. All notions of topology used in this chapter can be found in standard texts such as [78] or [64].

2.2 Adjunction Spaces

We start by presenting our formalism for general adjunction spaces. The focus is mainly on the situation in which topological spaces are glued along open sets, since this will be an important precursor to our later discussions of non-Hausdorff manifolds.

2.2.1 Basic Properties

There are at least two ways to glue together multiple topological spaces in a consistent way. These are:

1. to iterate a binary construction several times over, or
2. to glue a collection of spaces together simultaneously.

The first approach would amount to suitably modifying the standard adjunction spaces found in say [64] or [14]. Throughout this chapter we will instead focus on the latter case. Formally, gluing together multiple spaces can be achieved by fixing some index set I to enumerate the spaces that we would like to glue together, and by defining a triple of sets $\mathscr{F} := (\mathsf{X}, \mathsf{A}, \mathsf{f})$, where:

- the set X is a collection of topological spaces X_i,
- the set A is a collection of sets A_{ij} such that $A_{ij} \subseteq X_i$ for all $j \in I$, and
- the set f is a collection of continuous maps $f_{ij} : A_{ij} \to X_j$.

In order to yield a well-defined adjunction space, we need to impose some consistency conditions on the data contained within $\mathscr{F}$. These conditions are captured in the following definition.

Definition 2.2.1. *A triple $\mathscr{F} = (X, A, f)$, is called an adjunction system if it satisfies the following conditions for all $i, j \in I$.*

- ***A1)*** $A_{ii} = X_i$ *and* $f_{ii} = id_{X_i}$
- ***A2)*** $A_{ji} = f_{ij}(A_{ij})$, *and* $f_{ij}^{-1} = f_{ji}$
- ***A3)*** $f_{ik}(a) = f_{jk} \circ f_{ij}(a)$ *for each* $a \in A_{ij} \cap A_{ik}$.

Observe that the second condition above ensures that each f_{ij} is a homeomorphism. Given an adjunction system $\mathscr{F}$, we can then define the *adjunction space subordinate to* $\mathscr{F}$, denoted $\bigcup_{\mathscr{F}} X_i$, as the topological space obtained from quotienting the disjoint union

$$\bigsqcup_i X_i := \{(x,i) \mid x \in X_i\} \tag{2.1}$$

under the relation $\cong$, where $(x,i) \cong (y,j)$ iff $f_{ij}(x) = y$. The conditions of Definition 2.2.1 are precisely what is needed to ensure the relation $\cong$ is an equivalence relation. Points in the adjunction space $\bigcup_{\mathscr{F}} X_i$ can be described as equivalence classes of the form

$$[x,i] := \{(y,j) \mid f_{ij}(x) = y\}. \tag{2.2}$$

By construction we have a collection of canonical maps $\phi_i : X_i \to \bigcup_{\mathscr{F}} X_i$ which send each x in X_i to its equivalence class in $\bigcup_{\mathscr{F}} X_i$. By construction these maps are continuous and injective. Moreover, these maps will commute on the relevant overlaps, i.e. the equality

$$\phi_j \circ f_{ij} = \phi_i \tag{2.3}$$

holds for all i, j in I. Since the topology of an adjunction space is the quotient of a disjoint union, by construction we have the following useful characterisation of open sets.

Proposition 2.2.2. *A subset U of $\bigcup_{\mathscr{F}} X_i$ is open in the adjunction topology iff $\phi_i^{-1}(U)$ is open in X_i for all i in I.*

In the binary version of adjunction spaces, it is well-known that the adjunction of two spaces is the pushout of the diagram below [14].

$$Y \xleftarrow{\ f\ } A \xhookrightarrow{\ \iota_A\ } X \tag{2.4}$$

The following result shows that the adjunction space subordinate to $\mathscr{F}$ can be seen as the colimit of the diagram formed from $\mathscr{F}$.

Lemma 2.2.3. *Let $\psi_i : X_i \to Y$ be a collection of continuous maps from each X_i to some topological space Y, such that for every $i, j \in I$ it is the case that $\psi_i = \psi_j \circ f_{ij}$. Then there is a unique continuous map $g : \bigcup_{\mathscr{F}} X_i \to Y$ and $\psi_i = g \circ \phi_i$ for all i in I.*

Proof. We define the map g by $g([x,i]) = \psi_i(x)$, that is, $g = \psi_i \circ \phi_i^{-1}$. To see that this defines a function, we need to confirm that g preserves equivalence classes. Suppose that $[x,i] = [y,j]$, i.e. $x = f_{ij}(y)$. Then:

$$g([x,i]) = \psi_i(x) = \psi_j(f_{ij}(x)) = \psi_j(y) = g([y,j]) \tag{2.5}$$

as required. We now show that g is continuous. Let U be open in Y, and consider the set $g^{-1}(U) = \phi_i \circ \psi_i^{-1}(U)$. Recall the set $g^{-1}(U)$ is open in $\bigcup_{\mathscr{F}} X_i$ iff for each $i \in I$, the set $\phi_i^{-1}(g^{-1}(U))$ is open in X_i. Observe that:

$$\phi_i^{-1}(g^{-1}(U)) = \phi_i^{-1} \circ \phi_i \circ \psi_i^{-1}(U) = \psi_i^{-1}(U) \tag{2.6}$$

which is open since ψ_i is continuous. It follows that $g^{-1}(U)$ is open in $\bigcup_{\mathscr{F}} X_i$, and thus g is continuous. To see that g is unique, we can use a similar argument to that in [14]. □

Throughout the remainder of this chapter we will consider adjunction spaces formed by gluing open sets together. In this case, we make the following useful observation.

Lemma 2.2.4. *Let $\bigcup_{\mathscr{F}} X_i$ be an adjunction space formed from $\mathscr{F}$. If each A_{ij} is an open subset of X_i, then each ϕ_i is an open embedding.*

Proof. Fix some ϕ_i. By construction ϕ_i is injective and continuous, so it suffices to show that ϕ_i is an open map. So, let U be an open subset of X_i, and consider $\phi_i(U)$. By Prop. 2.2.2 this set is open in $\bigcup_{\mathscr{F}} X_i$ iff for every $j \in I$, the preimage $\phi_j^{-1} \circ \phi_i(U)$ is open in X_j. Observe that $\phi_j^{-1} \circ \phi_i(U) = f_{ij}(U \cap A_{ij})$. Since U is open in X_i, the set $U \cap A_{ij}$ is open in A_{ij} (equipped with the subspace topology). By construction $f_{ij} : A_{ij} \to X_j$ is a embedding. Moreover, each f_{ij} is an open map since we have assumed that each A_{ij} is open. Thus $f_{ij}(U \cap A_{ij})$ is also open in X_j, from which it follows that $\phi_j^{-1} \circ \phi_i(U)$ is open in X_j for all j. Since U was arbitrary, we may conclude that ϕ_i is an open map. The result then follows from the fact that every continuous, open injective map is an open embedding. □

2.2.2 Subsystems and Subspaces

Suppose that we have an adjunction space $\bigcup_{\mathscr{F}} X_i$ built from the adjunction system $\mathscr{F}$. It follows immediately from Definition 2.2.1 that any subset J of the underlying indexing set I will yield another adjunction system $\mathscr{G}$, which can be obtained by simply forgetting the parts of $\mathscr{F}$ that are not contained in $J \subset I$. As such, there is an associated adjunction space $\bigcup_{\mathscr{G}} X_j$, that consists of gluing only the component spaces X_j where j lies in the subset J. In this situation we will refer to $\mathscr{G}$ as an *adjunctive subsystem* of $\mathscr{F}$, and we will write $\mathscr{G} \subseteq \mathscr{F}$.

In order to distinguish objects in $\bigcup_{\mathscr{G}} X_j$ from objects in the full adjunction space, we will use the double-bracketed notation $[\![\cdot]\!]$ to denote objects defined in the adjunctive subspace. In particular, we will use

$$[\![x,j]\!] := \{(y,k) \mid k \in J \wedge f_{jk}(x) = y\} \tag{2.7}$$

to denote the points in $\bigcup_{\mathscr{G}} X_j$. Furthermore, we will denote the canonical maps of the adjunctive subspace by χ_j's.

The universal property 2.2.3 yields a continuous map $g : \bigcup_{\mathscr{G}} X_j \to \bigcup_{\mathscr{F}} X_i$ which will send each equivalence class $[\![x,j]\!]$ of $\bigcup_{\mathscr{G}} X_j$ into the (possibly larger) equivalence class $[x,j]$ of $\bigcup_{\mathscr{F}} X_i$. In general there is no guarantee that the map g acts as a homeomorphism between $\chi_j(X_j)$ and $\phi_j(X_j)$. However, this happens to be the case if we require the gluing regions A_{ij} to be open.

Theorem 2.2.5. *Let $\mathscr{F}$ be an adjunction system in which each all of the gluing regions A_{ij} are open in their respective spaces. For any adjunctive subsystem $\mathscr{G} \subseteq \mathscr{F}$, the continuous function $g : \bigcup_{\mathscr{G}} X_j \to \bigcup_{\mathscr{F}} X_i$ defined by $[\![x,i]\!] \mapsto [x,i]$ is an open embedding.*

Proof. By 2.2.3 g is continuous, so it suffices to show that g is both injective and open. For injectivity, let $[\![x,j]\!]$ and $[\![y,k]\!]$ be two distinct points in $\bigcup_{\mathscr{G}} X_j$. By construction the equivalence class $[\![x,j]\!]$ contains all elements of the form $(f_{jl}(x),l)$. Since $[\![y,k]\!]$ is distinct from $[\![x,j]\!]$, we may conclude that $f_{jk}(x)$ does not equal y, and thus $[x,j] \neq [y,k]$ as well. This confirms that g is injective.

Suppose now that U is some open subset of the adjunctive subspace $\bigcup_{\mathscr{G}} X_j$. By Prop 2.2.2, U is open in the adjunctive subspace iff all of the preimages $\chi_j^{-1}(U)$ are open in their respective spaces X_j. Since we have assumed all A_{ij} are open, Prop. 2.2.4 ensures that all of the canonical maps ϕ_i of the adjunction space $\bigcup_{\mathscr{F}} X_i$ are open embeddings. Thus the images $\phi_j \circ \chi_j^{-1}(U)$ are open in the spaces $\phi_j(X_j)$. Since these are open subspaces of $\bigcup_{\mathscr{F}} X_i$, we may conclude that each $\phi_j \circ \chi_j^{-1}(U)$ is an open subset of $\bigcup_{\mathscr{F}} X_i$. It follows that the union

$$\bigcup_{j \in J} \phi_j \circ \chi_j^{-1}(U) \tag{2.8}$$

is open in $\bigcup_{\mathscr{F}} X_i$. However, this set is precisely equal to the image $g(U)$. Since we chose U arbitrarily, it follows that g is open and therefore g is an open embedding. □

2.2.3 Preservation of Various Properties

We have seen that the adjunction of arbitrarily-many topological spaces is again a topological space. However, there is no guarantee that the gluing process will preserve any pre-existing structure. The following result is a collection of conditions that suffice to preserve topological features. We state these here without proof, since the arguments involved routinely follow from Lemma 2.2.4 and basic facts of topology.

Theorem 2.2.6. *Let $\mathscr{F} = (X, A, f)$ be an adjunction system with indexing set I, and denote by $\mathbf{X}$ the adjunction space subordinate to $\mathscr{F}$.*

1. *Suppose that for each i in I, the collection $\mathscr{B}_i$ forms a basis for X_i. If each ϕ_i is an open map, then the collection $\mathscr{B} = \{\phi_i(B) \mid B \in \mathscr{B}_i\}$ forms a basis for $\mathbf{X}$.*

2. *Let $\mathbf{X}$ be an adjunction space in which each ϕ_i is an open map. If each X_i is first-countable, then so is the adjunction space $\mathbf{X}$.*

3. *Let $\mathbf{X}$ be an adjunction space in which the indexing set I is countable. If every X_i is Lindelöf, then so is $\mathbf{X}$.*

4. *Let $\mathbf{X}$ be an adjunction space in which the indexing set I is countable. If each X_i is separable, then so is $\mathbf{X}$.*

5. *Suppose each X_i is second-countable. If I is countable and each ϕ_i is an open map, then $\mathbf{X}$ is also second-countable.*

6. *If each X_i is connected and each A_{ij} is non-empty, then $\mathbf{X}$ is connected.*

7. *Let $\mathbf{X}$ be an adjunction space in which every A_{ij} is non-empty. If each X_i is path-connected, then so is $\mathbf{X}$.*

8. *If I is finite and each X_i is compact, then so is $\mathbf{X}$.*

9. *Let $\mathbf{X}$ be an adjunction space formed from a collection of T_1 spaces. If the canonical maps ϕ_i are all open, then $\mathbf{X}$ is T_1.*

10. *Let $\mathbf{X}$ be an adjunction space in which each X_i is locally-Euclidean. If each ϕ_i is an open embedding, then $\mathbf{X}$ is also locally-Euclidean.*

2.3 Non-Hausdorff Manifolds

We will now use the adjunction formalism detailed in the previous section to construct non-Hausdorff manifolds. Observe that according to Lemma 2.2.4 and items 5 and 10 of Theorem 2.2.6 we already have the following.

Theorem 2.3.1. *Let $\mathscr{F}$ be an adjunctive system consisting of countably-many (Hausdorff) manifolds M_i, in which each A_{ij} is an open submanifold. Then the adjunction space subordinate to $\mathscr{F}$ is a locally-Euclidean second-countable space.*

Let us denote by $\mathbf{M}$ the adjunction space built according to the above. The open charts of $\mathbf{M}$ at the point $[x,i]$ are given by $(\phi_i(U), \varphi \circ \phi_i^{-1})$ where (U, φ) is any open chart of M_i. Thus the manifold $\mathbf{M}$ mirrors the local behaviour of the Hausdorff manifolds M_i. According to 2.2.4 we may interpret $\mathbf{M}$ as a locally-Euclidean, second-countable space that is covered by Hausdorff submanifolds.

2.3.1 Homeomorphic Boundaries

We will now identify conditions under which Hausdorff violation is guaranteed in the adjunction process. In order to do so, we will first recall a binary relation that formally encodes Hausdorff violation. We will opt for the notation used in [47, 77], though it should be noted that the same relation can also be found in [61] and [71, p. 67] in essentially equivalent forms.

Let $\mathbf{M}$ be a locally-Euclidean second-countable space, and consider the binary relation $\curlyvee$, defined as $x \curlyvee y$ if and only if every pair of open neighbourhoods of x and y necessarily intersect. The relation $\curlyvee$ is reflexive and symmetric by construction, but it need not be transitive.[1]

[1]Interestingly, the relation $\curlyvee$ is weaker than the Hausdorff relation used in [53]. It is likely that their relation is the transitive closure of $\curlyvee$, though a proof of this is beyond the scope of this chapter.

It is well-known that a topological space is Hausdorff whenever convergent sequences have unique limits. Although the converse is not always true, in a first countable space one can always construct a sequence that converges to both elements of a Hausdorff-violating pair. As such, in our context the relation $x \curlyvee y$ asserts the existence of a sequence that converges to both x and y in $\mathbf{M}$. Given a subset V of $\mathbf{M}$, we define

$$\curlyvee^{V} := \{x \in \mathbf{M} \mid \exists y (y \in V \wedge x \curlyvee y)\}, \tag{2.9}$$

thus $\curlyvee^{V}$ consists of all points in $\mathbf{M}$ that are Hausdorff-inseparable from V.

In what follows, we will use the $\curlyvee$-relation to describe the Hausdorff violating points of the canonical subspaces $\phi_i(M_i)$. In order to do so, we will make use of the following definition.

Definition 2.3.2. *Let $\mathscr{F}$ be an adjunction system as in Theorem 2.3.1, and let $\mathbf{M}$ be the corresponding adjunction space. We say that the gluing regions A_{ij} have homeomorphic boundaries whenever each A_{ij} is a proper subset of M_i and each gluing map f_{ij} can be extended to a homeomorphism $\tilde{f}_{ij} : Cl(A_{ij}) \to Cl(A_{ji})$.*

This additional requirement will allow us to describe the Hausdorff-violating points using the $\mathbf{M}$-relative boundaries of the canonical subspaces M_i. Before getting to the detailed description, we will first introduce some useful notation.

According to Theorem 2.2.4, we may view the subspaces M_i as embedded inside the larger space $\mathbf{M}$. To simplify notation, we will identify each M_i with its image $\phi_i(M_i)$. Under this convention we may view M_i as an honest Hausdorff (open) submanifold of $\mathbf{M}$. In what follows, it will be useful to describe the boundaries of the M_i in $\mathbf{M}$. We will use the notation $\boldsymbol{\partial}$ to denote the $\mathbf{M}$-relative boundaries of the M_i, that is, we define

$$\boldsymbol{\partial} M_i := \partial^{\mathbf{M}}(\phi_i(M_i)). \tag{2.10}$$

This boldface notation will similarly be used for the $\mathbf{M}$-relative closures and interiors of sets. We will also identify each A_{ij} in M_i with its images $\phi_i(A_{ij})$ and $\phi_j(A_{ji})$ in $\mathbf{M}$. In general, each A_{ij} may have several $\mathbf{M}$-relative boundary components. We denote these as follows:

$$\boldsymbol{\partial}^i A_{ij} := \left(\partial^{\mathbf{M}} \phi_i(A_{ij})\right) \cap \phi_i(M_i). \tag{2.11}$$

We will now show that requiring an adjunction system to have homeomorphic gluing regions establishes a clear relationship between the boundary operator $\boldsymbol{\partial}$ and the binary relation $\curlyvee$.

Lemma 2.3.3. *Suppose that $\mathbf{M}$ is a non-Hausdorff manifold built from an adjunction space in which the gluing regions have homeomorphic boundaries. Let $g_{ij} : \boldsymbol{\partial}^i A_{ij} \to \boldsymbol{\partial}^j A_{ij}$ be the map given by $g_{ij} := \phi_j \circ \tilde{f}_{ij} \circ \phi_i^{-1}$.*

1. *The map g_{ij} is a homeomorphism.*

2. *For distinct points $[x,i]$ and $[y,j]$ in $\mathbf{M}$, we have that $g_{ij}([x,i]) = [y,j]$ if and only if $[x,i] \curlyvee [y,j]$.*

3. $M_j \cap \partial M_i = M_j \cap \curlyvee^{M_i}$.

Proof. The first item is immediate from 2.2.4 and 2.3.2. Let $[x,i]$ and $[y,j]$ be distinct points in $\mathbf{M}$. Suppose first that $g_{ij}([x,i]) = [y,j]$. By construction this means that $y = \tilde{f}_{ij}(x)$. In $\mathbf{M}$, the gluing region $A_{ij} = A_{ji}$ equals the intersection $M_i \cap M_j$. Since the points $[x,i]$ and $[y,j]$ are distinct in $\mathbf{M}$, it must be the case that x and y are in the boundaries of A_{ij} and A_{ji} respectively. Let a_n be some sequence in A_{ji} that converges to y in M_j. Since $\tilde{f}_{ij}$ is a homeomorphism, it follows that $\tilde{f}_{ij}^{-1}(a_n)$ is a sequence in A_{ij} that converges to x in M_i. In the adjunction space $\mathbf{M}$, the sequences $[a_n, j]$ and $[\tilde{f}_{ij}^{-1}(a_n), i]$ will be equal, and will converge to two distinct limits $[x,i]$ and $[y,j]$. Thus $[x,i] \curlyvee [y,j]$. The converse follows from part (1) and the fact that each M_i is Hausdorff, thus has unique limits. For the third item, suppose that $[y,j]$ is some element of ∂M_i. Then there is a sequence in M_i which converges to $[y,j]$ in $\mathbf{M}$. Since M_j is an open neighbourhood of $[y,j]$, without loss of generality we may assume that this sequence lies in A_{ij}. Thus $[y,j]$ lies in $\partial^j A_{ij}$. It follows that $[y,j] \curlyvee g_{ij}([y,j])$, and thus $[y,j] \in \curlyvee^{M_i}$. The converse inclusion follows almost immediately from (1) and (2). □

The above result allows us to conclude that the Hausdorff-violating pairs of a non-Hausdorff manifold can be identified as the pairwise boundaries of open submanifolds that are glued together, provided that these boundaries exist and are homeomorphic to each other. The following result extends this idea.

Lemma 2.3.4. *Let $\mathbf{M}$ be a non-Hausdorff manifold built from an adjunctive system $\mathscr{F}$. If the regions A_{ij} have homeomorphic boundaries, then for each M_i we have that*

$$\curlyvee^{M_i} = \partial M_i = \bigcup_{j \neq i} \partial^j A_{ij}. \tag{2.12}$$

Proof. The inclusion $\curlyvee^{M_i} \subseteq \partial M_i$ follows from Prop. 2.3.3.3. Suppose that $[y,j]$ is some element of ∂M_i. Then there exists a sequence in M_i that converges to $[y,j]$. Since M_i and M_j are open, without loss of generality we may assume that the sequence sits in the intersection A_{ij}. It follows that $[y,j]$ lies in $\partial^j A_{ij}$. This shows that $\partial M_i \subseteq \bigcup_{j \neq i} \partial^j A_{ij}$. Suppose now that $[y,j]$ is in $\partial^j A_{ij}$. We can use Prop 2.3.3 to conclude that $[y,j] \curlyvee g_{ji}([y,j])$. It follows that $\bigcup_{j \neq i} \partial^j A_{ij} \subseteq \curlyvee^{M_i}$. □

2.3.2 Paracompactness and Partitions of Unity

It is well-known that Hausdorff manifolds are necessarily paracompact and admit partitions of unity subordinate to any open cover [65]. In this section we will explore such results in the non-Hausdorff setting.

According to our discussion thus far, we may build non-Hausdorff manifolds from adjunction spaces. Since all Hausdorff manifolds are paracompact, we may expect there to be an analogue to the results of Theorem 2.2.6 for paracompactness. Indeed this is true: if we consider finitely-built adjunction spaces and further restrict our attention to those adjunction spaces in which the gluing regions have homeomorphic boundaries, then we obtain the following result.

Theorem 2.3.5. *Let* $\mathbf{M}$ *be a non-Hausdorff manifold built from an adjunction space in which the gluing regions have homeomorphic boundaries. If the indexing set* I *of the adjunction system* $\mathscr{F}$ *is finite, then* $\mathbf{M}$ *is paracompact.*

Proof. (Sketch) Let $\mathscr{U}$ be some open cover of $\mathbf{M}$. Consider the open covers $\mathscr{U}_i := \{\phi_i^{-1}(U) \mid U \in \mathscr{U}\}$ of the M_i. Since each M_i is paracompact, each open cover $\mathscr{U}_i$ has a locally-finite refinement $\mathscr{V}_i$. We then define

$$\mathscr{V} = \bigcup_{i \in I} \{\phi_i(V) \mid V \in \mathscr{V}_i\}. \tag{2.13}$$

Clearly $\mathscr{V}$ is a refinement of $\mathscr{U}$. To show that $\mathscr{V}$ is locally-finite around some point $[x,i]$, we will construct an open neighbourhood W that intersects finitely-many members of $\mathscr{V}$. For any j in I, there are three scenarios.

1. $[x,i]$ lies in $M_i \cap M_j$. In this case we may use the local finiteness of $\mathscr{V}_j$ in M_j around $f_{ij}(x)$ to obtain some open neighbourhood X_j that intersects finitely-many elements of $\mathscr{V}_j$. The set $\phi_j(X_j)$ will then intersect finitely-many elements of $\mathscr{V}$ coming from $\mathscr{V}_j$.

2. $[x,i]$ lies in $M_i \backslash \mathbf{Cl}(M_j)$, that is, $[x,i]$ is in the set $\mathbf{Int}(M_i \backslash M_j)$. This is an open set that is disjoint from M_j.

3. $[x,i]$ lies in $M_i \cap \partial M_j$. In this case, we may use the local finiteness of M_j around the boundary element $\tilde{f}_{ij}(x)$. This maps homeomorphically into some open neighbourhood of x in $Cl^{M_i}(A_{ij})$. We may extend this to an open neighbourhood X_j of x in M_i. The set $\phi_i(X_j)$ then intersects finitely-many of the open sets in $\mathscr{V}$ coming from $\mathscr{V}_j$.

Intersecting all of the open sets mentioned above will yield an neighbourhood W of $[x,i]$ that intersects at most finitely-many elements of $\mathscr{V}$. Note that W is open since we have assumed that I is finite. □

The above result confirms that certain non-Hausdorff manifolds may be paracompact. In contrast to this, in the non-Hausdorff case there may be open covers that do not admit subordinate partitions of unity. This is immediate from standard results such as those in [65], however for the sake of completeness we will include a different argument.

Theorem 2.3.6. *Let* $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ *be an open cover of a non-Hausdorff manifold* $\mathbf{M}$*. If each* U_α *is Hausdorff, then the cover* $\mathscr{U}$ *does not admit a partition of unity subordinate to it.*

Proof. Let a and b be elements of $\mathbf{M}$ such that $a \curlyvee b$. Observe first that any continuous function $f : \mathbf{M} \to X$ to a Hausdorff space X will necessarily map $f(a) = f(b)$. Indeed, if it were the case that $f(a) \neq f(b)$, then we could apply the Hausdorff property to find two disjoint open sets U and V in X separating $f(a)$ and $f(b)$. The continuity of f would then cause the open sets $f^{-1}(U)$ and $f^{-1}(V)$ to contradict $a \curlyvee b$.
With this in mind, suppose towards a contradiction that there exists some partition of unity $\{\psi_\alpha\}$ subordinate to the cover $\{U_\alpha\}$. Consider the point a in $\mathbf{M}$. Since the partition of unity sums to 1 at a, there must be at least one function ψ_α such that $\psi_\alpha(a) > 0$. It follows that

$$a \in \{x \in \mathbf{M} \mid \psi_\alpha(x) \neq 0\} \subseteq \overline{\{x \in \mathbf{M} \mid \psi_\alpha(x) \neq 0\}} \subseteq U_\alpha. \tag{2.14}$$

However, by our observation will we have that $\psi_\alpha(b) = \psi_\alpha(a) > 0$. Therefore the above chain of inclusions applies equally to b, from which we may conclude that both a and b lie in the set U_α. This contradicts our assumption that U_α is Hausdorff. □

It follows from the above that the open cover consisting of the canonical submanifolds M_i cannot admit a subordinate partition of unity. It is likely that Theorem 2.3.6 will result in some non-trivial obstructions when attempting to recreate the standard theory of differential geometry in the non-Hausdorff regime, though that is beyond the scope of this chapter.

2.3.3 H-submanifolds

In the remainder of this section we will argue that all non-Hausdorff manifolds can be expressed as adjunction spaces. Before getting to the argument, we first need to review the notion of an H-submanifold. We recall the following definition, originally found in [47].

Definition 2.3.7. *Let $\mathbf{M}$ be a non-Hausdorff manifold. A subset V of $\mathbf{M}$ is called an H-submanifold if V is open, Hausdorff and connected, and is maximal with respect to these properties.*

Since non-Hausdorff manifolds are locally-Euclidean, in particular they are locally-Hausdorff. We may use this observation, together with an appeal to Zorn's Lemma to argue the following.

Proposition 2.3.8. *The collection of H-submanifolds of a non-Hausdorff manifold forms an open cover.*

It is not immediately clear whether a given Hausdorff submanifold is an H-submanifold. Fortunately this has been resolved by Hajicek [47, Thm. 2], who provides a useful criterion for determining whether or not a given subspace is an H-submanifold. In our notation, Hajicek's criterion can be stated as follows.

Theorem 2.3.9 (Hajicek's Criterion)**.** *A subset V of a non-Hausdorff manifold $\mathbf{M}$ is an H-submanifold if and only if the equality $\partial V = \mathbf{Cl}(Y^V)$ holds.*

We can use Hajicek's criterion together with Lemma 2.3.4 to conclude that any non-Hausdorff manifold $\mathbf{M}$ built according to 2.3.2 admits the canonical subspaces M_i as H-submanifolds. As we will see in Section 2.4, it is not always the case that the M_i are the only H-submanifolds of $\mathbf{M}$.

2.3.4 A Reconstruction Theorem

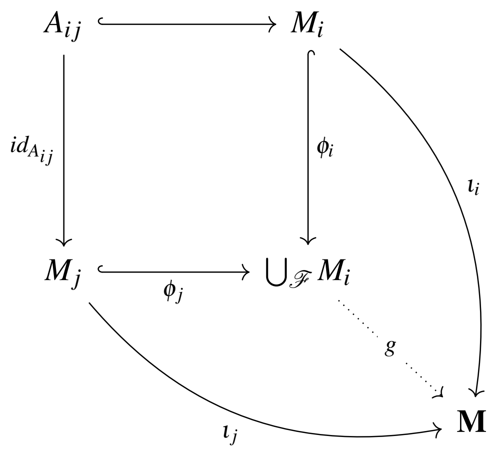
We will now use Proposition 2.3.8 to argue that all non-Hausdorff manifolds can be described via adjunction spaces. The idea behind this proof can be found in both [82] where it is proved for vector bundles, and [68], where it is proved for manifolds in general. For completeness we will provide a proof consistent with our notation.

Theorem 2.3.10. *If $\mathbf{M}$ is a non-Hausdorff manifold then $\mathbf{M}$ is homeomorphic to an adjunction space.*

Proof. Consider the family M_i of all H-submanifolds of $\mathbf{M}$. We know from Prop. 2.3.8 that the M_i form an open cover of $\mathbf{M}$. Since $\mathbf{M}$ is second-countable, in particular $\mathbf{M}$ is Lindelöf. Thus without loss of generality we may assume that the M_i form a countable open cover of $\mathbf{M}$. We can then define an adjunction system $\mathscr{F}$ using:

- the countable collection of M_i, with
- $A_{ij} := M_i \cap M_j$, and
- $f_{ij} : A_{ij} \to M_j$ the identity map.

Clearly this collection satisfies the conditions of Definition 2.2.1 and the criteria of Theorem 2.3.1. Thus the associated adjunction space $\bigcup_{\mathscr{F}} M_i$ is a well-defined locally-Euclidean second-countable space. We can invoke the universal property of adjunction spaces (Lemma 2.2.3) to conclude that there exists a unique continuous map g such that we have the following commutative diagram for every pair of M_i's:



(2.15)

where the $\iota.$ are the inclusion maps, and $g : \bigcup_{\mathscr{F}} M_i \to \mathbf{M}$ is the map that acts by $[x,i] \mapsto x$. This map is clearly open – this follows from the fact that ϕ_i and ι_i are open maps for all i. Moreover, f is a bijection – the inverse is given by $f^{-1}(x) = [x,i]$ where i is the index of any M_i that contains x. By construction, this map is well-defined. Thus we have obtained a bijective, continuous, open map from $\bigcup_{\mathscr{F}} M_i$ to $\mathbf{M}$. □

2.4 Examples

As mentioned in the introduction, the prototypical example of a non-Hausdorff manifold is the so-called line with two origins. We will now introduce several more examples, built according to our adjunction space formalism. The following spaces will motivate the discussion in Section 2.5, where we will study the prospect of characterising the types of H-submanifolds that a non-Hausdorff manifold might admit.

2.4.1 The n-branched Real Line

Take I to be an indexing set of size n, and consider the adjunction system $\mathscr{F}$ where:

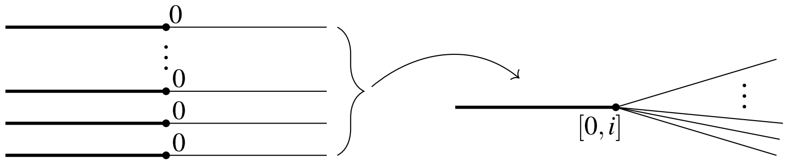


Figure 2.2: The construction of the n-branched real line

- each M_i equal to a copy of the real line equipped with the standard topology,
- each A_{ij} equal to the set $(-\infty, 0)$ for all i, j distinct, and
- each $f_{ij} : A_{ij} \to M_j$ is the identity map.

Once we require that each A_{ii} equals M_i, the data contained in $\mathscr{F}$ forms an adjunction system. The adjunction space subordinate to $\mathscr{F}$ will be a collection of n-many copies of the real line all glued to each other along the negative numbers, as pictured in Figure 2.2. According to Theorems 2.3.1 and 2.3.5 the n-branched real line is a paracompact non-Hausdorff manifold. Observe that the origins remain unidentified, thus there are n-many distinct equivalence classes $[0,i]$ in the n-branched real line. These points will violate the Hausdorff property. Moreover, there are n-many H-submanifolds equalling the canonically embedded M_i.

It should be noted that there are other interesting 1-manifolds that can be built from finitely-many copies of the real line. Indeed, consider three copies of the real line, in which $A_{12} = (-\infty, 0)$, $A_{23} = (0, \infty)$, and $A_{13} = \emptyset$, and all f-maps equal the identity. This data defines an adjunction system, and the resulting space will be a non-Hausdorff manifold in which there are three distinct copies of the origin in which the Y-relation is not transitive.

2.4.2 An Infinitely-Branching Real Line

We will now construct a non-Hausdorff manifold using countably-many copies of the real line. Suppose that the indexing set I coincides with the natural numbers, ordered linearly. Consider the triple $\mathscr{F} := (\mathbf{X}, \mathbf{A}, \mathbf{f})$, where:

- each M_i equals $\mathbb{R}$ with the standard topology,
- $A_{ij} = \begin{cases} (-\infty, i) & \text{if } i < j \\ (-\infty, j) & \text{if } i > j \\ \mathbb{R} & \text{if } i = j \end{cases}$, and
- each f_{ij} is the identity map on the appropriate domain.

The reader may verify that this collection $\mathscr{F}$ does indeed form an adjunction system. The resulting adjunction space, which we denote by $\mathbf{T}$, will be a countable collection of real lines, successively splitting in two at each natural number, as pictured in Figure 2.3.

In this example, the copies of $\mathbb{R}$ naturally sit inside $\mathbf{T}$ as H-submanifolds. In contrast to the previous example, there is an extra H-submanifold, given by

$$V = \bigcup_{i,j \in I} \phi_i(A_{ij}). \tag{2.16}$$

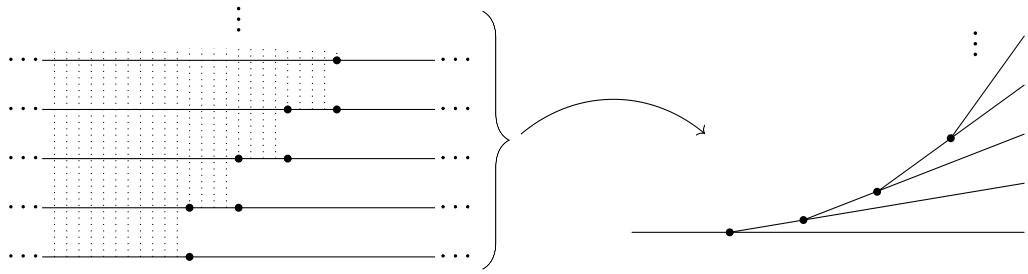


Figure 2.3: The construction of an infinitely-branching real line.

It should be noted that a similar space can be found as the rigid 1-manifold of [38].

2.4.3 A 2-Branched Euclidean Plane

We can form branched planes by taking the product of the n-branched real line with a copy of $\mathbb{R}$. The resulting spaces will still be non-Hausdorff manifolds, and can be seen as a collection of Euclidean planes that branch out from each other along x-axes.

Consider the 2-branched real plane. According to Theorem 2.3.10 there should be a way to construct this space in terms of adjunctions. The obvious choice is to form an adjunction system using:

1. $M_1 = M_2 = \mathbb{R}^2$,
2. $A_{12} = \{(x,y) \in \mathbb{R}^2 \mid y < 0\}$ is the open half-plane, and
3. $f : A_{12} \to M_2$ is the identity map.

This collection satisfies the conditions of 2.3.1, and the resulting adjunction space, denoted $\mathbf{R}^2$, is a paracompact non-Hausdorff manifold. The construction is depicted in Figure 2.4. The Hausdorff-violating points of $\mathbf{R}^2$ lie on the two copies of the x-axis. For convenience, we will denote the two x-axes by X_1 and X_2.

By Theorem 2.3.4 we see that the two copies of $\mathbb{R}^2$ that naturally sit inside $\mathbf{R}^2$ will be H-submanifolds. However, the space $\mathbf{R}^2$ admits many other H-submanifolds. We will now briefly describe one. Consider the subspace V of $\mathbf{R}$ defined by:

$$V = \mathbf{R}^2 \backslash \big(\{(x,0) \in X_1 \mid x \leq 0\} \cup \{(x,0) \in X_2 \mid x \geq 0\} \big), \tag{2.17}$$

that is, we remove from $\mathbf{R}^2$ the non-positive x-axis from X_1 and the non-negative x-axis from X_2. Observe that both copies of the origin are removed. The subset V is open in $\mathbf{R}^2$ since its complement is closed. Moreover, the Y-set of V is:

$$\mathsf{Y}^V = \{(x,0) \in X_1 \mid x > 0\} \cup \{(x,0) \in X_2 \mid x < 0\}. \tag{2.18}$$

By a simple analysis, one can see that

$$\partial V = \mathsf{Y}^V \cup \{[(0,0),1],[(0,0),2]\} = \mathbf{Cl}(\mathsf{Y}^V), \tag{2.19}$$

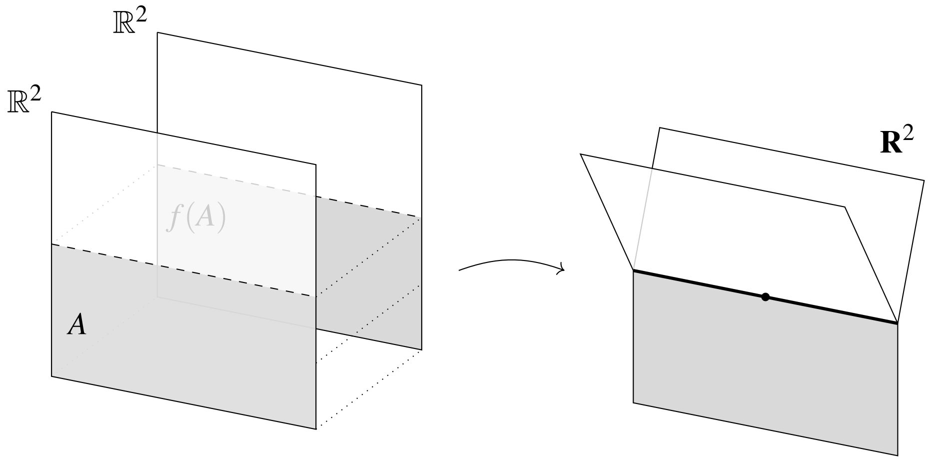


Figure 2.4: An adjunction construction of a 2-branched plane $\mathbf{R}^2$. Here the thick line denotes two Hausdorff-inseparable copies of the x-axis.

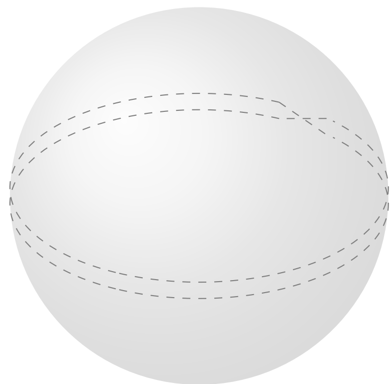


Figure 2.5: The non-Hausdorff sphere $\mathbf{S}^2$, taken from [55].

and thus Hajicek's criterion guarantees that V is an H-submanifold of $\mathbf{R}^2$.

2.4.4 A non-Hausdorff Sphere

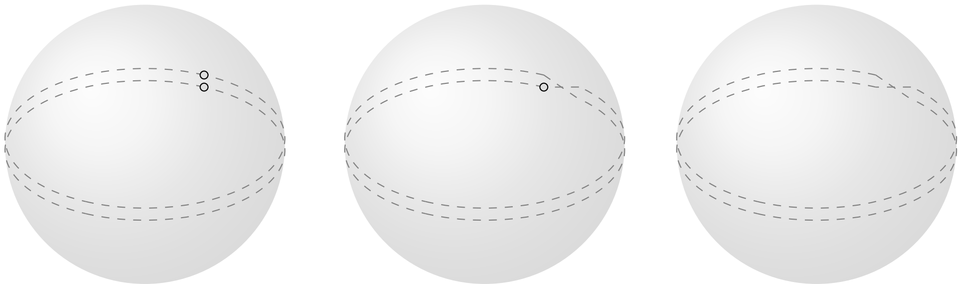
We finish this section with an example of a compact non-Hausdorff manifold. Roughly speaking, this space will be a 2-sphere in which the equatorial copy of S^1 is wrapped around itself so that the antipodal points become Hausdorff-inseparable, as pictured in Figure 2.5. We will denote this non-Hausdorff sphere by $\mathbf{S}^2$.

The original construction of $\mathbf{S}^2$ uses a modified torus to obtain the doubled equator [55]. A more succinct construct identifies antipodes of the 2-sphere everywhere outside the equatorial copy of S^1 [25]. Both of these constructions are not adjunction spaces, and it has even been suggested that such a colimit construction does not exist [94]. We will now remedy this by providing a construction of $\mathbf{S}^2$ in terms of an adjunction of punctured 2-spheres. We will start with the standard 2-sphere, embedded into $\mathbb{R}^3$ for convenience. Consider the equatorial copy of S^1 parameterised as

$$S^1 = \{(x,y,0) \mid x^2 + y^2 = 0\}. \tag{2.20}$$

	M_1	M_2	M_3	M_4
M_1	M_1	$M_1\backslash S^1$	$M^1\backslash L^1_-$	$M^1\backslash L^1_+$
M_2	$M^2\backslash S^1$	M_2	$M^2\backslash L^2_+$	$M^2\backslash L^2_-$
M_3	$M^3\backslash L^3_-$	$M^3\backslash L^3_+$	M_3	$M^3\backslash S^1$
M_4	$M^4\backslash L^4_+$	$M^4\backslash L^4_-$	$M^4\backslash S^1$	M_4

Figure 2.6: The adjunction data for the non-Hausdorff sphere



(a) (b) (c)

Figure 2.7: A successive construction of $\mathbf{S}^2$. Here (a) is the adjunction of the spaces M_1 and M_2. Figure (b) represents the gluing of M_1, M_2 and M_3. Figure (c) is the final space.

Let $a = (1,0,0)$ and $b = (-1,0,0)$ be antipodes on S^1. Consider the following punctured spheres:

- M_1 and M_2 are $S^2\backslash\{a\}$, and
- M_3 and M_4 are $S^2\backslash\{b\}$.

We will now glue the M_i in such a manner that halves of each S^1 eventually form a double-covered equator. Within each M_i there are two arcs connecting the antipodal points a and b. We denote these by L^i_+ and L^i_-, that is,

$$L^i_+ := \{(x,y,0) \in M_i \mid y \geq 0\} \text{ and } L^i_- := \{(x,y,0) \in M_i \mid y \leq 0\}. \tag{2.21}$$

Observe that in each M_i the segments $L^i_\pm$ are closed subsets. Figure 2.6 contains all of the pairwise gluing regions A_{ij}. This data indeed defines an adjunction system, and it is perhaps easiest to visualise the gluing as a successive process – Figure 2.7 depicts the stages in the construction.

2.5 The Characterisation of H-Submanifolds

In Section 2.4.3 we saw the 2-branched Euclidean plane, a non-Hausdorff manifold that admits infinitely-many H-submanifolds. Of these infinite H-submanifolds there are two that stand out – the images of the two copies of the plane. This raises an interesting question: given an adjunction space that forms a non-Hausdorff manifold, what topological properties characterise the images $\phi_i(M_i)$? In this section we will take steps towards this characterisation.

2.5.1 Simple non-Hausdorff Manifolds

We will restrict our attention to a basic type of non-Hausdorff manifold. Following the terminology of [77], we will refer to these as *simple* non-Hausdorff manifolds. The definition is as follows.

Definition 2.5.1. *A non-Hausdorff manifold* $\mathbf{M}$ *is called simple if it is homeomorphic to an adjunction space* $\bigcup_{\mathscr{F}} M_i$ *in which:*

- *each* M_i *is the same Hausdorff manifold* M*,*
- *the indexing set* I *is finite,*
- *each* A_{ij} *is the same connected open subspace* A *that has connected boundary, and*
- *each gluing map* f_{ij} *is the identity.*

Similarly, an adjunction system $\mathscr{F}$ *is called simple if its adjunction space yields a simple non-Hausdorff manifold.*

The above definition captures the idea of a non-Hausdorff manifold branching out finitely-many times from a single submanifold. Note that the n-branched real line and the 2-branched plane of Section 2.4 are both simple, whereas the infinitely-branching real line is not.

The main benefit of considering simple non-Hausdorff manifolds is that they admit a particularly easy description of Hausdorff violation. Observe that the component spaces of a simple non-Hausdorff manifold will have homeomorphic boundaries, since we may use the identity map to define each extension $\tilde{f}_{ij}$, and thus the boundary-maps g_{ij} equal $\phi_j \circ \phi_i^{-1}$ (cf. 2.3.2). We also make the following useful observation, which follows immediately from Definition 2.5.1.

Lemma 2.5.2. *If* $\mathbf{M}$ *is simple, then* $\partial^i A_{ij} = \partial^i A = \partial^i A_{ik}$ *for all* i, j, k *in* I*.*

It follows from the above that the Hausdorff-violating sets in a simple non-Hausdorff manifold will consist of the I-many (disjoint) connected components of the boundary of A. Thus we may conclude that the non-Hausdorff sphere of Section 2.4.4 is not simple, since the set $\curlyvee^{\mathbf{S}^2}$ is path-connected.

2.5.2 A Generalisation of Müller's Theorem

We will now argue that simple non-Hausdorff manifolds admit a characterisation of their canonical subspaces M_i. There is an argument due to Müller [77] that certain simple branched Minkowski spaces admit a characterisation in terms of existent limit points. We will extend this result by passing into a broader generality, and by using a weaker condition.

Recall that Hajicek's criterion demands that each H-submanifold satisfies $\partial V = \mathbf{Cl}(\curlyvee^V)$. We will argue that in a simple non-Hausdorff manifold, the canonical subspaces M_i are the unique H-submanifolds satisfying the stricter equality $\partial V = \curlyvee^V$. Our argument will be via induction on the size of the indexing set underlying $\mathscr{F}$. In order to do so, we will make use of the following key lemma.

Lemma 2.5.3. *Suppose that $\mathbf{M}$ is simple and V is an H-submanifold of $\mathbf{M}$ satisfying $\partial V = \curlyvee^V$. Then for any boundary component $\partial^i A$, either $\curlyvee^V$ and $\partial^i A$ are disjoint, or $\partial^i A$ is contained within $\curlyvee^V$.*

Proof. By 2.5.1 $\partial^i A$ is connected, so it suffices to show that the intersection $\curlyvee^V \cap \partial^i A$ is clopen in $\partial^i A$. Observe first that by assumption $\curlyvee^V$ coincides with the boundary ∂V, so it is closed in $\mathbf{M}$. Moreover, observe that Lemma 2.5.2 yields the equality

$$\curlyvee^V \cap \partial^i A = \bigcup_{k \neq i} \phi_i \circ \phi_k^{-1} \left(V \cap \partial^k A \right). \tag{2.22}$$

Since the sets $V \cap \partial^k A$ are all open and they map homeomorphically into $\partial^i A = \partial^i A$, we may conclude that $\curlyvee^V \cap \partial^i A$ is a union of open sets. □

We may use the above lemma together with Theorem 2.3.4 to obtain the following result.

Theorem 2.5.4. *Let $\mathbf{M}$ be a simple non-Hausdorff manifold that is homeomorphic to a binary adjunction space $M_1 \cup_f M_2$. If V is an H-submanifold of $\mathbf{M}$ that satisfies the equality $\partial V = \curlyvee^V$, then either $V = M_1$ or $V = M_2$.*

Proof. By definition the gluing regions in a simple non-Hausdorff manifolds have homeomorphic boundaries, so we may apply Lemma 2.3.4 to conclude that both M_i satisfy $\partial M_i = \curlyvee^{M_i}$. Suppose towards a contradiction that there is some H-submanifold V of $\mathbf{M}$ that satisfies $\partial V = \curlyvee^V$, but is distinct from both M_1 and M_2. Then there are points $[x,1]$ in $V \backslash M_2$ and $[y,2]$ in $V \backslash M_1$. Let γ be a path connecting $[x,1]$ to $[y,2]$ in V. Consider some element z in the γ-relative boundary $\partial^\gamma(M_1)$ that is distinct from $[x,1]$. Then z lies in ∂M_1 as well. Since $\partial M_1 = \curlyvee^{M_1}$, there exists some w in M_2 such that $z \curlyvee w$. Since z lies on the curve γ (which is a curve in V), it follows that the element w is a member of $\curlyvee^V$. Therefore $\curlyvee^V \cap \partial^2 A \neq \emptyset$, so $\partial^2 A \subseteq \curlyvee^V$ by Lemma 2.5.2. We may now repeat the same argument, this time starting with the subset $M_2 \cap \gamma$. We will obtain an element w' that lies in the set $\curlyvee^V \cap \partial^1 A$. This implies that both $\partial^1 A$ and $\partial^2 A$ are subsets of $\curlyvee^V$, which contradicts V as Hausdorff. □

In order to continue an inductive argument, we need to appeal to the adjunctive subspaces of Section 2.2.2. Observe first that any subsystem $\mathscr{G}$ of a simple adjunction system $\mathscr{F}$ will again be simple. We will also have the following collection of facts, which follow trivially from 2.2.5.

Lemma 2.5.5. *Let $\mathscr{F}$ be a simple adjunction system, and $\mathscr{G} \subset \mathscr{F}$. Denote by $g : \bigcup_{\mathscr{G}} M_j \to \bigcup_{\mathscr{F}} M_i$ the map sending each $[\![x,i]\!]$ to $[x,i]$. Let V be an H-submanifold of $\bigcup_{\mathscr{F}} M_i$. Then*

1. *if $V \subseteq g\left(\bigcup_{\mathscr{G}} M_j\right)$, then $[\![V]\!] := g^{-1}(V)$ is an H-submanifold of $\bigcup_{\mathscr{G}} M_j$, and*
2. *if additionally V satisfies $\partial V = \curlyvee^V$, then so does $[\![V]\!]$.*

We will now use the above results to prove the main theorem of this chapter.

Theorem 2.5.6. *Let $\mathbf{M} \cong \bigcup_{\mathscr{F}} M_i$ be a simple non-Hausdorff manifold. If V is an H-submanifold of $\mathbf{M}$ that satisfies the equality $\partial V = \curlyvee^V$, then $V = M_i$ for some i in I.*

Proof. We proceed via induction on the size of the adjunction space $\mathscr{F}$, that is, by induction on the size of the indexing set I. If $n = 2$, then the result follows from the argument detailed in Theorem 2.5.4. Suppose that the hypothesis holds for adjunction spaces of size n, and let $\mathbf{M}$ be an adjunction space of size $n+1$. Suppose towards a contradiction that there exists some H-submanifold V that is distinct from $M_1, ..., M_{n+1}$ yet satisfies the condition $\partial V = \curlyvee^V$.

Observe first that V cannot be contained within some finite union of n-many M_i's. Indeed – if it were then we could apply Lemma 2.5.5 and restrict V to $[\![V]\!]$. We may then apply the induction hypothesis to conclude that $[\![V]\!]$ equalled some M_i in the adjunctive subspace, and thus V would equal that same M_i in $\mathbf{M}$. It must therefore be the case that V is not a subset of some union of n-many M_i's. Moreover, by maximality it cannot be the case that $V \subseteq M_{n+1}$. Therefore, we may infer that:

1. there exists some element x in V that lies in the difference $M_{n+1} \setminus \bigcup_{i=1}^{n} M_i$, and
2. there exists some element y in V that lies in the difference $\mathbf{M} \setminus M_{n+1}$.[2]

We may now proceed in a manner similar to that of Theorem 2.5.4, with some key modifications for this more-complicated scenario. Let γ be a path in V that connects x to y in V. We will now use γ to yield a contradiction.

1. Consider the set $\gamma \cap M_{n+1}$. The path γ might enter and exit M_{n+1} several times. So, consider the connected component C of $\gamma \cap M_{n+1}$ that contains the endpoint x. The existence of the endpoints x and y ensure that the set C is an open subset of γ that is both non-empty and proper. As such, C has a γ-relative boundary. Let z be the element of $\partial^{\gamma} C$ distinct from x. Then z is also an element of ∂M_{n+1}. By Lemma 2.3.3, there must exist some z' in M_{n+1} such that $z \curlyvee z'$. Since z lies in γ, which lies in V, we may conclude that $z' \in \curlyvee^V$. Thus z' lies in both $\partial^{n+1} A$ and $\curlyvee^V$. According to Lemma 2.5.3, we may conclude that $\partial^{n+1} A$ is a subset of $\curlyvee^V$.
2. Consider the set $\gamma \setminus A$. This is a non-empty, closed proper subset of γ. Let D be the connected component of $\gamma \setminus A$ that contains x. The set D has two elements in its γ-relative boundary – the first is x, and the second will be some other element w in γ. Then $w \in \partial^{\gamma} A$, from which it follows that w also lies in the set $\partial^{n+1} A$. Since w is an element of γ, which is a path in V, it follows that $V \cap \partial^{n+1} A$ contains w.

We have thus arrived at our contradiction – according to items (1) and (2) above the intersection $V \cap \curlyvee^V$ is non-empty, which contradicts our assumption that V is Hausdorff. We may therefore conclude that there is no V in $\mathbf{M}$ that satisfies $\partial V = \curlyvee^V$ and is distinct from each $M_1, ..., M_{n+1}$. □

2.6 Conclusion

In this chapter we have introduced a general theory for simultaneously gluing arbitrarily-many topological spaces together along open subsets. In Section 2.3 we saw that if we consider a countable collection of Hausdorff manifolds that are glued along homeomorphic

[2] Throughout this proof we will suppress the equivalence classes of points in $\mathbf{M}$ for readability.

open subspaces, then we will always obtain a locally-Euclidean, second-countable space. Theorem 2.3.10 confirmed that all non-Hausdorff manifolds can be described in this manner. According to Theorems 2.2.6 and 2.3.5 we saw that such spaces are T_1, and may also be paracompact, provided that certain criteria is met. However, due to 2.3.6 they may have open covers which do not admit partitions of unity subordinate to them. Upon requiring that the gluing regions have pairwise homeomorphic boundaries, we saw in Theorems 2.3.3 and 2.3.4 that we can describe the Hausdorff-violating points of a non-Hausdorff manifold via these boundary components. This description can be summarised by the equality: $\partial M_i = \Upsilon^{M_i}$.

Finally, we extended and improved a result of Müller [77] to include general simple non-Hausdorff manifolds. Theorem 2.5.6 shows that any simple non-Hausdorff manifold $\mathbf{M}$ admits n-many H-submanifolds that satisfy the reduced Hajicek criterion $\partial V = \Upsilon^V$. These special H-submanifolds are given uniquely by the M_i, which were the spaces used to construct $\mathbf{M}$ in the first place.

It is not clear how to further generalise the result of 2.5.6. The requirement of simplicity was used in two essential ways: first, we restricted our attention to finite-sized adjunction systems in order to be able to perform an inductive argument, and we demanded the heavy restriction of 2.5.2 in order to use Lemma 2.5.3, which was an integral part of our eventual argument. It may be the case that the reduced Hajicek criterion is not an appropriate condition to uniquely characterise canonical subspaces of a non-simple non-Hausdorff manifold. Indeed – the infinitely-branched real line of Section 2.4.2 has an “extra" H-submanifold that still satisfies the property $\partial V = \Upsilon^V$. A more nuanced condition may be required for the general argument.

Summary of Chapter

In this chapter we introduced the basic topological properties of non-Hausdorff manifolds. We realised the original ideas of Haefliger and Reeb into a fully-fledged general formalism for gluing topological spaces together. In particular, we used this adjunction formalism to glue together Hausdorff manifolds. If this was done so that

1. the gluing regions were topologically open,
2. the gluing maps were open topological embeddings, and importantly
3. the boundaries of the gluing regions and the boundaries of their images under the gluing maps were homeomorphic,

then the resulting quotient space always assumes the structure of a non-Hausdorff manifold. In this quotient space, the Hausdorff-violating regions were characterised as the topological boundaries of the gluing regions involved in the construction. Later on, we saw a converse to the above: we showed that all non-Hausdorff manifolds can be realised as an adjunction of Hausdorff manifolds. These results pave the way for a description of the geometry of non-Hausdorff manifolds in a manner that circumvents the issue of non-arbitrary existence of partitions of unity in the non-Hausdorff case. In the next chapters, we will leverage our adjunction description of non-Hausdorff manifolds heavily.

Chapter 3

Vector Bundles over non-Hausdorff Manifolds

This chapter is published under the title "Vector Bundles over non-Hausdorff Manifolds", Topology and its Applications, page 108982, 2024 [85], and I am the sole author.

In this chapter we describe smooth non-Hausdorff manifolds and their vector bundles. Here we prove several important results that will be used later on:

- As an extension of Theorem 2.3.1, we show that smooth atlases can be defined on a non-Hausdorff manifold, provided that we glue *smooth* Hausdorff manifolds along pairwise *diffeo*morphic open submanifolds.
- In the case that we consider an adjunction space of finitely-many Hausdorff manifolds M_i, we show that the ring of smooth, real-valued functions on the resulting non-Hausdorff is isomorphic to a fibred product of the spaces $C^\infty(M_i)$.
- We show that vector bundles fibred over the Hausdorff manifolds M_i can be glued together to form a non-Hausdorff vector bundle fibred over the non-Hausdorff manifold. Conversely, we also show that all non-Hausdorff bundles arise in this manner, a generalisation of Theorem 2.3.10.
- We show that sections over non-Hausdorff bundles can be described as a fibre product of the sections $\Gamma(E_i)$ over Hausdorff subbundles.

Finally, we finish the chapter by describing the Cech cohomology of a non-Hausdorff manifold in terms of the corresponding cohomology groups associated to the Hausdorff M_i and the gluing regions A_{ij}. This is then used to count the inequivalent line bundles that one may fibre over a non-Hausdorff manifold.

3.1 Introduction

The standard theory of differential geometry assumes that any pair of points in a manifold can be separated by disjoint open sets. This, known as the Hausdorff property, is typically imposed for technical convenience. Indeed, it can be shown that any Hausdorff, locally-Euclidean, second-countable topological space necessarily admits partitions of unity subordinate to any open cover. In turn, these partitions of unity can then be used to construct some familiar features of a manifold.

Conversely, non-Hausdorff manifolds will always have open covers that cannot admit partitions of unity [84]. Despite this inconvenience, non-Hausdorff manifolds seem to arise within several areas of mathematics and theoretical physics. Within mathematics, one can find non-Hausdorff manifolds in the leaf spaces of foliations [39, 45], and in the spectra of certain C^* algebras [25]. Within theoretical physics, some general discussion of non-Hausdorff spacetimes can be found in [53, 89, 105], and non-Hausdorff manifolds can be found in the maximal extension of Taub-NUT spacetimes [46], in certain twistor spaces [108], and to various degrees in the study of branching spacetimes [9, 33, 47, 68, 77, 82].

Despite occurring in both mathematics and physics, a general theory of non-Hausdorff geometry is lacking. In [84], initial steps were made with a study of the topological properties of non-Hausdorff manifolds. The key idea underpinning their study were the observations of [45, 47, 68, 82], which suggest that non-Hausdorff manifolds can always be constructed by gluing together ordinary Hausdorff manifolds along open subspaces. This gluing operation is commonly known as an *adjunction space*, though it may also be seen as the topological colimit of a particular diagram. The goal of this chapter is to extend the formalism of [84] to include both smooth manifolds and the vector bundles fibred over them.

This chapter is organised as follows. In Section 3.2, we start by recalling some details of [84], and by making some important restrictions on the types of non-Hausdorff manifolds that we will consider. Once this is done, we will show that our non-Hausdorff manifolds naturally inherit smooth structures from the Hausdorff submanifolds that comprise them. We will then describe the ring of smooth real-valued functions, and use a modified version of partitions of unity to extend functions from Hausdorff submanifolds into the ambient non-Hausdorff space. We will show that, due to the contravariance of the C^∞ functor, the space of functions of a non-Hausdorff manifold can be seen as the *fibred product* of the functions defined on each of the Hausdorff submanifolds.

In Section 3.3 we will begin to consider the vector bundles that one may fiber over a non-Hausdorff manifold. In a direct analogy to the theorems of [68] and [84], we will show that every vector bundle over a non-Hausdorff manifold can be realised as a colimit of bundles fibred over the Hausdorff submanifolds. We will then use this result to describe the space of sections of a non-Hausdorff vector bundle. A similar fibred-product description will emerge, this time due to the contravariance of the Γ functor. This description will then be used to construct Riemannian metrics on our non-Hausdorff manifolds.

In Section 3.4 we will explore the prospect of classifying the line bundles over a non-

Hausdorff manifold. We will do this by appealing to the well-known result that real line bundles over Hausdorff manifolds can be counted by the first Čech cohomology group with $\mathbb{Z}_2$ coefficients. As such, we will first provide descriptions for the Čech cohomology groups of our non-Hausdorff manifolds as Mayer-Vietoris sequences built from the cohomologies of certain Hausdorff submanifolds. We will see that, despite the Čech functor $\check{H}$ being contravariant, in general there is no fibred product description for the cohomology groups of our non-Hausdorff manifolds. This result suggests that a general formula for the exact number of line bundles over a non-Hausdorff manifold is probably not possible. However, in the case that all of the spaces under consideration are connected, we will derive an alternating-sum formula for the number of line bundles that a non-Hausdorff manifold admits.

Throughout this chapter we will assume familiarity with differential geometry up to the level of [65] or [101]. In Section 3.4, unless otherwise stated, all results regarding the Čech cohomology are taken from [13]. We will assume that all manifolds, Hausdorff or otherwise, are locally-Euclidean, second-countable topological spaces, and as a convention we will use boldface notation to denote non-Hausdorff manifolds and their functions.

3.2 Smooth non-Hausdorff Manifolds

In this section we will introduce a formalism for smooth non-Hausdorff manifolds. We will start with the relevant topological features, and then we will use these to endow our manifolds with smooth structures. We will then discuss smooth real-valued functions on non-Hausdorff manifolds.

3.2.1 The Topology of non-Hausdorff Manifolds

We will now briefly review the topological properties of non-Hausdorff manifolds. All facts in this section are stated without proof, since they can already be found in [84], albeit in a slightly more general form.

We begin by generalising the adjunction spaces found in many standard texts such as [50] or [14]. Our data $\mathscr{F}$ will consist of three key components: a set $\mathsf{M} = \{M_i\}_{i\in I}$ of d-dimensional Hausdorff topological manifolds, a set $\mathsf{A} = \{M_{ij}\}_{i,j\in I}$ of bi-indexed submanifolds satisfying $M_{ij} \subseteq M_i$, and a set $\mathsf{f} = \{f_{ij}\}_{i,j\in I}$ of continuous maps of the form $f_{ij} : M_{ij} \to M_j$. For convenience, throughout this chapter we will assume that the indexing set I is some initial segment of the natural numbers. In order to ensure that this data induces a well-defined topological space, we will need to impose some consistency conditions. This is captured in the following definition.

Definition 3.2.1. *The triple $\mathscr{F} = (\mathsf{M},\mathsf{A},\mathsf{f})$ is called an adjunction system if it satisfies the following conditions for all $i, j \in I$.*

A1) $M_{ii} = M_i$ *and* $f_{ii} = id_{M_i}$

A2) $M_{ji} = f_{ij}(M_{ij})$, *and* $f_{ij}^{-1} = f_{ji}$

A3) $f_{ik}(x) = f_{jk} \circ f_{ij}(x)$ *for each* $x \in M_{ij} \cap M_{ik}$.

Given an adjunctive system $\mathscr{F}$, we define the *adjunction space subordinate to* $\mathscr{F}$ to be the quotient of the disjoint union:

$$\bigcup_{\mathscr{F}} M_i := \left(\bigsqcup_{i \in I} M_i \right) \Big/ \sim \tag{3.1}$$

where $\sim$ is the equivalence relation that identifies elements (x,i) and (y,j) of the disjoint union whenever $f_{ij}(x) = y$. Observe that the conditions of Definition 3.2.1 are precisely the conditions needed to ensure that $\sim$ is a well-defined equivalence relation. Points in the adjunction space are equivalence classes of elements of each of the spaces M_i. We will denote these classes by $[x,i]$, that is,

$$[x,i] := \left\{ (y,j) \in \bigsqcup_{i \in I} M_i \mid y = f_{ij}(x) \right\}. \tag{3.2}$$

By construction there exists a collection of continuous maps $\phi_i : M_i \to \bigcup_{\mathscr{F}} M_i$ that send each x in M_i to its equivalence class $[x,i]$ in the adjunction space.

In the case that an adjunction system $\mathscr{F}$ has an indexing set of size 2 we will refer to $\bigcup_{\mathscr{F}} M_i$ as a *binary* adjunction space. It is well-known that these spaces can be equivalently seen as the pushout of the diagram

$$M_2 \xleftarrow{f_{12}} M_{12} \xrightarrow{\iota_{12}} M_1 \tag{3.3}$$

in the category of topological spaces [14]. Similarly, it can be shown that general adjunction spaces are colimits of the diagram formed from the data in $\mathscr{F}$, with the maps ϕ_i satisfying the analogous universal property.

Lemma 3.2.2. *Let $\mathscr{F}$ be an adjunction system and let $\bigcup_{\mathscr{F}} M_i$ be the adjunction space subordinate to $\mathscr{F}$. Suppose that $\varphi_i : M_i \to X$ is a collection of continuous maps such that $\varphi_i = \varphi_j \circ f_{ij}$ for every i,j in I. Then there is a unique continuous map $\varepsilon : \bigcup_{\mathscr{F}} M_i \to X$ such that $\varphi_i = \varepsilon \circ \phi_i$ for all i in I.*

With this in mind, we will regularly borrow language from category theory and refer to an adjunctive system and its subordinate adjunction space as a *diagram* and its *colimit*, respectively.

The canonically-induced maps ϕ_i can be particularly well-behaved, provided that we make some extra assumptions on the data in $\mathscr{F}$. The following result makes this precise.

Lemma 3.2.3. *Let $\mathscr{F}$ be an adjunction system in which the gluing regions M_{ij} are all open submanifolds and the maps f_{ij} are open topological embeddings. Then*

1. *the maps ϕ_i are all open topological embeddings, and*
2. *the adjunction space subordinate to $\mathscr{F}$ is locally-Euclidean and second-countable.*

Due to the above, we will often refer to the ϕ_i as the *canonical embeddings*. We will also use $\mathbf{M}, \mathbf{N}, ...$ to denote adjunction spaces subordinate to any system satisfying the assumptions of Lemma 3.2.3. Given that each canonical embedding ϕ_i acts as a homeomorphism, we will often simplify notation and identify each M_i with its image $\phi_i(M_i)$.

According to Lemma 3.2.3 the colimit $\mathbb{M}$ resulting from a diagram $\mathscr{F}$ of Hausdorff manifolds may be a locally-Euclidean second-countable topological space. However, there is no guarantee that any such $\mathbb{M}$ will be non-Hausdorff. As a matter of fact, we will need to assume that the open submanifolds M_{ij} are proper, open submanifolds whose boundaries are pairwise homeomorphic. The following result summarises the consequences of this assumption.

Theorem 3.2.4. *Let $\mathscr{F}$ be an adjunctive system that satisfies the criteria of Lemma 3.2.3, and let $\mathbb{M}$ denote the adjunction space subordinate to $\mathscr{F}$. Suppose furthermore that each gluing map $f_{ij} : M_{ij} \to M_{ji}$ can be extended to a homeomorphism $\overline{f_{ij}} : Cl^{M_i}(M_{ij}) \to Cl^{M_j}(M_{ji})$ such that each $\overline{f_{ij}}$ satisfies the conditions of Definition 3.2.1. Then*

1. *The Hausdorff-violating points in $\mathbb{M}$ occur precisely at the $\mathbb{M}$-relative boundaries of the subspaces M_i.*
2. *Each M_i is a maximal Hausdorff open submanifold of $\mathbb{M}$.*
3. *If the indexing set I is finite, then $\mathbb{M}$ is paracompact.*
4. *If each M_i is compact and the indexing set I is finite, then $\mathbb{M}$ is compact.*

Throughout the remainder of this chapter we will take $\mathbb{M}$ to be a fixed but arbitrary non-Hausdorff manifold that is built as an adjunction of finitely-many Hausdorff manifolds M_i according to both Lemma 3.2.3 and Theorem 3.2.4. Consequently, $\mathbb{M}$ is a paracompact, locally-Euclidean second-countable space in which the manifolds M_i sit inside $\mathbb{M}$ as maximal Hausdorff open submanifolds.

Adjunctive Subspaces and Inductive Colimits

Generally speaking, given an adjunctive system $\mathscr{F}$, we may form other diagrams by selectively deleting data pertaining to particular indices in the set I. This procedure will define what is known as an *adjunctive subsystem*. Given some subset $J \subset I$, and the adjunctive subsystem $\mathscr{F}'$ formed by only considering the data in J, we may use Lemma 3.2.2 to construct a map between the adjunction spaces subordinate to $\mathscr{F}$ and $\mathscr{F}'$, given by:

$$\kappa : \bigcup_{\mathscr{F}'} M_i \to \bigcup_{\mathscr{F}} M_i \text{ where } \kappa([\![x,i]\!]) = [x,i]. \tag{3.4}$$

Note that here we are using the double-bracket notation to distinguish the two types of equivalence classes. It can be shown that this map κ is an open topological embedding whenever $\mathscr{F}$ satisfies the criteria of Lemma 3.2.3 [84, §1.2].

So far we have constructed our non-Hausdorff manifold $\mathbb{M}$ by gluing the subspaces M_i together simultaneously. However, it will also be useful to express $\mathbb{M}$ as a finite sequence of binary adjunction spaces. We will refer to this sequential construction as an *inductive colimit*. We will now justify the equivalence between these two points of view. The case for a colimit of three manifolds is shown below.

Lemma 3.2.5. *Let* M *be a non-Hausdorff manifold built from three manifolds* M_i*. Then* M *is homeomorphic to an inductive colimit.*

Proof. For readability we will only provide a sketch of the proof here – the full details can be found in the appendix. We may take the gluing region $A := M_{31} \cup M_{32}$ of M_3, and define the map $f_{31} \cup f_{32} : A \to M_1 \cup_{f_{12}} M_2$ by sending each element x of A into M_{3i} along the gluing map f_{i3}, and then into the equivalence class $[\![f_{3i}(x), i]\!]$ in $M_1 \cup_{f_{12}} M_2$. Note that by the gluing condition **(A3)** of Definition 3.2.1 there is no ambiguity here. We may thus glue M_3 to $M_1 \cup_{f_{12}} M_2$ along the map $f_{31} \cup f_{32}$. The universal properties of both $(M_1 \cup_{f_{12}} M_2) \cup_{f_{31} \cup f_{32}} M_3$ and M can then be invoked in order to create the desired homeomorphism. □

As one might expect, we may generalise the previous result to all finite adjunctive systems.

Theorem 3.2.6. *Let* M *be a non-Hausdorff manifold built from n-many manifolds* M_i*. Then* M *is homeomorphic to an inductive colimit.*

Proof. We will proceed by induction on the size of the indexing set I, for which we take to be an initial segment of the natural numbers without loss of generality . The case of $|I| = 3$ is already proved as the previous result. So, suppose that the hypothesis holds for all non-Hausdorff manifolds with indexing set I of size n. Let M be a non-Hausdorff manifold built from a diagram $\mathscr{F}$, with indexing set $I = \{1, ..., n, n+1\}$. Consider the subsystem $\mathscr{F}'$ formed from $\mathscr{F}$ by deleting all data pertaining to the space M_{n+1}. Then $\mathscr{F}'$ is a well-defined adjunctive system that yields a non-Hausdorff manifold N.

In analogy to Lemma 3.2.5, consider the gluing region of M_{n+1} defined by $A := \bigcup_{i \leq n} M_{(n+1)i}$ together with the map $f : A \to \mathrm{N}$ which sends each element x in A to its equivalence class $[\![f_{(n+1)i}(x), i]\!]$ in N. This yields a binary adjunction space $\mathrm{N} \cup_f M_{n+1}$. Again in analogy to Lemma 3.2.5, we may invoke the universal properties of both $\mathrm{N} \cup_f M_{n+1}$ and M to create a homeomorphism between the two spaces. The result then follows by applying the induction hypothesis to N. □

3.2.2 Smooth Structures

In Lemma 3.2.3, the topological structure of each manifold M_i was preserved by requiring that the gluing maps f_{ij} act as open embeddings. The consequence was that the canonical embeddings $\phi_i : M_i \to \mathrm{M}$ were also open, which ensured that the local Euclidean structure of the manifolds M_i can be transferred to M. Formally, this can be achieved by using a collection of atlases $\mathscr{A}_i$ of the manifolds M_i to define an atlas $\mathscr{A}$ on M:

$$\mathscr{A} := \bigcup_{i \in I} \{ (\phi_i(U_\alpha), \varphi_\alpha \circ \phi_i^{-1}) \mid (U_\alpha, \varphi_\alpha) \in \mathscr{A}_i \}. \tag{3.5}$$

We will now argue that this technique defines a smooth atlas of M, provided that the gluing maps f_{ij} are all smooth.

Lemma 3.2.7. *Let* M *be a non-Hausdorff manifold built according to Lemma 3.2.3. If, additionally, the* M_i *are smooth manifolds and the* f_{ij} *are all smooth maps, then the set* $\mathscr{A}$ *described above is a smooth atlas of* M.

Since any point $[x,i]$ in $\mathbb{M}$ is an equivalence class, in principle a pair of charts in $\mathscr{A}$ at this point may come from different atlases $\mathscr{A}_i$ and $\mathscr{A}_j$. Such charts will be of the form $(\phi_i(U_\alpha), \varphi_\alpha \circ \phi_i^{-1})$ and $(\phi_j(U_\beta), \varphi_\beta \circ \phi_j^{-1})$, where $(U_\alpha, \varphi_\alpha)$ is a chart of M_i at the point x, and (U_β, φ_β) is a chart of M_j at the point $f_{ij}(x)$. The transition maps in $\mathbb{M}$ will then be:

$$(\varphi_\beta \circ \phi_j^{-1}) \circ (\varphi_\alpha \circ \phi_i^{-1})^{-1} = \varphi_\beta \circ (\phi_j^{-1} \circ \phi_i) \circ \varphi_\alpha^{-1} = \varphi_\beta \circ f_{ij} \circ \varphi_\alpha^{-1}. \tag{3.6}$$

Observe that the smoothness of each f_{ij} ensures that these transition maps are smooth.[1] Moreover, since each smooth atlas has a unique maximal extension, we may consider the smooth structure induced from the atlas $\mathscr{A}$ described above. This allows us to effectively see our non-Hausdorff manifold $\mathbb{M}$ as smooth. We will now introduce some useful criteria for identifying smooth maps.

Lemma 3.2.8. *Let* $\mathbb{M}$ *and* $\mathbb{N}$ *be smooth (possibly non-Hausdorff) manifolds. A map* $f : \mathbb{M} \to \mathbb{N}$ *is smooth if and only if the restrictions* $f|_{M_i}$ *are smooth for all* i *in* I.

Proof. According to Lemma 3.2.3 the collection M_i forms an open cover of $\mathbb{M}$. The result then follows as an application of Prop. 2.6 of [65]. □

According to the above result, we may now view the canonical embeddings $\phi_i : M_i \to \mathbb{M}$ as *smooth* open embeddings. We may also argue for a universal property as in Lemma 3.2.2 and thus interpret $\mathbb{M}$ as the colimit of the diagram $\mathscr{F}$ in the category of smooth locally-Euclidean spaces. The following remark makes precise the primary object of study in this chapter.

Remark 3.2.9. *In addition to the conditions of Lemma 3.2.3 and Theorem 3.2.4, hereafter we will also assume that our non-Hausdorff manifold* $\mathbb{M}$ *is smooth in the sense of Lemma 3.2.7. Moreover, in order to study smooth objects defined on* $\mathbb{M}$*, we will also need to assume that the gluing regions* M_{ij} *have diffeomorphic boundaries, which in this context means that the closures* $Cl^{M+i}(M_{ij})$ *are all smooth closed submanifolds, and the extended maps* $\overline{f_{ij}}$ *of Theorem 3.2.4 are all smooth.*

3.2.3 Smooth Functions

Since smoothness is a local property, we may define smooth functions on $\mathbb{M}$ as in the Hausdorff case. Moreover, we may still appeal to the structure of the real line to view the space $C^\infty(\mathbb{M})$ as a unital associative algebra.[2] In this section we will study $C^\infty(\mathbb{M})$ in some detail. To begin with, we will discuss some techniques for constructing functions on $\mathbb{M}$. We will then use these techniques to establish a relationship between $C^\infty(\mathbb{M})$ and the algebras $C^\infty(M_i)$.

[1] In fact, due to Definition 3.2.1, the gluing maps f_{ij} are open *smooth* embeddings, meaning that we are gluing along diffeomorphisms.

[2] Throughout this chapter we will only consider the smooth functions $C^\infty(\mathbf{M})$, though it should be noted that one may readily consider weaker conditions such as $C^r(\mathbf{M})$ for $r \in \mathbb{N}$.

The Construction of Functions on $\mathbf{M}$

In the Hausdorff setting, there are two useful techniques for constructing smooth functions on a manifold. Roughly speaking, these are:

1. to glue together smooth functions defined on open subsets, and

2. to extend functions defined on a closed subset.

Will will refer to these two techniques are often known as the “Gluing Lemma" and the “Extension Lemma", respectively. For a more thorough discussion of these two constructions, the reader is encouraged to see Corollary 2.8 and Lemma 2.26 of [65].

We will now create versions of these two results for our non-Hausdorff manifold $\mathbf{M}$. To begin with, we show that the smooth functions on $\mathbf{M}$ can be built by gluing together a collection of smooth functions that are defined on the Hausdorff submanifolds M_i.

Lemma 3.2.10. *If $r_i : M_i \to \mathbb{R}$ is a collection of smooth functions such that $r_i = r_j \circ f_{ij}$ for all i, j in I, then the map $\mathbf{r} : \mathbf{M} \to \mathbb{R}$ defined by $\mathbf{r}([x,i]) = r_i(x)$ is a smooth function on $\mathbf{M}$.*

Proof. Observe first that $\mathbf{r}$ is well-defined – if we have $[x,i] = [y, j]$, then $x \in M_{ij}$ with $f_{ij}(x) = y$. So, we have that

$$\mathbf{r}[y, j] = r_j(y) = r_j(f_{ij}(x)) = r_i(x) = \mathbf{r}[x,i]. \tag{3.7}$$

Moreover, the restriction of $\mathbf{r}$ to each M_i equals r_i, which is smooth by assumption. The result then follows from an application of Lemma 3.2.8. □

The above result is a straightforward analogue of the Gluing Lemma. In contrast, an analogue of the Extension Lemma is more involved. The underlying complication is that the extension of a functions defined on a closed subset necessarily requires partitions of unity subordinate to any open cover. As proved in [84], in the non-Hausdorff setting we have the following obstruction to the existence of partitions of unity.

Lemma 3.2.11. *Any open cover of $\mathbf{M}$ by Hausdorff sets does not admit a partition of unity subordinate to it.*

Since each M_i is an open, Hausdorff submanifold of $\mathbf{M}$, we cannot directly use any partitions of unity subordinate to the cover $\{M_i\}$. However, our restrictions on the topology of $\mathbf{M}$ are stringent enough so as to allow certain techniques involving partitions of unity. Indeed, the requirement that the gluing regions M_{ij} have diffeomorphic boundaries may allow us to smoothly transfer objects between the submanifolds M_i, and the requirement that $\mathbf{M}$ be a finite colimit may allow this transfer to be performed inductively. We now illustrate this approach with a construction of non-zero functions on $\mathbf{M}$.

Theorem 3.2.12. *Any smooth function r_i on M_i can be extended to a smooth function on $\mathbf{M}$.*

Proof. We proceed by induction on the size of I. Suppose first that $\mathbf{M}$ is a binary adjunction space $M_1 \cup_{f_{12}} M_2$, and without loss generality suppose that $i = 2$. Let r_2 be any smooth function on M_2. The restriction of r_2 to the closed submanifold $Cl^{M_2}(M_{12})$ is also a smooth function, and moreover the composition

$$r_2 \circ \overline{f_{12}} : Cl^{M_1}(M_{12}) \to \mathbb{R} \tag{3.8}$$

is a smooth function on the copy of $Cl^{M_1}(M_{12})$ that sits inside M_1. We can now use a partition of unity argument on M_1 to extend $r_2 \circ \overline{f}_{12}$ to a function r_1 defined on all of M_1. By construction, the functions r_1 and r_2 satisfy the antecedent of Lemma 3.2.10, and consequently they define a global function $\mathbf{r}$ on $\mathbb{M}$ which restricts to r_2 on M_2.

Suppose now the hypothesis holds for all non-Hausdorff manifolds constructed as the colimit of n-many Hausdorff manifolds M_i, according to Theorem 3.2.4. Let $\mathbb{M}$ be a non-Hausdorff manifold defined as the colimit of $(n+1)$-many manifolds M_i. Without loss of generality, pick any smooth function r_n defined on M_n. According to Theorem 3.2.6, we may view $\mathbb{M}$ as the inductive colimit:

$$\mathbf{N} \cup_f M_{n+1}, \tag{3.9}$$

where we glue along the set $A := \bigcup_{i \leq n} M_{(n+1)i}$. By the induction hypothesis, there exists some non-zero function $\mathbf{r}$ defined on the adjunction space $\mathbf{N}$ that extends r_n.

Using the fact that each f_{ij} can be extended to a diffeomorphism of boundaries, we can extend the collective function f to a closed function $\overline{f} : Cl^{M_{n+1}}(A) \to Cl^{\mathbf{N}}(f(A))$. The map $\overline{f}$ is well defined since the extensions $\overline{f_{(n+1)i}}$ satisfy a cocycle condition as in Definition 3.2.1, and moreover $\overline{f}$ is a diffeomorphism since locally it equals $\overline{f_{(n+1)i}}$.

We can restrict $\mathbf{r}$ to $Cl^{\mathbf{N}}(A)$ and then the map $\mathbf{r} \circ \overline{f}$ will be a smooth function defined on $Cl^{M_{n+1}}(A)$. Using a partition of unity on M_{n+1}, we may extend the function $\mathbf{r} \circ \overline{f}$ to some function r' defined on all of M_{n+1}. We may then use Lemma 3.2.10 on $\mathbf{r}$ and r' to form a globally-defined function on all of $\mathbb{M}$, which by construction will restrict to r_n on M_n. □

Usefully, smooth functions can be pulled back along smooth maps via precomposition. In the case of the canonical embeddings ϕ_i, precomposition gives an algebra morphism $\phi_i^* : C^\infty(\mathbb{M}) \to C^\infty(M_i)$. As an immediate application of Theorem 3.2.12 we make the following observation.

Corollary 3.2.13. *For each M_i, the map $\phi_i^* : C^\infty(\mathbb{M}) \to C^\infty(M_i)$ is surjective.*

The Fibre Product Structure of $C^\infty(\mathbb{M})$

We saw previously that we can always create smooth functions on $\mathbb{M}$ by gluing together functions that are defined on the component spaces M_i, provided that they are compatible on the overlaps M_{ij}. The following result expresses this principle at the level of algebras.

Theorem 3.2.14. *The algebra $C^\infty(\mathbb{M})$ is isomorphic to the fibred product*

$$\prod_{\mathscr{F}} C^\infty(M_i) := \left\{(r_1, ..., r_n) \in \bigoplus_{i \in I} C^\infty(M_i) \middle| \, r_i = r_j \circ f_{ij} \text{ on } M_{ij} \text{ for all } i, j \in I\right\}. \tag{3.10}$$

Proof. Consider the map Φ^* that acts on each smooth function $\mathbf{r}$ on $\mathbb{M}$ by:

$$\Phi^*(\mathbf{r}) := (\phi_1^* r, \cdots, \phi_n^* r). \tag{3.11}$$

By the commutativity of the diagram $\mathscr{F}$, we have that

$$\phi_j^* \mathbf{r} \circ f_{ij}(x) = \mathbf{r}([f_{ij}(x), j]) = \mathbf{r}([x, i]) = \phi_i^* \mathbf{r}(x), \tag{3.12}$$

thus Φ^* takes image in the fibred product $\prod_{\mathscr{F}} C^\infty(M_i)$. Moreover, the map Φ^* is also an algebra homomorphism, since all the ϕ_i^* are. The map Φ^* is injective since any pair of distinct functions $\mathbf{r}$ and $\mathbf{r}'$ on $\mathbb{M}$ must differ on one of the M_i, and it is surjective by Corollary 3.2.13. Since every bijective algebra homomorphism is an isomorphism, this completes the proof. □

We saw in the form of Lemma 3.2.2 that the adjunction space $\mathbb{M}$ is the colimit $\mathscr{F}$ in the category of topological spaces. Subsequent remarks in Section 3.2.2 confirmed that this colimit also exists in the category of smooth locally-Euclidean manifolds. Since C^∞ is a contravariant functor, in principle we may apply it to all of $\mathscr{F}$ to obtain a diagram in the category of unital associative algebras. The following result confirms that $C^\infty(\mathbb{M})$ is the correct limit of this contravariant diagram.

Lemma 3.2.15. *Let A be a unital associative algebra together with a collection of I-many algebra morphisms $\rho_i : A \to C^\infty(M_i)$. If the maps ρ_i satisfy $\rho_i = \rho_j \circ f_{ij}$ for all i, j in I, then there is a unique algebra morphism $\alpha : A \to \prod_{\mathscr{F}} C^\infty(M_i)$.*

Proof. Consider the map ε defined by

$$\varepsilon(a) = (\rho_1(a), ..., \rho_n(a)). \tag{3.13}$$

This is clearly an element of the direct sum $\bigoplus_i C^\infty(M_i)$, and moreover the commutativity assumption of the ρ_i ensures that ε takes image in the fibred product. That ε is an algebra morphism follows from the fact that each ρ_i is. Finally, the uniqueness of ε is guaranteed since the morphisms from $\prod_{\mathscr{F}} C^\infty(M_i)$ to $C^\infty(M_i)$ are all projections. □

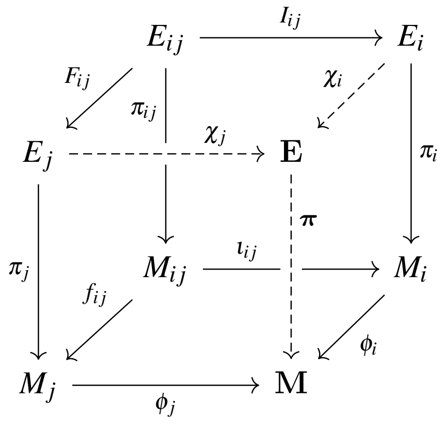
3.3 Vector Bundles

In this section we will construct vector bundles over our non-Hausdorff manifold $\mathbb{M}$. We will start by defining an adjunction of Hausdorff bundles E_i that are fibred over each of the submanifolds M_i. After this, we will argue that every vector bundle over $\mathbb{M}$ can be constructed in this manner. We then will generalise Theorem 3.2.14 by providing a description of sections of any vector bundle fibred over $\mathbb{M}$, eventually finishing with a discussion of Riemannian metrics in the non-Hausdorff setting. All of the basic details of vector bundles can be found in standard texts such as [65, 101].

3.3.1 Colimits of Bundles

Suppose that we are given a collection of rank-k real vector bundles $E_i \xrightarrow{\pi_i} M_i$. Since each M_{ij} is an open submanifold of M_i, we can always form the restricted bundle $E_{ij} := \iota_{ij}^* E_i$, which will be an open submanifold of E_i. In order to define a bundle structure on an adjunction space formed from these manifolds we need to assume a collection of bundle morphisms $F_{ij} : E_{ij} \to E_j$ that cover the gluing maps f_{ij}. According to Definition 3.2.1, these maps also need to satisfy the cocycle condition $F_{jk} \circ F_{ik} = F_{ij}$ on the triple intersections $E_{ij} \cap E_{ik}$.

With all of this data in hand, we may use Lemma 3.2.7 on the collection of bundles $E_i \xrightarrow{\pi_i} M_i$ to form a non-Hausdorff smooth manifold $\mathbb{E}$. We denote by χ_i the canonical embeddings of each E_i into $\mathbf{E}$. In order to describe a bundle structure on $\mathbf{E}$, we would first like to define a projection map $\boldsymbol{\pi} : \mathbf{E} \to \mathbf{M}$. This amounts to completing the commutative diagram



(3.14)

simultaneously for all i, j in I.

We will denote points in $\mathbf{E}$ by $[v,i]$, where $v \in E_i$. Strictly speaking this is an abuse of notation, since the equivalence classes of $\mathbf{E}$ are different from the equivalence classes used to define points in $\mathbf{M}$. However, in this notation the projection map $\boldsymbol{\pi} : \mathbf{E} \to \mathbf{M}$ can be easily defined as $\boldsymbol{\pi}([v,i]) = [\pi_i(v), i]$. Observe that $\boldsymbol{\pi}$ is well defined since our requirement that the bundle morphisms F_{ij} cover the gluing map f_{ij} ensures that

$$\boldsymbol{\pi}([F_{ij}(v), j]) = [\pi_j \circ F_{ij}(v), j] = [f_{ij} \circ \pi_i(v), j] = [\pi_i(v), i] \tag{3.15}$$

for all v in E_{ij}. Moreover, the map $\boldsymbol{\pi}$ is manifestly smooth since its local expression around any point $[v,i]$ of $\mathbf{E}$ will be the composition $\phi_i \circ \pi_i \circ \chi_i^{-1}$.

By construction the map $\boldsymbol{\pi}$ is surjective, and furthermore we may endow the preimages $\boldsymbol{\pi}^{-1}([x,i])$ with the structure of a rank-k vector space induced from the fibre $\pi_i^{-1}(x)$ of E_i. In our notation, addition and scalar multiplication are given by

$$[v,i] + [w,i] = [v+w, i] \text{ and } \lambda[v,i] = [\lambda v, i], \tag{3.16}$$

respectively. These operations are well-defined by our assumption that the F_{ij} are bundle morphisms, and consequently the fibres of $\mathbf{E}$ will indeed be k-dimensional vector spaces.

In direct analogy to the construction of smooth atlases in Section 3.2.2, we can describe local trivialisations of $\mathbf{E}$ using the bundles E_i. Suppose that we have a point $[x,i]$ in $\mathbf{M}$, and fix U to be a local trivialisation of the bundle E_i at the point x, with trivialising map Θ. Since ϕ_i is an open map, we can consider the set $\phi_i(U)$ as an open neighbourhood of $[x,i]$ in $\mathbf{M}$.

This data can be arranged into the following diagram

$$\begin{array}{ccccccc} U\times\mathbb{R}^k & \xleftarrow{\Theta} & \pi_i^{-1}(U) & \xrightarrow{\chi_i} & \pi^{-1}(\phi_i(U)) & \overset{\Psi}{\dashrightarrow} & \phi_i(U)\times\mathbb{R}^k \\ & \searrow^{p_1} & \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi} & {}^{p_1}\swarrow & \\ & & U & \xrightarrow[\phi_i]{} & \phi_i(U) & & \end{array} \tag{3.17}$$

where the p_1 are projections onto the first factor. The trivialising map Ψ can then be defined as the composition $\Psi := (\phi_i, \mathrm{id}) \circ \Theta \circ \chi_i^{-1}$. Transition functions for a pair of local trivialisations (U_α, Ψ_α) and (U_β, Ψ_β) around points in the gluing regions M_{ij} will be:

$$\Psi_\beta \circ \Psi_\alpha^{-1} = \left((\phi_j, \mathrm{id}) \circ \Theta_\beta \circ \chi_j^{-1}\right) \circ \left((\phi_i, \mathrm{id}) \circ \Theta_\alpha \circ \chi_i^{-1}\right)^{-1} \tag{3.18}$$

$$= \left((\phi_j, \mathrm{id}) \circ \Theta_\beta\right) \circ \left(\chi_j^{-1} \circ \chi_i\right) \circ \left(\Theta_\alpha^{-1} \circ (\phi_i^{-1}, \mathrm{id})\right) \tag{3.19}$$

$$= \left((\phi_j, \mathrm{id}) \circ \Theta_\beta \circ F_{ij} \circ \Theta_\alpha^{-1} \circ (\phi_i^{-1}, \mathrm{id})\right), \tag{3.20}$$

which mimic the local properties of the bundle morphisms F_{ij}.

According to our discussion thus far, we may consider $\mathbf{E}$ as a rank-k vector bundle fibred over the non-Hausdorff manifold $\mathbf{M}$ in which the maps χ_i of $\mathbf{E}$ are injective bundle morphisms that cover the canonical embeddings ϕ_i. This is summarised in the following result.

Theorem 3.3.1. *Let $\mathscr{G} := (E, B, F)$ be a triple of sets in which:*

1. *$E = \{(E_i, \pi_i, M_i)\}_{i \in I}$ is a collection of rank-k vector bundles,*
2. *$B = \{E_{ij}\}_{i,j \in I}$ consists of the restrictions of the bundles E_i to the intersections M_{ij}, and*
3. *$F = \{F_{ij}\}_{i,j \in I}$ is a collection of bundle isomorphisms $F_{ij} : E_{ij} \to E_{ji}$ that cover the gluing maps f_{ij} and satisfy the condition $F_{ik} = F_{jk} \circ F_{ij}$ on the intersection $M_{ij} \cap M_{ik}$, for all i, j, k in I.*

Then the resulting adjunction space $\mathbf{E} := \bigcup_{\mathscr{G}} E_i$ has the structure of a non-Hausdorff rank-k vector bundle over $\mathbf{M}$ in which the canonical inclusions $\chi_i : E_i \to \mathbf{E}$ are bundle morphisms covering the canonical embeddings $\phi_i : M_i \to \mathbf{M}$.

We will now confirm that the bundle $\mathbf{E}$ described satisfies a certain universal property.

Theorem 3.3.2. *Let $\mathbf{E}$ be a vector bundle over $\mathbf{M}$ as in Theorem 3.3.1, and let $\mathbf{F} \xrightarrow{\rho} \mathbf{M}$ be a vector bundle. Suppose there exist bundle morphisms $\xi_i : E_i \to \mathbf{F}$ covering the canonical maps ϕ_i satisfying $\xi_i = \xi_j \circ F_{ij}$ for all i, j in I. Then there exists a unique bundle morphism $\varepsilon : \mathbf{E} \to \mathbf{F}$ such that $\xi_i = \chi_i \circ \varepsilon$ for all i in I.*

Proof. Since all bundles are smooth manifolds and all bundle morphisms are smooth maps, we may apply the universal property of smooth non-Hausdorff manifolds to conclude that there exists a unique smooth map ε from $\mathbf{E}$ to $\mathbf{F}$ defined by

$$\varepsilon([v,i]) = \xi_i(v). \tag{3.21}$$

Observe that since the maps $\chi_i : E_i \to \mathbf{E}$ and the maps $\xi_i : E_i \to \mathbf{F}$ both cover the canonical embeddings ϕ_i, we have that

$$\rho \circ \varepsilon([v,i]) = \rho \circ \xi_i(v) = \phi_i \circ \pi_i(v) = \pi \circ \chi_i(v) = \pi([v,i]), \tag{3.22}$$

and thus the map ε covers the identity map on $\mathbf{M}$. Moreover, ε acts linearly on fibres of $\mathbf{E}$ since ε coincides with the map $\xi_i \circ \chi_i^{-1}$ on each E_i, from which we may conclude that ε is a bundle morphism. □

According to the above result, we may interpret any vector bundle $\mathbf{E}$ constructed according to Theorem 3.3.1 as a colimit in the category of smooth vector bundles over locally-Euclidean, second-countable spaces.

3.3.2 A Reconstruction Theorem

It is well-known that all non-Hausdorff manifolds can be constructed using adjunction spaces. In essence, this result follows from the fact that maximal Hausdorff submanifolds of a given non-Hausdorff manifold form an open cover [47]. It is then possible to fix a minimal open cover by Hausdorff submanifolds, and then to glue them along the identity maps defined on the pairwise intersections. The details of this result can be found in [68, 84] and [82] in different forms.

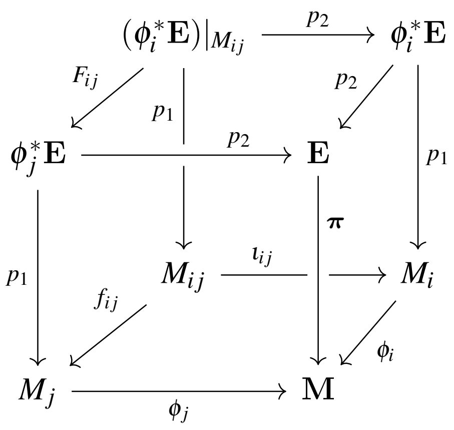
We will now argue that all vector bundles over $\mathbf{M}$ are colimits in the sense of Theorem 3.3.1. The argument is similar to the manifold case: we can always restrict a bundle down to the component spaces M_i to create a collection of Hausdorff bundles that can then be re-identified. Formally, this restriction is obtained by taking the pullbacks of the bundle $\mathbf{E}$ along the canonical embeddings ϕ_i.

Theorem 3.3.3. *Let* $\mathbf{E}$ *be some vector bundle over* $\mathbf{M}$*. Then* $\mathbf{E}$ *is isomorphic to a colimit bundle of the form detailed in Theorem 3.3.1.*

Proof. We would like to define a colimit bundle by gluing the pullback bundles $\phi_i^*\mathbf{E}$ along bundle morphisms that act by identity on each fiber. Formally, this can be achieved by the data $\mathscr{G} = (\mathsf{E}, \mathsf{B}, \mathsf{G})$, where:

- E consists of the pullback bundles $E_i := \phi_i^*\mathbf{E}$,
- B consists of the restricted bundles $E_{ij} := (\phi_i^*\mathbf{E})|_{M_{ij}} \cong (\phi_i \circ \iota_{ij})^*\mathbf{E}$, and
- G consists of the maps $F_{ij} : (\phi_i \circ \iota_{ij})^*\mathbf{E} \to \phi_j^*\mathbf{E}$ where $(x,v) \mapsto (f_{ij}(x), v)$.

This data satisfies the criteria of Theorem 3.3.1, thus we may conclude that $\mathbf{F} := \bigcup_{\mathscr{G}} E_i$ is a vector bundle over $\mathbf{M}$. By construction, the pullback bundles $\phi_i^*\mathbf{E}$ cover the canonical embeddings ϕ_i via the maps p_2 which project onto the second factor of the Cartesian product. This means that pairwise we have the following diagram.



(3.23)

This diagram commutes since the morphisms F_{ij} act by the identity on their second factors. According to the universal property of the colimit bundle $\mathbf{F}$ we may induce a (unique) bundle morphism ε from $\mathbf{F}$ to $\mathbf{E}$. Pointwise, the map ε acts by $[(x,v),i] \mapsto v$. This map is clearly bijective, from which it follows that ε is a bundle isomorphism. □

3.3.3 Sections

In Theorem 3.2.14 we saw that the ring of smooth functions of $\mathbf{M}$ is naturally isomorphic to the fibred product $\prod_{\mathscr{F}} C^\infty(M_i)$. In categorical terms, this product can be seen as the limit of a diagram that is formed by applying the C^∞ functor to all of the data in $\mathscr{F}$. We will now extend this result to sections of arbitrary bundles over $\mathbf{M}$. Throughout this section we take $\mathbf{E}$ to be an arbitrary but fixed vector bundle over $\mathbf{M}$, and we will denote by E_i the restricted (Hausdorff) bundles over the subspaces M_i. In analogy to Lemma 3.2.10, we will first show that every section of the vector bundle $\mathbf{E}$ can be described by gluing sections of E_i that are compatible on overlaps.

Lemma 3.3.4. *For each i in I, let s_i be a section of E_i. If the equality $F_{ij} \circ s_i = s_j \circ f_{ij}$ holds for all i, j in I then the function $\mathrm{s} : \mathbf{M} \to \mathbf{E}$ defined by $\mathrm{s}([x,i]) = [s_i(x), i]$ is a smooth section of $\mathbf{E}$.*

Proof. Observe first that s is well-defined, since:

$$\mathrm{s}([f_{ij}(x), j]) = [s_j \circ f_{ij}(x), j] = [F_{ij} \circ s_i(x), j] = [s_i(x), i]. \tag{3.24}$$

The map s is a right-inverse of the projection map π since $\pi \circ \mathrm{s}([x,i]) = \pi([s_i(x), i]) = [x,i]$ for all $[x,i]$ in $\mathbf{M}$. Finally, since the restriction of s to each M_i equals $\chi_i \circ s_i \circ \phi_i^{-1}$, the map s is smooth by Lemma 3.2.8. □

Following on from the approach of Section 3.2.3, we may now prove an analogue to Theorem 3.2.12.

Theorem 3.3.5. *Any section s_i of E_i can be extended to a section* s *of* $\mathbf{E}$.

Proof. We will proceed as in Theorem 3.2.12, that is, by induction on the size of indexing set I. Suppose first that $\mathbf{M}$ is a binary adjunction space $M_1 \cup_{f_{12}} M_2$, with E isomorphic to $E_1 \cup_{F_{12}} E_2$. As in Theorem 3.2.12, we may take a non-zero section on E_2, restrict it to the closure $Cl^{M_2}(M_{12})$, and then use the diffeomorphism $\overline{f_{12}}$ to pull back s_2 to some smooth section s_{12} defined on the submanifold $Cl^{M_1}(M_{12})$ of M_1. Using the fact that F_{12} is a bundle isomorphism onto its image, we may interpret s_{12} as a section of the closed subbundle $Cl^{\mathbf{E}}(E_{12})$ defined over $Cl^{M_1}(M_{12})$.

In order to extend s_{12} into the rest of M_1, we will need to apply a generalisation of the Extension Lemma for sections of vector bundles (cf. Lemma 10.12 of [65]). The idea behind this generalised Extension Lemma is essentially the same as in the case of smooth functions – we may always endow the closure $Cl^{M_1}(M_{12})$ with an outward-pointing collar neighbourhood U, and then use the flow of the associated vector field to extend s_{12} to all of U. Using a partition of unity subordinate to the open cover $\{U, M_1 \backslash Cl^{M_1}(M_{12})\}$ of M_1, we may then create a section s_1 of E_1.

Observe that by construction, any section s_1 created according to the above procedure will restrict to s_{12} on M_{12}. A global section s of $\mathbf{E}$ then exists by applying of Lemma 3.3.4 to s_1 and s_2. The inductive case follows the same structure as Theorem 3.2.12, this time using the Extension Lemma for vector bundles instead. □

Using the above, we can now create an argument similar to that of Theorem 3.2.14.

Theorem 3.3.6. *For any vector bundle* $\mathbf{E}$ *over* $\mathbf{M}$*, the space* $\Gamma(\mathbf{E})$ *of smooth sections of* $\mathbf{E}$ *is isomorphic to:*

$$\Gamma(\mathbf{E}) \cong \prod_{\mathscr{F}} \Gamma(E_i). \tag{3.25}$$

Proof. The argument is the same as that of Theorem 3.2.14, except that this time we use Lemma 3.3.4 and Theorem 3.3.5. □

3.3.4 Riemannian Metrics

In order to discuss Čech cohomology in the next section, we will first need to confirm that metrics exist on arbitrary vector bundles fibred over $\mathbf{M}$. The precise construction of such metrics will be similar to the approach of Lemmas 3.2.12 and 3.3.4. However, in order to use this technique we first need to confirm the following.

Lemma 3.3.7. *Let* $\mathbf{E}$ *be a vector bundle over* $\mathbf{M}$ *with colimit representation* $\bigcup_{\mathscr{G}} E_i$*. Then the* $(0,2)$ *tensor bundle* $T^{(0,2)}\mathbf{E}$ *is isomorphic to the colimit bundle*

$$\bigcup_{\mathscr{G}} T^{(0,2)} E_i. \tag{3.26}$$

Proof. For readability we will only provide a sketch, since the details of this argument can already be found in [82]. Let us denote by F_{ij} the bundle morphisms that are used to construct $\mathbf{E}$ from the E_i. Since the F_{ij} are diffeomorphisms from E_{ij} to E_{ji}, the differentials dF_{ij} will be bundle isomorphisms between the tangent bundles TE_{ij} and TE_{ji}. Moreover, a basic property of differentials confirms that

$$dF_{ik} = d(F_{jk} \circ F_{ij}) = dF_{jk} \circ dF_{ij}. \tag{3.27}$$

Consequently, we may glue each tangent bundle TE_i along the differentials dF_{ij} to create the bundle $\bigcup_{\mathscr{G}} TE_i$. The differentials $d\chi_i : TE_i \to T\mathbf{E}$ of the canonical embedding maps $\chi_i : E_i \to \mathbf{E}$ can then be used together with Theorem 3.3.2 to conclude that the bundle $\bigcup_{\mathscr{G}} TE_i$ is isomorphic to $T\mathbf{E}$. A similar argument can be made for the bundle $T^{(0,2)}\mathbf{E}$, except that this time we glue along the maps that pull back the $(0,2)$-tensors along the diffeomorphisms F_{ij}. □

A metric tensor on $\mathbf{E}$ may be seen as a global non-vanishing section of the bundle $T^{(0,2)}\mathbf{E}$ that is symmetric and positive-definite in its local expression. Using the above result, we may readily construct metrics on any vector bundle fibred over $\mathbb{M}$.

Theorem 3.3.8. *Any vector bundle* $\mathbf{E}$ *over* $\mathbb{M}$ *admits a metric.*

Proof. According to Theorem 3.3.3, we may view $\mathbf{E}$ as a colimit of bundles E_i that are fibred over the Hausdorff submanifolds M_i. Lemma 3.3.7 then allows us to express the tensor bundle $T^{(0,2)}\mathbf{E}$ as a colimit of the tensor bundles $T^{(0,2)}E_i$. We may then apply the construction of Theorem 3.3.5 and use an inductive series of partitions of unity defined on each M_i in order to construct a global section g of the bundle $T^{(0,2)}\mathbf{E}$. Note that we may guarantee that g is a bundle metric if we start with bundle metrics g_i of E_i and use the fact that bundles metrics are closed under convex combinations. □

In the next section we will need to appeal to the existence of Riemannian metrics defined on $\mathbb{M}$. Fortunately, these exist as an application of the previous result to $T\mathbb{M}$.

Corollary 3.3.9. $\mathbb{M}$ *admits a Riemannian metric.*

3.4 Čech Cohomology and Line Bundles

Theorems 3.3.1 and 3.3.2 tell us that any line bundle over $\mathbb{M}$ exists as a colimit of line bundles defined on each of the submanifolds M_i. In this section we will explore how this relationship manifests in the language of Čech cohomology. To begin with, we will proceed generally and study the Čech cohomology of $\mathbb{M}$ in terms of the cohomologies of the M_i.

Before getting to any results, we will first briefly recall the formalism of Čech cohomology. Aside from a slight change in notation, we will essentially follow [13, §8]. Consider an arbitrary topological space X, with an open cover $\mathscr{U} := \{U_\alpha \mid \alpha \in A\}$, and let G be an Abelian group. We will use index notation to abbreviate multiple intersections of open sets in $\mathscr{U}$, that is, we will write $U_{\alpha_0 \cdots \alpha_q} := U_{\alpha_0} \cap \cdots \cap U_{\alpha_q}$. A degree-$q$ Čech cochain $\check{a}$ consists of a choice of a constant function for each of the $(q+1)$-ary intersections of sets in $\mathscr{U}$ to the group G. In symbols:

$$\check{a} := \left\{ a_{\alpha_0 \cdots \alpha_q} : U_{\alpha_0 \cdots \alpha_q} \to G \;\middle|\; \alpha_0, \cdots, \alpha_q \in A \text{ and } a_{\alpha_0 \cdots \alpha_q} \text{ is constant} \right\}. \tag{3.28}$$

As a convention we assume skew symmetry $a_{\alpha_0 \cdots \alpha_i \cdots \alpha_j \cdots \alpha_q} = -a_{\alpha_0 \cdots \alpha_j \cdots \alpha_i \cdots \alpha_q}$ for all $0 \leq$

$i, j \leq q$.[3] The space of Čech q-cochains, which we will denote by $\check{C}^q(X, \mathscr{U}, G)$, consists of all sets $\check{a}$ of the above form. This space naturally inherits an Abelian group structure from G. We denote by δ the Čech differential, which raises the degree of each cochain by one. In additive notation, the map δ acts on individual functions as:

$$\delta a_{\alpha_0 \cdots \alpha_q \alpha_{q+1}}(x) = \sum_i (-1)^i a_{\alpha_0 \cdots \hat{\alpha}_i \cdots \alpha_{q+1}}(x), \tag{3.29}$$

where here the caret notation $\hat{\alpha}_i$ denotes exclusion of that index. On the level of cochains, the differential δ defines $\delta\check{a} := \{\delta a_{\alpha_0 \cdots \alpha_q \alpha_{q+1}} \mid a_{\alpha_0 \cdots \alpha_q \alpha_{q+1}} \in \check{a}\}$. The Čech differential is a group homomorphism that squares to zero, and we denote the resulting cohomology groups by $\check{H}^q(X, \mathscr{U}, G)$. We may define these groups for any open cover $\mathscr{U}$, and the collection of all $\check{H}^q(X, \mathscr{U}, G)$ can be made into a directed system of groups once ordered by refinement. The Čech cohomology of X is then defined as the direct limit:

$$\check{H}^q(X.G) := \lim_{\mathscr{U}} \check{H}^q(X, \mathscr{U}, G). \tag{3.30}$$

In what follows we will need to make use of the pullback of Čech cochains, so we recall this notion now. Suppose that $\varphi : X \to Y$ is a continuous map and $\mathscr{U}$ is an open cover of the topological space Y. We may define an open cover $\mathscr{V}$ of X by considering all sets of the form $\varphi^{-1}(U)$, where U is an element of $\mathscr{U}$. We may then define a map $\varphi^* : \check{C}^q(Y, \mathscr{U}, G) \to \check{C}^q(X, \mathscr{V}, G)$ by demanding that $\varphi^* a_{\alpha_0 \cdots \alpha_q}(x) = a_{\alpha_0 \cdots \alpha_q} \circ \varphi(x)$ for all locally-constant functions on Y. By construction, the pullback φ^* is a group homomorphism.

3.4.1 Čech Cohomology via a Mayer-Vietoris Sequence

We will now set about expressing the Čech cohomology $\check{H}^q(\mathrm{M}, G)$ in terms of the groups $\check{H}^q(M_i, G)$. We will obtain this relationship inductively, so throughout this section we will assume that M can be expressed as the colimit of two Hausdorff manifolds M_1 and M_2, in accordance with Remark 3.2.9.[4] We will also assume that M is endowed with a fixed but arbitrary open cover $\mathscr{U}$. We will not need to appeal to the particular structure of the Abelian group G, so we suppress this in our notation.

We start with a derivation of a Mayer-Vietoris sequence for M. In order to do so, we will need to make use of the pullbacks of cochains. According to our configuration, we have the following commutative diagram of pullbacks

[3] We remark that this convention is not strictly necessary, and indeed there is a formulation of Čech cohomology in which the indexing set A is taken to be linearly-ordered and the skew-symmetry condition is dropped. However, these “ordered Čech cochains" will still yield the same cohomology groups later on. See Exercise 8.4 of [13] for more details.

[4] Although we are working with non-Hausdorff manifolds primarily, it should be noted that the following derivation will also work for more general colimits of Hausdorff manifolds that do not assume the “homeomorphic boundary" condition.

$$\begin{array}{ccc} M_{12} & \xrightarrow{\iota_{12}} & M_1 \\ {\scriptstyle f_{12}}\downarrow & & \downarrow{\scriptstyle \phi_1} \\ M_2 & \xrightarrow[\phi_2]{} & \mathbb{M} \end{array} \quad \overset{\check{C}^q}{\longmapsto} \quad \begin{array}{ccc} \check{C}^q(M_{12},\mathscr{U}^{12}) & \xleftarrow{\iota_{12}^*} & \check{C}^q(M_1,\mathscr{U}^1) \\ {\scriptstyle f_{12}^*}\uparrow & & \uparrow{\scriptstyle \phi_1^*} \\ \check{C}^q(M_2,\mathscr{U}^2) & \xleftarrow[\phi_2^*]{} & \check{C}^q(\mathbb{M},\mathscr{U}) \end{array} \tag{3.31}$$

where here $\mathscr{U}^i = \{\phi_i^{-1}(U) \mid U \in \mathscr{U}\}$, and

$$\mathscr{U}^{12} = \{\iota_{12}^{-1}(V) \mid V \in \mathscr{U}^1\} = \{f_{12}^{-1}(V) \mid V \in \mathscr{U}^2\}. \tag{3.32}$$

As a convention we will index these three covers using the relevant subsets of the indexing set of $\mathscr{U}$.

We can combine the various pullback maps in order to create a single sequence from the above diagram. We will consider two maps: Φ^*, which acts on cochains on $\mathbb{M}$ by concatenating the pullbacks ϕ_i^* (as in 3.2.14, 3.3.6), and the map $\iota_{12}^* - f_{12}^*$, which pulls back a pair of cochains defined on the M_i to the subset M_{12} and then computes their difference. The following result confirms that this arrangement of functions forms a short exact sequence.

Lemma 3.4.1. *The sequence*

$$0 \to \check{C}^q(\mathbb{M},\mathscr{U}) \xrightarrow{\Phi^*} \check{C}^q(M_1,\mathscr{U}^1) \oplus \check{C}^q(M_2,\mathscr{U}^2) \xrightarrow{\iota_{12}^* - f_{12}^*} \check{C}^q(M_{12},\mathscr{U}^{12}) \to 0 \tag{3.33}$$

is exact for all q in $\mathbb{N}$.

Proof. The map Φ^* is injective since any two cochains on $\mathbb{M}$ must differ somewhere on M_1 or M_2, thus they will differ once pulled back by the appropriate ϕ_i map. The surjectivity of the difference map $\iota_{12}^* - f_{12}^*$ follows from a standard "extension by zero" argument, which we will now summarise. Observe first that any cochain $\check{a}$ on M_{12} has as elements functions of the form $a_{\alpha_0\cdots\alpha_q} : W_{\alpha_0\cdots\alpha_q} \to G$, where we define $W_{\alpha_0\cdots\alpha_q} := \iota_{12}^{-1}(U_{\alpha_0\cdots\alpha_q})$ in accordance with (3.32). In order to show that the map ι_{12}^* is surjective, we will need to define a cochain $\check{b}$ in $\check{C}^q(M_1,\mathscr{U}^1)$, which amounts to specifying functions for *all* intersections of sets in the open cover $\mathscr{U}^1$. With this in mind, for a given cochain $\check{a}$ in $\check{C}^q(M_{12},\mathscr{U}^{12})$ we define

$$\check{b}_{\alpha_0\cdots\alpha_q}(x) = \begin{cases} \check{a}_{\alpha_0\cdots\alpha_q}(x) & \text{if } U_{\alpha_0\cdots\alpha_q} \cap M_{12} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{3.34}$$

Simply put, the cochain $\check{b}$ will use the functions existing within $\check{a}$ where possible, and use the zero function on all other q-ary intersections of sets in $\mathscr{U}^1$. Observe that by construction the cochain $\check{b}$ will restrict to $\check{a}$ under a pullback by ι_{12}. This ensures that ι_{12}^* is surjective, and thus so is the difference map $\iota_{12}^* - f_{12}^*$.

Exactness may now be proved with an argument that $\mathrm{Im}(\Phi^*) = \ker(\iota_{12}^* - f_{12}^*)$. Observe first that the commutativity of the diagram preceding this Lemma ensures that $\iota_{12}^* \circ \phi_1^* = f_{12}^* \circ \phi_2^*$, and thus $\mathrm{Im}(\Phi^*) \subseteq \ker(\iota_{12}^* - f_{12}^*)$. For the converse inclusion, suppose that $\check{a}$ and $\check{b}$ are cochains on M_1 and M_2, respectively, such that $\iota_{12}^* \check{a} = f_{12}^* \check{b}$. Spelling this out, this means that

$$a_{\alpha_0 \cdots \alpha_q} \circ \iota_{12}(x) = b_{\alpha_0 \cdots \alpha_q} \circ f_{12}(x) \tag{3.35}$$

for all open sets $W_{\alpha_0}, \cdots, W_{\alpha_q}$ in $\mathscr{U}^{12}$. In order to define a cochain $\check{c}$ on $\mathbb{M}$ that satisfies $\Phi^*(\check{c}) = (\check{a}, \check{b})$, we need to specify a locally-constant function for all of the $(q+1)$-ary intersections of open sets in $\mathscr{U}$. So, suppose that $U_{\alpha_0}, \cdots, U_{\alpha_q}$ are arbitrary elements of $\mathscr{U}$. For any $[x,i]$ in the intersection $U_{\alpha_0 \cdots \alpha_q}$, we define the map $c_{\alpha_0 \cdots \alpha_q}$ as follows:

$$c_{\alpha_0 \cdots \alpha_q}([x,i]) = \begin{cases} a_{\alpha_0 \cdots \alpha_q} \circ \phi_1^{-1}([x,i]) & \text{if } U_{\alpha_0 \cdots \alpha_q} \cap M_1 \neq \emptyset \\ b_{\alpha_0 \cdots \alpha_q} \circ \phi_2^{-1}([x,i]) & \text{if } U_{\alpha_0 \cdots \alpha_q} \cap M_2 \neq \emptyset \end{cases}. \tag{3.36}$$

There is a potential ambiguity in this definition, so we must confirm that the value of $c_{\alpha_0 \cdots \alpha_q}$ does not depend on the choice of M_1 or M_2. So, suppose that the set $U_{\alpha_0 \cdots \alpha_q}$ intersects both M_1 and M_2. Then for any element in M_{12}, we have two representatives of the equivalent class $[x,1]$, namely x and $f_{12}(x)$. We then have that

$$a_{\alpha_0 \cdots \alpha_q} \circ \phi_1^{-1}([x,1]) = a_{\alpha_0 \cdots \alpha_q} \circ \iota_{12}(x) \overset{3.35}{=} b_{\alpha_0 \cdots \alpha_q} \circ f_{12}(x) = b_{\alpha_0 \cdots \alpha_q} \circ \phi_2^{-1}([f_{12}(x),2]) \tag{3.37}$$

as required. By construction, $\check{c}$ pulls back to $(\check{a}, \check{b})$ under the map Φ^*. We may therefore conclude that $\ker(\iota_{12}^* - f_{12}^*) \subseteq \mathrm{Im}(\Phi^*)$, from which the equality follows. □

As a consequence of the above result, we observe that the Čech cochains of $\mathbb{M}$ admit a fibred product structure.

Corollary 3.4.2. *The Čech cochains on* $\mathbb{M}$ *satisfy the equality*

$$\check{C}^q(\mathbb{M}, \mathscr{U}) \cong \check{C}^q(M_1, \mathscr{U}^1) \times_{\check{C}^q(M_{12}, \mathscr{U}^{12})} \check{C}^q(M_2, \mathscr{U}^2) \tag{3.38}$$

$$= \left\{ (\check{a}^1, \check{a}^2) \in \check{C}^q(M_1, \mathscr{U}^1) \oplus \check{C}^q(M_2, \mathscr{U}^2) \;\middle|\; \iota_{12}^* \check{a}^1 = f_{12}^* \check{a}^2 \right\}. \tag{3.39}$$

for all $q \in \mathbb{N}$.

Routine computations verify that the maps Φ^* and $\iota_{12}^* - f_{12}^*$ commute with the Čech differential δ. Consequently, we may expand the data of Lemma 3.4.1 into the long exact sequence

$$\cdots \xrightarrow{\delta^*} \check{H}^q(\mathbb{M}) \xrightarrow{\Phi^*} \check{H}^q(M_1) \oplus \check{H}^q(M_2) \xrightarrow{\iota_{12}^* - f_{12}^*} \check{H}^q(M_{12}) \xrightarrow{\delta^*} \check{H}^{q+1}(\mathbb{M}) \to \cdots \tag{3.40}$$

where here the maps δ^* are the connecting homomorphisms induced as an application of the Snake Lemma. It should be noted that here we have tacitly passed to the cover-independent version of Čech cohomology. In fact, the relationship above is induced from a combination of refinement maps and the universal property of direct limits – the details can be found in the Appendix.

According to the above sequence, it will not be the case that the Čech cohomology of M will equal a fibred product of the groups $\check{H}^q(M_1)$ and $\check{H}^q(M_2)$. Indeed, if we try and proceed in similar spirit to Theorems 3.2.14 and 3.3.6 and argue for the injectivity of the Φ^*, we will obtain an obstruction coming from the previous cohomology group of M_{12}. This obstruction can be read off of the Mayer-Vietoris sequence as:

$$\frac{\check{H}^q(\mathrm{M})}{\mathrm{Im}(\delta^*)} = \frac{\check{H}^q(\mathrm{M})}{Ker(\Phi)} = \mathrm{Im}(\Phi) = \ker(\iota_{12}^* - f_{12}^*) = \check{H}^q(M_1) \times_{\check{H}^q(M_{12})} \check{H}^q(M_2). \tag{3.41}$$

With an eye towards Section 3.4.3, we make the following observation.

Lemma 3.4.3. *Suppose that* M *is a binary adjunction space in which* M_1*,* M_2 *and* M_{12} *are all connected. Then*

$$\check{H}^1(\mathrm{M}) \cong \check{H}^1(M_1) \times_{\check{H}^1(M_{12})} \check{H}^1(M_2) \tag{3.42}$$

$$= \Big\{ \big([\check{a}^1],[\check{a}^2]\big) \in \check{H}^q(M_1) \oplus \check{H}^q(M_2) \ \Big| \ \iota_{12}^*[\check{a}^1] = f_{12}^*[\check{a}^2] \Big\}. \tag{3.43}$$

Proof. According to Theorem 1.6 of [84], M is connected. Since the 0th Čech cohomology group of a connected topological space always equals the globally-constant functions into G. These function are in one-to-one correspondence with the elements of G, and thus our Mayer-Vietoris sequence reduces to the following.

$$0 \to G \to G \oplus G \to G \xrightarrow{\delta^*} \check{H}^1(\mathrm{M}) \to \check{H}^1(M_1) \oplus \check{H}^1(M_2) \to \check{H}^1(M_{12}) \to \cdots \tag{3.44}$$

In this situation, the pullback map Φ^* will coincide with the map that restricts a globally-constant function on $\mathbf{M}$ to the corresponding functions on the M_i. On the level of groups, we may see the embedding of $H^0(\mathbf{M})$ into the sum $H^0(M_1) \oplus H^0(M_2)$ as the diagonal map $g \mapsto (g,g)$. Recall that the quotient of any group $G \oplus G$ by its diagonal will be canonically isomorphic to G itself. As such, in the above sequence, we may work from the left and use exactness to conclude that the labelled connecting homomorphism δ^* is the zero map, from which the result follows. □

3.4.2 The General Case

We will now discuss a version of Theorem 3.4.1 that is suitable for a non-Hausdorff manifold M expressed as the colimit of finitely-many manifolds M_i. We will start with a generalisation of Lemma 3.4.1 that takes into account the multiple intersections $M_{i_1 \cdots i_p}$.

To begin with, we need to generalise the difference maps $\iota_{12}^* - f_{12}^*$. Since there are now n-many submanifolds M_i, we will have multiple pairwise intersections M_{ij}. We may order the pairs of indices lexicographically and define a map

$$\tilde{\delta} : \bigoplus_i \check{C}^q(M_i, \mathscr{U}^i) \to \bigoplus_{i<j} \check{C}^q(M_{ij}, \mathscr{U}^{ij}) \tag{3.45}$$

which combines the difference maps $\iota_{ij}^* - f_{ij}^*$ in the obvious manner. Since there are now multiple intersections $M_{i_1 \cdots i_p}$, each with their own spaces of cochains $\check{C}^q(M_{i_1 \cdots i_q}, \mathscr{U}^{i_1 \cdots i_q})$, we will also have multiple pullbacks

$$\iota^*_{i_1\cdots\hat{i}\cdots i_p} : \check{C}^q(M_{i_1\cdots\hat{i}\cdots i_p}, \mathscr{U}^{i_1\cdots\hat{i}\cdots i_p}) \to \check{C}^q(M_{i_1\cdots i_p}, \mathscr{U}^{i_1\cdots i_p}), \tag{3.46}$$

where here $\iota_{i_1\cdots\hat{i}\cdots i_p} : M_{i_1\cdots i_p} \to M_{i_1\cdots\hat{i}\cdots i_p}$ is the inclusion map that forgets the i^{th} index. We can similarly define difference maps

$$\tilde{\delta} : \bigoplus_{i_1<\cdots<i_p} \check{C}^q(M_{i_1\cdots i_p}, \mathscr{U}^{i_1\cdots i_p}) \to \bigoplus_{i_1<\cdots<i_{p+1}} \check{C}^q(M_{i_1\cdots i_{p+1}}, \mathscr{U}^{i_1\cdots i_{p+1}}) \tag{3.47}$$

to act on cochains $\check{a}_{i_1\cdots i_{p+1}}$ defined on each $M_{i_1\cdots i_{p+1}}$ by[5]

$$(\tilde{\delta}\check{a})_{i_1\cdots i_{p+1}} = \sum_i (-1)^{i+1} \iota^*_{i_1\cdots\hat{i}\cdots i_{p+1}} \check{a}_{i_1\cdots\hat{i}\cdots i_{p+1}}. \tag{3.48}$$

In essence, the map $\tilde{\delta}$ plays the role of the differential that one would define for the Čech complex constructed from the open cover $\{M_i\}$. In a similar manner to the binary case of Section 3.4.1, this map $\tilde{\delta}$ can be shown to square to zero and to commute with the Čech differential δ defined for $\check{C}^q(\mathrm{M}, \mathscr{U})$.

We will now set about proving the general version of Lemma 3.4.1, this time taking into account the multiple intersections $M_{i_1\cdots i_p}$.

Theorem 3.4.4. *The sequence*

$$0 \to \check{C}^q(\mathrm{M}, \mathscr{U}) \xrightarrow{\Phi^*} \bigoplus_i \check{C}^q(M_i, \mathscr{U}^i) \xrightarrow{\tilde{\delta}} \cdots \xrightarrow{\tilde{\delta}} \check{C}^q(M_{1\cdots n}, \mathscr{U}^{1\cdots n}) \to 0 \tag{3.49}$$

is exact for all q in $\mathbb{N}$.

Proof. We will proceed by induction on the size of the set I. The case of $I = 2$ is already proved as Lemma 3.4.1. To spare notation, we will illustrate the inductive argument for $I = 3$, though it should be understood that the full inductive case is near-identical. With all the data available to us, we may construct a contravariant analogue of the diagram in [13, Pg.

[5] Note that the extra sum in the exponent of -1 is due to our convention that the indexing set I starts at one instead of zero.

187] as follows, where we have suppressed the open covers for readability.

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\kappa^*) & \xrightarrow{\phi_3^*} & \check{C}^q(M_3) & \xrightarrow{(-f_{13}^*,-f_{23}^*)} & \check{C}^q(M_{13})\oplus\check{C}^q(M_{23}) & \xrightarrow{\tilde{\delta}} & \check{C}^q(M_{123}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{C}^q(\mathrm{M},\mathscr{U}) & \xrightarrow{\Phi^*} & \bigoplus_i \check{C}^q(M_i) & \xrightarrow{\tilde{\delta}} & \bigoplus_{i<j}\check{C}^q(M_{ij}) & \xrightarrow{\tilde{\delta}} & \check{C}^q(M_{123}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \kappa^*} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \check{C}^q(M_1\cup M_2) & \longrightarrow & \check{C}^q(M_1)\oplus\check{C}^q(M_2) & \longrightarrow & \check{C}^q(M_{12}) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.50}$$

The map κ^* is the pullback of the inclusion $\kappa : M_1 \cup_{f_{12}} M_2 \to \mathrm{M}$ described in Section 3.2.1. The first row of the diagram is formed by taking kernels of the vertical maps, and the horizontal maps in the first row are defined so as to make the diagram commute. Since the columns of this diagram are all short exact sequences that split, in order to show the exactness of the center row it suffices to show that the first and third rows are exact. Observe first that the third row is a binary Mayer-Vietoris sequence, so is exact by Lemma 3.4.1.

Regarding the first row, we first observe that we may interpret the spaces $\check{C}^k(M_{13})$ and $\check{C}^k(M_{13})$ as genuine subspaces of $\check{C}^k(M_3)$ by taking their images under the isomorphisms f_{31}^* and f_{32}^*. These pullbacks form a commutative subdiagram:

$$\begin{array}{ccc}
 & \nearrow^{(-\iota_{31}^*,-\iota_{32}^*)} & \check{C}(M_{31})\oplus\check{C}(M_{32}) \\
\check{C}^q(M_3) & & f_{31}^*\oplus f_{32}^* \uparrow \quad \downarrow f_{13}^*\oplus f_{23}^* \\
 & \searrow_{(-f_{13}^*,-f_{23}^*)} & \check{C}(M_{13})\oplus\check{C}(M_{23})
\end{array} \tag{3.51}$$

which allows us to replace the terms in the first row of the previous diagram with cochain spaces pertaining to M_3 and its subsets.. Following the approach of [23], we may decompose this substituted first row into the sequences:

$$0 \to \ker(\kappa^*) \xrightarrow{\phi_3^*} \check{C}^q(M_3) \xrightarrow{-\iota^*} \check{C}^q(M_{31}\cup M_{32}) \to 0 \tag{3.52}$$

$$0 \to \check{C}^q(M_{31} \cup M_{32}) \to \check{C}^q(M_{31}) \oplus \check{C}^q(M_{32}) \to \check{C}^q(M_{123}) \to 0 \tag{3.53}$$

where here this decomposition is formed by taking the kernel of the $\tilde{\delta}$ map and the map ι is the inclusion of the union $M_{31} \cup M_{32}$ into M_3. The latter sequence is a Mayer-Vietoris sequence, so is exact by Lemma 3.4.1. Thus our proof is complete once we argue that the former is a short exact sequence.

Observe first that the map ϕ_3^* is injective since any two distinct elements of $\ker(\kappa^*)$ will be zero on $M_1 \cup_{f_{12}} M_2$, thus can only differ somewhere on M_3. The map $-\iota^*$ is surjective by a standard extension by zero argument (cf. Lemma 3.4.1). Finally, we will show that $\mathrm{Im}(\phi_3^*) = \ker(-\iota^*)$. The inclusion $\mathrm{Im}(\phi_3^*) \subseteq \ker(-\iota^*)$ is guaranteed since any element in $\ker(\kappa^*)$ will also vanish on $M_{31} \cup M_{32}$. For the converse inclusion, let $\check{a}$ be some cochain on M_3 that restricts to zero on $M_{31} \cup M_{32}$. We can extend $\check{a}$ to a full cochain $\check{b}$ on M by defining the functions of $\check{b}$ to be

$$b_{\alpha_0 \cdots \alpha_q}([x,i]) = \begin{cases} a_{\alpha_0 \cdots \alpha_q}(x) & \text{if } x \in M_3 \\ 0 & \text{otherwise} \end{cases}. \tag{3.54}$$

The cochain $\check{b}$ will then satisfy $\phi_3^* \check{b} = \check{a}$, from which we may conclude that $\ker(-\iota^*) \subseteq \mathrm{Im}(\phi_3^*)$, whence equality. □

The generalised Mayer-Vietoris long exact sequence of Theorem 3.4.4 can be equivalently rephrased as follows.

Corollary 3.4.5. *The Čech cochains of* M *satisfy the equality*

$$\check{C}^q(\mathrm{M}, \mathscr{U}) = \prod_{\mathscr{F}} \check{C}^q(M_i, \mathscr{U}^i) \tag{3.55}$$

$$= \left\{ \left(\check{a}^1, \cdots, \check{a}^n\right) \in \bigoplus_i \check{C}^q(M_i, \mathscr{U}^i) \;\middle|\; \iota_{ij}^* \check{a}^i = f_{ij}^* \check{a}^j \textit{ for all } i, j \in I \right\}. \tag{3.56}$$

In direct analogy to the construction of the Čech-de Rham bicomplex of [13], we may arrange the cochain data of all the M_i into a bicomplex as below.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
\bigoplus_i \check{C}^2(M_i, \mathscr{U}^i) & \xrightarrow{\tilde{\delta}} & \bigoplus_{i<j} \check{C}^2(M_{ij}, \mathscr{U}^{ij}) & \xrightarrow{\tilde{\delta}} & \bigoplus_{i<j<k} \check{C}^2(M_{ijk}, \mathscr{U}^{ijk}) & \longrightarrow & \cdots \\
\delta \uparrow & & -\delta \uparrow & & \delta \uparrow & & \\
\bigoplus_i \check{C}^1(M_i, \mathscr{U}^i) & \xrightarrow{\tilde{\delta}} & \bigoplus_{i<j} \check{C}^1(M_{ij}, \mathscr{U}^{ij}) & \xrightarrow{\tilde{\delta}} & \bigoplus_{i<j<k} \check{C}^1(M_{ijk}, \mathscr{U}^{ijk}) & \longrightarrow & \cdots \\
\delta \uparrow & & -\delta \uparrow & & \delta \uparrow & & \\
\bigoplus_i \check{C}^0(M_i, \mathscr{U}^i) & \xrightarrow{\tilde{\delta}} & \bigoplus_{i<j} \check{C}^0(M_{ij}, \mathscr{U}^{ij}) & \xrightarrow{\tilde{\delta}} & \bigoplus_{i<j<k} \check{C}^0(M_{ijk}, \mathscr{U}^{ijk}) & \longrightarrow & \cdots
\end{array} \tag{3.57}$$

Theorem 3.4.4 ensures that the rows of this bicomplex are exact. As such, we may employ standard spectral arguments (found in, say [13]) to conclude that the cohomology of this bicomplex coincides with the Čech cohomology of $\mathbb{M}$. We remark that it is possible to obtain some alternate descriptions of the Čech cohomology of $\mathbb{M}$ by computing the spectral sequence of the above bicomplex starting with the δ-cohomology of the columns. However, given the scope of this chapter we will not expand on this observation. For our purposes, we will only use the following result.

Theorem 3.4.6. *Suppose that* $\mathbb{M}$ *is built from n-many* M_i *in which all intersections* $M_{i_1\cdots i_p}$ *are connected for all* $p \leq n$*. Then* $\check{H}^1(\mathbb{M})$ *coincides with the fibred product* $\prod_{\mathscr{F}} \check{H}^1(M_i)$*.*

Proof. We will proceed by induction on the size of the indexing set I. This argument revolves around an inductive form of the finite fibred product, so the reader unfamiliar with this form is invited to read the appendix first.

The binary case is already proved in the form of Lemma 3.4.1. We will illustrate the case for $I = 3$, though it should be understood that the inductive argument is near-identical to the following. Let $\mathbb{M}$ be the colimit of the diagram formed from M_1, M_2 and M_3. We can use the inductive construction of $\mathbb{M}$ as in Lemma 3.2.5 to view $\mathbb{M}$ as an adjunction of the pair $M_1 \cup_{f_{12}} M_2$ to M_3 along the union $M_{31} \cup M_{32}$. By the Mayer-Vietoris argument of Lemma 3.4.1 we have the following portion of the long exact sequence:

$$\begin{array}{l} 0 \longrightarrow \check{H}^0(\mathbb{M}) \longrightarrow \check{H}^0(M_1 \cup M_2) \oplus \check{H}^0(M_3) \longrightarrow \check{H}^0(M_{31} \cup M_{32}) \xrightarrow{\delta^*} \\ \longrightarrow \check{H}^1(\mathbb{M}) \longrightarrow \check{H}^1(M_1 \cup M_2) \oplus \check{H}^1(M_3) \xrightarrow{\iota_a^* - \iota_b^*} \check{H}^1(M_{31} \cup M_{32}) \longrightarrow \cdots \end{array} \tag{3.58}$$

where we have used that $(M_1 \cup_{f_{12}} M_2) \cap M_3 \cong M_{31} \cup M_{32}$, and ι_a and ι_b are the inclusions of $M_{31} \cup M_{32}$ into $M_1 \cup_{f_{12}} M_2$ and M_3, respectively. Observe that again we have passed into the cover-independent Čech cohomology by using the direct limiting process found in the Appendix. By assumption all of the $\check{H}^0$ terms in this sequence equal G, so we may apply the same reasoning as that of Lemma 3.4.3 to conclude that the connecting homomorphism δ^* is the zero map. Thus $\check{H}^1(\mathbb{M}) = \ker(\iota_a^* - \iota_b^*)$, which can be equivalently stated as:

$$\check{H}^1(\mathbb{M}) = \left\{ ([\check{a}], [\check{b}]) \in \check{H}^1(M_1 \cup M_2) \oplus \check{H}^1(M_3) \;\middle|\; \iota_a^*[\check{a}] = \iota_b^*[\check{b}] \right\} \tag{3.59}$$

$$= \check{H}^1(M_1 \cup M_2) \times_{\check{H}^1(M_{31} \cup M_{32})} \check{H}^1(M_3). \tag{3.60}$$

By assumption the triple intersection M_{123} is also connected, so we apply the same reasoning to the union $M_{31} \cup M_{32}$ to conclude that

$$\check{H}^1(M_{31} \cup M_{32}) = \check{H}^1(M_{13}) \times_{H^1(M_{123})} \check{H}^1(M_{23}). \tag{3.61}$$

Putting this all together, we have that

$$\check{H}^1(\mathbf{M}) = \check{H}^1(M_1 \cup M_2) \times_{\check{H}^1(M_{31} \cup M_{32})} \check{H}^1(M_3) \tag{3.62}$$

$$= \check{H}^1(M_1 \cup M_2) \times_{\check{H}^1(M_{13}) \times_{H^1(M_{123})} \check{H}^1(M_{23})} \check{H}^1(M_3) \tag{3.63}$$

$$\cong \prod_{\mathscr{F}} \check{H}^1(M_i) \tag{3.64}$$

where the final isomorphism follows from Theorem A.1.2. □

3.4.3 Classifying Line Bundles

We will now discuss the prospect of classifying real line bundles over a fixed non-Hausdorff manifold $\mathbf{M}$. Continuing on from the previous section, we will assume that $\mathbf{M}$ may be expressed as the colimit of n-many Hausdorff manifolds M_i according to the requirements of Remark 3.2.9. We can describe any rank-k vector bundle $\mathbf{E}$ over $\mathbf{M}$ as a choice of transition functions $b_{\alpha\beta} : U_{\alpha\beta} \to GL(k)$ satisfying the conditions $b_{\alpha\alpha}([x,i]) = \text{id}$ and $b_{\alpha\beta} \circ b_{\beta\gamma} \circ b_{\gamma\alpha}([x,i]) = \text{id}$ for all $[x,i]$ in $\mathbf{M}$. When this is the case, we say that the open cover $\mathscr{U} = \{U_\alpha\}$ *trivialises* the bundle $\mathbf{E}$. As in the Hausdorff case, any two sets $\{b_{\alpha\beta}\}$ and $\{c_{\alpha\beta}\}$ of trivialisations will describe the same bundle $\mathbf{E}$ whenever there exists a collection of maps $\{k_\alpha : U_\alpha \to GL(k)\}$ satisfying $c_{\alpha\beta} = k_\beta^{-1} b_{\alpha\beta} k_\alpha$.

In the case that $\mathbf{E}$ is a line bundle, we may use Theorem 3.3.8 to conclude that $\mathbf{E}$ admits a bundle metric. This allows us to reduce the structure group of $\mathbf{E}$ from $GL(1)$ down to $O(1)$. As such, any transition function will now take image in the Abelian group $\mathbb{Z}_2$. We may then equivalently view the construction of $\mathbf{E}$ from local trivialisations as a cocycle in $\check{C}^1(\mathbf{M}, \mathscr{U}, \mathbb{Z}_2)$ whose cohomology class determines $\mathbf{E}$ up to isomorphism. Explicitly, for any open cover $\mathscr{U}$ of $\mathbf{M}$ there is a bijection between the set $\text{Line}_{\mathbb{R}}(\mathbf{M}, \mathscr{U})$ of real line bundles trivialised by $\mathscr{U}$ and the $\mathscr{U}$-dependent Čech cohomology $\check{H}^1(\mathbf{M}, \mathscr{U}, \mathbb{Z}_2)$. We will now use the direct limit construction of $\check{H}^1(\mathbf{M}, \mathbb{Z}_2)$ to prove the following.

Theorem 3.4.7. *There is a bijection between the set* $\text{Line}_{\mathbb{R}}(\mathbf{M})$ *of inequivalent real line bundles over* $\mathbf{M}$ *and the first Čech cohomology group* $\check{H}^1(\mathbf{M}, \mathbb{Z}_2)$.

Proof. For each open cover $\mathscr{U}$ of $\mathbf{M}$, denote by $\alpha_{\mathscr{U}}$ the bijection between $\text{Line}_{\mathbb{R}}(\mathbf{M}, \mathscr{U})$ and $\check{H}^1(\mathbf{M}, \mathscr{U})$. By construction, each group $\check{H}^1(\mathbf{M}, \mathscr{U}, \mathbb{Z}_2)$ maps into $\check{H}^1(\mathbf{M}, \mathbb{Z}_2)$ via some map $\psi_{\mathscr{U}}$, and this map commutes with any refinement maps. We define a map $\varepsilon : \text{Line}_{\mathbb{R}}(\mathbf{M}, \mathbb{Z}_2) \to \check{H}^1(\mathbf{M}, \mathbb{Z}_2)$ by sending each $\mathbf{L}$ to $\psi_{\mathscr{U}} \circ \varepsilon_{\mathscr{U}}(\mathbf{L})$, where $\mathscr{U}$ is some open cover that trivialises $\mathbf{L}$. To see that the map ε is well-defined, suppose that $\mathbf{L}$ is some line bundle that is simultaneously trivialised by two open covers $\mathscr{U}$ and $\mathscr{V}$. We may then select some open cover $\mathscr{W}$ that refines both $\mathscr{U}$ and $\mathscr{V}$ simultaneously. Consider two maps $\lambda_{\mathscr{U}}$ and $\lambda_{\mathscr{V}}$ that encode the refinements into $\mathscr{W}$. By the direct limit construction of $\check{H}^1(\mathbf{M}, \mathbb{Z}_2)$, we then have that

$$\psi_{\mathscr{U}} \circ \varepsilon_{\mathscr{U}}(\mathbf{L}) = \psi_{\mathscr{W}} \circ \lambda_{\mathscr{U}} \circ \varepsilon_{\mathscr{U}}(\mathbf{L}) = \psi_{\mathscr{W}} \circ \varepsilon_{\mathscr{W}}(\mathbf{L}) = \psi_{\mathscr{W}} \circ \lambda_{\mathscr{V}} \circ \varepsilon_{\mathscr{V}}(\mathbf{L}) = \psi_{\mathscr{V}} \circ \varepsilon_{\mathscr{V}}(\mathbf{L}), \tag{3.65}$$

as required. Finally, we observe that ε is bijective since every $\varepsilon_{\mathscr{U}}$ is. □

Given the results of Sections 3.1 and 3.2, it would be useful if there were formulas to express the order of $\check{H}^1(\mathrm{M},\mathbb{Z}_2)$ in terms of the orders of the groups $\check{H}^1(M_i,\mathbb{Z}_2)$. Unfortunately, a general formula may not exist. However, we may use Theorem 3.4.6 together with a basic property of fibred products to yield the following.

Theorem 3.4.8. *Let* M *be a non-Hausdorff manifold formed from Hausdorff manifolds* M_i*, according to 3.2.4. Suppose furthermore that:*

1. *the subspaces* $M_{i_1\cdots i_p}$ *are all connected, for all* $p \leq n$*, and*
2. *each of the descended difference maps*

$$\tilde{\delta} : \bigoplus_{i_1<\cdots<i_p} \check{H}^1(M_{i_1\cdots i_p},\mathbb{Z}_2) \to \bigoplus_{i_1<\cdots<i_{p+1}} \check{H}^1(M_{i_1\cdots i_{p+1}},\mathbb{Z}_2) \tag{3.66}$$

is surjective.

Then the number of inequivalent line bundles on M *can be expressed with the formula:*

$$|\check{H}^1(\mathrm{M},\mathbb{Z}_2)| = \sum_{k\leq n} \sum_{i_1,\cdots i_p \in I} (-1)^{(p+1)} |\check{H}^1(M_{i_1\cdots i_p},\mathbb{Z}_2)|. \tag{3.67}$$

3.5 Conclusion

In this chapter we have explored the prospect of fibering vector spaces over a base space which has the structure of a non-Hausdorff manifold. Using the topological criteria outlined in both Section 3.2.1 and [84], we saw that bundle structures can be naturally defined, reconstructed, and in some cases classified. Our initial observation was that of Theorem 3.2.7, which confirmed that smooth non-Hausdorff manifolds can be constructed by gluing together ordinary smooth manifolds along diffeomorphic open submanifolds. This, once combined with the contravariance of the C^∞ functor, then allowed us to express the algebra of smooth real-valued functions of a non-Hausdorff manifolds as a limit in the abelian category of unital associative algebras.

Throughout Section 3.3, we saw that vector bundles fibred over our non-Hausdorff manifold M can be constructed by taking a particular colimit of Hausdorff bundles. This was proved as Theorem 3.3.1, and it was later shown in Theorem 3.3.3 that every vector bundle over M can be described in this manner. Using this observation, we then showed that all sections of all bundles over M carry a description in terms of sections on each Hausdorff submanifold. In Theorem 3.3.8 this piecewise construction of sections was used to conclude that bundle metrics will exist for any non-Hausdorff bundle $\mathbf{E}$ fibred over M. In particular, Corollary 3.3.9 confirmed that Riemannian metrics always exist on any non-Hausdorff manifold M satisfying our particular topological criteria.

In Section 3.4 we studied the Čech cohomology of non-Hausdorff manifolds. In Sections 3.1 and 3.2 we related the Čech cohomology groups $\check{H}^q(\mathrm{M})$ to the groups $\check{H}^q(M_{i_1\cdots i_p})$ in the form of (generalised) Mayer-Vietoris sequences. These sequences will typically be non-trivial, and therefore the fibre-product structure seen in 3.2.14, 3.3.6 and 3.4.5 will not present itself in Čech cohomology. Despite that, we saw in the form of Theorem 3.4.6 that the first Čech cohomology group $\check{H}^1(\mathbb{M})$ *will* be a fibred product, provided that the Hausdorff submanifolds M_i and all their intersections are connected. Finally, in Theorem 3.4.8 we identified some conditions under which the inequivalent real lines bundles over $\mathbb{M}$ can be expressed as an alternating sum formula.

We will finish this chapter with some speculations regarding future developments. Of particular interest is the relationship between the Čech cohomology groups established here, and other theories such as de Rham cohomology. As an application of Theorem 3.3.6, the differential forms on $\mathbb{M}$ satisfy a fibred product structure, and therefore it will be interesting to compute the de Rham cohomology groups (à la [23]) and relate them to the groups $\check{H}^q(\mathbb{M},\mathbb{R})$. These relationships might then be used in conjunction with a sound theory of affine connections to establish a Chern-Weil Theory for non-Hausdorff manifolds.

Summary of Chapter

In the previous chapter, we saw that non-Hausdorff manifolds can be constructed by gluing together ordinary Hausdorff manifolds along particular open subsets. In this chapter we took this idea one step further, by considering the case of vector bundles fibred over these non-Hausdorff manifolds. Since a non-Hausdorff manifold can always be decomposed into a collection of (maximal) Hausdorff submanifolds, and throughout this chapter we saw the same story for their vector bundles. In particular, we saw that ordinary Hausdorff vector bundles can be glued together to form a non-Hausdorff vector bundle, and moreover, any non-Hausdorff vector bundle can be decomposed into Hausdorff vector bundles.

In the case of a finite adjunction, we also saw that the ring of smooth, real-valued functions of a smooth non-Hausdorff manifold can be realised as a categorical limit of functions on Hausdorff submanifolds, known as a *fibre product*. Heuristically, this description allows one to construct smooth functions on a non-Hausdorff manifold by “gluing" together functions defined on it’s Hausdorff subspaces. We saw that this description also held for sections of bundles – a corollary being that all tensor fields on a non-Hausdorff manifold can be described using Hausdorff tensor fields that are compatible in some precise sense. Looking forward towards certain physical applications, the results in this chapter suggest that non-Hausdorff manifolds are tractable from a naive geometric perspective, and display some of the important mathematical features of physical spaces.

Chapter 4

De Rham Cohomology for non-Hausdorff Manifolds

This chapter exists as a preprint under the name “A non-Hausdorff de Rham cohomology" arXiv:2310.17151, 2023 [83], and I am the sole author.

In this chapter we investigate the nature of differential forms on a non-Hausdorff manifold. Our aim is to build towards chapter 4 by describing all of the basic geometry that is needed for describing and evaluating gravitational actions in two dimensions. In particular, we will prove the following results:

- Using Theorem 5.2.3 of the previous chapter, we describe the differential forms on a non-Hausdorff manifold and determine their integral properties.
- We show that a well-defined exterior derivative operator exists in the non-Hausdorff case, and we use this to describe the de Rham cohomology of a non-Hausdorff manifold via certain Mayer-Vietoris-style sequences.
- We prove the Gauss-Bonnet theorem for Riemannian non-Hausdorff surfaces. This version of the theorem contains extra counterterms that compute geodesic curvature of the Hausdorff-violating submanifolds present.

Alongside these results, we also prove the non-Hausdorff version of de Rham's theorem, which gives an equivalence between de Rham cohomology and singular homology.

4.1 Introduction

The Hausdorff property says that any pair of distinct points can be separated by disjoint open sets. In standard formulations of differential geometry, one assumes that all manifolds are Hausdorff spaces. This is normally justified on technical grounds: it can be shown that any locally-Euclidean, second-countable topological space admits partitions of unity subordinate to an arbitrary open cover, provided that the space is Hausdorff. These partitions of unity can then be used to build various global structures by patching together locally-defined ones. Conversely, any non-Hausdorff, locally-Euclidean second-countable space cannot admit a partition of unity subordinate to any cover by Hausdorff open sets, even when paracompactness is assumed.

Observations such as the above often render non-Hausdorff manifolds as too bothersome to include within standard treatments of differential geometry. Nonetheless, non-Hausdorff manifolds still occur within certain domains of mathematics, despite their technical inconvenience. Indeed, non-Hausdorff manifolds can be found within foliation theory and/or non-commutative geometry [17, 23, 25, 37, 39, 45, 60], and also within certain areas of mathematical physics [33, 46, 47, 53, 68, 77, 89, 105, 108] .

In the seminal paper [45], Haefliger and Reeb observed that non-Hausdorff 1-manifolds may naturally arise as the leaf spaces of certain foliations. Of special importance is their construction of non-Hausdorff manifolds via adjunction spaces. Loosely speaking, the authors showed that certain non-Hausdorff 1-manifolds may be constructed by gluing together copies of the real line or along open subsets, whilst leaving the boundaries of these subsets unidentified, as pictured in Figure 4.1. In a somewhat different context, Hajicek showed that any non-Hausdorff manifold is naturally covered by maximal Hausdorff open submanifolds [47]. These observations motivated the work of [84], which developed a general treatment of non-Hausdorff topological manifolds in terms of generalised adjunction spaces. In [85] this notion of adjunction space was extended to include smooth non-Hausdorff manifolds and the vector bundles fibred over them.

In this chapter we will extend the formalism of [84] and [85] once more by studying the differential forms that a non-Hausdorff manifold may admit. Motivated by Hajicek's original result, we will derive descriptions of non-Hausdorff de Rham cohomology via certain Mayer-Vietoris sequences. We will mostly follow the standard derivation of these sequences (found in [13], say), with some key modifications ultimately due to certain technical difficulties surrounding the non-existence of arbitrary partitions of unity. After this, we use similar Mayer-Vietoris sequences for smooth singular cohomology in order to prove a non-Hausdorff version of de Rham's theorem. Finally, we will finish the chapter with a discussion of the Gauss-Bonnet theorem for non-Hausdorff manifolds.

This chapter is organised as follows. In Section 4.2 we will recall the basic formalism of non-Hausdorff manifolds, taken from [84] and [85]. Included are the basic notions of colimits of smooth manifolds, descriptions of the algebras of smooth functions, and a discussion of non-Hausdorff vector bundles. Of central importance is the concept that sections of any bundle will satisfy a certain *fibre product formula*. In this context, we mean that any section

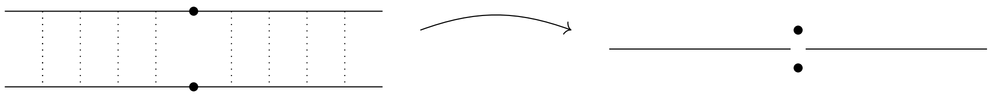


Figure 4.1: The line with two origins, the prototypical example of a non-Hausdorff manifold. Here it is constructed by gluing together two copies of $\mathbb{R}$ along the open subset $(\infty,0)\cup(0,\infty)$.

over a non-Hausdorff manifold can be uniquely described by a collection of sections defined over Hausdorff submanifolds, together with some consistency conditions on their mutual overlaps. We will spend due time on this observation, since it is a fundamental concept that underpins the remainder of the chapter.

In Section 4.3 we will describe the differential forms on a non-Hausdorff manifold. Before doing so, we will first take some time to explain a description of vector fields in function-theoretic terms. This interpretation of vector fields needs particular care, since the type of bump functions used in locality arguments will generally not exist in our setting. Once this is done, we will describe the space of non-Hausdorff differential forms, as well as their exterior derivative. We then finish the section with a discussion of integration over non-Hausdorff manifolds. As a small aside, we will see that Stoke's theorem fails in a particularly controlled manner for our non-Hausdorff manifolds.

In Section 4.4 we will discuss the de Rham cohomology of non-Hausdorff manifolds. After making some important assumptions on the intersection properties of Hausdorff submanifolds, we will derive the aforementioned Mayer-Vietoris sequences for de Rham cohomology. In the case that a non-Hausdorff manifold is built from two Hausdorff spaces, this sequence is similar to the Hausdorff setting (cf. [13]). However, in the more-general case, we will derive a Čech-de Rham bicomplex that relates the de Rham cohomology of a non-Hausdorff manifold to a particular cover-dependent Čech cohomology (cf. [85, §3]).

In Section 4.5 we will prove de Rham's theorem for non-Hausdorff manifolds. After some careful derivations of (smooth) singular homology, we may use our Mayer-Vietoris sequences together with and "integration over chains" pairing to describe an isomorphism between de Rham cohomology and singular homology. Our approach will generally follow that of [65], with some important modifications due to the failure of Whitney's Embedding theorem in the non-Hausdorff case.

Finally, in Section 4.6 we will prove a non-Hausdorff version of the Gauss-Bonnet theorem for closed 2-manifolds. As we will see, the integration formulas of Section 4.3, together with the Mayer-Vietoris sequences of Sections 3.4 and 3.5 will allow us to prove a Gauss-Bonnet theorem by decomposing the total scalar curvature into an integral over Hausdorff submanifolds. Applying the usual Gauss-Bonnet to each of these Hausdorff pieces will give the desired relationship. Although we can derive the result by reducing everything to the Hausdorff case, there is an important difference: for a non-Hausdorff 2-manifold there will be additional contributions to curvature coming from the Hausdorff-violating submanifolds that sit inside it.

Throughout this chapter we will assume that all manifolds, Hausdorff or otherwise, are locally-Euclidean second countable topological spaces. As a convention, we will use boldface to distinguish non-Hausdorff manifolds from their Hausdorff cousins.

4.2 Smooth non-Hausdorff Manifolds

We will now recall the basic topological properties of non-Hausdorff manifolds. The results here are taken from [84] and [85], so we will state them without proof.

4.2.1 The Topology of non-Hausdorff Manifolds

An adjunction of two topological spaces X and Y is formed from some subset A of X and a continuous map $f : A \to Y$. The adjunction space $X \cup_f Y$ is then defined to be the quotient of the disjoint union $X \sqcup Y$ by an equivalence relation that identifies every element in A with its image under f. In categorical terms, the adjunction space $X \cup_f Y$ can be shown to be the pushout of the diagram

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & X \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle \phi_X} \\ Y & \xrightarrow[\phi_Y]{} & X \cup_f Y \end{array} \tag{4.1}$$

in the category TOP [14, 50], where here the maps ϕ_X and ϕ_Y send each point in X or Y into its corresponding equivalence class in the quotient space.

Motivated by this construction, we will now use this idea to form non-Hausdorff manifolds. To begin with, we would like to consider more general diagrams than the above. Following [84], we will consider diagrams consisting of manifolds, open submanifolds, and maps between them. Formally, we will consider a tuple $\mathscr{F} = (\mathsf{M}, \mathsf{A}, \mathsf{f})$, where:

- $\mathsf{M} = \{M_i\}_{i\in I}$ is a finite set of Hausdorff topological manifolds,
- $\mathsf{A} = \{M_{ij}\}_{i,j\in I}$ is a set of bi-indexed open submanifolds, satisfying $M_{ij} \subseteq M_i$ for all i, j in I, and
- $\mathsf{f} = \{f_{ij}\}_{i,j\in I}$ is a set of continuous maps $f_{ij} : M_{ij} \to M_j$.

There is no guarantee that the data of $\mathscr{F}$ will yield a well-defined quotient space, so we will need to restrict our attention to the following triples.

Definition 4.2.1. *The triple $\mathscr{F} = (M, A, f)$ is called an adjunction system if it satisfies the following conditions for all $i, j \in I$.*

A1) $M_{ii} = M_i$ *and* $f_{ii} = id_{M_i}$

A2) $M_{ji} = f_{ij}(M_{ij})$, *and* $f_{ij}^{-1} = f_{ji}$

A3) $f_{ik}(x) = f_{jk} \circ f_{ij}(x)$ *for each* $x \in M_{ij} \cap M_{ik}$

The conditions listed above ensure that the resulting quotient space is well-defined. Given an adjunction space $\mathscr{F}$, we may define the adjunction space subordinate to it as:

$$\mathbf{M} := \bigcup_{\mathscr{F}} M_i = \left(\bigsqcup_{i \in I} M_i \right) \Big/ \sim \tag{4.2}$$

via the relation $\sim$ that identifies points (x,i) and (y,j) in the disjoint union whenever $f_{ij}(x) = y$. We will denote the points in $\mathbf{M}$ by tuples $[x,i]$, where $x \in M_i$. Since $\mathbf{M}$ is a quotient space, these points are actually equivalence classes of the form:

$$[x,i] := \left\{ (y,j) \in \bigsqcup_{i \in I} M_i \mid y = f_{ij}(x) \right\}. \tag{4.3}$$

We will denote by ϕ_i the continuous maps that send each x in M_i to its equivalence class $[x,i]$ in $\mathbf{M}$. By construction, the open sets of $\mathbf{M}$ can be usefully described with the following condition.

Lemma 4.2.2. *A subset U of $\mathbf{M}$ is open if and only if the set $\phi_i^{-1}(U)$ is open in M_i, for all i in I.*

Lemma 1.3 of [84] ensures that the above adjunction space together with the maps ϕ_i satisfy a certain universal property. This allows us to view $\mathbf{M}$ as the colimit of the diagram formed from the data in $\mathscr{F}$.

In principle, quotienting some collection of manifolds may also spoil local charts. However, we may preserve the local nature of the manifolds M_i by imposing some conditions on the gluing maps f_{ij} and the submanifolds M_{ij}. The following result captures the finer details.

Lemma 4.2.3. *Let $\mathscr{F}$ be an adjunction system in which the maps f_{ij} are all open topological embeddings. Then*

1. *the maps $\phi_i : M_i \to \mathbf{M}$ are all open topological embeddings, and*

2. *the adjunction space subordinate to $\mathscr{F}$ is locally-Euclidean and second-countable.*

The above result ensures that our adjunction spaces $\mathbf{M}$ are locally-Euclidean, second-countable spaces, thereby justifying our notation. Since the canonical maps ϕ_i are now embeddings, we may also view the Hausdorff manifolds M_i as genuine subspaces of $\mathbf{M}$. To simplify notation, we will regularly identify M_i with it's image $\phi_i(M_i)$, viewed as an open submanifold of $\mathbf{M}$.

Lemma 4.2.3 provides some conditions under which local charts of each M_i may be preserved in the adjunction process. However, the Hausdorff property may or may not be preserved under these conditions.[1] The key observation here revolves around the boundaries of the submanifolds M_i – if we identify M_{ij} and M_{ji} in the adjunction process, but leave their relative boundaries unidentified, then these boundaries (provided they exist) will violate the Hausdorff property in $\mathbf{M}$. The following result summarises some useful facts surrounding this observation.

[1]Consider, for example, a binary adjunction in which $M_1 = M_2 = M_{12} = \mathbb{R}$, with $f_{12} = id$. The resulting adjunction space is $\mathbb{R}$. Alternatively, we may take $A = (-\infty, 0)$, and the adjunction space will be non-Hausdorff.

Theorem 4.2.4. *Let $\mathscr{F}$ be an adjunctive system that satisfies the criteria of Lemma 4.2.3, and let **M** denote the adjunction space subordinate to $\mathscr{F}$. Suppose furthermore that each gluing map $f_{ij} : M_{ij} \to M_{ji}$ can be extended to a homeomorphism $\overline{f_{ij}} : Cl^{M_i}(M_{ij}) \to Cl^{M_j}(M_{ji})$ such that the $\overline{f_{ij}}$ satisfy the conditions of Definition 4.2.1. Then:*

1. *The Hausdorff-violating points in **M** occur precisely at the **M**-relative boundaries of the subspaces M_{ij}.*
2. *$[x,i]$ and $[y,j]$ violate the Hausdorff property in **M** if and only if $x \in \partial^{M_i}(M_{ij})$ and $y \in \partial^{M_j}(M_{ji})$ and $\overline{f_{ij}}(x) = y$.*
3. *Each M_i is a maximal Hausdorff open submanifold of **M**.*
4. ***M** is paracompact.*
5. *If each M_i is compact, then **M** is compact.*

As mentioned in the introduction, non-Hausdorff manifolds will not admit partitions of unity subordinate to an arbitrary open cover. The following is taken from [84, Thm 2.6].

Lemma 4.2.5. *Let $\mathscr{U}$ be an open set of a non-Hausdorff manifold **M** such that each element of $\mathscr{U}$ is Hausdorff. Then there is not a partition of unity subordinate to $\mathscr{U}$.*

The key observation here is that any continuous function $\mathbf{r} : \mathbf{M} \to \mathbb{R}$ will necessarily map Hausdorff-violating points to the same value in $\mathbb{R}$. As such, any proposed partition of unity will either not sum to one everywhere, or will not be supported in the Hausdorff open sets of the proposed cover.[2]

Inductive Construction of M

Generally speaking, many of the results that can be proved for binary adjunction spaces will also hold for a colimit of finitely-many manifolds. This is due to a certain inductive construction of colimits: if **M** a non-Hausdorff manifold built from $(n+1)$-many manifolds M_i, then we can equivalently view **M** as an adjunction of the two spaces **N** and M_{n+1}, where:

- **N** is the adjunction of the first n-many manifolds M_i,
- the subset $A = \bigcup_{i \leq n} M_{i(n+1)}$, viewed as an open submanifold of M_{n+1}, and
- the gluing map $F : A \to \mathbf{N}$ sends each a to the equivalence class in **N** containing the point $f_{(n+1)i}(a)$, for all $i \leq n$.

The universal properties of colimits, applied to both **M** and **N**, ensure the equivalence between these two representations of **M**. The details can be found in Section 1.1 of [85].

[2] Note that there are at least *some* open covers that admit partitions of unity – the singleton $\{\mathbf{M}\}$ being a somewhat trivial example.

4.2.2 Smooth Manifolds

We will now extend our discussion to the smooth setting. Suppose that we are given a non-Hausdorff manifold **M** built according to Definition 4.2.1 and Lemma 4.2.3. Given atlases $\mathscr{A}_i$ of the Hausdorff manifolds M_i, we may define a topological atlas $\mathscr{A}$ on **M** as:

$$\mathscr{A} := \bigcup_{i \in I} \{(\phi_i(U_\alpha), \varphi_\alpha \circ \phi_i^{-1}) \mid (U_\alpha, \varphi_\alpha) \in \mathscr{A}_i\}. \tag{4.4}$$

The antecedent conditions of Lemma 4.2.3 ensure that the transition maps of $\mathscr{A}$ will be continuous for charts coming from different M_i. As one might expect, this observation similarly holds for smooth manifolds. The following result is taken from [85, Lem 1.7].

Lemma 4.2.6. *Let* $\mathbf{M}$ *be a non-Hausdorff manifold built according to Lemma 4.2.3. If, additionally, the* M_i *are smooth manifolds and the* f_{ij} *are all smooth maps, then* $\mathbf{M}$ *admits a smooth atlas.*

The atlas $\mathscr{A}$ described above will suffice as a smooth atlas of **M**. Under this reading, we may view the canonical maps $\phi_i : M_i \to \mathbf{M}$ as smooth open embeddings. Moreover, arguments similar to that of [84, Lem. 1.3] ensure that **M** satisfies a universal property, effectively turning it into the colimit of the diagram $\mathscr{F}$ in the category of smooth locally-Euclidean, second-countable spaces.

Smoothness of maps between non-Hausdorff manifolds can be defined as in the Hausdorff case, that is, by appealing to local coordinate representations about each point in the manifold. Since the submanifolds M_i form an open cover of **M**, we see that any map $\psi : \mathbf{M} \to \mathbf{N}$ is smooth if and only if all of the maps $\psi \circ \phi_i : M_i \to \mathbf{N}$ are.

Remark 4.2.7. *In addition to the conditions of Lemma 4.2.3 and Theorem 4.2.4, hereafter we will also assume that our non-Hausdorff manifolds are smooth in the sense of Lemma 4.2.6. Moreover, we will also need to assume that the gluing regions* M_{ij} *have diffeomorphic boundaries, which in this context means that the closures* $Cl^{M_i}(M_{ij})$ *are all smooth closed submanifolds with boundary, and the extended maps* $\overline{f_{ij}}$ *of Theorem 4.2.4 are all diffeomorphisms from* $Cl^{M_i}(M_{ij})$ *to* $Cl^{M_j}(M_{ji})$*. In order to remove any potential ambiguity between the topological and manifold notion of boundary, we will also assume that the submanifolds* M_{ij} *are all regular domains in the sense of [65, Prop 5.46]*

Smooth functions on a non-Hausdorff manifold **M** can be defined in the same way as the Hausdorff case. In particular, the space $C^\infty(\mathbf{M})$ of smooth functions is still a unital, associative algebra over $\mathbb{R}$. In our context, the canonical maps $\phi_i : M_i \to \mathbf{M}$ are all smooth, so we might expect there to be some relationship between $C^\infty(\mathbf{M})$ the algebras $C^\infty(M_i)$. As a matter of fact there is such a relationship, however in order to motivate the exact description we first need to make the following observation regarding the construction of functions on **M**, taken from [85, §1.3].

Lemma 4.2.8. *Let* $\mathbf{M}$ *be a smooth non-Hausdorff manifold satisfying the criteria of Remark 4.2.7. Then:*

1. *If* $r_i : M_i \to \mathbb{R}$ *is a collection of smooth functions such that* $r_i = r_j \circ f_{ij}$ *for all* i, j *in* I*, then the map* $\boldsymbol{r} : \mathbf{M} \to \mathbb{R}$ *defined by* $\boldsymbol{r}([x,i]) = r_i(x)$ *is a smooth function on* $\mathbf{M}$*.*

2. *Any smooth function r_i on M_i can be extended to a smooth function $\mathbf{r}$ on $\mathbf{M}$.*

Item (1) above is a direct reformulation of the so-called “Gluing Lemma" for smooth maps, (cf [65, Cor. 2.8]). The second item above revolves around the so-called “extension by zero" construction for Hausdorff manifolds (cf. [65, Lem. 2.26]). The idea behind the proof is to transfer the smooth function r_i to each closed subset $\overline{M_{ji}}$ using the extended diffeomorphisms $\overline{f_{ij}}$, and then to inductively extend by zero into each M_j. This creates a collection of smooth functions defined on each M_j that agree on their pairwise intersections, and the result then follows from Item (1) above. The results of Lemma 4.2.8 may be used to prove the following [85, §1.3].

Theorem 4.2.9. *The algebra $C^\infty(\boldsymbol{M})$ is isomorphic to the fibred product*

$$\prod_{\mathscr{F}} C^\infty(M_i) := \left\{(r_1,...,r_n) \in \bigoplus_{i\in I} C^\infty(M_i) \middle|\, r_i = r_j \circ f_{ij} \text{ on } M_{ij} \text{ for all } i,j \in I\right\}. \tag{4.5}$$

This fibred-product structure has many useful consequences. Abstractly, it allows us to see $C^\infty(\mathbf{M})$ as the limit of the diagram formed by applying the contravariant functor $C^\infty(\cdot)$ to the data of $\mathscr{F}$. Concretely, Theorem 4.2.9 allows us to describe a smooth function $\mathbf{r}$ on $\mathbf{M}$ by any of the following equivalent descriptions:

$$\mathbf{r} = (\phi_1^*\mathbf{r}, \cdots, \phi_n^*\mathbf{r}) = (\mathbf{r}\circ\phi_1, \cdots, \mathbf{r}\circ\phi_n) = (\mathbf{r}|_{M_1}, \cdots, \mathbf{r}|_{M_n}) = (r_1, \cdots, r_n). \tag{4.6}$$

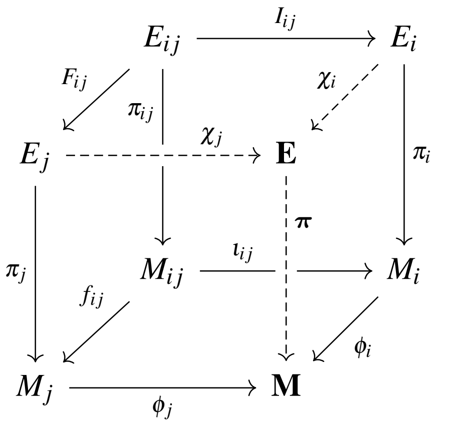
The above fibre product can be equivalently represented by a sequence. In the case of a binary adjunction space, we will have the following short sequence:

$$0 \to C^\infty(\mathbf{M}) \xrightarrow{\Phi^*} C^\infty(M_1) \oplus C^\infty(M_2) \xrightarrow{\iota_{12}^* - f_{12}^*} C^\infty(M_{12}) \to 0 \tag{4.7}$$

where here Φ^* is the map that sends any function $\mathbf{r}$ to the pair $(\phi_1^*\mathbf{r}, \phi_2^*\mathbf{r})$, and $\iota_{12} : M_{12} \to M_1$ is the inclusion map. Theorem 4.2.9 then states that the above sequence is exact at the first step, that is, $C^\infty(\mathbf{M}) \cong \ker(\iota_{12}^* - f_{12}^*)$. However, due to Lemma 4.2.5 there is no partition of unity subordinate to the open cover $\{M_1, M_2\}$, so in general we may not argue for the full exactness of this sequence by standard means, found in Prop. 2.3. of [13], say. This is an important issue, the resolution of which will be the subject of Section 4.4.

4.2.3 Vector Bundles

Vector bundles fibred over non-Hausdorff manifolds $\mathbf{M}$ can be constructed using colimits of Hausdorff bundles $E_i \xrightarrow{\pi_i} M_i$. The construction is very similar to that of Section 4.2.1, except that we additionally require some bundle maps F_{ij} covering the f_{ij} that specify the gluing of each fibre. Schematically, the colimit $\mathbf{E}$ of the bundles E_i should make the diagram:



(4.8)

commute for all i, j in I. This can all be confirmed formally, the details of which can be found in [85, §2]. For the purposes of this chapter, the relevant details are given below.

Theorem 4.2.10. *Let* $\mathscr{G} := (E, B, F)$ *be a triple of sets in which:*

1. $E = \{(E_i, \pi_i, M_i)\}_{i \in I}$ *is a collection of rank-k vector bundles,*
2. $B = \{E_{ij}\}_{i,j \in I}$ *consists of the restrictions of the bundles* E_i *to the intersections* M_{ij}*, and*
3. $F = \{F_{ij}\}_{i,j \in I}$ *is a collection of bundle isomorphisms* $F_{ij} : E_{ij} \to E_{ji}$ *that cover the gluing maps* f_{ij} *and satisfy the condition* $F_{ik} = F_{jk} \circ F_{ij}$ *on the intersections* M_{ijk}*, for all* i, j, k *in* I*.*

Then the resulting adjunction space $\mathbf{E}$ *has the structure of a non-Hausdorff rank-k vector bundle over* $\mathbf{M}$ *in which the canonical inclusions* $\chi_i : E_i \to \mathbf{E}$ *are bundle morphisms covering the canonical embeddings* $\phi_i : M_i \to \mathbf{M}$*.*

The bundle $\mathbf{E}$ described above can be shown to satisfy a certain universal property, effectively turning it into a colimit of the bundles E_i. Moreover, it can be shown that every bundle $\mathbf{E} \xrightarrow{\pi} \mathbf{M}$ is a colimit of Hausdorff bundles (cf [85, §2.2]). As an application of this idea, we may readily obtain descriptions of the tensor bundles over the non-Hausdorff manifold $\mathbf{M}$. The tangent bundle $T\mathbf{M}$ can be constructed as a colimit of the bundles TM_i, where here the gluing maps for the fibres are given by the differentials of the gluing maps for $\mathbf{M}$, i.e. $F_{ij} = df_{ij}$. According to Remark 4.2.7 each f_{ij} is a diffeomorphism, so their differentials will induce bundle isomorphisms on the TM_{ij}. The universal property of the colimit bundle can be applied (together with the maps $d\phi_i : TM_i \to T\mathbf{M}$) to conclude that $T\mathbf{M}$ is canonically isomorphic to a colimit of the bundles TM_i. According to Theorem 4.2.4, a pair of tangent vectors $[v, i]$ and $[w, j]$ in $T\mathbf{M}$ will violate the Hausdorff property if and only if they satisfy

$$v \in T_pM_i \text{ and } w \in T_{\overline{f_{ij}(p)}}M_j \text{ and } d\overline{f_{ij}}(v) = w \tag{4.9}$$

for some point p lying on the M_i-relative boundary of M_{ij}. Similarly, it can be shown that any higher tensor bundle $T^{(p,q)}\mathbf{M}$ is canonically isomorphic to a colimit of the same bundles $T^{(p,q)}M_i$ defined on each M_i.

Theorem 4.2.11. *Let $\mathbf{M}$ be a non-Hausdorff manifold satisfying the criteria of Remark 4.2.7. Then the rank (p,q) tensor bundle $T^{(p,q)}\mathbf{M}$ is canonically isomorphic to an adjunction of the bundles $T^{(p,q)}M_i$, glued along the maps*

$$F_{ij} := \underbrace{df_{ij} \otimes \cdots \otimes df_{ij}}_{p \text{ times}} \otimes \underbrace{df_{ji} \otimes \cdots \otimes df_{ji}}_{q \text{ times}}. \tag{4.10}$$

We finish this section with a pair of very useful observations regarding the sections of non-Hausdorff bundles. To begin with, we observe that the sections of a non-Hausdorff colimit bundle $\mathbf{E}$ can be described in a similar manner to that of smooth functions.

Theorem 4.2.12. *Let $\mathbf{E}$ be a vector bundle over $\mathbf{M}$. Then the space of sections $\Gamma(\mathbf{E})$ is isomorphic to the fibred product*

$$\prod_{\mathscr{F}} \Gamma(E_i) := \left\{ (s_1,...,s_n) \in \bigoplus_{i\in I} \Gamma(E_i) \,\middle|\, s_i = s_j \circ f_{ij} \text{ on } M_{ij} \text{ for all } i,j \in I \right\}. \tag{4.11}$$

In short, this means that any section on a bundle $\mathbf{E} \xrightarrow{\pi} \mathbf{M}$ canonically admits any of the following equivalent representations:

$$\mathbf{s} = (\phi_1^*\mathbf{s}, \cdots, \phi_n^*\mathbf{s}) = (\mathbf{s}\circ\phi_1, \cdots, \mathbf{s}\circ\phi_n) = (\mathbf{s}|_{M_1}, \cdots, \mathbf{s}|_{M_n}) = (s_1, \cdots, s_n). \tag{4.12}$$

Moreover, the isomorphism described in Theorem 4.2.12 is an isomorphism of $C^\infty(\mathbf{M})$-modules, so the linear operations on the space $\Gamma(\mathbf{E})$ translate into this representation as well.

For our final observation, we see that sections can be used to describe the Hausdorff-violating points in a colimit bundle.

Lemma 4.2.13. *Let $\mathbf{E}$ be a vector bundle over $\mathbf{M}$, with $\mathbf{s}$ a section of $\mathbf{E}$. If $[x,i]$ and $[y,j]$ are Hausdorff-inseparable in $\mathbf{M}$, then $\mathbf{s}([x,i])$ and $\mathbf{s}([y,j])$ are Hausdorff-inseparable in $\mathbf{E}$.*

Proof. Suppose first that $[x,i]$ and $[y,j]$ are Hausdorff-inseparable in $\mathbf{M}$. Then there is some sequence $[a_\alpha, i]$ in M_{ij} that converges to both points in $\mathbf{M}$. Since any continuous map between second-countable spaces will preserve the convergence of sequences, the sequence $\mathbf{s}([a_\alpha, i])$ will converge to both $\mathbf{s}([x,i])$ and $\mathbf{s}([y,j])$ in $\mathbf{E}$. Since $\mathbf{s}$ is injective, $\mathbf{s}([x,i])$ and $\mathbf{s}([y,j])$ are distinct points that are both limits of the same convergent sequence, so are Hausdorff-inseparable. □

4.3 Non-Hausdorff Differential Forms

In this section we will describe the differential forms on any non-Hausdorff manifold $\mathbf{M}$ satisfying the criteria of Remark 4.2.7. As we will see, the fibre-product formula of Theorem 4.2.12 will provide a reasonable description of non-Hausdorff differential forms. However, special care is required to prove the locality of any derivative operators defined on $\mathbf{M}$. To illustrate this idea, we will first provide a function-theoretic description of vector fields on $\mathbf{M}$.

4.3.1 Vector Fields

Since non-Hausdorff manifolds are locally-diffeomorphic to Hausdorff ones, there are at least some definitions of tangent vectors which directly apply to our setting. Under a geometric interpretation, we can always define tangent spaces using velocity vectors of curves passing through a given point. However, a function-theoretic definition of non-Hausdorff tangent vectors is currently lacking in the literature.

Naively, one might try to define tangent vectors as point-derivations of the ring $C^\infty(\mathbf{M})$. However, this will not make sense, since bump functions of the type used in [65, Prop. 3.8] to prove the locality of derivations will not exist in a non-Hausdorff manifold. Instead, we will consider derivations on the space $C^\infty_{[x,i]}(\mathbf{M})$ of germs of functions at a point. Locality is naturally incorporated into this definition since $C^\infty_{[x,i]}(\mathbf{M})$ will equal $C^\infty_{[x,i]}(U)$ for any open subset U of $\mathbf{M}$. We can therefore describe a basis of the tangent space using coordinate functions defined from open charts:

$$\sum_{a=1,\cdots,d} \lambda^a \frac{\partial}{\partial x_a}\bigg|_{[x,i]} \tag{4.13}$$

as in the Hausdorff case (cf [104, §14.1]). The geometric and function-theoretic definitions of tangent spaces can be shown to be equivalent, since they both have the same dimension as the tangent spaces of any M_i. An explicit isomorphism can be described using differentials $d\phi_i$, which either maps curves from M_i into $\mathbf{M}$, or maps the space $C^\infty_x(M_i)$ into $C^\infty_{[x,i]}(\mathbf{M})$.

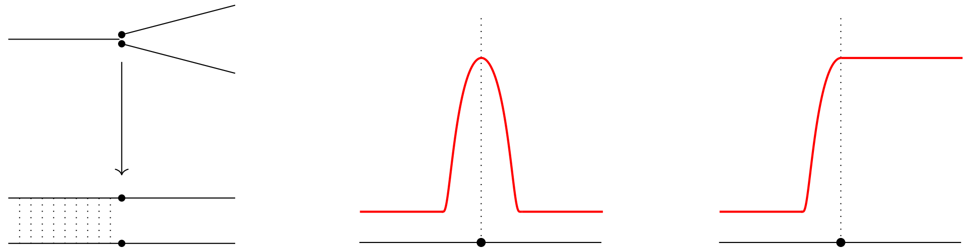
We may define the space $\mathfrak{X}(\mathbf{M})$ of vector fields on $\mathbf{M}$ as the global sections of the bundle $T\mathbf{M}$. Locally, we see that a vector field $\mathbf{X}$ in $\mathfrak{X}(\mathbf{M})$ is smooth if and only if the coefficients λ^a described above are elements of $C^\infty(U)$, where U is the local chart used to induce the coordinate expression. By appealing to the standard notions of local charts (i.e. with the Euclidean Leibniz rule), we see that $\mathbf{X}$ is indeed a derivation of the ring $C^\infty(\mathbf{M})$.

In order to describe an explicit bijection between the space of vector fields and the space of derivations, we must first prove that every derivation is a local operator on $\mathbf{M}$. We will approach this problem by first proving an auxilliary property, which we will call "semi-locality".

Lemma 4.3.1. *Let $\boldsymbol{D}$ be some derivation of $C^\infty(\boldsymbol{M})$. If $\boldsymbol{r}$ is a smooth function on $\boldsymbol{M}$ that vanishes on M_i, then $\boldsymbol{Dr}$ also vanishes on M_i.*

Proof. We first define a function $\boldsymbol{\psi} : \mathbf{M} \to \mathbb{R}$ such that $\boldsymbol{\psi}\mathbf{r} = \mathbf{r}$ everywhere, with $\boldsymbol{\psi} = 1$ for all Hausdorff-violating points in M_i. We construct this function in a few steps: first consider the set $B := \bigcup_{j\neq i} \partial^i M_{ij}$, which consists of all Hausdorff-violating points in $\mathbf{M}$ coming from M_i. Let $\psi_i : M_i \to \mathbb{R}$ to be a smooth bump function for the set B, constructed as in [65, §2]. We now define the smooth map $\boldsymbol{\psi} : \mathbf{M} \to \mathbb{R}$ as follows:

$$\boldsymbol{\psi}([x,j]) = \begin{cases} \psi_i \circ \phi_i^{-1}([x,j]) & \text{if } [x,j] \in M_i \\ 1 & \text{otherwise} \end{cases} \tag{4.14}$$



(a) The 2-branched real line (b) The bump function on M_1 (c) The cutoff function on M_2

Figure 4.2: Schematics of the smooth map ψ, defined for the 2-branched real line.

The map ψ will restrict to the bump function ψ_i on M_i, and will act as a cutoff function that equals 1 on M_j everywhere outside the set M_{ji}. Figure 4.2 depicts the function ψ for the branched real line. By construction, we have that $\mathbf{r}\psi = \mathbf{r}$ everywhere on $\mathbf{M}$. It follows that

$$\mathbf{Dr} = \mathbf{D}(\mathbf{r}\psi) = (\mathbf{Dr})\psi + \mathbf{r}(D\psi), \tag{4.15}$$

which reduces to the equality $\mathbf{Dr} = (\mathbf{Dr})\psi$ on M_i. We will now show that $\mathbf{Dr} = 0$ on M_i. There are two cases to consider.

Firstly, suppose that $[x,i]$ is some point in M_i that does not violate the Hausdorff property in $\mathbf{M}$. According to the construction of ψ, this means that $\psi([x,i]) \neq 1$. Thus the Leibniz property tells us that $(\mathbf{Dr})([x,i]) = (\mathbf{Dr})([x,i]) \cdot \psi([x,i])$, hence $(\mathbf{Dr})([x,i]) = 0$. Suppose now that $[x,i]$ does violate the Hausdorff property in $\mathbf{M}$. This means that x lies in the set B, and thus $\psi([x,i]) = 1$. Let a_n be some sequence in M_{ij} that converges to x in M_i. We may always arrange for this sequence to avoid the other boundary sets in B, so without loss of generality we may conclude that $\psi([a_n,i]) \neq 1$ for all n. By assumption the function $\mathbf{Dr}$ is smooth, so it follows that $\mathbf{Dr}([a_n,i])$ converges to $\mathbf{Dr}([x,i])$ in $\mathbb{R}$. However, we just showed that $\mathbf{Dr}$ vanishes for every element of M_i that does not violate the Hausdorff property. This means that $\mathbf{Dr}([a_n,i])$ is a constant sequence of zeroes in $\mathbb{R}$. The limit of this sequence is 0, that is,

$$\mathbf{Dr}([x,i]) = \mathbf{Dr}\left(\lim_{n\to\infty}[a_n,i]\right) = \lim_{n\to\infty}\mathbf{Dr}([a_n,i]) = \lim_{n\to\infty} 0 = 0. \tag{4.16}$$

We may thus conclude that $\mathbf{Dr} = 0$ on M_i, as required. □

This notion of semi-locality means that we can always restrict a derivation $\mathbf{D}$ of $C^\infty(\mathbf{M})$ down to a derivation D_i of $C^\infty(M_i)$ by defining

$$(D_i r)(x) = (\mathbf{Dr})([x,i]), \tag{4.17}$$

where $\mathbf{r}$ is any extension of r into $\mathbf{M}$, à la Lemma 4.2.8. Observe that D_i inherits its linearity and Leibniz property from $\mathbf{D}$. Moreover, the semi-locality property of Lemma 4.3.1 ensures that this map is well defined: any other extension $\tilde{\mathbf{r}}$ of r will cause the global function $\mathbf{r} - \tilde{\mathbf{r}}$ to vanish on M_i, which in turn means that $\mathbf{Dr} = \mathbf{D}\tilde{\mathbf{r}}$ on M_i. We will now argue that this notion of semi-locality can also be used to guarantee that an arbitrary derivation on $C^\infty(\mathbf{M})$ is a local operator.

Lemma 4.3.2. *Let $\mathbf{D} : C^\infty(\mathbf{M}) \to C^\infty(\mathbf{M})$ be a derivation. If $\mathbf{r}$ is a smooth function that vanishes on some open set U, then $\mathbf{Dr}$ also vanishes on U.*

Proof. Let p be some point in U. Since the M_i cover $\mathbf{M}$, p is of the form $[x,i]$ for some x in M_i. The restriction of the function $\mathbf{r}$ to M_i will vanish on the open set $U \cap M_i$. Since the restriction D_i is a derivation on $C^\infty(M_i)$, and all Hausdorff derivations are local, it follows that $D_i f_i$ vanishes on the subset $U \cap M_i$. In the construction of the derivation D_i, we are free to choose the extension of f_i to a global function in $\mathbf{M}$. In particular, we may choose the somewhat-trivial extension $\mathbf{r}$ to yield $0 = (D_i f_i)(x) = (\mathbf{Dr})([x,i])$. □

With a confirmed notion of locality for derivations on $C^\infty(\mathbf{M})$, an explicit isomorphism between the space of vector fields $\mathfrak{X}(\mathbf{M})$ and the space of derivations $\mathrm{Der}(C^\infty(\mathbf{M}))$ can now be derived from the procedure outlined in [104, Pg. 219]. Under this reading, we may meaningfully consider a vector field $\mathbf{X}$ on $\mathbf{M}$ as a derivation of $C^\infty(\mathbf{M})$. Any vector fields on M_i can be restricted to each M_{ij} by an analogue to the above, so we can further restrict any derivation D_i defined above to a derivation D_{ij} on $C^\infty(M_{ij})$. In fact from Theorems 4.2.11 and 4.2.12, we may actually view any derivation $\mathbf{D}$ of $C^\infty(\mathbf{M})$ as an element of the fibred product

$$\prod_{\mathscr{F}} \mathrm{Der}(M_i) := \{(D_1, \cdots, D_n) \mid D_i = D_j \text{ on } M_{ij} \text{ for all } i,j \in I\}. \tag{4.18}$$

In particular, this means that for any smooth function $\mathbf{r}$ we have $\mathbf{Dr} = (D_1 r_1, \cdots, D_n r_n)$.

The Lie Derivative

In general, vector fields on $\mathbf{M}$ will not admit global flows, ultimately due to the “splitting" of vector fields at Hausdorff-violating points. However, any vector field on $\mathbf{M}$ will be locally-equivalent to a Hausdorff vector field, so there will still be unique local flows. As such, we may still meaningfully define the Lie derivative $\mathscr{L}_{\mathbf{X}}\mathbf{r}$ of a function $\mathbf{r}$ with respect to a vector field $\mathbf{X}$ pointwise as $\mathscr{L}_{\mathbf{X}}\mathbf{r}([x,i]) = \mathscr{L}_{X_i} r_i(x)$. This is a manifesly local operator that will agree on the submanifolds M_{ij}, since both $\mathbf{X}$ and $\mathbf{r}$ do. As such, we may write Lie derivative as

$$\mathscr{L}_{\mathbf{X}}\mathbf{r} = (\mathscr{L}_{X_1} r_1, \cdots, \mathscr{L}_{X_n} r_n). \tag{4.19}$$

As an immediate consequence, we observe that the equalities $\mathscr{L}_{\mathbf{X}}\mathbf{r} = \mathbf{X}(\mathbf{r})$ and $\mathscr{L}_{\mathbf{X}}\mathbf{Y} = [\mathbf{X},\mathbf{Y}]$ both hold.

4.3.2 Differential Forms and the Exterior Derivative

In similar spirit to the previous section, we may again apply Theorems 4.2.11 and 4.2.12 to describe the algebra $\Omega^*(\mathbf{M})$ of differential forms on $\mathbf{M}$ via sections of the relevant exterior powers of $T^*\mathbf{M}$. A fibred product formula will again hold for differential forms: given a vector field $\mathbf{X}$ on $\mathbf{M}$, a differential 1-form $\boldsymbol{\omega}$ will act by

$$\begin{aligned}\boldsymbol{\omega}(\mathbf{X}) = (\phi_1^*\omega(\phi_1^*X), \cdots, \phi_n^*\omega(\phi_n^*X)) &= (\omega_1(X_1), \cdots, \omega_n(X_n)) \\ &= (\boldsymbol{\omega}(\mathbf{X})|_{M_1}, \cdots, \boldsymbol{\omega}(\mathbf{X})|_{M_n}),\end{aligned} \tag{4.20}$$

with higher-degree forms acting analogously. Since the wedge product distributes over pull-backs, we may readily extend it to differential forms on $\mathbf{M}$ using the equality:

$$\boldsymbol{\omega} \wedge \eta := (\omega_1 \wedge \eta_1, \cdots \omega_n \wedge \eta_n). \tag{4.21}$$

We may write an exterior derivative d on $\Omega^*(\mathbf{M})$ similarly:

$$d\boldsymbol{\omega} = (d\omega_1, \cdots, d\omega_n). \tag{4.22}$$

Observe that this is well-defined, since $\iota_{ij}^*(d\omega_i) = d(\iota_{ij}^*\omega_i) = d(f_{ij}^*\omega_j) = f_{ij}^*(d\omega_j)$ for all i, j in I. This definition of exterior derivative appears to be natural in that a semi-locality property similar to that of Lemma 4.3.1 is satisfied by construction. However, we ought to show that the operator d uniquely satisfies the usual desiderata of the exterior derivative.

Theorem 4.3.3. *The operator $\boldsymbol{D}$ is the unique operator on $\Omega^*(\boldsymbol{M})$ that satisfies the following properties.*

1. *d is local.*
2. *$d\boldsymbol{r}(X) = \boldsymbol{X}(\boldsymbol{r})$ for all $X \in \mathfrak{X}(\boldsymbol{M})$ and $\boldsymbol{r} \in C^\infty(\boldsymbol{M})$.*
3. *$d^2 = 0$*
4. *$d(\boldsymbol{\omega} \wedge \boldsymbol{\eta}) = (d\boldsymbol{\omega}) \wedge \boldsymbol{\eta} + (-1)^q \boldsymbol{\omega} \wedge d\boldsymbol{\eta}$ for all $\boldsymbol{\omega} \in \Omega^q(\boldsymbol{M})$ and $\boldsymbol{\eta} \in \Omega^*(\boldsymbol{M})$.*

Proof. The locality of d may be argued in a similar manner to that of Lemma 4.3.2. Indeed, suppose that we have some differential form $\boldsymbol{\omega}$ on $\mathbf{M}$ that vanishes on an open subset U. For any $[x,i]$ in $U \cap M_i$ we have that $\boldsymbol{\omega}([x,i]) = \omega_i(x)$. In particular, ω_i will vanish on the open set $U \cap M_i$. Since the exterior derivative on M_i is a local operator, we have that $d\boldsymbol{\omega}([x,i]) = d\omega_i(x) = 0$. Suppose now that $\mathbf{X}$ is some vector field on $\mathbf{M}$. Then:

$$(d\mathbf{r})(\mathbf{X}) = ((dr_1)(X_1), \cdots, (dr_n)(X_n)) = (X_1(r_1), \cdots X_n(r_n)) = \mathbf{X}(\mathbf{r}). \tag{4.23}$$

Moreover, the exterior derivative on $\mathbf{M}$ satisfies properties (iii) and (iv) above, since the exterior derivatives on M_i do. To see that d is unique with respect to these properties, suppose that there is some other operator $\tilde{d}$ on $\Omega^*(\mathbf{M})$ satisfying items (i)–(iv) above. Since $\tilde{d}$ is local, we may restrict it to each M_i to obtain an operator acting on $\Omega^*(M_i)$. The properties (ii)–(iv) are preserved under this restriction, and thus we may conclude that $d = \tilde{d}$ on each M_i. The global result then follows from the fact that $d\boldsymbol{\omega}([x,i]) = d\omega_i(x) = \tilde{d}\omega_i(x) = \tilde{d}\boldsymbol{\omega}([x,i])$ for all i in I and all $\boldsymbol{\omega}$ in $\Omega^*(\mathbf{M})$. □

4.3.3 Integration on non-Hausdorff Manifolds

Before getting to a definition of integration, we will first briefly comment on the existence of orientations on $\mathbf{M}$. Globally speaking, an orientation can be seen as a non-vanishing top form, which is a section of the bundle $\bigwedge^d T^*(\mathbf{M})$. According to Theorem 4.2.12, any orientation on $\mathbf{M}$ is induced from a series of compatible orientations on the submanifolds M_i. This idea can be restated as follows.

Lemma 4.3.4. *Let **M** be a colimit of Hausdorff manifolds, in accordance with 4.2.7. If each M_i is oriented, and the gluing maps f_{ij} are orientation-preserving, then **M** admits an orientation under which the maps $\phi_i : M_i \to \mathbf{M}$ become orientation-preserving, smooth open embeddings.*

On a Hausdorff manifold, the integration of compactly-supported differential forms is typically defined in terms of local charts, and then the total result is patched together using a partition of unity. According to Lemma 4.2.5 we do not have access to partitions of unity subordinate to arbitrary open covers, so we cannot use this definition in its desired generality. Instead, we will use the so-called "integration over parameterisations". In this approach an overall integral is broken down into smaller pieces that intersect on measure zero sets. The following definition is an adaptation of that found in [65, §16].

Definition 4.3.5. *Let **M** be a d-dimensional non-Hausdorff manifold defined according to 4.2.7 and oriented in the sense of 4.3.4, and let ω be a compactly-supported differential form on **M**. Suppose that we are given a finite collection $\{A_\alpha\}$ of open domains of integration in $\mathbb{R}^d$, together with a collection of maps $a_\alpha : \overline{A_\alpha} \to \mathbf{M}$ such that:*

1. *each a_α is an orientation-preserving diffeomorphism from A_α onto an open subset U_α in **M**,*
2. *the sets U_α are pairwise disjoint, and*
3. *$supp(\omega) := \overline{\{[x,i] \in \mathbf{M} \mid \omega([x,i]) = 0\}} \subseteq \bigcup_\alpha \overline{U_\alpha}$.*

*We then define the integral of ω in **M** to be*

$$\int_{\mathbf{M}} \omega = \sum_\alpha \int_{A_\alpha} a_i^* \omega. \tag{4.24}$$

Before deriving alternate descriptions of integration over **M**, we will first confirm that the above definition is well-defined.

Proposition 4.3.6. *The integral defined in Definition 4.3.5 is independent of the particular choice of the parameterizing sets.*

Proof. Suppose that $b_\beta : \overline{B_\beta} \to \mathbf{M}$ is another collection of sets and maps satisfying the conditions of Definition 4.3.5, and denote by V_β the image of B_β under b_β. For each U_α, we have that

$$\text{supp}(\omega) \cap \overline{U_\alpha} \subseteq \overline{U_\alpha} \cap \bigcup_\beta \overline{V_\beta} = \bigcup_\beta \overline{U_\alpha} \cap \overline{V_\beta}. \tag{4.25}$$

As such, we may consider the sets of the form $A_\alpha \cap B_\beta$ as a parametrized set covering A_α. This means that we can break down the integral of $\iota_\alpha^* \omega$ over A_α into the sum:

$$\int_{A_\alpha} a_i^* \omega = \sum_\beta \int_{A_\alpha \cap B_\beta} \iota_{\alpha\beta}^* a_i^* \omega, \tag{4.26}$$

where the map $\iota_{\alpha\beta}$ is the inclusion of the set $A_\alpha \cap B_\beta$ into A_α. We may repeat the same reasoning for fixed V_β and break the integral of $b_\beta^* \omega$ into a sum of integrals defined on the pairwise intersections $A_\alpha \cap B_\beta$. From this we may conclude that

$$\sum_\alpha \int_{A_\alpha} a_\alpha^* \omega = \sum_\alpha \sum_\beta \int_{A_\alpha \cap B_\beta} \iota_{\alpha\beta}^* a_\alpha^* \omega \tag{4.27}$$

$$= \sum_\beta \sum_\alpha \int_{A_\alpha \cap B_\beta} \iota_{\beta\alpha}^* b_\beta^* \omega \tag{4.28}$$

$$= \sum_\beta \int_{B_\beta} b_\beta^* \omega \tag{4.29}$$

as required. □

Proposition 16.8 of [65] ensures that Definition 4.3.5 recovers the ordinary integral for Hausdorff manifolds. Moreover, by some arguments of point-set topology it can be shown that the pullback of a compactly-supported form by the maps ϕ_i will still be compactly-supported in each M_i – see Lemma A.2.1 in the Appendices for the details. We will now use this observation in conjunction with Definition 4.3.5 to describe the total integral for $\mathbf{M}$ by decomposing it into ordinary integrals defined over the submanifolds M_i.

Theorem 4.3.7. *Let $\mathbf{M}$ be an oriented non-Hausdorff manifold, constructed as a binary adjunction of M_1 and M_2 according to 4.2.7 and 4.3.4. For all compactly-supported forms ω on $\mathbf{M}$, integration satisfies the sub-additive equality:*

$$\int_{\mathbf{M}} \omega = \int_{M_1} \omega_1 + \int_{M_2} \omega_2 - \int_{\overline{M_{12}}} \omega_{12} \tag{4.30}$$

where here $\overline{M_{12}}$ denotes the closure of M_{12} in M_1.

Proof. We would first like to describe a particular parametrisation of $\mathbf{M}$ in terms of parametrisations of the M_i and M_{12}. Suppose that:

- $\{a_i : A_i \to M_1\}$ is some parametrisation of the set $M_1 \backslash M_{12}$,
- $\{b_i : B_i \to M_2\}$ is some parametrisation of the set $M_2 \backslash M_{12}$, and
- $\{c_i : C_i \to \overline{M_{12}}\}$ is some parametrisation of the closed set $\overline{M_{12}}$, viewed as a subspace of M_1.

Observe that the collection $\{A_i\} \cup \{C_i\}$ forms a parametrisation of the entire manifold M_1, and similarly $\{B_i\} \cup \{\tilde{C}_i\}$ for M_2, where here $\tilde{C}_i$ denotes the transfer of the parametrization C_i into M_2 using the diffeomorphism $\overline{f_{12}}$. In particular, this means that

$$\int_{M_1} \eta = \sum \left(\int_{A_i} a_i^* \eta \right) + \sum \left(\int_{C_i} (\iota_{12} \circ c_i)^* \eta \right) = \sum \left(\int_{A_i} a_i^* \eta \right) + \sum \left(\int_{C_i} c_i^* \circ \iota_{12}^* \eta \right) \tag{4.31}$$

for any differential form η on M_1. Equivalently, this integral can be seen as a restriction of η into the two sets $\overline{M_{12}}$ and $M_1 \backslash M_{12}$, followed by an integration over the respective parametrizations of these two spaces. We remark at this stage that the inclusion of the boundary of M_{12} is necessary, as otherwise the restriction $\iota_{12}^*\eta$ need not be compactly-supported.[3] Similarly, we have that:

$$\int_{M_2}\mu=\sum\left(\int_{B_i}b_i^*\mu\right)+\sum\left(\int_{C_i}(\overline{f_{ij}}\circ c_i)^*\mu\right)=\sum\left(\int_{B_i}b_i^*\mu\right)+\sum\left(\int_{C_i}c_i^*\circ\overline{f_{ij}}^*\mu\right) \tag{4.32}$$

for any differential form μ on M_2. Since the maps ϕ_i are smooth, orientation-preserving open embeddings, the collection

$$\{\phi_1(A_i)\}\cup\{\phi_2(B_i)\}\cup\{\phi_1\circ\iota_{12}(C_i)\} \tag{4.33}$$

forms a parametrisation of $\mathbf{M}$. According to Proposition 4.3.6, the integral of any ω can be computed according to this particular parametrization. This yields:

$$\int_{\mathbf{M}}\omega=\sum\int_{A_i}(\phi_1\circ a_i)^*\omega+\sum\int_{B_i}(\phi_2\circ b_i)^*\omega+\sum\int_{C_i}(\phi_1\circ\iota_{12}\circ c_i)^*\omega \tag{4.34}$$

$$+\sum\int_{C_i}(\phi_1\circ\iota_{12}\circ c_i)^*\omega-\sum\int_{C_i}(\phi_1\circ\iota_{12}\circ c_i)^*\omega \tag{4.35}$$

$$=\sum\int_{A_i}(\phi_1\circ a_i)^*\omega+\sum\int_{B_i}(\phi_2\circ b_i)^*\omega+\sum\int_{C_i}(\phi_1\circ\iota_{12}\circ c_i)^*\omega \tag{4.36}$$

$$+\sum\int_{C_i}(\phi_2\circ\overline{f_{12}}\circ c_i)^*\omega-\sum\int_{C_i}(\phi_1\circ\iota_{12}\circ c_i)^*\omega \tag{4.37}$$

$$=\left(\sum\int_{A_i}(\phi_1\circ a_i)^*\omega+\sum\int_{C_i}(\phi_1\circ\iota_{12}\circ c_i)^*\omega\right) \tag{4.38}$$

$$+\left(\sum\int_{B_i}(\phi_2\circ b_i)^*\omega+\sum\int_{C_i}(\phi_2\circ\overline{f_{12}}\circ c_i)^*\omega\right)-\left(\sum\int_{C_i}(\phi_1\circ\iota_{12}\circ c_i)^*\omega\right) \tag{4.39}$$

$$=\int_{M_1}\phi_1^*\omega+\int_{M_2}\phi_2^*\omega-\int_{\overline{M_{12}}}\phi_{12}^*\omega \tag{4.40}$$

where here in the second equality we use the commutative property $\phi_1\circ\iota_{12}=\phi_2\circ\overline{f_{12}}$. □

We can extend this representation to manifolds $\mathbf{M}$ which are finite colimits. In this case, we will get a more-general alternating sum that keeps count of the number of sets being used in each possible intersection. As a shorthand notation, we will write $M_{i_1\cdots i_p}$ for the set $M_1\cap\cdots\cap M_p$.

Theorem 4.3.8. *Let $\mathbf{M}$ be an oriented non-Hausdorff manifold, built as the colimit of n-many manifolds M_i in accordance with 4.2.7. The integral of a differential form ω over $\mathbf{M}$ satisfies:*

$$\int_{\mathbf{M}}\omega=\sum_{i=1}^{n}\left(\int_{M_i}\phi_i^*\omega\right)-\sum_{p=2}^{n}(-1)^p\left(\sum_{\substack{i_1,\cdots,i_p\in I\\ i_1<\cdots<i_p}}\int_{\overline{M_{i_1\cdots i_p}}}\phi_{i_1\cdots i_p}^*\omega\right) \tag{4.41}$$

Proof. See Appendix 3.8.1. □

[3] For example, suppose that $M_1=\mathbb{R}$ and $M_{12}=(-\infty,0)$, and consider a bump 1-form in M_1 centered at the origin. The restriction of this form to M_{12} will take support on some interval $[-a,0)$, which is not compact in M_{12}. However, the interval $[-a,0]$ is compact in $\overline{M_{12}}=(\infty,0]$.

In the above arguments we had to include extra integrals over the boundaries of M_{ij}, since otherwise there is no guarantee that the restrictions $\omega|_{M_{ij}}$ maintain a compact support. This may appear to be an innocuous detail, since the inclusion of a boundary component of M_{12} will merely add an extra set of measure zero into the total integral. However, the inclusion of this boundary will have some-far reaching consequences regarding certain types of integral. To illustrate this, we present the following counterexample to Stoke's theorem.

Lemma 4.3.9. *Let $\mathbf{M}$ be a binary adjunction such that $\mathbf{M}$, and the M_i are compact, without boundary. Then*

$$\int_{\mathbf{M}} d\omega = -\int_{\partial M_{12}} \phi_{12}^* \omega. \tag{4.42}$$

Proof. Using Theorem 4.3.7 together with Stoke's Theorem for Hausdorff manifolds, we have:

$$\begin{aligned}
\int_{\mathbf{M}} d\omega &= \int_{M_1} \phi_1^* d\omega + \int_{M_2} \phi_2^* d\omega - \int_{\overline{M_{12}}} \phi_{12}^* d\omega && (4.43)\\
&= \int_{M_1} d\phi_1^* \omega + \int_{M_2} d\phi_2^* \omega - \int_{\overline{M_{12}}} d\phi_{12}^* \omega && (4.44)\\
&= \int_{\partial M_1} \phi_1^* \omega + \int_{\partial M_2} \phi_2^* \omega - \int_{\partial M_{12}} \phi_{12}^* \omega && (4.45)\\
&= -\int_{\partial M_{12}} \phi_{12}^* \omega && (4.46)
\end{aligned}$$

as required. □

In the Hausdorff setting, this particular integral will vanish. However, we see here that in the non-Hausdorff integral the internal Hausdorff-violating submanifold still has an influence. As a consequence of the above, we see that exact differential forms may not integrate to zero on a closed non-Hausdorff manifold.

4.4 De Rham Cohomology via Mayer-Vietoris Sequences

In this section we will describe the de Rham cohomology of a non-Hausdorff manifold $\mathbf{M}$ satisfying the conditions of Remark 4.2.7. According to Theorem 4.2.4, the (images of) the Hausdorff manifolds M_i sit inside $\mathbf{M}$ as open sets. Moreover, we saw in Section 4.3 that the space of differential forms $\Omega^*(\mathbf{M})$ and its exterior derivative d can be described in terms of the same data defined on each M_i. With this in mind, we will now set about deriving Mayer-Vietoris sequences for the de Rham cohomology of $\mathbf{M}$, computed according to the open cover $\{M_i\}$.

4.4.1 A Binary Mayer-Vietoris Sequence

We will derive our Mayer-Vietoris sequences inductively. So, throughout this section we fix a non-Hausdorff manifold $\mathbf{M}$ which is constructed from a pair of Hausdorff spaces M_1 and M_2 according to Remark 4.2.7. According to our discussion in the previous section, the differential forms on $\mathbf{M}$ display a fibred product structure. In a similar manner to our description of the smooth functions $C^\infty(\mathbf{M})$ in Section 4.2.2, this can be arranged into the following sequence.

$$0 \longrightarrow \Omega^q(\mathbf{M}) \xrightarrow{\Phi^*} \Omega^q(M_1) \oplus \Omega^q(M_2) \xrightarrow{\iota_{12}^* - f_{12}^*} \Omega^q(M_{12}) \longrightarrow 0 \tag{4.47}$$

Ideally we would like these sequences to be exact, since then we could apply the Snake Lemma to form a long exact sequence on cohomology. Although the fibre product structure of each $\Omega^q(\mathbf{M})$ guarantees that the above sequences are exact in the first term, without the existence of a partition of unity subordinate to $\{M_1, M_2\}$, there is no guarantee that the difference map $\iota_{12}^* - f_{12}^*$ is surjective. As such, the above sequences will not be short exact in general. To resolve this issue, we will instead exchange $\Omega^q(M_{12})$ for a subalgebra. Before doing so, we first make the following observation.

Lemma 4.4.1. *Let $\overline{M_{12}}$ be the closure of M_{12} in M_1. Then there is an equality*

$$\Omega^k(M_1) \times_{\Omega^k(M_{12})} \Omega^k(M_2) = \Omega^k(M_1) \times_{\Omega^k(\overline{M_{12}})} \Omega^k(M_2). \tag{4.48}$$

Proof. The inclusion from right-to-left is clear – any pair of forms that agree on $\overline{M_{12}}$ will also agree on M_{12}. For the converse, suppose that ω_i are a pair of q-forms on M_i such that $\iota_{12}^*\omega_1 = f_{12}^*\omega_2$ on M_{12}. By our assumption, it suffices to show that $\omega_1(x) = \omega_2(\overline{f_{12}}(x))$ for all elements x on the boundary of M_{12}. So, consider a point x in ∂M_{12}. By Theorem 4.2.4, the points $[x,1]$ and $[f_{ij}(x),2]$ violate the Hausdorff property in $\mathbf{M}$. According to Lemma 4.2.13, the global form ω constructed from ω_1 and ω_2 can be used to describe the Hausdorff-violating points of the q-form bundle over $\mathbf{M}$. In our situation, this means that $\omega([x,1])$ and $\omega([f_{12}(x),2])$ violate the Hausdorff property in the q-form bundle over $\mathbf{M}$. Again by Theorem 4.2.4, this means that $\omega_1 = \overline{f_{12}}^*\omega_2$, that is,

$$\iota_{12}^*\omega_1(x) = \omega_1 \circ \iota_{12}(x) = \omega_1(x) = \omega_2(\overline{f_{12}}(x)) = \overline{f_{12}}^*\omega_2(x), \tag{4.49}$$

as required. □

By restricting our attention to the differential forms on M_{12} that are finite on its boundary, we allow ourselves to extend any differential form ω in $\Omega^q(\overline{M_{12}})$ by zero into either M_1 or M_2. This ensures that both the pullback maps ι^* and f^* are surjective, and consequently so is the difference of the two. By Lemma 4.4.1 we obtain our desired short exact sequence,

$$0 \longrightarrow \Omega^k(\mathbf{M}) \xrightarrow{\Phi^*} \Omega^k(M_1) \oplus \Omega^k(M_2) \xrightarrow{\iota_{12}^* - (\overline{f_{12}})^*} \Omega^k(\overline{M_{12}}) \longrightarrow 0 \tag{4.50}$$

and thus we may form the following long exact sequence for de Rham cohomology.

$$\begin{array}{l} \cdots \longrightarrow H^q_{dR}(\mathbf{M}) \xrightarrow{\Phi^*} H^q_{dR}(M_1) \oplus H^q_{dR}(M_2) \xrightarrow{\iota_{12}^* - (\overline{f_{12}})^*} H^q_{dR}(\overline{M_{12}}) \\ \hookrightarrow H^{q+1}_{dR}(\mathbf{M}) \xrightarrow{\Phi^*} H^{q+1}_{dR}(M_1) \oplus H^{q+1}_{dR}(M_2) \xrightarrow{\iota_{12}^* - (\overline{f_{12}})^*} H^{q+1}_{dR}(\overline{M_{12}}) \longrightarrow \cdots \end{array} \tag{4.51}$$

In general we may not proceed any further than this. However, we observe that if **M** is built in such a way that the gluing region M_{12} is regular-open, then we may derive a Mayer-Vietoris sequence that is perhaps more familiar. Recall that a (sub)manifold with boundary is homotopically equivalent to its interior (cf. [65, Thm 9.26]). Moreover, for a regular open set, it is always the case that the interior of its closure equals itself. Combining these two facts, we see that requiring M_{12} to be regular-open amounts to requiring that M_{12} is homotopically equivalent to its closure $\overline{M_{12}}$. Since de Rham is homotopy invariant for Hausdorff manifolds, our extra requirement ensures that $H^q_{dR}(M_{12})$ and $H^q(\overline{M_{12}})$ are isomorphic. This means that we can replace terms in the above sequence to obtain the following.

$$\begin{array}{l} \cdots \longrightarrow H^q_{dR}(\mathbf{M}) \xrightarrow{\Phi^*} H^q_{dR}(M_1)\oplus H^q_{dR}(M_2) \xrightarrow{\iota_{12}^*-f_{12}^*} H^q_{dR}(M_{12}) \longrightarrow \\ \longrightarrow H^{q+1}_{dR}(\mathbf{M}) \xrightarrow{\Phi^*} H^{q+1}_{dR}(M_1)\oplus H^{q+1}_{dR}(M_2) \xrightarrow{\iota_{12}^*-f_{12}^*} H^{q+1}_{dR}(M_{12}) \longrightarrow \cdots \end{array} \tag{4.52}$$

4.4.2 The Čech-de Rham Bicomplex

Suppose now that **M** is covered by finitely-many M_i. In this case, we will proceed with an argument similar to that of [13, 23, 85] to construct a bicomplex which will describe the de Rham cohomology of **M**.

Before getting to the desired result, we first need to comment on some additional assumptions required in the proceeding arguments. In what follows, we will supplement each higher intersection $M_{i_1\cdots i_p}$ with its Hausdorff boundary, and compute the Mayer-Vietoris long exact sequence according to these sets. However, our derivation will not hold in the case that the closed submanifolds $\overline{M_{i_1\cdots i_p}}$ carry additional cohomological information that is not already present in their interiors. In order to avoid such situations, we will assume that the higher intersections satisfy the closure property:

$$\overline{M_{i_1\cdots i_m}} = \bigcap_{p=1}^{m} \overline{M_{i_p}} \tag{4.53}$$

for all m in I. Of course, this property does not hold for arbitrary closed sets. However, it removes the possibility of sets M_{ij} intersecting only on the codimension-1 boundary components.

With these extra assumptions in mind, we will now derive a Mayer-Vietoris long exact sequence for **M**. In doing so, we will need to use the Čech differential δ, the details of which can be found in [13, §8]. In our context, the map δ generalises the difference maps $\iota_{ij}^* - f_{ij}^*$ as follows:

$$\delta\omega_{i_1\cdots i_{p+1}} = \sum_{\alpha=1}^{p} (-1)^{\alpha} \iota^*_{i_1\cdots i_{\hat\alpha}\cdots i_p} \omega_{i_1\cdots i_{\hat\alpha}\cdots i_p}, \tag{4.54}$$

where here the map $\iota_{i_1\cdots i_{\hat\alpha}\cdots i_p} : M_{i_1\cdots i_p} \to M_{i_1\cdots i_{\hat\alpha}\cdots i_p}$ is the inclusion map, and the caret denotes the omission of that particular index. Note that by virtue of its combinatorics, the Čech differential always squares to zero (cf [13, Prop. 8.3]).

Theorem 4.4.2. *Let $\boldsymbol{M}$ be a colimit of n-many manifolds M_i in accordance with 4.2.7. In addition, suppose that each M_{ij} is regular-open in M_i and that the closure-intersection property $\overline{M_{i_1\cdots i_m}} = \bigcap_{p=1}^{m} \overline{M_{i_p}}$ holds for all $m \leq n$. Then the sequence*

$$0 \to \Omega^q(\boldsymbol{M}) \xrightarrow{\Phi^*} \bigoplus_i \Omega^q(M_i) \xrightarrow{\delta} \bigoplus_{i<j} \Omega^q\left(\overline{M_{ij}}\right) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Omega^q\left(\overline{M_{1\cdots n}}\right) \to 0 \tag{4.55}$$

is exact for all q in $\mathbb{N}$.

Proof. We will argue this by induction on the size of the indexing set I. Observe first that the binary case is already satisfied via the discussion in Section 4.4.1. For readability we will illustrate the inductive argument for $I = 3$, since the general case is identical. According to our assumptions on the topology of $\mathbf{M}$, we may view $\mathbf{M}$ as an inductive colimit in which we glue M_3 to the union $M_1 \cup M_2$ along the subspace $M_{13} \cup M_{23}$. We denote by κ the inclusion of the adjuction $M_1 \cup_{f_{12}} M_2$ into $\mathbf{M}$. Note that according to Section 1.1.1 of [85], κ is an open topological embedding. Diagrammatically, the differential forms on all the spaces in involved may be arranged as follows.

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker(\kappa^*) & \xrightarrow{\phi_3^*} & \Omega^q(M_3) & \xrightarrow{(-\overline{f_{13}}^*,-\overline{f_{23}}^*)} & \Omega^q(\overline{M_{13}}) \oplus \Omega^q(\overline{M_{23}}) & \xrightarrow{\delta} & \Omega^q(\overline{M_{123}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^q(\mathbf{M}) & \xrightarrow{\Phi^*} & \bigoplus_i \Omega^q(M_i) & \xrightarrow{\delta} & \bigoplus_{i<j} \Omega^q(\overline{M_{ij}}) & \xrightarrow{\delta} & \Omega^q(\overline{M_{123}}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \kappa^*} & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^q(M_1 \cup M_2) & \longrightarrow & \Omega^q(M_1) \oplus \Omega^q(M_2) & \longrightarrow & \Omega^q(\overline{M_{12}}) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array} \tag{4.56}$$

In the above diagram, the objects in the first row are kernels of the vertical maps, and the horizontal maps are defined so as the make the diagram commute. Observe that the third row is an exact sequence. Thus, in order to argue for the exactness of the central row, it suffices to argue that the first row is exact. In principle we may argue for this directly. However, in order to illustrate the inductive nature of this argument, we will instead decompose the first row into the sequences:

$$0 \to \ker(\kappa^*) \xrightarrow{\phi_2^*} \Omega^q(M_3) \xrightarrow{-\iota^*} \Omega^q(\overline{M_{13}} \cup \overline{M_{23}}) \to 0 \tag{4.57}$$

$$0 \to \Omega^q(\overline{M_{13}} \cup \overline{M_{23}}) \to \Omega^q(\overline{M_{13}}) \oplus \Omega^q(\overline{M_{23}}) \to \Omega^q(\overline{M_{123}}) \to 0 \tag{4.58}$$

where here this splicing is formed by evaluating the kernel of the δ map. Regarding the former sequence, we will argue for exactness directly. Observe first that the map ϕ_3^* is manifestly injective, since any pair of distinct forms lying in $\ker \kappa^*$ will differ somewhere on M_3. Moreover, any form that lies in the image of the map ϕ_3^* is the restriction of a global form on $\mathbf{M}$ that vanishes on $M_1 \cup M_2$. Since $\overline{M_{13}} \cup \overline{M_{23}}$ is a subset of $M_1 \cup M_2$, the restriction of such forms will also vanish. This ensures that $Im(\phi_3^*) \subseteq \ker(-\iota^*)$. For the converse, suppose that ω is some form on M_3 that maps to zero under ι^*. This means that ω vanishes on M_1 and M_2. As such, the collection $(0,0,\omega)$ will induce a global form η on $\mathbf{M}$ such that $\phi_3^*\eta = \omega$ and $\eta \in \ker(-\iota^*)$. Finally, since the set $\overline{M_{13}} \cup \overline{M_{23}}$ is closed, an extension by zero into M_3 will ensure the surjectivity of the restriction map $-\iota^*$.

In general, the latter sequence may not be exact. However, our assumptions on the intersections ensures that $\overline{M_{123}} = \overline{M_{13}} \cap \overline{M_{23}}$, which turns this sequence into a Hausdorff Mayer-Vietoris sequence for closed subsets. Exactness of this sequence then follows by an argument analogous to that of Lemma 3.8.3 in the Appendix.[4] Since the two aforementioned sequences are exact, and they splice to form the first row of the diagram, we may conclude that the first row of the diagram is exact, from which the result follows. □

According to the previous result, we may organise the de Rham cohomology for $\mathbf{M}$ in terms of the bicomplex detailed in Figure 4.3. This bicomplex relates the de Rham cohomology to the Čech cohomology of $\mathbf{M}$ computed from the open cover $\{M_i\}$. According to Theorem 4.4.2 the rows of this bicomplex are exact, so we may use standard arguments (found in e.g. [13, §8]) to conclude that the cohomology of this bicomplex coincides with the de Rham cohomology for $\mathbf{M}$.

It is important to note that at this stage we have made no assumptions on the cohomology of each M_i. In particular, we did not assume that each M_i (and their interesections) are contractible. This means that generally speaking, there is no guarantee that the columns of this bicomplex are exact, and thus we cannot prove a Čech-de Rham equivalence as in the Hausdorff case.

4.5 A non-Hausdorff de Rham Theorem

De Rham's theorem is a fundamental result that states an equivalence between de Rham cohomology and singular homology with real coefficients. In this section we will extend the result to non-Hausdorff manifolds, essentially by an appeal to the Mayer-Vietoris sequences of Section 4.4.

Throughout this section we will follow [65, Chpt. 18], and denote singular n-simplices by $\sigma : \Delta_q \to \mathbf{M}$. We denote the space of singular q-chains by $C_q(\mathbf{M})$, where here we view these as infinite-dimensional vector spaces over $\mathbb{R}$. An arbitrary singular q-chain is then a

[4] In the general inductive step, this sequence will expand into a longer sequence of the form of that found in Appendix 3.8.2, and exactness will instead follow from that result.

$$
\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & \Omega^3(\mathbf{M}) & \longrightarrow & \bigoplus_i \Omega^3(M_i) & \longrightarrow & \bigoplus_{i<j} \Omega^3(\overline{M_{ij}}) & \longrightarrow & \bigoplus_{i<j<k} \Omega^3(\overline{M_{ijk}}) & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & \Omega^2(\mathbf{M}) & \longrightarrow & \bigoplus_i \Omega^2(M_i) & \longrightarrow & \bigoplus_{i<j} \Omega^2(\overline{M_{ij}}) & \longrightarrow & \bigoplus_{i<j<k} \Omega^2(\overline{M_{ijk}}) & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & \Omega^1(\mathbf{M}) & \longrightarrow & \bigoplus_i \Omega^1(M_i) & \longrightarrow & \bigoplus_{i<j} \Omega^1(\overline{M_{ij}}) & \longrightarrow & \bigoplus_{i<j<k} \Omega^1(\overline{M_{ijk}}) & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
0 \longrightarrow & \Omega^0(\mathbf{M}) & \longrightarrow & \bigoplus_i \Omega^0(M_i) & \longrightarrow & \bigoplus_{i<j} \Omega^0(\overline{M_{ij}}) & \longrightarrow & \bigoplus_{i<j<k} \Omega^0(\overline{M_{ijk}}) & \longrightarrow \cdots \\
 & & & \uparrow & & \uparrow & & \uparrow & \\
 & & & \check{C}^0(\mathbf{M},\{M_i\}) & \longrightarrow & \check{C}^1(\mathbf{M},\{M_i\}) & \longrightarrow & \check{C}^2(\mathbf{M},\{M_i\}) & \longrightarrow \cdots
\end{array}
$$

Figure 4.3: The Čech-de Rham Bicomplex for **M**.

formal linear combination of continuous maps from Δ_q to $\mathbf{M}$. We will denote the boundary operator for simplices by ∂, defined as per usual.

Ordinarily, the Hausdorff de Rham theorem is proved by defining an “integration over chains", which relates differential forms to singular chains. However, there is a subtely here: in order to integrate over chains, differential forms need to be pulled back to Δ_q along *smooth* maps. As such, integration over chains actually induces a pairing between de Rham cohomology and the “smooth singular homology", which *a priori* may differ from singular homology. This issue may be resolved by arguing that smooth singular homology is equivalent to singular homology via some homotopic approximation of continuous maps by smooth ones. In this section we will follow a similar approach, with some key modifications.

4.5.1 Smooth Singular Homology

We start our derivation of de Rham’s theorem with a confirmation that smooth singular homology coincides with singular homology in the non-Hausdorff regime. In this context, “smooth" means that any simplex $\sigma : \Delta_q \to \mathbf{M}$ has a smooth extension in some neighbourhood of each point in Δ_q. By standard arguments it can be shown that the boundary of a smooth chain is again smooth, and thus the space $C_q^\infty(\mathbf{M})$ of smooth q-chains forms a subcomplex of $C_q(\mathbf{M})$, with the inclusion $i : C_q^\infty(\mathbf{M}) \to C_q(\mathbf{M})$ being a chain map.

For Hausdorff manifolds, the equivalence between the smooth and singular homologies is typically argued by appealing to Whitney’s Embedding theorem, a consequence of which is that any continuous map between manifolds is homotopic to a smooth one. This general observation can be used to construct a homotopy equivalence between $C_q(\mathbf{M})$ and $C_q^\infty(\mathbf{M})$ (cf. [65, Thm 18.7]). Since Hausdorffness is a hereditary property, there are no circumstances under which a non-Hausdorff manifold may be embedded into Euclidean space, of any dimension. Moreover, it is unclear whether all continuous maps between non-Hausdorff manifolds can be approximated by smooth maps. Instead of addressing the problem head-on, we will instead prove the equivalence between $H_p^\infty(\mathbf{M})$ and $H_p(\mathbf{M})$ by alternate means.

Since the standard Hausdorff argument is not available to us, instead we will derive a Mayer-Vietoris sequence for smooth singular homology from first principles. For a binary adjunction, this would amount to a proof of the exactness of the sequence

$$0 \to C_q^\infty(M_{12}) \xrightarrow{((\iota_{12})_*, -(f_{12})_*)} C_q^\infty(M_1) \oplus C_q^\infty(M_2) \xrightarrow{(\phi_1)_* + (\phi_2)_*} C_q^\infty(\mathbf{M}) \to 0 \tag{4.59}$$

where the maps involved are analogues of the pushforwards found in [65]. As a matter of fact, the above sequence is trivially exact if we replace the last non-zero term with the space $C_q^\infty(M_1 + M_2)$, which is the space of smooth singular chains in $\mathbf{M}$ that take image in either M_1 or M_2. This is a subspace of $C_q^\infty(\mathbf{M})$, and thus there is an inclusion $\iota : C_q^\infty(M_1 + M_2) \to C_q^\infty(\mathbf{M})$. In the singular case, there is a well-known construction which guarantees that the inclusion ι is a homotopy equivalence [50][Prop. 2.21]. As noted in [35, Pg. 135], the same argument will hold for smooth singular chains as well. With an eye towards Theorem 4.5.2, we will state the result in slightly more general language.

Lemma 4.5.1. *Let $\mathscr{U} = \{U_1, U_2\}$ be an open cover of $\boldsymbol{M}$, and denote by $C_q^\infty(U_1 + U_2)$ the space of smooth singular chains $c = \sum_\alpha c_\alpha \sigma_\alpha$ such that each σ_α has image contained entirely in either U_1 or U_2. Then the inclusion $\iota : C_q^\infty(U_1 + U_2) \to C_q^\infty(\boldsymbol{M})$ is a homotopy equivalence.*

We may apply the Snake Lemma to the short exact sequences

$$0 \to C_q^\infty(M_1 \cap M_2) \xrightarrow{((\iota_{12})_*, -(f_{12})_*)} C_q^\infty(M_1) \oplus C_q^\infty(M_2) \xrightarrow{(\phi_1)_* + (\phi_2)_*} C_q^\infty(M_1 + M_2) \to 0 \tag{4.60}$$

and then use Lemma 4.5.1 to replace each $H_p^\infty(M_1 + M_2)$ with $H_p^\infty(\mathbf{M})$ in the resulting Mayer-Vietoris sequence. We will now use this observation to prove an equivalence between the groups $H_p^\infty(\mathbf{M})$ and $H_p(\mathbf{M})$.

Theorem 4.5.2. *Let $\boldsymbol{M}$ be a non-Hausdorff manifold satisfying the properties of Remark 4.2.7. Then the smooth singular homology group $H_p^\infty(\boldsymbol{M})$ is isomorphic to the singular homology group $H^p(\boldsymbol{M})$ for all p in $\mathbb{N}$.*

Proof. We proceed by induction on the size of the indexing set I defining $\mathbf{M}$. Suppose first that $\mathbf{M}$ is a binary colimit. By the discussion preceeding this theorem, there is a Mayer-Vietoris long exact sequence for smooth singular homology. The standard derivation of the Mayer-Vietoris sequence for singular homology (cf. [50, §2.2]) also holds in this setting. The inclusions $i : C_q^\infty(\cdot) \to C_q(\cdot)$ are all homomorphisms that commute with the relevant boundary maps, as depicted below.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_q^\infty(M_{12}) & \longrightarrow & C_q^\infty(M_1) \oplus C_q^\infty(M_2) & \longrightarrow & C_q^\infty(M_1 + M_2) & \longrightarrow & 0 \\ & & \downarrow i & & \downarrow i & & \downarrow i & & \\ 0 & \longrightarrow & C_q(M_{12}) & \longrightarrow & C_q(M_1) \oplus C_q(M_2) & \longrightarrow & C_q(M_1 + M_2) & \longrightarrow & 0 \end{array} \tag{4.61}$$

As such, we may use the naturality of the Snake Lemma to construct the following commutative diagram.

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_q^\infty(M_{12}) & \longrightarrow & H_q^\infty(M_1) \oplus H_q^\infty(M_2) & \longrightarrow & H_q^\infty(M_1 + M_2) & \longrightarrow & H_{q-1}^\infty(M_{12}) & \longrightarrow & H_{q-1}^\infty(M_1) \oplus H_{q-1}^\infty(M_2) & \longrightarrow & \cdots \\ & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* & & \downarrow i_* & & \\ \cdots & \longrightarrow & H_q(M_{12}) & \longrightarrow & H_q(M_1) \oplus H_q(M_2) & \longrightarrow & H_q(M_1 + M_2) & \longrightarrow & H_{q-1}(M_{12}) & \longrightarrow & H_{q-1}(M_1) \oplus H_{q-1}(M_2) & \longrightarrow & \cdots \end{array} \tag{4.62}$$

Theorem 18.7 of [65] ensures that the inclusion maps between Hausdorff components of this diagram are isomorphisms. Since the diagram commutes, we may apply the Five Lemma to conclude that the central i_*-map depicted above is also an isomorphism. By applying Lemma 4.5.1 and the analogue for singular chains, it follows that

$$H_q^\infty(\mathbf{M}) \cong H_q^\infty(M_1 + M_2) \cong H_q(M_1 + M_2) \cong H_q(\mathbf{M}), \tag{4.63}$$

as required.

For the inductive case, suppose that smooth singular homology coincides with singular homology for all non-Hausdorff manifolds of size n. Suppose now that we have a non-Hausdorff manifold built as a colimit of $(n+1)$-many Hausdorff manifolds M_i. We may use the inductive representation of $\mathbf{M}$ and rewrite it as a binary adjunction space $\mathbf{M} \cong \mathbf{N} \cup_F M_{n+1}$, where $\mathbf{N}$ is the colimit of the first n-many spaces in $\mathbf{M}$, and $\mathbf{N}$ is glued to M_{n+1} along the subspace $A := \bigcup_{i \leq n} M_{i(n+1)}$. Since the subsets $\mathbf{N}$ and M_{n+1} form an open cover of $\mathbf{M}$, we may apply the result of Lemma 4.5.1 to conclude that

$$0 \to C_q^\infty(A) \xrightarrow{((\iota_A)_*, -F_*)} C_q^\infty(\mathbf{N}) \oplus C_q^\infty(M_{n+1}) \xrightarrow{(\varphi_1)_* + (\varphi_2)_*} C_q^\infty(\mathbf{N} + M_{n+1}) \to 0 \tag{4.64}$$

is a short exact sequence, where here the maps φ_i are the canonical embeddings of $\mathbf{N}$ and M_{n+1} into $\mathbf{M}$. The inclusions i into singular cochains will still commute with boundary maps, and thus the descended map i_* will still be a chain map for the homologies. Since A and M_{n+1} are Hausdorff manifolds, the i_* map on their cochains will be isomorphisms by [65]. Moreover, the inductive hypothesis tells us that the map $i_* : H_q^\infty(\mathbf{N}) \to H_q(\mathbf{N})$ is an isomorphism for all q. As such, we may again apply the Five Lemma to conclude that $H_q^\infty(\mathbf{N} + M_{n+1}) \cong H_q(\mathbf{N} + M_{n+1})$. The result then follows as an application of Lemma 4.5.1.

□

4.5.2 Integration Over Chains

With Theorem 4.5.2 in mind, we will now restrict our attention to the smooth singular homology of $\mathbf{M}$. As with the Hausdorff case, we may define a notion of *integration over chains*, defined by pulling back differential forms on $\mathbf{M}$ to the simplex Δ_q, and then evaluating the integral there. Given a smooth singular chain $c = \sum_{\alpha=1}^{m} c_\alpha \sigma_\alpha$ and a differential form ω on $\mathbf{M}$, we define the integral of ω over c as:

$$\mathscr{I}(\omega, c) := \sum_{\alpha=1}^{m} c_\alpha \int_{\Delta_q} \sigma_\alpha^* \omega. \tag{4.65}$$

In Section 4.3, we saw that Stoke's theorem will generally not hold when integrating forms on $\mathbf{M}$. However, integration over chains circumvents this issue by evaluating the integral over the simplex Δ_q, interpreted as a submanifold of $\mathbb{R}^q$. In particular, the Hausdorff version of Stoke's theorem applies (cf [65][Thm. 18.12]), and we may conclude the following.

Lemma 4.5.3. *If c is a smooth singular q-chain and ω is a $(p-1)$-form on $\mathbf{M}$, then*

$$\int_c d\omega = \int_{\partial c} \omega. \tag{4.66}$$

As with the Hausdorff case, it can be shown that the map $\mathscr{I}$ descends to cohomology. We will hereafter interpret integration as a map

$$\mathscr{I} : H^p(\mathbf{M}) \to H^p_\infty(\mathbf{M};\mathbb{R}), \tag{4.67}$$

where the image is the smooth singular cohomology, interpreted as the canonical dual space of $H_p^\infty(\mathbf{M})$.

4.5.3 The de Rham Theorem

We will now combine the results of Section 4.4 with our discussion of smooth singular homology to construct a non-Hausdorff version of de Rham's theorem. The full argument is detailed below.

Theorem 4.5.4. *Suppose that $\mathbf{M}$ is a non-Hausdorff manifold satisfying the criteria of 4.2.7 and Theorem 4.4.2, and suppose furthermore that all unions of the sets M_{ij} are regular-open sets. Then the integral map $\mathscr{I} : H^p_{dR}(\mathbf{M}) \to H^p_\infty(\mathbf{M};\mathbb{R})$ is an isomorphism.*

Proof. Our argument will be similar in structure to that of Theorem 4.5.2, that is, we will proceed by induction on the size of the indexing set I. Suppose first that $\mathbf{M}$ is a colimit of two Hausdorff manifolds M_1 and M_2. According to our discussion in Section 4.4, we may construct a Mayer-Vietoris sequence for the de Rham cohomology of $\mathbf{M}$ by using the cohomologies of the M_i, and the region M_{12}. Following [65][§18.4], we note that the descended integration maps $\mathscr{I}$ commute with smooth maps and the connecting homomorphisms of de Rham and smooth singular cohomologies. As such, we have the following commutative diagram.

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^{q-1}_{dR}(M_1)\oplus H^{q-1}_{dR}(M_2) & \longrightarrow & H^{q-1}_{dR}(M_{12}) & \longrightarrow & H^{q}_{dR}(\mathbf{M}) & \longrightarrow & H^{q}_{dR}(M_1)\oplus H^{q}_{dR}(M_2) & \longrightarrow & H^{q}_{dR}(M_{12}) & \longrightarrow \cdots \\
& \downarrow \mathscr{I} & & \downarrow \mathscr{I} & & \downarrow \mathscr{I} & & \downarrow \mathscr{I} & & \downarrow \mathscr{I} & \\
\cdots \longrightarrow & H^{q-1}_{\infty}(M_1;\mathbb{R})\oplus H^{q-1}_{\infty}(M_2;\mathbb{R}) & \longrightarrow & H^{q-1}_{\infty}(M_{12};\mathbb{R}) & \longrightarrow & H^{q}_{\infty}(\mathbf{M};\mathbb{R}) & \longrightarrow & H^{q}_{\infty}(M_1;\mathbb{R})\oplus H^{q}_{\infty}(M_2;\mathbb{R}) & \longrightarrow & H^{q}_{\infty}(M_{12};\mathbb{R}) & \longrightarrow \cdots
\end{array} \tag{4.68}$$

We may then apply the Hausdorff version of de Rham's theorem to all of the Hausdorff columns of this commutative diagram. It then follows from the Five Lemma that the map $\mathscr{I} : H^q_{dR}(\mathbf{M}) \to H^q_{\infty}(\mathbf{M};\mathbb{R})$ is an isomorphism.

Suppose now that the map $\mathscr{I}$ is an isomorphism for all non-Hausdorff manifolds constructed as a colimit of n-many Hausdorff manifolds M_i. Let $\mathbf{M}$ be a non-Hausdorff manifold with indexing set of size $(n+1)$. Similarly to Theorem 4.5.2, we will consider the inductive colimit $\mathbf{M} \cong \mathbf{N} \cup_F M_{n+1}$. As with the binary case, we again have a Mayer-Vietoris sequence for the de Rham and smooth singular cohomologies of $\mathbf{M}$, this time using the open cover $\{\mathbf{N}, M_{n+1}\}$.[5] Again by Theorem 18.12 of [65], the descended integral map $\mathscr{I}$ commutes with pullbacks and connecting homomorphisms, so we may proceed as in the binary case and interpret $\mathscr{I}$ as a chain map between the two Mayer-Vietoris sequences. The result then follows from an application of the induction hypothesis applied to $\mathbf{N}$ and A, together with the Five Lemma. □

Using the above argument, together with Theorem 4.5.2, we obtain the non-Hausdorff de Rham's theorem as an immediate corollary.

The Line with Two Origins

In the de Rham theorem of 4.5.4 we assumed that our non-Hausdorff manifolds were formed according to the assumptions of 4.4.2. We will now illustrate the necessity of these assumptions with an example. Consider the line with two origins, constructed from two copies of $\mathbb{R}$ glued along the open subset $(-\infty,0)\cup(0,\infty)$ as in Figure 4.1. Observe that the open set M_{12} is not regular open in $\mathbb{R}$. In particular, this means that a Mayer-Vietoris sequence for the de Rham cohomology will take the form

$$0 \to \mathbb{R} \to \mathbb{R}\oplus\mathbb{R} \to \mathbb{R} \to H^1_{dR}(\mathbf{M}) \to 0 \to 0 \tag{4.69}$$

where here in the third term we have used that $\overline{M_{12}} = \mathbb{R}$. Exactness of this sequence ensures that the $H^1_{dR}(\mathbf{M}) = 0$.

On the other hand, we may compute the singular homology for this space by alternate means. By inspection it should be clear that the first singular homology of $\mathbf{M}$ should carry some extra term, since $H^S_1(\mathbf{M})$ is the abelianization of the first fundamental group of $\mathbf{M}$, and here there are non-contractible loops that wrap around the two copies of the origin. Alternatively, we may use a Mayer-Vietoris sequence for singular homology and then dualize it to

[5]Technically, the Mayer-Vietoris sequence that we will obtain will have terms of the form $H^q(\overline{A})$, however, due to our assumption that A is regular-open, we may replace such terms with $H^q(\overline{A})$.

express the group $H^1_S(\mathbf{M},\mathbb{R})$. We do not need to use the boundary of M_{12} in this sequence, so the simplicial cohomology of $\mathbf{M}$ may be described with the following long exact sequence

$$0 \to \mathbb{R} \to \mathbb{R}\oplus\mathbb{R} \to \mathbb{R}\oplus\mathbb{R} \to H^1_S(\mathbf{M},\mathbb{R}) \to 0 \to 0 \tag{4.70}$$

where we observe an extra copy of $\mathbb{R}$ in the third term, due to the two connected components of M_{12}. By exactness, it follows that $H^1_S(\mathbf{M},\mathbb{R}) = \mathbb{R}$, which differs from the associated de Rham cohomology.

Interestingly, the line with two origins may also serve as a counterexample for the homotopy invariance of non-Hausdorff de Rham cohomology. Indeed, we may consider a manifold $\mathbf{N}$ in which the two copies of the real line are glued along the sets $B = (-\infty,-1)\cup(1,\infty)$. This will yield a non-Hausdorff manifold that is homotopy equivalent to $\mathbf{M}$, yet is constructed by gluing along regular open sets. As such, the manifold $\mathbf{N}$ falls under the scope of the non-Hausdorff de Rham theorem, from which we may deduce that $H^1_{dR}(\mathbf{N}) = \mathbb{R} \neq 0 = H^1_{dR}(\mathbf{M})$.

4.6 The Gauss-Bonnet Theorem

For a compact Hausdorff surface Σ with boundary, the Gauss-Bonnet theorem relates the total scalar curvature to the Euler characteristic via the equality

$$2\pi\chi(\Sigma) = \frac{1}{2}\int_\Sigma R dA + \int_{\partial\Sigma} \kappa d\gamma, \tag{4.71}$$

where here the latter term computes the geodesic curvature of the boundary of Σ. In this section we will prove a non-Hausdorff version of this statement. In order to emphasise that we are working with surfaces, throughout this section we will denote our non-Hausdorff manifold by $\mathbf{\Sigma}$, with its Hausdorff submanifolds denoted similarly. We will see that in a manner similar to that of Theorem 4.5.3, even if the manifold $\mathbf{\Sigma}$ has no boundary, the total curvature of $\mathbf{\Sigma}$ will take additional contributions from the internal Hausdorff-violating submanifolds. In order to properly understand this theorem, we will start with a brief discussion of curvature in non-Hausdorff manifolds.

4.6.1 Scalar Curvature for non-Hausdorff Manifolds

According to Theorem 4.2.12 the sections of the tensor bundle $T^{(2,0)}\mathbf{\Sigma}$ are a fibre product of the spaces $\Gamma(T^{(2,0)}\Sigma_i)$. As such, we may describe a Riemmanian metric on $\mathbf{\Sigma}$ using a collection of metrics g_i defined on each Σ_i, such that the isometry condition $g_i = f^*_{ij}\circ g_j$ holds on Σ_{ij} for all i, j in I. Explicitly, we have the following result, paraphrased from [85].

Lemma 4.6.1. *Let $\mathbf{\Sigma}$ be a non-Hausdorff manifolds built according to 4.2.7. If each Σ_i is equipped with a Riemannian metric g_i such that $f_{ij} : \Sigma_{ij} \to M_j$ is an isometric embedding, then $\mathbf{\Sigma}$ admits a Riemannian metric $\mathbf{g}$ such that the canonical maps $\phi_i : \Sigma_i \to \mathbf{\Sigma}$ are open isometric embeddings.*

In coordinate-free notation, a connection on $\mathbf{\Sigma}$ may be defined as an operator

$$\nabla : \mathfrak{X}(\mathbf{\Sigma}) \times \mathfrak{X}(\mathbf{\Sigma}) \to \mathfrak{X}(\mathbf{\Sigma}),\ (\mathbf{X},\mathbf{Y}) \mapsto \nabla_{\mathbf{X}}\mathbf{Y} \tag{4.72}$$

that is $C^\infty(\Sigma)$-linear in the first argument, and satisfies the Liebniz property

$$\nabla_{\mathbf{X}}(\mathbf{rY}) = \mathbf{r}\nabla_{\mathbf{X}}(\mathbf{Y}) + \mathbf{X}(\mathbf{r})\mathbf{Y} \tag{4.73}$$

in the second argument. Given a metric $\mathbf{g}$ on Σ, we may define the Levi-Civita connection as:

$$\boldsymbol{\nabla}(\mathbf{X},\mathbf{Y})([x,i]) := \nabla^i(X_i,Y_i)(x), \tag{4.74}$$

where ∇^i is the Levi-Civita connection for each (Σ_i, g_i). Note that the isometries $\overline{f_{ij}}$ together with the uniqueness of Hausdorff Levi-Civita connections ensure that the above definition is well-defined and unique. The following result summarises some relevant properties of $\boldsymbol{\nabla}$.

Lemma 4.6.2. *Let $\boldsymbol{\nabla}$ be the connection defined above. Then*

1. *$\boldsymbol{\nabla}$ is a local operator*
2. *$\boldsymbol{\nabla}$ is compatible with the metric $\mathbf{g}$*
3. *$\boldsymbol{\nabla}$ is torsion-free.*

Proof. Semi-locality of $\boldsymbol{\nabla}$ is established by construction, and locality follows from an argument similar to the exterior derivative (cf Theorem 4.3.3). For readability we will illustrate the latter two properties for the case that $I = 2$, since the general case simply requires more indices. For compatibility with the metric, we have

$$\begin{aligned}
\mathbf{X}(\mathbf{g}(\mathbf{Y},\mathbf{Z})) &= \big(X_1(g_1(Y_1,Z_1)), X_2(g_2(Y_2,Z_2))\big) && (4.75)\\
&= \big(g_1(\nabla^1_{X_1}Y_1,Z_1) + g_1(Y_1,\nabla^1_{X_1}Z_1), g_2(\nabla^2_{X_2}Y_2,Z_2) + g_2(Y_2,\nabla^2_{X_2}Z_2)\big) && (4.76)\\
&= \big(g_1(\nabla^1_{X_1}Y_1,Z_1), g_2(\nabla^2_{X_2}Y_2,Z_2)\big) + \big(g_1(Y_1,\nabla^1_{X_1}Z_1), g_2(Y_2,\nabla^2_{X_2}Z_2)\big) && (4.77)\\
&= \mathbf{g}(\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Y},\mathbf{Z}) + \mathbf{g}(\mathbf{Y},\boldsymbol{\nabla}_{\mathbf{X}}\mathbf{Z}), && (4.78)
\end{aligned}$$

where in the second line we have used the metric compatibility of the Hausdorff Levi-Civita connections. To see that $\boldsymbol{\nabla}$ is torsion-free, we may use the expression for the Lie bracket

$$[\mathbf{X},\mathbf{Y}] = ([X_1,Y_1],[X_2,Y_2]) \tag{4.79}$$

derived in Section 4.3.1. We may then use the Hausdorff Levi-Civita connections to deduce that

$$\begin{aligned}
\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} &= (\nabla^1_{X_1}Y_1 - \nabla^1_{Y_1}X_1, \nabla^2_{X_2}Y_2 - \nabla^2_{Y_2}X_2) && (4.80)\\
&= ([X_1,Y_1],[X_2,Y_2]) && (4.81)\\
&= [\mathbf{X},\mathbf{Y}], && (4.82)
\end{aligned}$$

as required. □

Now that we have established a Levi-Civita connection for Σ, we may define the Riemann curvature tensor as in the Hausdorff case. Given the expressions for $\boldsymbol{\nabla}$ and the Lie derivative on Σ, we see that the Riemann curvature tensor for Σ will again be a fibred product of the same tensors defined on each Σ_i. Again, this will be a local operator since both $\boldsymbol{\nabla}$

and the Lie derivative are. Since the maps f_{ij} are isometries and the connection ∇ is local, we may compute the Ricci tensor, and consequently the Ricci scalar $\mathbf{R}$ on Σ will be locally equivalent to the same value computed within each Hausdorff manifold Σ_i.

By a similar line of reasoning, the volume form of Σ can be described according to Lemma 4.3.4. We may thus write the total scalar curvature of Σ as

$$\int_{\Sigma} \mathbf{R} dA = \int_{\Sigma} \mathbf{R} \sqrt{\det(g)} d^2 x. \tag{4.83}$$

Using Theorem 4.3.8, we may always decompose this integral into a sub-additive collection of scalar curvatures defined on each Σ_i and their various intersections.

4.6.2 The Euler Characteristic

Before getting to a proof of the Gauss-Bonnet theorem, we will first describe the Euler characteristic of the manifold Σ. Since simplicial homology is not available for non-Hausdorff manifolds, we will instead define the Euler characteristic as

$$\chi(\Sigma) := \sum_{q=0}^{d} (-1)^q \, b_q(\Sigma), \tag{4.84}$$

where we define these Betti numbers via singular homology groups. According to this definition, the Euler characteristic will satisfy the following sub-additive property.

Theorem 4.6.3. *The Euler characteristic of* Σ *equals*

$$\chi(\Sigma) = \sum_{p \in I} \sum_{i_1, \ldots, i_p \in I} (-1)^{p+1} \chi(\Sigma_{i_1, \ldots, i_p}). \tag{4.85}$$

Proof. We argue by induction on the size of the indexing set I. Suppose first that $I = 2$, that is, that Σ is a binary colimit. According to the results of Section 4.5.1, we may construct a long exact sequence that relates the spaces $H^q(\Sigma)$ to the cohomologies of Σ_1, Σ_2 and Σ_{12}. Since this sequence is a long exact sequence of vector spaces, the alternating sum of the dimensions of the entries always equals zero. Spelling this out, we have:

$$0 = \sum_q (-1)^q \big(b_q(\Sigma) - b_q(\Sigma_1) - b_q(\Sigma_2) + b_q(\Sigma_{12})\big) \tag{4.86}$$

$$= \sum_q (-1)^q b_q(\Sigma) - \sum_q (-1)^q \big(b_q(\Sigma_1) + b_q(\Sigma_2)\big) + \sum_q (-1)^q b_q(\Sigma_{12}) \tag{4.87}$$

$$= \chi(\Sigma) - \chi(\Sigma_1) - \chi(\Sigma_2) + \chi(\Sigma_{12}) \tag{4.88}$$

from which the equality follows.

Suppose now that the hypothesis holds for all colimits of size $I = n$, and let Σ have indexing set size $n+1$. As with the proofs of Section 4.5, we may view Σ as an inductive colimit $\tilde{\Sigma} \cup_F \Sigma_{n+1}$. The cohomologies of these spaces can be arranged into the following

Mayer-Vietoris sequence

$$\cdots \longrightarrow H^q(\Sigma) \xrightarrow{\varphi^*} H^q(\tilde{\Sigma}) \oplus H^q(\Sigma_{n+1}) \xrightarrow{\iota_A^* - F^*} H^q(A) \longrightarrow H^{q-1}(\Sigma) \xrightarrow{\varphi^*} H^{q-1}(\tilde{\Sigma}) \oplus H^{q-1}(\Sigma_{n+1}) \xrightarrow{\iota_A^* - F^*} H^{q-1}(A) \longrightarrow \cdots \tag{4.89}$$

where here A is the union of the subspaces Σ_{in+1}. Since this a long exact sequence of vector spaces, we may again use that the alternating sum of dimensions vanish to deduce that:

$$0 = \sum_q (-1)^q \big(b_q(\Sigma) - b_q(\tilde{\Sigma}) - b_q(\Sigma_{n+1}) + b_q(A)|\big) \tag{4.90}$$

$$= \sum_q (-1)^q b_q(\Sigma) - \sum_q (-1)^q \big(b_q(\tilde{\Sigma}) + b_q(\Sigma_{n+1})\big) + \sum_q (-1)^q b_q(A) \tag{4.91}$$

$$= \chi(\Sigma) - \chi(\tilde{\Sigma}) - \chi(\Sigma_{n+1}) + \chi(A). \tag{4.92}$$

We may then apply the inductive hypothesis to both $\tilde{\Sigma}$ and $A = \bigcup_i \Sigma_{in+1}$ to yield the following

$$\chi(\Sigma) = \chi(\tilde{\Sigma}) + \chi(\Sigma_{n+1}) - \chi(A) \tag{4.93}$$

$$= \left(\sum_{k \leq n} \sum_{i_1,\ldots,i_k \in I} (-1)^{k+1} \chi(\Sigma_{i_1,\ldots,i_k}) \right) + \chi(\Sigma_{n+1}) \tag{4.94}$$

$$- \left(\sum_{k \leq n} \sum_{i_1,\ldots,i_k \in I} (-1)^{k+1} \chi(\Sigma_{i_1,\ldots,i_k,n+1}) \right) \tag{4.95}$$

$$= \sum_{k \leq n+1} \sum_{i_1,\ldots,i_k \in I} (-1)^{k+1} \chi(\Sigma_{i_1,\ldots,i_k}), \tag{4.96}$$

as required. □

4.6.3 Proof of the Theorem

We will prove the Gauss-Bonnet theorem in a similar manner to the non-Hausdorff Stoke's theorem of Section 4.3. We start with the binary version of the theorem.

Lemma 4.6.4. *Suppose that Σ is a non-Hausdorff manifold that can be constructed as a binary colimit of Hausdorff manifolds Σ_1 and Σ_2 according to 4.2.7 and 4.4.2. If Σ_1 and Σ_2 are compact and without boundary, then:*

$$2\pi\chi(\Sigma) = \frac{1}{2}\int_{\Sigma} \boldsymbol{R} dA - \int_{\partial\Sigma_{12}} \kappa d\gamma. \tag{4.97}$$

Proof. According to our discussion of curvature in Section 4.6.1, we may decompose the total scalar curvature of Σ into the scalar curvatures of the Σ_i by using Theorem 4.3.7. Using the Hausdorff Gauss-Bonnet theorem where possible, we have the following string of equivalences:

$$2\pi\chi(\Sigma) = 2\pi\left(\chi(\Sigma_1) + \chi(\Sigma_2) - \chi(\overline{\Sigma_{12}})\right) \tag{4.98}$$
$$= \frac{1}{2}\int_{\Sigma_1} RdA + \frac{1}{2}\int_{\Sigma_2} RdA - \frac{1}{2}\int_{\overline{\Sigma_{12}}} RdA - \int_{\partial\Sigma_{12}} \kappa d\gamma \tag{4.99}$$
$$= \int_{\boldsymbol{\Sigma}} RdA - \int_{\partial\Sigma_{12}} \kappa d\gamma, \tag{4.100}$$

where here we have used the fact that $\overline{\Sigma_{12}}$ is a manifold with boundary. □

By an inductive argument, we may obtain the following general version of the Gauss-Bonnet theorem for non-Hausdorff manifolds. Again for simplicity we restrict our attention to a colimit in which the Σ_i have no boundary.

Theorem 4.6.5. *Suppose that Σ is a non-Hausdorff manifold that can be constructed as a colimit of n-many Hausdorff manifolds Σ_i according to Remark 4.2.7. If each Σ_i is compact and without boundary, and each Σ_{ij} and their unions are regular-open sets, then*

$$2\pi\chi(\Sigma) = \frac{1}{2}\int_{\boldsymbol{\Sigma}} \boldsymbol{R}dA + \sum_{p=2}^{n}(-1)^{p+1}\sum_{i_1,\cdots,i_p\in I}\left(\int_{\partial M_{i_1\cdots i_p}} \kappa d\gamma\right). \tag{4.101}$$

Proof. Using Theorems 4.6.3 and 4.3.8, together with the assumption that each Σ_{ij} is regular open, we have that:

$$2\pi\chi(\Sigma) = 2\pi\left(\sum_{p=1}^{n}(-1)^{p+1}\sum_{i_1,\ldots,i_p\in I}\chi(\overline{\Sigma_{i_1,\ldots,i_p}})\right) \tag{4.102}$$
$$= \sum_{p=1}^{n}(-1)^{p+1}\sum_{i_1,\ldots,i_p\in I}2\pi\chi(\overline{\Sigma_{i_1,\ldots,i_p}}) \tag{4.103}$$
$$= \sum_{p=1}^{n}(-1)^{p+1}\sum_{i_1,\ldots,i_p\in I}\left(\frac{1}{2}\int_{\overline{\Sigma_{i_1,\ldots,i_p}}} RdA + \int_{\partial\Sigma_{i_1,\ldots,i_p}} \kappa d\gamma\right) \tag{4.104}$$
$$= \sum_{p=1}^{n}(-1)^{p+1}\sum_{i_1,\ldots,i_p\in I}\left(\frac{1}{2}\int_{\overline{\Sigma_{i_1,\ldots,i_p}}} RdA\right) + \sum_{p=2}^{n}(-1)^{p+1}\sum_{i_1,\ldots,i_p\in I}\left(\int_{\partial\Sigma_{i_1,\ldots,i_p}} \kappa d\gamma\right) \tag{4.105}$$
$$= \frac{1}{2}\int_{\boldsymbol{\Sigma}} \mathbf{R}dA + \sum_{p=2}^{n}(-1)^{p+1}\sum_{i_1,\ldots,i_p\in I}\left(\int_{\partial\Sigma_{i_1,\ldots,i_p}} \kappa d\gamma\right), \tag{4.106}$$

as required. □

4.7 Conclusion

The goal of this chapter was to describe some basic features of de Rham cohomology for non-Hausdorff manifolds. Using the colimit description of **M** already available to us, we constructed vector fields, differential forms and their integrals in Section 4.3. Naturally, we related these to the various Hausdorff formulas for the submanifolds M_i. Of central interest at this stage was the subadditivity of integration, which required a compactification of the pairwise intersections M_{ij}. In turn, this caused Stoke's theorem to fail in a controlled manner.

De Rham cohomology itself was described in Section 4.4. Through an assumption of regular-open sets satisfying certain intersection properties, we saw that the de Rham cohomology for our non-Hausdorff manifold **M** can be related to the cohomologies of the M_i using Mayer-Vietoris sequences. In the binary case, a long exact sequence emerged, and the more-general colimit birthed a Čech-de Rham bicomplex whose cohomology coincided with de Rham by standard means.

When discussing singular homology in Section 4.5, we saw that we could derive an equivalence with the smooth theory without appealing to Whitney's embedding theorem. Instead, the equivalence followed from some derivations of Mayer-Vietoris sequence for smooth singular homology, arising from first principles. We then proved de Rham's theorem by a similar approach, this time by appealing to the standard pairing via integration over chains.

Finally, we combined the results of Sections 2,3 and 4 into a proof of a non-Hausdorff Gauss Bonnet theorem. After a brief discussion of curvature in 2-manifolds, we derived a convenient subadditivity property of the Euler Characteristic. In combination with Theorem 4.3.8, we then proved the desired Gauss-Bonnet theorem. Importantly, the compactifications of the sets M_{ij} within our definition of integration forced us to invoke the Gauss-Bonnet theorem for manifolds with boundary. In this context, these boundaries were "internal", in the sense that they were characterised by the Hausdorff-violating submanifolds sitting inside **M**. As a final result, we proved this in further generality.

Finally, we finish the chapter with a few comments regarding further work. As the astute reader may notice, throughout this chapter we discuss neither compactly-supported forms nor their de Rham cohomology. There is at least some discussion of these topics within the literature, however it appears that their notion of smooth function differs from that used in this chapter. Nonetheless, it will be interesting to compute the Poincaré Duality of our non-Haudorff manifolds, and compare it to that found in [23] and [60]. As well as this, it will be interesting to see whether a Čech-de Rham isomorphism can be approached in a similar manner to that of Theorem 4.5.4. According to [85], there is a Mayer-Vietoris sequence that relates the Čech cohomology of **M** to the cohomologies of M_1 and M_2. Although an explicit Čech-de Rham isomorphism cannot be constructed on **M** due to the non-existence of good open covers, it nonetheless seems reasonable to construct homomorphisms between the Hausdorff spaces in both bicomplexes, and then to establish an equivalence via arguments involving the associated spectral sequences. This idea can be captured within the following.

Conjecture 4.7.1. *Let **M** be a non-Hausdorff manifold constructed according to 4.2.7 and 4.4.2. Suppose furthermore that each gluing region M_{ij} is regular-open. Then*

$$\check{H}^q(\boldsymbol{M},\mathbb{R}) \cong H^q_{dR}(\boldsymbol{M}) \cong H^q_S(\boldsymbol{M},\mathbb{R}) \tag{4.107}$$

for all q in $\mathbb{N}$.

Summary of Chapter

Throughout Chapter 3 we saw that non-Hausdorff manifolds may admit smooth structures, and their vector bundles may be described as colimits of Hausdorff ones. Those results suggest that non-Hausdorff manifolds have lots of the mathematical structure required for physical analyses. In this chapter, we refined this observation by studying differential forms and their integral properties. Importantly, we derived descriptions of non-Hausdorff differential forms, and derived expressions for their integration. Our derivations took particular care, as standard Hausdorff versions of these descriptions employ partitions of unity throughout. Nonetheless, we confirmed the existence of a well-defined integral without appealing to arbitrarily-existent partitions of unity.

As a mathematical curiosity, we then used our description of differential forms to derive a non-Hausdorff de Rham cohomology. This cohomology was then related to the associated cohomologies of the Hausdorff submanifolds via a certain type of Mayer-Vietoris sequence. We also proved a non-Hausdorff version of de Rham's theorem, this time proving an equivalence with smooth singular homology. Finally, we derived some curvature expressions and proved a non-Hausdorff version of the Gauss-Bonnet theorem in Euclidean signature. Interestingly, due to the previously-derived integral properties, we saw that the non-Hausdorff Gauss-Bonnet theorem requires and additional term that computes the geodesic curvatures of the extra Hausdorff-violating submanifolds sitting inside the overall non-Hausdorff space.

Throughout chapters 2,3 and 4 we have confirmed the geometric expressivity of non-Hausdorff manifolds, and have confirmed their potential consideration within physics.

Chapter 5

Non-Hausdorff Spacetimes within a Path Integral for 2d Gravity

This chapter is published under the name *Topology Change from Pointlike Sources*, Physical Review D, 110(6):064026, 2024 [80]. It was written in collaboration with my supervisor Prof. Yasha Neiman.

Throughout the our work thusfar, we have investigated the topological and geometric properties of non-Hausdorff manifolds. In this chapter we will use this understanding to properly phrase Penrose's idea of topology-changing spacetimes. Our focus will be on the two-dimensional setting, and we will recreate the analysis of [67] for the Trousers space.

In order to evaluate a non-Hausdorff gravitational action in two dimensions, we first modify the Gauss-Bonnet theorem of Section 5.6 to the Lorentzian case. As with the Riemannian version, this theorem is proved by collating the Hausdorff version of the theorem together in a particular way. We will do this, and along the way will review and make minor improvements to the Hausdorff Gauss-Bonnet theorem in Lorentzian signature.

Due to our construction, the gravitational action for non-Hausdorff transitions will take extra contributions coming from the Hausdorff-violating points. These contributions will amount to computing turning angles between adjacent nullrays. A sign convention is needed to determine the imaginary part of the angle, and the choice of sign will determine whether a transition is suppressed or enhanced in the exponent of the path integral. With this in mind, we successfully derive the sign convention that will entail suppression of the non-Hausdorff transitions, in a manner similar to that of the Trousers space.

5.1 Introduction

In this chapter we will concern ourselves with topology changing spacetimes and their transition amplitudes within a path integral for gravity. We consider perhaps the simplest non-trivial setting, that is, a transition from one circle into two.[1] The customary model for this type of topology change is the so-called *Trousers space*, which is a smooth two-dimensional manifold that traces out the splitting process, as pictured in Figure 5.1. The Trousers space has long served as the prototypical example of topology change, and has been discussed in various physical contexts [4, 12, 16, 24, 28, 29, 31, 32, 36, 48, 49, 67, 70, 97–99, 107].

Despite a broad discussion of topology change within physics, there exists an interesting gap in the literature that has remained unexplored for over half a century. When discussing time-asymmetry in [89], Penrose sketches a particular type of topology-change that is markedly different from the Trousers space. In his image, the manifold does not change its topology at a single point in time, but at a single point in *spacetime*. This means that a single point changes its topology from one connected component to two, and then this change is allowed to grow along null rays. If taken within a compact universe, this will develop into a full topology change of spacelike slices within finite time, and thus may be used to model a possible transition between S^1 and $S^1 \sqcup S^1$.

Although a potentially interesting model for topology change, Penrose correctly identifies an important technical issue in his spacetimes: if one wants the pointlike splitting to remain a manifold, then models such as those pictured in Figure 5.2 are necessarily non-Hausdorff. At that time it was not clear how expressive a non-Hausdorff differential geometry could be, and thus after musing for some time and deviating wildly from the original scope of his paper, he finishes his discussion with the now-famous quote:[2]

> I have, in any case, strayed far too long from my avowed conventionality in this discussion, and no new insights as to the origin of time-asymmetry have, in any case, been obtained. I must therefore return firmly to sanity by repeating to myself three times: 'spacetime is a *Hausdorff* differentiable manifold; spacetime is a *Hausdorff*...'

In the past half century a theory of non-Hausdorff manifolds has emerged [17, 23, 37, 83–85], and these discussions have occasionally extended into non-Hausdorff spacetimes [46, 47, 53, 68, 72, 77]. In this chapter we will leverage recent mathematical developments in order to take seriously Penrose's splitting spacetimes. As a guiding reference, we will mimic the standard analysis of the Trousers space found in [67, 100] by defining and evaluating the gravitational action for a non-Hausdorff version of the Trousers space. We will then compare transition amplitudes for the Hausdorff and non-Hausdorff Trousers spaces, leading to the eventual conclusion that the latter admits a similar imaginary-strength action that yields a

[1] Our reason for this will become clear throughout the chapter: we will consider some unconventional types of topology change, and their novelty should already manifest in the two-dimensional regime where gravity is famously topological.

[2] We have decided to include it in full, since the latter half of this quote is often misquoted as an argument against non-Hausdorff manifolds.

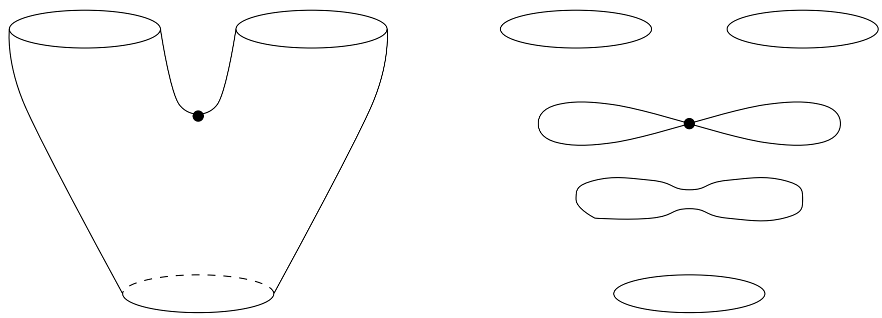


Figure 5.1: The Trousers space, together with a sample of its spacelike slices.

dampening in the resulting path integral.

In the remainder of this introduction we will briefly review some details of Hausdorff topology change and provide an informal discussion of non-Hausdorff topology, before outlining the chapter in detail.

5.1.1 Topology Change and Path Integrals

An n-dimensional spacetime M with boundary $\partial M = \Sigma_1 \sqcup \Sigma_2$ exhibits *topology change* whenever the initial boundary Σ_1 and the final boundary Σ_2 are not homeomorphic. In such a model, the initial data Σ_1 is assumed to smoothly evolve and change its topology through time. Mathematically speaking, such interpolating manifolds are known as *cobordisms*, and have been studied extensively in the literature.

Following [28, 51, 99], in a naive sum-over-histories approach to quantum gravity we consider a path integral whose sum may include multiple distinct geometries and topologies. For topology change in Lorentzian signature, the transition amplitude between a pair of non-homeomorphic spacelike hypersurfaces (Σ_1, h_1) and (Σ_2, h_2) could be represented symbolically as

$$\langle \Sigma_1, h_1 | \Sigma_2, h_2 \rangle = \sum_M \int \mathscr{D}[g] \exp\{i\mathscr{S}(M,g)\},^3 \tag{5.1}$$

where here we sum over all physically-reasonable Lorentz cobordisms interpolating between Σ_1 and Σ_2. With a naive prescription such as the above, we are met with an immediate principled question: which interpolating manifolds should we sum over in the domain of the path integral?

Before considering such an question, it makes sense to determine whether or not the sum is non-empty in the first place. In general dimensions, there are several known topological obstructions to the existence of Lorentz cobordisms, which are usually articulated as relationships between Σ_1 and Σ_2. It is well-known that any pair of $(n-1)$-dimensional manifolds will permit a smooth n-dimensional cobordism provided that they are related by a procedure known as *Morse surgery* [73, 91]. Moreover, any smooth compact manifold admits a well-defined Lorentzian metric provided that it admits a globally non-vanishing vector field [86].

[3] Here we assume $\hbar = 1$.

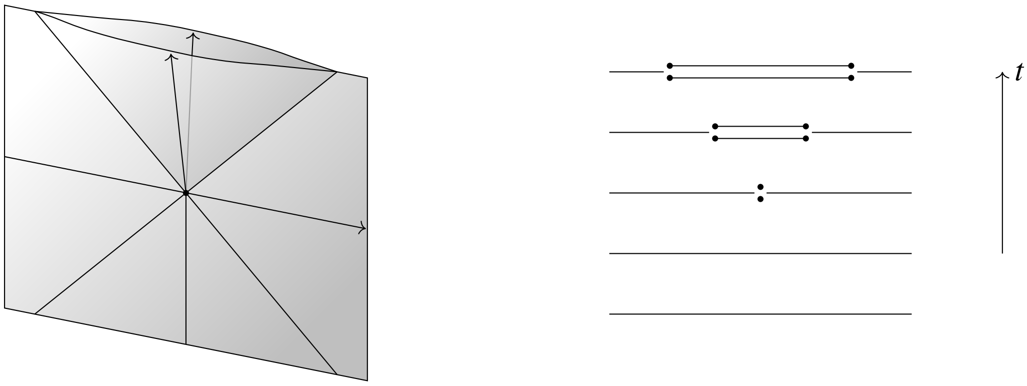


Figure 5.2: A simplification of Penrose's non-Hausdorff topology changing spacetime (left), together with a sample of its level sets. Although impossible to depict in ordinary Euclidean space, here there are two copies of the origin superimposed on top of each other, together with two copies of the future null cone.

This pair of observations may be taken in combination to explicitly describe Lorentz cobordisms via Morse theory [1, 29, 31, 110, 111]. Note that in particular, Lorentz cobordisms exist between any pair of manifolds with $dim(\Sigma_i) \leq 3$.

After verifying the existence of Lorentz cobordisms, the next step might be to impose some top-down causal desiderata for the types of spacetimes we are willing to sum over. The exact specifications of this causal behaviour are still subject to debate, but at the very least it seems as though our spacetimes ought to admit a global time function, since this is implicitly used in the formation of (5.1). The existence of global time functions is known to be equivalent to the property of *stable causality* [52]. Roughly put, a spacetime is stably causal provided it contains no closed timelike curves, and neither does any small perturbation of its metric. Such spacetimes lie relatively high up in the causal hierarchy of [75, 76].

At the apex of this causal hierarchy are the *globally hyperbolic* spacetimes. These spacetimes are just about as causally well-behaved as possible, in that they always admit Cauchy surfaces. However, it can be shown that any globally hyperbolic spacetime is necessarily cylindrical, in that both its topological and smooth manifold structure are isomorphic (in the appropriate sense) to a product of the Cauchy surface with either the real line or the unit interval [10, 40]. As such, we see that topology-changing spacetimes may never be globally-hyperbolic. Nonetheless, it appears as though we may be able to include topology-changing spacetimes within (5.1) if relax our causality requirements slightly and allow for stably-causal spacetimes. However, we are then met with the following result of Geroch (adapted from [42]).

Theorem 5.1.1 (Geroch)**.** *Let M be a compact spacetime with initial boundary Σ_1 and final boundary Σ_2. If Σ_1 and Σ_2 are not homeomorphic, then M admits a closed timelike curve.*

We are thus prompted to exclude topology-changing spacetimes on causal grounds, since they are not stably causal. A possible circumvention of this issue is to relax the assumption that M admits a globally-defined Lorentzian metric. Instead, we may allow an *almost* Lorentzian manifold in which the metric is allowed to degenerate at select points in the space. Within the context of topology change, this approach seems quite reasonable, as it naturally aligns with the Morse-theoretic view of smooth cobordisms. Under this reading, we may consider what is known as a *Morse function*, which has critical points along which the topology of space will change. The gradient of this Morse function will then provide us with a vector field that vanishes only on the critical points, and this vector field may be used to define an almost-Lorentzian manifold.

Another causal requirement might be to suggest that the almost-Lorentz cobordism induces a *causal poset* structure on its lightcones. According to the causality theory of [28, 75, 76, 88], we may induce a binary *causal precedence* relation $\leq$ on any almost-Lorentzian manifold. This relation states a point p *causally precedes* a point q, written $p \leq q$ if q lies in the future lightcone of p. Causality properties may then be paraphrased as order-theoretic properties of the relation $\leq$. In particular, one may suggest including into (5.1) all almost-Lorentz cobordisms in which the binary relation $\leq$ is reflexive, transitive and antisymmetric [28]. Note that the requirement of antisymmetry excludes those spacetimes admitting closed timelike curves.

5.1.2 The Trousers Space

Topologically, the Trousers space is homeomorphic to the 3-punctured sphere, and may be seen as a cobordism from S^1 to $S^1 \sqcup S^1$. This manifold is almost-Lorentzian, in the sense that it admits a non-degenerate Lorentzian metric everywhere except at a single point (commonly called the *crotch singularity*). Away from this point, the Trousers space may be furnished with a Lorentzian metric that is locally isometric to the flat cylinder [4, 67].

It seems reasonable to suggests that one may avoid Geroch's theorem by simply removing this troublesome point and allowing the manifold to be non-compact. However, with transition amplitudes such as (5.1) in mind, it seems that we would like to preserve compactness as best as we can. An alternate resolution involves what is known as a *causal closure* construction [41]. In this method one chooses to maintain compactness, at the cost of an allowed degeneracy in the metric at the crotch singularity. A lightcone structure may still be placed at the crotch singularity, however this structure will be irregular in the sense that there will be double the amount of distinct lightcones – two future-directed and two past-directed [48]. Suggestive depictions of the Trousers space and the origin of the irregular causal structure of its crotch singularity can be found in Figure 2 of the recent paper [36].

With a causally-closed Trousers space, we may still evaluate the transition amplitude of (5.1). In two dimensions, the gravitational action in vacuum is given by the total scalar curvature of the manifold:

$$\mathscr{S}(M,g) = \frac{1}{2\kappa}\int_M R dA + \frac{1}{\kappa}\int_{\partial M} k d\gamma, \tag{5.2}$$

where here the latter term computes the geodesic curvature of the boundary and κ is the gravitational constant. An analysis of Louko and Sorkin shows that the above action may still be evaluated for the Trousers space [67]. In their work they employ an $i\varepsilon$-regularisation in which the Lorentzian metric is perturbed into a complex one in a controlled manner. Their conclusion is that the Trousers space has a δ-like curvature localised to the crotch singularity of strength $\pm 2\pi i$, with the ambiguity being controlled by the sign of the regulariser.

Alternatively, the curvature of the Trousers space may be obtained via a certain Lorentzian Gauss-Bonnet theorem [100]. We will discuss the details of the Lorentzian Gauss-Bonnet theorem and its subtleties at length in Section 5.3 of this chapter, so for now we will merely deliver a brief summary. In short: the notion of Lorentzian angle does not make sense for vectors of different signatures, again it is commonplace to employ the $i\varepsilon$-regularisation of [67] and complexify the Minkowski metric. This gives a notion of Lorentzian angle that is well-defined, yet complex-valued [5, 58, 79, 100]. The Lorentzian Gauss-Bonnet theorem then loosely states that:

$$\frac{1}{2}\int_M R dA + \int_{\partial M} k d\gamma = \mp 2\pi i \chi(M). \tag{5.3}$$

The imaginary coefficient on the right-hand-side arises from our complexification of the metric, with $\mp 2\pi i$ being the periodicity of angles around a point in flat two-dimensional Minkwoski space, and the sign ambiguity again provided by the $i\varepsilon$-regulariser. For the Trousers space the Euler characteristic equals -1, so we may conclude that its total scalar curvature equals $\pm 2\pi i$, in agreement with the prior analysis of [67].

The sign ambiguity in the $i\varepsilon$-regulariser may be resolved in several ways. It is commonly argued that the correct sign of the imaginary periodicity should be $-2\pi i$, since then

$$\exp\{i\mathscr{S}(M,g)\} = \exp\left\{i\left(\frac{-2\pi i\chi(M)}{\kappa}\right)\right\} = \exp\left\{\frac{i(+2\pi i)}{\kappa}\right\} = \exp\left\{\frac{-2\pi}{\kappa}\right\} < 1, \tag{5.4}$$

which would in turn cause the Trousers space to be suppressed relative to other spacetimes like the cylinder or the 2-disk [5, 67, 79, 100]. In this chapter we will only focus on transitions from S^1 to $S^1 \sqcup S^1$. Within our context, a similar argument is valid: creating some more complicated Trousers-like transition between circles by adding extra genera to the bulk will always decrease the Euler characteristic. The Lorentzian Gauss-Bonnet theorem roughly stated above then implies that adding more and more genera to the Trousers space will continually increase the imaginary part of the action, and thus the transition amplitudes of (5.1) will become exponentially small. In contrast, if we were to choose the other sign convention, then we are left with a periodicity of $+2\pi i$ in which higher genus Trousers-like spaces would be exponentially enhanced. In this sense, the sign convention advocated in [5, 67, 79, 100] indeed appears to be the correct one.

5.1.3 A Primer on non-Hausdorff Topologies

The Hausdorff property states that any pair of distinct points in a topological space may be separated by disjoint open sets. Conversely, a topological space is called *non-Hausdorff* whenever there exists a pair of points whose open neighbourhoods always intersect. Hausdorffness is usually assumed in the definition of a manifold, however there still exist non-Hausdorff locally-Euclidean spaces. For example, the right-hand-side of Figure 5.3 depicts a simple one dimensional non-Hausdorff manifold, commonly called the *branched line*. In this space there are two copies of the origin, and the topology is defined to be locally-equivalent

Figure 5.3: Hausdorff (left) and non-Hausdorff (right) topology change in one dimension. The Hausdorff topology change is necessarily singular, whereas the non-Hausdorff model blows up the singularity into two distinct points and adjusts their separability in order to ensure that the space remains locally-Euclidean.

to the real line. One can see that any pair of open intervals around the Hausdorff-violating pair of origins will necessarily intersect throughout the negative numbers.

In the manifold setting, Hausdorff violation may be seen in many different ways. Perhaps the most instructive is via the non-uniqueness of limits: it is well-known that any convergent sequence in a manifold has a unique limit, provided that manifold is Hausdorff. In the non-Hausdorff setting this is no longer true – in fact, a pair of points will violate the Hausdorff property if and only if they can be realised as distinct limits of the same sequence [84].

Usually we include the Hausdorff property in the definition of a manifold for technical convenience. In particular, it can be shown that any open cover of a Hausdorff manifold admits a partition of unity subordinate to it [65]. These arbitrarily-existent partitions of unity are used frequently in order to construct various geometric structures of interest. In the non-Hausdorff case, such partitions of unity do not exist in full generality, and thus the various enjoyable features of Hausdorff manifolds may appear to be in jeopardy. Put differently: without the Hausdorff property we do not have access to the usual constructive techniques, and *prima facie* it is not clear whether non-Hausdorff manifolds are as expressive as their Hausdorff counterparts.

Despite this issue with partitions of unity, it is nonetheless possible to describe a differential geometry of non-Hausdorff manifolds. Underpinning this study is the observation that non-Hausdorff manifolds may be constructed by gluing together ordinary Hausdorff manifolds along open sets [45, 47, 68, 84]. Intuitively, if one glues together Hausdorff manifolds along an open subset but leaves the boundary of this subset unidentified, then this boundary may become Hausdorff-violating in the quotient space. As an example: we may realise the branched line of Figure 5.3 by gluing together two copies of the real line along the subset $A := (-\infty, 0)$. In the resulting quotient space, any sequence of negative numbers that would ordinarily converge to the origin will now have two distinct limits, thereby realising Hausdorff-violation.

5.1.4 Outline of Chapter

In Section 5.2 we will provide a formal overview of non-Hausdorff differential geometry. We will start with matters topological, and then move on to smooth structures, bundles, integrals and eventually curvature. Underpinning our discussion is the aforementioned gluing concept – essentially all of these geometric structures may be defined on non-Hausdorff spaces by first defining them on Hausdorff submanifolds, and then by imposing some consistency conditions on overlapping submanifolds. The non-Hausdorff manifolds that we define will be locally-isomorphic to Hausdorff ones, however their global features will differ. In particular, we will see that their notion of integration needs to include the extra Hausdorff-violating data in order to be well-defined, and this in turn will have some far-reaching consequences for the rest of the chapter.

In Section 5.3 we will discuss various extensions of the Gauss-Bonnet theorem. We start with the standard statement for Riemannian manifolds found in say [27], and we will then modify it in two orthogonal directions: firstly, we will pass from Riemannian metrics to Lorentzian metrics, and secondly, we will pass from Hausdorff surfaces to non-Hausdorff ones. The result of our discussion will be a non-Hausdorff version of the Gauss-Bonnet theorem in Lorentzian signature. Here we will see a crucial novelty – due to integration results of Section 5.2, the non-Hausdorff result will require an extra counterterm that computes the geodesic curvature of the Hausdorff-violating submanifold sitting inside the manifold. This counterterm is a sort-of "internal boundary" term that has no analogue in the Hausdorff regime.

In Section 5.4 we provide the primary contribution of this chapter. Here, we will study a non-Hausdorff version of the Trousers space. In essence, this "non-Hausdorff Trousers space" can be see as a version of Penrose's spacetime of Figure 5.2 that has been compactified so that its initial and final surfaces equal S^1 and $S^1 \sqcup S^1$, respectively. To begin with, in Section 5.4.1 we will analyse the causal properties of the non-Hausdorff Trousers space. Using the results of Section 5.2, we will argue that this space cannot be excluded from the path integral (5.1) on the basis that it is not a rich-enough geometric structure. We will then analyse its causal properties by confirming the existence of a global time function, the non-existence of closed timelike curves, the compactness of its causal diamonds, and the poset structure of its causality relation $\leq$.

In the remainder of Section 5.4 we will then determine the gravitational action for the non-Hausdorff Trousers space. As an organisational choice, we will first motivate the Lorentzian action from its Euclidean cousin. In line with the Gauss-Bonnet theorems of Section 5.3, we will see that the non-Hausdorff gravitational action requires another Gibbons-Hawking-York term for the extra Hausdorff-violating surface. We will argue that the non-Hausdorff Trousers space has zero curvature, meaning that in the Euclidean theory there is no inherent mechanism that would enable its suppression. In the Lorentzian theory, however, we will see that the presence of corner terms in the action, together with the freedom to choose signs of the $i\varepsilon$-regulator, will allow us to suppress the non-Hausdorff Trousers space as desired. Finally, we finish with some brief remarks regarding more elaborate non-Hausdorff branching.

5.2 Non-Hausdorff Differential Geometry

We start with a review of non-Hausdorff manifolds. Throughout this section and the remainder of this chapter, we will reserve the term "manifold" for its ordinary usage, that is, manifolds are taken to be Hausdorff, locally-Euclidean and second-countable. In distinction to this, we will use the term "non-Hausdorff manifold" to mean a non-Hausdorff, locally-

Euclidean, second-countable space. Regarding notation, we will mostly follow the notation of Lee in [65] for ordinary differential geometry, and [83–85] for non-Hausdorff variants. In particular, we will use boldface letters to denote non-Hausdorff manifolds and their geometric structures – for example **M,N**, .. would denote non-Hausdorff manifolds whereas $M, N, \ldots$ would denote Hausdorff ones.

5.2.1 Topological Structure

To begin, we will describe a general technique for gluing together manifolds. This formalism is known as an *adjunction space* in the literature, though may take subtly different forms depending on the context. We assume as input two Hausdorff manifolds M_1 and M_2 of the same dimension, a subset A of M_1, and a continuous map $f : A \to M_2$. We may then glue M_1 to M_2 along the map f by quotienting the disjoint union $M_1 \sqcup M_2$ according to the equivalence relation that identifies each point in A to its image under f.

This notion of adjunction space is far too general, and may spoil the topological structure of A in the gluing process. The following result identifies some conditions under which the adjunction space described above will yield a non-Hausdorff manifold.

Theorem 5.2.1 ([84]). *Let M_1 and M_2 be two Hausdorff manifolds of the same dimension, and let A be an open subset of M_1. Suppose that $f : A \to M_2$ is an open topological embedding. If f can be extended to a closed embedding $f : \overline{A} \to M_2$,[4] then the quotient space*

$$\boldsymbol{M} := \frac{M_1 \sqcup M_2}{a \sim f(a)} = M_1 \cup_f M_2 \tag{5.5}$$

is a non-Hausdorff manifold in which the Hausdorff-violating points occur precisely at the boundary of the image of A in the quotient space.

At first glance, the above appears to be very similar to the connected sum of manifolds (cf. [65]). However, there is an important distinction: the connected sum assumes that the gluing region A is topologically *closed*, which is necessary in order to preserve the Hausdorff property. However, in our context, we want to take our gluing region to be an *open* subset with a non-empty boundary. This assumption intentionally spoils the Hausdorff property, since the boundaries of these glued open sets remain unidentified, thus serve as distinct limits to the same sequences.

It should be noted at this point that the adjunction space construction may also be phrased categorically – the quotient construction of **M** in the above result can be viewed as the colimit of the upper-left corner of the diagram below in the category of topological spaces.

$$\begin{array}{ccc} A & \xrightarrow{\iota} & M_1 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \phi_1} \\ M_2 & \xrightarrow[\phi_2]{} & \mathbf{M} \end{array} \tag{5.6}$$

[4]Here we use the notation $\overline{A}$ to denote the topological closure of A within M_1.

In the above the map $\iota : A \to M_1$ is the inclusion map, and the $\phi_i : M_i \to \mathbf{M}$ are the canonical maps that send each point to its equivalence class in $\mathbf{M}$. It can be shown that the maps ϕ_i are open topological embeddings, provided that f is open and A itself is [84], and in fact this is precisely what is used in order to transfer the local charts from M_i into $\mathbf{M}$. By construction, any local chart (U, φ) of M_i defines a chart $(\phi_i(U), \varphi \circ \phi_i^{-1})$ on $\mathbf{M}$, and it is in this sense that the non-Hausdorff manifold $\mathbf{M}$ of Theorem 5.2.1 is locally equivalent to the manifolds M_1 and M_2. Given that the ϕ_i maps are open topological embeddings, we may see M_1 and M_2 as sitting inside $\mathbf{M}$ as maximal Hausdorff open submanifolds.

The idea that the maps ϕ_i will be as equally well-behaved as the gluing map f may be extended beyond topology alone. As the next result illustrates, we may actually pass this entire adjunction construction into an appropriate smooth category.

Theorem 5.2.2 ([85]). *Suppose in addition to the criteria of Theorem 5.2.1 that the M_i and A are all smooth manifolds, and $f : A \to M_2$ is a smooth map. If f can be extended to a smooth embedding $\overline{f} : \overline{A} \to M_2$, then $\mathbf{M}$ can be endowed with a smooth atlas.*

Once endowed with a smooth atlas, the canonical embeddings $\phi_i : M_i \to \mathbf{M}$ now become *smooth* open embeddings. Consequently, we may view the Hausdorff manifolds M_i as smooth open submanifolds of $\mathbf{M}$, with the ϕ_i acting as local diffeomorphisms.

It should be noted at this stage that the colimit formulation of [84, 85] is far more general than what we have presented here, in that it may also be extended to colimits of more than two manifolds. However, since we will only be considering transitions from S^1 to $S^1 \sqcup S^1$, we will not require this formalism in full generality. Throughout the remainder of this section, we will assume that $\mathbf{M}$ is a non-Hausdorff manifold built according to Theorems 5.2.1 and 5.2.2.

5.2.2 Vector Bundles

Smooth vector bundles over a non-Hausdorff manifold can be described with an analogue of the colimit construction of Theorems 5.2.1 and 5.2.2. The only major difference is that we must also require the existence of a gluing map for the fibres of the part of the bundle that lies over the gluing region A. Once this is correctly done, we may indeed glue bundles along their fibres in order to form a non-Hausdorff vector bundle. In a manner similar to that of [68, 84], a converse to this construction holds: any vector bundle $\mathbf{E}$ fibred over $\mathbf{M}$ is in fact a colimit of ordinary Hausdorff bundles E_i fibred over the M_i.

Intuitively speaking, we can represent any smooth section $\mathbf{s}$ of a non-Hausdorff bundle $\mathbf{E} \xrightarrow{\pi} \mathbf{M}$ by pulling it back to the Hausdorff bundles $\phi_i^*\mathbf{E} \to M_i$ and describing it piecewise. Provided that the two pulled-back sections $\phi_i^*\mathbf{s}$ agree once mutually restricted to the gluing region $\mathbf{A}$, it is then possible to canonically reconstruct $\mathbf{s}$ from Hausdorff data. Figure 5.4 depicts a semi-local representation of a non-Hausdorff section.

The pullback correspondence of Figure 5.4 may also be phrased algebraically. For any non-Hausdorff vector bundle $\mathbf{E}$ over $\mathbf{M}$, we may use pointwise addition and multiplication by

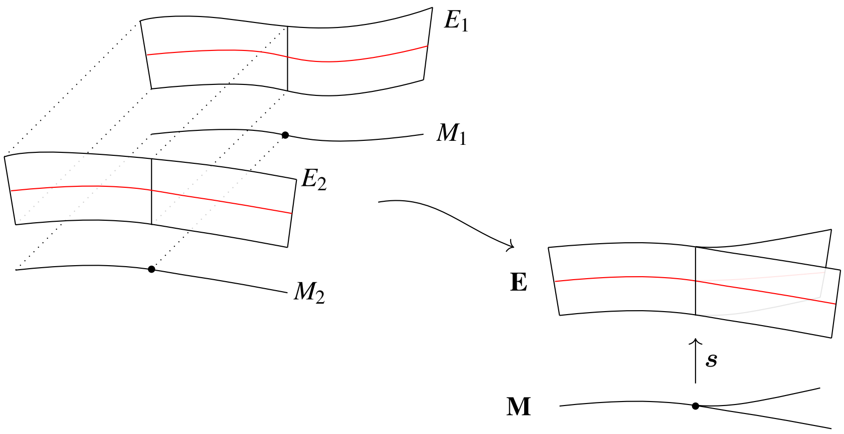


Figure 5.4: A semi-local depiction of the formation of sections of non-Hausdorff bundles.

scalar functions to endow the space of sections $\Gamma(\mathbf{E})$ with the structure of a $C^\infty(\mathbf{M})$-module. This can then be related to the spaces of sections of the Hausdorff submanifolds as follows.

Theorem 5.2.3 ([84]). *Let $\boldsymbol{E}$ be a vector bundle over $\boldsymbol{M}$ with colimit representation $E \cong E_1 \cup_F E_2$ where $F : E_A \to E_2$ is a bundle isomorphism covering $f : A \to M_2$. Then the space of smooth sections $\Gamma(\boldsymbol{E})$ is canonically isomorphic to the fibre product:*

$$\Gamma(\boldsymbol{E}) \cong \Gamma(E_1) \times_{\Gamma(E_A)} \Gamma(E_2) = \left\{(s_1, s_2) \in \Gamma(E_1) \oplus \Gamma(E_2) \;\middle|\; \iota_A^* s_1 = f^* s_2\right\}. \tag{5.7}$$

On the level of an individual section, the above result is stating that defining a smooth section $\mathbf{s}$ on $\mathbf{E}$ amounts to defining a pair of sections s_i on the restricted bundles E_i that agree once mutually pulled back to the bundle E_A. On the categorical level, the pullback of sections by smooth maps allows us to see $\Gamma(\cdot)$ as a contravariant functor. Once applied to the colimit diagram used to construct/describe the bundle $\mathbf{E}$, we obtain a new diagram in a particular (abelian) category of modules over rings. Theorem 5.2.3 then states that the contravariant functor $\Gamma(\cdot)$ sends our colimit $\mathbf{E}$ into the limit $\Gamma(\mathbf{E})$.

These abstract bundle-theoretic arguments can be applied to the tangent bundle $T\mathbf{M}$ in order to describe the vector fields over $\mathbf{M}$. To begin with, one can show that the tangent bundle $T\mathbf{M}$ is canonically isomorphic to the colimit of the bundles TM_1 and TM_2, glued along TA via the gluing (bundle) map $df : TA \to TM_2$. Since f is an open embedding, its differential df is a bundle embedding, and thus falls under the scope of Theorem 5.2.3. We may then conclude that any vector field $\mathbf{v}$ in $\Gamma(T\mathbf{M})$ can be uniquely described by a pair of vector fields v_i in $\Gamma(TM_i)$ that agree once restricted to A. The higher-rank tensorial bundles also admit a similar colimit construction (cf. [85, Sec. 2]), and the tensor fields on the non-Hausdorff manifold $\mathbf{M}$ may therefore be described with the fibre product formula of Theorem 5.2.3.

5.2.3 Integration

In our discussions thus far we have been identifying conditions under which locally-defined data may be described in the non-Hausdorff case, ultimately by a transfer of the Hausdorff data under the canonical maps ϕ_i. Once the correct gluing conditions were identified, our discussion was somewhat intuitive and unproblematic. However, despite being locally-equivalent to Hausdorff manifolds, there is a significant issue when passing from local to global structures.

In Hausdorff differential geometry, a useful method for pasting together locally-defined objects is via partitions of unity. The precise definition is not necessary for our purposes, but their utility should not be understated: partitions of unity are used in various constructions and arguments for manifolds, including the locality of derivative operators, the construction of Riemannian metrics and their Levi-Civita connections, and various arguments involving de Rham cohomology. In the non-Hausdorff case, we have the following inconvenient fact.

Theorem 5.2.4 ([83])**.** *If $\mathbf{M}$ be a non-Hausdorff manifold built according to Theorems 5.2.1 and 5.2.2, then there is no partition of unity subordinate to any cover of $\mathbf{M}$ by Hausdorff open sets.*

In Hausdorff differential geometry, the integral of a compactly-supported differential form is performed by decomposing the form into several local charts, performing the integrals in Euclidean space, and then summing over the results via a partition of unity. In this context, the partition of unity is required in order to avoid overcounting the integral on overlapping charts.

In the non-Hausdorff setting we do not have access to partitions of unity in full generality, so we will need to define integration on a non-Hausdorff manifold by alternate means. Instead of appealing to partitions of unity, we will follow the so-called "integration over parametrizations", found in say [65, Chpt. 13]. Roughly put, in this scheme a total integral is broken down into integrals over certain open sets whose closures cover the support of the differential form, in such a way that adjacent regions only intersect at their measure-zero boundaries. With this intersection property there is no risk of overcounting the integral, and thus partitions of unity are not required.

Suppose that we have some compactly-supported differential form ω on $\mathbf{M}$, and consider a collection $\{r_i : U_i \to \mathrm{M}\}$ of finitely-many open domains of integration U_i that are mapped diffeomorphically into $\mathbf{M}$ such that orientations are preserved. Provided that the union $\bigcup_i r_i(U_i)$ cover the support of ω and the sets $r_i(U_i)$ pairwise intersect on at most their boundaries in $\mathbf{M}$, we may define the integral of ω in $\mathbf{M}$ to be

$$\int_{\mathbf{M}} \omega = \sum_i \int_{U_i} r_i^* \omega. \tag{5.8}$$

Although the above is technically a well-defined notion of integration, it is not particularly useful for our needs. What is more helpful for us is the following, which relates the integral of a form on $\mathbf{M}$ to the ordinary integrals over the Hausdorff submanifolds M_1, M_2 and A.

Theorem 5.2.5 ([83, Thm. 2.6])**.** *Let ω be a compactly-supported differential form on $\mathbf{M}$. Then*

$$\int_{\boldsymbol{M}} \omega = \int_{M_1} \omega_1 + \int_{M_2} \omega_2 - \int_{\bar{A}} \omega_A, \tag{5.9}$$

where $\overline{A}$ is the topological closure of A within M_1, and $\omega_i := \phi_i^*\omega$ and $\omega_{\overline{A}} := \iota_{\overline{A}} \circ \phi_1^*\omega = \overline{f} \circ \phi_2^*\omega$.

An important distinction between Theorem 5.2.5 and the standard subadditivity property of Hausdorff integration is the inclusion of the additional boundary of the subspace A. Heuristically, we need to ensure that the restriction of ω to A is compactly supported, and the only way to do this is to include the boundary of A within the integral. It may seem that the inclusion of this extra component is an innocuous prescription, given that boundaries are of measure zero. However, as we will see in Section 5.3, this extra boundary component will have some far reaching consequences for the non-Hausdorff Gauss-Bonnet theorems.

5.2.4 Metrics and Curvature

We may construct metrics of arbitrary signature on a non-Hausdorff manifold by gluing together the spaces M_1 and M_2 along an isometry. In a global picture, we may view two metrics on the M_i as sections of the appropriate tensor bundle for which the overlap condition of Theorem 5.2.3 manifests as an isometric equivalence on the gluing region A. It should be noted that there is no issue regarding the regularity of the resulting non-Hausdorff metric – Theorem 5.2.3 ensures that any Lorentzian metric, viewed as a global section of the appropriate tensor bundle, is indeed smooth everywhere.

Despite the non-existence of partitions of unity, affine connections may still be constructed in the non-Hausdorff setting. In global notation, an affine connection on $\mathbf{M}$ is defined as per usual, that is, as a bilinear operator

$$\boldsymbol{\nabla} : \Gamma(T\mathbf{M}) \times \Gamma(T\mathbf{M}) \to \Gamma(T\mathbf{M}),\ \boldsymbol{\nabla}(\mathbf{v},\mathbf{w}) \mapsto \boldsymbol{\nabla}_{\mathbf{v}}\mathbf{w} \tag{5.10}$$

that is $C^\infty(\mathbf{M})$-linear in the first argument, and satisfies the Leibniz rule: $\nabla_{\mathbf{v}}(\mathbf{fw}) = \mathbf{f}\nabla_{\mathbf{v}}(\mathbf{w}) + \mathscr{L}_{\mathbf{v}}(\mathbf{f})\mathbf{w}$ for all $\mathbf{f} \in C^\infty(\mathbf{M})$ and $\mathbf{v},\mathbf{w} \in \Gamma(T\mathbf{M})$.[5] As with the non-Hausdorff sections of Section 5.2.2, it can be shown that any connection $\boldsymbol{\nabla}$ on $\mathbf{M}$ will restrict to a pair affine connections $\nabla^i := \phi_i^*\boldsymbol{\nabla}$ on M_i that agree once mutually pulled back to A. As the following result states, the converse is also true: a pair of connections on M_i may be "glued" together to define a connection on $\mathbf{M}$.

Lemma 5.2.6 ([83]). *Suppose ∇_i are a pair of affine connections defined on the manifolds M_i. If $\iota_A^*\nabla_1 = f^*\nabla_2$ on A, then $\boldsymbol{\nabla}$ defined by*

$$(\boldsymbol{\nabla}_{\boldsymbol{v}}\boldsymbol{w})(\boldsymbol{x}) = \begin{cases} \phi_1\left((\nabla_1)_{(\phi_1^*\boldsymbol{v})}(\phi_1^*\boldsymbol{w})(\phi_1^{-1}(\boldsymbol{x}))\right) & \text{if } \boldsymbol{x} \in \phi_1(M_1) \subseteq \boldsymbol{M} \\ \phi_2\left((\nabla_2)_{(\phi_2^*\boldsymbol{v})}(\phi_2^*\boldsymbol{w})(\phi_2^{-1}(\boldsymbol{x}))\right) & \text{if } \boldsymbol{x} \in \phi_2(M_2) \subseteq \boldsymbol{M} \end{cases} \tag{5.11}$$

is an affine connection on $\boldsymbol{M}$.

Observe that in the above, the assumption $\iota_A^*\nabla_1 = f^*\nabla_2$ is precisely what is needed to ensure that $\boldsymbol{\nabla}$ is well-defined, since by construction $\phi_1(M_1) \cap \phi_2(M_2) = \phi_1(A) \cap \phi_2(f(A))$.

[5]The Lie derivative used here is defined as in the Hausdorff case, that is, via local flows of the vector fields. It can be shown that in the non-Hausdorff case, the Lie derivative is still a local operator that satisfies $\mathscr{L}_{\mathbf{v}}\mathbf{w} = [\mathbf{v},\mathbf{w}]$ – see Section 3.1 of [83] for a detailed discussion.

Using the above prescription, it can be shown that a Levi-Civita connection exists for any metric on $\mathbf{M}$. Heuristically, although we don't have full access to partitions of unity for $\mathbf{M}$, we *do* have full access for the Hausdorff manifolds M_i. Therefore, we may construct pieces of the Levi-Civita connection on each M_i, and requiring that $f: A \to M_2$ be an isometric embedding ensures that these Hausdorff Levi-Civita connections may be transferred into $\mathbf{M}$. Following on from this, we may define familiar geometric quantities such as the Riemann curvature tensor, the Ricci tensor and the Ricci scalar in the non-Hausdorff case, ultimately by appealing to the fact that $\mathbf{M}$ is locally isometric to both M_i [83].

In a similar spirit, orientations of the manifolds M_i may be glued in a manner consistent with Theorem 5.2.3 to yield an orientation of the non-Hausdorff manifold $\mathbf{M}$. In such a situation, the canonical maps $\phi_i : M_i \to \mathbf{M}$ become orientation-preserving isometries.

With an eye towards Section 5.4, we finish this section with a brief application of these ideas. Suppose for a moment that $\mathbf{M}$ is a two-dimensional non-Hausdorff manifold with Riemannian metric $\mathbf{h}$. The canonical maps $\phi_i : M_i \to \mathbf{M}$ act as isometric embeddings, which means that $\mathbf{M}$ may be locally isometric to either M_i, depending on where you are in the manifold. According to the integral formula of Theorem 5.2.5, the total scalar curvature of $\mathbf{M}$ may be written as

$$\int_{\mathbf{M}} R\, dA = \int_{M_1} R\, dA + \int_{M_2} R\, dA - \int_{\bar{A}} R\, dA \tag{5.12}$$

where here the metrics on the Hausdorff manifolds are the pullbacks of $\mathbf{h}$ by the relevant maps, and each Ricci scalar and area form is computed via the pulled-back versions of $\mathbf{h}$.

5.3 Gauss-Bonnet in Various Forms

The Gauss-Bonnet theorem is a powerful result that relates the total scalar curvature of a two-dimensional manifold to its Euler characteristic. As explained in the introduction, our main strategy for evaluating the gravitational action of the non-Hausdorff Trousers space will be via this particular theorem. As such, we will now spend some time discussing various versions of the Gauss-Bonnet theorem. In distinction to Section 5.2, throughout this section we will assume that all manifolds, Hausdorff or otherwise, are two-dimensional.

In Euclidean signature, the Gauss-Bonnet theorem for a manifold (M,h) with boundary may be stated as the equality

$$2\pi\chi(M) = \frac{1}{2}\int_M R dA + \int_{\partial M} k d\gamma + \sum \theta_{ext}, \tag{5.13}$$

where here $\chi(M)$ is the Euler characteristic of M,[6] and the boundary ∂M is assumed to consist of finitely many piecewise-smooth connected components [27]. The expressions θ_{ext} denote the exterior angles between adjacent smooth segments of a boundary compo-

[6] Here we may define the Euler characteristic to be the alternating sum of the ranks of the simplicial homology groups of M, or equivalently as $\chi(M) = V_{\mathscr{T}} - E_{\mathscr{T}} + F_{\mathscr{T}}$ for any triangulation $\mathscr{T}$ of M.

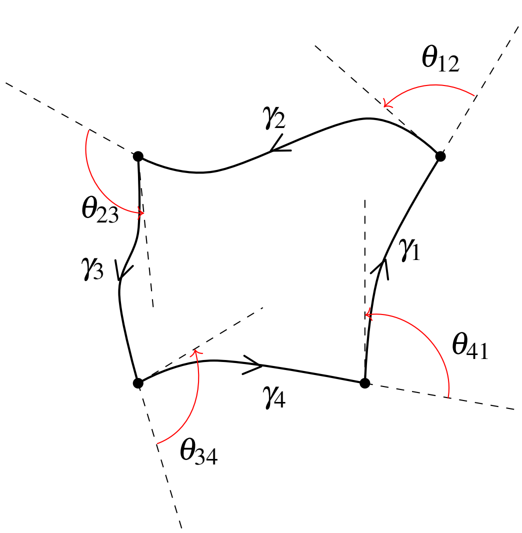


Figure 5.5: Turning angles for a piecewise-smooth, oriented closed curve γ that bounds a flat disk embedded in $\mathbb{R}^2$. Here the total geodesic curvature of γ equals $\int_\gamma k d\gamma = \sum_{i=1}^4 \int_{\gamma_i} k d\gamma + \sum \theta_{ext}$. The exterior angles (red) are computed using tangent vectors at each marked vertex. Hopf's Umlaufsatz states that the sum of these exterior angles equals 2π, and the Gauss-Bonnet theorem confirms that the Euler characteristic of the disk is equal to 1.

nent. Geometrically, these are the angles by which a vector must instantaneously turn at the non-smooth corners, as pictured in Figure 5.5. We will often borrow from physics parlance and refer to these exterior angles as *corner terms*. Given that the boundary ∂M is a piecewise-smooth curve, the corner term θ_{ext} at a vertex p is computed by taking the one-sided derivatives of the adjacent curves and computing the angle between the corresponding vectors lying in the tangent space T_pM.

We will now set about modifying this theorem in two orthogonal directions: firstly, we will review the so-called *Lorentzian Gauss-Bonnet theorem*, and then we will generalise everything to the non-Hausdorff setting. As mentioned in the introduction, there is a conceptual difficulty when proving the Lorentzian Gauss-Bonnet theorem, ultimately stemming from the ill-defined notion of angle within Minkowski space. So, before getting to any modifications of the Gauss-Bonnet theorem, we will first spend some time reviewing the literature on Lorentzian angles.

5.3.1 Lorentzian Angles

In a two-dimensional vector space with a fixed metric, the convex angle between a pair of normalised vectors may be determined from the parameter of the isometry transformation that sends one vector into the other. When working with the usual Euclidean metric, the isometry transformation lying in $SO(2)$ is an honest rotation of the unit circle. Up to a preferred orientation, the convex angle between two vectors in Euclidean space is uniquely determined – ultimately because the action of $SO(2)$ on $\mathbb{R}^2$ is free and transitive.

This perspective also partially applies to Minkowski space. When passing to the $(-,+)$-signature of the Minkowski metric, the orbit spaces under $SO(1,1)$ of non-null vectors divide

$\mathbb{R}^{1,1}$ into four disjoint quadrants. As such, a pair of Minkowski vectors u and v will admit an unambiguous and well-defined angle θ_{uv} provided that they are both non-null and are related by a boost. We will label the orbit space of the unit spacelike vector $v = (0,1)$ as Q1, and count the quadrants anticlockwise from there. Following [100], we introduce

$$Z(u,v) := u \cdot v + \sqrt{(u \cdot v)^2 - (u \cdot u)(v \cdot v)} \text{ and } \overline{Z}(u,v) = u \cdot v - \sqrt{(u \cdot v)^2 - (u \cdot u)(v \cdot v)} \tag{5.14}$$

as useful shorthands. We may then express the angle θ_{uv} between two spacelike vectors lying in the same quadrant as follows:

$$\theta_{uv} = \log \frac{Z(u,v)}{|u||v|} \qquad \text{if } u,v \text{ both spacelike and in the same quadrant.} \tag{5.15}$$

This formula may be related to the familiar trigonometric expression of boosts by standard identities – see [5] for the alternate form. Note that for spacelike vectors, the norm $|u| = \sqrt{u \cdot u}$ is real.

Since no boost can change the signature of a Minkowski vector, it may appear as though there is no meaningful notion of angle between vectors lying in different quadrants. The now-standard remedy for this issue is to analytically continue the meaningful fragments of angular formulae into the complex plane. With such a procedure, the result is a complex-valued notion of angle. Treatments of complex-valued Minkowski angles exist in various forms in the literature [5, 58, 79, 100], though they typically differ in both scope and convention. For the purposes of this chapter, we will opt for Sorkin's approach [100], since his treatment is sufficiently general so as to include both null vectors and a Gauss-Bonnet theorem.

As an illustration, we will now outline the derivation of a complex angle between two Lorentzian vectors $a := (0,1)$ and $b := (1,0)$. Using the null basis $m := (\frac{1}{2}, -\frac{1}{2})$ and $n := (\frac{1}{2}, \frac{1}{2})$, we may write $a = n - m$ and $b = n + m$. An interpolating vector c lying in the convex wedge between a and b may be described as $c = m + \lambda n$, where $\lambda \in [-1,1]$. Allowing λ to smoothly vary from -1 to 1 will trace out a continuous transformation of a into b. As we do this, we see that the expression $\frac{Z(a,c)}{|a||c|}$ will become singular as c becomes null when crossing quadrants at $\lambda = 0$. This pole may be avoided by endowing the Minkowski metric with a small positive-definite imaginary part (cf. [67]), which adjusts the dot product by $c \cdot c \to c \cdot c \pm i\varepsilon$ for vectors c with norm close to zero. This allows us to circumvent the singularity at $\lambda = 0$ and continue into the adjacent quadrant without issue.

There are two subtleties to consider here. Firstly, we are trying to complexify the ratio $\frac{Z(a,c)}{|a||c|}$, which means that we need to select a branch of the complex logarithm that will eventually be used as in (5.15). Following [100], we select the principle branch of the logarithm, so that $log(\frac{1}{i}) = \frac{-\pi i}{2}$. Secondly, when performing the circumvention of the pole at $\lambda = 0$, the sign of the $i\varepsilon$-regulariser will dictate the sign of the (imaginary) norm for timelike vectors: $|u| := \sqrt{u \cdot u} = \pm i\sqrt{|u \cdot u|}$. With this in mind, the Lorentzian angle between our chosen

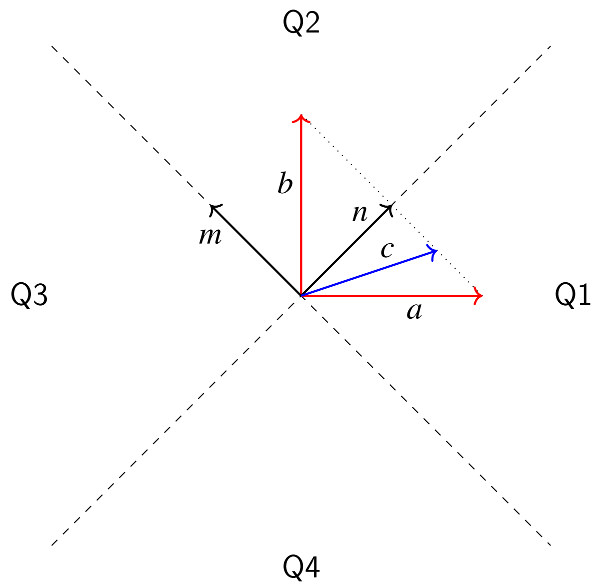


Figure 5.6: Sorkin's derivation of the Lorentzian angle between a spacelike and a timelike vector. Here we pick the lightcone basis $m := (\frac{1}{2}, -\frac{1}{2})$ and $n = (\frac{1}{2}, \frac{1}{2})$, for which $a = n - m$ and $b = n + m$, and $c = n + \lambda m$. We may then continuously trace from a to b, using a parameter $\lambda \in [-1, 1]$. This parameter will yield a singularity in $\frac{Z(a,c)}{|a|\,|c|}$ at $\lambda = 0$, which we may avoid via an $i\varepsilon$-regularisation.

vectors a and b, will be purely imaginary:

$$\theta_{ab} = \log\left(\frac{Z(a,b)}{|a|\;|b|}\right) = \log\left(\frac{1}{\pm i}\right) = \mp\frac{i\pi}{2}, \tag{5.16}$$

the sign of which depends on the choice of $\pm i\varepsilon$. For a general spacelike vector u in Quadrant 1 and a timelike vector v lying in Quadrant 2, there may also be a real part of the Lorentzian angle. The general angular formula will be

$$\theta_{uv} = \log\left(\frac{Z(u,v)}{|u|\;|v|}\right) = \log\left(\frac{Z(u,v)}{||u||\;||v||}\right) \mp \frac{\pi i}{2}, \tag{5.17}$$

where here in the latter term we isolate the imaginary component of the angle by taking $||\cdot||$, the absolute value of the norm $|\cdot|$.

In similar spirit, the Lorentzian angle between a pair of timelike vectors lying in the same quadrant may be determined once we have fixed a sign for the complex quantity $|u| = \sqrt{u \cdot u}$. In contrast to (5.15), we have:

$$\theta_{uv} = -\log\frac{\overline{Z}(u,v)}{|u||v|} \qquad \text{if } u, v \text{ both timelike and in the same quadrant.} \tag{5.18}$$

Using the above, one can interpret the imaginary contribution of $\mp\frac{\pi i}{2}$ in the formula (5.17) as a discrete "rotation" of the spacelike vector into Quadrant 2, followed by an application of formula (5.18) to determine the remaining real part of the angle.

Generally speaking, the $i\varepsilon$-regularisation causes a discrete contribution of $\mp\frac{i\pi}{2}$ whenever we compute angles between non-null vectors in adjacent quadrants of Minkowski space. One can imagine a Euclidean rotation in which we trace one vector through null rays into adjacent quadrants (cf. Figure 5.6). Every time a vector passes through a null ray it receives a discontinuous contribution of $\mp\frac{\pi i}{2}$. The total angle around the origin will then equal $\mp 2\pi i$, as depicted in [5, 79].

The $i\varepsilon$-regularisation of [5, 58, 67, 79, 100] may also be used to define angles between null vectors. Combining equation (5.17) with the additivity of angles, and proceeding via a case distinction, the angle involving null vectors may take any of the following forms:

$$\theta_{m,u} = \log\left(\frac{m\cdot u}{l_0||u||}\right) \mp \frac{i\pi}{4} \qquad \text{if } u \text{ spacelike} \tag{5.19}$$

$$\theta_{m,v} = \log\left(\frac{m\cdot v}{l_0||v||}\right) \mp \frac{i\pi}{4} \qquad \text{if } v \text{ timelike} \tag{5.20}$$

$$\theta_{m,n} = \log\left(\frac{m\cdot n}{l_0^2}\right) \mp \frac{i\pi}{2} \qquad \text{if } n \text{ null and } m,n \text{ bound spacelike quadrant} \tag{5.21}$$

$$\theta_{m,n} = -\log\left(\frac{m\cdot n}{l_0^2}\right) \mp \frac{i\pi}{2} \qquad \text{if } n \text{ null and } m,n \text{ bound timelike quadrant} \tag{5.22}$$

Here we follow [100] and introduced two more conventions. Firstly, due to the additivity of angles, we need to choose how to divvy up the imaginary contribution of $\mp\frac{\pi i}{2}$ arising from (5.17) into the two formulae (7) and (8). In the above we have opted for a balanced contribution of $\mp\frac{\pi i}{4}$ for either side of the null vector. Secondly, we commit to an additional variable l_0, which is an arbitrary length scale that is required in order to make the angle formulae dimensionless (cf. the $4d$ corner ambiguites in [59, 66]). Although both of these conventions are choices, they will ultimately not affect the Gauss-Bonnet theorem once we keep them consistently fixed.

5.3.2 A Gauss-Bonnet Theorem for Surfaces with Null Boundaries

A Lorentzian version of the Gauss-Bonnet theorem was originally proved implicitly by Chern in [21], who extended his famous Chern-Gauss-Bonnet theorem to closed, even-dimensional Lorentzian manifolds. Various alternate versions of the result exist in the literature, and each varies in scope and generality. Older resources such as [2, 7, 57, 63] prove the theorem by considering manifolds with either empty or non-null boundary. This was generally due to an underdeveloped notion of of Lorentzian angles at their time of writing. In recent years, however, Sorkin [100] extended the Lorentzian Gauss-Bonnet theorem to include null boundary components. We will now briefly overview his argument, and flesh out some of the more rigorous details.

We start by proving a Gauss-Bonnet theorem for the Lorentzian triangle. This version, commonly called the “local Gauss-Bonnet theorem" in Euclidean terminology, ultimately follows from an analogue of Hopf’s *Umlaufsatz* – cf. the Euclidean version in Figure 5.5. In this context, the Umlaufsatz states that the sum of turning angles around the oriented boundary of a Lorentzian triangle will always equal $\mp 2\pi i$. Heuristically speaking, we can imagine parallel transporting a vector around the boundary of the triangle and summing up the discrete jumps at the corners. Since the triangle is assumed to be flat, the vector will return to itself with no real angular defect, however it will have collected a full rotation of Minkowski angles along its journey.

Formally, the Lorentzian Umlaufsatz may be proved via a case distinction on all the different types of triangles. For triangles with non-null edges, the Umlaufsatz was discussed by both Jee [57] and Law [63]. For triangles involving null edges, the discussion was continued in [100]. For the purposes of illustration, we consider the case of a triangle Δ with two null edges m and n, and a spacelike edge w. After orienting $\partial\Delta$, we may consider the turning angles by fixing null vectors with an affine length equal to that of the corresponding edge of Δ. Using the additivity of Lorentzian angles, we can represent the sum of turning angles as a sum between purely null edges:

$$\begin{aligned}\theta_{mn} + \theta_{nw} + \theta_{wm} &= \theta_{mn} + (\theta_{n(-m)} + \theta_{(-m)w}) + (\theta_{w(-n)} + \theta_{(-n)m}) \\ &= \theta_{mn} + \theta_{n(-m)} + \theta_{(-m)(-n)} + \theta_{(-n)m}.\end{aligned} \tag{5.23}$$

Heuristically, we may imagine parallel transporting all three vectors to the same point, and then using additivity of angles to cancel out the terms containing the spacelike vector w. This intuition is depicted in Figure 5.7. From this perspective, the Umlaufsatz for Δ is clear, provided that the ambiguous length scale l_0 is kept fixed throughout the sum of the turning angles. We should also note that this version of the Umlaufsatz is equivalent to Sorkin's observation that the interior angles of a Minkowskian triangle will sum up to the flat half-value $\mp i\pi$, since:

$$\mp 2\pi i = \sum_{i=1}^{3} \theta_{ext} = \sum_{i=1}^{3} (\mp i\pi - \theta_{int}) = \mp 3\pi i - \sum_{i=1}^{3} \theta_{int}. \tag{5.24}$$

Using this result for triangles in Minkowski space, we may then triangulate a given almost-Lorentzian surface, making sure to arrange any causal irregularities onto vertices of the triangulation, and then derive a discrete version of the Gauss-Bonnet by the same arguments found in the Euclidean case – for instance those of [27, Sec 4.5]. Provided that we pick the same sign of the $i\varepsilon$-regulariser for all the vertices in a given triangulation,[7] we can then express curvature via a Regge action which sums over Lorentzian defect angles. This yields the following version of the Gauss-Bonnet theorem, amenable to two-dimensional Regge calculus in Lorentzian signature.

Theorem 5.3.1 (Adapted from [100]). *Let M be an almost-Lorentzian surface with boundary, and let $\mathscr{T}$ be a triangulation of M in which the degenerate points of M lie on vertices of $\mathscr{T}$. Denote by V_M and $V_{\partial M}$ the set of vertices of $\mathscr{T}$ that lie in the bulk and boundary of M, respectively. Then*

$$\mp 2\pi i \chi(M) = \sum_{p\in V_M} \delta^2(p) + \sum_{q\in V_{\partial M}} \delta^1(q), \tag{5.25}$$

[7] This assumption is implicit within [100], though is explicitly called the "global Wick rotation" in [5].

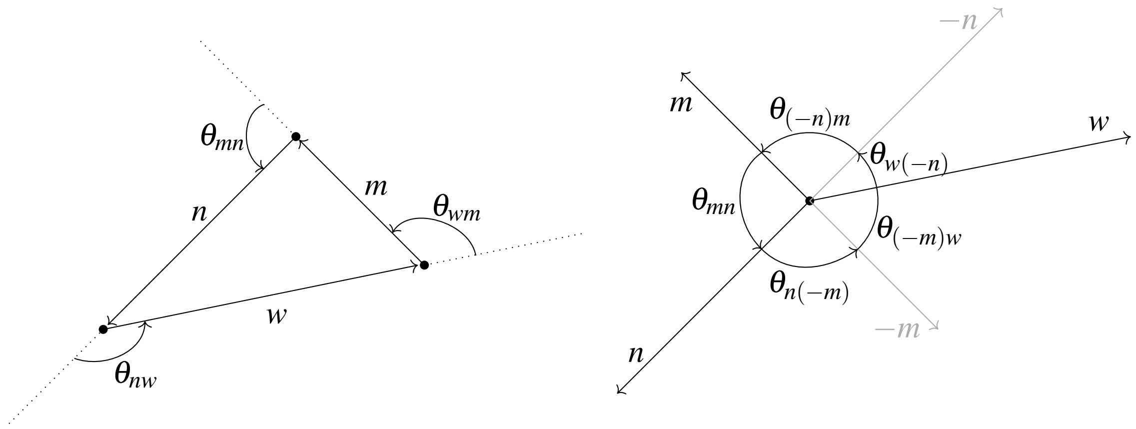


Figure 5.7: The intuition behind the Umlaufsatz of a Lorentzian triangle with two null edges. Here we use the point-vector correspondence of flat space to transport all vectors to the same point, and observe the manifest value of $\mp 2\pi i$.

where $\delta^2(p) = \mp 2\pi i - \sum\theta(p)$ *and* $\delta^1(q) = \mp\pi i - \sum\theta(q)$ *measure the defect angles in* 2*d and* 1*d, respectively.*

We may derive a continuum version of the above by measuring curvature by different means. In the triangulation argument of Regge calculus we flatten the triangles, thereby localising any curvature to vertices. Alternatively, we could also determine the curvature of a triangle Δ by measuring the defect in the sum of its interior angles. Following [20], by taking a limit of smaller and smaller triangles around a point, we may define an infinitesimal notion of curvature which may then be integrated to determine the total scalar curvature of the bulk. Similarly, for each smooth boundary component, we may determine its geodesic curvature via a limit of defect angles of small lines surrounding points. Thus we obtain the following (smooth) version of the Lorentzian Gauss-Bonnet theorem.

Theorem 5.3.2. *Let M be an almost-Lorentzian manifold with boundary. Then*

$$\frac{1}{2}\int_M RdA + \int_{\partial M} kd\gamma + \sum\theta_{ext} = \mp 2\pi i\chi(M). \tag{5.26}$$

5.3.3 Non-Hausdorff Gauss-Bonnet Theorems

We can repeat the philosophy of Section 5.2 in order to prove a Gauss-Bonnet theorem for non-Hausdorff manifolds. Our approach will be to first relate the Euler characteristic of a non-Hausdorff manifold to the Euler characteristics of its Hausdorff submanifolds. After this, we may apply either the Euclidean or Lorentzian Gauss-Bonnet Theorem to all of these Hausdorff pieces, and then collate the resulting integrals into a global non-Hausdorff curvature term using the subadditivity formula of Theorem 5.2.5.

In our discussion thus far we have assumed some simplicial definitions of the Euler characteristic. However, since all simplicial complexes are by construction Hausdorff topological spaces, we do not have access to simplicial homology for non-Hausdorff manifolds. There are several ways to bypass this issue. It is reasonable to suggest that there may be a non-Hausdorff version of simplicial homology, with a well-defined boundary operator that may in turn yield some kind of topological invariants. However, we will avoid this line of inquiry and instead appeal to a broader definition of Euler characteristic that holds for more-general topological spaces. We define the Euler characteristic of a non-Hausdorff n-dimensional manifold $\mathbf{M}$ as follows:

$$\chi(\mathbf{M}) = \sum_{i=1}^{n} (-1)^i \mathrm{rank}(H_i^S(\mathbf{M})), \tag{5.27}$$

where here we use the singular homology groups $H_i^S(\mathbf{M})$. In the Hausdorff setting, this definition of Euler characteristic coincides with the ordinary definition via triangulations, due to the equivalence between simplicial and singular homology [50]. In the case of smooth non-Hausdorff manifolds, singular cohomology is known to be isomorphic to de Rham cohomology for any manifold satisfying the conditions of Theorem 5.2.2 [83].

Under this reading, we may now relate the Euler characteristic of a non-Hausdorff manifold to those of its Hausdorff constituents. The following result confirms that the familiar "inclusion-exclusion principle" (cf. pg. 221 of [93]) holds for our non-Hausdorff manifolds.

Theorem 5.3.3 ([83])**.** *Let $\boldsymbol{M}$ be a non-Hausdorff manifold, defined according to Theorem 5.2.1. Then*

$$\chi(\boldsymbol{M}) = \chi(M_1) + \chi(M_2) - \chi(A). \tag{5.28}$$

Proof. We provide a sketch of the proof; for details, see [83]. Firstly, we may observe that the singular homology groups of $\mathbf{M}$ may be related to the homology groups of the Hausdorff submanifolds by the Mayer-Vietoris sequence:

$$0 \longrightarrow \cdots \longrightarrow H_{q+1}^S(A) \xrightarrow{i_* \oplus f_*} H_{q+1}^S(M_1) \oplus H_{q+1}^S(M_2) \xrightarrow{\phi_{1*} - \phi_{2*}} H_{q+1}^S(\mathbf{M}) \longrightarrow$$

$$\longrightarrow H_q^S(A) \xrightarrow{i_* \oplus f_*} H_q^S(M_1) \oplus H_q^S(M_2) \xrightarrow{\phi_{1*} - \phi_{2*}} H_q^S(\mathbf{M}) \longrightarrow \cdots \longrightarrow 0 \tag{5.29}$$

The precise description of these maps is not too important for our purposes, but can be found in [83] (they are essentially a reformulation of the pushforward maps found in [50][Sec. 2.2], descended to homology). The important observation is that the above is a long exact sequence of vector spaces that terminates. Crucially, for any long exact sequence of vector spaces, the alternating sum of the dimensions of all the entries of the sequence must be zero. The result then follows by a rearrangement of this vanishing alternating sum. □

In order to prove a non-Hausdorff Gauss-Bonnet theorem, we need to make a subtle yet crucial observation. In the integral formula of Theorem 5.2.5, we must integrate over the topological closure of A, which in this context means that we are integrating a manifold with boundary. However, in the subadditive formula of $\chi(\mathbf{M})$ listed above, we instead have an expression for $\chi(A)$, and in general it is not the case that $\chi(A) = \chi(\overline{A})$.[8] However, when we work with the non-Hausdorff Trousers space in Section 5.4 we will not need to worry about

[8] In the case that the M_i are manifolds without boundaries, this tension may be broken by assuming that the open subset A is *regular-open*, meaning that the interior of $\overline{A}$ equals A itself. This allows a homotopy equivalence between A and $\overline{A}$, which induces an isomorphism of singular homologies and guarantees the equality $\chi(A) = \chi(\overline{A})$.

this potential issue, so in what follows we will simply assume the equality $\chi(A) = \chi(\overline{A})$.

According to the remark we made at the end of Section 5.2.4, we may apply the subadditivity principle of Theorem 5.2.5 to a $2d$ curvature form to yield the equality (5.12). If we apply the Gauss-Bonnet theorem to each Hausdorff component of $\mathbf{M}$, we will obtain the equality:

$$\frac{1}{2}\int_{\mathbf{M}} R\, dA = \frac{1}{2}\int_{M_1} R\, dA + \frac{1}{2}\int_{M_2} R\, dA - \frac{1}{2}\int_{\overline{A}} R\, dA \tag{5.30}$$

$$= \left(2\pi\chi(M_1) - \int_{\partial M_1} k d\gamma\right) + \left(2\pi\chi(M_2) - \int_{\partial M_2} k d\gamma\right) - \left(2\pi\chi(\overline{A}) - \int_{\partial\overline{A}} k dx\right) \tag{5.31}$$

$$= 2\pi\chi(\mathbf{M}) - \left(\int_{\partial M_1} k d\gamma + \int_{\partial M_2} k d\gamma - \int_{\partial\overline{A}} k d\gamma\right). \tag{5.32}$$

Here we assume that the boundary of $\mathbf{M}$ consists of the images of ∂M_i under the canonical maps ϕ_i. Note that it's possible for the two sets $\phi_i(\partial M_i)$ to intersect, however this may only occur if M_1 and M_2 have a common manifold boundary along the gluing region A. Thus, the closure $\overline{A}$ may have two different types of boundary: a *manifold* boundary that already exists within A, and a *topological* boundary that forms the Hausdorff-violating points once mapped into $\mathbf{M}$. We thus write $\partial\overline{A} = \partial A \sqcup \mathsf{Y}$, where here we use the special symbol Y to denote the Hausdorff-violating piece of $\overline{A}$. By the usual subadditivity of Hausdorff integration, we see that

$$\int_{\partial\mathbf{M}} k d\gamma = \int_{\partial M_1} k d\gamma + \int_{\partial M_2} k d\gamma - \int_{\partial A} k d\gamma \quad \text{and} \quad \int_{\partial\overline{A}} k d\gamma = \int_{\partial A} k d\gamma + \int_{\mathsf{Y}} k d\gamma. \tag{5.33}$$

By substituting these equalities into the previous equation, we obtain a non-Hausdorff Gauss-Bonnet in Euclidean signature:

$$\frac{1}{2}\int_{\mathbf{M}} R\, dA = 2\pi\chi(\mathbf{M}) - \left(\int_{\partial\mathbf{M}} k d\gamma - \int_{\mathsf{Y}} k d\gamma\right). \tag{5.34}$$

We may the apply essentially the same reasoning as the above, this time using the Lorentzian Gauss-Bonnet theorem of Theorem 5.3.2, to deduce the following.

Theorem 5.3.4. *Let $(\boldsymbol{M}, \boldsymbol{g})$ be a non-Hausdorff two-dimensional spacetime built from Hausdorff spacetimes M_1 and M_2 according to Theorem 5.2.2. Suppose furthermore that the M_i are manifolds with boundary, and A satisfies $\chi(A) = \chi(\overline{A})$. Then*

$$\mp 2\pi i\chi(\boldsymbol{M}) = \frac{1}{2}\int_{\boldsymbol{M}} R dA + \int_{\partial\boldsymbol{M}} k d\gamma - \int_{\mathsf{Y}} k d\gamma. \tag{5.35}$$

5.4 The non-Hausdorff Trousers Space

We will now define and evaluate the gravitational action for a non-Hausdorff version of the Trousers space. As outlined in the introduction, we will consider a compactified version of Penrose's spacetime of Figure 5.2, taken to be long enough so that the initial surface Σ_1 is homeomorphic to S^1 and the final surface Σ_2 is homeomorphic to $S^1 \sqcup S^1$. According to our discussion in Section 5.2, we can construct such a space by gluing two copies of the cylinder together everywhere outside the causal future of a point. Formally, we take

- $M_1 = M_2$ to be cylinders $S^1 \times D^1$ endowed with the same flat Lorentzian metric,
- a preferred point p in M_1 whose causal future $J^+(p)$ contains the final boundary $S^1 \times \{1\}$ of M_1 as a subset,
- $A = M_1 \backslash J^+(p)$ endowed with the open submanifold atlas and metric induced from M_1, and
- $f : A \to M_2$ to be the identity map.

We denote by $\mathbf{T}$ the topological space formed according to the above. Since the gluing map f is taken to be the identity map, the data above falls within the scope of Theorems 5.2.1, 5.2.2 and 5.2.3, and thus we may view $\mathbf{T}$ as a smooth non-Hausdorff manifold that contains the two cylinders M_i as maximal Hausdorff open submanifolds. Moreover, the map f is also a time-orientation preserving isometric embedding, so we may effectively transfer the Lorentzian metrics of the cylinders M_i into the non-Hausdorff manifold $\mathbf{T}$ according to our discussion in Section 5.2.4. We denote the resulting non-Hausdorff spacetime by $(\mathbf{T}, \mathbf{g})$, and we will hereafter refer to this as the *non-Hausdorff Trousers space*.

By construction, the M_1-relative closure $\overline{A}$ of the gluing region A will include the lightlike future of the point p, as depicted in Figure 5.8. Once the quotient is performed to construct $(\mathbf{T}, \mathbf{g})$, the Hausdorff-violating submanifold of $\mathbf{T}$ will be the two lightlike futures of the distinct points $\phi_1(p)$ and $\phi_2(p)$. The height function $\mathbf{f} : \mathbf{T} \to \mathbb{R}$ that maps each point in $\mathbf{T}$ to its D^1-coordinate in either M_i will be a well-defined smooth function that allows us to interpret $(\mathbf{T}, \mathbf{g})$ as a non-Hausdorff transition from S^1 to $S^1 \sqcup S^1$. Indeed – on the spacelike slice of $\mathbf{T}$ that contains the points $\phi_i(p)$, the topology will begin splitting by changing from $\Sigma_1 = S^1$ to a circle with two Hausdorff-violating points. These points will then propagate along null rays, and finish splitting into two circles at some later time. At every point in time after this, the spacelike slices of $(\mathbf{T}, \mathbf{g})$ will have the topology of $S^1 \sqcup S^1$.

5.4.1 Causal Properties

As mentioned in the introduction, one may justify the inclusion of the Hausdorff Trousers space into a path integral scheme such as Equation 5.1 on causal grounds. Despite being non-globally hyperbolic, the Hausdorff Trousers space, aside from the causal irregularity at the crotch, is a well-behaved spacetime. In particular, it exhibits no closed timelike curves, admits a global time function, and determines a causal structure whose relation $\leq$ is a partial order.

We will now show that the non-Hausdorff Trousers space $(\mathbf{T}, \mathbf{g})$ enjoys similar, and even superior, causal features. To begin, we recall the relation $\mathbf{x} \leq \mathbf{y}$ if and only if $\mathbf{y}$ lies in the future lightcone of $\mathbf{x}$, which in this context means that there is a future-directed causal curve connecting $\mathbf{x}$ to $\mathbf{y}$. The following result characterises the causal relation $\leq$ of $\mathbf{T}$ in terms of the Hausdorff spacetimes M_i.

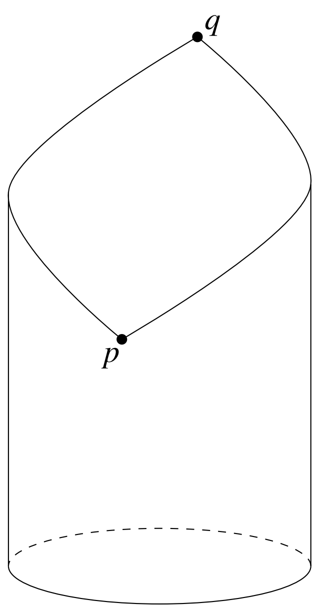


Figure 5.8: The region $\overline{A}$ for the non-Hausdorff Trousers space **T**. When constructing **T** we glue two cylinders together everywhere in A, yet leave the future component of ∂A unidentified. In the quotient space **T**, this yields two Hausdorff-inseparable copies of the lightlike future of the point p.

Lemma 5.4.1. *For any two points $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\boldsymbol{T}$, we have that $\boldsymbol{x} \leq \boldsymbol{y}$ if and only if $\phi_1^{-1}(\boldsymbol{x}) \leq \phi_1^{-1}(\boldsymbol{y})$ or $\phi_2^{-1}(\boldsymbol{x}) \leq \phi_2^{-1}(\boldsymbol{y})$, or both.*

In this sense, we can see that the non-Hausdorff Trousers space **T** naturally inherits the causal structures of M_1 and M_2. In particular, we see that the causal structure $(\mathbf{T}, \leq)$ naturally inherits a poset structure from the causality of the cylinders M_i. [9]

Theorem 5.4.2. *The causal relation $\leq$ on $\boldsymbol{T}$ is a transitive, reflexive and antisymmetric relation.*

As a brief aside, we remark that the above appears to hold more generally – if we consider some category of non-necessarily-Hausdorff spacetimes with isometric embeddings as morphisms, then the mapping of spacetimes to causal sets may be considered as a covariant functor into the category of partially-ordered sets. In this case, it is likely that the above argument generalises directly to conclude that this “causal functor" takes colimits to colimits, so that we can describe any causal structure on a non-Hausdorff spacetime as a colimit of posets.

Aside from the poset structure of $(\mathbf{T}, \leq)$, the non-Hausdorff Trousers space satisfies other nice causal properties, which do *not* hold in the Hausdorff Trousers case. First, recall from pointset topology that the compactness of a set is preserved under continuous maps. Applied to our context, we may use the canonical maps $\phi_i : M_i \to \mathbf{T}$ to send causal diamonds $J^+(x) \cap J^-(y)$ from either M_i into **T**. Since the Lorentzian cylinder is globally hyperbolic, Lemma 5.4.1 allows us to conclude that the causal diamonds $J^+(\mathbf{x}) \cap J^-(\mathbf{y})$ are compact

[9] In fact, it appears that a more precise categorisation of the features of the causal space $(\mathbf{T}, \leq)$ is that it manifests as a model of the logico-mathematical theory BST_{NF} of [9, 82, 109]. This is an order-theoretic axiomatisation of a certain type of indeterminism, and runs somewhat parallel to the considerations of the causal set theorists.

in **T**. Moreover, one can readily find spacelike slices of **T** whose domain of dependence equals the whole space. In these two senses, the non-Hausdorff Trousers space is causally better-behaved than the Hausdorff Trousers. In fact, these properties are strongly reminiscent of global hyperbolicity – a connection that should be more fully explored by studying field equations on the non-Hausdorff spacetime.

5.4.2 The Gravitational Action

We will continue our discussion in a similar spirit to Section 5.3.3, that is, we will first describe a non-Hausdorff action in Euclidean signature, and then we will move on to the subtleties of Lorentzian signature. In each case we will begin with a general treatment of non-Hausdorff manifolds, before evaluating the derived actions for the case of the non-Hausdorff manifold **T**, equipped with the metric of appropriate signature. As such, throughout this section we will we assume that **M** is a non-Hausdorff two-dimensional manifold satisfying the topological assumptions of Theorem 5.3.4 regarding the Euler characteristics of A and $\overline{A}$.

Euclidean Gravity

For a two dimensional Hausdorff manifold M with possible boundary ∂M, vacuum Euclidean gravity may be fully described via the Gauss-Bonnet action:

$$\mathscr{S}(M,h) = \frac{1}{2\kappa}\int_M R\, dA + \frac{1}{\kappa}\int_{\partial M} k d\gamma, \tag{5.36}$$

where here any potential corner contributions have been absorbed into the Gibbons-Hawking-York boundary term. According to the Gauss-Bonnet theorem for Riemannian surfaces, we know that $\mathscr{S}(M,h) = \frac{2\pi}{\kappa}\chi(M)$, and it is in this sense that two-dimensional gravity is a purely topological theory.

If we were to proceed in a similar manner for a non-Hausdorff action, then it would seem natural to maintain the topological nature of the theory and define the action according to a Euclidean version of the non-Hausdorff Gauss-Bonnet theorem. For a non-Hausdorff manifold **M** constructed via Theorem 5.2.2 and endowed with a Riemannian metric **h**, we write:

$$\mathscr{S}(\mathbf{M},\mathbf{h}) := \frac{1}{2\kappa}\int_{\mathbf{M}} R dA + \frac{1}{\kappa}\int_{\partial\mathbf{M}} k d\gamma - \frac{1}{\kappa}\int_{\mathsf{Y}} k d\gamma. \tag{5.37}$$

By construction, the set $\mathsf{Y} = \overline{A}\backslash A$ is precisely one-half of the Hausdorff-violating submanifold sitting inside **M**. This can be seen as a sort-of “interior surface term" or perhaps an “internal boundary component", the dynamics of which are captured by an extra Gibbons-Hawking-York term of the appropriate sign.

Aside from a recreating a topological theory, the action $\mathscr{S}$ may also be justified according to a variational principle. Indeed, if we were to vary $\mathscr{S}$ with respect to the Riemannian metric

$\mathbf{h}$, then we are left with the following:

$$\delta_{\mathbf{h}}\mathscr{S} = \delta_{\mathbf{h}}\left(\int_{\mathbf{M}} RdA + \int_{\partial\mathbf{M}} kd\gamma - \int_{\mathsf{Y}} kd\gamma\right) \tag{5.38}$$

$$= \delta_{\mathbf{h}}\Big(\int_{M_1} RdA + \int_{M_2} RdA - \int_{\overline{A}} RdA + \int_{\partial M_1} kd\gamma + \int_{\partial M_2} kd\gamma - \int_{\partial A} kd\gamma - \int_{\mathsf{Y}} kd\gamma\Big) \tag{5.39}$$

$$= \delta_{h_1}\left(\int_{M_1} RdA + \int_{\partial M_1} kd\gamma\right) + \delta_{h_2}\left(\int_{M_2} RdA + \int_{\partial M_2} kd\gamma\right) - \delta_{h_{\overline{A}}}\left(\int_{\overline{A}} RdA + \int_{\partial\overline{A}} kd\gamma\right) \tag{5.40}$$

$$= 0 + 0 - 0 \tag{5.41}$$

where here we have used the same delineations of the boundary as in Section 5.3.3 and have suppressed the overall factors of κ^{-1} for readability. Observe that in the above expression we needed to include the additional geodesic curvature of the submanifold Y, since otherwise we are left with incomplete boundary data for the subspace $\overline{A}$.

Now, consider the non-Hausdorff Trousers space $\mathbf{T}$ as defined previously, but this time equipped with some Riemannian (rather than Lorentzian) metric $\mathbf{h}$, defined according to our general discussion in Section 5.2.4. Topologically, $\mathbf{T}$ consists of a pair of cylinders M_i glued along a semi-open cylinder $A \cong S^1 \times [0,1)$. These spaces all retract onto the circle, so their Euler characteristics vanish. According to Theorem 5.3.3, the Euclidean action (5.37) for $(\mathbf{T},\mathbf{h})$ may then be evaluated as

$$S(\mathbf{T},\mathbf{h}) = \frac{1}{2\kappa}\int_{\mathbf{M}} RdA + \frac{1}{\kappa}\int_{\partial\mathbf{M}} kd\gamma - \frac{1}{\kappa}\int_{\mathsf{Y}} kd\gamma \tag{5.42}$$

$$= \frac{2\pi}{\kappa}\chi(\mathbf{T}) \overset{(5.3.3)}{=} \frac{2\pi}{\kappa}(\chi(M_1) + \chi(M_2) - \chi(\overline{A})) = \frac{2\pi}{\kappa}(0+0-0) = 0. \tag{5.43}$$

In short, this means that the non-Hausdorff Trousers space is flat, which agrees with the fact that $(\mathbf{T},\mathbf{h})$ is locally isometric to the flat cylinder by construction. We may conclude from this that in the Euclidean theory, the non-Hausdorff Trousers space will have a larger contribution to the path integral than the Euclidean version of the Hausdorff Trousers space.

In fact, if we consider a broader path integral that sums over all topologies interpolating between all types of boundary components, then we are left with the observation that the non-Hausdorff Trousers space contributes to the Euclidean path integral as strongly as the cylinder. Moreover, it appears as though one can make arbitrarily-complicated configurations of non-Hausdorff cylinders that remain globally flat, e.g. by adding extra legs to the Trousers, which will similarly contribute with equal strength to the path integral. It seems that the only way to ensure the suppression of non-Hausdorff Trousers in Euclidean gravity (relative to the cylinder) would be to artificially introduce an extra term that tracks Hausdorff-violation:

$$\exp\{-\mathscr{S}(\mathbf{M},\mathbf{g})\} \longrightarrow \exp\{-\mathscr{S}(\mathbf{M},\mathbf{g}) - \alpha(\mathbf{M})\}, \tag{5.44}$$

where here α ought to vanish for Hausdorff manifolds and be strictly positive otherwise.

Lorentzian Gravity

In Lorentzian gravity, we would like to define the action as in (5.37), where here we compute the curvature and volume form according to the Lorentzian metric instead. We may again motivate this choice by the topological nature of the action, and again the variation of this action will vanish provided we include the additional Gibbons-Hawking-York term for the Hausdorff-violating submanifold. However, in contrast to the Euclidean setting, we now have the additional subtlety that the corner terms of the Hausdorff-violating submanifold need to be computed using Lorentzian turning angles.

We will illustrate this for the non-Hausdorff Trousers space $(\mathbf{T},\mathbf{h})$. Here there are two special points in $\overline{A}$ that are used in the construction of $\mathbf{T}$. These are p, the initial pointlike source of topology change, and q, which is the final point of topology change. Observe that there are two future-directed null rays connecting p to q, as pictured in Figure 5.8, which makes the subset Y of $\overline{A}$ a boundary consisting of piecewise-geodesic null rays. Expanding out the Lorentzian version of equation (5.37), we have:

$$\mathscr{S}(\mathbf{T},\mathbf{g}) = \frac{1}{2\kappa}\int_{\mathbf{M}} RdA + \frac{1}{\kappa}\int_{\partial\mathbf{M}} kd\gamma - \frac{1}{\kappa}\int_{\mathsf{Y}} kd\gamma \tag{5.45}$$

$$= \frac{1}{2\kappa}\left(\int_{M_1} RdA + \int_{M_2} RdA - \int_{\overline{A}} RdA\right) + \frac{1}{\kappa}\left(\int_{S^1} kd\gamma + \int_{S^1} kd\gamma + \int_{S^1} kd\gamma\right) \tag{5.46}$$

$$-\frac{1}{\kappa}\left(\int_{\gamma_1} kd\gamma + \int_{\gamma_2} kd\gamma + \sum\theta_{ext}\right), \tag{5.47}$$

where here the γ_i denote the two null rays connecting p to q in $\overline{A} \subset M_1$. The total scalar curvature terms above will all vanish, since each of the spaces involved is a flat cylinder. Moreover, all of the geodesic curvature terms will also vanish, since the copies of S^1 on the boundary of $\mathbf{T}$ are all spacelike geodesic, and both the γ_i are null geodesics from p to q in $\overline{A}$. Thus the total action reduces to the term $\sum\theta_{ext}$, which here is a sum of the turning angles of $\mathsf{Y} = \overline{A}\backslash A$ at p and q.

Since both curves in Y are null rays, we are left with a computation of turning angles using the null angular formulae of (5.19)–(5.22). Assuming that we may select the sign of the $i\varepsilon$-regulariser for the tangent spaces $T_p\overline{A}$ and $T_q\overline{A}$ of the corners p and q independently, we are left with four possible combinations. These are:

(i) $T_p\overline{A}$ and $T_q\overline{A}$ both use regulariser $+i\varepsilon$,

(ii) $T_p\overline{A}$ and $T_q\overline{A}$ both use regulariser $-i\varepsilon$,

(iii) $T_p\overline{A}$ uses regulariser $+i\varepsilon$ and $T_q\overline{A}$ uses regulariser $-i\varepsilon$,

(iv) $T_p\overline{A}$ uses regulariser $-i\varepsilon$ and $T_q\overline{A}$ uses regulariser $+i\varepsilon$.

For the first two cases, the two corner contributions to $\mathscr{S}(\mathbf{T},\mathbf{g})$ have equal strength but opposite sign, so they cancel and the total action reduces to zero. Put differently, cases (i) and (ii) will correspond to global Wick rotations for a triangulation of $\overline{A}$, and thus fall within the scope of the Gauss-Bonnet theorems 5.3.1 and 5.3.2. The Euler characteristic of $\mathbf{T}$ being zero then ensures that $\mathscr{S}(\mathbf{T},\mathbf{g})$ vanishes, and we again encounter the same issue as in the Euclidean case.

Alternatively, we may also consider cases (iii) and (iv): if we were to pick regularisers with opposite signs, the sum of corner terms would not cancel. Thus we may *not* apply the Gauss-Bonnet theorem in order to evaluate the action. Instead, we are left with actions taking the values

$$\mathscr{S}^{(iii)}(\mathbf{T},\mathbf{g}) = +\frac{\pi i}{\kappa} \text{ and } \mathscr{S}^{(iv)}(\mathbf{T},\mathbf{g}) = -\frac{\pi i}{\kappa}. \tag{5.48}$$

The former will lead to a suppression of the non-Hausdorff Trousers space in the schematic path integral (5.1), and the latter will lead to an enhancement. It seems reasonable, therefore, to suggest that we select regularisers of opposite sign, according to case (iii) above. For future reference, we state this here in more general form.

$$\operatorname{sgn}\big(i\varepsilon(x)\big) = \begin{cases} +i\varepsilon & \text{if } x \text{ corresponds to an initial point of topology change} \\ -i\varepsilon & \text{if } x \text{ corresponds to a final point of topology change} \end{cases} \tag{5.49}$$

There is an obvious and intentional similarity between our angular prescription and that of Sorkin/Louko [67]. Perhaps of interest is the difference in overall value of the resulting action – in [67] the action of the Hausdorff Trousers space is computed to be $\frac{2\pi i}{\kappa}$ (with appropriate sign convention), and here, we obtain an overall strength of $\frac{\pi i}{\kappa}$. Thus, if simultaneously included within the same path integral, it appears as though the non-Hausdorff Trousers space will enjoy a weaker suppression factor relative to its Hausdorff counterpart.

5.4.3 More S^1 Transitions

As mentioned in the introduction, a generally unsung feature of Sorkin's angular convention is that any more transitions other than the Trousers spaces will yield a further dampening within the path integral. Indeed, adding an extra genus (which topologically amounts to taking a connected sum with the torus) changes the Euler characteristic by:

$$\chi(M\#T^2) = \chi(M) + \chi(T^2) - \chi(S^2) = \chi(M) - 2, \tag{5.50}$$

so adding any extra genera to the bulk of a Trousers space will continually decrease its total contribution to the path integral. In this sense, according to the same-sign convention for $i\varepsilon$, any more elaborate branching will be further suppressed in the path integral. Alternatively, if we were to pick the other sign of the regulator, then we are left with the observation that transitions from Σ_1 to Σ_2 with arbitrarily-many genera will have the highest probability of occurring. With this in mind, it makes sense to pick the sign that entails the suppression of these higher-genus transitions.

In the non-Hausdorff case, we still have this genus complication, as well as the possibility

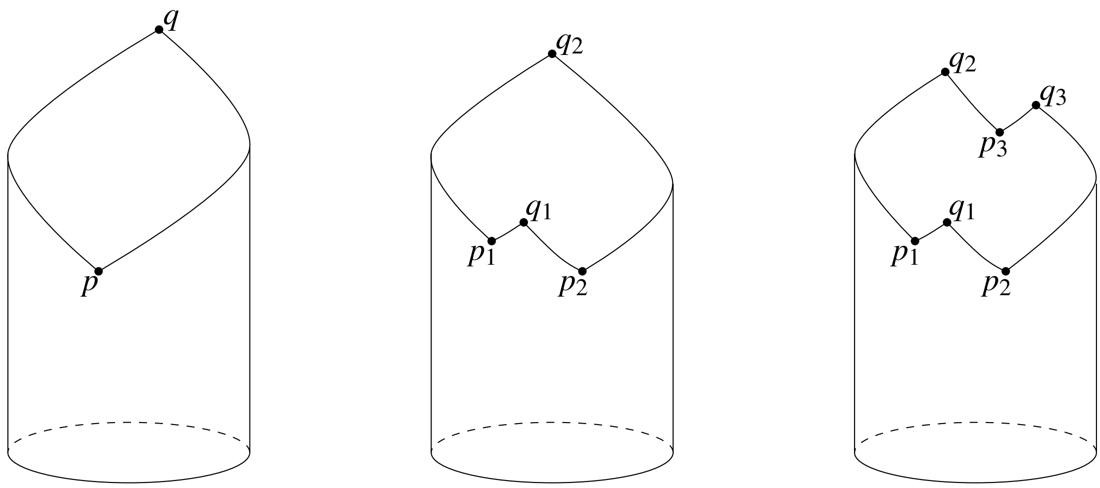


Figure 5.9: The region $\overline{A}$ (left), together with more elaborate models. Based on Euler characteristic alone, there is no distinction between these three spaces. However, due to our angular convention (5.49) the total action will increase in multiples of πi.

that the topology of Σ_1 may change from multiple spacelike-separated sources. We will now treat these two possible complications separately.

Transitions with extra pointlike sources

For a collection $p_1, \cdots, p_n$ of spacelike separated points on the cylinder, we may construct a non-Hausdorff manifold by gluing two cylinders together along the open subset

$$B := M_1 \Big\backslash \bigcup_{\alpha=1}^{n} J^+(p_\alpha) \tag{5.51}$$

by taking f to be the identity map. The result will be a manifold that is homeomorphic to the non-Hausdorff Trousers space $\mathbf{T}$, but not isometric to it. The only difference is with the causal structure of the Hausdorff-violating submanifold Y – we will obtain a null boundary with a sort-of "zigzag" structure, as depicted in Figure 5.9.

When evaluating the action for this spacetime, we are again forced to consider some sort of angular convention for the turning angles of the extra null boundary Y. In this case, consistently applying the angular convention (5.49) will force an additional suppression of these spaces relative to $\mathbf{T}$. Precisely, if $\mathbf{T}_n$ is the non-Hausdorff Trousers space arising from n-many pointlike sources, then under our sign convention $\mathscr{S}(\mathbf{T}_n)$ will take value $n\pi i/\kappa$. Thus our more-elaborately branching models will naturally enjoy a larger suppression in the path integral (5.1).

Transitions with extra genera

It is also possible that the non-Hausdorff Trousers space may contain extra genera in the bulk, and these ought to be accounted for within the theory. In this situation we are combining both Morse-theoretic and Penrosian topology change, and there are many possible combinations to consider. Perhaps much more may be said about this particular regime, and there may well be limiting cases of interest, for instance, a non-Hausdorff branching point occurring precisely at the crotch singularity of a Trousers space. For simplicity's sake, we will assume that the future null cones of any pointlike sources of topology change do not contain any critical points such as the crotch singularity. Moreover, we will assume that **M** is a non-Hausdorff manifold built from Hausdorff manifolds M_1 and M_2 glued along order-preserving isometries, so that the Gauss-Bonnet theorem of 5.3.4 and the causal properties of Section 5.4.1 are still maintained.[10]

We will now rewrite the Lorentzian non-Hausdorff action in such a manner that we obtain a purely topological piece, as well as a piece that spoils the Gauss-Bonnet theorem. The former will arise from the action evaluated with the global $+i\varepsilon$ angular convention, while the latter will encode the effect of switching to our chosen angular convention (5.49).

Let $\tilde{\theta}_{ext}$ denote the turning angles of Y that would follow from the $+i\varepsilon$ convention, and let θ_{ext} denote the angles that follow from our convention (5.49). We can then write the Y boundary term as:

$$\int_{\mathsf{Y}} k d\gamma = \sum \int_{\gamma} k d\gamma + \sum \theta_{ext} \tag{5.52}$$

$$= \left(\sum \int_{\gamma} k d\gamma + \sum \tilde{\theta}_{ext} \right) + \left(\sum \theta_{ext} - \sum \tilde{\theta}_{ext} \right). \tag{5.53}$$

Now, at each corner, the quantity $\theta_{ext} - \tilde{\theta}_{ext}$ is either 0 or $-i\pi$, depending on whether the corner corresponds to an initial or final point of topology change, respectively. Thus, we have:

$$\int_{\mathsf{Y}} k d\gamma = \left(\sum \int_{\gamma} k d\gamma + \sum \tilde{\theta}_{ext} \right) - n\pi i\,, \tag{5.54}$$

where n is the number of pointlike sources of topology change. Plugging this into the action and using the Gauss-Bonnet theorem (which applies for the angles $\tilde{\theta}_{ext}$), we get:

$$\mathscr{S}(\mathbf{M},\mathbf{g}) = \frac{1}{2\kappa} \int_{\mathbf{M}} R dA + \frac{1}{\kappa} \int_{\partial \mathbf{M}} k d\gamma - \frac{1}{\kappa} \int_{\mathsf{Y}} k d\gamma \tag{5.55}$$

$$= \frac{1}{\kappa} \left(\frac{1}{2} \int_{\mathbf{M}} R dA + \int_{\partial \mathbf{M}} k d\gamma - \sum \int_{\gamma} k d\gamma - \sum \tilde{\theta}_{ext} \right) + \frac{n\pi i}{\kappa} \tag{5.56}$$

$$= -\frac{2\pi i}{\kappa} \chi(\mathbf{M}) + \frac{n\pi i}{\kappa}\,. \tag{5.57}$$

We see that in distinction to our Euclidean discussion of Section 5.4.2, our convention (5.49) for Lorentzian turning angles at corners naturally introduces a term that suppresses non-Hausdorff topology changes, in addition to the usual suppression of Hausdorff ones (i.e. of

[10]Note that the desired poset structure and global time function still exist, provided that we glue along isometries that preserve time-orientation. In such a situation, the causal orderings of the M_i will be preserved under the canonical maps ϕ_i, and we may transfer the global time functions of the M_i to **M** using Theorem 5.2.3.

higher genera) via the Euler characteristic.

5.5 Conclusion

In this chapter we have provided a basic analysis of non-Hausdorff transitions between various copies of the circle, in a manner consistent with the original idea of Penrose [89]. According to the colimit constructions of Section 5.2, we saw that non-Hausdorff manifolds may be readily endowed with a smooth structure, as well as all tensor fields that one may require in differential geometry. All of these notions were defined without issue according to local definitions together with consistency conditions on the map $f : A \to M_2$ which provided the specifics of the gluing construction.

A central issue for non-Hausdorff manifolds is the passing from local to global data. This manifests in the non-existence of arbitrary partitions of unity (cf. Theorem 5.2.4), and in the inability to use the usual notion of integration. However, we circumvented this issue and saw in the integral formula of Theorem 5.2.5 that the global integral of a compactly-supported top form on a non-Hausdorff manifold satisfies a particular subadditivity formula. The formula 5.2.5 is almost identical to the standard subadditivity for Hausdorff manifolds, except that it required the crucial inclusion of the extra boundary term of $\partial\overline{A}$. According to the colimit construction of Section 5.2, this term integrates the extra piece of the Hausdorff-violating submanifold sat inside **M**.

Although innocuous from a measure-theoretic perspective, the inclusion of this "internal boundary" term into the global integral had some important consequences for the non-Hausdorff Gauss-Bonnet theorem. Our main observation of Section 5.3 was an extension of the usual Gauss-Bonnet formula into the non-Hausdorff Lorentzian setting. Theorem 5.3.4 ultimately showed that the total scalar curvature alone that does not equate to the Euler characteristic, but instead requires the extra integral counterterm, which in this context may be interpreted as the geodesic curvature of the extra Hausdorff-violating submanifold sitting inside the space.

As we saw in Section 5.4, this Gauss-Bonnet theorem ultimately suggests that there is no inherent way to guarantee the suppression of the non-Hausdorff Trousers space within a Euclidean path integral that sums over topologies. However, in Lorentzian signature we had the additional subtlety that now, in order to properly evaluate the action, we needed to compute the particular turning angles between adjacent null geodesics. Given that each of these turning angles is subject to its own convention for Lorentzian angles, we argued that there indeed exists a method for suppression of non-Hausdorff Trousers spaces in Lorentzian signature: we needed to intentionally spoil the Gauss-Bonnet theorem by selecting opposite sign conventions depending on the nature of these turning angles. Once this was done, the gravitational action took the correct sign in Lorentzian signature, which accounted the desired suppression.

In distinction to the Hausdorff case, there are two possible types of further branching: the addition of extra sources of pointlike change, and the inclusion of Hausdorff branching as per usual. In either case, we saw that more elaborate branching caused a further dampening in the path integral.

Future Work

We will now finish with some discussions of future work. As we saw in Sections 1.4 and 3.1, the non-Hausdorff Trousers space naturally inherits any Lorentzian metric placed on the ordinary cylinder. Moreover, we saw in Section 5.4.1 that the causal structure arising from the metric **g** on **T** is relatively well-behaved, thus on causal grounds alone one may suggest that the non-Hausdorff Trousers space should be reasonably included in the schematic path integral of Equation (5.1). However, it seems natural to suggest that a“physically reasonable" spacetime ought to be a suitable background upon which to define both spinors and quantum fields.

Regarding possible spin structures: it is well-known in the Hausdorff case that questions regarding the existence and uniqueness of spin structures on a given manifold may be best articulated within the language of Čech cohomology. Specifically, it can be shown that a given (Hausdorff) manifold admits a spin structure if and only the second Steifel-Whitney number w_2 vanishes. In addition, it can be shown that the number of inequivalent spin structures on a given manifold may be classified with the Čech cohomology group $\check{H}^1(M,\mathbb{Z}_2)$. For the non-Hausdorff case, the Čech cohomology is partially understood [85], though a full theory of non-Hausdorff spin geometry is yet to exist.

Regarding quantum fields: in the non-Hausdorff case the causal irregularities of the crotch singularity do not exist, so the abnormalities present in the Trousers space (discussed in [4, 16, 70]) will probably not exist either. Moreover, it appears as though the categorical phrasing of non-Hausdorff manifolds detailed in Section 5.2 may readily be applied in conjunction with the locally-covariant algebraic quantum field theory introduced in [15] to construct at least *some* class of quantum fields. However, it is not clear what the exact properties of such fields ought to be, and this should be an interesting avenue of inquiry.

Broadly speaking, our discussion in this chapter suggest that Penrose’s topology changing spacetimes may be appropriately furnished with the mathematical structures required for an inquiry into physics. From a causal perspective, these non-Hausdorff spacetimes appear to be better behaved than the ordinary Trousers space, ultimately due to the absence of any metric singularities. This good behaviour suggests that non-Hausdorffness may be a more appropriate model of topology change in Lorentzian signature. Once the spin geometry and quantum field theory of non-Hausdorff manifolds is well-understood, it would be very interesting to study the inclusion of these spacetimes within pre-existing theories that sum over topologies. In particular, our non-Hausdorff transitions may provide a geometric realization of interacting (i.e. splitting and joining) strings within Lorentzian signature.

Summary of Chapter

Throughout chapters 2,3 and 4 we developed the mathematical machinery needed to properly analyze non-Hausdorff manifolds. In particular, we saw that non-Hausdorff manifolds can be made smooth, and even endowed with metrics, the curvatures of which were integrable quantities. In this chapter, we extended these ideas into the setting of Lorentzian non-Hausdorff manifolds. Physically, we were motivated by the possibility of non-Hausdorff transitions within a path integral that sums over topologies interpolating between one circle and two. The standard picture employed the Trousers space, a two-dimensional almost-Lorentzian manifold exhibiting a causal irregularity at a single point. Motivated by this object, we defined the so-called *non-Hausdorff Trousers space*, which was formed by gluing together a pair of Lorentzian cylinders along the complement of the causal future of a single point.

Throughout this chapter we analysed various properties of the non-Hausdorff trousers. Interestingly, we saw that since it was formed as an adjunction of such causally well-behaved spaces in the first place, the non-Hausdorff trousers inherited many of the causal features of the Lorentzian cylinder. In particular, we saw that the non-Hausdorff trousers admits a globally-defined Lorentzian metric, in stark contrast to the Hausdorff trousers. After making some refinements to the Lorentzian version of the Gauss-Bonnet theorem, we then generalised this result to the setting of non-Hausdorff spacetimes. Again as in the Riemannian case seen in Chapter 4, here we saw that there is an additional term corresponding to the geodesic curvature of the Hausdorff-violating submanifolds inside our space. Ultimately, it was this total Lorentzian scalar curvature that motivated the correct non-Hausdorff gravitational action – the additional counterterm acted as a kind-of Gibbons-Hawking boundary term, and was required to make the variation of the action vanish.

Motivated by these geometric results, we then studied the potential inclusion of the non-Hausdorff trousers within a path integral summing over the aforementioned transitions. The result was positive: we provided a sign convention for the turning angles between Lorentzian null rays that would entail desired suppresion of the non-Hausdorff trousers within the path integral. This result mimicked the Hausdorff argument of Sorkin and Louko. Finally, as a consistency check, we showed that under our sign convention, more elaborate branching enjoyed a stronger suppression within the path integral.

Overall, the results in this chapter suggest that from a causal perspective, the non-Hausdorff trousers space is at least as well-behaved as its Hausdorff counterpart, if not superior due to the lack of causal irregularities. Although encouraging, we note that a further analysis is required before making bold claims regarding the overall superiority of non-Hausdorff topology change in Lorentzian signature.

Chapter 6

Conclusion

Throughout this thesis we have developed a working theory of non-Hausdorff manifolds, with an eye towards models of Penrosian topology-changing transition amplitudes within a naive $2d$ quantum gravity. Given the large gap in the literature, we first started with a description of the underlying topological features of non-Hausdorff manifolds, and then slowly walked through a development of the geometry required for the meaningful descriptions of non-Hausdorff transitions between copies of the circle, as a combination of the ideas of Penrose and Sorkin/Louko. We will now finish this thesis with a technical summary of our work, followed by a brief observation regarding future work.

Technical Summary

We began our inquiry by introducing a general formalism for non-Hausdorff manifolds in terms of adjunction spaces. This formalism generalised the observation of Haefliger/Reeb found in Figure 1.6 by providing a method for gluing together arbitarily-many topological spaces X_i along open subsets A_{ij} via continuous maps f_{ij} which encode the gluing. Importantly, we showed that the resulting quotient space is always the colimit of the diagram formed from the X_i, A_{ij} and f_{ij} in the usual topological category.

Throughout Sections 1.2 and 1.3 we showed that the gluing maps f_{ij} may preserve other topological features of the spaces X_i, provided some additional assumptions that depend on the features in question. In particular, we showed that whenever the X_i are topological manifolds, the resulting quotient space preserves the properties of being locally-Euclidean and second-countable, provided the gluing maps f_{ij} are open topological embeddings and there are countably-many spaces being glued. This prescription was slightly too general in order to guarantee a non-Hausdorff structure in the quotient space, so we provided an additional condition in Section 1.3.1 that ensures non-Hausdorffness.

In Section 1.3.2 we also showed a technical limitation to non-Hausdorff manifolds, that is, the non-existent of certain partitions of unity. As we demonstrated, the Hausdorff condition is rather crucial for the arbitrary existence of partitions of unity on a manifold, and consequently our non-Hausdorff manifolds are not subject to the expected level of expressivity. This is an important technical nuisance of non-Hausdorff manifolds, and one that will always need to be kept in mind when working with them.

Using the adjunction formalism of Chapter 1, we saw in the form of Theorem 2.3.10 that non-Hausdorff topological manifolds can always be formed by gluing together ordinary Hausdorff ones along homeomorphic open submanifolds. Moreover, we also confirmed that the images of these Hausdorff manifolds will exist as maximal open, Hausdorff, connected submanifolds of the non-Hausdorff quotient space. These subspaces, known as H-submanifolds in the literature [46, 77], were shown to satisfy the refined criterion $\partial V = \mathsf{Y}^V$, provided some additional assumptions on their simplicity. Overall, Chapter 1 can be seen as a completion of the observation of Haefliger and Reeb to the setting of non-Hausdorff topological manifolds in general dimension.

In the second chapter we extended our adjunction formalism into a suitable smooth category. In particular, in Section 2.2.2 we showed that *smooth* manifolds M_i can be glued along *smooth* open submanifolds M_{ij} via *smooth* gluing maps f_{ij} to yield a *smooth* non-Hausdorff manifold $\mathbf{M}$. This space $\mathbf{M}$ is again a colimit of the appropriate diagram, this time in the category of smooth, not-necessarily-Hausdorff manifolds. Thus armed with an appropriate notion of calculus, we then studied the resulting ring of smooth, real-valued functions $C^\infty(\mathbf{M})$. In the case of a finite colimit $\mathbf{M}$, in Section 2.2.3 we used an inductive argument to show that the space $C^\infty(\mathbf{M})$ is a *limit* of the algebras $C^\infty(M_i)$, known commonly as a fibre product.

Vector bundles over a non-Hausdorff manifold $\mathbf{M}$ were then described in two, essentially equivalent, manners: firstly, in Theorem 3.3.1 we saw that colimits of vector bundles $E_i \xrightarrow{\pi_i} M_i$ are well-defined, provided some extra bundle gluing maps $F_{ij} : E_{ij} \to E_j$ are provided. A sort-of converse was also proved: it was shown in Theorem 3.3.3 that *any* vector bundle $\mathbf{E}$ over a non-Hausdorff manifold $\mathbf{M}$ can be realised as a colimit of bundles in the previous sense. We then extended the result of Theorem 3.2.14 to smooth cross-sections of an arbitrary bundle $\mathbf{E}$ over a finite colimit $\mathbf{M}$. Heuristically, it is shown in Theorem 3.3.6 that any section of any vector bundle over $\mathbf{M}$ can be seen as a collection of sections over Hausdorff subbundles that are "compatibly glued" on the regions E_{ij}. We then applied this to the particular case of metrics, and proved that our smooth non-Hausdorff manifolds admit Riemannian metrics, provided that we glue along isometries.

Results such as Theorems 3.2.14 or 3.3.6 appear to require some form of partition of unity in order to verify. This is somewhat true, but in a refined sense. Throughout Chapter 2 we assumed that our non-Hausdorff manifolds we built by gluing along regions A_{ij} that have pairwise diffeomorphic boundaries. Using the fact that continuous maps from a non-Hausdorff space into a Hausdorff one must send Hausdorff-violating points to the same value, this allowed us to confirm that functions (or sections) on non-Hausdorff manifold took the same value on Hausdorff-violating points. We could then apply partitions of unity *within each Hausdorff submanifold* at will, and use a different method of extension (namely 3.2.12 or 3.3.5) to describe functions on the whole non-Hausdorff manifold. This procedure allowed us to effectively circumvent the partition of unity problem for non-Hausdorff manifolds.

In the final section of Chapter 2 we were first introduced to an important theme regarding cohomology theories: it is not the case that cohomology groups can be “glued" like manifolds, bundles, functions or sections. Instead, the cohomology of a non-Hausdorff manifold exists in a particular relationship to the cohomologies of its Hausdorff constituents, known familiarly as *Mayer-Vietoris sequences*. In Section 2.4.1 we saw the first version of such a sequence, namely the Mayer-Vietoris sequence for non-Hausdorff Čech cohomology, and then generalised the setting in Section 2.4.2. Throughout the remainder of Chapter 2 we saw what these descriptions of cohomology bought us: in certain cases we were able to use non-Hausdorff Čech cohomology to classify the inequivalent line bundles that a non-Hausdorff manifold permits.

In Chapter 3 we took these cohomological considerations one step further by describing de Rham cohomology for our non-Hausdorff manifolds. Given the general non-existence of partitions of unity on a non-Hausdorff manifold, the standard descriptions via Mayer-Vietoris sequences were particularly intractable. In order to resolve this, we restricted our attention to the case in which the non-Hausdorff manifolds have gluing regions A_{ij} whose topological closures are smooth manifolds with boundary equipped with a small outward-pointing collar neighbourhood. From this assumption, the non-Hausdorff Mayer-Vietoris sequence took a particular form involving the de Rham cohomology groups of the form $H^q_{dR}(\overline{A_{ij}})$ instead of the usual ones. In the case that each of these regions A_{ij} were regular-open, we found that the expected Mayer-Vietoris sequences could be recovered. In this case, we then used a five-lemma argument to prove the non-Hausdorff version of de Rham's theorem.

Alongside our treatment of de Rham cohomology, in Chapter 3 we also derived the correct form of integration on a non-Hausdorff manifold. We opted for the so-called *integration over parameterizations*, which integrates differential forms over patches of space that intersect on measure-zero boundaries. Of particular interest was the expression in Theorem 5.2.5, where we expressed the non-Hausdorff integral of a differential form in terms of the Hausdorff submanifolds M_i and gluing regions A_{ij}. Importantly, such an integral satisfied a particular sub-additive equality, where in particular the integral over gluing regions included the boundaries ∂A_{ij}.

Although the inclusion of extra boundary terms ∂A_{ij} into the integral over $\mathbf{M}$ may appear to be innocuous by virtue of boundaries being measure zero sets, we saw that there were non-trivial consequences for certain topological results. In particular, in Lemma 4.5.3 we derived the correct version of Stoke's Theorem for non-Hausdorff manifolds, and saw a clear counterterm manifest. Finally, we saw a similar counterterm manifest in the Riemannian version of the Gauss-Bonnet theorem of 4.6.5. Again, we saw that the Euler characteristic of a non-Hausdorff manifold $\mathbf{M}$ is connected to the total scalar curvatures of its Hausdorff submanifolds subadditively, together with an additional counterterm for the boundaries ∂A_{ij}. Recalling from Chapter 1 that these boundary sets comprise half of the Hausdorff-violating portions of the manifold $\mathbf{M}$, the extra counterterm in the Euclidean Gauss-Bonnet theorem may be interpreted as the geodesic curvature of the extra Hausdorff-violating submanifolds sitting inside the non-Hausdorff surface $\mathbf{M}$.

Finally, in Chapter 4 we finished the thesis by including particular $2d$ non-Hausdorff manifolds within the path integrals of naive quantum gravity in Lorentzian signature. We started by once again extending our adjunction formalism, this time into Lorentzian geometry. In particular, we argued that any Lorentzian structures can be extended into a non-Hausdorff manifold **M**, provided that the gluing regions A_{ij} are open Lorentzian submanifolds and the gluing maps f_{ij} are isometric open embeddings. We also argued in Section 4.4.1 that certain causal properties of Hausdorff spacetimes will be preserved in this gluing process. Of particular interest was the preservation of global time functions, which could be "glued" to form a global time function on the overall non-Hausdorff spacetime via the exposition detailed in Chapter 2.

As a consequence of this causal analysis, we observed that a non-Hausdorff spacetime built in this manner will admit a globally non-vanishing Lorentzian metric. This is in stark contrast to the Trousers spacetime, and in this sense the non-Hausdorff transitions from one circle to two are strictly superior to their Hausdorff counterparts. Extending this observation, we also argued that the causal ordering on a non-Hausdorff spacetime may exhibit a poset structure, and thus ought to be included within any sum-over-histories-style path integral that includes Trousers spaces on causal grounds.

Later in Chapter 4 we derived and evaluated a gravitational action for non-Hausdorff spacetimes. Since we were working in $2d$, this action was essentially topological, and connected to the Euler characteristic via a Lorentzian version of the Gauss-Bonnet theorem. Our technique for proving the non-Hausdorff Gauss-Bonnet theorem in Lorentzian signature was a direct application of the techniques illustrated in Chapter 3 – we used a Mayer-Vietoris-style sequence to express the de Rham cohomology of the non-Hausdorff spacetime in terms of the cohomology groups of its Hausdorff submanifolds, applied the Hausdorff Gauss-Bonnet theorem to all of these pieces, and then collated the result into a total integral that expressed a total scalar curvature. As with the Euclidean case discussed in Chapter 3, the Lorentzian Gauss-Bonnet theorem included an additional counterterm computing the geodesic curvature of the Hausdorff-violating submanifold sitting inside our spacetime. In Lorentzian signature, we interpreted this counterterm as an extra Gibbons-Hawking piece of the action, attributed to an "interior boundary" of Hausdorff-violating null curves.

In the Hausdorff regime there are already some interesting subtleties regarding the Lorentzian version of the Gauss-Bonnet theorem – the overall sign of the action (which in turn encoded the suppression rules in the path integral) was actually a *choice*, that amounted to picking the correct convention for Lorentzian angles. This was known to Sorkin and Louko in [67], and the correct sign choice was used to justify the desired suppression of topology-changing spacetimes within their path integral. In the final portion of Chapter 4 we recreated this analysis in the non-Hausdorff case, by finding the correct sign angle that would entail the desired suppression. In the non-Hausdorff analogue of the Trousers space the bulk is flat, so the total strength of the action was determined by a sign convention on the corners of the Hausdorff-violating interior boundaries. After selecting the right sign conventions for all of the corners, we obtained the suppression rules (5.49). We then finished the chapter by confirming that any more-elaborate branching would yield a stronger suppression in the resulting path integral, provided our sign convention was respected.

Future Work

Throughout this thesis we have progressively demonstrated the mathematical and physical legitimacy of non-Hausdorff manifolds as a viable candidate for topology change in Lorentzian signature. As we saw in Chapter 4, there are non-Hausdorff transitions between copies of the circle which possess a globally non-singular Lorentzian metric. This metric, in turn, had some surprisingly well-behaved causal properties, as was seen in Section 4.4.1. In particular, we saw that the causal precedence relation for some non-Hausdorff spacetimes may be a poset, whilst possessing global time functions and compact causal diamonds, without being globally hyperbolic in the usual sense of the term.

Quantum Fields on a non-Hausdorff Background

With these desirable causal properties in mind, it will be interesting to perform a further qualitative comparison between the Hausdorff and non-Hausdorff versions of the Trousers space. As was mentioned in the introduction to this thesis, the ordinary Trousers space possesses a certain causal oddity due to the crotch singularity, which has four lightcones instead of the usual two. As was argued, this is essentially the best possible option for the Trousers space, since its Euler characteristic doesn't vanish. However, the non-Hausdorff Trousers space avoids these issues and therefore may be more physically-reasonable with regards to its resulting properties. In the future, it will be very interesting to investigate other physical properties, so as to build a more holistic comparison between the Hausdorff and non-Hausdorff Trousers.

In [4, 16, 70] quantum fields are placed on the Trousers space and their basic properties studied. These three analyses differ in their approach, however they all arrive to the same conclusion: the causal irregularity and the crotch singularity will cause an infinite burst of energy in the quantum field. In some sense this is to be expected, as quantum field theory in curved spaces typically generalise from Minkowski spacetime into globally-hyperbolic spacetimes. The reason for this is that globally-hyperbolic spacetimes possess a Cauchy surface, and field equations on this surface are particularly amenable to quantization. The famous *splitting theorem* of Geroch [40] tells us that any globally-hyperbolic spacetime cannot exhibit topology change, and conversely the Trousers space cannot be globally-hyperbolic. With this in mind, one may expect at least something to go wrong with quantum fields placed on the Trousers space.

In contrast to this, the non-Hausdorff Trousers space of Section 4.4 is built from two globally-hyperbolic spacetimes by gluing them together along a subspace A which is also globally-hyperbolic. Taking this as motivation, it seems as though there may be a method of studying QFT on the non-Hausdorff Trousers by "gluing" quantum fields defined on all of these globally-hyperbolic Hausdorff submanifolds. As we have seen several times throughout this thesis, this notion of gluing is best described categorically. Therefore, it seems as though a natural starting point for a non-Hausdorff quantum field theory ought to be the *locally covariant QFT* of [15]. In this framework, a quantum field theory is defined to be a functor that associates a C^*-algebra of operators to each (Hausdorff) globally-hyperbolic spacetime of a fixed dimension. This association, being functorial, is taken to be compatible with causal embeddings in such a way that the overall definition of a locally covariant QFT mimics the Haag-Kastler axioms of algebraic quantum field theory.

Given that the non-Hausdorff Trousers space is defined to be a colimit of Hausdorff globally-hyperbolic spacetimes, it seems as though the most appropriate thing to do would be to extend a fixed locally-covariant QFT to include these non-Hausdorff colimits. This would amount to finding an appropriate colimit of the C^*-algebras associated to the Lorentzian cylinder and the gluing region A, in such a way that the usual axioms of [15] are satisfied as best as they can be. Supposing that such an investigation is successful, it will be very interesting to determine whether or not quantum fields on the non-Hausdorff Trousers avoid this "infinite burst of energy" by virtue of avoiding the causal irregularities present in the ordinary Trousers space.

If successful, this investigation could provide compelling evidence for the superiority of Penrosian topology change over the Morse-theoretic Hausdorff analogue. In turn, this could suggest that non-Hausdorffness would be a more appropriate model for any two-dimensional Lorentzian theory of quantum gravity which sums over distinct topologies. This may include the worldsheets of closed bosonic string theory in Lorentzian signature, therefore suggesting that non-Hausdorffness is a better picture of interacting strings in a Lorentzian string theory. However, the validity of this speculation is unclear at present.

Discrete Quantum Gravity

Simplicial complexes, being Hausdorff, do not directly apply to the non-Hausdorff manifolds we have described throughout this thesis. However, it seems nonetheless plausible to appropriately generalise the notion of discreteness to include our non-Hausdorff spaces. In a sense, a prototype of this idea was discussed in Section 5.3, where we proved a non-Hausdorff Lorentzian Gauss-Bonnet theorem by decomposing everything into Hausdorff pieces, performing integrals there, and then recomposing everything into a meaningful global quantity. It may well be the case that a similar approach may work in general: perhaps the correct notion of discretization of a non-Hausdorff manifold is a simultaneous discretizations of its Hausdorff submanifolds, up to some consistency conditions on overlaps and Hausdorff-violating boundaries.

From a homological perspective, one would expect the correct notion of discretization to admit an equivalence to the smooth singular homology detailed in Section 4.5. Presumably, such an equivalence would arise via a derived Mayer-Vietoris sequence followed by a Five Lemma argument. In that sense, the correct space of "non-Hausdorff q-chains" ought to relate to ordinary simplicial complexes via a short exact sequence. In its simplest form, we may expect some kind of space $\Delta_q(\mathbf{M})$ of q-chains such that

$$0 \to \Delta_q(A_{12}) \xrightarrow{((\iota_{12})_*, -(f_{12})_*)} \Delta_q(M_1) \oplus \Delta_q(M_2) \xrightarrow{(\phi_1)_* + (\phi_2)_*} \Delta_q(\mathbf{M}) \to 0 \tag{6.1}$$

forms a short exact sequence. Presumably then, in direct analogy to the techniques used in Section 4.5, one might justify a non-Hausdorff simplicial homology via a duality with the de Rham cohomology described in Section 4.4.

Perhaps most interesting for physics would be an explicit description of the gluing procedures one needs in order to construct non-Hausdorff simplicial complexes from Hausdorff building blocks. Most likely, the space $\Delta_k(\mathbf{M})$ conjectured above ought to be constructible by relaxing the notion of gluing simplices by allowing for the case in which simplices may be glued along their interiors but not their boundaries. Alternatively, perhaps a better description could come from defining a new fundamental "non-Hausdorff simplex" which may be glued according to more familiar-looking rules. In either case, such a development, if successful, would be the correct starting point for a discrete version of the ideas outlined in Chapter 5. This could, in turn, serve as an interesting modification of the usual theory of Causal Dynamical Triangulations.

Higher-Dimensional Gravity

As mentioned in the introduction to Chapter 5, throughout this thesis we considered the simplest possible case of non-Hausdorff topology change, namely, that of two dimensions. An obvious extension of these ideas would be to extend the ideas of Chapter 5 into higher dimensions. Generally speaking, one can imagine a non-Hausdorff manifold $\mathbf{M}$ that is built from a pair of n-dimensional globally-hyperbolic spacetimes M_1 and M_2, glued together along a common region $A := M_1 \backslash J^+(p)$, in direct analogy to the non-Hausdorff trousers space of Section 5.4. Classically, the gravitational action for such a space ought to follow similarly to that of Equation (5.37), that is, some kind of schematic decomposition:

$$\mathscr{S}(\mathbf{M},\mathbf{g}) = \mathscr{S}_{Bulk}(\mathbf{M},\mathbf{g}) + \mathscr{S}_{Bdry}(\mathbf{M},\mathbf{g}) - \mathscr{S}_{NH}(\mathbf{M},\mathbf{g}) \tag{6.2}$$

where presumably the third, Hausdorff-violating piece of the action would be needed in order to ensure the variation of the overall action vanish – cf. (5.38). The final term in the above ought to be an action corresponding to a null internal boundary, so presumably the precise form would be similar to those found in say [87] or [66].

Perhaps most interesting is the potential recreation of suppression rules similar to those found in Chapter 5. Presumably, there would be a corner term present in the evaluation of the null piece $\mathscr{S}_{NH}(\mathbf{M},\mathbf{g})$, so perhaps a bespoke analytic argument is needed to guarantee the potential suppression of such models relative to their ordinary, unbranched comrades. That being said, such observations are speculative at best, and thus a further inquiry is required in order to settle the matter.

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Appendix A

A.1 Appendix to Chapter 3

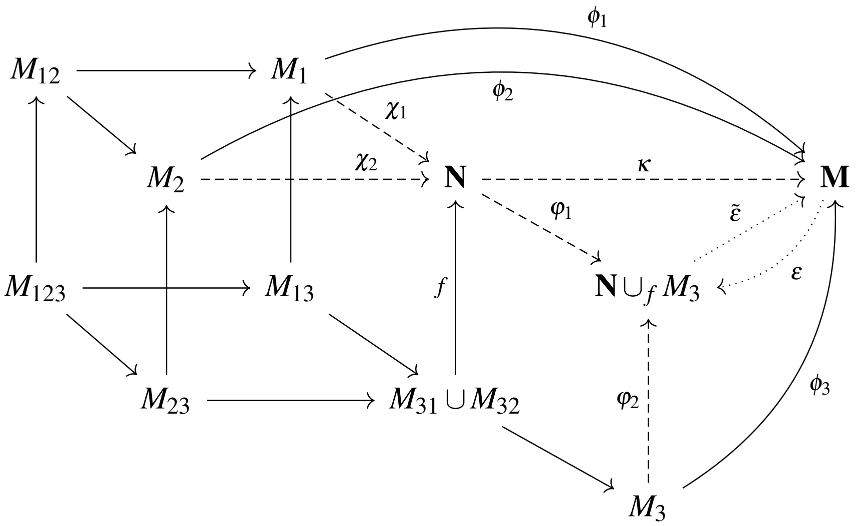
Lemma A.1.1. *Let* M *be a non-Hausdorff manifold built from three manifolds M_i. Then* M *is homeomorphic to the two-step adjunction space*

$$(M_1 \cup_{f_{12}} M_2) \cup_f M_3. \tag{A.1}$$

Proof. Let N denote the adjunction of M_1 and M_2 along the open subspace M_{12}. We will use φ_1 and φ_2 to denote the canonical embeddings of M_1 and M_2, and brackets $[\![\cdot,\cdot]\!]$ to denote points in N. Consider the subspace $A := M_{31} \cup M_{32}$, which is a subspace of M_3. We define a function $f : A \to \mathrm{N}$ by

$$f(x) = \begin{cases} [\![f_{31}(x), 1]\!] & \text{if } x \in M_{31} \\ [\![f_{32}(x), 2]\!] & \text{if } x \in M_{32} \end{cases} \tag{A.2}$$

The function f is well-defined since any element x in the intersection $M_{31} \cap M_{32}$ will satisfy the equality $f_{31}(x) = f_{21} \circ f_{32}(x)$ by condition **(A2)** of Definition 3.2.1. The broad schematics of our argument may be depicted as follows,



(A.3)

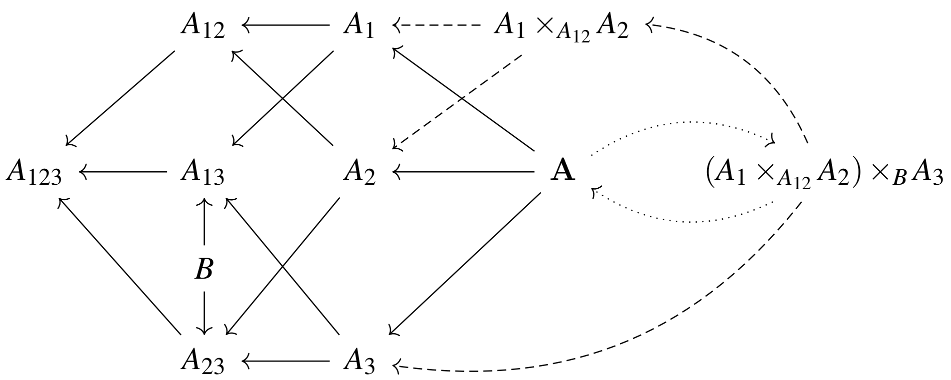
where here the maps χ_i and φ_i are all the canonical embedding maps of the various (binary) adjunction spaces, and κ is the map defined as in Section 3.2.1. Observe that all three M_i naturally embed into the adjunction space $\mathrm{N} \cup_f M_3$. Indeed: for M_1 and M_2 we may use the compositions $\varphi_1 \circ \chi_i$, and for M_3 we may simply use φ_2. Moreover, by construction these maps commute with all of the ι_{ij} and f_{ij} maps. As such, we may employ the universal property 3.2.2 of $\mathbf{M}$ to conclude that there exists a unique continuous map $\varepsilon : \mathbf{M} \to \mathbf{N} \cup_f M_3$ making the diagram commute. Explicitly, this map will act as:

$$\varepsilon([x,i]) = \begin{cases} \big[[x,1],a\big] & \text{if } x \in M_1 \\ \big[[x,2],a\big] & \text{if } x \in M_2 \\ \big[[x,3],b\big] & \text{if } x \in M_3 \end{cases}. \tag{A.4}$$

Similarly, we can use the maps $\kappa : \mathbf{N} \to \mathbf{M}$ and $\phi_3 : M_3 \to \mathbf{M}$ together with the universal property of the adjunction space $\mathbf{N} \cup_f M_3$ to conclude that there is a unique map $\tilde{\varepsilon} : \mathbf{N} \cup_f M_3 \to \mathbf{M}$. A simple exercise confirms that the maps ε and $\tilde{\varepsilon}$ are inverses of each other, which yields the desired homeomorphism. □

Theorem A.1.2. *Suppose that the diagram $\mathscr{F}$ appears in an Abelian category in inverse form. Then the general fibred product $\prod_{\mathscr{F}} A_i$ is isomorphic to an inductive limit.*

Proof. We proceed via induction. Suppose first that the indexing set I has size 3. We will show that $\mathbf{A}$ is isomorphic to the two-step construction $(A_1 \times_{A_{12}} A_2) \times_B A_3$, where $B := A_{13} \times_{A_{123}} A_{23}$. The following diagram depicts the schematics of the result.



(A.5)

We may use the universal property of B to construct two maps $\varepsilon : A_1 \times_{A_{12}} A_2 \to B$ and $\varepsilon : A_3 \to B$. Firstly, ε is defined by applying the universal property to the two maps that following the sequence $A_1 \times_{A_{12}} A_2 \to A_i \to A_{i3} \to A_{123}$. The map ε is defined by applying the universal property of B to the two maps following the sequence $A_3 \to A_{i3} \to A_{123}$. The two-step fibred product may then be defined as

$$(A_1 \times_{A_{12}} A_2) \times_B A_3 := \{(a,b) \in A_1 \times_{A_{12}} A_2) \times A_3 \mid \varepsilon(a) = \varepsilon(b)\}. \tag{A.6}$$

We may then use the same argument as that of the previous Lemma and use the universal properties of both $\mathbf{A}$ and $(A_1 \times_{A_{12}} A_2) \times_B A_3$ to construct the desired isomorphism. For the inductive argument, we can proceed in essentially the same manner as that of Theorem 3.2.6, except that this time we reverse all the arrows and take limits instead. □

The Cover-Independent Mayer-Vietoris Sequence

In Section 3.4.1 we determined the Mayer-Vietoris sequence for Čech cohomology relative to an open cover $\mathscr{U}$. We will now briefly justify a similar relationship between the Čech cohomologies that arise via direct limits. Suppose that $\mathscr{V} = \{V_\beta\}_{\beta \in B}$ is a refinement of the open cover $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$. This refinement can be formalised using a map $\lambda : B \to A$ defined by the property $V_\beta \subseteq U_{\lambda(\beta)}$ for all β in B. The refinement map may be passed onto the level of cochains by defining $\lambda(a_{\alpha_0 \cdots \alpha_q}) := a_{\lambda(\alpha_0) \cdots \lambda(\alpha_q)}$ for all locally-constant functions in a given cochain. There may be many such maps for the same refinement, however the following argument is independent of the particular choice of λ.

Lemma A.1.3. *The refinement map* $\lambda : \check{C}^q(\mathrm{M}, \mathscr{V}) \to \check{C}^q(\mathrm{M}, \mathscr{U})$ *commutes with both* Φ^* *and* $\iota_{12}^* - f_{12}^*$.

Proof. Let $\check{a}$ be a cochain in $\check{C}^q(\mathrm{M}, \mathscr{V})$. For any function $a_{\beta_0 \cdots \beta_q}$ in $\check{a}$, we have

$$\lambda(\phi_i^* \check{a})_{\beta_0 \cdots \beta_q} = \lambda(a_{\beta_0 \cdots \beta_q} \circ \phi_i) = a_{\lambda(\beta_0) \cdots \lambda(\beta_q)} \circ \phi_i = \phi_i^*(\lambda \check{a}). \tag{A.7}$$

Then

$$\lambda \circ \Phi^* \check{a} = (\lambda \circ \phi_1^* \check{a}, \lambda \circ \phi_2^* \check{a}) = (\phi_1^* \circ \lambda \check{a}, \phi_2^* \circ \lambda \check{a}) = \Phi^* \circ \lambda(\check{a}). \tag{A.8}$$

The case for ι_{12}^* and f_{12}^* are similar. □

A standard computation confirms that the refinement map λ commutes with the Čech differential [13]. This, together with the previous result, allows us to view λ as a chain map. We may then apply the Snake Lemma, together with the descended map λ, to obtain the following commutative relationship between the two cover-dependent Mayer-Vietoris sequences.

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\delta^*} & \check{H}^q(\mathrm{M}, \mathscr{U}) & \xrightarrow{\Phi^*} & \check{H}^q(M_1, \mathscr{U}^1) \oplus \check{H}^q(M_2, \mathscr{U}^2) & \xrightarrow{\iota_{12}^* - f_{12}^*} & \check{H}^q(M_{12}, \mathscr{U}^{12}) & \xrightarrow{\delta^*} & \cdots \\ & & \uparrow \lambda & & \uparrow \lambda & & \uparrow \lambda & & \\ \cdots & \xrightarrow[\delta^*]{} & \check{H}^q(\mathrm{M}, \mathscr{V}) & \xrightarrow[\Phi^*]{} & \check{H}^q(M_1, \mathscr{V}^1) \oplus \check{H}^q(M_2, \mathscr{V}^2) & \xrightarrow[\iota_{12}^* - f_{12}^*]{} & \check{H}^q(M_{12}, \mathscr{V}^{12}) & \xrightarrow[\delta^*]{} & \cdots \end{array} \tag{A.9}$$

Note that the map λ commutes with the connecting homomorphisms δ^* due to the naturality of the Snake Lemma. The cover-independent Čech cohomologies are direct limits that satisfy certain universal properties. Since the refinement maps λ form commutative diagrams pictured above, we may invoke these universal properties of directed limits (by using the right choice of compositions) in order to construct a unique Mayer-Vietoris sequence

$$\cdots \xrightarrow{\delta^*} \check{H}^q(\mathrm{M}) \xrightarrow{\Phi^*} \check{H}^q(M_1) \oplus \check{H}^q(M_2) \xrightarrow{\iota_{12}^* - f_{12}^*} \check{H}^q(M_{12}) \xrightarrow{\delta^*} \check{H}^{q+1}(\mathrm{M}) \to \cdots \tag{A.10}$$

as required.

A.2 Appendix to Chapter 4

A.2.1 Some Results Regarding Integrals

Lemma A.2.1. *Let ω be a compactly supported form on $\mathbf{M}$. Then $\phi_i^*\omega$ is compactly supported on M_i.*

Proof. Let ω be some compactly-supported form in $\mathbf{M}$. Since ϕ_i is continuous, the preimage of a closed set will still be closed in M_i. By definition, the pullback ω_i into M_i will have support

$$\operatorname{supp}(\omega_i) = \phi_i^{-1}(\operatorname{supp}(\omega)). \tag{A.11}$$

We now argue that this set is compact. Suppose that $\mathscr{U}$ is some open cover of the set $\operatorname{supp}(\omega_i)$ in M_i. We would like to construct some open cover $\mathscr{V}$ of $\operatorname{supp}(\omega)$ in $\mathbf{M}$ such that $\mathscr{V}$ coincides with $\phi_i(\mathscr{U})$ on M_i. This can be achieved by considering the sets

$$\{\phi_i(U) \mid U \in \mathscr{U}\} \cup \bigcup_{j\neq i} \{\phi_j(V) \mid V \cap M_{ij} = \overline{f_{ij}}(U) \text{ for some } U \in \mathscr{U}\} \cup \bigcup_{j\neq i} Int^{\mathbf{M}}(M_j \backslash M_i). \tag{A.12}$$

The above forms an open cover of the set $\operatorname{supp}(\omega)$, so by compactness there exists some finite subcover. Moreover, a smaller subcover of this set will cover the set $\phi_i(\operatorname{supp}(\omega_i))$, and we may pull this back to a finite subcover of $\mathscr{U}$. □

Theorem A.2.2. *The integral of a differential form ω over $\mathbf{M}$ satisfies:*

$$\int_{\mathbf{M}} \omega = \sum_{i=1}^{n} \left(\int_{M_i} \phi_i^* \omega \right) - \sum_{p=2}^{n} (-1)^p \left(\sum_{\substack{i_1,\cdots,i_p \in I \\ i_1<\cdots<i_p}} \int_{\overline{M_{i_1\cdots i_p}}} \phi_{i_1\cdots i_p}^* \omega \right) \tag{A.13}$$

Proof. Proceeding via induction, we may write $\mathbf{M}$ as the binary adjunction $\mathbf{N} \cup M_{n+1}$. Running through the same argument as Lemma 4.3.7, we have that

$$\int_{\mathbf{M}} \omega = \int_{\mathbf{N}} \omega + \int_{M_{n+1}} \omega - \int_{\overline{A}} \omega \tag{A.14}$$

Applying the induction hypothesis to **N** and A (viewed as a subspace of M_{n+1}) we have that

$$\int_{\mathbf{M}} \omega = \int_{\mathbf{N}} \omega + \int_{M_{n+1}} \omega - \int_{\overline{A}} \omega \tag{A.15}$$

$$= \sum_{i=1}^{n} \left(\int_{M_i} \phi_i^* \omega \right) - \sum_{p=2}^{n} (-1)^p \left(\sum_{\substack{i_1,\cdots,i_p \\ i_1<\cdots<i_p}} \int_{\overline{M_{i_1\cdots i_p}}} \phi_{i_1\cdots i_p}^* \omega \right) + \int_{M_{n+1}} \omega \tag{A.16}$$

$$- \sum_{i=1}^{n} \left(\int_{M_{i(n+1)}} \phi_i^* \omega \right) + \sum_{p=2}^{n} (-1)^p \left(\sum_{\substack{i_1,\cdots,i_p \\ i_1<\cdots<i_p}} \int_{\overline{M_{i_1\cdots i_p(n+1)}}} \phi_{i_1\cdots i_p(n+1)}^* \omega \right) \tag{A.17}$$

$$= \sum_{i=1}^{n+1} \left(\int_{M_i} \phi_i^* \omega \right) - \sum_{p=2}^{n+1} (-1)^p \left(\sum_{\substack{i_1,\cdots,i_p \\ i_1<\cdots<i_p}} \int_{\overline{M_{i_1\cdots i_p}}} \phi_{i_1\cdots i_p}^* \omega \right) \tag{A.18}$$

as required. □

A.2.2 Exactness of the Mayer-Vietoris Sequence for Manifolds with Boundary

In the proof of Theorem 4.4.2, we argued for the exactness of a Mayer-Vietoris sequence of the form:

$$0 \to \Omega^q(\overline{M_{13}} \cup \overline{M_{23}}) \to \Omega^q(\overline{M_{13}}) \oplus \Omega^q(\overline{M_{23}}) \to \Omega^q(\overline{M_{123}}) \to 0 \tag{A.19}$$

where we use that $\overline{M_{123}} = \overline{M_{13}} \cap \overline{M_{23}}$.

Lemma A.2.3. *Let $\overline{A}$ and $\overline{B}$ be codim-0 submanifolds with boundary, embedded within a Hausdorff manifold M. Suppose furthermore that*

- *A and B satisfy $\overline{A} \cap \overline{B} = \overline{A \cap B}$, and*
- *$A \cap B$ is a codim-0 submanifold of M with boundary.*

Then the following is a short exact sequence.

$$0 \to \Omega^q(\overline{A} \cup \overline{B}) \xrightarrow{r} \Omega^q(\overline{A}) \oplus \Omega^q(\overline{B}) \xrightarrow{\iota_A^* - \iota_B^*} \Omega^q(\overline{A \cap B}) \to 0 \tag{A.20}$$

Proof. The injectivity of r and the inclusion $Im(r) \subseteq \ker(\iota_A^* - \iota_B^*)$ follow as in the standard Mayer-Vietoris. Moreover, an extension by zero from $\overline{A \cap B}$ into $\overline{A}$ and $\overline{B}$ will ensure the surjectivity of the inclusions ι_A^* and ι_B^*.

The non-trivial step is in the proof that $\ker(\iota_A^* - \iota_B^*) \subseteq Im(r)$. Suppose that ω_A and ω_B are two forms on A and B respectively, such that $\iota_A^* \omega_A = \iota_B^* \omega_B$. We define the form ω on the union $A \cup B$ as:

$$\omega_{A\cup B}(x) = \begin{cases} \omega_A(x) & \text{if } x \in A \\ \omega_B(x) & \text{if } x \in B \end{cases}. \tag{A.21}$$

The equality $\iota_A^*\omega_A = \iota_B^*\omega_B$ guarantees that ω defines a function. We will now argue that ω is smooth. Recall first that a form on an embedded submanifold will be smooth if and only if it is the restriction of some smooth form defined on some larger open set. By assumption ω_A and ω_B are both smooth, which means that there exists

- some open set U containing A, and a form $\omega_U \in \Omega^q(U)$ such that $\omega_U = \omega_A$ on A, and
- some open set V containing B, and a form $\omega_V \in \Omega^q(V)$ such that $\omega_V = \omega_B$ on B.

We will now glue these two forms together to define a form ω on the union $U \cup V$. In order to do so, we need to confirm that ω_U and ω_V agree on the intersection $U \cap V$. In general this may not be true, so we will now modify our forms accordingly. Let ρ_U and ρ_V be a partition of unity of the union $U \cup V$, subordinate to the open cover $\{U, V\}$. We now define two forms

$$\tilde{\omega}_U := \omega_U - \rho_V(\omega_U - \omega_V), \text{ and } \tilde{\omega}_V := \omega_V + \rho_U(\omega_U - \omega_V). \tag{A.22}$$

These two forms will remain unchanged on the intersection $A \cap B$, however, they will cancel their differences on the set $(U \cap V)\backslash(A \cap B)$. Indeed, on the intersection $U \cap V$ we have:

$$\begin{aligned} \tilde{\omega}_U - \tilde{\omega}_V &= \omega_U - \rho_V(\omega_U - \omega_V) - (\omega_V - \rho_U(\omega_U - \omega_V)) && \text{(A.23)} \\ &= \omega_U - \omega_V - (\rho_V + \rho_U)(\omega_U - \omega_V) && \text{(A.24)} \\ &= \omega_U - \omega_V - (1)(\omega_U - \omega_V) && \text{(A.25)} \\ &= 0. && \text{(A.26)} \end{aligned}$$

Thus $\tilde{\omega}_U$ and $\tilde{\omega}_V$ agree on the intersection $U \cap V$. As such, we may glue these forms together to define a form $\omega_{U\cup V}$. By construction $\omega_{U\cup V}$ is a smooth form on the neighbourhood $U \cup V$ that restricts to $\omega_{A\cup B}$ on $A \cup B$. Thus $\omega_{A\cup B}$ is a member of $\Omega^q(A \cup B)$, and consequently $\ker(\iota_A^* - \iota_B^*) \subseteq Im(r)$. ☐

Theorem A.2.4. *Let $\{A_i\}_{i=1}^n$ be a collection of closed, codim-0 submanifolds of a Hausdorff M. Suppose furthermore that the equalities $\overline{A_{1\cdots m}} = \cap_{p=1}^m \overline{A_p}$ hold for all $m \leq n$. Then the sequence*

$$0 \to \Omega^q(A) \to \bigoplus_i \Omega^q(A_i) \to \bigoplus_{i<j} \Omega^q\left(\overline{A_{ij}}\right) \to \cdots \to \Omega^q\left(\overline{A_{1\cdots n}}\right) \to 0 \tag{A.27}$$

is exact for all q in $\mathbb{N}$.

Proof. Observe first that r^* is injective as usual, $\delta^2 = 0$ via combinatoric means (found in say [13]), and the last δ map is surjective via an extension by zero argument as before. Again the non-trivial step is to ensure that $\ker(\delta) \subseteq Im(\delta)$. Let ω be some element of $\bigoplus \Omega^q\left(\overline{A_{i_0\cdots i_p}}\right)$ such that $\delta\omega = 0$. Let U_i be a collection of open sets that cover the A_i, and let ρ_i be a partition of unity for the union $\bigcup U_i$, subordinate to the open cover $\{U_i\}$. We may extend each component $\omega_{i_0\cdots i_p}$ into a differential form

$$\tilde{\omega}_{i_0\cdots i_p} \in \bigoplus_{i_0<\cdots<i_p} \Omega^q(U_{i_0\cdots i_p}). \tag{A.28}$$

As with the previous Lemma, there is no guarantee that $\tilde{\omega}$ will be δ-closed as a differential form over the $U_{i_0\cdots i_p}$. To remedy this, we will modify each component of $\tilde{\omega}$ with a counterterm, by defining:

$$\eta_{i_0\cdots i_p} := \tilde{\omega}_{i_0\cdots i_p} - \sum_{r\neq i_0\cdots i_p} \rho_r(\delta\omega)_{ri_0\cdots i_p}. \tag{A.29}$$

Observe that each of these new forms will still restrict to $\omega_{i_0\cdots i_p}$ on the intersections $A_{i_0\cdots i_p}$, since the counterterm will vanish there. We now have that

$$\delta\eta_{i_0\cdots i_{p+1}} = \sum_\alpha (-1)^\alpha \eta_{i_0\cdots\hat{i_\alpha}\cdots i_p} \tag{A.30}$$

$$= \sum_\alpha (-1)^\alpha \left(\tilde{\omega}_{i_0\cdots\hat{i_\alpha}\cdots i_p} - \sum_{r\neq i_0\cdots i_p} \rho_r(\delta\omega)_{r\ i_0\cdots\hat{i_\alpha}\cdots i_p}\right) \tag{A.31}$$

$$= \sum_{\alpha=0}^{p} (-1)^\alpha \left(\tilde{\omega}_{i_0\cdots\hat{i_\alpha}\cdots i_p}\right) - \sum_\alpha (-1)^\alpha \left(\sum_{r\neq i_0\cdots i_p} \rho_r(\delta\omega)_{ri_0\cdots\hat{i_\alpha}\cdots i_p}\right). \tag{A.32}$$

Unpacking this latter term, we see that there are two possible cases:

1. if $r \neq i_{\hat{\alpha}}$. In this case, we see may unpack the summation of $\delta\tilde{\omega}$:

$$(\delta\omega)_{ri_0\cdots\hat{i_\alpha}\cdots i_p} = \sum_{\beta=0,\cdots,p+1 \text{ or}\beta=r} \omega_{ri_0\cdots\hat{i_\alpha}\cdots\hat{i_\alpha}\cdots i_p}. \tag{A.33}$$

   The only non-zero contribution will be when $\beta = r$, since otherwise we are omitting indices $i_{\hat{\alpha}}$ and $i_{\hat{\beta}}$ twice in the overall sum, with opposite sign.

2. if $r = i_{\hat{\alpha}}$. In this case we are adding back in the deleted index, so $\rho_r(\delta\omega)_{ri_0\cdots\hat{i_\alpha}\cdots i_p} = \rho_{i_{\hat{\alpha}}}(\delta\omega)_{i_0\cdots\hat{i_\alpha}\cdots i_p}$.

Putting this all together, we have that

$$\sum_\alpha (-1)^\alpha \left(\sum_{r\neq i_0\cdots i_p} \rho_r(\delta\omega)_{ri_0\cdots\hat{i_\alpha}\cdots i_p}\right) = \sum_{\alpha=0}^{p+1} \rho_{i_\alpha}\delta\tilde{\omega}_{i_0\cdots i_p} + \sum_{r\neq i_0\cdots i_{p+1}} \rho_r\delta\tilde{\omega}_{i_0\cdots i_p} \tag{A.34}$$

$$= \left(\sum_{i\in I}\rho_i\right)\delta\tilde{\omega}_{i_0\cdots i_p}. \tag{A.35}$$

We may thus conclude that $\delta\eta = 0$. Using the exactness of the Mayer-Vietoris sequence for open coverings (cf. [13, §8]), there exists some form θ in $\bigoplus\Omega^q(U_{i_0\cdots i_{p-1}})$ such that $\delta\theta = \eta$. The restriction of θ to the intersections $A_{i_0\cdots i_{p-1}}$ will then yield a smooth q-form in $\bigoplus\Omega^q(A_{i_0\cdots i_{p-1}})$ that maps to ω under δ. This confirms that $\ker(\delta) \subseteq Im(\delta)$, from which the result follows. □